$\boxed{\text{Example}}$ **Variation of Parameters** 11/2020

12a prob 4, $b = 4$. We want to solve

$u_t = u_{xx} - 4u$ on $[0, \pi]$, $t \geqq 0$; $u(0,t) = 0$, $u(\pi, t) = 1$;
$u(x, 0) = 0$.

We must convert boundary values at $x = 0$, $x = \pi$ to homogeneous type. Can put $\boxed{u = \frac{x}{\pi} + v}$ as one very simple choice. Notice that $v(0,t) = 0$, $v(\pi, t) = 0$.
Also notice that:

$$u_t = u_{xx} - 4u \iff v_t = 0 + v_{xx} - 4\left(\frac{x}{\pi} + v\right)$$
$$\iff v_t = (v_{xx} - 4v) - \frac{4}{\pi}x \, .$$

This is a non-homogeneous linear PDE:

$$v_t - v_{xx} + 4v = -\frac{4}{\pi}x \quad \text{for } \left\{\begin{matrix} 0 \leqq x \leqq \pi \\ t \geqq 0 \end{matrix}\right\}.$$

Since we have $v(0,t) = 0$, $v(\pi, t) = 0$, this is a situation amenable to the variation of param. method!!
"We simply blast through and see what we get."

Let
$$v(x,t) = \sum_{n=1}^{\infty} B_n(t) \underline{\underline{\sin(nx)}} \, . \qquad \{\text{FSS style}\}$$

Want

$$\sum_n B_n'(t)\sin(nx) - \sum_n B_n(t)(-n^2)\sin(nx) + 4\sum_n B_n(t)\sin(nx)$$
$$= \sum_{n=1}^{\infty} \left[B_n'(t) + (n^2+4)B_n(t)\right] \underline{\underline{\sin(nx)}} = -\frac{4}{\pi}x \, .$$

$\left\{\begin{array}{l}\text{The above is a formal calculation, \underline{presuming} that} \\ \text{everything is nicely convergent at least for } 0<x<\pi \\ \text{(and } t>0\text{). See famous footnote p. 117 (*).}\end{array}\right\}$

Know: $x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)$ on $0<x<\pi$, by 382.

Get:

$$\sum_{n=1}^{\infty}\left[B_n'(t)+(n^2+4)B_n(t)\right]\sin(nx) = \sum_{n=1}^{\infty}\frac{8}{\pi n}(-1)^n\sin(nx).$$

Deduce that $\boxed{\text{KEY ODE}}$

$$B_n'(t)+(n^2+4)B_n(t) = \frac{8}{\pi n}(-1)^n, \quad n \geqq 1.$$

Easy ODE solution gives

$$B_n(t) = \frac{8}{\pi n}(-1)^n\frac{1}{n^2+4} + C e^{-(n^2+4)t}$$

for each $n \geqq 1$.

Recall that $v(x,0) = -\frac{x}{\pi} \quad \left(= \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi n}\sin(nx)\right).$

Thus $B_n(0) = \frac{2}{\pi n}(-1)^n$. Hence, for each $n \geqq 1$,

$$C = \frac{2}{\pi n}(-1)^n - \frac{8(-1)^n}{\pi n(n^2+4)}.$$

Plug in to get:

$$\underline{\underline{v}} = \sum_{n=1}^{\infty}\left[\frac{8(-1)^n}{\pi n(n^2+4)} + \left(\frac{2(-1)^n}{\pi n} - \frac{8(-1)^n}{\pi n(n^2+4)}\right)e^{-(n^2+4)t}\right]\underline{\underline{\sin(nx)}}$$

$$= \sum_{n=1}^{\infty}\frac{8(-1)^n}{\pi n(n^2+4)}\underline{\underline{\sin(nx)}} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi n}\left\{1-\frac{4}{n^2+4}\right\}e^{-(n^2+4)t}\underline{\underline{\sin(nx)}}$$

$$= (\text{sum \#1}) + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi}\frac{n}{n^2+4}e^{-(n^2+4)t}\underline{\underline{\sin(nx)}}.$$

Finally, then,

$$u = \frac{x}{\pi} + v \quad \Longrightarrow$$

sum #1

$$u = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \sin(nx) + \sum_{n=1}^{\infty} \frac{8(-1)^n}{\pi n(n^2+4)} \sin(nx)$$

$$+ \sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi} \frac{n}{n^2+4} e^{-(n^2+4)t} \sin(nx) .$$

But, note that [as on preceding page]

$$\frac{2(-1)^{n+1}}{\pi n} + \frac{8(-1)^n}{\pi n(n^2+4)} = \frac{2(-1)^{n+1}}{\pi n}\left[1 - \frac{4}{n^2+4}\right]$$

$$= \frac{2(-1)^{n+1}}{\pi} \frac{n}{n^2+4} .$$

So, V.P. leads to:

$$u = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi} \frac{n}{n^2+4} \sin(nx)$$

$$+ \sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi} \frac{n}{n^2+4} e^{-(n^2+4)t} \sin(nx)$$

$\boxed{\text{this is } \dfrac{\sinh(2x)}{\sinh(2\pi)} \text{ by p. 382}}$

which <u>agrees with</u> p. 122 prob 4 ($b = 4$).

"math is consistent", GREAT!!