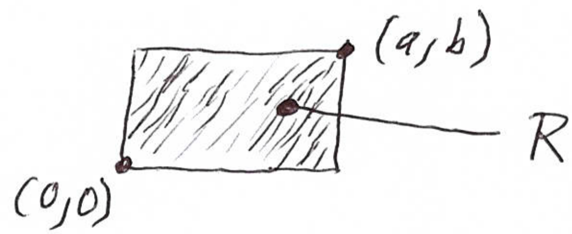
Consider regions D in $\mathbb{R}^2$. One writes

$$\langle f, g\rangle = \iint_D f(x,y)\, g(x,y)\, dA$$

to get the <u>standard</u> inner product for functions defined on D.

Consider the rectangle $R = [0,a] \times [0,b] \subseteq \mathbb{R}^2$ and keep $n \geq 1$, $\ell \geq 1$.



One easily checks that the double-indexed functions

$$\varphi_{n\ell} = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{\ell\pi y}{b}\right)$$

are <u>orthogonal</u> on rectangle R. That is,

$$\langle \varphi_{n\ell}, \varphi_{\tilde{n}\tilde{\ell}}\rangle = 0 \quad \text{anytime } (n,\ell) \neq (\tilde{n},\tilde{\ell}).$$

Notice too that

$$\langle \varphi_{n\ell}, \varphi_{n\ell} \rangle = \left(\frac{a}{2}\right)\left(\frac{b}{2}\right) .$$

It is evident that double-indices can be arranged in "single file" in <u>many</u> ways.
E.g.

$$\begin{matrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{matrix}$$

The <u>generalized</u> Fourier series of a general function $f(x,y)$ on R is,
BY DEFINITION,

$$f \sim \sum_{n\ell} b_{n\ell} \varphi_{n\ell} ,$$

where

$$b_{n\ell} = \frac{\langle f, \varphi_{n\ell} \rangle}{\langle \varphi_{n\ell}, \varphi_{n\ell} \rangle} .$$

on double Fourier sine series

Theorem

Let $f(x,y)$ be any C^4 function on R such that $f(x,y) = 0$ along the edge (i.e., boundary) of R. Define the Fourier coefficient

$$b_{n\ell} = \frac{\langle f, \varphi_{n\ell} \rangle}{\langle \varphi_{n\ell}, \varphi_{n\ell} \rangle} = \frac{2}{a}\frac{2}{b} \iint_R f(x,y) \sin\frac{n\pi x}{a} \sin\frac{\ell \pi y}{b}\, dA .$$

Introduce the constant $M_{xxyy} = \max_R |f_{xxyy}|$. We then have

$$|b_{n\ell}| \leq \left(\frac{a^2}{\pi^2}\right)\left(\frac{b^2}{\pi^2}\right) \frac{4\, M_{xxyy}}{n^2 \ell^2} \qquad \{n \geq 1, \ell \geq 1\}$$

and

$$\boxed{f(x,y) = \sum_{n,\ell} b_{n\ell} \sin\frac{n\pi x}{a} \sin\frac{\ell \pi y}{b} ,}$$

the latter with nice majorized (hence, UNIFORM + ABSOLUTE) convergence on R.

Proof

We only discuss the case $a=\pi$, $b=\pi$ since the general case can then be gotten by an elementary change of variable

$$x = \frac{a}{\pi}\mathscr{X}, \quad y = \frac{b}{\pi}\mathscr{Y} .$$

To prove the theorem (with $a=b=\pi$), we use 2 lemmas. very basic

Lemma 1.

Let $T(x)$ be any C^2 function on $[0,\pi]$. Assume that $T(0)=T(\pi)=0$. Let $\mathfrak{M} = \max_{[0,\pi]} |T''(x)|$. Then:

$$\int_0^\pi T(x)\sin(nx)\,dx = -\frac{1}{n^2}\int_0^\pi T''(x)\sin(nx)\,dx ;$$

$$\left|\int_0^\pi T(x)\sin(nx)\,dx\right| \leq \frac{\pi}{n^2}\mathfrak{M} .$$

Pf of Lemma

"Just integrate by parts twice." Alternatively, use

$$\int_a^b [uv'' - vu'']\,dx = [uv' - vu']_a^b$$

for C^2 fcns u and v. Take $u = \sin(nx)$, $v = T(x)$. Get:

$$\int_0^\pi [\sin(nx)T'' - T(-n^2)\sin(nx)]\,dx$$

$$= [\sin(nx)T' - Tn\cos(nx)]_0^\pi$$

$$= 0 - 0 = 0 \implies$$

$$\int_0^\pi T(x)\sin(nx)\,dx = -\frac{1}{n^2}\int_0^\pi T''(x)\sin(nx)\,dx.$$

OK Of course, we then get

$$\left|\int_0^\pi T(x)\sin(nx)\,dx\right| \leq \frac{1}{n^2}\int_0^\pi \mathcal{M}\,dx = \frac{\pi}{n^2}\mathcal{M}. \blacksquare$$

Keeping $T(x)$ as above, note that $T_{odd}(x)$ on $[-\pi,\pi]$ is plainly of type (abc). We therefore know that $FS(T_{odd}) \equiv FSS(T)$ is <u>very nice</u>. How nice??

Lemma 2.

Let $T(x)$ be as in Lemma 1. Form the FSS of T on $[0,\pi]$:

$$(*) \quad T(x) = \sum_{n=1}^{\infty} B_n \sin(nx) \,.$$

We then have

$$|B_n| \leqq \frac{2}{n^2}\mathfrak{M}$$

$\max_{[0,\pi]} |T''(x)|$

and a very explicit form of majorized convergence in the FSS (*).

Pf of Lemma

$B_n = \frac{2}{\pi}\int_0^{\pi} T(x)\sin(nx)\,dx$, so just apply Lemma 1. ▥

With Lemmas 1 and 2 in place, we are now ready the begin the "actual" proof.

Given our C^4 function $f(x,y)$ which vanishes on the edge of R. Freeze y for a second and form the FSS with respect to x. Get:

$$f(x,y) = \sum_{n=1}^{\infty} B_n(y)\sin(nx) \quad , \quad 0 \leq x \leq \pi$$

with

$$B_n(y) = \frac{2}{\pi}\int_0^{\pi} f(x,y)\sin(nx)\,dx \ .$$

Think Lemma 2 with $T(x) = f(x,y)$ $\{y$ held fixed$\}$.

Unfreeze y. For each $n \geq 1$, the function $B_n(y)$ is plainly a nice continuous function of y. In fact, Leibnitz's rule can be used to give integral formulas for $B_n'(y)$, $B_n''(y)$. Notice that $B_n(0) = 0$, $B_n(\pi) = 0$.

Notice further that, by Lemma 1,

$$B_n(y) = \frac{2}{\pi}\left(-\frac{1}{n^2}\right)\int_0^{\pi} f_{xx}(x,y)\sin(nx)\,dx \ .$$

This is an alternate formula for $B_n(y)$. Notice that Leibnitz's rule can also be applied twice in this formula! As such, we have both

$$B_n''(y) = \frac{2}{\pi}\int_0^{\pi} f_{yy}(x,y)\sin(nx)\,dx \quad \text{and}$$

$$B_n''(y) = \frac{2}{\pi}\left(-\frac{1}{n^2}\right)\int_0^{\pi} f_{xxyy}\sin(nx)\,dx \,.$$

In the second formula, notice that f_{xxyy} means $(f_{xx})_{yy}$. This is a specific continuous function of (x,y). Similarly f_{yy} is a specific continuous function of (x,y). Both formulas for $B_n''(y)$ therefore manifest a continuity in y {when y is treated as a parameter}.

The function $B_n(y)$ is therefore C^2 on $[0,\pi]$ in addition to having $B_n(0) = B_n(\pi) = 0$.

With n still fixed, we (now) form the FSS of $B_n(y)$ with respect to y in accordance with Lemma 2. [Compare p. 7 line 5.] This gives

$$B_n(y) = \sum_{\ell=1}^{\infty} C_{n\ell} \sin(\ell y) \quad , \quad 0 \leq y \leq \pi$$

with

$$C_{n\ell} = \frac{2}{\pi}\int_0^\pi B_n(y)\sin(\ell y)\,dy \qquad \boxed{n \text{ fixed}}$$

AND

$$|C_{n\ell}| \leq \frac{2}{\ell^2} \max_{[0,\pi]} |B_n''(y)| \quad .$$

The "fun" now starts!! Notice first that

$$C_{n\ell} = \frac{2}{\pi}\int_0^\pi B_n(y)\sin(\ell y)\,dy$$

p. 7

$$= \frac{2}{\pi}\int_0^\pi \left[\frac{2}{\pi}\int_0^\pi f(x,y)\sin(nx)\,dx\right]\sin(\ell y)\,dy$$

$$= \left(\frac{2}{\pi}\right)\left(\frac{2}{\pi}\right)\int_0^\pi\int_0^\pi f(x,y)\sin(nx)\sin(\ell y)\,dx\,dy$$

$$\Downarrow$$

$$c_{n\ell} = \left(\frac{2}{\pi}\right)\left(\frac{2}{\pi}\right)\iint_R f(x,y)\sin(nx)\sin(\ell y)\,dA$$

$a = \pi$
$b = \pi$

$$c_{n\ell} = b_{n\ell} \quad \text{on p. 3}.$$

Excellent!

Next,

$$B_n''(y) = \frac{2}{\pi}\left(-\frac{1}{n^2}\right)\int_0^\pi f_{xxyy}\sin(nx)\,dx \qquad \text{p. 8}$$

$$\Rightarrow \quad |B_n''(y)| \leq \frac{2}{\pi}\frac{1}{n^2}\int_0^\pi M_{xxyy}\cdot 1\,dx$$

$$|B_n''(y)| \leq \frac{2}{\pi}\frac{1}{n^2}(\pi M_{xxyy}) = \frac{2M_{xxyy}}{n^2}$$

$$\{\text{each } y\}$$

$$\Downarrow \qquad \text{p. 9}$$

$$|c_{n\ell}| \leq \frac{2}{\ell^2}\max_{[0,\pi]}|B_n''(y)| \leq \left(\frac{2}{\ell^2}\right)\frac{2M_{xxyy}}{n^2}$$

$$\Downarrow$$

$$|b_{n\ell}| \leq \frac{4M_{xxyy}}{\ell^2 n^2}.$$

p. 3
$a = b = \pi$

Great!

We now have:

$$f(x,y) = \sum_{n=1}^{\infty} B_n(y)\sin(nx) \qquad \text{p. } 7$$

$$= \sum_{n=1}^{\infty}\left(\sum_{\ell=1}^{\infty} c_{n\ell}\sin(\ell y)\right)\sin(nx) \qquad \text{p. } 9$$

$$= \sum_{n=1}^{\infty}\left(\sum_{\ell=1}^{\infty} b_{n\ell}\sin(\ell y)\right)\sin(nx) \qquad \text{p. } 10 \text{ top}$$

wherein

$$|b_{n\ell}| \leqq \frac{4\mathfrak{M}_{xxyy}}{\ell^2 n^2}.$$

Since

$$\sum_{n,\ell}\frac{1}{\ell^2 n^2} = \left(\sum_{\ell}\frac{1}{\ell^2}\right)\left(\sum_{n}\frac{1}{n^2}\right) = \left(\frac{\pi^2}{6}\right)^2 < \infty,$$

the double summation [[with any order of addition]]

p. 2 line 4

$$\sum_{n,\ell} b_{n\ell}\sin(\ell y)\sin(nx)$$

is convergent (indeed, absolutely convergent) and numerically equal to the iterated summation

$$\sum_{n=1}^{\infty}\left(\sum_{\ell=1}^{\infty} b_{n\ell}\sin(\ell y)\right)\sin(nx) . \qquad (*)$$

Thus, from p. 11, we get

$$f(x,y) = \sum_{n,\ell} b_{n\ell}\sin(nx)\sin(\ell y)$$

exactly as promised on p. 3. ▮

(*) This is a general property of absolutely convergent double summations $\sum_{n,\ell} w_{n\ell}$. It is akin to Fubini's theorem for double integrals $\iint_Q w(x,y)\,dA$ taken over Q, the first quadrant in $\mathbb{R}^2$.

The theorem on p. 3 is a kind of "foot-in-the-door" theorem.

By approximating 2-dimensional step functions on R (in the mean square sense!) by use of [finite sums of] C^4 functions which vanish on the edge of R, one is able to exploit the uniform convergence on p. 3 to establish the completeness of $\{\varphi_{n\ell}\}$ on p. 1 fairly quickly. $\boxed{\text{Hence: Parseval's formula.}}$

Following this, one can then work to establish the equality

$$f(x,y) = \sum_{n,\ell} b_{n\ell} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{\ell \pi y}{b}\right)$$

for classes of functions f more extensive than the one used in the Theorem on p. 3. There are a host of subtleties that arise in this.