

MATH 8280/5990 – SPRING 2016
INTROD. TO ANALYTIC NUMBER THEORY
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BRIEF TABLE OF CONTENTS

LECTURE 1: Introduction; very basic theorems and definitions; Chebyshev's theorem (part 1); statement of the prime number theorem.

LECTURE 2: $\pi(x) \log(x) \sim \psi(x) \sim \theta(x)$; Chebyshev's theorem (part 2); comments about improvements; list of dates concerning the PNT.

LECTURES 3 + 4: Review of some complex analysis, plus several high points involving Riemann-Stieltjes integrals.

LECTURE 5: More on Riemann-Stieltjes integrals; Abel's lemma; Abel's theorem on power series; complex logarithms; infinite products and their convergence properties; first results on the Riemann zeta function $\zeta(z)$ and its analytic continuation.

LECTURE 6: More on infinite products; Euler's identity; beginning estimates for $\zeta(z)$ in $\{\operatorname{Re}(z) > 0\}$; nonvanishing of $\zeta(z)$ along $\{\operatorname{Re}(z) = 1\}$.

LECTURE 7: The Abel summation lemma and related convergence tests; derivation of the Fourier series development of $x - [x] - \frac{1}{2}$; improved estimates for $\zeta(z)$ and $1/\zeta(z)$; Riemann's formula for $\psi_1(x)$; proof that $\psi_1(x) \sim x^2/2$.

LECTURE 8: Proof that $\psi(x) \sim x$, hence of the PNT; Fourier integral approach to $\psi_1(x)$; Euler-Maclaurin (version 1); boundedness of partial sums in the Fourier series of $x - [x] - \frac{1}{2}$.

LECTURES 9 + 10: Euler's formula for $\zeta(2k)$; the expansions of $\pi \operatorname{ctn}(\pi z)$ and $\sin(\pi z)$; Euler-Maclaurin (version 2); use of E-M to prove the analyticity of $\zeta(z)$ on $\mathbb{C} - \{1\}$; basic properties of $\Gamma(z)$, including Stirling's formula.

LECTURE 11: Review of Fourier series; Poisson summation formula; the Riemann ξ -function; the functional equation for ξ and ξ_0 ; related estimates.

LECTURE 12: Review of some complex function theory, including Jensen's formula, the Hadamard-Borel-Caratheodory lemma, canonical products, and the Hadamard factorization theorem.

LECTURE 13: More function theory; application to $\xi(s)$; infinite number of zeros; Riemann's formula for $\zeta'(s)/\zeta(s)$; the classical zero-free region for $\zeta(s)$.

LECTURE 14: Classical “big oh” estimates for $\psi(x) - x$ and $\pi(x) - \text{li}(x)$.

LECTURE 15: Preparations for deriving Riemann’s explicit formula for $\psi_1(x)$; the sliding partial fraction development of $\zeta'(s)/\zeta(s)$; the function $S(T)$ and its use in counting (à la Riemann) the number of zeros ρ up to height T .

LECTURE 16: The explicit formula for $\psi_1(x)$; relation to the prime number theorem; standard estimates for $\psi(x) - x$ and $\pi(x) - \text{li}(x)$ involving the “maximum abscissa” Θ .

LECTURES 17+18: The explicit formula for $\psi(x)$; applications to the size of $\psi(x) - x$.

LECTURES 19+20: The Perron summation formula with error term; the Möbius μ -function; an estimate for the summatory function $M(x)$; convergence of certain Dirichlet series involving $\mu(n)$; the Möbius inversion formula; elementary equivalence of $M(x) = o(x)$ and the prime number theorem.

LECTURE 21: Completion of the proof that PNT and $M(x) = o(x)$ are equivalent; the classical Dirichlet divisor problem bound; review of some basic properties of generalized Dirichlet series and “Dirichlet integrals”; Landau’s theorem on singular points; elementary Ω_{\pm} estimates for $\psi(x) - x$ and $\pi(x) - \text{li}(x)$ referencing $\Theta - \varepsilon$.

LECTURE 22: Preparations for Landau’s proof of Hardy’s theorem asserting an infinite number of zeros of $\zeta(s)$ along the critical line $\{\text{Re}(s) = \frac{1}{2}\}$.

LECTURE 23: Landau’s proof of the Hardy theorem. (Consult the Addendum in Lecture 24 for an improvement.)

LECTURE 24: Recollection of some complex function theory involving estimates of Phragmén-Lindelöf type; the Lindelöf μ -function for $\zeta(s)$; Littlewood’s formula concerning the zero-counting function $N(\sigma; T_1; T_2)$.

LECTURE 25: More on Lindelöf’s function $\mu(\sigma)$, also on generalized Dirichlet series; a quick rehash of Fourier transforms; development of a simple L_2 estimate for Dirichlet polynomials in the spirit of Hilbert’s inequality.

LECTURE 26: Some van der Corput-type estimates of exponential sums; the 1921 Hardy-Littlewood theorem on approximating $\zeta(s)$ by its partial sums.

LECTURE 27: The Bohr-Landau theorem that all but an infinitesimal proportion of the zeros of $\zeta(s)$ are located within $\{|\text{Re}(s) - \frac{1}{2}| < \varepsilon\}$.

LECTURE 28: D. J. Newman’s quick proof of the PNT; attached, for reference, Landau’s two-page 1932 *Göttinger Nachrichten* paper (giving the shortest known proof of the Wiener-Ikehara Tauberian theorem).

LECTURE 29: J. E. Littlewood’s 1914 Ω_{\pm} estimate for $\psi(x) - x$ proved using a variant of Ingham’s 1936 Fourier transform technique.

LECTURE 30: Connecting Euler's 1748 assertion that $\sum \frac{\mu(n)}{n} = 0$ to $M(x) = o(x)$ and the prime number theorem; some concluding remarks on $S(T)$ and Turing's Law (for checking the Riemann Hypothesis without ever leaving the line $\text{Re}(s) = \frac{1}{2}$).

ADDENDUM A: Pertaining to Lecture 30 ("an alternate ending").

ADDENDUM B: Pertaining to Lecture 28.

CLOSING COMMENTS: Assorted remarks concerning Lectures 1–30.