I began with a quick development of basic Fourier series—in a nonstandard way, i.e., via E-M summation.

\[ \langle f, g \rangle = \int_{a}^{b} f(x) \overline{g(x)} \,dx \]

On \([0,1]\) or any \([a_j, a_{j+1}]\):

\[ \langle \varphi_m, \varphi_n \rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \]

For

\[ \varphi_n(x) = e^{2\pi i nx}, \quad n \in \mathbb{Z}. \]

So, we have the usual idea of trying to write \(f\) "most of the time" as \( \sum_{n} c_n \varphi_n \), \( c_n = \langle f, \varphi_n \rangle \).

**Lemma**

\( f \in C[0,N] \). Assume \( f \) is only piecewise \( \leq 1 \) (not \( \equiv 1 \)).

Then we still have

\[
\frac{1}{2} f(0) + f(1) + \cdots + f(N-1) + \frac{1}{2} f(N) = \int_{0}^{N} f(x) \, dx + \int_{0}^{N} f(x) \beta(x) \, dx
\]

\[ \beta(x) = x - \lfloor x \rfloor - \frac{1}{2}. \]
\[ \text{Pr} \]

Begin as before

\[
f(1) + \cdots + f(N) = \int_0^N f(x) \, d\beta(x) \\
= \int_0^N f(x) \, d\left(x - \frac{1}{2} - \beta(x)\right) \quad (R-S) \\
= \int_0^N f(x) \, dx - \int_0^N f(x) \, d\beta(x).
\]

Split \( \int_0^N f(x) \, d\beta \) into chunks corresponding to corners of \( \mathcal{F}_0 \).

Then do the integration by parts and recombine. Ambiguous \( f' \) at a finite \( N \) of corners does not affect \( \int_0^N \beta \, f' \, dx \).

\[ \Rightarrow \text{All is fine.} \]

Take \( N=1 \). Assume \( f \in C([0,1]) \), piecewise \( C^1 \).

Hence, by lemma,

\[
\frac{1}{2} f(0) + \frac{1}{2} f(1) = \int_0^1 f \, dx + \int_0^1 f' \left( -\sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{2\pi n} \right) \, dx \\
= \int_0^1 f \, dx + \sum_{n=1}^{\infty} \int_0^1 f' \left(-\frac{\sin 2\pi nx}{2\pi n}\right) \, dx,
\]

the 2nd line by Lec 9, p. 14, Baby Fact.
Note that the error term after $N$ is

$$
\pm \int_0^1 f'(\beta - sn) \, dx
$$

i.e.,

$$
\text{ABS VALUE} \leq MN \int_0^1 |\beta - sn| \, dx
$$

$$
MN = \sup_{[0,1]} |f'|
$$

The $|\beta - sn|$ integral is an absolute expression, say $\omega_N$, and $\omega_N \to 0$ so $\mathcal{O}$

$$
|\text{Error}| \leq MN \omega_N
$$

Note too that

$$(n \geq 1)$$

$$
\int_0^1 f' \sin 2\pi nx \, dx = \frac{1}{2\pi in} \int_0^1 f'(e^{-2\pi i nx} - e^{2\pi i nx}) \, dx
$$

Write the last expr. as

$$
\frac{1}{2\pi in} \int_0^1 f' e^{-2\pi in x} \, dx + \frac{1}{2\pi in(-n)} \int_0^1 f' e^{2\pi i(-n)x} \, dx
$$
\[
\frac{1}{2\pi i} \int_{0}^{1} f(e^{-2\pi i n x}) \, dx
\]
\[
= \frac{1}{2\pi i} \int_{0}^{1} e^{-2\pi i n x} \, df \quad \text{(standard parts)}
\]
\[
= \frac{1}{2\pi i} \int_{0}^{1} e^{-2\pi i n x} f(x) \, dx
\]
\[
- \frac{1}{2\pi i} \int_{0}^{1} f(x) \, d(e^{-2\pi i n x})
\]
\[
= \frac{f(1) - f(0)}{2\pi i} + \int_{0}^{1} f(e^{-2\pi i n x}) \, dx.
\]

Similarly for \(-n\). Now add! Get:

\[
(\text{term } u) + (\text{term } -u) \equiv C_u + C_{-u}
\]

where

\[
C_k = \int_{0}^{1} f(e^{-2\pi i k x}) \, dx.
\]

So,

\[
\frac{1}{2} f(0) + \frac{1}{2} f(1) = C_0 + \sum_{n \to \infty} \sum_{n} C_n \left( \frac{\xi}{1} \right)^{n+2}.
\]
\[ \frac{1}{2} F(0) + \frac{i}{2} F(1) = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n \]

any \( f \in C[0,1] \), piecewise \( C^1 \).

\( \text{example} \)

**Ah, ah! This is really a Fourier series!**

\[ \lim_{N \to \infty} \sum_{n=-N}^{N} c_n e^{2\pi in} = 0 \]

The proof was just basic Egorov version II.

and

\[ x - \|x\| - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{\pi n} \]

\( x \in \mathbb{Z} \).

**NOTE THAT** error term for \( |x| > N \) is

\[ \leq M N \|

\]
**Initial Thm**

Given \( f \in C([0,1]), \) piecewise \( C^1 \).

Let \( c_k = \int_0^1 f(x) e^{-2\pi i k x} dx = \langle f, q_k \rangle \).

Then:
\[
\frac{1}{2} f(0) + \frac{1}{2} f(1) = \lim_{N \to \infty} \left\{ \sum_{k=-N}^{N} c_k e^{2\pi i k 0} \right\}
\]

\[|\text{Error}| \leq MwN.\]

**pf**
As above.

---

**Thm**

Let \( f \in C(\mathbb{R}), \) periodic 1, piecewise \( C^1 \).

Then:
\[
\lim_{N \to \infty} \sum_{k=-N}^{N} c_k e^{2\pi i k x} \rightarrow f(x) \text{ on } \mathbb{R}.
\]

**pf**
Fix any \( x_0 \in \mathbb{R} \). Consider \( g(x) = f(x + x_0) \) on \([0,1]\) in previous Thm. Note
\[
c_k(g) = \int_0^1 g(x) e^{-2\pi i k x} dx = \int_0^1 f(x + x_0) e^{-2\pi i k x} dx
\]
\[
\{ y = x + x_0 \}
\]
\[
= \int_{x_0}^{x_0+1} f(y) e^{-2\pi i k y} e^{2\pi i k x_0} dy
\]

\[ e^{2\pi ikx_0} \int_{x_0}^{x_0 + 1} f(y) e^{-2\pi iky} \, dy = e^{2\pi ikx_0} \int_{0}^{1} f(y) e^{-2\pi iky} \, dy \]

\[ \{ \text{by periodicity of } f \} \]

\[ \{ \text{integrand} \} \]

\[ = e^{2\pi ikx_0} \mathcal{F}(f) \]

So \( f(x_0) = \lim_{N \to \infty} \frac{1}{N} \sum_{-N}^{N} \mathcal{F}(f) e^{2\pi ikx_0} \)

\[ \text{|Error|} \leq M \max_{N} \, , \quad M = \sup_{\mathbb{R}} |f'| \]

Qed. \[ \square \]

The next theorem is a commonly used augmentation of Theorem 6 on [6] bottom.

**Theorem**

Let \( f \) belong to \( C^2(\mathbb{R}) \) and be periodic 1. We then have

\[ |c_k| \leq \frac{1}{(2\pi k)^2} \int_{0}^{1} |f''(x)| \, dx \quad k \neq 0 \]

This ensures that, on [6] bottom, \( \sum_{k \neq 0} c_k e^{2\pi ikx} \)

conv both uniformly and absolutely to \( f(x) \) on \( \mathbb{R} \).
Simply integrate by parts twice:

$$c_k = \frac{1}{(2\pi ik)^2} \int_0^1 f(x) e^{-2\pi i k x} \, dx, \quad k \neq 0.$$ 

The following is our main assertion in this approach to FS based on $e^{-M}$.

**Theorem (Standard Fourier series that in undergrad)**

Let $f$ be given on $[a, b]$ and be periodic $2L$.

Assume $f$ is piecewise $C^1$. (See picture.)

Let $c_k = \int_0^1 f(x) e^{-2\pi i k x} \, dx$ and

$$FS(f) \equiv \sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$$

as a formal sum.

We then have $\Sigma_{-N}^{N} c_k e^{2\pi i k x} \rightarrow f(x)$ as $N \rightarrow \infty$

away from the discontinuities of $f$. At the points of discontinuity, we have

$$\sum_{-N}^{N} c_k e^{2\pi i k x} \rightarrow \frac{1}{2} \left[f(x+0) + f(x-0)\right].$$

Here $f(x+0)$, $f(x-0)$ are the one-sided limits.

(cont'd)
In addition, the partial sums \( \sum_{-N}^{N} a_k e^{2\pi i k x} \) will be uniformly bounded on \( \mathbb{R} \).

**Proof**

The thm is certainly correct if \( f \) has no discontinuities on \( \mathbb{R} \). See (6) bottom.

We now do a trick. (using \( \beta \))

---

**Baby Lemma**

Let \( H(x) \equiv \lim_{N \to \infty} \sum_{-N}^{N} a_k e^{2\pi i k x} \), where the limit exists pointwise on all of \( \mathbb{R} \). Assume that the partial sums \( \sum_{-N}^{N} \) are uniformly bounded on \( \mathbb{R} \). Finally, assume that the partial sums \( \sum_{-N}^{N} \) converge uniformly away from \( \{c_1, \ldots, c_m \} \mod \mathbb{Z} \) (in finite). Then:

(A) \( H(x) \) is Riemann integrable on \( [0,1] \);

(B) \( a_k \approx \int_0^1 H(x) e^{-2\pi i k x} \, dx \), each \( k \in \mathbb{Z} \).

No! (B) is not a tautology!
Pf of Lemma

The discontinuities of \( H \) are contained in \( \{ c_1, \ldots, c_m \} \mod \mathbb{Z} \) by the uniform convergence of \( H(x) \) is also bounded by the uniform boundedness of

\[
S_N(x) = \sum_{N}^{\infty} a_k e^{2\pi i k x}.
\]

By baby calculus, \( H \) is Riemann integrable on any finite \( [a, b] \). Hence \( [0, 1] \).

As we saw earlier, baby analysis \( \Rightarrow \)

\[
\int_0^1 |H(x) - S_N(x)| \, dx \to 0 \quad \text{as} \quad N \to \infty.
\]

See Lec 9 p. 9.

By that same idea, we have:

\[
\int_0^1 e^{-2\pi i m x} S_N(x) \, dx \to \int_0^1 e^{-2\pi i m x} H(x) \, dx
\]

for each \( m \in \mathbb{Z} \). But LHS = \( am + O(1) \) for large \( N \).

Hence,

\[
am = \int_0^1 H(x) e^{-2\pi i m x} \, dx.
\]
Before continuing, observe that: \( l \in \mathbb{N} \)
\[
\frac{e^{x + \pi i l}}{-2\pi i l} + \frac{e^{-x - \pi i l}}{-2\pi i l} = -\frac{1}{2\pi i l} (e^{x + \pi i l} - e^{-x - \pi i l})
\]
\[= \frac{\sin(2\pi l x)}{\pi l}.
\]

Also write
\[
\tilde{\beta}(y) = \begin{cases} 0, & y \in \mathbb{Z} \\ \beta(y), & y \notin \mathbb{Z} \end{cases}.
\]

We already know that
\[
\tilde{\beta}(x) = \sum_{m=1}^{\infty} \frac{\sin 2\pi mx}{\pi m} = \sum_{n=0}^{\infty} -\frac{1}{2\pi i n} e^{2\pi i n x}
\]
all \( x \in \mathbb{R} \). Unit conv away from \( \mathbb{Z} \)'s partial sums unit bounded. Similarly
\[
\tilde{\beta}(x-c) = \sum_{n=0}^{\infty} -\frac{e^{-2\pi i n c}}{2\pi i n} e^{2\pi i n x}
\]
all \( x \in \mathbb{R} \). By Baby Lemma on \( \theta \)'s automatically
\[
\int \tilde{\beta}(x)e^{-2\pi i k x} \, dx = \begin{cases} 0, & k = 0 \\ -\frac{1}{2\pi i n}, & k \neq 0 \end{cases}
\]
\[ \int_0^1 \beta(x-c) e^{-2\pi inx} dx = \begin{cases} 0, & n = 0 \\ \frac{-e^{-2\pi inc}}{2\pi in}, & n \neq 0 \end{cases}. \]

Thus:

\[ \mathcal{F} \left[ \beta(x) \right] = \sum_{n \neq 0} -\frac{1}{2\pi in} e^{2\pi inx}. \]

\[ \mathcal{F} \left[ \tilde{\beta}(x-c) \right] = \sum_{n \neq 0} \frac{e^{-2\pi inc} e^{2\pi inx}}{2\pi in}. \]

Obviously, the "\( n \)" can be removed from \( \beta \).

These Fourier series can of course be checked directly, but we prefer the slick approach.

We now return to the proof of p. 8 THM.

Let \( f(x) \) have nontrivial discontinuities at points \( c_1, \ldots, c_m \mod \mathbb{Z} \). Let the "right-left" jump be \( J_{i^*} \). Saying \( J_{i^*} = 0 \) means \( f(c_{i^*} + 0) = f(c_{i^*} - 0) \)

but \( f(c_{i^*}) \neq f(c_{i^*} + 0) \).

Recall that

\[ \beta(0^+) - \beta(0^-) = -\frac{1}{2} - \frac{1}{2} = -1. \]
Define:

\[ g(x) = f(x) + \sum_{i=1}^{m} J_i \beta(x - c_i), \quad x \in \mathbb{R}. \]

For \( g \) is very interesting! It is obviously periodic 1. Also, it is obviously piecewise \( C^1 \). It may have discontinuities, but these lie in \( \{ c_1, \ldots, c_m \} \mod \mathbb{Z} \).

Note however that

\[ g(c_i + 0) - g(c_i - 0) = J_i^+ - J_i^- + 0 = 0 \quad \text{each} \quad 1 \leq i \leq m. \]

The points \( c_i \) are thus "removable discontinuities" if \( g \) is redefined correctly at these points.

Apply p. 6 bottom THM to this modified \( g \).

We conclude that \( FS(g) \) converges uniformly over \( \mathbb{R} \) to \[ \frac{1}{2} \left[ g(x + 0) + g(x - 0) \right]. \] The partial sums are automatically uniformly bounded on \( \mathbb{R} \).

By linearity, however, as series,

\[ FS(f) = FS(g) - \sum_{i=1}^{m} J_i \cdot FS[\beta(x - c_i)]. \]
At once, the partial sums of $FS(f)$ are uniformly bounded on $R$ (by the corresponding fact for $\beta$).

Also, $FS(f)$ converges uniformly away from the $\{c_0, \ldots, c_m \mod Z \}$ (by the corresponding fact for $\beta$).

At points $x \neq c_1, \ldots, c_m \mod Z$, clearly $FS(f)$ converges to

$$g(x) = \sum_{i=1}^{m} \beta_i (x - c_i^*) = F(x).$$

(Big surprise!!)

At $c_i^*$, $FS(f)$ converges to

$$\frac{1}{2} \left[ g(c_i^*+0) + g(c_i^*-0) \right] - 0 = \sum_{l \neq i} J_l \beta(c_i^* - c_l).$$

But:

$$g(c_i^* + 0) = f(c_i^* + 0) + \int_{-\frac{1}{2}}^{0} \beta(c_i^* - c_l) \ dx + \sum_{l \neq i} J_l \beta(c_i^* - c_l),$$

$$g(c_i^* - 0) = f(c_i^* - 0) + \int_{\frac{1}{2}}^{0} \beta(c_i^* - c_l) \ dx + \sum_{l \neq i} J_l \beta(c_i^* - c_l).$$

$$\frac{g(c_i^* + 0) + g(c_i^* - 0)}{2} = \frac{f(c_i^* + 0) + f(c_i^* - 0)}{2} + \sum_{l \neq i} J_l \beta(c_i^* - c_l).$$
\[ \text{FS}(f) \text{ conv to} \quad \frac{f(c_i^+ + 0) + f(c_i^+ - 0)}{2} \]

at each \( c_i \). (Again, big surprise!!)

Thus, all is now proved.

---

**Famous Formula**

**THM (Parseval's formula)**

Let \( f \) be periodic 1, piecewise \( C^1 \) as in p.8 THM.

We then have:

\[
\int_0^1 |f(x)|^2 \, dx = \sum_{k=-\infty}^{\infty} |c_k|^2 .
\]

**PF**

\( f \) is uniformly bounded on \( \mathbb{R} \). We know \( S_N(x) \) is uniformly bounded on \( \mathbb{R} \) too. We also have \( S_N(x) \to f(x) \) away from \( \{c_1, \ldots, c_m\} \mod \mathbb{Z} \). Apply the idea of Lec 9 p.6 again! (See p.10 above.)
We get:
\[
\int_0^1 f(x) \overline{S_N(x)} \, dx \to \int_0^1 f(x) f(x) \, dx \quad (N \to \infty)
\]

but
\[
LHS = \int_0^1 f(x) \left( \sum_{-N}^{N} c_k e^{2\pi i k x} \right) \, dx
\]
\[
= \sum_{-N}^{N} c_k \overline{c_k} = \sum_{-N}^{N} |c_k|^2
\]

The Fourier theory so far has been a kind of $L_0 \times L_1$ theory. In traditional real analysis courses, one investigates to see if an $L_2 \times L_2$ theory might be better (or more natural).

We will not bother to pursue the latter beyond a few remarks.

Use of completing the square on integrals like
\[
\int_0^1 |f(x) - S_N(x)|^2 \, dx \quad \int_0^1 |f(x) - \sum_{-N}^{N} a_k e^{2\pi i k x}|^2 \, dx
\]

for a general piecewise continuous, periodic $f(x)$ leads to
\[ \sum_{-N}^{N} |c_k|^2 \leq \int_{-\infty}^{\infty} |f(x)|^2 \, dx \] (each \( N \))

\[ \sum_{-\infty}^{\infty} |c_k|^2 \leq \int_{-\infty}^{\infty} |f(x)|^2 \, dx \]  

(\text{i.e., Bessel's inequality})

Here \( c_k = \int_{0}^{1} e^{-2\pi i k x} \, dx \).

(Actually, equality holds — but this is a harder theorem. One uses Thm and "approximates" \( f \) by piecewise \( \mathcal{C}^1 \) functions. \text{ SEE ANY STANDARD BOOK ON F.S.})

Our 2\text{nd} remark is a theorem.

\textbf{THM} (slight strengthening of p. 6 bottom)

Let \( f \in \mathcal{C}(\mathbb{R}) \), periodic 1, and be piecewise \( \mathcal{C}^1 \).

Let \( c_k = \int_{0}^{1} e^{-2\pi i k x} \, dx \). The Fourier series

\[ \sum_{-\infty}^{\infty} c_k e^{2\pi i k x} \]

then converges \underline{uniformly} to \( f(x) \) on \( \mathbb{R} \)

AND we also have

\[ \sum_{-\infty}^{\infty} |c_k|^2 < \infty \].
pf
Take $k \neq 0$. By standard integ by parts,

$$c_k = \frac{1}{2\pi i k} \int_0^1 f'(x)e^{-2\pi i k x} \, dx.$$  \hspace{1cm} (4)

Again, NOTE THAT RHS is not affected by a few ambiguities in $f'$. Write the foregoing as

$$c_k = \frac{1}{2\pi i k} c_k(f')$$

and recall \boxed{(Bessel's ineq)}. By Cauchy-Schwarz inequality,

$$\sum_{k=1}^{N} |c_k| = \frac{1}{2\pi} \sum_{k=1}^{N} \frac{1}{k} |c_k(f')|$$

$$\leq \frac{1}{2\pi} \sqrt{\sum_{k=1}^{N} \frac{1}{k^2}} \sqrt{\sum_{k=1}^{N} |c_k(f')|^2} < + \infty.$$ 

Similarly for $k < 0$. \boxed{\ }
Next topic: Poisson summation formula.

\[ \text{THM} \]

Given \( \varphi \in C^2(\mathbb{R}) \) such that, say, \( |\varphi(x)|, |\varphi'(x)|, |\varphi''(x)| \) all = \( O\left[ (1+|x|)^{-2} \right] \).

Let
\[ \hat{\varphi}(p) = \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i px} \, dx \quad \text{for } p \in \mathbb{R}. \]

Let
\[ F(x) = \sum_{n=-\infty}^{\infty} \varphi(x+n) \quad x \in \mathbb{R}. \]

We then have
\[ F(x) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k) e^{2\pi i kx} \quad \text{Poisson summation formula} \]

for \( x \in \mathbb{R} \), with absolute and uniform conv on both sides over every interval \([-A,A]\).

\[ \text{Proof} \]

The series \( \sum_{n=-\infty}^{\infty} \varphi^{(j)}(x+n), 0 \leq j \leq 2 \) are clearly conv both abs and uniformly on every \([-A,A]\).

As such, we immediately get \( F \in C^2(\mathbb{R}) \). It is also apparent that \( F(x+1) = F(x) \).

Apply Thm \( \text{7} \) bottom. Get :
\[ F(x) = \sum_{n=-\infty}^{\infty} A_k e^{2\pi i k x} \quad \text{nicely,} \]

\[ A_k = \int_{0}^{1} F(x) e^{-2\pi i k x} \, dx \]

But:

\[ A_k = \int_{0}^{1} \left( \sum_{n=-\infty}^{\infty} \varphi(x+n) \right) e^{-2\pi i k x} \, dx \]

\[ = \sum_{n=-\infty}^{\infty} \int_{0}^{1} \varphi(x+n) e^{-2\pi i k x} \, dx \quad \text{by unif. conv} \]

\[ = \sum_{n=-\infty}^{\infty} \int_{n}^{n+1} \varphi(y) e^{-2\pi i k y} \, dy = \int_{\mathbb{R}} \varphi(y) e^{-\varphi i k y} \, dy \]

\[ = \varphi(k) . \]

**Example**

\[ \varphi(x) = e^{-ax^2} \quad a > 0 \]

\[ \int_{-\infty}^{\infty} e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}} \quad \Rightarrow \]

\[ \int_{-\infty}^{\infty} e^{-ax^2} e^{-2\varphi i k x} \, dx = \sqrt{\frac{\pi}{a}} e^{-\frac{\pi k^2}{a}} \]

(by elementary contour shift).

Hence, by Poisson summation formula,

\[ \sum_{n=-\infty}^{\infty} e^{-a(x+n)^2} = \sqrt{\frac{\pi}{a}} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi^2 k^2}{a}} e^{2\pi ikx} . \]  \hspace{1cm} (21)

**Special Case:**
\[ \sum_{n=-\infty}^{\infty} e^{-\pi\beta n^2} = \sqrt{\frac{1}{\beta}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{\beta}} \quad (\beta > 0) . \]

The famous "\( \Theta \)" relation of Jacobi:
\[ \Theta(\beta) = \frac{1}{\sqrt{\beta}} \Theta(\frac{1}{\beta}) . \]

We are now ready to derive (following Riemann) a slick formula for \( \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s) \).

**Easily check:**
\[ \Gamma(\frac{s}{2}) = \int_{0}^{\infty} x^{\frac{s}{2}-1} e^{-x} \frac{dx}{x} \quad \text{Re}(s) > 1 \quad \text{say} \]

\[ \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \int_{0}^{\infty} y^{\frac{s}{2}-1} e^{-\pi \gamma y} \frac{dy}{y} . \]

\[ \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \int_{0}^{\infty} y^{\frac{s}{2}} \left[ \sum_{n=1}^{\infty} e^{-\pi \gamma y} \right] \frac{dy}{y} . \]

A nice positive \( \zeta \) for \( y \to 0^+ \) clearly \( O(\sqrt{y}) \) for \( y \to 0^+ \) by \( \Theta \)-relation.
Note: the foregoing integral is nicely convergent near \( y = 0 \) because

\[
\int_0^1 y^{\alpha \over 2} \frac{1}{\sqrt{y}} \frac{dy}{y} < \infty \quad \text{for} \quad \sigma > 1
\]

Write

\[
\Psi(y) = \sum_{n=1}^{\infty} e^{-\pi n^2 y} \quad \text{and} \quad \Theta(y) = 2\Psi(y) + 1
\]

So:

\[
\Psi(y) + \frac{1}{\alpha} = \frac{1}{\sqrt{y}} \left[ \Psi(y) + \frac{1}{\alpha} \right] \quad \text{for} \quad y > 0
\]

\[
\Psi(y) = -\frac{1}{\alpha} + \frac{1}{\alpha} \sqrt{y} + y^{-\alpha / 2} \Psi(1 / y)
\]

Get:

\[
\pi^{-\alpha \over 2} \Gamma(\frac{\alpha}{2}) \Gamma(\frac{\sigma}{2}) = \int_0^1 y^{\alpha \over 2} \Psi(y) dy + \int_1^\infty y^{\alpha \over 2} \Psi(y) dy
\]

put \( y = \frac{1}{\sqrt{y}} \)
here

\( \$ \) now grind! \( \$ \)
\[ = \int_1^{\infty} \frac{v^{-\frac{s}{2}}}{\sqrt{2}} \left[ -\frac{1}{2} + \frac{1}{2} v^{\frac{1}{2}} + v^{\frac{1}{2}} \psi(v) \right] \frac{dv}{v} \]

\[ + \int_1^{\infty} y^{\frac{s}{2}} \psi(y) \frac{dy}{y} \]

\[ = -\frac{1}{s} - \frac{1}{1-s} + \int_1^{\infty} \frac{1-s}{2} \psi(v) \frac{dv}{v} \]

\[ + \int_1^{\infty} y^{\frac{s}{2}} \psi(y) \frac{dy}{y} \]

\[ = - \left[ \frac{1}{s} + \frac{1}{1-s} \right] + \int_1^{\infty} \left( y^{\frac{1-s}{2}} + y^{\frac{s}{2}} \right) \psi(y) \frac{dy}{y} \]

\[ = -\frac{1}{s(1-s)} + \int_1^{\infty} \left( y^{\frac{1-s}{2}} + y^{\frac{s}{2}} \right) \psi(y) \frac{dy}{y} \]

So, for \( \Re(s) > 1 \),

\[ \pi^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} \left( y^{\frac{1-s}{2}} + y^{\frac{s}{2}} \right) \psi(y) \frac{dy}{y} \]

\[ O(e^{-\pi y}) \quad \text{as} \quad y \to +\infty \]

\[ \text{The integral is analytic for all} \quad s \in \mathbb{C}. \]

\[ \frac{1}{s(s-1)} \quad \text{is trivially analytic on} \quad \mathbb{C} \setminus \{0, 1\}. \]
**Theorem (Functional Equation)**

\[ \Xi(s) \equiv \pi^{-s/2} \Gamma(s) \zeta(s) \] is analytic on 
\( \Re(s) > 0 \) and satisfies

\[ \Xi(s) = \Xi(1-s). \]

We also have for \( \Xi : \)

- \( s = 1 \) simple poles, residue 1
- \( s = 0 \) simple poles, residue \(-1\)

**Pf**

The first part is just \( \Theta \) bottom. \( \Theta \)

By \( \Theta \) bottom, with \( s = 1 + h \), we get

\[ \Xi(1+h) = \frac{1}{(1+h)h} + O(1) \]

\[ = \frac{1}{h} + O(1) \]

And, similarly, with \( s = h \)

\[ \Xi(h) = \frac{1}{h(h-1)} + O(1) = -\frac{1}{h} + O(1) \]
Cor 1

\[ \xi_0(s) = s(s-1) \xi_0(s) = s(s-1) \pi^{-3/2} \Gamma(\frac{3}{4}) \xi(s) \] is an entire fcn which satisfies

\[ \xi_0(s) = \xi_0(1-s), \quad \xi_0(1) = 1. \]

PF

\[ \xi_0(s) = 1 + s(s-1) \int_1^\infty \left( \frac{1-s}{y^2} + \frac{s}{y^2} \right) \Psi(y) \frac{dy}{y} \] by (23).

Cor 2

\[ \gamma(-ak) = 0 \text{ for } k \geq 1 \text{ (simple zero).} \]

PF

\[ \xi(x) = \pi^{-x/2} \Gamma(\frac{x}{2}) I(x) > 0 \text{ for } x > 1. \] But \( \xi(x) = \xi(1-x) \).

Hence \( \xi(x) > 0 \text{ for } x < 0 \).

Let \( x \to -2k \).

Since \( \Gamma(\frac{x}{2}) \to \Gamma(-k) \) simple poles get \( I(x) \to 0 \) à la simple zero."
Lemma

\[(1 - 2^{1-s}) J(s) = \sum_{k=1}^{\infty} \left( (2k+1)^{-s} - (2k)^{-s} \right) \]

For \(\text{Re}(s) > 1\) and the RHS is actually analytic for \(\{\text{Re}(s) > 0\}\).

PF

\(\text{Re}(s) > 1 \Rightarrow \)

\[J(s) = \sum_{k=1}^{\infty} \left( (2k-1)^{-s} \right) + \sum_{k=1}^{\infty} \left( (2k)^{-s} \right) \text{ trivially} \]

\[Q^{1-s} J(s) = 2 \sum_{m=1}^{\infty} \left( (2m)^{-s} \right) \]

Difference = \(\sum_{k=1}^{\infty} \left( (2k-1)^{-s} - (2k)^{-s} \right) \) \(\{\text{nice abs conv}\}\).

Keep \(s \in K\) where \(K\) is a compact subset of \(\{\text{Re}(s) > 0\}\). Observe that:

\[ (2k-1)^{-s} - (2k)^{-s} = (2k)^{-s} \left[ \left(1 - \frac{1}{2k} \right)^{-s} - (2k)^{-s} \right] \]

\[
\left\{ \begin{array}{l}
(1+u)^{-s} = 1 + (-s) u + O_K(1) u^2 \quad \text{for } |u| \leq \frac{3}{4} \\
(2k)^{-s} \left( \frac{s}{2k} + O(1) \frac{1}{k^2} \right) \\
(2k-1)^{-s} - (2k)^{-s} = O(1) k^{-s-1} \text{ for } s \in K.
\end{array} \right.
\]
Corollary

In the sense of analytic continuation,

\[ \xi(x) \neq 0 \quad \text{for} \quad x \in \mathbb{R} \]
\[ \xi_0(x) \neq 0 \quad \text{for} \quad x \in \mathbb{R} \]
\[ I(x) < 0 \quad \text{for} \quad 0 < x < 1 \]

Proof
That \( I(x) < 0 \) on \( 0 < x < 1 \) is obvious from \( \mathcal{A} \).

Hence \( \xi(x) \neq 0 \) on \( 0 < x < 1 \). The points \( x = 0, 1 \)
are poles and take care of themselves.

For \( x > 1 \) and \( x < 0 \), we have \( \xi(x) > 0 \) \( \text{à la} \)
\( \mathcal{A} \). Since \( \xi_0(x) = x(x-1)\xi(x) \), the assertions
for \( \xi_0 \) are immediate.
We wish to bound the size of

\[ F(z) = z(z-1) e^{-\frac{z}{2}} H(z) J(z) \]

(roughly) using Stirling \( F(z) = F(1-z) \), and basic properties of \( J(z) \).

Because of \( F(z) = F(1-z) \), we can clearly restrict to \( \text{Re}(z) \geq \frac{1}{2} \).

We had

\[ |S(x+iy)| \leq \frac{C}{\delta(1-\delta)} |y|^{1-\delta} \left\{ \begin{array}{l} x \geq \delta \\ |y| \geq 2 \end{array} \right\} \]

any \( 0 < \delta < 1 \) \( \text{Lec 6 page 9} \) \( \text{EG } \delta = \frac{1}{2} \).

Also, we had

\[ |J(z)-1| < \frac{3}{4} \text{ for } \text{Re}(z) > 2 \]

by Lec 5 page 10.
\[ F(z) = z(z-1)^{\pi - \frac{x}{2}} \Gamma\left(\frac{x}{2}\right) \Gamma\left(\frac{z}{2}\right) \]

\[ |F(z)| = |z||z-1| \pi^{-\frac{x}{2}} |\Gamma\left(\frac{x}{2}\right)| |\Gamma\left(\frac{z}{2}\right)| \]

\[ |F(z)| \approx |z|^3 \left[ 1 + O\left(\frac{1}{x}\right) \right] \pi^{-\frac{x}{2}} \Gamma\left(\frac{z}{2}\right) |\Gamma\left(\frac{z}{2}\right)| \]

Know:

\[ |\Gamma(z) - 1| < \frac{3}{4} \quad \text{for} \quad x > 2 \]

\[ |\Gamma(x+iy)| \approx O(1/y^{1/2}) \quad x \approx \frac{y}{2}, \quad y \approx 2. \]

Also:

\[ \ln |\Gamma\left(\frac{z}{2}\right)| = \Re \left\{ \log \Gamma\left(\frac{z}{2}\right) \right\} \]

\[ \text{Stirling's} \quad \text{Lec 10} \quad \text{p. 16} \]
\[
\log \frac{\pi}{\alpha} = \left(\frac{\pi}{2} - \frac{1}{2}\right) \log \left(\frac{\pi}{2}\right) - \frac{\pi}{2} + \frac{1}{2} \ln(\alpha r)
\]
\[
+ O\left(\frac{1}{\pi r}\right)
\]

\{ For, say, \( |z| = r \), \( r \) large, \( |\text{Arg } z| \leq \frac{3}{4} \pi \) \}

As in Ingham 56-57, we get

\[
\ln \Gamma\left(\frac{z}{\alpha}\right) \approx \frac{r}{2} \ln r + A_1 r
\]

\{ For, say, \( |z| = r \), \( |\text{Arg } z| \leq \frac{3}{4} \pi \) \}

\[
\ln |F(re^{i\theta})| \approx \frac{r}{2} \ln r + A_2 r
\]

for \( |z| = r \), \( \text{Re}(z) \geq \frac{1}{\alpha} \)

then, using \( F(z) = F(1 - z) \), similarly for \( \text{Re}(z) \leq \frac{1}{2} \).
Also, looking at $\Theta = 0$,

$$F(R) = R^2 \left[ 1 + O \left( \frac{1}{R} \right) \right] \pi^{-\frac{R}{2}} \Gamma \left( \frac{R}{2} \right) \mathcal{I}(R)$$

$$\geq \text{(constant)} \ R^2 \pi^{-\frac{R}{2}} \Gamma \left( \frac{R}{2} \right)$$

\[ \begin{align*}
\left\{ \begin{array}{l}
\text{but} \\
\ln \Gamma \left( \frac{R}{2} \right) \sim \left( \frac{R}{2} - \frac{1}{2} \right) \ln \left( \frac{R}{2} \right) - \frac{R}{2}
\end{array} \right\}
\end{align*} \]

\[ \Downarrow \]

$$\ln F(r) \geq \frac{r}{2} \ln r - A_2 r \quad (r \text{ large}).$$

**THM**

Let $F(z) = z(z-1) \pi^{-z/2} \Gamma \left( \frac{z}{2} \right) \mathcal{I}(z)$. Let

$$M(r) = \max_{|z|=r} |F(z)| \quad (r \text{ large}).$$

Then:

$$\ln M(r) \sim \frac{r}{2} \ln r.$$
Proof

As above.

For any entire function $g(z)$, $g \not\equiv 0$, we write

$$M(r) = \max_{|z| = r} |g(z)|.$$

Then put:

$$\rho = \inf \left\{ \omega \mid \ln M(r) \leq r^\omega, \text{ all large } r \right\},$$

$$\tau = \inf \left\{ \beta \mid \ln M(r) \leq \beta r^\rho, \text{ all large } r \right\}.$$

Herein $\omega \geq 0$ and $\beta \geq 0$. Empty braces mean $\inf = +\infty$. We call:

$$\rho = \text{ORDER of } g(z),$$

$$\tau = \text{TYPE of } g(z).$$

For our $F(z)$, clearly $\rho = 1$ and $\tau = +\infty$. 