Lecture 11
(Feb 24)

I began with a quick development of basic Fourier series — in a nonstandard way, i.e., via E–M summation.

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

On $[0,1]$ or <u>any</u> $[a, a+1]$:

$$\langle \varphi_m, \varphi_n\rangle = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

for

$$\varphi_n(x) = e^{2\pi i n x}, \quad n \in \mathbb{Z}.$$

So, we have the usual <u>idea</u> of trying to write f "most of the time" as $\sum_n c_n \varphi_n$, $c_n = \langle f, \varphi_n\rangle$.

<u>Lemma</u>

$f \in C[0,N]$. Assume f is only <u>piecewise</u> $\underline{C}^1$. not C^1

Then we still have

$$\frac{1}{2}f(0) + f(1) + \cdots + f(N-1) + \frac{1}{2}f(N) = \int_0^N f\,dx + \int_0^N f'\beta(x)\,dx,$$

$$\beta(x) \equiv x - [\![x]\!] - \frac{1}{2}.$$

Pf

Begin as before Lec 8 p. 14

$$f(1) + \cdots + f(N) = \int_0^N f(x)\, d[\![x]\!]$$

$$= \int_0^N f(x)\, d\left(x - \tfrac{1}{2} - \beta(x)\right) \qquad (R-S)$$

$$= \int_0^N f(x)\,dx - \int_0^N f(x)\, d\beta(x) .$$

Split $\int_0^N f\, d\beta$ into chunks corresponding to corners of f.

Then do the integration by parts and recombine. Ambiguous f' at a finite # of corners does not affect $\int_0^N \beta f'\,dx$.

$\Rightarrow$ All is fine. ∎

Take $N = 1$. Assume $f \in C[0,1]$, piecewise C^1.

Hence, by Lemmas

$$\tfrac{1}{2}f(0) + \tfrac{1}{2}f(1) = \int_0^1 f\,dx + \int_0^1 f'\left(-\sum_n \frac{\sin 2\pi n x}{\pi n}\right) dx$$

$$= \int_0^1 f\,dx + \sum_{n=1}^{\infty} \int_0^1 f' \frac{-\sin 2\pi n x}{\pi n}\, dx ,$$

the 2^{nd} line by Lec 9, p. 9, Baby Fact.

Note that the error term after N is

$$\pm \int_0^1 f'(\beta - S_N)\,dx$$

i.e.

$$\text{ABS VALUE} \lesssim \mathfrak{M} \int_0^1 |\beta - S_N|\,dx,$$

$$\mathfrak{M} \equiv \sup_{[0,1]} |f'| .$$

The $|\beta - S_N|$ integral is an absolute expression, say ω_N, and $\omega_N \to 0$. So:

$$|\text{Error}| \lesssim \mathfrak{M}\omega_N .$$

Note TOO that $(n \geq 1)$

$$\int_0^1 f' \frac{\sin 2\pi n x}{-\pi n}\,dx = \frac{1}{2\pi i n} \int_0^1 f'(e^{-2\pi i n x} - e^{2\pi i n x})\,dx .$$

Write the last expr. as

$$\frac{1}{2\pi i n} \int_0^1 f' e^{-2\pi i n x}\,dx + \frac{1}{2\pi i(-n)} \int_0^1 f' e^{-2\pi i(-n)x}\,dx .$$

But,

$$\frac{1}{2\pi i n}\int_0^1 f' e^{-2\pi i n x}\,dx$$

$$= \frac{1}{2\pi i n}\int_0^1 e^{-2\pi i n x}\,df \qquad \text{(standard parts)}$$

$$= \frac{1}{2\pi i n}\left[e^{-2\pi i n x} f(x)\right]_0^1$$

$$- \frac{1}{2\pi i n}\int_0^1 f\,d\left(e^{-2\pi i n x}\right)$$

$$\simeq \frac{f(1)-f(0)}{2\pi i n} + \int_0^1 f e^{-2\pi i n x}\,dx .$$

Similarly for $-n$. Now add! Get:

$$(\text{term } n) + (\text{term } -n) \cong c_n + c_{-n}$$

where

$$c_k = \int_0^1 f e^{-2\pi i k x}\,dx .$$

(2) bottom

So,

$$\frac{1}{2}f(0) + \frac{1}{2}f(1) = c_0 + \lim_{N\to\infty} \sum_{1\le |n| \le N} c_n$$

$$\Downarrow$$

$$\frac{1}{2} f(0) + \frac{1}{2} f(1) = \lim_{N \to \infty} \sum_{-N}^{N} c_n ,$$

any $f \in C[0,1]$, piecewise C^1.

AHA! This is really a Fourier series (example);

i.e.,

$$\lim_{N \to \infty} \sum_{-N}^{N} c_n e^{2\pi i n \cdot 0} \; !$$

The proof was *just* basic E-M, version I, and

$$x - [\![x]\!] - \frac{1}{2} = - \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{\pi n} , \quad x \notin \mathbb{Z} .$$

NOTE THAT error term for $|n| > N$ is

$$\lesssim \mathcal{M} \omega_N .$$

Initial Thm

Given $f \in C[0,1]$, piecewise C^1.

Let
$$c_k = \int_0^1 f e^{-2\pi i k x} dx = \langle f, \varphi_k \rangle .$$

Then:
$$\frac{1}{2} f(0) + \frac{1}{2} f(1) = \lim_{N\to\infty} \sum_{-N}^{N} c_k e^{2\pi i k 0},$$
$$|\text{Error}| \leq M\omega_N .$$

Pf

As above. ∎

Thm

Let $f \in C(\mathbb{R})$, periodic 1, piecewise C^1.

Then:

unif. conv.
$$\sum_{-N}^{N} c_k e^{2\pi i k x} \rightrightarrows f(x) \quad \text{on } \mathbb{R} .$$

Pf

periodic 1
$C[0,1]$, piecewise C^1

Fix any $x_0 \in \mathbb{R}$. Consider $g(x) = f(x+x_0)$ on $[0,1]$ in previous Thm. Note

$$c_k(g) = \int_0^1 g(x) e^{-2\pi i k x} dx = \int_0^1 f(x+x_0) e^{-2\pi i k x} dx$$
$$\{y = x + x_0\}$$
$$= \int_{x_0}^{x_0+1} f(y) e^{-2\pi i k y} e^{2\pi i k x_0} dy$$

$$= e^{2\pi i k x_0} \int_{x_0}^{x_0+1} f(y) e^{-2\pi i k y}\, dy$$

$$= e^{2\pi i k x_0} \int_0^1 f(y) e^{-2\pi i k y}\, dy$$

$\left\{ \text{by periodicity of } \underline{\text{integrand}} \right\}$

$$= e^{2\pi i k x_0} c_k(f).$$

So:

$$f(x_0) = \lim_{N \to \infty} \sum_{-N}^{N} c_k(f) e^{2\pi i k x_0},$$

$$|\text{Error}| \leq M \omega_N, \quad M = \sup_{\mathbb{R}} |f'|.$$

Qed. ▮

The next thm is a commonly used augmentation of thm on ⑥ bottom.

Theorem

Let f belong to $C^2(\mathbb{R})$ and be periodic 1. We then have

$$|c_k| \leq \frac{1}{(2\pi k)^2} \int_0^1 |f''| dx, \quad k \neq 0.$$

This ensures that, on ⑥ bottom, $\sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$ conv both uniformly and absolutely to $f(x)$ on $\mathbb{R}$.

Pf compare 4

Simply integrate by parts twice:

$$c_k = \frac{1}{(2\pi i k)^2} \int_0^1 f'' e^{-2\pi i k x}\,dx, \quad k \neq 0.$$ ∎

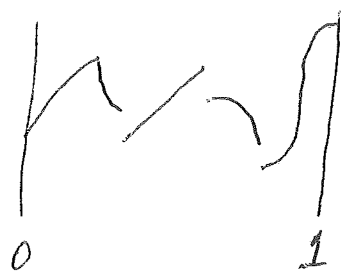
The following is our MAIN assertion in this approach to FS based on E–M.

THEOREM (standard Fourier series thm in undergrad analysis)

Let f be given on $\mathbb{R}$ and be periodic 1.
Assume f is piecewise C^1. (See picture)
Let $c_k = \int_0^1 f e^{-2\pi i k x}\,dx$ and

$$FS(f) \equiv \sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$$

as a formal sum.



We then have

$$\sum_{-N}^{N} c_k e^{2\pi i k x} \overset{\boxed{\text{unif conv}}}{\rightrightarrows} f(x) \quad \text{as } N \to \infty$$

away from the discontinuities of f. At the points of discontinuity, we have

$$\sum_{-N}^{N} c_k e^{2\pi i k x} \longrightarrow \frac{1}{2}\left[f(x+0) + f(x-0)\right].$$

Here $f(x+0)$, $f(x-0)$ are the one-sided limits.

(cont'd)

In addition, the partial sums $\sum_{-N}^{N} c_k e^{2\pi i k x}$ will be <u>uniformly</u> <u>bounded</u> on $\mathbb{R}$.

<u>Proof</u>

The thm is certainly correct if f has <u>no</u> discontinuities on $\mathbb{R}$. See ⑥ bottom.

We now do a trick. (using β)

<u>Baby Lemma</u>

Let $H(x) \equiv \lim_{N\to\infty} \sum_{-N}^{N} a_k e^{2\pi i k x}$, where the limit exists pointwise on all of $\mathbb{R}$. Assume that the partial sums $\sum_{-N}^{N}$ are <u>uniformly</u> <u>bounded</u> on $\mathbb{R}$. Finally, assume that the partial sums $\sum_{-N}^{N}$ <u>converge</u> <u>uniformly</u> away from $\{c_1, \ldots, c_m\}$ mod $\mathbb{Z}$ (m finite). Then:

(A) $H(x)$ is Riemann integrable on $[0,1]$;

(B) $a_k = \int_0^1 H(x) e^{-2\pi i k x}\, dx$, each $k \in \mathbb{Z}$.

$\boxed{\text{No! (B) is not a tautology!}}$

Pf of Lemma

The discontinuities of H are contained in $\{c_1, \dots, c_m\}$ mod $\mathbb{Z}$ by the unif conv. $H(x)$ is also bounded by the unif boundedness of

$$S_N(x) = \sum_{-N}^{N} a_k e^{2\pi i k x}.$$

By baby calculus, H is Riemann integrable on any finite $[a,b]$. Hence $[0,1]$.

As we saw earlier, baby analysis $\Rightarrow$

$$\int_0^1 |H(x) - S_N(x)|\, dx \to 0 \quad \text{as } N \to \infty.$$

See Lec 9 p. 9.

By that same idea, we have:

$$\int_0^1 e^{-2\pi i m x} S_N(x)\, dx \longrightarrow \int_0^1 e^{-2\pi i m x} H(x)\, dx$$

for each $m \in \mathbb{Z}$. But LHS $= a_m + 0$!!

for large N

Hence,

$$a_m = \int_0^1 H(x) e^{-2\pi i m x}\, dx. \quad \blacksquare$$

Before continuing, observe that: $(\ell \geq 1)$

$$\frac{e^{2\pi i \ell x}}{-2\pi i \ell} + \frac{e^{-2\pi i \ell x}}{-2\pi i(-\ell)} = -\frac{1}{2\pi i \ell}\left(e^{2\pi i \ell x} - e^{-2\pi i \ell x}\right)$$

p. 3 line 11

$$= \frac{\sin(2\pi \ell x)}{-\pi \ell} .$$

Also write

$$\tilde{\beta}(y) = \left\{ \begin{array}{l} 0, \; y \in \mathbb{Z} \\ \beta(y), \; y \notin \mathbb{Z} \end{array} \right\} .$$

We already know that

$$\tilde{\beta}(x) = \sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{-\pi m} = \sum_{n \neq 0} -\frac{1}{2\pi i n} e^{2\pi i n x},$$

all $x \in \mathbb{R}$. Unif conv away from $\mathbb{Z}$; partial sums unif bounded. Similarly

$$\tilde{\beta}(x-c) = \sum_{n \neq 0} -\frac{e^{-2\pi i n c}}{2\pi i n} e^{2\pi i n x},$$

all $x \in \mathbb{R}$. By Baby Lemma on 9, automatically,

$$\int_0^1 \tilde{\beta}(x) e^{-2\pi i n x} dx = \left\{ \begin{array}{l} 0, \; n = 0 \\ -\frac{1}{2\pi i n}, \; n \neq 0 \end{array} \right\}$$

$$\int_0^1 \tilde{\beta}(x-c)\, e^{-2\pi i n x}\, dx = \left\{ \begin{array}{ll} 0, & n = 0 \\ -\dfrac{e^{-2\pi i n c}}{2\pi i n}, & n \neq 0 \end{array} \right\}.$$

THUS:

$$FS[\tilde{\beta}(x)] \equiv \sum_{n\neq 0} -\frac{1}{2\pi i n} e^{2\pi i n x}$$

$$FS[\tilde{\beta}(x-c)] \equiv \sum_{n\neq 0} -\frac{e^{-2\pi i n c}}{2\pi i n} e^{2\pi i n x}.$$

Obviously, the "$\sim$" can be removed from β.

[[These Fourier series can of course be checked directly, but we prefer the slick approach.]]

We now return to the PROOF of p. (8) THM.
Let $f(x)$ have nontrivial discontinuities at points $\{c_1, \ldots, c_m\}$ mod $\mathbb{Z}$. Let the "right$-$left" jump be J_i. Saying $J_i = 0$ means $f(c_i+0) = f(c_i-0)$ but $f(c_i) \neq f(c_i+0)$.

Recall that

$$\beta(0+) - \beta(0-) = -\frac{1}{2} - \frac{1}{2} = -1.$$

Define: $\boxed{\text{THIS IS THE TRICK}}$

$$g(x) = f(x) + \sum_{i=1}^{m} J_i \beta(x - c_i) \,, \quad x \in \mathbb{R} .$$

Fcn g is <u>very</u> interesting! It is obviously periodic 1. Also, it is obviously piecewise C^1. It <u>may</u> have discontinuities, but these lie in $\{c_1, \ldots, c_m\}$ mod $\mathbb{Z}$.

Note however that

$$\begin{aligned} g(c_i + 0) - g(c_i - 0) &= J_i - J_i + 0 \\ &= 0 \,, \quad \text{each } 1 \leq i \leq m . \end{aligned}$$

The points c_i are thus "removable discontinuities" <u>if</u> g is redefined correctly at these points.

Apply p. 6 bottom THM to this modified g. We conclude that $FS(g)$ converges <u>uniformly</u> <u>over</u> $\underline{\mathbb{R}}$ to $\frac{1}{2}[g(x+0) + g(x-0)]$. The partial sums are automatically uniformly bounded on $\mathbb{R}$.

By linearity, however, as series,

$$FS(f) \equiv FS(g) - \sum_{i=1}^{m} J_i \, FS[\beta(x - c_i)] .$$

At once, the partial sums of $FS(f)$ are unif bounded on $\mathbb{R}$ (by the corresponding fact for β).

Also, $FS(f)$ conv uniformly away from the $\{c_i\}$ mod $\mathbb{Z}$ (by the corr fact for β).

At points $x \not\equiv c_1, \ldots, c_m$ mod $\mathbb{Z}$, clearly $FS(f)$ converges to

$$g(x) - \sum_{i=1}^{m} J_i \beta(x - c_i) = f(x).$$

(Big surprise!!)

At c_i, $FS(f)$ converges to

$$\frac{1}{2}[g(c_i+0) + g(c_i-0)] - 0 - \sum_{l \neq i} J_l \beta(c_i - c_l).$$

But:

$$g(c_i+0) = f(c_i+0) + J_i(-\tfrac{1}{2}) + \sum_{l \neq i} J_l \beta(c_i - c_l)$$

$$g(c_i-0) = f(c_i-0) + J_i(\tfrac{1}{2}) + \sum_{l \neq i} J_l \beta(c_i - c_l)$$

$$\frac{g(c_i+0) + g(c_i-0)}{2} = \frac{f(c_i+0) + f(c_i-0)}{2} + \sum_{l \neq i} J_l \beta(c_i - c_l)$$

$\Downarrow$

$$FS(f) \text{ conv to } \frac{f(c_i+0)+f(c_i-0)}{2}$$

at each c_i. (Again, big surprise!!)

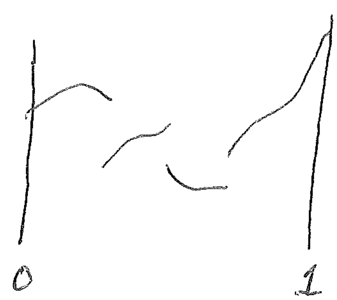
Thus, all is now proved.

---

famous formula

<u>THM</u> (Parseval's formula)

Let f be periodic 1, piecewise C^1 as in p. 8 THM.



We then have:

$$\int_0^1 |f(x)|^2\,dx = \sum_{k=-\infty}^{\infty} |c_k|^2 .$$

$\sum_{-N}^{N} c_k e^{2\pi i k x}$

<u>PF</u>

f is unif bdd on $\mathbb{R}$. We know $S_N(x)$ is unif bdd on $\mathbb{R}$ too. We also have $S_N(x) \rightrightarrows f(x)$ away from $\{c_1, \ldots, c_m\}$ mod $\mathbb{Z}$. Apply the idea of Lec 9 p. 9 again! (See p. 10 above.)

We get:

$$\int_0^1 f(x)\overline{S_N(x)}\,dx \to \int_0^1 f(x)\overline{f(x)}\,dx \qquad (N\to\infty)$$

but

$$LHS = \int_0^1 f(x)\overline{\left(\sum_{-N}^{N} c_k e^{2\pi i k x}\right)}\,dx$$

$$= \sum_{-N}^{N} c_k \bar{c}_k = \sum_{-N}^{N} |c_k|^2 . \qquad \blacksquare$$

The Fourier theory so far has been a kind of $L_\infty \times L_1$ theory. In traditional real analysis courses, one investigates to see if an $L_2 \times L_2$ theory might be better (or more natural).

We will not bother to pursue the latter beyond 2 very quick remarks.

Use of completing the square on integrals like

$$\int_0^1 |f(x)-S_N(x)|^2\,dx \quad , \quad \int_0^1 \left|F(x)-\sum_{-N}^{N} A_k e^{2\pi i k x}\right|^2 dx$$

for a general piecewise continuous, periodic 1, $F(x)$ leads to

$$\sum_{-N}^{N} |c_k|^2 \leq \int_0^1 |f(x)|^2 dx \qquad \text{(each } N\text{)}$$

$$\Downarrow$$

$$\boxed{\sum_{-\infty}^{\infty} |c_k|^2 \leq \int_0^1 |f(x)|^2 dx}$$

(i.e. Bessel's inequality).

Here $c_k = \int_0^1 f e^{-2\pi i k x} dx$.

(Actually, equality holds — but this is a harder theorem. One uses 15 Thm and "approximates" f by piecewise $\underline{\underline{C^1}}$ functions.) SEE ANY STANDARD BOOK ON F.S.

Our 2^{nd} remark is a theorem.

<u>THM</u> (slight strengthening of p. 6 bottom)

Let $f \in C(\mathbb{R})$, periodic 1, and be piecewise C^1.
Let $c_k = \int_0^1 f e^{-2\pi i k x} dx$. The Fourier series

$$\sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$$

then converges <u>uniformly</u> to $f(x)$ on $\mathbb{R}$

<u>AND</u> we also have

$$\sum_{-\infty}^{\infty} |c_k| < \infty.$$

Ie, have nice ABS conv!

Pf

Take $k \neq 0$. By standard integ by parts,

$$c_k = \frac{1}{2\pi i k} \int_0^1 f'(x) e^{-2\pi i k x} dx . \quad (4)$$

Again: NOTE THAT RHS is not affected by a few ambiguities in f'. Write the foregoing as

$$c_k = \frac{1}{2\pi i k} c_k(f')$$

and recall (17) box (Bessel's ineq). By Cauchy-Schwarz inequality,

$$\sum_{k=1}^{\infty} |c_k| = \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k} |c_k(f')|$$

$$\leq \frac{1}{2\pi} \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \sqrt{\sum_{k=1}^{\infty} |c_k(f')|^2} < +\infty .$$

Similarly for $k < 0$. ∎

Next topic: Poisson summation formula.

THM

Given $\varphi \in C^2(\mathbb{R})$ such that, say,

$$|\varphi(x)|, \ |\varphi'(x)|, \ |\varphi''(x)| \ \text{all} = O[(1+|x|)^{-2}].$$

Let
$$\hat{\varphi}(p) = \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i p x}\, dx, \quad p \in \mathbb{R}.$$

Let
$$F(x) \equiv \sum_{n=-\infty}^{\infty} \varphi(x+n), \quad x \in \mathbb{R}.$$

We then have
$$F(x) = \sum_{-\infty}^{\infty} \hat{\varphi}(k) e^{2\pi i k x}$$

Poisson summation formula

for $x \in \mathbb{R}$, with absolute and uniform conv on both sides over every interval $[-A, A]$.

Proof

The series $\sum_{-\infty}^{\infty} \varphi^{(j)}(x+n)$, $0 \leq j \leq 2$, are clearly conv. both abs and uniformly on every $[-A, A]$. As such, we immediately get $F \in C^2(\mathbb{R})$. It is also apparent that $F(x+1) = F(x)$.

Apply Thm 7 bottom. Get:

$$F(x) = \sum_{-\infty}^{\infty} A_k e^{2\pi i k x} \quad \text{nicely},$$

$$A_k = \int_0^1 F(x) e^{-2\pi i k x}\,dx .$$

But:

$$A_k = \int_0^1 \left(\sum_{-\infty}^{\infty} \varphi(x+n) \right) e^{-2\pi i k x}\,dx$$

$$= \sum_{-\infty}^{\infty} \int_0^1 \varphi(x+n) e^{-2\pi i k x}\,dx \quad \text{by unif conv}$$

$$= \sum_{-\infty}^{\infty} \int_n^{n+1} \varphi(y) e^{-2\pi i k y}\,dy = \int_{\mathbb{R}} \varphi(y) e^{-2\pi i k y}\,dy$$

$$= \hat{\varphi}(k) . \quad \blacksquare$$

Example

$$\varphi(x) = e^{-ax^2}, \quad a > 0$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \quad \Rightarrow$$

$$\int_{-\infty}^{\infty} e^{-ax^2} e^{-2\pi i p x}\,dx = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2 p^2}{a}}$$

(by elementary contour shift).

Hence, by Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} e^{-a(x+n)^2} = \sqrt{\frac{\pi}{a}} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi^2 k^2}{a}} e^{2\pi i k x} .$$

Special Case:

$$\sum_{n=-\infty}^{\infty} e^{-\pi\beta n^2} = \sqrt{\frac{1}{\beta}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{\beta}} \qquad (\beta > 0) .$$

The famous "θ" relation of Jacobi.

$$\theta(\beta) = \frac{1}{\sqrt{\beta}}\,\theta\left(\frac{1}{\beta}\right)$$

We are now ready to derive (following Riemann) a slick formula for $\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

Easily check:

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty x^{\frac{s}{2}} e^{-x} \frac{dx}{x} , \qquad Re(s) > 1 \text{ say}$$

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty y^{\frac{s}{2}} e^{-\pi n^2 y} \frac{dy}{y}$$

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty y^{\frac{s}{2}} \left[\sum_{n=1}^{\infty} e^{-\pi n^2 y}\right] \frac{dy}{y}$$

A nice positive fcn. Clearly $O(1/\sqrt{y})$ for $y \to 0^+$ by Θ-relation.

{ Note: the foregoing integral is nicely convergent near $y = 0$ because

$$\int_0^1 y^{\frac{\sigma}{2}} \frac{1}{\sqrt{y}} \frac{dy}{y} < \infty \quad \text{for } \sigma > \underline{\underline{1}}$$
}.

DO NOT CONFUSE WITH PNT ψ and θ

Write

$$\psi(y) = \sum_{n=1}^{\infty} e^{-\pi n^2 y} \quad \text{and} \quad \theta(y) = 2\psi(y) + 1 .$$

So:

$$\psi(y) + \frac{1}{2} = \frac{1}{\sqrt{y}} \left[\psi\left(\frac{1}{y}\right) + \frac{1}{2} \right], \quad y > 0$$

$$\psi(y) = -\frac{1}{2} + \frac{1}{2} y^{-1/2} + y^{-1/2} \psi\left(\frac{1}{y}\right) .$$

Get:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^1 y^{\frac{s}{2}} \psi(y) \frac{dy}{y} + \int_1^{\infty} y^{\frac{s}{2}} \psi(y) \frac{dy}{y}$$

put $y = \frac{1}{v}$ here

{ now grind! }

$$= \int_1^\infty v^{-\frac{s}{2}} \left[-\tfrac{1}{2} + \tfrac{1}{2} v^{1/2} + v^{1/2} \psi(v) \right] \underline{\frac{dv}{v}}$$

$$+ \int_1^\infty y^{\frac{s}{2}} \psi(y) \frac{dy}{y}$$

$$= -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty v^{\frac{1-s}{2}} \psi(v) \underline{\frac{dv}{v}}$$

$$+ \int_1^\infty y^{\frac{s}{2}} \psi(y) \frac{dy}{y}$$

$$= -\left[\frac{1}{s} + \frac{1}{1-s} \right] + \int_1^\infty \left(y^{\frac{1-s}{2}} + y^{\frac{s}{2}} \right) \psi(y) \frac{dy}{y}$$

$$= -\frac{1}{s(1-s)} + \int_1^\infty \left(y^{\frac{1-s}{2}} + y^{\frac{s}{2}} \right) \psi(y) \frac{dy}{y} .$$

So, for $\mathrm{re}(s) > 1$,

$$\pi^{-\frac{s}{2}} \Gamma\!\left(\tfrac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(y^{\frac{1-s}{2}} + y^{\frac{s}{2}} \right) \psi(y) \frac{dy}{y} .$$

NOTE
invariance under
$s \leftrightarrow 1-s$

$O(e^{-\pi y})$
as $y \to +\infty$

The integral is analytic for <u>all</u> $s \in \mathbb{C}$.

The $\dfrac{1}{s(s-1)}$ is trivially analytic on $\mathbb{C} - \{0,1\}$.

<u>Theorem</u> (Functional Equation)

$\xi(s) \equiv \pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s)$ is analytic on $\mathbb{C} \setminus \{0,1\}$ and satisfies

$$\xi(s) = \xi(1-s).$$

We also have for ξ:

$s=1$ simple pole, residue 1 ;
$s=0$ simple pole, residue -1.

<u>Pf</u>

The first part is just 23 bottom. OK

By 23 bottom, with $s = 1+h$, we get

$$\xi(1+h) = \frac{1}{(1+h)h} + O(1)$$

$$= \frac{1}{h} + O(1).$$

And, similarly, with $s = h$,

$$\xi(h) = \frac{1}{h(h-1)} + O(1) = -\frac{1}{h} + O(1).$$ ∎

Cor 1

$\xi_0(s) = s(s-1)\xi(s) = s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ is an entire fcn which satisfies

$$\xi_0(s) = \xi_0(1-s), \quad \xi_0(1) = 1.$$

Pf

$\xi_0(s) = 1 + s(s-1)\int_1^\infty (y^{\frac{1-s}{2}} + y^{\frac{s}{2}})\psi(y)\frac{dy}{y}$ by 23. ∎

Cor 2

$\zeta(-2k) = 0$ for $k \geq 1$ (simple zero).

Pf

$\xi(x) = \pi^{-x/2}\Gamma(\frac{x}{2})\zeta(x) > 0$ for $x > 1$. But $\xi(x) = \xi(1-x)$. Hence $\xi(x) > 0$ for $x < 0$. Let $x \to -2k$. Since $\Gamma(\frac{x}{2}) \to \Gamma(-k)$ simple pole, get $\zeta(x) \to 0$ à la simple zero. ∎

Lemma

$$(1-2^{1-s})\zeta(s) = \sum_{k=1}^{\infty}\left((2k+1)^{-s}-(2k)^{-s}\right)$$

for $\mathrm{Re}(s)>1$ and the RHS is actually analytic for $\{\mathrm{Re}(s)>0\}$.

Pf

$\mathrm{Re}(s)>1 \Rightarrow$

$$\zeta(s) = \sum_{k=1}^{\infty}(2k-1)^{-s} + \sum_{k=1}^{\infty}(2k)^{-s} \quad \text{trivially}$$

$$2^{1-s}\zeta(s) = 2\sum_{m=1}^{\infty}(2m)^{-s}$$

$$\text{difference} = \sum_{k=1}^{\infty}\left((2k-1)^{-s}-(2k)^{-s}\right) \quad \left\{\begin{matrix}\text{nice}\\ \text{abs}\\ \text{conv}\end{matrix}\right\}.$$

Keep $s\in K$ where K is a compact subset of $\{\mathrm{Re}(s)>0\}$. Observe that:

$$(2k-1)^{-s}-(2k)^{-s} = (2k)^{-s}\left[1-\frac{1}{2k}\right]^{-s} - (2k)^{-s}$$

$$\left\{ (1+u)^{-s} = 1+(-s)u + \underline{O_K}(1)u^2, \; |u|\leq \frac{3}{4} \right\}$$

$$(2k-1)^{-s}-(2k)^{-s} = (2k)^{-s}\left[\frac{s}{2k} + O(1)\frac{1}{k^2}\right]$$

$$= O(1)k^{-s-1} \quad \text{for } s\in K. \quad \blacksquare$$

Corollary

In the sense of analytic continuation,

$$\xi(x) \neq 0 \quad \text{for} \quad x \in \mathbb{R}$$

$$\xi_0(x) \neq 0 \quad \text{for} \quad x \in \mathbb{R}$$

$$\zeta(x) < 0 \quad \text{for} \quad 0 < x < 1.$$

Proof

That $\zeta(x) < 0$ on $0 < x < 1$ is obvious from (26). Hence $\xi(x) \neq 0$ on $0 < x < 1$. The points $x = 0, 1$ are poles and take care of themselves. For $x > 1$ and $x < 0$, we have $\xi(x) > 0$ à la (25). Since $\xi_0(x) = x(x-1)\xi(x)$, the assertions for ξ_0 are immediate. ∎

We wish to bound the size of

$$F(z) \equiv z(z-1)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

(roughly) using Stirling, $F(z) = F(1-z)$, and basic properties of $\zeta(z)$.

Because of $F(z) = F(1-z)$, we can clearly restrict to $\mathrm{Re}(z) \gtrsim \frac{1}{2}$.

We had

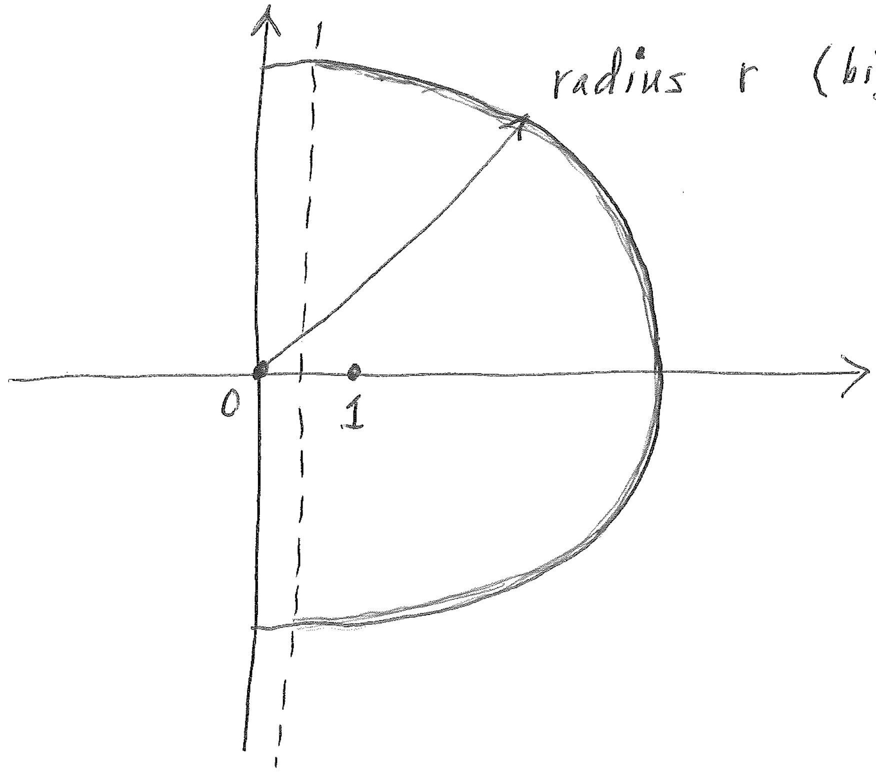
$$|\zeta(x+iy)| \lesssim \frac{C}{\delta(1-\delta)}|y|^{1-\delta} \quad \left\{\begin{matrix} x \gtrsim \delta \\ |y| \gtrsim 2 \end{matrix}\right\}$$

any $0 < \delta < 1$. Lec 6 page 9. EG $\delta = \frac{1}{2}$.

Also, we had

$$|\zeta(z) - 1| < \frac{3}{4} \quad \text{for} \quad \mathrm{Re}(z) > 2$$

by Lec 5 page 10.



$|z| = r$

$$F(z) = z(z-1)\,\pi^{-\frac{z}{2}}\,\Gamma\!\left(\tfrac{z}{2}\right)\zeta(z)$$

$$|F(z)| = |z|\,|z-1|\,\pi^{-\frac{x}{2}}\left|\Gamma\!\left(\tfrac{z}{2}\right)\right|\,|\zeta(z)|$$

$$|F(z)| = |z|^2\left[1+O\!\left(\tfrac{1}{r}\right)\right]\pi^{-\frac{x}{2}}\left|\Gamma\!\left(\tfrac{z}{2}\right)\right|\,|\zeta(z)|$$

Know:

$$|\zeta(z)-1| < \frac{3}{4} \quad \text{for } x > 2$$

$$|\zeta(x+iy)| \lesssim O(|y|^{1/2}), \quad x \gtrsim \tfrac{1}{2},\ |y| \gtrsim 2 .$$

Also:

$$\ln\left|\Gamma\!\left(\tfrac{z}{2}\right)\right| = \mathrm{Re}\left\{\mathcal{L}og\,\Gamma\!\left(\tfrac{z}{2}\right)\right\}$$

↑ Stirling Lec 10 p. 42

and

$$\operatorname{Log} \Gamma\left(\frac{z}{2}\right) = \left(\frac{z}{2}-\frac{1}{2}\right)\operatorname{Log}\left(\frac{z}{2}\right) - \frac{z}{2} + \frac{1}{2}\ln(2\pi) + O(1/|z|)$$

$\{$ for, say, $|z| = r$, r large, $|\operatorname{Arg} z| \leqq \frac{3}{4}\pi \}$.

As in Ingham 56–57, we get

$$\ln|\Gamma(\tfrac{z}{2})| \leqq \frac{r}{2}\ln r + A_1 r$$

$\{$ for, say, $|z| = r$, $|\operatorname{Arg} z| \leqq \frac{3}{4}\pi \}$

$$\Downarrow$$

$$\ln|F(re^{i\theta})| \leqq \frac{r}{2}\ln r + A_2 r$$

for $|z| = r$, $Re(z) \geqq \frac{1}{2}$

$$\boxed{\text{then, } \underline{\text{using}}\ F(z) = F(1-z), \text{ similarly for } Re(z) \leqq 1/2}$$.

Also, looking at $\theta = 0$,

$$F(R) = R^2\left[1 + O\left(\frac{1}{R}\right)\right]\pi^{-\frac{R}{2}}\Gamma\left(\frac{R}{2}\right)\zeta(R)$$

$$\gtrapprox (\text{constant})\, R^2\, \pi^{-\frac{R}{2}}\,\Gamma\left(\frac{R}{2}\right)$$

$$\left\{ \begin{array}{l} \text{but} \\ \underline{\underline{\ln}}\, \Gamma\left(\frac{R}{2}\right) \sim \left(\frac{R}{2} - \frac{1}{2}\right)\ln\left(\frac{R}{2}\right) - \frac{R}{2} \end{array} \right\}$$

$$\Downarrow$$

$$\underline{\underline{\ln}}\, F(r) \gtrapprox \frac{r}{2}\ln r - A_3 r \qquad (r \text{ large}).$$

<u>THM</u>

Let $F(z) = z(z-1)\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)$. Let

$$M(r) = \max_{|z|=r} |F(z)| \qquad (r \text{ large}).$$

Then:

$$\underline{\underline{\ln}}\, M(r) \sim \frac{r}{2}\ln r.$$

Proof

As above. ▥

For any entire fcn $g(z)$, $g \not\equiv 0$, we write

$$M(r) = \max_{|z|=r} |g(z)| .$$

Then put:

$$\rho = \inf\{\omega : \underline{\ln} M(r) \leqq r^{\omega}, \text{ all large } r\}$$

$$\tau = \inf\{\beta : \underline{\ln} M(r) \leqq \beta r^{\rho}, \text{ all large } r\}.$$

Herein $\omega \geqq 0$ and $\beta \geqq 0$. Empty braces mean $\inf = +\infty$. We call:

ρ = ORDER of $g(z)$

τ = TYPE of $g(z)$.

$$\boxed{\text{For our } F(z), \text{ clearly } \rho = 1 \text{ and } \tau = +\infty.}$$