Lecture 12

(26 Feb)

2 Notes!

Lec 10 p. 42 Stirling (Corollary).

We also have:

Thm (Stirling)

$$\frac{\Gamma'(z)}{\Gamma(z)} = \operatorname{Log} z - \frac{1}{2z} + \sum_{k=1}^{R} \left(-\frac{B_{2k}}{2k}\right) z^{-2k} + O_{R\delta}\left(\frac{1}{|z|^{2R+1}}\right)$$

as $z \to \infty$ in $|\operatorname{Arg} z| \leq \pi - \delta$.

Pf

Call the $\tilde{B}_{2R+1}(t)$ integral term in 42 Thm $r(z)$.
Note $r(z)$ is nicely analytic and $r(z) = O(z^{-2R-1})$
by the Cor on 42. But:

$$r'(z) = \frac{1}{2\pi i} \oint_{|w-z|=1} \frac{r(w)}{(w-z)^2}\,dw .$$

Just use $|\operatorname{Arg} z| \leq \pi - 2\delta$ in place of $|\operatorname{Arg} z| \leq \pi - \delta$.
Done. ∎

About $\Gamma(z) \neq 0$, Lec 10 p. 26. One can avoid Hurwitz's thm.

Thm
Let $f_n(z) \rightrightarrows f(z)$ on $|z| < R$ compacta. We assume f_n, f are analytic. Let $f_n(z) \neq 0$ for all z and $f(z) \not\equiv 0$. Then: $f(z) \neq 0$.

Pf

Zeros of f are isolated. Hence finite # on each $|z| \leq R - \varepsilon$. Find $R_n \uparrow R$ so $f(R_n e^{i\theta}) \neq 0$.

Fix any N. Find $m, M > 0$ so $m \leq |f(R_N e^{i\theta})| \leq M$.
By unif conv,

$$\frac{m}{2} \lesssim |f_n(R_N e^{i\theta})| \lesssim 2M \quad , \quad n \gtrsim \mathcal{N}.$$

Apply max mod principle to f_n AND $1/f_n$. Get

$$|f_n(z)| \lesssim 2M \qquad\qquad n \gtrsim \mathcal{N}$$

$$\left|\frac{1}{f_n(z)}\right| \lesssim \frac{2}{m}$$

on $|z| \leq R_N$. So,

$$\frac{m}{2} \lesssim |f_n| \lesssim 2M.$$

Let $n \to \infty$ to get $\frac{m}{2} \leq |f| \leq 2M$ on $|z| \leq R_N$.
Now let $N \to \infty$. Done. ■

$\boxed{\text{END OF NOTES}}$

We then turned to the issue of the entire fcn

$$\xi_0(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$\xi_0(s) = \xi_0(1-s)$$

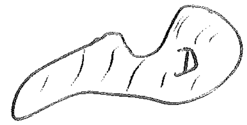
Lec 11 p. 25 31

order 1, type ∞

and trying to get a product expansion over the zeros. Ie, trying to get a "Hadamard factorization" of ξ_0 to justify Riemann's unproved assertion. [from 1859]

Standard lemmas.

<u>Lemma 1</u>



D = simply-connected domain.
Let $f = u + iv$ be analytic on D.
Then: u is harmonic on D (i.e. C^2 and $u_{xx} + u_{yy} = 0$).

Conversely, given real-valued U harmonic on D. We can cook up V, harmonic on D, so $F = U + iV$ is analytic on D.

<u>Cor</u>

Every harmonic U on D is actually C^∞.

<u>Lemma 2</u> (mean-value property)

Let u be harmonic on D (as above).
Let $|z - z_0| \leq R$ be contained in D.
Then
$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\varphi})\,d\varphi .$$

<u>Lemma 3</u>

D as above. Let g be analytic on D and $g(z) \neq 0$. We can always find an <u>analytic</u> function $\phi(z)$ on D such that $\exp(\phi) = g$.

[ϕ is unique up to $+2\pi i k$]

Theorem (Jensen's formula) ← Lemma 4

D as above. Let $\{|z| \leq R\} \subseteq D$. Let f be analytic on D, $f \neq 0$ on $|z| = R$, $f(0) \neq 0$.

Then:

$$\ln|f(0)| + \sum_{j=1}^{m} \ln \frac{R}{|a_j|} = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\varphi})|\,d\varphi .$$

Here $\{a_1, \ldots, a_m\}$ are the zeros of f in $0 < |z| < R$ listed with multiplicity.

Pf

Wlog $D = \{|z| < R + \varepsilon\}$, ε tiny.

Wlog $f \neq 0$ on $\{R \leq |z| < R + \varepsilon\}$. Form analytic fcn:

$$F(z) = f(z) \prod_{j=1}^{m} \frac{R^2 - \bar{a}_j z}{R(z - a_j)} .$$

Get $|F| = |f|$ on $|z| = R$, $F(z) \neq 0$ on $|z| < R + \varepsilon$.

Apply Lemma 3 to get $\operatorname{Log} F(z)$. By lemma 2+1,

$$\ln|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|F(Re^{i\varphi})|\,d\varphi .$$

Done. ∎

If $f(0) = 0$, people usually just pass to $\frac{f(z)}{z^N}$.

<u>Thm</u> (Lemma 5)

Let f be entire of order $\leq \rho$. $(f \not\equiv 0)$

Then, counting with multiplicity,

$$n(r) \equiv N[\underline{\text{zeros}} \text{ of } f \text{ in } |z| \leq r] = O(r^{\rho+\varepsilon})$$

for all r large. Here $\varepsilon > 0$.

<u>Pf</u>

$f(0) = 0 \Rightarrow$ pass to $g = \dfrac{f(z)}{z^N}$. g is still entire and has order $\leq \rho$.

WLOG $f(0) = 1$. Know $\ln M(R; f) \leq R^{\rho+\varepsilon}$, large R.

Perturb R slightly to make $f(Re^{i\theta}) \neq 0$.

Apply Lemma 4 (Jensen):

$$0 + \sum_{j=1}^{m} \ln \frac{R}{|a_j|} = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\varphi})|\, d\varphi \leq R^{\rho+\varepsilon}.$$

Hence:

$$n\left(\frac{R}{2}\right) \ln 2 \leq R^{\rho+\varepsilon}$$

$$\Rightarrow n(r) = O(r^{\rho+\varepsilon}) \quad \text{all large } r. \quad \blacksquare$$

KEY THM (Lemma 6, Hadamard/Borel/Caratheodory)

D as above. f analytic on D.
Suppose $\{|z-z_0| \leq R\} \subseteq D$. Let $f = \sum_0^\infty c_n (z-z_0)^n$
on the closed disk.

Assume further that

$$\operatorname{Re} f(z) \leq \mathcal{M} \quad \longleftarrow \text{KEY}$$

on the closed disk. Then:

(A) $$|c_n| \leq \frac{2}{R^n}(\mathcal{M} - \operatorname{Re} c_0), \quad n \geq 1$$

(B) $$|f(z) - f(z_0)| \leq \frac{2r}{R-r}\{\mathcal{M} - \operatorname{Re} c_0\}, \quad |z-z_0| \leq r < R$$

(C) $$\left|\frac{f^{(k)}(z)}{k!}\right| \leq \frac{2R}{(R-r)^{k+1}}\{\mathcal{M} - \operatorname{Re} c_0\}, \quad k \geq 1.$$

<u>Pf</u>

See Ingham 50-51. (There is a) Famous trick in this proof (starts 50 bottom). ▨

Lemma 7

Suppose we have an entire fcn f such that $f(0) \neq 0$. Let its zeros (listed with multiplicity) be $\{a_j\}$. WLOG $|a_j| \leq |a_{j+1}|$. Assume that we know $n(r) = O(r^\beta)$ for large r. Then:

(5) $$\sum_n \frac{1}{|a_n|^\gamma} < \infty \quad \text{for each } \gamma > \beta.$$

Pf

Take δ tiny. Look at $\int_\delta^T r^{-\gamma} dn(r)$ and integrate by parts. ■

Corollary

Let f be entire and $f(0) \neq 0$. Then:

$$\sum \frac{1}{|a_n|^{\rho+\varepsilon}} < \infty, \quad \text{each } \varepsilon > 0.$$

Pf

Lemma 5 + 7. ■

Thus, we always have (for $f(z)$ entire)

$$\sum \frac{1}{|a_n|^{[\![\rho]\!]+1}} < \infty.$$

We let

$$\boxed{p = [\![\rho]\!]}$$

(Do not confuse p with a prime!)

when we play with a given f.

When p = non-neg integer, following Weierstrass it is customary to define:

$$E(u;p) = \left\{ \begin{array}{ll} 1-u, & p = 0 \\ & \\ (1-u)\exp\left[u + \frac{u^2}{2} + \dots + \frac{u^p}{p}\right], & p \geq 1 \end{array} \right\}.$$

Note that $E(z;p)$ is entire.

Take $|u| \leq h < 1$. With <u>some</u> branch of $\log$,

$$\log E(u;p) = \mathrm{Log}(1-u) + u + \dots + \frac{u^p}{p}$$

$$= -\sum_{n=1}^{\infty} \frac{u^n}{n} + u + \dots + \frac{u^p}{p}$$

$$= -\sum_{n=p+1}^{\infty} \frac{u^n}{n}.$$

Clearly,

$$|\log E(u;p)| \leq \frac{|u|^{p+1}}{1-|u|} \qquad (p=0 \text{ ok too}).$$

Hence:

$$\ln |E(u;p)| \leq \frac{|u|^{p+1}}{1-h}, \qquad |u| \leq h < 1.$$

Given $p \geq 0$. Also given $a_n \in \mathbb{C} \sim \{0\}$, $a_n \to \infty$, and

$$\sum_n \frac{1}{|a_n|^{p+1}} < \infty .$$

We call

$$\prod_{n=1}^{\infty} E\left(\frac{z}{a_n}; p\right)$$

a CANONICAL PRODUCT of genus p.

THM

In the above, the canonical product of genus p converges uniformly & absolutely on $\mathbb{C}$-compacta. Hence it is an entire function with zeros exactly at $\{a_n\}$.

Pf

We use our standard reduction to the "$\sum_{k=1}^{\infty} \operatorname{Log}(1+b_k(z))$ theorem".

Take $K = \{|z| \leq R\}$. Restrict attention to $|a_n| > 1000R$. Hence, in product, each term has

$$\left|\frac{z}{a_n}\right| < \frac{1}{1000} \quad \text{for } z \in K .$$

Get:

$$\left|\log E\left(\frac{z}{a_n};p\right)\right| \leq \frac{\left|\frac{z}{a_n}\right|^{p+1}}{1-\frac{1}{1000}}$$

$$\leq (1.01)\left|\frac{z}{a_n}\right|^{p+1}$$

$$\leq (1.01)\left(\frac{1}{1000}\right)^{p+1}$$

$$\leq .002 \quad .$$

Therefore, the "log" is actually <u>Log</u>.

And:

$$\left|\operatorname{Log} E\left(\tfrac{z}{a_n};p\right)\right| \leq .002$$

$$\left|E\left(\tfrac{z}{a_n};p\right)-1\right| \leq .01$$

i.e. " $|b_n(z)| \leq .01$ " (on K).

Must look at

$$\sum_n \left|\operatorname{Log}(1+b_n(z))\right|$$

on K. This sum will be

$$\leq \sum_n (1.01)\left(\frac{R}{|a_n|}\right)^{p+1} \quad \{\text{by the above}\}$$

for all $z \in K$. $\Rightarrow$ all is OK. ∎

When we study canonical products, it is helpful to conceptualize them as

$$\prod E\left(\tfrac{z}{a_n}; p\right) \equiv \prod_{|a_n| \leq 1000R} E\left(\tfrac{z}{a_n}; p\right) \cdot \prod_{|a_n| > 1000R} E\left(\tfrac{z}{a_n}; p\right)$$

THIS PART IS NONZERO

over $\{|z| \leq R\}$.

<u>THEOREM</u> (preliminary factorization)

Let f be entire. Let the order be $\rho < \infty$. Put $p = [\![\rho]\!]$. Let the <u>zeros</u> of f in $\mathbb{C} \sim \{0\}$ be $\{a_n\}$. [This set could be empty!] We then have:

$$f(z) = z^N \exp[\phi(z)] \prod_{a_n \neq 0} E\left(\tfrac{z}{a_n}; p\right),$$

where ϕ is <u>some</u> entire fcn and where the product (if infinite) is abs + uniformly conv on $\mathbb{C}$ compacta.

Pf

Pass first to $g(z) \equiv \frac{f(z)}{z^N}$, as usual.

The fcn g is entire, order ρ, $g(0) \neq 0$.

Now review 9 and form

$$h(z) \equiv \frac{g(z)}{\prod_{a_n \neq 0} E(\frac{z}{a_n}; p)} .$$

See 11 top. Get $h(z) \neq 0$ for all $z \in \mathbb{C}$.

By Lemma 3 applied to $h(z)$, we can write $h = \exp(\phi(z))$. Done. ■

Hadamard realized, in studying Riemann's work, that he needed some way of controlling $\phi(z)$ using <u>only</u> information about

<u>Re</u> $\phi(z)$.

This is what led to p. 6 Key Thm!

Hadamard's Factorization Theorem ~1893

Given the situation of p. 11 THM.
We then have that $\phi(z)$ must be a **polynomial** of degree $\leqq p$.

(Recall that $p = [\![\rho]\!]$.)

In the case of $\Xi_0(s)$, we had $\rho = 1$ and type $\tau = \infty$. So, here,

$$\Xi_0(s) = e^{As+B} \prod_n E\left(\frac{s}{a_n}; 1\right).$$

Lec 11 p. 25 31 32

The (current-day) proof of the HFT either follows an approach of Landau or else one based on the so-called Poisson – Jensen formula [a very common identity used in Nevanlinna theory]. The proof is **function theory**, not number theory.

The Landau approach is remarkable for its simplicity. See:

Landau, <u>Vorlesungen</u> <u>über</u> <u>Zahlentheorie</u>, Satz 423 (from 1927)

OR

Landau, <u>Math. Zeitschrift</u> 26 (1927) 170–175.

INGHAM, pages 54 (bottom) – 55 (bottom) is compressed, but follows LANDAU.

I presented the details of this in Lecture #12 and the first half of Lecture #13. I <u>OMIT</u> them here!