Lecture 13

(2 Mar 2016)

I began by finishing the proof of the Hadamard factorization thm.

I then turned to some simple function-theoretic facts and some corollaries of HFT.

Simple Facts

(I) f order ρ, g order $\rho' < \rho$ $\Rightarrow$ $f+g$ order ρ

(II) f order ρ_1, g order ρ_2 $\Rightarrow$ fg order $\leq \max\{\rho_1, \rho_2\}$

(III) $p(z) \not\equiv 0$ polynomial, f order ρ

$\Rightarrow p(z)f(z)$ order ρ too

(IV) f order ρ. Zeros at $\{a_1, \ldots, a_M\}$. Then

$$g(z) = \frac{f(z)}{(z-a_1)\cdots(z-a_M)} \quad \text{order } \rho.$$

(V) Let f have order ρ. Let $z_0 \in \mathbb{C}$. Then:

$h(z) = f(z+z_0)$ has order ρ.

{Must fiddle and use max mod principle.}

(VI) Let f be entire, even, and order ρ.
Then
$$g(z) = f(z^{1/2}) \quad \text{has order } \rho/2.$$

Cor 1 to HFT

Let f be entire, order ρ, $\rho \notin \mathbb{Z}$.
Let $a \in \mathbb{C}$. Then $f(z) = a$ has infinitely many roots.

Pf

$\rho \notin \mathbb{Z} \Rightarrow \rho > 0$. Let $g = f(z) - a$.
g has order ρ. Suppose $g(z)$ has only finitely many zeros. $\rho(g) = \rho$, $p = [\![\rho]\!]$.
Apply HFT.
$$g(z) = z^N e^{\phi(z)} \prod_{n=1}^{\mathcal{N}} E\left(\tfrac{z}{a_n}; p\right), \quad \mathcal{N} < \infty$$

But $p < \rho < p+1$. RHS has order AT MOST p since $\deg \phi \leq p$. {Recall def of E, Lec 12 p. 8.}
Contradiction! ■

Cor 2 to HFT (a form of the baby Picard thm)

Let f be entire, order $\rho > 0$, $\rho \in \mathbb{Z}$.
Then $f(z)$ assumes every $a \in \mathbb{C}$ with AT MOST ONE exception.

Pf
Suppose 2 exceptions: $f \neq \alpha$, $f \neq \beta$. Write

$$g(z) = \frac{f(z) - \alpha}{\beta - \alpha} .$$

Order ρ again. And $g \neq 0, 1$.

Apply HFT. Get $g = \exp(\phi)$, ϕ = polynomial, degree $\leq p$. HERE $p = \rho$.

But, order of $g(z)$ is $\rho = p$, so $\deg \phi = p$.
Since $p \geq 1$, we can solve $\phi(z) = 2k\pi i$ for any integer k. Thus, we get many points z_k where $g(z_k) = 1$. Contrad! ∎

WE NOW GO TO $\xi_0(s)$!!

THM

Recall $\xi_0(s) = s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ àla Lec 11 p. 25 + 31. We have $\xi_0(s) = \xi_0(1-s)$, $\xi_0(1) = 1$. Also order $\rho = 1$ and type $\tau = \infty$.

$\xi_0(s)$ has infinitely many zeros; these lie exclusively in $\{0 < Re(s) < 1\}$.

Pf

Let $f(z) = \xi_0(\frac{1}{2} + z)$ so f has order 1 AND $f(z)$ IS EVEN. Form $g(z) = f(z^{1/2})$, which has order $1/2$. Page 2 top.

By Cor 1, g has infinitely many zeros! Hence so does f, and hence ξ_0.

By Lec 11 p. 27, $\xi_0 \neq 0$ along the real axis.

For $Re(s) > 1$, we know $s(s-1) \neq 0$, $\pi^{-s/2} \neq 0$, $\Gamma(\frac{s}{2}) \neq 0$, $\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \neq 0$.

The same is true for $s = 1 + it$, t real $\neq 0$.

(Recall $\zeta(1+it) \neq 0$ was proved Lec 6 p. 7.)

Hence, $\xi_0(s) \neq 0$ for all $\mathrm{Re}(s) \geqq 1$.

By $\xi_0(s) = \xi_0(1-s)$, get same for $\mathrm{Re}(s) \leqq 0$.

So, all zeros lie in $\{0 < \mathrm{Re}(s) < 1\}$. ∎

$\xi_0(1) = \xi_0(0) = 1$, $\xi_0(s) = \xi_0(1-s)$.

HFT now implies

$$\boxed{\xi_0(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}}$$

$$0 < \mathrm{Re}(\rho) < 1.$$

It might be better to use a letter other than ρ (to avoid confusion with order ρ). But "everyone" uses ρ for these zeros, Riemann, Landau, etc etc.

$$\text{Order} = 1, \quad [\![\text{order}]\!] + 1 = 2 \quad \text{for } \xi_0.$$

So,

$$\sum_{\rho} \frac{1}{|\rho|^2} < \infty.$$

We had $\xi_0(x) \neq 0$ for $x \in \mathbb{R}$, so

$$\boxed{\mathrm{Im}(\rho) \neq 0}.$$

Recall that:

$$\xi_0(s) = 1 + s(s-1) \int_1^\infty \left[y^{\frac{1-s}{2}} + y^{\frac{s}{2}} \right] \psi(y) \frac{dy}{y}$$

$$= \sum_1^\infty e^{-\pi n^2 y}$$

àla Lec 11 p. 25. Clearly

$$\xi_0(\bar{s}) = \overline{\xi_0(s)}, \quad \text{all } s \in \mathbb{C}.$$

Hence any zeros occur in conjugate pairs. The CANONICAL PRODUCT on 5 is thus real-valued for $s \in \mathbb{R}$. So is $\xi_0(s)$. As such, we conclude $\underline{\underline{B}}$ must be real.

Letting $s \to 0$ on 5 middle, we get $e^A = 1$, so WLOG $A = 0$.

With <u>some</u> branches,

$$\log \xi_0(s) = A + Bs + \sum_\rho \log \left\{ \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \right\} .$$

Keep s on some $\{|s| \leq R\}$ at first!

Following Riemann, get:

- $$\frac{\xi_0'(s)}{\xi_0(s)} = B + \sum_\rho \left\{ \frac{1}{s-\rho} + \frac{1}{\rho} \right\}$$

$\left\{ \begin{array}{l} \text{with nice convergence on} \\ \mathbb{C}\text{-compacta away from the } \rho\text{'s} \\ \text{(Weierstrass conv theorem)} \end{array} \right\}.$

But, ④ line 2,

$$\log \xi_0(s) = \log s + \log(s-1) - \frac{s}{2} \ln \pi + \log \Gamma\left(\frac{s}{2}\right) + \log \zeta(s)$$

$$\Downarrow$$

$$\frac{\xi_0'(s)}{\xi_0(s)} = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2} \ln \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{\zeta'(s)}{\zeta(s)}$$

$$\Gamma(z+1) \equiv z\,\Gamma(z)$$

$$\frac{\Gamma'}{\Gamma}(z+1) \equiv \frac{1}{z} + \frac{\Gamma'}{\Gamma}(z)$$

$$\Downarrow$$

$$\frac{\xi_0'(s)}{\xi_0(s)} = \frac{1}{s-1} - \frac{1}{2}\ln\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \frac{\zeta'(s)}{\zeta(s)} .$$

On 7 bottom, if desired, one could also substitute

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right)$$

from Lec 10 p. 30. We'll <u>skip</u> this for now.

Combine 7 line 5 with line 3 above. Get:

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \left(B + \frac{1}{2}\ln\pi\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)$$
$$+ \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

Thm (Riemann)

Away from the set $\{1\} \cup \{\rho\} \cup \{-2k\}_{k=1}^{\infty}$ we have

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \left(B + \frac{1}{2}\ln\pi\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)$$

$$+ \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Pf

As above. ∎

$$3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2$$

$$\geq 0$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{2}^{\infty}\frac{\Lambda(n)}{n^s} = \sum_{2}^{\infty}\frac{\Lambda(n)}{n^\sigma}n^{-it}$$

$$\mathrm{Re}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] = \sum_{2}^{\infty}\frac{\Lambda(n)}{n^\sigma}\cos(t\ln n)$$

$$\sigma = \mathrm{Re}(s) > 1.$$

Fact

$\sigma > 1$, $t \in \mathbb{R}$, $t \neq 0$

$$\mathrm{Re}\left[3\frac{\zeta'}{\zeta}(\sigma) + 4\frac{\zeta'}{\zeta}(\sigma+it) + \frac{\zeta'}{\zeta}(\sigma+2it) \right] \leqq 0.$$

Pf

As above. ▨

Abbreviate Thm on p. 9 as

$$\frac{\zeta'(s)}{\zeta(s)} = -g(s) + f(s)$$

$$\uparrow$$

$$\sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

For a few moments. Keep $\sigma > 1$.

We immediately get:

$$\mathrm{Re}[3f(\sigma) + 4f(\sigma+it) + f(\sigma+2it)]$$

$$\leqq \mathrm{Re}[3g(\sigma) + 4g(\sigma+it) + g(\sigma+2it)],$$

$t \neq 0$ real, $\sigma > 1$.

THM (classical zero-free region)

There exists an absolute constant $a > 0$ so that

$$\zeta(s) \neq 0 \quad \text{on} \quad \left\{ \sigma > 1 - \frac{a}{\log(|t|+2)} \right\}.$$

Proof

Essentially following INGHAM 59–60.

WLOG $t > 0$.

We can always re-adjust "a" to take care of bounded t. So, WLOG, we need only look at the case of $t > G$, $G =$ giant.

We play with (10) bottom for $1 < \sigma < 3$, $t > G$. Remember that

$$g(s) = \frac{1}{s-1} - b + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)$$

for some real constant b. Hence $g(s)$ is rather explicit.

Recall Stirling for $\frac{\Gamma'}{\Gamma}$; Lec 12 p. 1.

We clearly get

$$g(s) = O(\ln t) \qquad \underline{1 < \mathrm{Re}(s) < 3}$$

anytime $\mathrm{Im}(s) \geqq 100$, say.

Of course, for $1 < \sigma < 3$,

$$g(\sigma) = \frac{1}{\sigma - 1} + O(1).$$

On 10 bottom, we get:

$$\mathrm{Re}\left[3f(\sigma) + 4f(\sigma + it) + f(\sigma + 2it)\right]$$

$$\leqq \frac{3}{\sigma - 1} + A \ln t \qquad \left\{\begin{matrix} 1 < \sigma < 3 \\ t > G \end{matrix}\right\}.$$

Here:

$$f(s) \equiv \sum_{\rho} \left\{\frac{1}{s-\rho} + \frac{1}{\rho}\right\}. \quad \leftarrow \text{9 line 5}$$

For $\sigma > 1$ and $\rho = \beta + i\gamma$, note that

$$\mathrm{Re}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = \frac{\sigma - \beta}{(\sigma-\beta)^2 + (t-\gamma)^2} + \frac{\beta}{\beta^2 + \gamma^2} \geqq 0 \;!!!$$

Consider now <u>any</u> zero $\rho_0 = \beta_0 + i\gamma_0$ of ξ_0
with $\gamma_0 > G$.

last line

Apply 12 lines <u>8 + 9</u> keeping 12 bottom in mind.

KEY

Get:

$$4 \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (t - \gamma_0)^2} \lesssim \frac{3}{\sigma - 1} + A \log t \;,$$

all $1 < \sigma < 3$, $t > G$.

Notice that

$$LHS \lesssim \frac{4}{\sigma - \beta_0}$$

<u>trivially</u> for ANY $\sigma > 1$. For $\sigma \gtrsim 3$, get

$$LHS \lesssim \frac{4}{3 - \beta_0} \lesssim 2 \,.$$

By revising A, we can thus say

$$4 \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (t - \gamma_0)^2} \lesssim \frac{3}{\sigma - 1} + \tilde{A} \ln t$$

for <u>ANY</u> $\sigma > 1$ and $t > G$. (And any ρ_0 with $\gamma_0 > G$)

Put $t = \gamma_0$ to see that

$$\frac{4}{\sigma-\beta_0} \leqq \frac{3}{\sigma-1} + \tilde{A} \ln \gamma_0$$

for all $\sigma > 1$.

Let $\sigma \to 1$ to see that $\beta_0 = 1$ is impossible
(a fact we already know].

One expects that $\sigma - \beta_0$ and $\sigma - 1$ of comparable size will be most illuminating. Take:

$$\sigma = 1 + \lambda(1-\beta_0), \quad \lambda > 0$$
$$\sigma - \beta_0 = \sigma - 1 + 1 - \beta_0 = (\lambda+1)(1-\beta_0)$$

$$\Downarrow$$

get

$$\frac{4}{(1+\lambda)(1-\beta_0)} - \frac{3}{\lambda(1-\beta_0)} \leqq \tilde{A} \ln \gamma_0$$

any $\lambda > 0$.

For large λ, obviously $\frac{4}{1+\lambda} - \frac{3}{\lambda} > 0$
since $4 > 3$.

$\lambda = 4$ gives

$$\frac{.80 - .75}{1-\beta_0} \lesssim \tilde{A} \ln \gamma_0$$

$$\Downarrow$$

$$1-\beta_0 \gtrapprox \frac{.05}{\tilde{A}(\ln \gamma_0)} .$$

Hence,

$$\beta_0 \lessapprox 1 - \frac{.05}{\tilde{A}(\ln \gamma_0)} .$$

This is sufficient to prove Thm on p. 11. ∎