Lecture 15 Synopsis

(9 Mar 2016)

Before starting, I noted a simple lemma having relevance to $\xi_0(s)$.

Lemma

Let $f(z)$ be entire, with order $\rho \in [1,2)$. Let $\{a_n\}$ be the nonzero zeros of f (listed with multiplicity). We must then have

$$\sum \frac{1}{|a_n|} = +\infty$$

if either

(a) $1 < \rho < 2$

(b) $\rho = 1$ but type $\tau = +\infty$.

Pf

Apply Hadamard factorization. $p = [\![\rho]\!] = 1$.

$$f(z) = z^{q} e^{Q(z)} \prod_n E\left(\frac{z}{a_n};1\right)$$

$$\deg Q \leq 1$$

$\left(\begin{array}{c}\text{Lec 12}\\ \text{p. 13}\end{array}\right)$

Know $\sum \frac{1}{|a_n|^2} < \infty$ by Lec 12 p. 7.

Assume that $\sum \frac{1}{|a_n|} < \infty$. Take R large.

Observe that, for $|z| = R$,

recall Lec 5
16 Corollary

$$|f(Re^{i\theta})| = R^q \left|e^{Az+B}\right| \cdot \prod_n \left|1 - \frac{z}{a_n}\right| \left|e^{\frac{z}{a_n}}\right|$$

$$\lesssim R^q e^{O(R)} \prod_n \left(1 + \frac{|z|}{|a_n|}\right) e^{\frac{|z|}{|a_n|}}$$

$$\lesssim R^q e^{O(R)} \prod_n e^{\frac{|z|}{|a_n|}} e^{\frac{|z|}{|a_n|}}$$

$$\{\text{since } 1+u \lesssim e^u,\ u \gtrsim 0\}$$

$$\lesssim R^q e^{O(R)} \prod_n e^{2R \frac{1}{|a_n|}}$$

$$= R^q e^{O(R)} e^{2R\left(\sum_n \frac{1}{|a_n|}\right)}$$

$$\lesssim e^{O(R)} \quad \Longrightarrow$$

$$M(R;f) \lesssim e^{O(R)}$$

$$\ln M(R;f) \lesssim O(R).$$

This is a <u>contradiction</u> to both (a) and (b). We conclude that

$$\sum \frac{1}{|a_n|} = +\infty \quad \text{whenever (a) or (b) holds.}$$

$\blacksquare$

Lec 11 pp. 25, 31
Lec 13 p. 4

Since $\xi_0(z)$ had $\rho = 1$, $\tau = \infty$, we conclude at once that

$$\sum \frac{1}{|a_n|} = +\infty ,$$

whereupon $\xi_0(z)$ must have <u>infinitely many</u> zeros (each lying in $0 < x < 1$).

IE

There is no need to use the $\rho = \frac{1}{2}$ trick following from $\xi_0(z+\frac{1}{2}) = \text{EVEN}$. Lec 13 p. 4

One might also recall that, for $0 < \rho \underline{\leq 1}$, an entire f has infinitely many zeros by Lec 13, p. 2. Here, of course, $\sum \frac{1}{|a_n|} < \infty$.

I then remarked that I. M. Vinogradov showed (with method of trigonometric sums) that

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma \geq 1 - c(\ln t)^{-2/3}$$

t large. [As noted in Ingham, 2nd edition, p. xii, there is some question about this; only $(\ln t)^{-2/3}(\ln \ln t)^{-1/3}$ is properly justified.]

Taking $\eta(t) = \dfrac{1}{G(\ln t)^{2/3}}$ $(t \geq e)$, Lec 14

leads to

$$\omega(x) \simeq \min_{t \geq e} \left\{ \frac{\ln x / G}{(\ln t)^{2/3}} + \ln t \right\},$$

i.e. trivial calc for $u \geq 1$ on the fcn.

$$g(u) = \frac{\ln x / G}{u^{2/3}} + u \qquad \text{(compare Lec 14, 11)}$$

$\Rightarrow$ min for $u = (\text{const})(\ln x)^{3/5}$

if x = large

$\Rightarrow \omega(x) \simeq (\text{const})(\ln x)^{3/5}$

$$\Downarrow$$

$$\boxed{\begin{aligned} \psi(x) - x &= O\left[x e^{-c(\ln x)^{3/5}}\right] \\ \pi(x) - li(x) &= O\left[x e^{-c(\ln x)^{3/5}}\right] \end{aligned}}$$

with suitable $c > 0$. Compare Lec 14, 12.

The box needs a slight adjustment if only

$$\sigma \gtrapprox 1 - c(\ln t)^{-2/3}(\ln\ln t)^{-1/3} \quad \text{zerofree}$$

holds rigorously.

The "3/5" is about where things still lie in 2016 unconditionally. On this point, note that:

$$x^{1/2} = x \cdot x^{-\frac{1}{2}} = x e^{-\frac{1}{2}(\ln x)^{1}}.$$

There is obviously "some" distance yet to go!!!

$$\begin{array}{ccccc} \frac{1}{2} & \longrightarrow & \frac{3}{5} & \longrightarrow & 1 \\ \sim 1900 & & \sim 1958 & & \text{when??} \end{array}$$

The rest of the lecture was devoted to some important preparations for getting an explicit formula for $\psi_1(x)$.

— — —

Know

$$\xi_0(s) = e^{Bs} \prod_\rho \left(1-\frac{s}{\rho}\right) e^{s/\rho} \qquad B \in \mathbb{R}$$

$$\xi_0(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$\frac{\Gamma'(z)}{\Gamma(z)} = \operatorname{Log} z - \frac{1}{2z} - \sum_{k=1}^{R} \frac{B_{2k}}{2k} z^{-2k} + O_{R\delta}\left(|z|^{-2R-1}\right)$$

$$|\operatorname{Arg} z| \leqq \pi - \delta$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s} - \frac{1}{s-1} + \left(B + \frac{1}{2}\ln\pi\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

$$\operatorname{Im}(\rho) \neq 0, \quad 0 < \operatorname{Re}(\rho) < 1, \quad \rho = \beta + i\gamma$$

by virtue of

Lec 12 1 Stirling Γ'/Γ

Lec 13 4 ~ 9

and $\frac{\Gamma'}{\Gamma}(1+z) = \frac{1}{z} + \frac{\Gamma'}{\Gamma}(z)$.

We also have:

$$\frac{\xi_0'(s)}{\xi_0(s)} = B + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

$$\frac{\xi'(s)}{\xi(s)} = -\left[\frac{1}{s} + \frac{1}{s-1}\right] + B + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

$$\left\{ \xi(s) \equiv \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \right\}$$

$$\sum \frac{1}{|\rho|^2} < \infty \,, \qquad \sum \frac{1}{\gamma^2} < \infty$$

by Lec 13 7, 5 (bottom), and $\mathrm{Im}(\rho) = \gamma \neq 0$.

THM

For large t,

$$N[\rho : |\gamma - t| \leqq 1] = O(\ln t).$$

Pf

We follow von Mangoldt's method.

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{2}^{\infty} \frac{\Lambda(n)}{n^s} \qquad \operatorname{Re}(s) > 1$$

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leqq \sum_{2}^{\infty} \frac{\Lambda(n)}{n^\sigma} \qquad \operatorname{Re}(s) > 1$$

Keep $s = \sigma + it$, $t \geqq$ big G.

Keep $1 < \sigma \leqq 10$, say.

Apply Riemann's formula for $\frac{\zeta'}{\zeta}(s)$ on 6 bottom.

TAKE REAL PART ONLY!!!

Get

$$O(1) = O(1) + \overset{\text{Stirling}}{O(\ln t)} + \sum_{\rho} \left\{ \operatorname{Re}\left(\frac{1}{s-\rho}\right) + \operatorname{Re}\left(\frac{1}{\rho}\right) \right\}$$

$$O(\ln t) = \sum_{\text{all } \rho} \frac{\sigma - \beta}{(\sigma-\beta)^2 + (t-\gamma)^2} + \sum_{\text{all } \rho} \frac{\beta}{\beta^2 + \gamma^2}$$

but $\sigma - \beta > 0$ since $0 < \beta < 1$

$\Downarrow$

put $\sigma = 2$ to get

$$O(\ln t) = \sum_{\text{all } \rho} \frac{2-\beta}{(2-\beta)^2 + (t-\gamma)^2}$$

$\Downarrow$

$$\boxed{\sum_{\text{all } \rho} \frac{1}{1+(t-\gamma)^2} = O(\ln t).}$$

Restrict box to $\overbrace{\ }^{\text{those}}$ terms with $|\gamma - t| \leqq 1$.

Thus,

$$N[|\gamma - t| \leqq 1] = O(\ln t),$$

as promised. {Valid for $\underline{\underline{t \geqq 2}}$ by inflation of constant.}

<u>Corollary</u>

$$N[\rho : 0 < \gamma \leqq T] = O(T \ln T), \quad T \geqq 2.$$

Pf

WLOG T = giant.

By theorem, know that:

$$N[\tfrac{u}{2} < \gamma \leqq u] = O(u \ln u)$$

for all $u \geqq 4$ say. Now just apply the standard

$$\left\{ \frac{T}{2^{k+1}} < \gamma \leqq \frac{T}{2^k} \right\}$$

summation. ▨

We now go back to $\frac{\zeta'(s)}{\zeta(s)}$ but <u>don't</u> take the real part!

The formula on 6 bottom is valid at $3+it$ and also for any $s = \sigma + it$ with $-1 \leqq \sigma \leqq 2$, $t \geqq 6$, $t \neq$ any γ. Get:

$$\frac{\zeta'(s)}{\zeta(s)} = O\left(\frac{1}{t}\right) + O(\ln t) + \sum_\rho \left(\frac{1}{\sigma+it-\rho} + \frac{1}{\rho}\right)$$

$$\frac{\zeta'}{\zeta}(3+it) = O\left(\frac{1}{t}\right) + O(\ln t) + \sum_\rho \left(\frac{1}{3+it-\rho} + \frac{1}{\rho}\right)$$

{ Subtract }

$\Downarrow$

$$\frac{\zeta'(s)}{\zeta(s)} + O(1) = O\left(\frac{1}{t}\right) + O(\ln t) + \sum_{\rho}\left(\frac{1}{\sigma+it-\rho} - \frac{1}{3+it-\rho}\right)$$

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln t) + \sum_{|\gamma-t|>1}\left[\frac{3-\sigma}{(\sigma+it-\rho)(3+it-\rho)}\right] + \sum_{|\gamma-t|\leqq 1}\left[\frac{1}{\sigma+it-\rho} - \frac{1}{3+it-\rho}\right]$$

$$\left\{\begin{array}{l} \underline{\text{but}}, \text{ for } |\gamma-t|\leqq 1, \\ \left|\dfrac{1}{3+it-\rho}\right| = \dfrac{1}{|(3-\beta)+i(t-\gamma)|} \leqq \dfrac{1}{3-\beta} \\ \qquad \leqq \dfrac{1}{2} \end{array}\right\}$$

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln t) + \sum_{|\gamma-t|>1}\left[\frac{3-\sigma}{(\sigma+it-\rho)(3+it-\rho)}\right] + \sum_{|\gamma-t|\leqq 1}\frac{1}{\sigma+it-\rho}$$

but, for $|\gamma - t| > 1$, $-1 \leqq \sigma \leqq 2$,

$$\left| \frac{3-\sigma}{(\sigma-\beta+i(t-\gamma))((3-\beta)+i(t-\gamma))} \right| \leqq \frac{4}{|t-\gamma|^2}$$

while, by 9 box,

$$\sum_{\text{all } \rho} \frac{1}{1+(t-\gamma)^2} = O(\ln t)$$

$$\Downarrow$$

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln t) + \sum_{\substack{\rho \\ |\gamma - t| \leqq 1}} \frac{1}{s-\rho} .$$

One remarks here that the "1" can be replaced (if convenient) by any positive constant. Just review the earlier steps!

<u>Theorem</u> (Very Important and Basic)

Let $-1 \leqq \sigma \leqq 2$ and t be large, with $t \neq$ all γ. We then have

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln t) + \sum_{\substack{\rho \\ |\gamma - t| \leqq 1}} \frac{1}{s-\rho}$$

for $s = \sigma + it$. The "1" can be replaced by any positive constant (as convenient).

<u>Pf</u>

As above. ∎

On p. 7, recall that $\xi(s) = \pi^{-s/2}\,\Gamma\!\left(\frac{s}{2}\right)\zeta(s)$ had $\xi(s) = \xi(1-s)$ and <u>simple poles</u> at $s=1$ and $s=0$.

Lec 11, p. ~~24~~ 23

Recall too:

$$\xi(s) = (s-q)^M \phi(s), \quad \phi(s) = \text{power series in } (s-q) \text{ with } \phi(q) \neq 0$$

$$\Rightarrow \frac{\xi'(s)}{\xi(s)} = \frac{M}{s-q} + [\text{analytic near } s = q].$$

And, remember that

$$\xi(x) \neq 0, \quad x \in \mathbb{R} \qquad \text{Lec 11 } 27 .$$

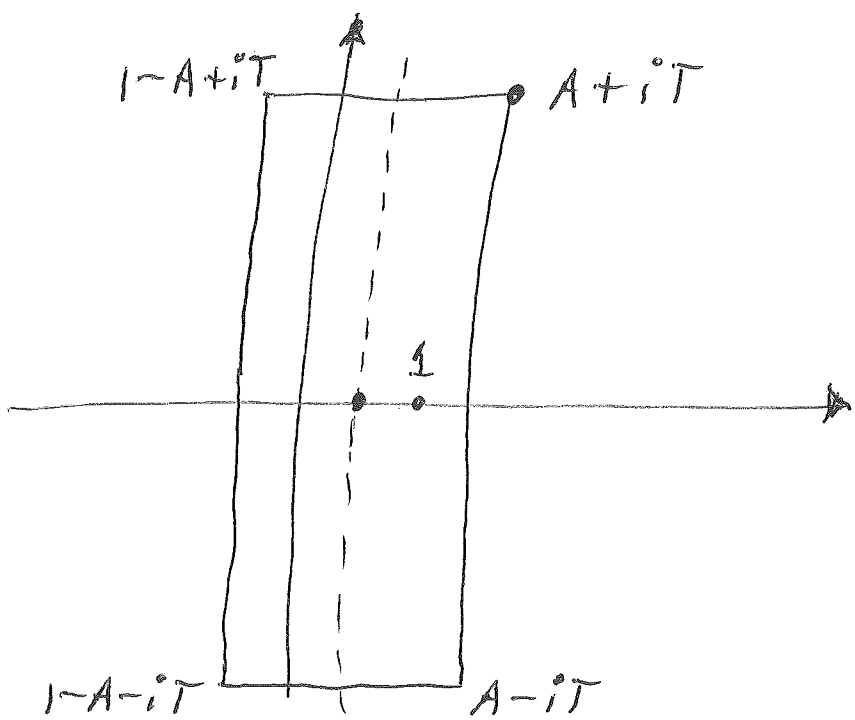
We propose to try to count the zeros of $\xi(s)$ — following Riemann.

Let $T \geqq 2$, say, $T \neq$ all γ.

Let $A > 1$.

Draw the rectangle

$$R(A,T) = [1-A, A] \times [-T, T]$$



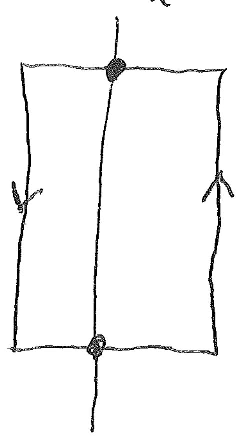
Put
$$N(T) = N[\rho : 0 < \gamma \le T] .$$

By 13 bottom and CRT, we get

$$\frac{1}{2\pi i}\oint_{\partial R(A,T)} \frac{\xi'(s)}{\xi(s)}\, ds = 2N(T) - 2 .$$

$\xi(\bar{s}) = \overline{\xi(s)}$
by Lec 11, 23

from $s = 0, 1$
simple poles
of ξ



Note:

$$\frac{1}{2\pi i}\int_{\text{left}} \frac{\xi'(s)}{\xi(s)}\, ds \overset{s = 1-u}{=} \frac{1}{2\pi i}\int_{\substack{u\text{-image} \\ \text{of left}}} \frac{\xi'}{\xi}(1-u)\,(-du)$$

$$= -\frac{1}{2\pi i}\int_{\substack{u\text{-image} \\ \text{of left}}} \frac{\xi'}{\xi}(1-u)\, du$$

{ easily check u-image of "left" is exactly "right" (in the correct direction) }

$$= -\frac{1}{2\pi i}\int_{\text{right}} \frac{\xi'}{\xi}(1-u)\,du \,.$$

But,

$$\xi(z) = \xi(1-z)$$

$$\Rightarrow \log \xi(z) = \log \xi(1-z) \qquad \text{suitable branches}$$

$$\Rightarrow \frac{\xi'}{\xi}(z) = -\frac{\xi'}{\xi}(1-z) \,.$$

So:

$$\frac{1}{2\pi i}\int_{\text{left}} \frac{\xi'}{\xi}(s)\,ds = \frac{1}{2\pi i}\int_{\text{right}} \frac{\xi'(u)}{\xi(u)}\,du \,.$$

Hence: (15) line 4

$$2N(T) - 2 = \frac{2}{2\pi i}\int_{\text{right}} \frac{\xi'(s)}{\xi(s)}\,ds$$

$$N(T) = 1 + \frac{1}{2\pi i}\int_{\text{right}} \frac{\xi'(s)}{\xi(s)}\,ds \,.$$

Notice that $A>1$ has <u>not</u> been specified yet in $R(A,T)$.

Put $\xi(s) = G(s)\zeta(s)$, $G(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$.

$$\Rightarrow \frac{\xi'(s)}{\xi(s)} = \frac{G'(s)}{G(s)} + \frac{\zeta'(s)}{\zeta(s)} \,.$$

The function $G(s)$ is analytic on $Re(s)>0$ and <u>not</u> <u>zero</u>.

By CIT,

$$\frac{1}{2\pi i}\int_{\text{right}} \frac{G'}{G}(s)\,ds = \frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{G'}{G}(s)\,ds \,.$$

So:

$$N(T) = 1 + \frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{G'}{G}(s)\,ds + \frac{1}{2\pi i}\int_{\text{right}} \frac{\zeta'(s)}{\zeta(s)}\,ds \,.$$

To handle the $\frac{G'}{G}$ integral, recall our use of the uniquely determined branch $\mathcal{L}og\, \Gamma(z)$ near p. 42 of Lec 10. We had $\mathcal{L}og\, \Gamma(x) = \mathrm{Log}\, \Gamma(x) = \ln \Gamma(x)$ for $x > 0$.

We can therefore unambiguously declare

$$\log G(s) = -\frac{s}{2} \ln \pi + \mathcal{L}og\, \Gamma\left(\frac{s}{2}\right)$$

for $\underline{\mathrm{Re}(s) > 0}$. At once:

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{G'}{G}(s)\, ds$$

$$= \frac{1}{2\pi i} \Big[\log G(s) \Big]_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}$$

$$\left\{ \begin{array}{l} \text{but} \qquad\qquad \boxed{\sigma > 0} \\ G(\bar{s}) = \overline{G(s)} \\ \log G(\bar{s}) = \overline{\log G(s)} \\ \log G(u) = \ln |G(u)| + i \,\mathrm{Arg}\, G(u) \end{array} \right\}$$

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\frac{G'}{G}(s)\,ds$$

$$= \frac{1}{2\pi i}\,2i\,\operatorname{Arg} G(\tfrac{1}{2}+iT)$$

$$= \frac{1}{\pi}\operatorname{Arg} G(\tfrac{1}{2}+iT) \qquad \text{see (18) middle}$$

$$= \frac{1}{\pi}\left[-\frac{T}{2}\ln\pi + \operatorname{Arg}\Gamma(\tfrac{1}{4}+i\tfrac{T}{2})\right].$$

So far, then, we have:

$$\boxed{N(T) = 1 + \frac{1}{\pi}\operatorname{Arg} G(\tfrac{1}{2}+iT) + \frac{1}{2\pi i}\int_{\text{right}}\frac{\zeta'(s)}{\zeta(s)}\,ds.}$$

This box clearly holds for any $T > 0$, $T \neq$ all γ. There was nothing used about $T \geqq 2$ yet.

We now PAUSE to apply Stirling to

$$\frac{1}{\pi}\left[-\frac{T}{2}\ln\pi + \operatorname{Arg}\Gamma\left(\frac{1}{4}+i\frac{T}{2}\right)\right].$$

Here we keep $T \gtrsim 2$ and imagine $T \gtrsim 6$ if necessary (along the way).

$$\operatorname{Arg}\Gamma\left(\frac{1}{4}+i\frac{T}{2}\right) = \operatorname{Im}\operatorname{Log}\Gamma\left(\frac{1}{4}+i\frac{T}{2}\right)$$

$$= \operatorname{Im}\left[\left(\frac{1}{4}+i\frac{T}{2}-\frac{1}{2}\right)\operatorname{Log}\left(\frac{1}{4}+i\frac{T}{2}\right) - \left(\frac{1}{4}+i\frac{T}{2}\right) + \ln\sqrt{2\pi} + O\left(\frac{1}{T}\right)\right]$$

{Lec 10 p. 42}

$$= \operatorname{Im}\left[\left(-\frac{1}{4}+i\frac{T}{2}\right)\left\{\operatorname{Log}\left(i\frac{T}{2}\right)\left(1+\frac{1}{2iT}\right)\right\} - \frac{1}{4} - i\frac{T}{2} + \ln\sqrt{2\pi} + O\left(\frac{1}{T}\right)\right]$$

$$= \operatorname{Im}\left[\left(-\frac{1}{4}+i\frac{T}{2}\right)\left\{\ln\left(\frac{T}{2}\right)+i\frac{\pi}{2} + \operatorname{Log}\left(1+\frac{1}{2iT}\right)\right\} - \frac{1}{4} - i\frac{T}{2} + \frac{1}{2}\ln(2\pi) + O(T^{-1})\right]$$

$$\left\{ \operatorname{Log}\left(1+\frac{1}{2iT}\right) = \frac{1}{2iT} + O(T^{-2}) \right\}$$

$$\simeq \frac{T}{2}\ln\frac{T}{2} - \frac{\pi}{8} + O(T^{-1})$$

$$- \frac{T}{2} + O(T^{-1})$$

$$= -\frac{\pi}{8} + \frac{T}{2}\ln\left(\frac{T}{2e}\right) + O(T^{-1}).$$

This is valid for $T \geqq 6$, then by constant inflation for $T \geqq 2$. Compare: Ingham 69 line 6.

Get:

$$\frac{1}{\pi}\operatorname{Arg} G\left(\tfrac{1}{2}+iT\right)$$

$$= \frac{1}{\pi}\left[-\frac{T}{2}\ln\pi + \operatorname{Arg}\Gamma\left(\tfrac{1}{4}+i\tfrac{T}{2}\right)\right] \quad (19)$$

$$\simeq -\frac{T}{2\pi}\ln\pi + \left(-\frac{1}{8}\right) + \frac{T}{2\pi}\ln\frac{T}{2e} + O(T^{-1})$$

$$= \frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) - \frac{1}{8} + O(T^{-1}),$$

$$T \geqq 2.$$

On 19, for $T \geqq 2$ ($T \neq$ all γ), we therefore have

$$\boxed{N(T) = \frac{T}{2\pi} \ln\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + O(T^{-1}) + \frac{1}{2\pi i} \int_{\text{right}} \frac{\zeta'(s)}{\zeta(s)}\, ds} .$$

To address $\frac{1}{2\pi i} \int_{\text{right}} \frac{\zeta'(s)}{\zeta(s)}\, ds$, we proceed in 2 ways.

First, recall that:

$$|\zeta(z) - 1| < 3 \cdot 2^{-x}, \quad x \geqq 2;$$

{ Lec 5 p. 10 }

$$\text{Log}\, \zeta(z) = \sum_{2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-z}, \quad x > 1$$

{ Lec 6, p. 4 + 3 }

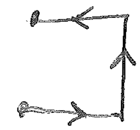
$$\text{Log}\, \zeta(z) = O(2^{-x}), \quad x \text{ large};$$

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{2}^{\infty} \frac{\Lambda(n)}{n^z} \quad , \quad x > 1$$

$\{$ Lec 6, p. 6 $\}$

$$\frac{\zeta'(z)}{\zeta(z)} = O(2^{-x}) \quad , \quad x \text{ large} .$$

We can now <u>freeze</u> T and let $A \to \infty$ in $R(A,T)$ to get



$$\frac{1}{2\pi i} \int_{\text{right}} \frac{\zeta'}{\zeta}(s)\,ds = \frac{1}{2\pi i} \int_{\frac{1}{2}}^{\infty} \frac{\zeta'}{\zeta}(u - iT)\,du$$

$$+ \frac{1}{2\pi i} \int_{\infty}^{\frac{1}{2}} \frac{\zeta'}{\zeta}(u + iT)\,du$$

$$+ \; 0 \longleftarrow \{ 2^{-A} \to 0 \}$$

$$\left\{ \text{but } \frac{\zeta'}{\zeta}(u - iT) = \overline{\frac{\zeta'}{\zeta}(u + iT)} \right\}$$

$$= \frac{1}{2\pi i} \int_{\infty}^{\frac{1}{2}} \left[\frac{\zeta'}{\zeta}(u + iT) - \overline{\frac{\zeta'}{\zeta}(u + iT)} \right] du$$

$$= \frac{1}{2\pi i} \int_{\infty}^{\frac{1}{2}} 2i \,\mathrm{Im}\, \frac{\zeta'}{\zeta}(u + iT)\,du$$

$$= -\frac{1}{\pi}\int_{1/2}^{\infty} \operatorname{Im}\frac{\zeta'}{\zeta}(u+iT)\,du \,.$$

Thus,

$$\boxed{\frac{1}{2\pi i}\int_{\text{right}} \frac{\zeta'}{\zeta}(s)\,ds = -\frac{1}{\pi}\operatorname{Im}\int_{\frac{1}{2}}^{\infty} \underbrace{\frac{\zeta'}{\zeta}(u+iT)}_{O(2^{-u})}\,du \,.}$$

with A fixed

The 2^{nd} way is more basic. One starts with $\operatorname{Log}\zeta(s)$ on $\operatorname{Re}(s) > 1$ (see 22) and forms an <u>analytic continuation</u> along the line segments $[\frac{1}{2}+iT, A+iT]$ and $[\frac{1}{2}-iT, A-iT]$ in an obvious way (starting at $A \pm iT$).

THIS IS LEGAL SINCE $T \neq$ all γ. No zeros of $\zeta(s)$ will be hit.

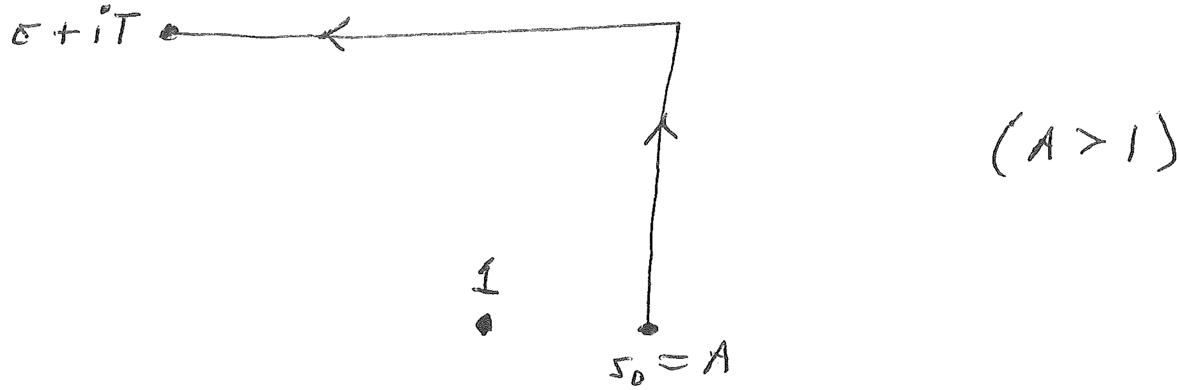
This branch of $\log\zeta(s)$ clearly satisfies

$$\log\zeta(\bar{s}) = \overline{\log\zeta(s)} \,.$$

SEE Lec 6, 12 line 10

[Analytic $f(z)$ with $f(x) \in \mathbb{R} \Rightarrow f(\bar{z}) \equiv \overline{f(z)}$.]

One typically says $\mathcal{L}og\, \zeta(s)$ has been found by continuing "up" from $s_0 = A$, then "across" to s from $A \pm iT$.



With this convention,

$$\frac{1}{2\pi i}\int_{\text{right}} \frac{\zeta'}{\zeta}(s)\,ds = \frac{1}{2\pi i}\int_{\text{right}} d\left[\mathcal{L}og\,\zeta(s)\right]$$

$$= \frac{1}{2\pi i}\left[\mathcal{L}og\,\zeta(\tfrac{1}{2}+iT) - \mathcal{L}og\,\zeta(\tfrac{1}{2}-iT)\right]$$

$$= \frac{1}{2\pi i}\,\underline{2i}\,\mathrm{Im}\left[\mathcal{L}og\,\zeta(\tfrac{1}{2}+iT)\right] \Rightarrow$$

$$\boxed{\frac{1}{2\pi i}\int_{\text{right}} \frac{\zeta'}{\zeta}(s)\,ds = \frac{1}{\pi}\,\mathrm{Arg}\,\zeta(\tfrac{1}{2}+iT)}$$

in an obvious "up and across" sense.

THM (Very important and basic) ← Essentially Riemann

Let $T > 0$, $T \neq$ any γ.

Let

$$\xi(s) = G(s)\zeta(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s).$$

We then have:

$$N(T) = N[\rho : 0 < \gamma \leq T]$$

$$= 1 + \frac{1}{\pi}\arg G(\tfrac{1}{2}+iT) + \underline{\underline{S}}(T)$$

wherein

$$\underline{\underline{S}}(T) = \frac{1}{\pi}\arg\zeta(\tfrac{1}{2}+iT) \quad \text{"up and across"}$$

$$= -\frac{1}{\pi}\,\mathrm{Im}\int_{1/2}^{\infty}\frac{\zeta'}{\zeta}(\sigma+iT)\,d\sigma$$

and $\arg G(\frac{1}{2}+iT)$ is defined à la Stirling.

For $T \gtrsim 2$, we have:

$$\frac{1}{\pi}\arg G(\tfrac{1}{2}+iT) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) - \frac{1}{8} + O\left(\frac{1}{T}\right).$$

C^{∞} fcn of T

Pf

As above. ▮

THM (Very Important and Basic)

Introduce $S(T)$ for $T > 0$, $T \neq$ all γ as on p. 26. We then have

$$S(T) = O(\ln T), \quad T \geqq 2.$$

Pf

Apply Thm on 13. Remember that

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \mathrm{Re}(s) > 1$$

$$= O(2^{-\sigma}), \quad \sigma \geqq 2.$$

See 23 top. Get:

$$S(T) = -\frac{1}{\pi} \mathrm{Im} \int_{1/2}^{2} \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma$$

$$-\frac{1}{\pi} \mathrm{Im} \int_{2}^{\infty} \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma$$

$$= -\frac{1}{\pi}\,\mathrm{Im}\int_{1/2}^{2}\Big[O(\ln T) + \sum_{|\gamma - T|\leqq 1}\frac{1}{s-\rho}\Big]\,d\sigma$$

$$+\ O(1)\int_{2}^{\infty} 2^{-\sigma}\,d\sigma$$

$\{$ here $s = \sigma + iT$, $\rho = \beta + i\gamma$ $\}$

$$= O(\ln T)$$

$$-\frac{1}{\pi}\,\mathrm{Im}\sum_{|\gamma - T|\leqq 1}\left(\int_{1/2}^{2}\frac{1}{s-\rho}\,d\sigma\right)$$

$$+\ O(1)$$

$\Big\{$ **but** $\displaystyle\int_{\frac{1}{2}+iT}^{2+iT}\frac{1}{s-\rho}\,ds$, $\quad T \neq$ all γ

$$= \mathrm{Log}(2 + iT - \rho) - \mathrm{Log}(\tfrac{1}{2} + iT - \rho)$$

$\Rightarrow$ imaginary part has absolute $\leqq \pi$ $\Big\}$

$$= O(\ln T) + O(1)\sum_{|\gamma - T|\leqq 1} 1 \;+\; O(1)$$

$$= O(\ln T)$$ by ⑧ THM. ▨

THM ← stated by Riemann

$$N(T) = N[\rho : 0 < \gamma \leq T] \quad \leftarrow \text{definition}$$

$$= \frac{T}{2\pi} \ln\left(\frac{T}{2\pi e}\right) + O(\ln T)$$

for all $T \geq 2$. {Ingham p. 68 thm 25}

Proof
For $T \neq$ all γ, just combine 26 + 27.
If $T =$ some γ, just use the right continuity of $N(t)$ as a counting function. ▨

For later use, notice that:

$t = 2\pi u$
$u = t/2\pi$

$$\frac{d}{dt}\left(\frac{t}{2\pi} \ln\left(\frac{t}{2\pi e}\right)\right) = \frac{1}{2\pi} \frac{d}{du}\left(u \ln\left(\frac{u}{e}\right)\right)$$

$$= \frac{1}{2\pi} \ln u$$

$$= \frac{1}{2\pi} \ln\left(\frac{t}{2\pi}\right).$$