Lectures 17 and 18

(23 and 25 March)

Before proceeding to the explicit formula for $\psi(x)$, we treat an important connection between Θ (Lec 16, p. 19) and the PNT.

Let $\Theta' = \inf\{\omega > 0 : \psi(x) - x = O(x^{\omega}),\ x \geq 2\}$.

<u>Thm</u>

$$\Theta' = \Theta.$$

$\boxed{\text{Ingham 84}}$

<u>Pf</u>

By Lec 16 p. 19 2nd Thm, $\Theta' \leq \Theta$.

Suppose now that $\psi(x) = x + O(x^{\omega})$, $x \geq 2$. $\boxed{\omega > 0}$

For $\mathrm{Re}(s) > 1$, we immediately check

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_1^{\infty} u^{-s}\, d\psi(u) = s\int_1^{\infty} \psi(u)\, u^{-s-1}\, du$$

$$\frac{s}{s-1} = s\int_1^{\infty} u \cdot u^{-s-1}\, du$$

so

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} - 1 = s\int_1^{\infty} \frac{\psi(u) - u}{u^{s+1}}\, du.$$

see Lec 8 p. 11

The RHS is analytic wrt s for $Re(s) > \omega$.

Note that $\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}$ has a removable singularity at $s = 1$. Lec 7 p. 17 We thus find that

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \text{analytic for } Re(s) > \omega .$$

Clearly, for any ρ with $\Xi_0(\rho) = 0$,* we must then get $Re(\rho) \leqq \omega$. Hence: $\Theta \leqq \omega$. Hence $\Theta \leqq \Theta'$. ▥

In the near future, we will improve the theorem on p. 1.

One interprets p. 1 THM as saying (loosely) that Θ controls the size of the remainder term in $\psi(x) - x$ or $\pi(x) - li(x)$. See here Lec 14 p. 8 BOX and 10 bottom. Also Lec 16 pp. 26 - 27. All of this will (soon) be improved/sharpened.

---

* Recall Lec 13, p. 4 THM. Also: Lec 8, p. 10.

The discussion that I gave (in Lec 17+18) of the explicit formula for $\psi(x)$ can be seen as something having 2 basic stages:

(A) an initial "fleshing it out" in the style of Landau; (Landau, Vorlesungen über Z...)

(B) tightening that up – and strengthening it.

I follow the same procedure in these notes, but make some slight changes to streamline (AND strengthen) things.

We had

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$

à la Lec 7. Here $c>1$, $x>0$. By a purely formal differentiation wrt x, one expects that

$$\psi(x) \text{ is } \underbrace{\text{“}}_{\text{basically}} \text{associated with } \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\left(-\frac{\zeta'}{\zeta}\right) ds.$$

Danger exists here since, for each x,

$$\int_{c-i\infty}^{c+i\infty} \frac{x^c}{|s|}(1)\,|ds| = +\infty .$$

Ie absolute convergence fails! $\boxed{:(}$

By Lec 7, we expect to study

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds$$

for $y > 0$.

<u>Fundamental Lemma</u> (Perron + others)

Keep $0 < y < \infty$, $0 < c \leqq 2$, $T \geqq 3$.

Let

$$\eta(y) = \left\{ \begin{array}{ll} 1, & y > 1 \\ 1/2, & y = 1 \\ 0, & 0 < y < 1 \end{array} \right\}.$$

We then have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds = \eta(y) + \text{Remainder},$$

where

$$|\text{Remainder}| \lesssim \begin{cases} \dfrac{y^c}{\pi T\,|\ln y|}, & y \neq 1 \\[2ex] \dfrac{c}{\pi T}, & y = 1 \end{cases}.$$

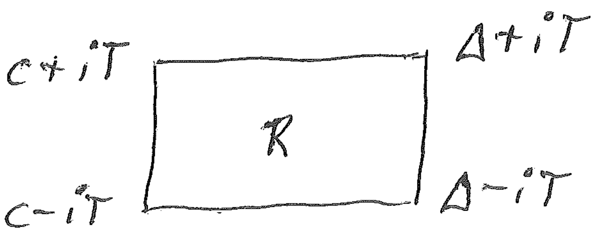
One also has, if one is willing to be quite crude,

$$|\text{Remainder}| \lesssim O_A(1)$$

for $0 < y < A$, with an "implied" constant which depends solely on A.

Proof

Take $0 < y < 1$ first. Look at $\frac{1}{2\pi i}\int_{\partial R} \frac{y^s}{s}\,ds$ on



$(s = \sigma + it)$

with T frozen and let $\Delta \to \infty$. Note that $y^\Delta \to 0$ along $[\Delta - iT, \Delta + iT]$. By the Cauchy Integral Thm, we get

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} = \frac{1}{2\pi i}\int_{c-iT}^{\infty - iT} + \frac{1}{2\pi i}\int_{\infty + iT}^{c+iT}.$$

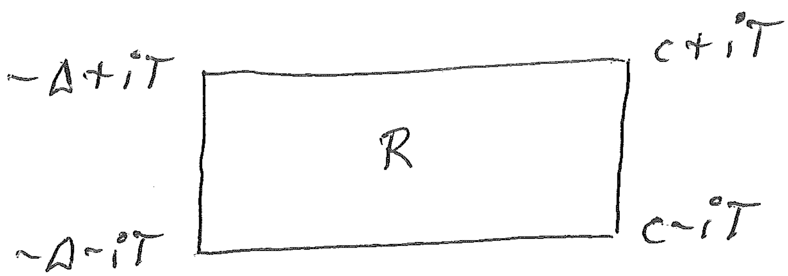
At once,

$$|LHS| \lesssim \frac{2}{2\pi}\int_c^{\infty} \frac{y^{\sigma}}{|s|}\,d\sigma \qquad (s=\sigma\pm iT)$$

$$\lesssim \frac{1}{\pi T}\int_c^{\infty} e^{-\sigma|\ln y|}\,d\sigma$$

$$= \frac{1}{\pi T}\,\frac{e^{-c|\ln y|}}{|\ln y|} = \frac{1}{\pi T}\,\frac{y^{c}}{|\ln y|}\,.$$

Take $1<y<\infty$ next. Here we use $\frac{y^s}{s}$ and



$(s=\sigma+it)$.

Notice that $\operatorname{Res}\left\{\frac{y^s}{s};\ s=0\right\}=1$ (simple pole). Freeze T and let $\Delta\to\infty$ again. By the Cauchy Residue Theorem (or Cauchy Integral Formula), get:

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} = 1 - \frac{1}{2\pi i}\int_{-\infty-iT}^{c-iT} - \frac{1}{2\pi i}\int_{c+iT}^{-\infty+iT}$$

$$= 1 - \frac{1}{2\pi i}\int_{-\infty-iT}^{c-iT} + \frac{1}{2\pi i}\int_{-\infty+iT}^{c+iT}\,.$$

At once,

$$|\text{Remainder}| \lesssim \frac{2}{2\pi}\int_{-\infty}^{c} \frac{y^{\sigma}}{|s|}\,d\sigma \qquad (s=\sigma \pm iT)$$

$$\lesssim \frac{1}{\pi T}\int_{-\infty}^{c} e^{\sigma(\ln y)}\,d\sigma \qquad (y>1)$$

$$= \frac{1}{\pi T}\,\frac{e^{c(\ln y)}}{\ln y} = \frac{y^{c}}{\pi T(\ln y)} .$$

For $y=1$, we notice that $(0<c\lesssim 2)$

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1^{s}}{s}\,ds = \frac{1}{2\pi i}\left[\operatorname{Log} s\right]_{c-iT}^{c+iT}$$

$$= \frac{1}{2\pi i}\left[\operatorname{Log}(c+iT) - \overline{\operatorname{Log}(c+iT)}\right]$$

$$= \frac{1}{\pi}\operatorname{Arg}(c+iT)$$

$$= \frac{1}{\pi}\arctan\frac{T}{c}$$

$$= \frac{1}{\pi}\left[\int_{0}^{\infty}\frac{dq}{1+q^{2}} - \int_{T/c}^{\infty}\frac{dq}{1+q^{2}}\right]$$

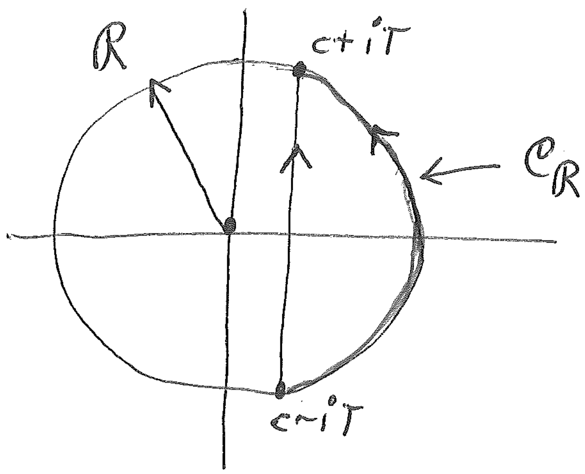
$$= \frac{1}{2} - \frac{1}{\pi}\int_{T/c}^{\infty}\frac{dq}{1+q^{2}}$$

$$\Downarrow$$

$$|\text{Remainder}| < \frac{1}{\pi}\,\frac{1}{(T/c)} = \frac{c}{\pi T} .$$

To conclude, we assume that $0 < y < A$ for some $A \geqq 2$ (wlog).

For $0 < y < 1$, look at



radius $R = \sqrt{c^2 + T^2}$

and get

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} = \frac{1}{2\pi i}\int_{C_R} \frac{y^s}{s}\,ds \qquad \text{by CIT}$$

$\Longrightarrow$

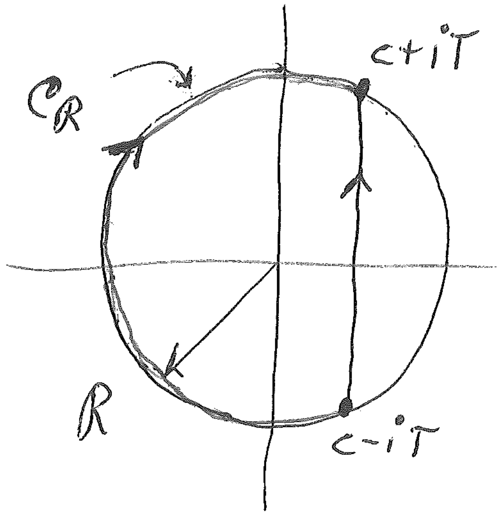
$$|\text{Remainder}| \lessapprox \frac{1}{2\pi}\int_{C_R} \frac{y^\sigma}{|s|}\,|ds|$$

$$\lessapprox \frac{1}{2\pi}\, y^c \,\frac{1}{R}\,(\pi R)$$

$$\lessapprox \frac{1}{2}$$

since $0 < c \lessapprox 2$ (and $0 < y < 1$).

Next, consider $1 \leq y < A$. Use



$$R = \sqrt{c^2+T^2}$$

and get

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} = 1 + \frac{1}{2\pi i}\int_{C_R} \frac{y^s}{s}\,ds$$

$\Rightarrow$

$\eta(1) = \frac{1}{2}!$

$$|\text{Remainder}| \leq \frac{1}{2} + \frac{1}{2\pi}\int_{C_R} \frac{y^\sigma}{|s|}\,|ds|$$

$$\leq \frac{1}{2} + \frac{1}{2\pi}\, y^c \frac{1}{R}\,(2\pi R)$$

$$\leq \frac{1}{2} + y^c$$

$\{$but $0 < c \leq 2$ and $1 \leq y < A\}$

$$\leq \frac{1}{2} + A^2.$$

∎

Corollary of Lemma 4

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds = \eta(y) = \begin{Bmatrix} 1, & y>1 \\ 1/2, & y=1 \\ 0, & y<1 \end{Bmatrix}$$

For $0<y<\infty$, $0<c\leqq 2$.

Guided by p. 3 bottom, we now turn our attention to

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]ds$$

under the hypothesis that

$$x \geqq 1+\delta_0 \quad \text{AND} \quad 1<c\leqq 2.$$

Here δ_0 is some small positive constant (e.g. $1/2$) which we fix once and for all.

N.B. Landau, Vorlesungen über Z... likes $c=2$ and arbitrary $x>1$.

Since we plan to use p. 4 Fund Lemma, we also insist that

$$T \geqq 3.$$

Conceptually, in phase (A) on p. 3, it is simplest to *declare* $c = 2$ [for instance] like Landau does, and then stay with that.

To facilitate making improvements, we prefer however [*in contrast* to what was done in the lectures] to keep c *free* for the time being.

Let $\mathcal{F}$ be the set $\{p^m : p = \text{prime}, m = \text{pos. integer}\}$. Obviously $\mathcal{F} \subseteq \mathbb{Z} \cap [2, \infty)$.

Let $\|u\|' = \left\{ \begin{matrix} |u|, & \text{if } u \neq 0 \\ \infty, & \text{if } u = 0 \end{matrix} \right\}$. Here $u \in \mathbb{R}$.

For $x \geqq 1 + \delta_0$, let $\xi(x) = \min\limits_{\lambda \in \mathcal{F}} |x - \lambda|$. Also write

$$\langle x \rangle = \min \left\{ \frac{1}{100}, \|\xi(x)\|' \right\}.$$

Notice that $\langle x \rangle = \frac{1}{100}$ unless $\|\xi(x)\|' < \frac{1}{100}$, which would mean that $x \notin \mathcal{F}$ but x lies LESS THAN $\frac{1}{100}$ units from $\mathcal{F}$.

In particular, we see that:

$$\langle x \rangle = \frac{1}{100} \quad \text{anytime } x \in \mathbb{Z}.$$

In all cases, obviously:

$$0 < \langle x \rangle \leqq \frac{1}{100}.$$

Lemma 2

Keep $x \geqq 1+\delta_0$, $1 < c \leqq 2$, $T \geqq 3$.

Let
$$\psi^*(x) = \frac{\psi(x+0) + \psi(x-0)}{2}.$$

We then have:

(i) $$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds = \psi^*(x) +$$

$$O\left[\frac{x^c \ln x}{T(c-1)}\right] + O\left[\frac{x \ln x}{T\langle x\rangle}\right];$$

(ii) $$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds = \psi^*(x) +$$

$$O\left[\frac{x^c \ln x}{T(c-1)}\right] + O(\ln x).$$

The implied constants will depend on [at most] δ_0.

Proof

We use p. 4 LEMMA. Get:

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right) ds$$

$$= \sum_{1}^{\infty}\Lambda(n)\,\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{(x/n)^s}{s}\,ds$$

$$= \sum_{1}^{\infty}\Lambda(n)\left\{\eta\left(\frac{x}{n}\right) + \begin{bmatrix} \dfrac{\Omega}{\pi T}\,\dfrac{(x/n)^c}{|\ln x - \ln n|}, & n \neq x \\[2ex] \dfrac{2\Omega}{\pi T}, & n = x \\[2ex] O_A(1), & \dfrac{x}{A} < n < \infty \end{bmatrix}\right\}.$$

Here $|\Omega| \leq 1$. Clearly

$$\sum_{1}^{\infty}\Lambda(n)\,\eta\left(\frac{x}{n}\right) = \psi^*(x).$$

We first look at (i). This will use chunks #1 and #2 in the big bracket.

Make the obvious

$$1 \leqq n \leqq [\![x]\!] - 1$$

$$n = [\![x]\!] \longleftarrow n_1$$

$$n = [\![x]\!] + 1 \longleftarrow n_2$$

$$n \geqq [\![x]\!] + 2$$

splitting.

I

$$\sum_{n=[\![x]\!]+2}^{\infty} \Lambda(n) \frac{\Omega}{\pi T} \frac{(x/n)^c}{|\ln x - \ln n|}$$

$$= \sum_{n=[\![x]\!]+2}^{2[\![x]\!]+4} + \sum_{n=2[\![x]\!]+5}^{\infty}$$

Must bound the expressions in absolute value!

$n \geqq 2[\![x]\!] + 5 \implies n \geqq 2([\![x]\!]+1) > 2x$.
So, $\ln(n/x) > \ln 2$. Get:

$$\leqq \frac{x^c}{T} \sum_{n=2[\![x]\!]+5}^{\infty} \frac{\Lambda(n)}{n^c} \frac{1}{\ln 2}$$

$$= O\left(\frac{x^c}{T}\right)\left[\frac{1}{c-1} + O(1)\right] = O\left(\frac{x^c}{T}\right)\frac{1}{c-1} \quad ||$$

since $-\frac{\zeta'(\sigma)}{\zeta(\sigma)} = \sum_{1}^{\infty} \frac{\Lambda(n)}{n^\sigma} = \frac{1}{\sigma-1} + O(1)$ for $1 < \sigma \leqq 2$. ② top.

Next, $[\![x]\!]+2 \leqq n \leqq 2[\![x]\!]+4$. Get:

$$\lesssim \frac{1}{T} \sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \Lambda(n)\left(\frac{x}{n}\right)^c \frac{1}{\ln n - \ln x}$$

$$\lesssim \frac{1}{T} O(\ln(10x)) \sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{1}{\ln n - \ln x}$$

$$\left\{ \begin{array}{l} \text{apply mean value thm to } \ln n - \ln x; \\ \text{get final sum} \\ \qquad \lesssim O(1) \sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{1}{\frac{1}{x}(n-x)} \\ \qquad \lesssim O(1)\, x \cdot O(1) \ln(10x) ; \\ \text{note here that this estimate is} \\ \text{trivial if, say, } x \leqq 1000 \end{array} \right\}$$

$$\lesssim \frac{x}{T} O(1) (\ln 10x)^2 \lesssim \frac{x}{T} O(1) \ln^2 x .$$

$x \geqq 1+\delta_0$

But:

$$\frac{x^c}{c-1} \gtrsim (\text{const})\, x \ln x \quad \text{for } x \geqq 1+\delta_0 .$$

$\boxed{\text{TRICK}}$

This follows by elem calculus ; the "const" will depend on δ_0.

$$\left\{ \begin{array}{l} \text{For } x > e, \text{ the fcn } \dfrac{x^v}{v} \text{ on } 0 < v \leqq 1 \\ \text{has its MIN at } v = \dfrac{1}{\ln x}. \end{array} \right\}$$

just like Lec 6 p. 21 lines 2–10

Accordingly : for $x \geqq 1 + \delta_0$

$$\frac{x^c \ln x}{T(c-1)} \gtrsim (\text{const}) \frac{x \ln^2 x}{T}.$$

Hence,

$$\sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{1}{T} \Lambda(n) \left(\frac{x}{n}\right)^c \frac{1}{\ln n - \ln x}$$

$$= O(1) \frac{x^c \ln x}{T(c-1)}. \quad \|$$

The "implied" constant will depend on δ_0.

Pause For A Moment!

In Lec 17 with $c = 2$, what I remarked following Landau was that :

(proceeding somewhat crudely...)

$$\sum_{[\![x]\!]+2}^{\infty} \frac{1}{T}\Lambda(n)\left(\frac{x}{n}\right)^2 \frac{1}{\ln n - \ln x}$$

$$= O\left(\frac{x^2}{T}\right) + \sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{1}{T}\Lambda(n)\frac{x^2}{n^2}\frac{1}{\ln n - \ln x}$$

$$= O\left(\frac{x^2}{T}\right) + \frac{x^2}{T}\sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{\Lambda(n)}{n^2}\frac{1}{(\text{const})\frac{1}{x}(n-x)}$$

$$= O\left(\frac{x^2}{T}\right) + \frac{x^2}{T}\sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{\ln(10x)}{x^2}\frac{(\text{const})x}{n-x}$$

$$\leqq O\left(\frac{x^2}{T}\right) + (\text{const})\frac{x^2}{T}\ln(10x)\cdot O(1)$$

$$= O(1)\frac{x^2 \ln x}{T} \qquad \text{for } x \geqq 1+\delta_0$$

END of PAUSE. { See Landau, <u>Vorlesungen</u>, proof of Satz 451 near (564). }

Step II.
$$\sum_{n=1}^{[\![x]\!]-1} \Lambda(n)\frac{\Omega}{\pi T}\frac{(x/n)^c}{(\ln x - \ln n)}$$

$$= \sum_{n<\frac{1}{2}[\![x]\!]} + \sum_{\frac{1}{2}[\![x]\!] \leqq n \leqq [\![x]\!]-1}$$

Expect things to be similar to step I.

At once,

$$\sum_{n<\frac{1}{2}[\![x]\!]} \Lambda(n)\,\frac{1}{T}\,\frac{(x/n)^c}{(\ln\frac{x}{n})}$$

$$\leqq \sum_{n<\frac{1}{2}[\![x]\!]} \Lambda(n)\,\frac{1}{T}\,\frac{x^c}{n^c}\,\frac{1}{\ln 2}$$

$$= O\left(\frac{x^c}{T}\right) \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^c}$$

$$= O\left(\frac{x^c}{T}\right)\left[\frac{1}{c-1} + O(1)\right] \leqq O\left(\frac{x^c}{T(c-1)}\right).$$

Like (15) bottom.

Also,

$$\sum_{\frac{1}{2}[\![x]\!] \leqq n \leqq [\![x]\!]-1} \Lambda(n)\,\frac{1}{T}\,\frac{(x/n)^c}{\ln x - \ln n}$$

$$\leqq \frac{O(\ln(10x))}{T}\,(\text{const}) \sum_{\frac{1}{2}[\![x]\!] \leqq n \leqq [\![x]\!]-1} \frac{x}{x-n}$$

$$\leqq O(1)\,\frac{x\ln(10x)}{T}\cdot \ln(10x)$$

$$\leqq \frac{x}{T}\,O(1)\ln^2 x \qquad \text{for } x \geqq 1+\delta_0 .$$

Line 6 on (17) is still valid, so we get:

$$\sum_{\frac{1}{2}[\![x]\!] \lessapprox n \lessapprox [\![x]\!]-1} \frac{1}{T}\Lambda(n)\left(\frac{x}{n}\right)^{c}\frac{1}{\ln x - \ln n}$$

$$= O(1)\frac{x^{c}\ln x}{T(c-1)} . \quad \|$$

Recall $n_1 = [\![x]\!]$, $n_2 = [\![x]\!]+1$ on (15) top.

By (14), we must still check that:

LHS

$$\sum_{n\in\{n_1,n_2\}} \Lambda(n)\begin{bmatrix} \frac{1}{T}\frac{(x/n)^{c}}{|\ln x - \ln n|} & , \; n \neq x \\ \frac{1}{T} & , \; n = x \end{bmatrix}$$

$$= O\left[\frac{x^{c}\ln x}{T(c-1)}\right] + O\left[\frac{x\ln x}{T\langle x\rangle}\right] .$$

Step III A

Suppose first that $x =$ integer $(\gtrapprox 2)$.

Here $\langle x\rangle = \frac{1}{100}$ by (12) ; $n_1 = x$, $n_2 = x+1$;

and

$$LHS = \Lambda(n_1)\frac{1}{T} + \Lambda(n_2)\frac{1}{T}\frac{(x/n_2)^{c}}{|\ln x - \ln n_2|}$$

(21)

$$\approx \frac{O(\ln x)}{T} + \frac{O(\ln(x+1))}{T} \frac{1}{\ln(x+1) - \ln x}$$

$$\approx \frac{O(\ln x)}{T} + \frac{O[x \ln(x+1)]}{T} \quad \left\{ \begin{matrix} \text{by mean} \\ \text{value thm} \end{matrix} \right\}$$

$$\approx \frac{O(x \ln x)}{T},$$

which is subsumed by **both** $\frac{x^c \ln x}{T}$ and $\frac{x \ln x}{T\langle x \rangle}$. OK III A

Step III B

Suppose next that $x \neq$ integer $(x \gtrsim 1 + \delta_0)$.

For **each** j, notice that (in LHS on (20)):

$$\Lambda(n_j^\circ) \frac{1}{T} \left(\frac{x}{n_j^\circ}\right)^2 \frac{1}{|\ln x - \ln n_j^\circ|}$$

$$\approx \Lambda(n_j^\circ) \frac{O(1)}{T} \frac{x}{|x - n_j^\circ|} \quad \left\{ \begin{matrix} \text{by mean} \\ \text{value thm} \end{matrix} \right\}.$$

If $|x - n_j^\circ| \gtrsim \frac{1}{100}$, the foregoing bound is

ie, TERM

$$O(\ln x)\,\frac{x}{T}\,(100) = O\left[\frac{x\ln x}{T}\right].$$

This is obviously subsumed by $O\left[\frac{x\ln x}{T\langle x\rangle}\right]$.

$$\langle x\rangle \lessapprox \frac{1}{100} \quad \text{by } 12$$

On the other hand, suppose $|x - n_j| < \frac{1}{100}$.
The TERM on 21 bottom is <u>either</u>

$$0 \qquad \text{or else} \qquad O(\ln x)\,\frac{O(1)}{T}\,\frac{x}{\langle x\rangle}$$

by recalling the def of $\langle x\rangle$ again. Here, then, we <u>again</u> get something subsumed by $O\left[\frac{x\ln x}{T\langle x\rangle}\right]$.

Both cases in the "either" can occur so the $O\left[\frac{x\ln x}{T\langle x\rangle}\right]$ is essentially <u>sharp</u>.

Bottom Line:

$$\left[\begin{array}{c}\text{for } x \neq \text{integer} \\ \text{LHS on } 20\end{array}\right] = O(1)\,\frac{x\ln x}{T\langle x\rangle}. \qquad \text{OK}$$

III B

By the 2 OK's, we get: in III

$$\sum_{n \in \{n_1, n_2\}} \Lambda(n) \begin{bmatrix} \frac{1}{T} \frac{(x/n)^c}{|\ln x - \ln n|} & , & n \neq x \\ \frac{1}{T} & , & n = x \end{bmatrix}$$

$$= O(1) \frac{x \ln x}{T\langle x \rangle} .$$ ||||

All told, chunks #1 and #2 in the big bracket on 14 lead to an error term [à la 15, 17, 19, 20, and line 3 above] of

$$O(1) \frac{x \ln x}{T\langle x \rangle} + O(1) \frac{x^c \ln x}{T(c-1)} \qquad \text{for } x \geqq 1 + \delta_0 ,$$

where the "implied constants" depend on δ_0. Assertion (i) on p. 13 is thus proved.

13

One expects that (ii) will be VERY similar — with use of chunk #3 in the big bracket on 14 at an appropriate point.

Note that steps I and II are OK as is!

We need only fiddle with step III. See 20 and 14 bracket.

$$n_1 = [\![x]\!], \quad n_2 = [\![x]\!]+1 \qquad \left\{\begin{array}{l} n_j \text{ is relevant} \\ \underline{\text{only}}\ \underline{\text{if}}\ \Lambda(n_j) \neq 0 \end{array}\right\}$$

Want $\frac{x}{A} < n_j < \infty$

$$\Downarrow$$

$A = 1$ OK for $j = 2$

for $j=1$, n_1 is <u>relevant</u> only if $[\![x]\!] \geqq 2$
whereupon $A = 2$ is adequate

$$\Downarrow$$

NEW step III
$$= \sum_{n \in \{n_1, n_2\}} \Lambda(n) \left[O_{\underline{2}}(1) \text{ wlog} \right]$$

$$\leqq O(1) \ln(x+1) \leqq O(1) \ln x, \qquad \|$$

for $x \geqq 1 + \delta_0$.

Page 13 assertion (ii) follows at once. ▨

<u>In</u> Lec <u>17</u>, following Landau $(c=2)$, we got $O\left(\frac{x^2 \ln x}{T}\right) + O(\ln x)$ for assertion (ii).

To continue, consider $\{x, c, T\}$ as on 10 + 11 top.
Assume that

$$T \neq \text{all } \gamma . \qquad \boxed{\rho = \beta + i\gamma}$$

Form (rectangle) R as on p. 6 with $A = 2m+1$. Treat $\{x, c, T\}$ as frozen for a few moments. By letting $m \to \infty$, we immediately get

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds + \frac{1}{2\pi i}\int_{c+iT}^{-\infty+iT}$$

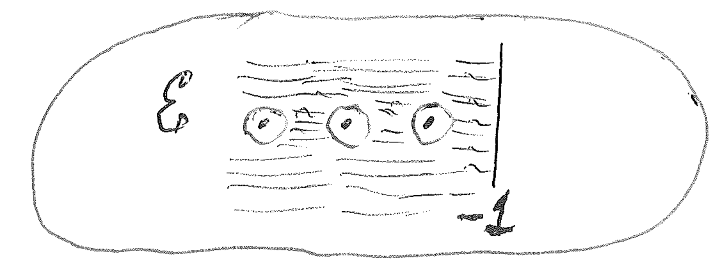
$$+ \frac{1}{2\pi i}\int_{-\infty-iT}^{c-iT} = \sum \text{Res}$$

$$= x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{k=1}^{\infty} \frac{x^{-2k}}{(-2k)} - \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho}$$

thanks to

$$\frac{\zeta'(s)}{\zeta(s)} = O(1) \ln(|s| + 10) \quad \text{for } s \in \mathcal{E},$$

a fact proved in Lec 16, pp. 5 ~ 9.



$\boxed{\text{Unfreeze } x, c, T\,!}$

The horizontal integrals have $s = \sigma \pm iT$ and are <u>absolutely</u> convergent (since $x > 1$).

To estimate them, it suffices to look at

$$\left| \frac{1}{2\pi i} \int_{c+iT}^{-\infty + iT} \frac{x^s}{s} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds \right| .$$

For $-1 \leqq \sigma \leqq 2$, we know that

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln T) + \sum_{|\gamma - T| \leqq 1} \frac{1}{s - \rho}$$

when $s = \sigma + iT$. See Lec 15 p. 13. One inflates the implied constant in $O(\ln T)$ to handle <u>moderate</u> T.

The portion of the horiz integral arising from $(-\infty + iT, \underline{-1} + iT]$ is clearly

$$O(1) \int_{-\infty}^{-1} \frac{x^\sigma}{|s|} O(\ln |s|)\, d\sigma \qquad \left\{ \begin{array}{l} \text{by } \frac{\zeta'}{\zeta} \text{ estimate} \\ \text{over } \mathcal{E} \end{array} \right\}$$

$$\leqq O(1) \frac{\ln T}{T} \int_{-\infty}^{-1} x^\sigma \, d\sigma$$

$$= O(1) \frac{\ln T}{T} \frac{x^{-1}}{\ln x} \leqq O(1) \frac{\ln T}{T} x^{-1} \qquad \{ x \geqq 1 + \delta_0 \}.$$

To treat the stretch $[-1+iT, c+iT]$, we must look at

$$\left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s} \left[O(\ln T) + \sum_{|\gamma - T| \leq 1} \frac{1}{s-\rho} \right] ds \right|$$

wherein $s = \sigma + iT$. We stress that the bracket is a <u>continuous</u> function of σ.

The portion with $O(\ln T)$ clearly gives:

$$O(\ln T) \int_{-1}^{c} \frac{x^\sigma}{T} d\sigma$$

$$= O\left(\frac{\ln T}{T}\right) \int_{-1}^{c} x^\sigma d\sigma$$

$$= O\left(\frac{\ln T}{T}\right) \int_{-\infty}^{c} x^\sigma d\sigma$$

$$= O\left(\frac{\ln T}{T}\right) \frac{x^c}{\ln x}$$

$$= O\left(\frac{\ln T}{T}\right) x^c \quad /\!/ \quad \{x \geq 1 + \delta_0\}.$$

Compare (26) bottom.

To handle the rest, we look at

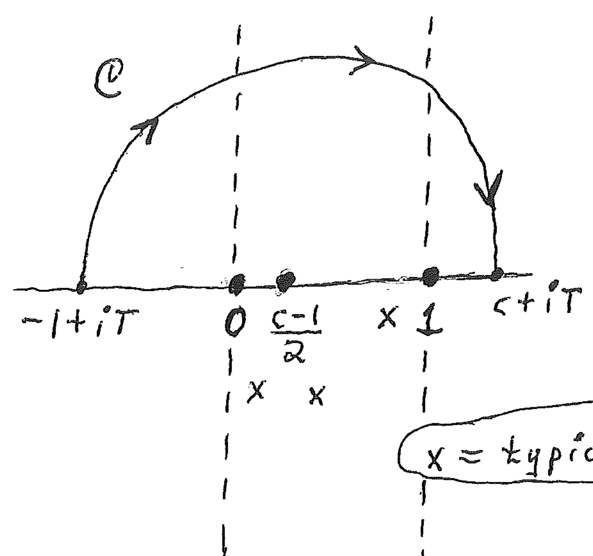
$$\sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s(s-\rho)}\, ds \right|$$

using the Cauchy integral theorem on each integral separately.

> We will not seek the best possible estimate (especially for c very close to 1), just something that leads to an explicit formula for $\psi(x)$ of a quality comparable to the best that is currently known for practical use.

Remember that $1 < c \leqq 2$. Also that $\rho = \beta + i\gamma$ has $0 \leqq \beta \leqq 1$. We propose to deform $[-1+iT, c+iT]$.

$T-1 \leqq \gamma < T \Rightarrow$



radius $\frac{c+1}{2}$
center $\frac{c-1}{2}$

x = typical $\beta + i\gamma$'s

> A rectangle of height $\frac{1}{2}(c+1)$ can also be used.

$T < \gamma \leqq T+1 \Rightarrow$ make similar $\mathcal{C}$, but go down.

For $s \in \mathcal{C}$, notice first that:

$$|s-\rho| \gtrapprox |s-(\beta+iT)|.$$

Elem geometry shows that the sliding circle

$$\{\, |s-(\eta+iT)| = \tfrac{1}{2}(c-1) \,\}$$

never intersects $\mathcal{C}$ for $0 \leqq \eta \leqq 1$. [At $\eta=0$, $\frac{c-1}{2} < 1$.]

Accordingly, $s \in \mathcal{C} \Rightarrow |s-\rho| \gtrapprox \frac{1}{2}(c-1)$ for $1 < c \leqq 2$.

For EACH ρ, get:

$$\left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s(s-\rho)}\, ds \right|$$

$$= \left| \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{x^s}{s(s-\rho)}\, ds \right| \qquad \text{(by CIT)}$$

$$\lessapprox \frac{1}{2\pi} \int_{\mathcal{C}} \frac{x^\sigma}{|s||s-\rho|} |ds|$$

$$\lessapprox \frac{1}{2\pi} \int_{\mathcal{C}} \frac{x^c}{(T-\frac{3}{2})\,\frac{1}{2}(c-1)} |ds|$$

$$\lessapprox (\text{const}) \frac{x^c}{T} \frac{1}{c-1} (\pi c + \pi)$$

$$\lessapprox (\text{const}) \frac{x^c}{T} \frac{1}{c-1} \quad //$$

Since $N[\rho : |\gamma - T| \leqq 1] = O(\ln T)$ by Lec 15 p. 8, we now get:

$$\left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s} \Big[\sum_{|\gamma - T| \leqq 1} \frac{1}{s-\rho} \Big] ds \right| \leftarrow \text{on } 27$$

$$\leqq O(\ln T) \frac{x^c}{T} \frac{1}{c-1} .$$

By 27 bottom, we get:

$$\left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s} \Big[O(\ln T) + \sum_{|\gamma - T| \leqq 1} \frac{1}{s-\rho} \Big] ds \right|$$

$$= O\left(\frac{\ln T}{T}\right) \frac{x^c}{c-1} \quad /\!/ .$$

By 26, we thus see that:

$$\frac{1}{2\pi i} \int_{-\infty \pm iT}^{c \pm iT} \frac{x^s}{s} \left(- \frac{\zeta'(s)}{\zeta(s)} \right) ds = O\left(\frac{\ln T}{T}\right) \frac{x^c}{c-1} \quad /\!/$$

for $1 < c \leqq 2$, $x \geqq 1 + \delta_0$, $T \geqq 3$, $T \neq$ all γ .

Referring to 25 middle and 13, we find that:

$$\psi^*(x) + O\left[\frac{x^c \ln x}{T(c-1)}\right] + \left\{\begin{array}{c} O(\ln x) \\ \\ O\left[\frac{x \ln x}{T\langle x\rangle}\right] \end{array}\right\}$$

$$+ \ O\left(\frac{\ln T}{T}\right)\frac{x^c}{c-1} \quad \longleftarrow \text{by } 30$$

$$= x - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{x^{-2k}}{k} - \sum_{|\gamma|\leqq T}\frac{x^\rho}{\rho}$$

for $x \geqq 1+\delta_0$, $T \geqq 3$, $T \neq$ all γ, $1 < c \leqq 2$.

Notice that, for <u>any</u> small $h \in (0,1)$,

$$\left|\sum_{|\gamma - t|\leqq h}\frac{x^\rho}{\rho}\right| \leqq O(\ln t)\frac{x}{t}$$

$$\leqq O\left(\frac{\ln t}{t}\right)x$$

anytime $t \geqq \frac{5}{2}$. Similarly for $|\gamma + t| \leqq h$.

Here $x \geqq 1+\delta_0$, as usual.

Holding $\{x, c\}$ fixed for a moment, we can now apply right continuity in T to address cases on 31 top where $T \geq 3$, but $T =$ some γ.

GET: (for any $T \geq 3$)

$$\psi^*(x) = x - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln(1 - x^{-2}) - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho}$$

$$+ O\left(\frac{\ln T}{T}\right)\frac{x^c}{c-1}$$

$$+ O\left(\frac{x^c \ln x}{T(c-1)}\right) + \left\{ \begin{array}{c} O(\ln x) \\ \\ O\left[\frac{x \ln x}{T\langle x\rangle}\right] \end{array} \right\}$$

for $x \geq 1 + \delta_0$, $1 < c \leq 2$.

$\boxed{\text{NB}}$

Compare: Ingham, p. 77, wherein $c = 2$,

AND Landau, Vorlesungen, Satz 452 (wherein $c = 2$). Note how our formula above is better than both, via taking $c = 3/2$.

To optimize 32 middle, one basically wants to <u>minimize</u>

$$\frac{x^c}{c-1},$$

i.e., put $c \approx 1 + \frac{b}{\ln x}$ where b = a tiny constant. Here $x \geqq 1+\delta_0$ and we need to have $1 < c \leqq 2$. See 17 top. Since c is free in 32 middle, this choice of c is <u>completely legal</u>.

For $x \geqq 1+\delta_0$, $T \geqq 3$, we thus get:

$$\psi^*(x) = x - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln(1-x^{-2}) - \sum_{|\gamma| \leqq T} \frac{x^\rho}{\rho}$$

$$+ O\left(\frac{x \ln T \cdot \ln x}{T}\right) + O\left(\frac{x \ln^2 x}{T}\right)$$

$$+ O(\ln x) \min\left\{1, \frac{x}{T\langle x\rangle}\right\}.$$

The implied constants will depend [solely] on δ_0.

The "trivial terms"

$$-\frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln(1-x^{-2}) \quad \leftarrow \text{ on } 33$$

define an obvious power series in x^{-2} and are often simply replaced by $O(1)$. Insofar as that is done, once any given remainder term on 33 $\overset{\text{bot}}{\vee}$ drops below Ωx^{-6}, say, (with $|\Omega| \leqq 1$) that term takes on a patently secondary role vis à vis x. ← $\boxed{\text{especially if } x \to \infty}$

That being said, we now observe that:

(A) for $x \geqq 1+\delta_0$ and $T \geqq 3x^{10}$ (say),

$$\frac{x \ln T \cdot \ln x}{T} + \frac{x \ln^2 x}{T} \leqq \frac{x(\ln T + \ln x)^2}{T}$$

$$= O(x^{-8}) \quad ;$$

(B) for $x \geqq 1+\delta_0$ and $3 \leqq T \leqq 3x^{10}$,

$$\underline{c(\delta_0)} \frac{x}{T}(\ln T + \ln x)^2 \leqq \frac{x}{T}(\ln T \cdot \ln x + \ln^2 x) \leqq \frac{x}{T}(\ln T + \ln x)^2,$$

$\boxed{\text{this final chunk scales like } \langle 1, 144 \rangle \frac{x \ln^2 x}{T} \text{ if } x \geqq 3}$

where $c(\delta_0)$ is some appropriately tiny positive constant.

Theorem (Standard statement of Explicit Formula for $\psi(x)$)

p^m set

Let $x \geqq 1+\delta_0$, $T \geqq 3$. Define $\exists$ and $\langle x \rangle$ as on 12. We then have:

$$\psi^*(x) = x - \sum_{|\gamma| \leqq T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln(1-x^{-2})$$

$$+ O\left[\frac{x}{T}(\ln T + \ln x)^2\right]$$

$$+ O(\ln x)\min\left\{1, \frac{x}{T\langle x \rangle}\right\},$$

wherein

$$\psi^*(x) = \frac{\psi(x+0) + \psi(x-0)}{2}.$$

The implied constants will depend on [at most] δ_0. In addition, one has:

$$\frac{\zeta'(0)}{\zeta(0)} = \ln(2\pi).$$

Proof

See 33 (bottom) and, then, the obvious relation

$$\frac{x \ln T \cdot \ln x}{T} + \frac{x \ln^2 x}{T} \leq \frac{x(\ln T + \ln x)^2}{T}$$

used on 34. This proves the formula for $\psi^*(x)$. OK

We'll verify $\dfrac{\zeta'(0)}{\zeta(0)} = \ln(2\pi)$ in a theorem stated several pages below. ← $\boxed{\text{See 41.}}$

∎

N. B.

The formula on 35 middle can be found many places; e.g., in Davenport, Mult. Number Theory, 2^{nd} ed., p. 109 (9)(10) OR Prachar, Primzahlverteilung, Satz 4.5 on pp. 231–2.

We define:

$$\sum_\rho \frac{x^\rho}{\rho} \equiv \lim_{T\to\infty} \sum_{|\gamma|\leq T} \frac{x^\rho}{\rho}.$$

Recall 31 bottom concerning slight "sloppiness" in T.

Riemann

Corollary 1.

For each $x \geq 1+\delta_0$, we have

$$\psi^*(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln(1-x^{-2}).$$

In this regard, we also have (in an obvious sense)

$$\sum_{|\gamma|>T} \frac{x^\rho}{\rho} = O\left[\frac{x}{T}(\ln T + \ln x)^2\right] + O(\ln x)\min\left\{1, \frac{x}{T\langle x\rangle}\right\}.$$

If desired, the 1st term on the RHS can be replaced by

$$O\left[\frac{x}{T}(\ln T\cdot\ln x + \ln^2 x)\right].\quad *$$

Pf

Straightforward. See 33 bottom for the last assertion.

∎

---

* As already hinted in the two boxes on 28, the term $\frac{x\ln T\cdot\ln x}{T}$ can in fact be improved slightly. This will not affect the estimates for $\psi(x)-x$ though. See 34. Also p. 39.

Corollary 2.

Let $[x_1, x_2]$ be any closed interval in $[1+\delta_0, \infty)$.

(a) If $[x_1, x_2] \cap \mathcal{F} = \phi$, then $\sum \frac{x^\rho}{\rho}$ converges uniformly on $[x_1, x_2]$ as a symmetric limit in T.

(b) In every instance, the partial sums $\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho}$ are uniformly bounded on $[x_1, x_2]$ for all $T \geqq 3$.

Pf

For (a), use corollary 1.

For (b), rearrange (35) middle and use the "1" in the minimum. ▮

Thm (recall Lec 16, p. (19))

The explicit formula for $\psi(x)$ immediately gives

$$\psi(x) = x + O(x^\Theta \ln^2 x). \qquad (x \geqq 2)$$

Pf

By Lec 16, p. (1), assertion (a), we have:

$$\sum_{|\gamma| \leq T} \frac{1}{|\gamma|} = O(\ln^2 T), \quad T \geqq 3.$$

We stress that the foregoing bound is essentially sharp due to

$$\int_3^T \frac{1}{t}\, d\left(\frac{t}{2\pi}\ln\frac{t}{2\pi e}\right) = \int_3^T \frac{1}{t}\left[\frac{1}{2\pi}\ln\frac{t}{2\pi}\right] dt$$

Lec 15 p. 29

$$\{t = 2\pi u\}$$

$$= \frac{1}{2\pi}\int_{3/2\pi}^{T/2\pi} \frac{\ln u}{u}\, du$$

$$\sim \frac{1}{4\pi}\left(\ln\frac{T}{2\pi}\right)^2 \sim \frac{1}{4\pi}(\ln T)^2 .$$

Apply (35) with, say, $x \gtrapprox 100$ and $T = x^2$. Get:

$$|\psi^*(x) - x| \lessapprox \sum_{|\gamma| \leqq x^2} \frac{x^\Theta}{|\gamma|} + O(1) + O\left[\frac{x\ln^2 x}{x^2}\right]$$
$$+ O(\ln x)$$

$$\lessapprox O(1)\, x^\Theta \ln^2 x .$$

But $\psi(x) = \psi^*(x) + O(\ln x)$. Hence,

$$|\psi(x) - x| \lessapprox O(1)\, x^\Theta \ln^2 x$$

as promised. ▮

We now PAUSE for some elementary facts (better late than never) related to Lec 5 and Lec 16, p. 7.

THM

Let $\gamma \approx$ the Euler constant. Near $z=1$, we then have:

$$\zeta(z) = \frac{1}{z-1} + \gamma + O(|z-1|) .$$

Proof

Recall Lec 5, pp. 8 - 10 with $r(t) \equiv t - [\![t]\!]$.

The function $G(z) \equiv \zeta(z) - \frac{1}{z-1}$ is analytic on $\{x>0\}$.

$$G(z) = 1 - z \int_1^\infty \frac{r(t)}{t^{z+1}}\, dt$$

Apply Lec 5 p. 8 but take $z=1$. Get:

$$\sum_{n=1}^{N} \frac{1}{n} = 1 + \ln N - \int_1^N \frac{r(t)}{t^2}\, dt$$

$$\Downarrow$$

$$\gamma = 1 - \int_1^\infty \frac{r(t)}{t^2}\, dt .$$

Accordingly,

$$G(1) = 1 - \int_1^{\infty} \frac{r(t)}{t^2}\, dt = \gamma .$$

By Taylor series,

$$G(z) = \sum_{k=0}^{\infty} \frac{G^{(k)}(1)}{k!} (z-1)^k$$

$$= \gamma + O(z-1) \qquad \text{near } z = 1,$$

and we are done. $\blacksquare$

<u>THEOREM</u>

$$\zeta'(0) = -\frac{1}{2} \ln(2\pi), \qquad \frac{\zeta'(0)}{\zeta(0)} = \ln(2\pi) .$$

<u>PF</u>

Know $\zeta(0) = -\frac{1}{2}$ by Lec 9, pp. 18 + 20.

Now have:

$$\zeta(s) = \frac{1}{s-1} \left[1 + \gamma(s-1) + O(s-1)^2 \right]$$

by 40. Accordingly:

$$\log \zeta(s) = -\log(s-1) + \gamma(s-1) + O(s-1)^2$$

$$\Downarrow$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + O(s-1). \quad ////$$

This <u>sharpens</u> Lec 7, p. 17. We can now apply {the functional equation}

$$\frac{\zeta'(z)}{\zeta(z)} = \ln 2\pi + \frac{\pi}{2}\operatorname{ctn}\frac{\pi z}{2} - \frac{\Gamma'}{\Gamma}(1-z) - \frac{\zeta'}{\zeta}(1-z)$$

from Lec 16 p. 7. Recall

$$\Gamma(1) = 1, \quad \Gamma'(1) = -\gamma$$

by Lec 10, p. 30 assertion (e) ⟦and p. 22⟧. Take $z \to 0$ to get:

$$\frac{\zeta'(0)}{\zeta(0)} = \ln 2\pi - \frac{\Gamma'(1)}{\Gamma(1)}$$

$$+ \lim_{z\to 0}\left[\frac{1}{z} + O(z) + \left\{\frac{1}{-z} - \gamma + O(z)\right\}\right]$$

$$= \ln 2\pi + \gamma - \gamma + 0 = \ln 2\pi.$$

Multiply by $\zeta(0)$ to get $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$. ▨

$$\boxed{\text{END-OF-PAUSE.}}$$

We closed Lec 18 with a statement of, and very brief sketch-of-the-proof for, the so-called PERRON SUMMATION FORMULA associated with a general Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and $\sum_{n \leq x} a_n \quad (x \geq 10)$.

The formula is based on p. 4 Lemma [above] and its verification parallels pp. 14 ~ 24.

Due to the (already excessive!) length of these notes for Lec 17 + 18, we postpone this "Perron matter" until the notes for Lec 19.

~