Lectures 19 and 20

(30 March and 1 April)

SYNOPSIS

The development in Lectures 17+18, pp. 4 – 24 is highly suggestive and capable of a major generalization.

We touched on this already in Lec 18 at the very end. See Lec 18 p. 43.

This brought us to the Perron summation formula with error term.

We'll sketch this formula emphasizing the similarity to Lec 17+18, pp. 4 – 24.

We begin with a slight revision of Lec 17, p. 4 Lemma. Compare Ingham p. 75.

FACT

In Lec 17, p. 4 Lemma, things can be broadened to have

$$0 < y < \infty, \quad 0 < c \leqq 3, \quad T \geqq 3$$

getting

$$|\text{Remainder}| \leqq \left\{ \begin{array}{ll} O(1)\dfrac{y^c}{T|\ln y|}, & y \neq 1 \\ O(1)\dfrac{1}{T}, & y = 1 \\ O(1)\, y^c, & \text{any } y \end{array} \right\}.$$

The "implied" constant in the $O(1)$ is absolute. One also has [the somewhat cruder]

$$|\text{Rem.}| \leqq O(1)\,\frac{y^c}{1 + T|\ln y|}.$$

Proof

Simply review pp. 5 ~ 9 and modify several lines. For the final [cruder] assertion, divide into $T|\ln y| > 1$ and $T|\ln y| \leqq 1$. ▥

Recall Lec 17 p. 12. We generalize this!

Let $\{a_n\}_{n=1}^{\infty}$ be given. Assume that $a_n \neq 0$ infinitely often as $n \to \infty$.

Let $\mathcal{F}$ be any subset of $\mathbb{Z}^+$ which <u>includes</u> the set $\{n : a_n \neq 0\}$.

Define $\|u\|' = \begin{cases} |u|, & u \neq 0 \\ \infty, & u = 0 \end{cases}$ for $u \in \mathbb{R}$.

For $x \geqq \frac{3}{2}$, let $\xi(x) = \min_{\lambda \in \mathcal{F}} |x - \lambda|$. Also write

$$\langle x \rangle = \min\left\{\frac{1}{100}, \|\xi(x)\|'\right\}.$$

Notice that $\langle x \rangle = \frac{1}{100}$ unless $\|\xi(x)\|' < \frac{1}{100}$, which would mean $x \notin \mathcal{F}$, but lies <u>LESS</u> THAN $\frac{1}{100}$ units from $\mathcal{F}$.

We clearly have:

$$\langle x \rangle = \frac{1}{100} \quad \text{anytime } x \in \mathbb{Z} \quad (x \geqq \tfrac{3}{2})$$

$$0 < \langle x \rangle \leqq \frac{1}{100} \quad \text{always.}$$

THEOREM (Perron summation formula with error term)

Given a Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which is absolutely convergent on $\{Re(s) > 1\}$. Assume that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^w} = O\left(\frac{1}{(w-1)^\alpha}\right) \quad \text{for } 1 < w \leq 2.$$

Here $\alpha \geq 0$. Assume further that $|a_n| \leq \Phi(n)$, where $\Phi(v)$ is some continuous, nonnegative, monotonic increasing function on $\{1 \leq v < \infty\}$.

Consider $\{c, x, T\}$ such that

$$1 < c \leq 2, \quad x \geq 10, \quad T \geq 3.$$

Taking $\sigma = Re(s)$ and $\sigma + c > 1$, we then have the following relations insofar as $-9 \leq \sigma \leq 10$ (say):

(1) in a style reminiscent of the explicit formula,

$$\sum_{n<x} a_n n^{-s} + \begin{Bmatrix} 0, & x \notin \mathbb{Z} \\ \frac{1}{2} a_x x^{-s}, & x \in \mathbb{Z} \end{Bmatrix}$$

$$= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} dw + O\left[\frac{x^c}{T(\sigma+c-1)^\alpha}\right]$$

$$+ O\left[\frac{\Phi(2x)\, x^{1-\sigma} \ln x}{T}\right] \quad + \text{(see next page)}$$

$$+ \, O(1) \, \frac{\Phi(2x)}{x^{\sigma}} \min\left\{1, \frac{x}{T\langle x \rangle}\right\}$$

[see (3) for $\langle x \rangle$] ;

(2) in a style closer to that of an __apriori__ bound,

$$\sum_{n<x} a_n n^{-s} + \begin{Bmatrix} 0, & x \notin \mathbb{Z} \\ \frac{1}{2} a_x x^{-s}, & x \in \mathbb{Z} \end{Bmatrix}$$

$$= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} \, dw$$

$$+ \, O(1) \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \frac{(x/n)^c}{1 + T|\ln \frac{x}{n}|} .$$

In (1), the "implied" constants are absolute apart from a __mild dependence__ on α and the implied constant associated with $O[(\omega-1)^{-\alpha}]$. In (2), the implied constant is absolute.

__Proof__

It will be convenient to let $N =$ the integer nearest to x (with $x = k + \frac{1}{2} \Rightarrow N = k$).

We look first at:

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\,dw$$

$$= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \sum_{n=1}^{\infty} \frac{a_n}{n^{s+w}}\frac{x^w}{w}\,dw$$

$$= \sum_{n=1}^{\infty}\frac{a_n}{n^s}\left(\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{(x/n)^w}{w}\,dw\right)$$

in the setting of p. 2 FACT. Think

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}\left\{\eta\left(\frac{x}{n}\right) + \text{REM.}\right\},$$

where REM. takes various formats.

Assertion (2) follows immediately. $\boxed{\text{OK}}$

For (1), we need to adapt Lec 17 pp. 14 – 24. We merely sketch this.

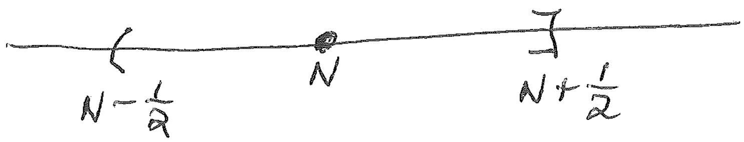
First of all, corresponding to Lec 17 p. 15 top, we split things according to

$$1 \leqq n \leqq N-1$$

$$n = N$$

$$N+1 \leqq n < \infty .$$

Note that, by hypothesis, $N - \frac{1}{2} < x \leqq N + \frac{1}{2}$.
(When $x = k + \frac{1}{2}$, $N = k$ and all is OK.)



For $1 \leqq n < \frac{x}{2}$, one easily checks

$$\sum_{1 \leqq n < \frac{x}{2}} \frac{a_n}{n^s} \{\text{REM.}\} = O\left[\frac{x^c}{T(\sigma+c-1)^q}\right] .$$

<u>N.B.</u> $-9 \leqq \sigma \leqq 10$, $1 < c \leqq 2 \Rightarrow -8 < \sigma + c \leqq 12$ apriori.
All is OK if $1 < \sigma + c \leqq 2$. For $2 < \sigma + c \leqq 12$, one easily adjusts the implied constant in $O[(\omega-1)^{-q}]$ to accommodate $2 < \omega \leqq 12$. This clearly involves q. Inflation by 11^q is sufficient.

For <u>$2x < n < \infty$</u>, we also get

$$O\left[\frac{x^c}{T(\sigma+c-1)^q}\right] .$$

For $\frac{x}{2} \lesssim n \leqq N-1$, we write $n = N-r$ and IMITATE Lec 17 p. 19 bottom. We get

$$\sum_{\frac{x}{2} \lesssim n \leqq N-1} \frac{a_n}{n^s} \{\text{REM.}\}$$

$$= O(1) \frac{\Phi(N)}{x^\sigma} \frac{x \ln x}{T} \qquad \left\{ \begin{array}{l} \text{recall } x \geqq 10, \\ -9 \leqq \sigma \leqq 10 \end{array} \right\}$$

$$= O(1) \frac{\Phi(2x)}{x^\sigma} \frac{x \ln x}{T} \quad ////$$

via trivial insertion of absolute values.

Not surprisingly, noting Lec 17 p. 16 middle, we find that

$$\sum_{N+1 \leqq n \lesssim 2x} \frac{a_n}{n^s} \{\text{REM.}\}$$

$$= O(1) \frac{\Phi(2x)}{x^\sigma} \frac{x \ln x}{T} \quad ////$$

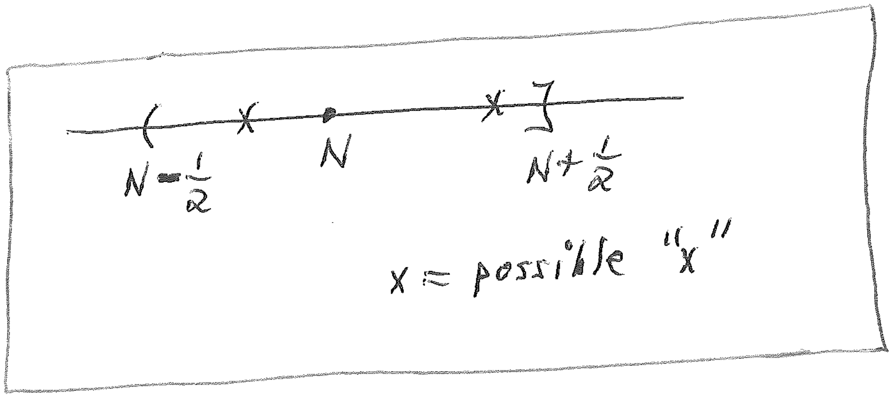
as well.

It remains to discuss

$$\frac{|a_N|}{N^\sigma}\,|\mathrm{REM}_\circ| \qquad \{n=N\}.$$

$$\boxed{\begin{array}{c} x \geq 10 \\ \Downarrow \\ N \geq 10 \end{array}}$$

If $a_N = 0$, we get 0 which is subsumed <u>by anything</u>. Suppose therefore that $a_N \neq 0$. Hence $N \in \mathcal{F}$.



If $|x-N| \geq \frac{1}{100}$, we have:

$$\frac{|a_N|}{N^\sigma}\,|\mathrm{REM}_\circ| = O(1)\,\frac{\Phi(N)}{N^\sigma}\,\frac{1}{T}\,\frac{1}{|\ln x - \ln N|}$$

$$\leq O(1)\,\frac{\Phi(N)}{x^\sigma}\,\frac{1}{T}\,\frac{1}{\frac{1}{x}|x-N|}$$

$$= O(1)\,\frac{\Phi(N)}{x^\sigma}\,\frac{x}{T}$$

$$= O(1)\,\frac{\Phi(2x)}{x^\sigma}\,\frac{x}{T}$$

$$= O(1)\,\Phi(2x)\,\frac{x^{1-\sigma}}{T}\,.$$

This is subsumed by $O(1)\,\Phi(2x)\,\frac{x^{1-\sigma}\ln x}{T}$ without further ado.

OK

$\boxed{\text{cf. page 8 above; also 4 bottom.}}$

To finish up, we therefore take $|x-N| < \frac{1}{100}$ wlog.
We still have $a_N \neq 0$ and $N \in \mathcal{F}$.

Must consider 2 cases.

Case I $N = x$.

Here $\langle x \rangle = \frac{1}{100}$ (cf. 3) and

$y = \frac{x}{N}$

$$\frac{|a_N|}{N^\sigma}\,|\mathrm{REM}_0| = O(1)\,\frac{|a_N|}{N^\sigma}\min\left\{1, \frac{1}{T}\right\} \quad \text{by } 2$$
$$= O(1)\,\frac{\Phi(2x)}{x^\sigma}\min\left\{1, \frac{1}{T}\right\}.$$

Of course, $T \geq 3$. In any event, this last expression is safely subsumed by

$$O(1)\,\frac{\Phi(2x)}{x^\sigma}\min\left\{1, \frac{x}{T\langle x\rangle}\right\}. \;/\!/$$

OK

Though not really necessary, we remark that:

$$\frac{|a_N|}{N^\sigma}|REM_0| = O(1)\frac{\Phi(2x)}{x^\sigma}\frac{1}{T}$$

$$= O(1)\,\Phi(2x)\frac{x^{-\sigma}(x\ln x)}{T};$$

i.e. matters are also subsumed by the term

$$O(1)\,\Phi(2x)\frac{x^{1-\sigma}\ln x}{T}$$

in case I

à la ⑧ above (cf. also ④).

Case II $N \neq x$ but $|x-N| < \frac{1}{100}$.

By ②,

$$\frac{|a_N|}{N^\sigma}|REM_0| = O(1)\frac{|a_N|}{N^\sigma}\min\left\{\left(\frac{x}{N}\right)^c, \frac{(x/N)^c}{T|\ln x - \ln N|}\right\}$$

$$\boxed{1<c\leqq 2,\ -9\leqq\sigma\leqq 10,\ x\gtrapprox 10,\ T\gtrapprox 3}$$

$$= O(1)\frac{|a_N|}{N^\sigma}\left(\frac{x}{N}\right)^c\min\left\{1, \frac{1}{\frac{T}{x}|x-N|}\right\}$$

$$= O(1)\frac{|a_N|}{x^\sigma}\min\left\{1, \frac{x}{T\langle x\rangle}\right\}$$

$\langle x\rangle = |x-N|$ here; see ③

$$= O(1)\frac{\Phi(2x)}{x^{\sigma}}\min\left\{1, \frac{x}{T\langle x\rangle}\right\}.$$

OK

By combining the three OK's [on 10 + here], we deduce that:

$$\frac{|a_N|}{N^{\sigma}}|REM_0| = O(1)\,\Phi(2x)\frac{x^{1-\sigma}\ln x}{T}$$

$$+ O(1)\frac{\Phi(2x)}{x^{\sigma}}\min\left\{1, \frac{x}{T\langle x\rangle}\right\}$$

exactly as needed.

This completes the proof of (1) on 4.

Page 4 THM is clearly little more than a pedestrian revamp of Lec 17+18 pp. 4 – 24. Note that:

$$\sum_{n<x} a_n n^{-s} + \left\{\begin{matrix} 0, & x \notin \mathbb{Z} \\ \frac{1}{2}a_x x^{-s}, & x \in \mathbb{Z}\end{matrix}\right\} = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\,dw$$

pointwise w.r.t. x. This is the original version of the Perron summation formula.

On the matter of assertion (2), we leave it as an easy exercise to obtain

*

$$|\text{remainder term}| = O\left[\frac{x^c}{T(\sigma+c-1)^\alpha}\right]$$
$$+ O(1)\,\Phi(2x)\,x^{-\sigma}\,\frac{x\ln[\min(T, x^{1/2})]}{T}$$
$$+ O(1)\,\Phi(2x)\,x^{-\sigma}\min\left\{1, \frac{x}{T\,\xi_0(x)}\right\}$$

where

$$\xi_0(x) \equiv \min\left\{\frac{1}{100}, \xi(x)\right\}$$

and $\xi(x) = \min_{\lambda\in\mathcal{F}} |x-\lambda|$ as on (3). Of course,

$$\min(T^{1/2}, x^{1/2}) \lesssim \min(T, x^{1/2}) \lesssim \min(T, x)$$
$$\Downarrow$$
$$\tfrac{1}{2}\ln[\min(T,x)] \lesssim \ln[\min(T, x^{1/2})] \lesssim \ln[\min(T,x)].$$

This allows line 6 to be cleaned up slightly.

---

* As suggested, treat $T < x^{1/2}$ and $T \gtrsim x^{1/2}$ separately. Note that $\xi_0(x)$ can be 0.

For $T > x^{1/2}$, assertion (1) is better than lines 5–7 on 13. Cf. $\langle x \rangle$ versus $\xi_0(x)$. For $T < x^{1/2}$, lines 5–7 will sometimes give the better result. EG take $x \notin \mathbb{Z}$ and $T = \exp[(\ln x)^\delta]$, δ tiny.

$\boxed{\text{NEW TOPIC.}}$

Let

$$\mu(n) = \left\{ \begin{array}{ll} 0, & \text{if } n \text{ is NOT squarefree} \\ (-1)^\nu, & \text{if } n = p_1 \cdots p_\nu \quad (\text{distinct primes}) \end{array} \right\}.$$

Of course, $\nu = 0$ for $n = 1$. It is completely standard by Euler's identity (see Ingham 16 + Lec 6 p. 4) to verify that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$. It is also standard to verify that

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$

wherein $d(n) \equiv N[k \geq 1 : k | n]$. Thus $d(6) = 4$ via $\{1, 2, 3, 6\}$. Elementary number theory gives

$$d(n) = (1+E_1)\cdots(1+E_r) \quad \text{for} \quad n = p_1^{E_1}\cdots p_r^{E_r}.$$

Here $p_1 < p_2 < \cdots < p_r$ are primes. An easy analysis shows

$$d(n) = O(n^{\varepsilon})$$

for every $\varepsilon > 0$. Indeed: $\{\text{wlog } n \geq 4\}$

$$E_1 + \cdots + E_r \leq (1.5) \ln n$$

$$1 + m \leq (p^{\varepsilon})^m \quad \text{for all} \quad p \geq 2^{1/\varepsilon} \quad (m \geq 0)$$

$$\prod_{j=1}^{r} (1+E_j) = \prod_{p_j < 2^{1/\varepsilon}} (1+E_j) \cdot \prod_{p_j \geq 2^{1/\varepsilon}} (1+E_j)$$

$$\leq \prod_{p_j < 2^{1/\varepsilon}} (1+E_j) \cdot \prod_{p_j \geq 2^{1/\varepsilon}} p_j^{\varepsilon E_j}$$

$$\leq \prod_{p_j < 2^{1/\varepsilon}} (3(\ln n)) \cdot n^{\varepsilon}$$

$$\leq O(n^{2\varepsilon}). \quad \{\text{EACH } \varepsilon\}$$

For real $y \geq 1$, we define

$$M(y) = \sum_{n \leq y} \mu(n)$$

$$T(y) = \sum_{n \leq y} d(n).$$

Theorem

The flip-flopping function $\mu(n)$ has average 0 in the sense that

$$\lim_{x \to \infty} \frac{M(x)}{x} = 0 .$$

IN FACT:

$$M(x) = O\left[x e^{-c(\ln x)^{1/10}} \right]$$

with some suitably small $c > 0$.

Pf

We use (Perron) p. 4 version (1) with $x = m + \frac{1}{2}$, m large. We take

$$a_n = \mu(n),\ \Phi(v) = 1,\ q = 1,\ s = 0,\ \sigma = 0 .$$

We note that

$$\frac{1}{\zeta(s)} = O[\ln^7 t] \quad \text{for } \sigma \geqq 1 - \lambda (\ln t)^{-9}$$

whenever λ is appropriately small and $t \geqq 3$.
In checking this, wlog $t =$ giant. We remember that:

$$\frac{1}{\zeta(\sigma+it)} = O(\ln^7 t)\,,\quad \sigma \geqq 1,\ t \geqq 3 \qquad \text{Lec 7 p. } 5$$

$$|\zeta(\sigma+it)| \geqq A(\ln t)^{-7} \quad \text{here}$$

$$|\zeta(\sigma+it)| \leqq \frac{C}{\delta(1-\delta)}|t|^{1-\delta} \qquad \sigma \geqq \delta,\ |t| \geqq 2 \qquad \text{Lec 6 p. } 9$$

$$|\zeta(\sigma+it)| \leqq A_2 \ln|t|\,,\quad \sigma \geqq 1-\frac{5}{\ln|t|}\,,\quad |t| \geqq t_0 \qquad \text{Lec 6 } 14\ 20$$

$$|\zeta'(\sigma+it)| \leqq A_3 \ln^2|t|\,,\quad \sigma \geqq 1-\frac{5}{\ln|t|}\,,\quad |t| \geqq t_0 \qquad \text{Lec 6 } 20$$

and

$$s_1 = 1+it\,,\quad s_2 = \sigma_2 + it\,,\quad \tfrac{1}{2} < \sigma_2 < 1 \implies$$

$$\begin{aligned}|\zeta(s_2)| &\geqq |\zeta(s_1)| - |\zeta(s_2) - \zeta(s_1)| \\ &\geqq |\zeta(s_1)| - \int_{s_2}^{s_1} |\zeta'(w)|\,d\sigma \qquad \{w = \sigma + it\}.\end{aligned}$$

Get: $(t \geqq t_0)$

$$\begin{aligned}|\zeta(\sigma_2+it)| &\geqq A(\ln t)^{-7} - \int_{\sigma_2}^{1} A_3 \ln^2 t\, d\sigma \\ &\geqq A(\ln t)^{-7} - A_3 \ln^2 t\,(1-\sigma_2)\end{aligned}$$

KEEP $\sigma_2 > 1 - \dfrac{\lambda}{(\ln t)^9}$ $\qquad$ <u>λ tiny</u>

$$\Downarrow$$

$$\geqq (A - \lambda A_3)(\ln t)^{-7} \geqq \tfrac{1}{2}A(\ln t)^{-7}.$$

In other words,

$$\frac{1}{\zeta(\sigma+it)} = O(\ln^7 t), \quad \sigma \gtrapprox 1 - \frac{\lambda}{(\ln t)^9}, \quad t \gtrapprox 3$$

for <u>some</u> tiny λ.

By p. ④ (1), with $\mathcal{F} = \mathbb{Z}^+$ and $c \in (1,2]$,

$$M(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{1}{\zeta(w)}\frac{x^w}{w}\,dw$$
$$+ O\left[\frac{x^c}{T(c-1)}\right] + O\left[\frac{x\ln x}{T}\right]$$
$$+ O(1)\min\left\{1, \frac{x}{T(\frac{1}{2})}\right\}.$$

To optimize $O\left[\frac{x^c}{T(c-1)}\right]$ in regard to c, we put

$$c = 1 + \frac{1}{\ln x}.$$

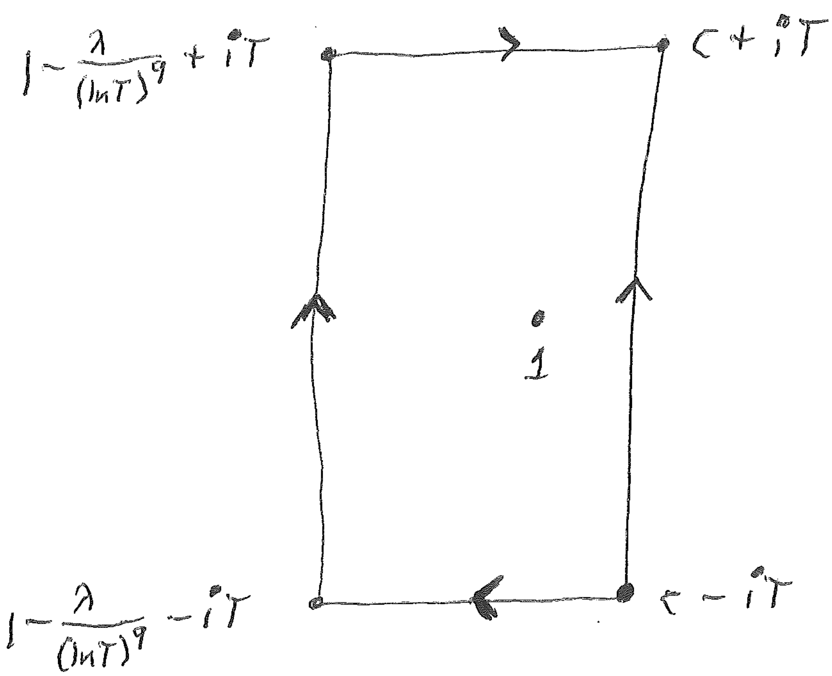
Since $x = m + \frac{1}{2}$ is big, this is legal. Get:

$$M(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} + O\left[\frac{x\ln x}{T}\right] + O\left[\frac{x\ln x}{T}\right]$$
$$+ O(1)\frac{x}{T}$$

$$\Downarrow$$

$$M(x) \approx \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \quad + \quad O\left[\frac{x}{T}(\ln x)\right] .$$

We now deform $[c-iT, c+iT]$ remembering pp. 17, 18 top. Of course, $1/\zeta(w)$ has a __simple zero__ at $w=1$ (in the sense of a removable singularity).



use a rectangle!

By either reducing λ or inflating T, we (and do) can hypothesize that $1/\zeta(w)$ is nicely __analytic__ on this closed rectangle. By 18 top, the issue occurs only for $|\mathrm{Im}\, w| \lessapprox 3$.

By the Cauchy Integral Thm, looking at $\frac{1}{\zeta(w)} \frac{x^w}{w}$, we get:

$$|M(x)| \lesssim O(1)(\ln T)^7 x^{1-\frac{\lambda}{(\ln T)^9}} \int_3^T \frac{1}{v}\,dv$$

$$+ O(1)(\text{const})\, x^{1-\frac{\lambda}{(\ln T)^9}} \int_0^3 (\text{const})\,dv$$

$$+ O(1)\left|(c-1) + \frac{\lambda}{(\ln T)^9}\right| (\ln T)^7 \frac{x^c}{T} \qquad (c>1)$$

$$+ O\left[\frac{x \ln x}{T}\right] \leftarrow \text{(19) top}$$

$$= O(1)(\ln T)^8 x^{1-\frac{\lambda}{(\ln T)^9}}$$

$$+ O(1) \frac{1}{\ln x} \frac{x(\ln T)^7}{T} \leftarrow \boxed{c = 1 + \frac{1}{\ln x}}$$

$$+ O(1) \frac{\lambda}{(\ln T)^2} \frac{x}{T} + O\left[\frac{x \ln x}{T}\right]$$

$$= O(1)\, x \cdot (\ln T)^8 x^{-\frac{\lambda}{(\ln T)^9}}$$

$$+ O(1)\, x \cdot \frac{(\ln T)^7}{T \ln x}$$

$$+ O(1)\, x \cdot \frac{\lambda}{T(\ln T)^2}$$

$$+ O(1)\, x \cdot \frac{\ln x}{T}$$

$$= O(x)\left[(\ln T)^8 x^{-\frac{\lambda}{(\ln T)^9}} + \frac{(\ln T)^7}{T \ln x} + \frac{\lambda}{T(\ln T)^2} + \frac{\ln x}{T} \right].$$

Continue to keep x large. Also keep $T \gtrapprox e^{10}$.
Since λ is tiny, obviously

$$\frac{\ln x}{T} \gtrapprox \frac{\lambda}{T} \gtrapprox \frac{\lambda}{T(\ln T)^2}.$$

We can therefore erase the term $\frac{\lambda}{T(\ln T)^2}$.

ASSUME NOW THAT $\underline{T \lessapprox X}$.

At once:

$$|M(x)| \lessapprox O(x)(\ln x)^8 x^{-\frac{\lambda}{(\ln T)^9}} + O(x)(\ln x)^6 \frac{1}{T} + O(x)(\ln x)\frac{1}{T}$$

$$\leqq O(x)(\ln x)^8\left[x^{-\frac{\lambda}{(\ln T)^9}} + \frac{1}{T}\right]$$

$$= O(x)(\ln x)^8\left[e^{-\frac{\lambda \ln x}{(\ln T)^9}} + e^{-\ln T}\right].$$

But,

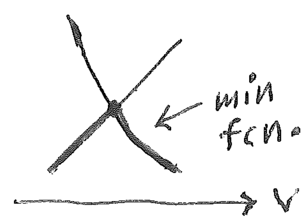
$$e^{-\min\{A,B\}} \leqq e^{-A} + e^{-B} \leqq 2e^{-\min\{A,B\}}$$

for $A>0$, $B>0$.

So,

$$|M(x)| = O(x)(\ln x)^8 e^{-\min\left\{\frac{\lambda \ln x}{(\ln T)^9},\, \ln T\right\}}.$$

Optimize by graphing "$\frac{\lambda \ln x}{v}$ versus v", and thus setting



$$\frac{\lambda \ln x}{(\ln T)^9} = \ln T$$

I.E.

$$\ln T = (\lambda \ln x)^{\frac{1}{10}}$$

I.E.

$$T = \exp\left[(\lambda \ln x)^{\frac{1}{10}}\right].$$

$\boxed{\text{clearly } T \leqq X}$

Hence:

$$|M(x)| = O(x)(\ln x)^8 e^{-(\lambda \ln x)^{\frac{1}{10}}}$$

$$= O(x)\, e^{\underline{8 \ln\ln x}}\, e^{-\lambda^{\frac{1}{10}}(\ln x)^{\frac{1}{10}}}$$

$$= O(x)\, e^{-\frac{1}{2}(\lambda \ln x)^{1/10}} \quad \text{for large } x.$$

∎

<u>Corollary</u>

For $x \geq 3$ and <u>any</u> big Δ, we have

$$M(x) = O\left(\frac{x}{\ln^{\Delta} x}\right).$$

<u>Pf</u>

$\Delta \ln\ln x < \frac{1}{2}(\lambda \ln x)^{\frac{1}{10}}$ once x is big enough.

∎

We now give some easy corollaries of p. 23 Corollary.

Proposition 1

The series $\sum_{1}^{\infty} \frac{\mu(n)}{n^s}$ converges at each point of $\{Re(s)=1\}$. The convergence is uniform on compact subsets.

Pf

Consider a general point $s_0 = 1 + it$ $(t \in \mathbb{R})$. Know $M(x) = 1$ for $1 \leq x < 2$. Keep U large and in $\mathbb{Z}^+$.

$$\sum_{n=1}^{U} \frac{\mu(n)}{n^{s_0}} = 1 + \int_1^U x^{-s_0}\, dM(x)$$

$$= 1 + \left[x^{-s_0} M(x)\right]_1^U - \int_1^U M(x)\, d(x^{-s_0})$$

$$= U^{-s_0} M(U) + s_0 \int_1^U \frac{M(x)}{x^{s_0+1}}\, dx$$

$$= O(1)\, U^{-1} \frac{U}{\ln^4 U} + s_0 \int_1^U \frac{M(x)}{x^{2+it}}\, dx$$

$$= O(1) \frac{1}{\ln^4 U} + s_0 \int_1^U \frac{M(x)}{x^{2+it}}\, dx$$

But $\int_3^\infty \frac{M(x)}{x^{2+it}}\, dx$ is nicely majorized by $\int_3^\infty \frac{O(1)}{x(\ln x)^4}\, dx$.

A moment's thought now gives the 2 statements in the proposition. ▥

> Notice that we get
> $$\sum_{1}^{\infty} \frac{\mu(n)}{n^{s_0}} = s_0 \int_1^{\infty} \frac{M(x)}{x^{s_0+1}}\, dx .$$

Prop 2

In connection with $\sum_{1}^{\infty} \frac{\mu(n)}{n^s}$, we have uniform conv on compact subsets on $\{Re(s) \geq 1\}$.

Pf

Easy modification of Prop 1. ▥

> Again,
> $$\sum_{1}^{\infty} \frac{\mu(n)}{n^s} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}}\, dx = \frac{1}{\zeta(s)},$$
> this time for $Re(s) \geq 1$.

Prop 3

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 .$$

(Euler, 1748 not rigorously)

Pf

Let $s \to 1$ in the 2nd box and use unif conv. ∎

Prop 4

Let $\ell \geqq 1$. The series $\sum_1^\infty \frac{\mu(n)(\ln n)^\ell}{n^s}$ conv unif on compact subsets of $\{\mathrm{Re}(s) \geqq 1\}$.

Pf

Imitate Prop 1 + 2 with $\Delta \geqq \ell + 4$ in 23 Corollary.

EG

$$\sum_1^u \frac{\mu(n)(\ln n)^\ell}{n^{s_0}} = \int_1^u \frac{(\ln x)^\ell}{x^{s_0}}\, dM(x)$$

$$= \left[M(x) \frac{(\ln x)^\ell}{x^{s_0}} \right]_1^u - \int_1^u M(x)\, d\left[\frac{(\ln x)^\ell}{x^{s_0}} \right]$$

$$= \text{etc.} \quad ∎$$

Using the Weierstrass conv thm (for <u>analytic</u> fcns), notice that

$$-\frac{\zeta'(s)}{\zeta(s)^2} = -\sum_{n=1}^\infty \frac{\mu(n)\ln n}{n^s}, \quad \mathrm{Re}(s) > 1.$$

(Clean up by erasing the minus signs.)

By virtue of the unif conv in Prop 4, we immediately get

$$\frac{\zeta'(1+it)}{\zeta(1+it)^2} = \sum_{n=1}^{\infty} \frac{\mu(n)\ln n}{n^{1+it}}, \quad t \neq 0.$$

Letting $t \to 0$ gives

$$\boxed{\sum_{n=1}^{\infty} \frac{\mu(n)\ln n}{n} = -1}.$$

Indeed:

$$\zeta(s) = \frac{1}{s-1}\left[1+\gamma(s-1) + O(s-1)^2\right] \qquad \text{Lec 18 p. 40}$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + O(s-1) \qquad \text{Lec 18 p. 42}$$

$$\frac{\zeta'(1+w)}{\zeta(1+w)} = -\frac{1}{w} + \gamma + O(w) \qquad w \to 0$$

$$\frac{1}{\zeta(1+w)} = w\left[1-\gamma w + O(w^2)\right] \qquad w \to 0$$

$$\frac{\zeta'(1+w)}{\zeta(1+w)^2} = (-1)\left[1-2\gamma w + O(w^2)\right] \qquad w \to 0.$$

OK

These facts using 23 Corollary are nice and are of interest because of several facts (very old ones) which we will not prove fully at this time.

$\boxed{A}$ The Riemann Hypothesis is equivalent to the statement that $M(x) = O(x^{\frac{1}{2}+\varepsilon})$.

$\boxed{B}$ The RH is equivalent to the statement that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ converges for all $\{\mathrm{Re}(s) > \frac{1}{2}\}$.

$\boxed{C}$ By elementary techniques (no use of complex variable), one can show

$M(x) = o(x)$ is equiv to $\psi(x) \sim x$.

$\boxed{D}$ By elementary techniques, one can show that the following are equivalent

(i) $\psi(x) \sim x$;
(ii) $M(x) = o(x)$;
(iii) $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$.

so Euler "knew" the PNT

And, supplementing this, that 27 box $\Rightarrow$ (iii)(ii)(i).

In Lec 20, I proposed to look at (just) $\boxed{C}$.
There is a certain amount of "fun" in doing so. Plus being instructive!

It is convenient to first review some preliminaries involving elementary number theory and $\mu(n)$. We do so quickly.

Given $f(n)$ for $n \in \mathbb{Z}^+$. Recall that f is <u>multiplicative</u> when

$$f(n_1 n_2) = f(n_1) f(n_2) \quad \text{if } (n_1, n_2) = 1.$$

To avoid trivialities, we assume $f(1) = 1$.

Note that $\mu(n)$ and $d(n)$ are multiplicative. Cf. 14 + 15 top.

Review proposition

Prop R1

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}.$$

Pf

$1 = \frac{1}{\zeta(s)} \zeta(s)$. Think of s real, $s > 1$. So,

$$1 = \sum_1^\infty \frac{\mu(k)}{k^s} \cdot \sum_1^\infty \frac{1}{\ell^s} .$$

By absolute conv of both series, get *

$$\left\{ \begin{matrix} 1, & n=1 \\ 0, & n>1 \end{matrix} \right\} = \sum_{k\ell=n} \mu(k)\cdot 1 \qquad \left\{ \begin{matrix} k \geq 1 \\ \ell \geq 1 \end{matrix} \right\}$$

$$= \sum_{k|n} \mu(k) . \qquad \blacksquare$$

(ABS CONV ensures that an infinite series can be summed/rearranged in <u>any</u> order.)

<u>Prop R2</u> (Möbius inversion)

Let f be <u>any</u> fcn defined on $\mathbb{Z}^+$. Let

$$g(n) = \sum_{d|n} f(d) .$$

Then:

$$f(n) = \sum_{d|n} \mu(d)\, g\left(\frac{n}{d}\right) .$$

<u>Pf</u>

For an elementary proof, see R1 and any basic

---

* Think $\sum_1^\infty c_n n^{-s} = \left(\sum_1^\infty a_k k^{-s}\right)\left(\sum_1^\infty b_\ell \ell^{-s}\right) \Leftrightarrow c_n = \sum_{k\ell=n} a_k b_\ell .$

book on number theory. Eg Hardy and Wright.
The "slick informal" proof goes like so:

$$\sum_1^\infty \frac{g(n)}{n^s} = \zeta(s) \sum_1^\infty \frac{f(\ell)}{\ell^s} \qquad s > G \quad (G = \text{giant})$$

$$\Rightarrow \quad \frac{1}{\zeta(s)} \sum_1^\infty \frac{g(m)}{m^s} = \sum_1^\infty \frac{f(n)}{n^s}$$

$$\Rightarrow \quad f(n) = \sum_{\ell m = n} \mu(\ell) g(m) = \sum_{\ell | n} \mu(\ell) g\left(\frac{n}{\ell}\right).$$

$\boxed{n \text{ frozen}}$

∎

Prop R3 (converse of Möbius inversion)

f, g on $\mathbb{Z}^+$. Assume $f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$.

Then:

$$g(n) = \sum_{d|n} f(d).$$

Pf

Look in, eg, Hardy and Wright (using Prop R1).

Slick/informal proof:

$$\sum_1^\infty \frac{f(n)}{n^s} = \frac{1}{\zeta(s)} \sum_{k=1}^\infty \frac{g(k)}{k^s} \qquad s > G \text{ giant}$$

$$\Rightarrow \quad \sum_1^\infty \frac{g(n)}{n^s} = \zeta(s) \sum_1^\infty \frac{f(k)}{k^s}$$

$$\Rightarrow \quad g(n) = \sum_{\ell k = n} 1 \cdot f(k) = \sum_{k|n} f(k). \quad ∎$$

Prop R4

Let f be multiplicative (on $\mathbb{Z}^+$). Then, so is

$$g(n) = \sum_{d|n} f(d).$$

Pf

$g(1) = f(1) = 1$ OK. Given $n_j \geqq 2$ for $j = 1, 2$.

Suppose $n_1 n_2$ has $(n_1, n_2) = 1$. Use elem numb th.

$$g(n_1 n_2) = \sum_{d|n_1 n_2} f(d)$$

THINK prime factorizations of n_1 and n_2

$\{$ such d are uniquely $d_1 d_2$ with $d_1 | n_1$, $d_2 | n_2 \}$

$$= \sum_{d_j | n_j} f(d_1 d_2) = \sum_{d_j | n_j} f(d_1) f(d_2)$$

$$= \left(\sum_{d_1 | n_1} f(d_1) \right) \left(\sum_{d_2 | n_2} f(d_2) \right)$$

$$= g(n_1) g(n_2).$$

The case $n_1 = 1, n_2 \geqq 2$ is trivial. ∎
$n_2 = 1, n_1 \geqq 2$

~

OK, then. To prove $\boxed{C}$, there are $\underline{2}$ halves:

$$\psi(x) \sim x \implies M(x) = o(x) \quad \text{and}$$

$$M(x) = o(x) \implies \psi(x) \sim x .$$

We begin with $\psi(x) \sim x \implies M(x) = o(x)$.

FACT

$$\sum_{n \leq x} \mu(n) \ln \frac{x}{n} = M(x) \ln x + \sum_{k\ell \leq x} \Lambda(k)\mu(\ell) .$$

Here $x \in \mathbb{R}$, $x \geq 2$.

Pf

$$\left(\frac{1}{\zeta}\right)' = -\frac{1}{\zeta^2}\zeta' = \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{1}{\zeta(s)} \implies$$

$$-\sum_{n=1}^{\infty} \frac{\mu(n)\ln n}{n^s} = \left[\sum_{k=1}^{\infty} \frac{\Lambda(k)}{k^s}\right]\left[\sum_{\ell=1}^{\infty} \frac{\mu(\ell)}{\ell^s}\right]$$

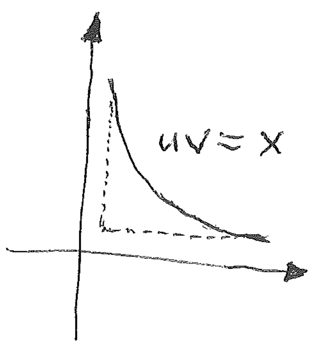
$$\boxed{-\mu(n)\ln n = \sum_{k\ell = n} \Lambda(k)\mu(\ell)} .$$

$$\mu(n)\ln\frac{x}{n} = \mu(n)\ln x + \sum_{k\ell=n} \Lambda(k)\mu(\ell)$$

{now sum!}

$$\sum_{n\leqq x} \mu(n)\ln\frac{x}{n} = M(x)\ln x + \sum_{n\leqq x}\sum_{k\ell=n} \Lambda(k)\mu(\ell)$$

$k \geqq 1$
$\ell \geqq 1$
$(k,\ell) \in \mathbb{R}^2$
is a lattice point



$$= M(x)\ln x + \sum_{\substack{\text{all } (k,\ell) \\ \text{lattice points} \\ k\ell \leqq x}} \Lambda(k)\mu(\ell)$$

$\left\{ \begin{array}{l} \text{view in first quadrant} \\ \text{of } uv\text{-plane, and} \\ \text{under hyperbola } v = \frac{x}{u} \end{array} \right\}$.

KEY

∎

FACT

$x \in \mathbb{R}$, $x \geq 2$. Then:

$$\sum_{n \leq x} \ln \frac{x}{n} = \underline{\underline{x}} + O(\ln x).$$

Pf

By Lec 1, $\ln [\![x]\!]! = x \ln x - x + O(\ln x)$.

See Lec 1 p. 11, Thm 6. But, in the above,

$$
\begin{aligned}
\text{LHS} &= [\![x]\!] \ln x - \ln [\![x]\!]! \\
&= [\![x]\!] \ln x - x \ln x + x + O(\ln x) \\
&= ([\![x]\!] - x) \ln x + x + O(\ln x) \\
&= O(\ln x) + x + O(\ln x) \\
&= x + O(\ln x). \quad \blacksquare
\end{aligned}
$$

FACT

$$M(x)\ln x = O(x) - \sum_{\ell \leq x} \mu(\ell)\,\psi\!\left(\tfrac{x}{\ell}\right). \qquad \begin{matrix} x \in \mathbb{R} \\ x \geq 2 \end{matrix}$$

PF

Use (33) Fact. Note there that

$$\left| \sum_{n \leq x} \mu(n) \ln \tfrac{x}{n} \right| \leq \sum_{n \leq x} \ln \tfrac{x}{n} = O(x)$$

by (35). Get:

$$M(x)\ln x = O(x) - \sum_{k\ell \leq x} \Lambda(k)\,\mu(\ell)$$

view the hyperbola region in first quadrant (34)

given any $\ell \leq x$, note that k then must range in $[1, x/\ell]$

$$M(x)\ln x = O(x) - \sum_{\ell \leq x} \mu(\ell) \left(\sum_{k \leq x/\ell} \Lambda(k) \right)$$

$$= O(x) - \sum_{\ell \leq x} \mu(\ell)\,\psi\!\left(\tfrac{x}{\ell}\right). \qquad ▨$$

FACT

$$\left| \sum_{m \leqq x} \frac{\mu(m)}{m} \right| \leqq 1 \quad \text{for all } \begin{matrix} x \in \mathbb{R} \\ x \geqq 2 \end{matrix}.$$

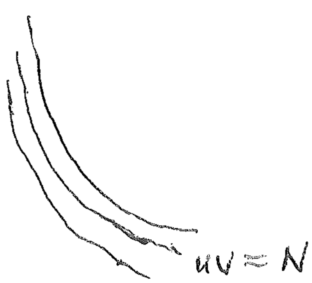
Pf

WLOG $x \in \mathbb{Z}$. $x = 1$ gives Sum $= 1$.
$x = 2$ OK; SUM $= \frac{1}{2}$. So, wlog, $x \geqq 3$.

Note:

$$\sum_{m \leqq x} \mu(m) \left[\!\left[\frac{x}{m} \right]\!\right] = \sum_{mk \leqq x} \mu(m) \cdot 1$$

as in the p. 34 hyperbola region



$\left\{ \begin{array}{l} \text{write } N = mk \\ \text{for } \underline{\text{each}}\ N, \text{ note how } m \\ \text{ranges over the divisors of } N \end{array} \right\}$ and $k = \frac{N}{m}$!

$$= \sum_{N \leqq x} \left\{ \sum_{m | N} \mu(m) \right\}$$

$$= \sum_{N \leqq x} \left\{ \begin{matrix} 1, & N = 1 \\ 0, & N > 1 \end{matrix} \right\} \quad \leftarrow 29 \text{ R1}$$

$$= 1. \quad ////$$

Accordingly,

$$\sum_{m=1}^{x} \mu(m)\left(\frac{x}{m} - r\left(\frac{x}{m}\right)\right) = 1$$

$$\{\, r(t) = t - [\![t]\!] \,\}$$

$$x \sum_{m=1}^{x} \frac{\mu(m)}{m} = 1 + \sum_{1}^{x} \mu(m)\, r\left(\frac{x}{m}\right)$$

$$\left\{ \begin{array}{l} m=1 \Rightarrow \mu(1)r(x) = 0 \\ m=x \Rightarrow \mu(x)r(1) = 0 \\ \text{in general, } 0 \leq r\left(\frac{x}{m}\right) < 1 \end{array} \right\}$$

$$x \left| \sum_{1}^{x} \frac{\mu(m)}{m} \right| \leq 1 + (x-2) = x-1 < x$$

$$\Downarrow$$

$$\left| \sum_{m=1}^{x} \frac{\mu(m)}{m} \right| < 1 \quad \text{all } x \geq 2 . \quad \blacksquare$$

Recall 36 : get

$$M(x) \ln x = O(x) - \sum_{\ell \leq x} \mu(\ell)\, \psi\left(\frac{x}{\ell}\right) \qquad x \in \mathbb{R},\ x \geq 2$$

$$\{ \text{use } 37 \}$$

$$= O(x) + \sum_{\ell \leq x} \mu(\ell)\left(\frac{x}{\ell} - \psi\left(\frac{x}{\ell}\right)\right) .$$

$\boxed{\text{Very Slick}}$

FACT

$$\psi(x) \sim x \quad \text{as} \quad x \to \infty \quad \Longrightarrow$$

$$M(x) = o(x).$$

Pf

Use 38 bottom.

Choose any tiny $\varepsilon > 0$. Let $|\psi(y) - y| < \varepsilon y$ for all $y \geq A \geq 5$, say. (A depends on ε.)

Assume $x \geq 1000A$. Get:

$$|M(x)| \ln x \leq O(x) + \sum_{l \leq \frac{x}{A}} |\mu(l)| \left| \frac{x}{l} - \psi\left(\frac{x}{l}\right) \right|$$

$$+ \sum_{\frac{x}{A} < l \leq x} |\mu(l)| \left| \frac{x}{l} - \psi\left(\frac{x}{l}\right) \right|$$

$$\left\{ \begin{array}{l} \text{since } \psi(x) \sim x \text{ as } x \to \infty, \\ \text{one knows trivially that} \\ \qquad |\psi(y)| \leq \mathfrak{M} y \\ \text{for ALL } y \geq 1 \text{ with some } \mathfrak{M} \\ \hline \text{also recall Chebyshev estimate} \\ \text{for } \psi \text{ from Lec 1 p. 18} \end{array} \right\}$$

$$\lesssim O(x) + \sum_{l \lesssim \frac{x}{A}} \varepsilon \frac{x}{l}$$

$$+ (1 + \mathfrak{M}) \sum_{\frac{x}{A} < l \lesssim x} \frac{x}{l}$$

$$\lesssim O(x) + (1)\varepsilon x \left[\ln \frac{x}{A} + O(1) \right]$$

$$+ O(1)\, x \left[\ln \frac{x}{x/A} + O(1) \right]$$

KEY

$$\Downarrow$$

$$|M(x)| \ln x \lesssim O(x) + O(x) + \varepsilon x \ln x$$

$$+ O(x)$$

$$\Downarrow$$

$$|M(x)| \lesssim O\left(\frac{x}{\ln x}\right) + \varepsilon x .$$

yes!

Hence $M(x) = o(x)$ as $x \to \infty$. ∎

We'll treat $M(x) = o(x) \Rightarrow \psi(x) \sim x$ recall p. 33

in the next [redacted] set of notes. (40 pages is LONG ENOUGH!) for a synopsis