Lecture 2 (22 Jan 2016)

Our primary goal today is to improve Theorem A from Lecture 1. We also wish to address some preliminary stuff as well.

Theorem 1

$x \geq 2$. We have

$$\frac{\theta(x)}{\ln x} \leq \pi(x) \leq x^{1-\delta} + \frac{\theta(x)}{\ln x}\frac{1}{1-\delta}$$

for any $0<\delta<1$.

Pf

$\pi(x)\ln x = \sum_{p\leq x} \ln x \geq \sum_{p \leq x} \ln p = \theta(x)$ is obvious. On the other hand,

$$\theta(x)-\theta(x^{1-\delta}) = \sum_{x^{1-\delta}<p\leq x} \ln p \geq \ln(x^{1-\delta})\left[\pi(x)-\pi(x^{1-\delta})\right]$$

$$\Downarrow$$

$$\pi(x)-\pi(x^{1-\delta}) \leq \frac{\theta(x)-\theta(x^{1-\delta})}{(1-\delta)\ln x}$$

$$\pi(x) \leq \pi(x^{1-\delta}) + \frac{\theta(x)}{(1-\delta)\ln x} \qquad \{\text{since } \theta(y) \geq 0\}$$

$$\pi(x) \leq x^{1-\delta} + \frac{\theta(x)}{(1-\delta)\ln x} \qquad \{\text{trivially: } \pi(y) \leq y\}$$

OK ∎

Corollary 1

As $x \to \infty$,

$$\pi(x) \sim \frac{\theta(x)}{\ln x} \sim \frac{\psi(x)}{\ln x} .$$

Pf

By Lec 1 thm A, we know $c_1 x < \theta(x) < c_2 x$. By Lec 1 thm 3, we then get

$$\frac{\psi(x)}{\theta(x)} \to 1 \quad \text{as} \quad x \to \infty .$$

We need to show $\frac{\pi(x) \ln x}{\theta(x)} \to 1$ too. But here we can apply Thm 1 above to get:

$$\theta(x) \leqq \pi(x) \ln x \lesssim x^{1-\delta} \ln x + \frac{\theta(x)}{1-\delta}$$

$$1 \leqq \frac{\pi(x) \ln x}{\theta(x)} \lesssim \frac{x^{1-\delta} \ln x}{\theta(x)} + \frac{1}{1-\delta} .$$

Just take δ smaller and smaller! Clearly

$$\limsup_{x \to \infty} \frac{\pi(x) \ln x}{\theta(x)} \lesssim \frac{1}{1-\delta}$$

(again using $c_1 x < \theta(x) < c_2 x$), so we are done. ▨

This successively reducing the δ seems a bit ugly. We can junk it.

Corollary 2

For $x \gtrapprox 2$,

$$1 \lessapprox \frac{\pi(x)\ln x}{\theta(x)} \lessapprox 1 + \frac{O(1)}{\sqrt{\ln x}} .$$

Pf

By inflating the constant in $O(1)$, we can assume $x \gtrapprox x_0$ (suff. large) so that

$$3\,\frac{\ln\ln x}{\ln x} < \frac{1}{2} \text{, say.}$$

This is legal.

We propose to simply take $\delta = 3\,\frac{\ln\ln x}{\ln x}$ in Theorem 1 above. Note that

$$\frac{1}{1-u} < 1 + 2u \quad \text{for} \quad 0 < u < \frac{1}{2} .$$

Plug in page ② line 11 [which is just a rewrite of Thm 1]. Get:

$$1 \lessapprox \frac{\pi(x)\ln x}{\theta(x)} \lessapprox \frac{1}{\theta(x)}\,x \cdot e^{-\delta \ln x} \cdot \ln x + 1 + 2\delta$$

$$= \frac{1}{\theta(x)}\,x \cdot \frac{\ln x}{(\ln x)^3} + 1 + 6\,\frac{\ln\ln x}{\ln x}$$

$$= \frac{x}{\theta(x)}\,\frac{1}{(\ln x)^2} + 1 + \frac{6\ln\ln x}{\ln x} .$$

Remember that $c_1 x < \theta(x) < c_2 x$ and $x \gtrapprox x_0$. The last expression is clearly $\lessapprox 1 + \frac{O(1)}{\sqrt{\ln x}}$. ▨

Theorem A′ (Chebyshev ≈ 1850)

For $x \gtrsim 2$,

$$x(\ln 2) + O(\log x) \lesssim \psi(x) \lesssim x(\ln 4) + O(\ln^2 x)$$

$$x(\ln 2) + O(x^{1/2}) \lesssim \theta(x) \lesssim x(\ln 4) + O(x^{1/2})$$

$$[\ln 2 + o(1)]\frac{x}{\ln x} \lesssim \pi(x) \lesssim [\ln 4 + o(1)]\frac{x}{\ln x} .$$

Here $o(1)$ means "bounded but tends to <u>zero</u> as $x \to \infty$".

Pf

For the first 2 lines, see Lec 1 Thm A. The third line, pertaining to $\pi(x)$, follows from corollary 1 or 2. ▨

$$\ln 2 \approx .693147^{+}$$
$$\ln 4 \approx 1.386294^{+}$$

We would like to reduce the spread between there numbers!

still using elementary techniques...

Chebyshev *also* played with the combination

$$\frac{(30m)!\, m!}{(15m)!\,(10m)!\,(6m)!}\,.$$

It is NOT obvious this is an integer!

Note that

$$30-15-10-6+1 = 30\left[1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{30}\right]=0\,.$$

Correspondingly, let

$$\underline{\underline{\psi_\sigma(x)}} = L(x)-L\left(\frac{x}{2}\right)-L\left(\frac{x}{3}\right)-L\left(\frac{x}{5}\right)+L\left(\frac{x}{30}\right)$$

$$\left\{ \text{recall } L(x)=\ln [\![x]\!]! = \sum_{n\le x} [\![\tfrac{x}{n}]\!]\Lambda(n) \right\} \longleftarrow \text{Lec 1}$$

$$= \sum_{n\le x}\left(\left[\frac{x}{n}\right]-\left[\frac{x}{2n}\right]-\left[\frac{x}{3n}\right]-\left[\frac{x}{5n}\right]+\left[\frac{x}{30n}\right]\right)\Lambda(n)$$

$$\equiv \sum_{n\le x}\sigma\left(\frac{x}{n}\right)\Lambda(n)\,. \quad \|\|$$

$\sigma(x)$ has interesting properties (as one soon discovers). First of all,

$$\sigma(x) = [\![x]\!]-[\![\tfrac{x}{2}]\!]-[\![\tfrac{x}{3}]\!]-[\![\tfrac{x}{5}]\!]+[\![\tfrac{x}{30}]\!]\,.$$

Note that $\llbracket y \rrbracket$ and $\sigma(y)$ are right continuous on $\mathbb{R}$. Also,

$$
\begin{aligned}
\sigma(x+30) = \ & \llbracket x \rrbracket + 30 \\
& - \llbracket \tfrac{x}{2} \rrbracket - 15 \\
& - \llbracket \tfrac{x}{3} \rrbracket - 10 \\
& - \llbracket \tfrac{x}{5} \rrbracket - 6 \\
& + \llbracket \tfrac{x}{30} \rrbracket + 1 \quad = \sigma(x).
\end{aligned}
$$

So, $\sigma(x)$ is periodic 30. To appreciate σ, one simply breaks down and computes it by hand or via a computer.

$$[0,30) = [0,1) \cup [1,2) \cup [2,3) \cup \cdots \cup [29,30).$$

GET:

[0,1)	0	10	0	20	0
[1,2)	1	11	1	21	0
[2,3)	1	12	0	22	0
[3,4)	1	13	1	23	1
[4,5)	1	14	1	24	0
[5,6)	1	15	0	25	0
[6,7)	0	16	0	26	0
[7,8)	1	17	1	27	0
[8,9)	1	18	0	28	0
[9,10)	1	19	1	29	1

•

We thus find that $\sigma = 0$ or 1 for all x.

(It was not obvious apriori that, e.g., $\sigma \geqq 0$.)

N.B. Notice that the original factorial quotient on ⑤ line 2 has logarithm $\psi_\sigma(30m)$. Since $\sigma \in \{0,1\}$, the formula on ⑤ line 9 makes it clear that the original quotient is a positive integer!

Clearly,

$$\psi_\sigma(x) = \sum_{n \leqq x} \sigma\left(\frac{x}{n}\right) \Lambda(n) \leqq \sum_{n \leqq x} \Lambda(n) = \psi(x). \quad \blacksquare$$

Also:

$$\sigma\left(\frac{x}{n}\right) = 1 \quad \text{for} \quad 1 \leqq \frac{x}{n} < 6 \quad \text{is VERY convenient}$$

IE get $\sigma\left(\frac{x}{n}\right) = 1$ for all $\frac{x}{6} < n \leqq x$.

Notice that $\sigma\left(\frac{x}{n}\right) = 1$ in some other portions of $n \leqq x$ TOO. But, for now, we don't use this.

As a tautology,

$$\psi_\sigma(x) = \sum_{\frac{x}{6}<n\le x} \sigma\left(\frac{x}{n}\right)\Lambda(n) + \underbrace{\sum_{n\le\frac{x}{6}} \sigma\left(\frac{x}{n}\right)\Lambda(n)}_{\text{non-negative!}}$$

$$\Downarrow$$

$$\psi_\sigma(x) \ge \sum_{\frac{x}{6}<n\le x} 1\,\Lambda(n) = \psi(x) - \psi\left(\frac{x}{6}\right)$$

$$\Downarrow$$

$$\psi(x) \le \psi\left(\frac{x}{6}\right) + \psi_\sigma(x) .$$

So,

$$\psi_\sigma(x) \le \psi(x) \le \psi_\sigma(x) + \psi\left(\frac{x}{6}\right) .$$

Recall (Lec 1, thm 6)

$$L(y) = y \ln y - y + O(\ln y)$$ "Stirling"

for all $y \gtrapprox 2$.

We substitute into

$$\psi_0(x) = L(x) - L\left(\tfrac{x}{2}\right) - L\left(\tfrac{x}{3}\right) - L\left(\tfrac{x}{5}\right) + L\left(\tfrac{x}{30}\right),$$

keeping $x \gtrapprox 60$ for safety. Get:

$$\begin{aligned}
&x \ln x - x + O(\ln x) \\
&- \tfrac{x}{2} \ln\left(\tfrac{x}{2}\right) + \tfrac{x}{2} + O(\ln x) \\
&- \tfrac{x}{3} \ln\left(\tfrac{x}{3}\right) + \tfrac{x}{3} + O(\ln x) \\
&- \tfrac{x}{5} \ln\left(\tfrac{x}{5}\right) + \tfrac{x}{5} + O(\ln x) \\
&+ \tfrac{x}{30} \ln\left(\tfrac{x}{30}\right) - \tfrac{x}{30} + O(\ln x) \\
\hline
= \;&\left(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{30}\right) x \ln x \quad \longleftarrow 0 \\
&+ x\left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} - 1 - \tfrac{1}{30}\right) \quad \longleftarrow 0 \\
&+ x\left(\tfrac{1}{2}\ln 2 + \tfrac{1}{3}\ln 3 + \tfrac{1}{5}\ln 5 - \tfrac{1}{30}\ln 30\right) \\
&+ O(\ln x)
\end{aligned}$$

$$\frac{1}{2}\ln 2 + \frac{1}{3}\ln 3 + \frac{1}{5}\ln 5 - \frac{1}{30}\ln 30$$

$$\approx .9212920229^+$$

$$\Downarrow$$

$$\boxed{\psi_0(x) \approx (.9212920229^+)x + O(\ln x)}.$$

By $\psi_0(x) \leqq \psi(x)$ on ⑦ + ⑧, we get:

$$(.921^+)x + O(\ln x) \lesssim \psi(x).$$

By $\psi(x) - \psi(\frac{x}{6}) \lesssim \psi_0(x)$ on ⑧, we get:

$$\psi(x) - \psi\left(\frac{x}{6}\right) \lesssim (.921^+)x + B\ln x$$

for some B whenever $x \geqq 60$. We can now iterate this to get:

$$\psi(x) - \psi\left(\tfrac{x}{6}\right) \lesssim (.921^{+})x + B \ln x \qquad \text{step } 1$$

$$\psi\left(\tfrac{x}{6}\right) - \psi\left(\tfrac{x}{36}\right) \lesssim (.921^{+})\tfrac{x}{6} + B \ln \tfrac{x}{6} \qquad \text{step } 2$$

$$\vdots$$

$$\psi\left(\tfrac{x}{6^r}\right) - \psi\left(\tfrac{x}{6^{r+1}}\right) \lesssim (.921^{+})\frac{x}{6^r} + B \ln \frac{x}{6^r} \qquad \text{step } r+1$$

$$\left\{ \begin{array}{l} \text{take } \frac{x}{6^r} \in [216, 1296] \text{ for safety} \\ \text{hence } r = \frac{\ln x - \Omega}{\ln 6} \ , \quad \ln 216 \lesssim \Omega \lesssim \ln 1296 \end{array} \right\}$$

ADD

$$\Downarrow$$

$$\psi(x) + O(1) \lesssim \frac{6}{5}(.921^{+})x + B(r+1)\ln x$$

$$\Downarrow$$

$$\psi(x) \lesssim \frac{6}{5}(.921^{+})x + \frac{B}{\ln 6}(\ln x)^2 .$$

Note:

$$\frac{6}{5}(.9212920229^{+}) = 1.105550427^{+} .$$

Theorem B (harder Chebyshev ≈ 1850) $\left(\begin{matrix}\text{using}\\ \psi_\sigma\end{matrix}\right)$

For $x \geqq 2$, we have

$$(.921292^+)x + O(\ln x) \leqq \psi(x) \leqq (1.105550^+)x + O(\ln^2 x)$$

$$(.921292^+)x + O(\sqrt{x}) \leqq \theta(x) \leqq (1.105550^+)x + O(\sqrt{x})$$

$$[.921292^+ + o(1)]\frac{x}{\ln x} \leqq \pi(x) \leqq [1.105550^+ + o(1)]\frac{x}{\ln x} .$$

Here

$$1.105550^+ \approx \frac{6}{5}(.921292^+) .$$

Proof

See ⑩ + ⑪ and inflate the (implied) constants (as necessary) for $\psi(x)$. The rest is like Theorem A'. ▨

Addendum (for completeness)

On ⑤, $\psi_\sigma(x) = \sum_{n \le x} \sigma(\frac{x}{n})\Lambda(n)$. Let the value of $\sigma(y)$ on $[k, k+1)$ be σ_k. Here $k \ge 0$. Note that index $k \ge 1$ in $\psi_\sigma(x)$. corresponds to $k \le \frac{x}{n} < k+1$ hence $\frac{x}{k+1} < n \le \frac{x}{k}$

Hence:

$$\psi_\sigma(x) = \sum_{k=1}^{\infty} \sigma_k \left\{ \sum_{\frac{x}{k+1} < n \le \frac{x}{k}} \Lambda(n) \right\} = \sum_{k=1}^{\infty} \sigma_k \left[\psi(\tfrac{x}{k}) - \psi(\tfrac{x}{k+1})\right]$$

$$= \sigma_1(\psi_1 - \psi_2) + \sigma_2(\psi_2 - \psi_3) + \sigma_3(\psi_3 - \psi_4) + \cdots$$

{in shorthand}

$$= \psi_1\sigma_1 + \psi_2(\sigma_2 - \sigma_1) + \psi_3(\sigma_3 - \sigma_2) + \cdots$$

{notice that $\sigma_n - \sigma_{n-1}$ is periodic $\underline{\underline{30}}$}

[apply ⑥ bottom]

$$\begin{aligned} = {} & \psi(x) + \psi(\tfrac{x}{2})(0) + \psi(\tfrac{x}{3})(0) + \psi(\tfrac{x}{4})(0) + \psi(\tfrac{x}{5})(0) \\ & - \psi(\tfrac{x}{6}) + \psi(\tfrac{x}{7}) + \psi(\tfrac{x}{8})(0) + \psi(\tfrac{x}{9})(0) + \psi(\tfrac{x}{10})(-1) \\ & + \psi(\tfrac{x}{11}) + \psi(\tfrac{x}{12})(-1) + \psi(\tfrac{x}{13}) + \psi(\tfrac{x}{14})(0) + \psi(\tfrac{x}{15})(-1) \\ & + \psi(\tfrac{x}{16})(0) + \psi(\tfrac{x}{17}) + \psi(\tfrac{x}{18})(-1) + \psi(\tfrac{x}{19}) + \psi(\tfrac{x}{20})(-1) \\ & + \psi(\tfrac{x}{21})(0) + \psi(\tfrac{x}{22})(0) + \psi(\tfrac{x}{23}) + \psi(\tfrac{x}{24})(-1) + \psi(\tfrac{x}{25})(0) \\ & + \psi(\tfrac{x}{26})(0) + \psi(\tfrac{x}{27})(0) + \psi(\tfrac{x}{28})(0) + \psi(\tfrac{x}{29})(1) + \psi(\tfrac{x}{30})(-1) \\ & + \psi(\tfrac{x}{31})(1) + \cdots \end{aligned}$$

$$\begin{aligned} \underline{\underline{\psi_\sigma(x)}} = {} & (1) - (6) + (7) - (10) + (11) - (12) + (13) - (15) \\ & + (17) - (18) + (19) - (20) + (23) - (24) + (29) - (30) \\ & + (31) \pm \cdots \end{aligned}$$

NOTE THE ALTERNATE SIGNS

Some Additional Remarks.

It's not immediately clear what other combinations of $L(x/j)$ can be used — and how much of an improvement can be gained.

The standard reference is:

> Diamond and Erdős, On sharp elementary prime number estimates, L'Enseignement Math. 26 (1980) 313-321.

This reference is usually regarded as saying that if one already knows that the PNT is true, then there is in principle no obstruction to building better and better combinations that lead to "$1-\varepsilon$ and $1+\varepsilon$".

But, the assertion does not take into account the possibility of exploiting recursive relations and additional positive terms like ⑦ bottom line (and ⑧ top, for $n \lesssim \frac{x}{6}$).

J. J. Sylvester found improvements based on use of recursive relations. See:

Sylvester, Amer. J. Math. 4 (1881) 230–247

; Messenger of Math. 21 (1892) 1–19, and 87–120.

$.9569^+$
1.0442^+

Also Mathews, Th. of Numbers, pp. 287–294 from 1892.

It seems fair to say the overall status of things is not as clear as one would like.

Incidentally, see:

a classic!

Landau, Handbuch der Lehre von der Verteilung der Primzahlen, vol. 1, pp. 94–95 (1909)

for a tiny improvement in $\frac{6}{5}(.921292^+)$ based just on ⑦ (bottom line) + ⑧ (top). He got:

$$\frac{171}{175}\,\frac{6}{5}(.921292^+)$$

$$\approx 1.\underline{080}280^+ .$$

We'll drop this stuff temporarily: it seems obvious that some essentially new idea would be needed to reach $1-\varepsilon$, $1+\varepsilon$ via "elementary reasoning".

of Thm B

Corollary (related to Bertrand's Postulate)

There exists a positive c so that, for large x,

$$\sum_{x<p\leqq 2x} \ln p \geqq cx .$$

Pf

LHS $\approx \theta(2x)-\theta(x)$. By Thm B,

$$\theta(2x)-\theta(x) \geqq 2(.921292^{+})x - (1.105550^{+})x + O(\sqrt{x})$$

$$\approx \frac{4}{5}(.921292^{+})x + O(\sqrt{x})$$

$$\geqq (.737)x + O(\sqrt{x}) . \quad \blacksquare$$

Hence,

$$\sum_{x<p\leqq 2x} 1 \geqq \frac{cx}{\log(2x)} .$$

the Bertrand issue

Theorem 2

For $x \geq 2$,

$$\sum_{p \leq x} \frac{\ln p}{p} = \ln x + O(1)$$

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \ln x + O(1) .$$

Proof

$$L(x) = \sum_{n \leq x} \llbracket \tfrac{x}{n} \rrbracket \Lambda(n) \quad \text{for all} \quad x \geq 2 . \qquad \left\{ \begin{matrix} \text{Lec 1} \\ \text{Thm 7} \end{matrix} \right\}$$

Hence,

$$x \ln x + O(x) = \sum_{n \leq x} \llbracket \tfrac{x}{n} \rrbracket \Lambda(n)$$

by Lec 1 Thm 6. Temporarily write

$$\llbracket \tfrac{x}{n} \rrbracket = \frac{x}{n} - \varphi_n \quad , \quad 0 \leq \varphi_n < 1 .$$

Get

$$x \ln x + O(x) = \sum_{n \leq x} \frac{x}{n} \Lambda(n) - \sum_{n \leq x} \varphi_n \Lambda(n)$$

This term is non-neg and $O(x)$ by thm A′

$$\Downarrow$$

$$x\ln x + O(x) \approx x \sum_{n \lesssim x} \frac{\Lambda(n)}{n} + O(x)$$

$$\Downarrow$$

$$\sum_{n \lesssim x} \frac{\Lambda(n)}{n} \approx \ln x + O(1) .$$

Notice however that

$$\sum_{p^2 \lesssim x} \frac{\ln p}{p^2} + \sum_{p^3 \lesssim x} \frac{\ln p}{p^3} + \cdots$$

$$\lesssim \sum_{p^2 \lesssim x} \frac{\ln p}{p^2} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right)$$

$$\lesssim \sum_{\text{all } p} \frac{\ln p}{p^2} \frac{1}{1-\frac{1}{p}} \lesssim 2 \sum_{\text{all } p} \frac{\ln p}{p^2} < +\infty .$$

At once,

$$\sum_{p \lesssim x} \frac{\ln p}{p} \approx \ln x + O(1) \quad \text{too.}$$

$\blacksquare$

(Thm 2 ~ Mertens ≈ 1874)

Theorem 3

We have

$$\liminf_{x\to\infty} \frac{\pi(x)\ln x}{x} = \liminf_{x\to\infty} \frac{\psi(x)}{x} \leqq 1$$

$$\limsup_{x\to\infty} \frac{\pi(x)\ln x}{x} = \limsup_{x\to\infty} \frac{\psi(x)}{x} \geqq 1 .$$

Hence, if $\psi(x) \sim cx$, we must have $c=1$.

Pf

By ② corollary 1, need only treat $\psi(x)$.

Recall

$$L(x) = x\ln x + O(x) = \sum_{k\leqq x} \psi\left(\frac{x}{k}\right)$$

for $x \geqq 2$. See Lec 1, thms 6 + 7.

Assume $\liminf \frac{\psi(x)}{x} = \alpha > 1$. Hence $\psi(y) \geqq (1+\overset{\overset{\text{pos}}{\downarrow}}{h})y$ for all $y \geqq x_0$. For large x, we have:

$$x\ln x + O(x) = \sum_{1\leqq k \leqq \frac{x}{x_0}} \psi\left(\frac{x}{k}\right) + \sum_{\frac{x}{x_0} < k \leqq x} \psi\left(\frac{x}{k}\right)$$

$$\geqq (1+h) \sum_{1\leqq k\leqq \frac{x}{x_0}} \frac{x}{k} + \sum_{\frac{x}{x_0}<k\leqq x} O(1)\psi(x_0)$$

$$= (1+h)x \sum_{k=1}^{x/x_0} \frac{1}{k} + O[\psi(x_0)]x$$

$$= (1+h)x\left[\ln\frac{x}{x_0} + O(1)\right] + O[\psi(x_0)]x$$

$$\eqsim (1+h)x\ln x + O_{x_0}(1)x \;,$$

$O_{x_0}(1)$ *meaning* the implied constant depends on $\underline{\underline{x_0}}$. As $x \to \infty$, we get $1 \gtrsim 1+h$, an obvious contradiction. (OK)

The lim sup case is similar (again by contrad!). Just take $\psi(y) \lesssim (1-h)y$ for $y \gtrsim x_0$. (OK)

■

Date Highlights

Euler suggests PNT $\approx 1740 \sim 1760$

The boy → **Gauss** suggests PNT ≈ 1790
(counts in tables)

Legendre looks like $\frac{x}{\log x - c}$ ≈ 1800

Gauss letter to Encke $\int_2^x \frac{dt}{\ln t}$ 1849

Chebyshev rigorous $c_1 \frac{x}{\ln x} < \pi(x) < c_2 \frac{x}{\ln x}$ ≈ 1850

Riemann **1859** introd. of complex variable
(etc)

Sylvester elementary refinements 1880~1892
à la Chebyshev

de la Vallée-Poussin, Hadamard proof of PNT 1896