Lecture 21 Synopsis
(6 April)

We must now prove that

$$M(x) = o(x) \implies \psi(x) \sim x$$

by elementary methods. See Lec 20 p. 33.

We need to make a detour into the Dirichlet divisor problem.

$$T(x) = \sum_{n \leq x} d(n) = \sum_{\substack{(k,\ell) \\ \text{lattice point} \\ k\ell \leq x}} 1$$

Trivially,

$$r(t) = t - [\![t]\!] \qquad 0 \leq r(t) < 1$$

$$\begin{aligned} T(x) &= \sum_{m \leq x} [\![\tfrac{x}{m}]\!] \\ &= \sum_{m \leq x} \frac{x}{m} - \sum_{m \leq x} r\left(\frac{x}{m}\right) \\ &= x \ln x + O(x) + O(x) \\ &= x \ln x + O(x). \end{aligned}$$

One checks that

$$\mathrm{card}\{(m_1, m_2) : m_1 m_2 \leqq x\}$$

$$= \mathrm{card}\{m_1 \leqq [\![x^{1/2}]\!],\ m_2 \leqq [\![x^{1/2}]\!]\}$$

$$+ \mathrm{card}\{m_1 \leqq [\![x^{1/2}]\!],\ m_2 > [\![x^{1/2}]\!],\ m_1 m_2 \leqq x\}$$

$$+ \mathrm{card}\{m_1 > [\![x^{1/2}]\!] \text{ and } m_1 m_2 \leqq x\}$$

but $m_2 \leqq \frac{x}{m_1}$ and $m_1 > [\![x^{1/2}]\!]$
$\Rightarrow m_1 > x^{1/2} \Rightarrow m_2 \leqq \frac{x}{m_1} < x^{1/2}$
$\Rightarrow m_2 \leqq [\![x^{1/2}]\!]$ <u>automatically</u>

$$= A + B + \overset{m_1 \leftrightarrow m_2}{B} \quad \text{(in obvious notation)}$$

$$= 2(A + B) - A$$

$$= 2 \sum_{m_1 \leqq [\![x^{1/2}]\!]} [\![\tfrac{x}{m_1}]\!] - [\![\sqrt{x}]\!]^2$$

$$= 2 \sum_{k \leqq [\![\sqrt{x}]\!]} \frac{x}{k} - 2 \sum_{k \leqq [\![\sqrt{x}]\!]} r\left(\frac{x}{k}\right) - x + O(\sqrt{x})$$

Lec 18 p. 40 bottom re: γ

$$= 2x\left[\ln [\![\sqrt{x}]\!] + \gamma + O\left(\tfrac{1}{\sqrt{x}}\right)\right] + O(\sqrt{x}) - x$$

$$= 2x \ln [\![\sqrt{x}]\!] + 2\gamma x + O(\sqrt{x}) - x$$

$$\left\{ \begin{aligned} \ln(\sqrt{x} - \theta) &= \ln \sqrt{x}\left(1 - \frac{\theta}{\sqrt{x}}\right) \qquad 0 \leqq \theta < 1 \\ &= \frac{1}{2}\ln x + O\left(\frac{1}{\sqrt{x}}\right) \end{aligned} \right\}$$

$$= x \ln x + \underline{O(\sqrt{x})} + (2\gamma - 1)x .$$

Thm (Dirichlet)

For $x \geqq 1$,

$$T(x) = \sum_{n \leqq x} d(n) = x \ln x + (2\gamma - 1)x + O(\sqrt{x}) .$$

Dirichlet divisor problem
= best exponent α
$\alpha = \frac{1}{4} + \varepsilon$??

Pf

As above. ∎

Fact

Let $\Delta(x) = \sum_{n \leqq x} (\ln n - d(n) + 2\gamma)$, $x \geqq 1$.

Then:

$$\Delta(x) = O(\sqrt{x}) .$$

Pf

WLOG x = integer. Just use THM and $\ln x!$.

∎

Note that:

$$\zeta = \zeta^2 \cdot \frac{1}{\zeta}$$

$$\Rightarrow \boxed{1 = \sum_{k\ell = n} d(k)\mu(\ell)}$$

$$-\frac{\zeta'}{\zeta} = (-\zeta') \cdot \frac{1}{\zeta}$$

$$\Rightarrow \boxed{\Lambda(n) = \sum_{k\ell = n} (\ln k)\mu(\ell)}$$

---

and, as before,

$$\left(\frac{1}{\zeta}\right)' = -\frac{\zeta'}{\zeta^2} = \left(-\frac{\zeta'}{\zeta}\right) \cdot \frac{1}{\zeta}$$

$$\Rightarrow \boxed{-\mu(n)\ln n = \sum_{k\ell = n} \Lambda(k)\mu(\ell)}$$

Lec 20 33 .

<u>Fact</u>

$$\psi(x) - x + 2\gamma = \sum_{\ell k \leq x} \mu(\ell)\left\{\ln k - d(k) + 2\gamma\right\},$$

$x \in \mathbb{Z}^+$.

Proof

$$\sum_{n \lessapprox x} \Lambda(n) \approx \sum_{n \lessapprox x} \left(\sum_{kl \approx n} (\ln k)\mu(l) \right) \qquad \text{by } 4$$

$$\psi(x)$$

$$\approx \sum_{\substack{(k,l) \\ kl \lessapprox x}} \mu(l) \ln k$$

$$\approx \sum_{lk \lessapprox x} \mu(l) \ln k$$

$$-\sum_{n \lessapprox x} 1 \approx -\sum_{n \lessapprox x} \left(\sum_{kl \approx n} d(k)\mu(l) \right) \qquad \text{by } 4$$

$$-x$$

$$\approx -\sum_{lk \lessapprox x} \mu(l) d(k)$$

$$2\gamma \sum_{lk \lessapprox x} \mu(l) \approx 2\gamma \sum_{\substack{N \lessapprox x \\ \text{with} \\ N \approx kl}} \mu(l)$$

$\longleftarrow$ Lec 20 37 middle (hyperbola)

$$\approx 2\gamma \sum_{N \lessapprox x} \sum_{l \mid N} \mu(l)$$

$$\approx 2\gamma \sum_{N \lessapprox x} \begin{Bmatrix} 1, & N \approx 1 \\ 0, & N > 1 \end{Bmatrix}$$

$$\approx 2\gamma .$$

Add together. ALL IS FINE! ∎

FACT

$$M(x) = o(x) \Rightarrow \psi(x) \sim x .$$

Compare Lec 20 p. 39

Proof

Keep $x \in \mathbb{Z}^+$. Choose any large integer G, $100 \leq G \leq x$. Recall 4 bottom.

$$\psi(x) - x + 2\gamma = \sum_{\substack{\ell k \leq x \\ k \leq G}} \mu(\ell) \{\ln k - d(k) + 2\gamma\}$$

$$+ \sum_{\substack{\ell k \leq x \\ k > G}} \mu(\ell) \{\ln k - d(k) + 2\gamma\} .$$

For the $k \leq G$ piece, note:

$$\sum_{\substack{\ell k \leq x \\ 1 \leq k \leq G}} = \sum_{k=1}^{G} \sum_{1 \leq \ell \leq \frac{x}{k}} \mu(\ell) \{\ln k - d(k) + 2\gamma\}$$

$$= \sum_{k=1}^{G} (\ln k - d(k) + 2\gamma) M\left(\frac{x}{k}\right) .$$

G is held fixed. As $x \to \infty$, by hypothesis, the RHS $= o(x)$. //

on 6

Next, for $k > G$, notice that:

$$\ell k \leqq x \quad \Rightarrow \quad 1 \leqq \ell < \frac{x}{G} \quad \text{a priori}$$

$$\Downarrow$$

for each $\ell \in [1, \frac{x}{G})$, we look at

$$G < k \leqq \frac{x}{\ell}$$

$$\Downarrow$$

$$\sum_{\substack{\ell k \leqq x \\ k > G}} = \sum_{1 \leqq \ell < \frac{x}{G}} \sum_{k \in (G, \frac{x}{\ell}]} \mu(\ell)\left(\ln k - d(k) + 2\gamma\right)$$

$$= \sum_{1 \leqq \ell < \frac{x}{G}} \mu(\ell)\left[\Delta\left(\frac{x}{\ell}\right) - \Delta(G)\right]$$

see 3 bottom

$$= \sum_{1 \leqq \ell < \frac{x}{G}} \mu(\ell)\left[O(1)\sqrt{\frac{x}{\ell}} + O(1)\sqrt{G}\right]$$

$$\{|\mu(\ell)| \leqq 1\}$$

$$= O(1) \sum_{\ell < \frac{x}{G}} \sqrt{\frac{x}{\ell}} + O(1)\frac{x}{G}\sqrt{G}$$

$$= O(1)\, x^{1/2} \left\{ 2\sqrt{\tfrac{x}{G}} + O(1) \right\} + O(1) \frac{x}{\sqrt{G}}$$

$$\uparrow \tfrac{x}{G} \geqq 1$$

$$= O(1) \frac{x}{\sqrt{G}} + O(x^{1/2}) \quad // \, .$$

Hence: 6 middle

$$|\psi(x) - x + 2\gamma| \lesssim o(x) + O(1) \frac{x}{\sqrt{G}} \quad \text{as } x \to \infty .$$

By moving G upward in successive jumps, we get:

$$\psi(x) - x = o(x), \quad \text{i.e.,}$$

$$\psi(x) \sim x . \quad \blacksquare$$

OK

Remark:

Recall Perron's formula $x \geq 10$, $x \notin \mathbb{Z}$

$$\sum_{n<x} a_n n^{-s} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s+w) \frac{x^w}{w} \, dw$$

$$\sigma + c > 1$$

Lec 19 p. 4 etc.

$s = 0 \Rightarrow$

$$\sum_{n<x} d(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(w)^2 \frac{x^w}{w} \, dw.$$

An easy manipulation* gives:

$$\operatorname{Res}\left[\zeta(w)^2 \frac{x^w}{w} ; w = 1 \right] = x \ln x + (2\gamma - 1)x.$$

BIG SURPRISE!!! (See THM on 3.)

$\boxed{\begin{array}{l} * \text{ Lec 18 p. 40} \\ \text{note the } \gamma \end{array}}$

In the remainder of Lec 21, I pretty much followed Ingham 86~89 (middle), 90~91 (top).

There's no point repeating that discussion here except for a couple of special items that I plan to refer to later.

Definitions:

Dirichlet Series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^s}$

Generalized Dirichlet Series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$

with $0 < \lambda_1 < \lambda_2 < \cdots \to \infty$

[usually $\lambda_1 = 1$]

one often puts $\lambda_n = e^{\mu_n}$ $\quad\boxed{\displaystyle\sum_1^{\infty} a_n e^{-s\mu_n}}$

"Dirichlet integral" $\displaystyle\int_1^{\infty} \frac{a(x)}{x^s}\,dx$

Here $a(x)$ = a piecewise continuous fcn on $[1, \infty)$ with a discrete set of (possible) jump points which run off to $+\infty$. Also

$$\int_1^{\infty} \equiv \lim_{R \to \infty} \int_1^R . \qquad (R = \text{real})$$

Fact 1 (for Dirichlet series)

Let $\sum_1^\infty a_n n^{-s}$ conv at s_0. Then:

(a) the series conv absolutely on $\{Re(s) > Re(s_0) + 1\}$

(b) there is uniform conv on any half-plane $\{Re(s) \geq Re(s_0) + 1 + \delta\}$.

$\boxed{\text{uses } \zeta(\sigma - \sigma_0)}$

Fact 2 (for generalized D.S.)

Given $\sum_1^\infty a_n \lambda_n^{-s}$ with, say, $\lambda_1 = 1$. Assume that we have convergence at s_0. Then:

(a) we have pointwise conv on $\{Re(s) > Re(s_0)\}$;

(b) we have uniform conv on sectors of the form $\{|Arg(s - s_0)| \leq \frac{\pi}{2} - \delta\}$.

Proof (sketch)

WLOG $s_0 = 0$.

(a) Let $T = \sum_{n=1}^\infty a_n$. Let $F(x) = \sum_{\lambda_n \leq x} a_n$. $F(x)$ is a right continuous step function. $\lim_{x \to \infty} F(x) = T$.

Fix any s with $Re(s) > 0$. Integrate by parts in the standard way.

$$\lim_{N\to\infty} \left\{ a_1 + \int_1^N x^{-s}\,dF(x) \right\} \qquad a_1 = 1$$

$$= \lim_{N\to\infty} \left\{ a_1 + [x^{-s}F(x)]_1^N + s\int_1^N F(x)x^{-s-1}dx \right\}$$

$$= \lim_{N\to\infty} \left\{ N^{-s}F(N) + s\int_1^N F(x)x^{-s-1}dx \right\}$$

$$= s\int_1^\infty \frac{F(x)}{x^{s+1}}dx \quad \longleftarrow \text{ integ is absolutely conv}$$

(b) For uniform conv, by modifying a_1, WLOG $T = \underline{\underline{0}}$. Note that $N^{-s}F(N)$ [above] tends uniformly to 0 on $\{Re(s) \geq 0\}$. The issue is how fast

$$s\int_1^N \frac{F(x)}{x^{s+1}}dx \longrightarrow s\int_1^\infty \frac{F(x)}{x^{s+1}}dx$$

on $\{ |Arg(s)| \leq \frac{\pi}{2} - \delta,\ s \neq 0 \}$. Write $\alpha = \frac{\pi}{2} - \delta$.

Assume that $|F(x)| < \varepsilon$ for $x \geq N_\varepsilon$ ($N_\varepsilon \in \mathbb{Z}^+$).
We <u>know</u> $\lim_{x\to\infty} F(x) = 0$ since $T = 0$.

Get:

$$\left|\text{relevant remainder}\right| = |s| \left| \int_N^\infty \frac{F(x)}{x^{s+1}}\, dx \right|$$

$\{\text{keep } N \geq N_\varepsilon\}$

$$\leq |s| \int_N^\infty \frac{\varepsilon}{x^{\rho\cos\theta + 1}}\, dx$$

$\{\text{we write } s = \rho e^{i\theta},\ r > 0,\ |\theta| \leq \varphi\}$

$$= \rho\, \varepsilon \int_N^\infty x^{-\rho\cos\theta - 1}\, dx$$

$$= \rho\, \varepsilon\, \frac{N^{-\rho\cos\theta}}{\rho\cos\theta}$$

$$\leq \frac{\varepsilon}{\cos\theta}\, N^{-\rho\cos\theta}$$

$\{\cos t \text{ decreases on } [0, \varphi]\}$

$$\leq \frac{\varepsilon}{\cos\varphi} \cdot 1 .$$

By recalibrating ε for our given $\varphi = \frac{\pi}{2} - \delta$, we are done. ∎

Fact 3 (for generalized D.S.)

In Fact 2, the associated summation fcn $f(s)$ is analytic on $\{Re(s) > Re(s_0)\}$, with unif conv on compacta. Hence,

$$f^{(k)}(s) = \sum_{n=1}^{\infty} \frac{a_n(-\ln \lambda_n)^k}{\lambda_n^s} \qquad \boxed{k \geq 1}$$

with unif conv on compacta too.

Fact 4 (for generalized D.S.)

Every $\sum_1^\infty a_n \lambda_n^{-s}$ with, say, $\lambda_1 = 1$ has an abscissa of convergence $\sigma_c \in [-\infty, +\infty]$ so that

$$\sum_1^\infty a_n \lambda_n^{-s} \text{ conv on } \{Re(s) > \sigma_c\}$$

$$\sum_1^\infty a_n \lambda_n^{-s} \text{ diverges on } \{Re(s) < \sigma_c\}.$$

[No assertion about $Re(s) = \sigma_c$.]

FACTS 2 – 4 have easy analogs for Dirichlet <u>integrals</u>

10 bottom

$$f(s) = \int_1^\infty \frac{a(x)}{x^s}\,dx \,.$$

$s_0 = 0$. Define $F(x) = \int_1^x a(y)\,dy$. Note that $F(x)$ is continuous and <u>piecewise</u> C^1 on $[1, \infty)$.

Also:

$$\int_1^R x^{-s} \underbrace{a(x)\,dx} = \int_1^R x^{-s}\,dF(x) \qquad (R > 1)$$

$$\Downarrow$$

$$f(s) = s\int_1^\infty \frac{F(x)}{x^{s+1}}\,dx \qquad \text{etc etc}$$

insofar as $\mathrm{Re}(s) > 0$.

IMPORTANT NOTE:

Lec 11 p. 26

The standard example

$$f(s) = (1 - 2^{1-s})\,\zeta(s) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s}$$

has $\sigma_c = 0$ [via $s = \varepsilon$] yet <u>no</u> singularity anywhere along $\mathrm{Re}(s) = 0$.

Not like a Taylor series.

USEFUL REMARK (not stated in Lec)

$f(s)$ = generalized D.S. or Dirichlet integral

Assume $\sigma_c \neq +\infty$. Take any $A > \sigma_c$.

Then:

$$|f(\sigma+it)| = O(1+|s|)$$

on $\{\operatorname{Re}(s) \geqq A\}$.

PF

Do a trivial translation to position things so that the point $s_0 = 0$ is a point of convergence. Look at

$$s\int_1^\infty \frac{F(x)}{x^{s+1}}\,dx$$

on 12 + 15. Keep $\operatorname{Re}(s) \geqq A > 0$. Obviously

$$\begin{aligned}|\text{previous expression}| &\lessapprox |s|\int_1^\infty \frac{|F(x)|}{x^{\sigma+1}}\,dx \\ &\lessapprox |s|\int_1^\infty \frac{C}{x^{\sigma+1}}\,dx \\ &= |s|\frac{C}{\sigma} \lessapprox |s|\frac{C}{A}.\end{aligned}$$

OK

Landau's Thm

$f(s)$ = generalized D.S. or Dirichlet integral. Assume $\sigma_c \in \mathbb{R}$. Assume also that the a_n or $a(x)$ are real and eventually of fixed sign. Then, as an analytic function (cf. 14), $f(s)$ must have a true singularity at the point $s = \sigma_c$.

Proof

Famous. As in Ingham 88-89. ∎

THM

Introduce $\Theta = \sup \mathrm{Re}(\rho)$ for ζ as usual.

Then:

$$\psi(x) - x = \Omega_{\pm}(x^{\Theta - \varepsilon})$$

$$\Pi(x) - \mathrm{li}(x) = \Omega_{\pm}(x^{\Theta - \varepsilon})$$

for each tiny $\varepsilon > 0$ (as $x \to +\infty$). Here

$$\Pi(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\ln n} = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \cdots .$$

The "implied constant" in each of the foregoing can be taken arbitrarily large.

Also

$$\mathrm{li}(x) \equiv \int_2^x \frac{dt}{\ln t}.$$

This uses $\zeta(x) \neq 0$ on $0 < x < 1$.

Lec 11 p. 27

__Proof__

Ingham 90–91 top. $\blacksquare$

plus baby calculus

STANDARD DEFINITION.

Recall: Ingham 86

$h(x)$ real;

$g(x) > 0$.

$h(x) = \Omega_+[g(x)] \iff$

$h(x) > c\,g(x)$ infinitely often as $x \to +\infty$

for __some__ constant $c > 0$. Similarly for $h(x) = \Omega_-[g(x)]$ and $\Omega[g(x)]$.