Lecture 22 Synopsis

(8 April)

We will basically <u>not</u> repeat Ingham 91(middle) – 92 (top) here. The reasoning in the book is quite clear. We get:

$\gamma_1 \cong 14.134725$

$$\limsup_{x \to \infty} \frac{\psi(x) - x}{x^{1/2}} \geqq \frac{m_1}{|\frac{1}{2} + i\gamma_1|}$$

$$\liminf_{x \to \infty} \frac{\psi(x) - x}{x^{1/2}} \leqq - \frac{m_1}{|\frac{1}{2} + i\gamma_1|}$$

if RH is true [$\frac{1}{2} + i\gamma_1$ = first zero on critical line; m_1 = its multiplicity]. If RH is false, $\Theta > \frac{1}{2}$ and the aforementioned lim sup / lim inf are $+\infty$ and $-\infty$; see Lec 21 p. 17.

Littlewood proved in 1914 that (actually) a much better result can be obtained if RH holds. See Ingham p. 100. If time permits, we will discuss his result later.

We turn now to a proof of Hardy's theorem that $\zeta(s)$ has infinitely many zeros along $\mathrm{Re}(s) = \frac{1}{2}$. (1914)

We follow the approach of Landau $\approx$ 1927 in his Vorlesungen.

We (must) begin with some calculus lemmas.

Fact 1

Let $f \in C^1[a,b]$ and $\alpha(x)$ be monotonic on $[a,b]$ (either increasing or decreasing).

Then:

$$\int_a^b f(x)\,d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x) f'(x)\,dx,$$

the "dx" integral existing as a nice Riemann integral.

Pf

See Lec 3 p. 8 bottom; also (Lec 3) 7 middle – 8 middle.

$\blacksquare$

<u>Fact 2</u> (1st mean-value thm)

Let $g \in C[a,b]$ (and <u>real</u>). Let $\alpha(x) \nearrow$ on $[a,b]$. Then

$$\int_a^b g(x)\,d\alpha(x) = g(\xi)\int_a^b d\alpha(x)$$

for <u>some</u> $\xi \in [a,b]$.

<u>Pf</u>

If $\alpha(b) = \alpha(a)$, matter is trivial.
If $\alpha(b) > \alpha(a)$, take $m = \min g$, $M = \max g$ on $[a,b]$. Note

$$LHS \in [m(\alpha(b)-\alpha(a)),\ M(\alpha(b)-\alpha(a))].$$

So, we can write $LHS = C\,[\alpha(b)-\alpha(a)]$ for a unique $C \in [m, M]$. Apply intermediate value thm to g. Get $g(\xi) = C$. ∎

<u>Fact 3A</u> (rudimentary 2nd mean value thm)

Let f be <u>monotonic</u> on $[a,b]$. Let α be <u>real</u> and in $C'[a,b]$. There then exists $\xi \in [a,b]$ so that

$$\int_a^b f(x)\,d\alpha(x) = f(a)\int_a^{\xi} d\alpha(x) + f(b)\int_{\xi}^b d\alpha(x).$$

here $d\alpha(x) = \alpha'(x)\,dx$

Pf

The ideas in Lec 3 7 - 8 assure us that

$$(R\text{-}S) \int_a^b H(x)\,d\alpha(x) = (R) \int_a^b H(x)\,\alpha'(x)\,dx$$

holds whenever H is either continuous or monotonic. To be strictly correct, one writes

$$\alpha = \alpha_1 - \alpha_2$$

with $\alpha_j \in C^1$ and $\alpha_j \nearrow$.

To prove the relation stated in Fact 3A, we flip f to $-f$ if need be and declare $f \nearrow$ wlog. By Fact 1, (i.e. integ by parts)

$$\int_a^b f(x)\,d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\,df(x).$$

f increasing

By Fact 2,

$$\int_a^b \alpha(x)\,df(x) = \alpha(\xi)\int_a^b df = \alpha(\xi)\,[f(b) - f(a)].$$

Hence:

$$\int_a^b f\,dg = f(b)\,g(b) - f(a)\,g(a)$$
$$\qquad - g(\xi)\,[f(b) - f(a)]$$

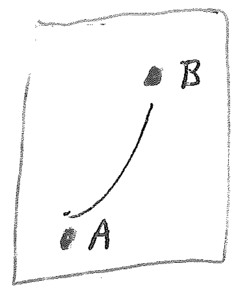
$$= f(a)\,[g(\xi) - g(a)] + f(b)\,[g(b) - g(\xi)]$$

$$= f(a)\int_a^\xi dg + f(b)\int_\xi^b dg \,. \qquad \blacksquare$$

<u>Fact 3B</u> (2^{nd} mean-value thm)

Let f be monotonic $\nearrow$ on $[a,b]$. Let $g \in$
Let $A \leqq f(a+)$, $B \geqq f(b-)$.
$C'[a,b]$ and <u>real</u>.
We can then find $\xi \in [a,b]$ so that

$$\int_a^b f\,dg = A\int_a^\xi dg + B\int_\xi^b dg \,.$$



SIMILARLY for $f \downarrow$ on $[a,b]$.

<u>Pf</u>

Let $f_0(x) = \left\{ \begin{array}{l} A, \ x = a \\ f(x), \ a < x < b \\ B, \ x = b \end{array} \right\}$. Note that $f_0 \nearrow$.

Apply Fact 3A to get

$$\int_a^b f_0\,dg = A\int_a^\xi dg + B\int_\xi^b dg \,.$$

But,

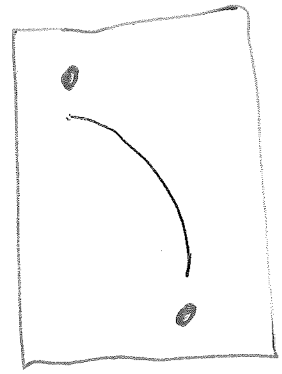
$$\int_a^b f_0\,d\varphi = \int_a^b f_0(x)\varphi'(x)\,dx = \int_a^b f(x)\varphi'(x)\,dx$$

$$= \int_a^b f\,d\varphi$$

by 4 top. ∎

NOTE: for $f \downarrow$, one takes

$A \geq f(a+)$, $B \leq f(b-)$.



Lemmas I – IV are in the spirit of van der Corput ≈ 1921

Lemma I

Let $F \in C'[a,b]$ and real. Assume that $F'(x)$ is monotonic on $[a,b]$. Assume further that

$$F'(x) \geq m > 0 \quad \text{OR} \quad F'(x) \leq -m < 0$$

for all $x \in [a,b]$. Then:

$$\left| \int_a^b e^{iF(x)}\,dx \right| \leq \frac{4}{m}.$$

⑦

Pf

WLOG $F'(x)$ is monotonic $\nearrow$ (simply flip F to $-F$ if need be).

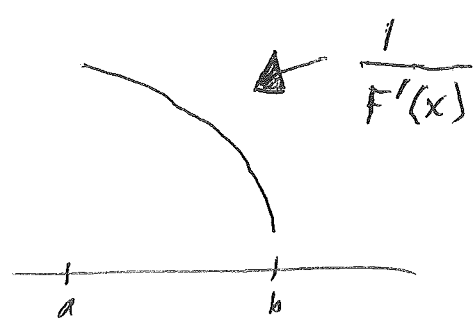
Suppose first that $F'(x) \geq m > 0$.

$$\int_a^b \cos F \, dx = \int_a^b \frac{1}{F'} (\cos F \cdot F') \, dx$$

$$= \int_a^b \frac{1}{F'} \, d(\sin F)$$

$\left\{ \frac{1}{F'} \text{ is monotonic } \downarrow \text{ and positive} \right\}$

Recall Fact 3B.



$\left\{ \text{Put } \underline{A} = \frac{1}{F'(a)} \text{ and } \underline{B} = 0. \right\}$

$$\int_a^b \frac{1}{F'} d(\sin F) = \frac{1}{F'(a)} \int_a^\xi d(\sin F) + 0 \int_\xi^b d(\sin F)$$

$$\Rightarrow \left| \int_a^b \cos F \, dx \right| \leq \frac{2}{|F'(a)|} \leq \frac{2}{m}$$

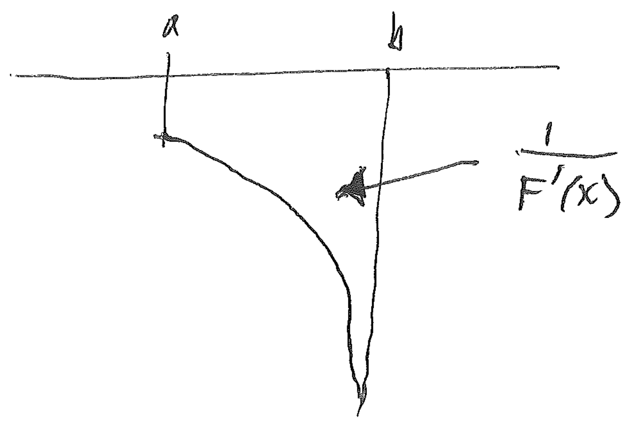
Similarly:

$$\left|\int_a^b \sin F\,dx\right| \lessapprox \frac{2}{m}.$$

Now suppose $F'(x) \lessapprox -m < 0$ on $[a,b]$.

$$\int_a^b \cos F\,dx = \int_a^b \frac{1}{F'}\,d(\sin F) \quad \text{as before}$$

$$\left\{\begin{array}{l} F' \text{ is negative, monotonic } \nearrow \\ \frac{1}{F'} \text{ is negative, monotonic } \downarrow \end{array}\right\}$$



Fact 3B $\left\{\begin{array}{l} A = 0 \\ B = \frac{1}{F'(b)} \end{array}\right\}$

$$\int_a^b \frac{1}{F'}\,d(\sin F) = 0\int_a^\xi d(\sin F) + \frac{1}{F'(b)}\int_\xi^b d(\sin F)$$

$$\Rightarrow \left|\int_a^b \cos F\,dx\right| \lessapprox \frac{2}{|F'(b)|} \lessapprox \frac{2}{m}.$$

Similarly:

$$\left| \int_a^b \sin F \, dx \right| \leq \frac{2}{m} . \quad \blacksquare$$

Lemma II

F, G real. $G(x) > 0$. F, G in $C^1[a,b]$.

Assume $\frac{F'(x)}{G(x)}$ is monotonic on $[a,b]$.

Assume further that

$$\frac{F'(x)}{G(x)} \geq m > 0 \quad \text{OR} \quad \frac{F'(x)}{G(x)} \leq -m < 0$$

for all $x \in [a,b]$. Then:

$$\left| \int_a^b G(x) e^{iF(x)} dx \right| \leq \frac{4}{m} .$$

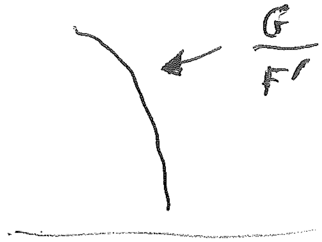
Pf

Flip F to $-F$ if need be; (so) take $\frac{F'}{G}$ monotonic $\uparrow$ on $[a,b]$ wlog.

Suppose first that $\frac{F'(x)}{G(x)} \geq m > 0$.

$$\int_a^b G(x) \cos F(x) \, dx = \int_a^b \frac{G(x)}{F'(x)} \, d(\sin F)$$

Mimic 7.



$\frac{G}{F'}$ pos.

$\Downarrow$

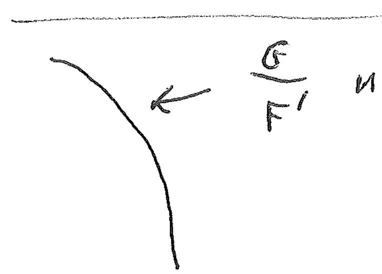
$$\left| \int_a^b G \cos F \, dx \right| \leq \frac{2}{m}.$$

Then:

$$\left| \int_a^b G \sin F \, dx \right| \leq \frac{2}{m}.$$

Suppose next that $\frac{F'(x)}{G(x)} \leq -m < 0$.

Mimic 8.



$\frac{G}{F'}$ neg.

$\Downarrow$

$$\left| \int_a^b G \cos F \, dx \right| \leq \frac{2}{m}.$$

Then

$$\left| \int_a^b G \sin F \, dx \right| \leq \frac{2}{m}. \qquad \blacksquare$$

Lemma III

F real. $F \in C^2[a,b]$. $F''(x) \geqq r > 0$ OR $F''(x) \leqq -r < 0$ for all $x \in [a,b]$. Then:

$$\left| \int_a^b e^{iF(x)}\,dx \right| \leqq \frac{8}{\sqrt{r}} .$$

Pf

Note that $\left| \int_a^b e^{iF}dx \right| \leqq (b-a)$ trivially.

Hence, wlog,

$$\frac{8}{\sqrt{r}} < b-a . \qquad |||$$

Case 1

$F'(x) > 0$ on $a < x < b$.

Know $F''(x) \neq 0$ on $[a,b]$ and $|F''(x)| \geqq r > 0$.

Hence $F'(x)$ is monotonic on $a \leqq x \leqq b$.

Suppose, e.g., $F'(x)$ is decreasing.

Clearly

$$F'(x) - F'(b) \geqq r\delta \quad \text{for } a \leqq x \leqq b-\delta .$$

We assume here $\delta < b-a$! Hence, $F'(x) \geqq r\delta$.

Write

$$I = \int_a^b e^{iF}dx = \int_a^{b-\delta} e^{iF}dx + \int_{b-\delta}^{b} e^{iF}dx$$

to get

$$|I| \lessapprox \left| \int_a^{b-\delta} e^{iF}dx \right| + \delta$$

$\{$ apply Lemma I $\}$

$$|I| \lessapprox \frac{4}{r\delta} + \delta .$$

We propose to take $\delta = \frac{2}{\sqrt{r}}$. We have

$$b - a > \frac{8}{\sqrt{r}} > \frac{2}{\sqrt{r}} = \delta$$

for (11) bottom

as needed. So,

$$|I| \lessapprox \frac{4}{\sqrt{r}} .$$ OK

Similarly for $F'(x)$ increasing.

<u>Case 2</u>

$F'(x) < 0$ on $a < x < b$.

Here, just flip F to $-F$ and use Case 1. OK

For all <u>other</u> cases, we must then have
$F'(c) = 0$ for some $c \in (a, b)$.

We know $F''(x) \neq 0$ on $[a, b]$ and $|F''(x)| \geq r > 0$.
Hence $F'(x)$ is <u>strictly</u> <u>monotonic</u> on $[a, b]$.
The point c is therefore unique.

<u>Case 3</u>

$F'(x) > 0$ on $[a, c)$, $F'(x) < 0$ on $(c, b]$.

We maintain that

$$\left| \int_a^c e^{iF} dx \right| \leq \frac{4}{\sqrt{r}} \quad \underline{\text{and}} \quad \left| \int_c^b e^{iF} dx \right| \leq \frac{4}{\sqrt{r}}.$$

Take, say, $[c, b]$. If $b - c \leq \dfrac{4}{\sqrt{r}}$, we are fine. Suppose therefore

$$b - c > \frac{4}{\sqrt{r}}.$$

Know $F'(c) - F'(x) \geq r(x - c)$ for $c < x \leq b$.
Hence

$$F'(c) - F'(x) \geq r\delta \quad \text{for } c + \delta \leq x \leq b.$$

We <u>assume</u> here $\delta < b - c$. Get: $F'(x) \leq -r\delta$.
See (11) bottom.

Get:

$$\left|\int_c^b e^{iF}dx\right| \lesssim \left|\int_c^{c+\delta} e^{iF}dx\right| + \left|\int_{c+\delta}^b e^{iF}dx\right|$$

$$\lesssim \delta + \frac{4}{r\delta} \quad \text{by Lemma I.}$$

Propose to put $\delta = \frac{2}{\sqrt{r}}$ to get (the bound) $\frac{4}{\sqrt{r}}$.

We need:

$$\frac{2}{\sqrt{r}} < b-c, \quad \underline{\underline{\text{but}}} \quad b-c > \frac{4}{\sqrt{r}}$$

(by hypothesis). Hence:

$$\left|\int_c^b e^{iF}dx\right| \lesssim \frac{4}{\sqrt{r}}.$$

it's essentially case 1

The case $[a,c]$ is similar, of course.

GET:

$$\left|\int_a^b e^{iF}dx\right| \lesssim \frac{4}{\sqrt{r}} + \frac{4}{\sqrt{r}} = \frac{8}{\sqrt{r}}.$$

<u>Case 4</u>

$F'(x) < 0$ on $[a,c)$, $F'(x) > 0$ on $(c,b]$.

Here, just flip F to $-F$ and use Case 3. ∎

Lemma IV

F real, $G > 0$. $F \in C^2$, $G \in C^1$ on $[a, b]$.
Assume $F''(x) \geqq r > 0$ or $F''(x) \leqq -r < 0$, all x.
Assume also that $\frac{F'}{G}$ is monotonic on $[a, b]$
and $|G(x)| \leqq M$. Then:

$$\left| \int_a^b G(x) e^{iF(x)} dx \right| \leqq \frac{8M}{\sqrt{r}} .$$

Pf

Imitate proof of Lemma III. WLOG, $\frac{8}{\sqrt{r}} < b - a$.
Use Lemma II. Etc.

$$\text{EG case 1} \Longrightarrow \quad M\delta + \frac{4}{\left(\frac{r\delta}{M}\right)} = M\left[\delta + \frac{4}{r\delta}\right]$$

$$\Rightarrow \text{etc.}$$

In case 3, p. 13 middle, refer to:

$$\frac{4M}{\sqrt{r}} \quad \text{and} \quad \frac{4M}{\sqrt{r}} .$$

$\blacksquare$