# Lecture 25 Synopsis

(Wed, 20 Apr)

The lecture covered a variety of topics.

First, regarding Lindelöf's $\mu$-function for $\zeta(s)$. Cf. Lec 24 p. 11 ff.

note Lec 24, p. 13 lines 4-8

**Thm**

Consider $f(s) = \zeta(s)$ for $\mathrm{Im}(s) \gtrsim 1$, say.

(a) $\mu(\sigma) + (\sigma - \frac{1}{2}) = \mu(1-\sigma)$

(b) $\mu(\sigma) = 0, \quad \sigma > 1$

(c) $\mu(\sigma) = \frac{1}{2} - \sigma, \quad \sigma < 0$

(d) $\mu(\sigma)$ is convex on every $[a,b]$

(e) $\mu(\sigma)$ is continuous on $\mathbb{R}$

(f) $\mu(\sigma) \geqq 0$

(g) $\mu(\sigma)$ is monotonic decreasing

(h) $\mu(\frac{1}{2}) \lesssim \frac{1}{4}$

(i) Lindelöf's conjecture is true $\Leftrightarrow \mu(\frac{1}{2}) = 0$.

**Pf**

(a) Lec 24 15.

(b) Lec 24 13 bot + 14.

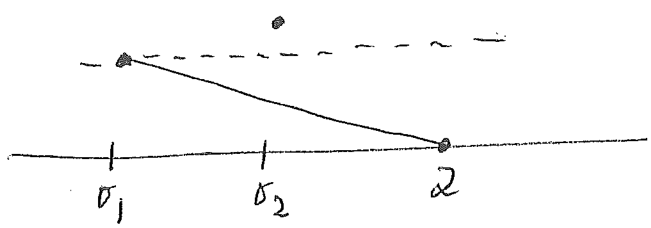
(c) Combine (a) + (b). See Lec 24 16.

(d) Lec 24 12.

(e) Lec 24 13 top.

(f) Know $\mu(\sigma) = 0$, $\sigma > 1$. Hence $\mu(\sigma) = 0$, $\sigma \geqq 1$.
**Suppose** $\mu(\sigma_1) < 0$ with some $\sigma_1 < 1$. Take $\sigma_2 = 2$ and apply convexity over $[\sigma_1, \sigma_2]$.
Get $\mu(\frac{3}{2}) < 0$. Contrad!

(g) Know $\mu(\sigma) \geqq 0$. And $\mu(\sigma) = 0$, $\sigma \geqq 1$.
**Suppose** $\sigma_1 < \sigma_2$ has $0 \leqq \mu(\sigma_1) < \mu(\sigma_2)$.
So, $\sigma_2 < 1$. Look at convexity over $[\sigma_1, 2]$.




This violates convexity (at $\sigma_2$).

(h) Lec 24 p. 16 line 4, put $\sigma = 1/2$.

(i) Lec 24 p. 17. ▨

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Recall **Lindelöf's Conjecture**

$$\mu(\sigma) = \left\{ \begin{array}{ll} 0, & \frac{1}{2} \leqq \sigma < \infty \\ \frac{1}{2} - \sigma, & -\infty < \sigma \leqq \frac{1}{2} \end{array} \right\}.$$

It is known that $RH \Rightarrow$ Lindelöf Conjecture. Lec 24 16

## 2nd topic.

I briefly discussed the following thm.

**Thm**

Let $f(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ be a given generalized Dirichlet series with $1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$.

Suppose the series converges at $s_0 \in \mathbb{C}$.

Then:

(a) the series conv uniformly on every Stolz angle

$$\{ |\operatorname{Arg}(s - s_0)| \leqq \frac{\pi}{2} - \delta \} ;$$

(b) the series conv uniformly on every "super" Stolz angle

$$\{ |t - t_0| \leqq e^{M(\sigma - \sigma_0)} - 1 \}$$

$(M > 0)$.

The proof (omitted here) is an interesting exercise. Of course, (a) is known already by Lec 21, p. 11 Fact 2. Concerning (b), I simply remark: just study Lec 21, p. (13), line 7 when (wlog) $s_0 = 0$. For $\sigma > A$ big [but frozen], notice that:

$$\sigma \lesssim e^{M\sigma} \qquad (M \geq 1 \text{ wlog})$$

$$|t| \lesssim e^{M\sigma} - 1 \lesssim e^{M\sigma}$$

$$|s| \lesssim 2e^{M\sigma} \quad \text{a priori}$$

$$\Downarrow$$

get a $\frac{\varepsilon \cdot 2}{A} e^{-\sigma(\ln N - M)}$ term!

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Needless to say, by a minor expungement and insertion (of a new "$\lambda$"), we can actually allow ANY $\lambda_1$ in the above Thm; we do not need $\lambda_1 = 1$ ONLY $\lambda_1 > 0$.

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Because of (3) Thm, Stolz angles or "super" Stolz angles are natural vehicles on which to discuss, e.g., identity theorems of the sort $f_1(\xi_k) = f_2(\xi_k)$, all $k \geq 1 \Rightarrow a_{n1} \equiv a_{n2}$.

3rd topic.

We did a quick review of basic Fourier transforms and related analysis.

$$\hat{f}(p) \equiv \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i p x}\, dx \qquad p \in \mathbb{R}$$

$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$ , $f$ piecewise $C^1$ basically

$$\frac{f(x+0)+f(x-0)}{2} = \int_{-\infty}^{\infty} \hat{f}(p)\, e^{2\pi i p x}\, dp$$

$$RHS \equiv \lim_{R\to\infty} \int_{-R}^{R} \hat{f}(p)\, e^{2\pi i p x}\, dp$$

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$$\tilde{f}(u) \equiv \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx$$

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(u)\, e^{iux}\, du$$

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For "nice" functions $f, g$ (real or complex) on $\mathbb{R}$, we define the <u>convolution</u>

$$H(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\, dt .$$

$H(x)$ is a reasonable function, due to

$$|H(x)| \leq \int_{-\infty}^{\infty} |f(t)|\,|g(x-t)|\, dt$$

$$\int_{-\infty}^{\infty} |H(x)|\, dx \leq \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |f(t)|\,|g(x-t)|\, dt \right) dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t)|\,|g(x-t)|\, dx\, dt$$

$\{$ Fubini $\}$

$$= \int_{-\infty}^{\infty} |f(t)| \left( \int_{-\infty}^{\infty} |g(\xi)|\, d\xi \right) dt$$

$$= \left( \int_{-\infty}^{\infty} |f(t)|\, dt \right) \left( \int_{-\infty}^{\infty} |g(\xi)|\, d\xi \right)$$

$$< \infty .$$

Often, $f$ and $g$ are initially kept in the Schwartz class $\mathcal{S}$.

One easily checks that $H(x)$ is continuous and bounded if <u>either</u> $f$ or $g$ is known to be bounded. This is true in all "sensible" cases.

review ⑥ middle

What is also checked quite easily by Fubini and the key fact that

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)} \qquad \begin{matrix}\theta \in \mathbb{R} \\ \phi \in \mathbb{R}\end{matrix}$$

is the <u>relation</u> from Fourier transform theory

$$\hat{H}(p) = \hat{f}(p)\hat{g}(p)$$

$$\left(\text{also } \tilde{H}(u) = \tilde{f}(u)\tilde{g}(u)\right).$$

This is WHY the convolution $H = f * g$ is so useful!

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Another useful property goes as follows. Assume $f, g, \hat{f}, \hat{g}$ are all "nice". Then, observe that:

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\overline{\left[\int_{-\infty}^{\infty} \hat{g}(p)e^{2\pi i p x}\,dp\right]}\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\left[\int_{-\infty}^{\infty} \overline{\hat{g}(p)}\,e^{-2\pi i p x}\,dp\right]dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{\hat{g}(p)}\,e^{-2\pi i p x}\,\underline{dx}\,dp$$

(by Fubini)

$$= \int_{-\infty}^{\infty} \overline{\hat{g}(p)}\left[\int_{-\infty}^{\infty} f(x)e^{-2\pi i p x}\,dx\right]dp$$

$$= \int_{-\infty}^{\infty} \hat{f}(p)\overline{\hat{g}(p)}\,dp \,.$$

Thus,

$$\int_{-\infty}^{\infty} |f_1(x)-f_2(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}_1(p)-\hat{f}_2(p)|^2\,dp$$

for nice $f_j$. In particular:

$$\int_{-\infty}^{\infty} |f_1(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}_1(p)|^2\,dp \,.$$

This is the <u>Plancherel</u> <u>formula</u>.

Let $\chi_E(x) = \left\{ \begin{matrix} 1, & x \in E \\ 0, & x \notin E \end{matrix} \right\}$. One easily checks:

$$\widehat{\chi_{[-c,c]}(x)} = 2\,\frac{\sin cu}{u}$$

$$\widehat{\max(0, b-|x|)} = 2\,\frac{1-\cos bu}{u^2} = \frac{4\sin^2(\frac{b}{2}u)}{u^2}$$

$$\{u = 2\pi p\}$$

$$\widehat{\chi_{[-c,c]}(x)} = \frac{\sin 2\pi p c}{\pi p}$$

$$\widehat{\max(0, b-|x|)} = \frac{1-\cos 2\pi p b}{2\pi^2 p^2} = \frac{\sin^2(\pi p b)}{\pi^2 p^2}.$$

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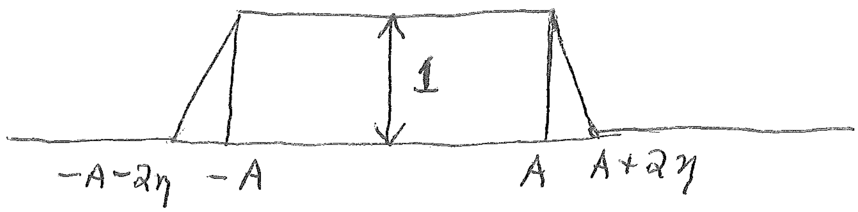
It will be convenient to consider the convolution

$$T(x) \equiv \frac{1}{2\eta}\, \chi_{[-\eta,\eta]}(x) * \chi_{[-A-\eta, A+\eta]}(x).$$

## THM

$A > 0$, $\eta > 0$. $T(x)$ as above. Then:

(1) $T(x)$ is the trapezoid




(2)
$$\widetilde{T(x)} = \frac{\cos(Au) - \cos((A+2\eta)u)}{\eta u^2}$$

(3)
$$\widetilde{T(x)} = 2\,\frac{\sin(\eta u)\sin((A+\eta)u)}{\eta u^2} .$$

Pf

By 7 + 9,

$$\widetilde{T(x)} = \frac{1}{2\eta}\, 2\,\frac{\sin \eta u}{u}\, 2\,\frac{\sin (A+\eta)u}{u}$$

$$= 2\,\frac{\sin(\eta u)\sin((A+\eta)u)}{\eta u^2}, \qquad (11)$$

so (3) is OK. Of course,

$$\cos(\theta-\phi) - \cos(\theta+\phi) = 2\sin\theta\sin\phi$$

$$\Downarrow$$

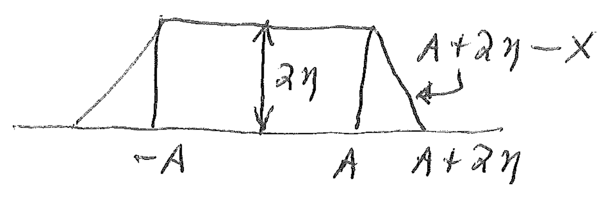
$$\cos(Au) - \cos((A+2\eta)u) = 2\sin((A+\eta)u)\sin(\eta u)$$

$$= 2\sin(\eta u)\sin((A+\eta)u),$$

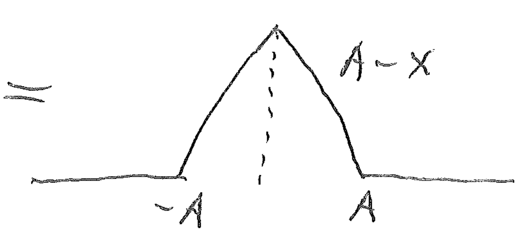
so (2) is OK too.

For (1), define $g(x)$ to be the trapezoid (shown) on (10). Look at:

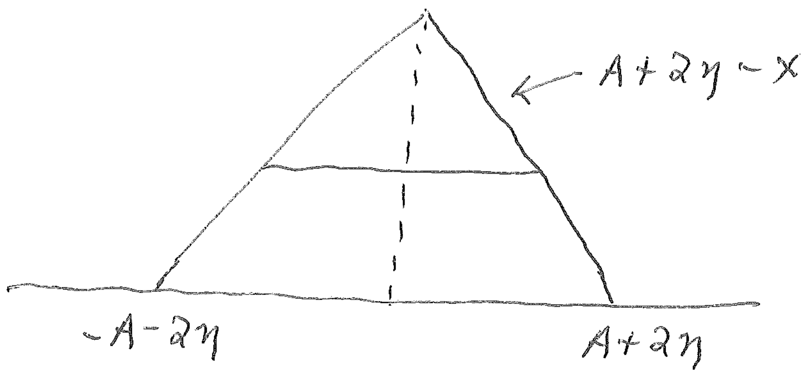
$2\eta\, g(x) =$




$\max(0, A-|x|) =$



$$2\eta\, q(x) + \max(0, A-|x|) =$$




$$= \max(0, A+2\eta-|x|)$$

$$\Downarrow$$

$$\boxed{2\eta\, q(x) = \max(0, A+2\eta-|x|) - \max(0, A-|x|)}$$

$$2\eta\, \widetilde{q(x)} = 2\left[\frac{1-\cos((A+2\eta)u)}{u^2}\right] - 2\left[\frac{1-\cos(Au)}{u^2}\right]$$

$$= 2\left[\frac{\cos(Au) - \cos((A+2\eta)u)}{u^2}\right]$$

$$\Rightarrow \widetilde{q(x)} = \frac{\cos(Au) - \cos((A+2\eta)u)}{\eta u^2} .$$

By (2) on p. 10,

$$\widetilde{q(x)} = \widetilde{T(x)} \quad . \qquad (\text{all } u \in \mathbb{R})$$

Apply 5 last line to this situation. Get

$$q(x) = T(x), \quad \text{each } x \in \mathbb{R},$$

since $q$ and $T$ are continuous on $\mathbb{R}$. Cf. also the $\tilde{f}_j$ counterpart of 8 line −4. In any case, (1) is now true. ∎

## $4^{th}$ Topic.

**THM** (an important $L_2$ **estimate** for Dirichlet polynomials).

We have:

$$\int_{\alpha}^{\alpha+H} \left| \sum_{j=1}^{N} b_j e^{-i\lambda_j t} \right|^2 dt = H \sum_{j=1}^{N} |b_j|^2 + \frac{O(1)}{\delta} \sum_{j=1}^{N} |b_j|^2$$

anytime

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N$$

$$|\lambda_k - \lambda_j| \geq \delta, \quad \text{all } k \neq j$$

$$b_j \in \mathbb{C}, \quad \alpha \in \mathbb{R}, \quad H > 0.$$

The "implied" constant in $O(1)$ is absolute; it can be taken to be $\frac{4\pi}{\sqrt{3}}$.

Pf

Some easy wlog's (giving same implied constant in $O(1)$).

First one: $\alpha = -\frac{H}{2}$. Second one: $H = 2$.

Must prove:

$$\int_{-1}^{1} \Big| \sum_{j=1}^{N} b_j e^{-i\lambda_j t} \Big|^2 dt = 2 \sum_{j=1}^{N} |b_j|^2 + \frac{O(1)}{\delta} \sum_{j=1}^{N} |b_j|^2 .$$

Take $T(t)$ on (10) with $A = 1$, $\eta = \eta$. Let

$$\mathcal{J} = \int_{-1}^{1} \Big| \sum b_j e^{-i\lambda_j t} \Big|^2 dt .$$

Clearly

$$\mathcal{J} \lesseqqgtr \int_{\mathbb{R}} \Big| \sum b_j e^{-i\lambda_j t} \Big|^2 T(t)\, dt$$

$$\mathcal{J} \lesseqqgtr \sum_{j,k} b_j \bar{b}_k \int_{\mathbb{R}} T(t) e^{-i(\lambda_j - \lambda_k)t} dt$$

$\{$put $d_{jk} = \lambda_j - \lambda_k$ $\}$

$$\mathcal{I} \leqq \sum_{j,k} b_j \bar{b}_k \frac{2}{\eta} \frac{\sin(\eta d_{jk})}{d_{jk}} \frac{\sin[(1+\eta) d_{jk}]}{d_{jk}}$$

by 10 (3)

$$\mathcal{I} \leqq \sum_{j=1}^{N} |b_j|^2 \left\{ \frac{2}{\eta} \eta (1+\eta) \right\}$$

$$+ \frac{2}{\eta} \sum_{j \neq k} b_j \bar{b}_k \frac{\sin(\eta d_{jk})}{d_{jk}} \frac{\sin[(1+\eta) d_{jk}]}{d_{jk}}$$

$$\leqq (2+2\eta) \sum_{1}^{N} |b_j|^2$$

$$+ \frac{2}{\eta} \sum_{j \neq k} Re(b_j \bar{b}_k) \boxed{*} \boxed{*}$$

$$\left\{ \text{but } 2Re(b_j \bar{b}_k) \leqq |b_j|^2 + |b_k|^2 \right\}$$

$$\leqq (2+2\eta) \sum_{1}^{N} |b_j|^2 + \frac{1}{\eta} \sum_{j \neq k} \frac{|b_j|^2 + |b_k|^2}{d_{jk}^2}$$

$$= (2+2\eta) \sum_{1}^{N} |b_j|^2 + \frac{2}{\eta} \sum_{j \neq k} \frac{|b_j|^2}{d_{jk}^2}$$

$$\leq (2+2\eta) \sum_1^N |b_j|^2$$

$$+ \frac{2}{\eta} \sum_1^N |b_j|^2 \left( 2 \sum_{m=1}^{\infty} \frac{1}{m^2 \delta^2} \right)$$

$$\left\{ \begin{array}{l} \text{somewhat crudely} \\ \text{via } |\lambda_j - \lambda_k| \geq \delta > 0 \\ \text{for } j \neq k \end{array} \right\}$$

$$= (2+2\eta) \sum_1^N |b_j|^2 + \frac{4}{\eta \delta^2} \sum_1^N |b_j|^2 \frac{\pi^2}{6}$$

$$\left\{ \text{let } D = \frac{\pi}{\sqrt{3}} \right\}$$

$$= (2+2\eta) \sum_1^N |b_j|^2 + \frac{2D^2}{\eta \delta^2} \sum_1^N |b_j|^2$$

$$= \left( 2 + 2\eta + \frac{2D^2}{\eta \delta^2} \right) \sum_1^N |b_j|^2 \quad .$$

To minimize RHS, take

$$\eta = \frac{D}{\delta} \quad .$$

Get:

$$\mathcal{J} \leqq \left(2 + \frac{4D}{\delta}\right) \sum |b_j|^2$$

or

$$\int_{-1}^{1} \left| \sum_{j=1}^{N} b_j e^{-i\eta_j t} \right|^2 dt \leqq 2 \sum_{1}^{N} |b_j|^2 + \frac{4\pi/\sqrt{3}}{\delta} \sum_{1}^{N} |b_j|^2 .$$

This, of course, is the <u>upper</u> bound posited on page 14.

[The remainder of the proof was (actually) done in Lec 26, but we include it here!]

The <u>lower</u> bound for $\mathcal{J}$ is similar but slightly harder. Must prove:

$$\mathcal{J} \geqq \left(2 - \frac{4D}{\delta}\right) \sum_{1}^{N} |b_j|^2 .$$

If $\delta \leq 2D = \frac{2\pi}{\sqrt{3}}$, matters are trivial.
So, wlog, $\delta > 2D$. Hence, $\frac{1}{2} > \frac{D}{\delta}$.

We consider $T(t)$ on 10 with $A = 1 - 2\eta$, $0 < \eta < \frac{1}{2}$, and observe that $A + 2\eta = 1$. Here

$$\mathcal{J} \geq \int_{-\infty}^{\infty} T(t) \left| \sum_1^N b_j e^{-i\lambda_j t} \right|^2 dt$$

$$= \sum b_j \bar{b}_k \int_{-\infty}^{\infty} T(t) e^{-i(\lambda_j - \lambda_k)t} \, dt$$

$$= \sum b_j \bar{b}_k \, 2 \frac{\sin(\eta d_{jk})}{\eta d_{jk}} \frac{\sin((1-\eta) d_{jk})}{d_{jk}}$$

$$= \sum_{j=1}^N |b_j|^2 (2 - 2\eta) + \sum_{j \neq k} \mathrm{Re}(b_j \bar{b}_k) \, 2 \frac{\sin \eta d_{jk}}{\eta d_{jk}} \frac{\sin((1-\eta) d_{jk})}{d_{jk}}$$

{ as on 16 }

$$\geq (2 - 2\eta) \sum_1^N |b_j|^2 - \sum_{j \neq k} \left( |b_j|^2 + |b_k|^2 \right) \frac{1}{\eta d_{jk}} \frac{1}{d_{jk}}$$

$$= (2 - 2\eta) \sum_1^N |b_j|^2 - \frac{2}{\eta} \sum_1^N |b_j|^2 \left( \sum_{k \neq j} \frac{1}{d_{jk}^2} \right)$$

{ as on 16 bottom }

$$\geq (2-2\eta)\sum_1^N |b_j|^2 - \frac{2}{\eta}\sum_1^N |b_j|^2 \frac{1}{\delta^2}\, 2\left(\frac{\pi^2}{6}\right)$$

$\{$ cf. 17 middle $\}$ $\{ D = \frac{\pi}{\sqrt{3}} \}$

$$= \sum_1^N |b_j|^2 \left(2 - 2\eta - \frac{2D^2}{\eta\delta^2}\right).$$

Take $\eta = \frac{D}{\delta}$. Since $\frac{1}{2} > \frac{D}{\delta}$, $\eta$ is admissible.

Get

$$\mathcal{J} \geq 2\sum_1^N |b_j|^2 - \frac{4D}{\delta}\sum_1^N |b_j|^2,$$

with $4D = \frac{4\pi}{\sqrt{3}}$, parallel to 18 lines 4–5. This is the lower bound promised. ▯

Let $\mathcal{C}$ be the constant in the $O(1)$ on 14. The theorem on 14 is very closely related to the generalized Hilbert inequality

$$(*) \qquad \left| \sum_{j \neq k} \frac{z_j \bar{z}_k}{\lambda_j - \lambda_k} \right| \leq \frac{\mathcal{C}/2}{\delta} \sum_{j=1}^N |z_j|^2 \qquad (z_j \in \mathbb{C}).$$

One readily checks that $(*) \Rightarrow$ thm on 14. Selberg has noted [in a very slick proof] that the thm on 14 $\Rightarrow (*)$.

By choosing majorant/minorant functions more sophisticated than trapezoids, one finds that the best $C$ is $2\pi$.

Note:

$$\frac{4\pi}{\sqrt{3}} = 2\pi(1.1547^{+}),$$

which is not too bad!

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