Lecture 26 Synopsis

(Fri, 22 Apr)

Recall E–M version 2 from Lec 9, pp. 12 + 14.
Taking $R = 0$ led to

$$\tilde{B}_1(x) = -2 \sum_{1}^{\infty} \frac{\sin 2\pi n x}{2\pi n} = x - [\![x]\!] - \frac{1}{2}$$

for $x \notin \mathbb{Z}$. See Lec 9, pp. 13 bottom, 14 top.

In Lec 9, p. 19, we saw that

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1} - z \int_1^{\infty} \frac{\tilde{B}_1(u)}{u^{z+1}}\, du \quad , \quad x > 1 .$$

In fact, the derivation on 18 bottom, with $N \hookrightarrow N-1$ and $u = 1+t$, produced:

$$\sum_{k=1}^{N} k^{-z} = \frac{1}{2} + \frac{1}{z-1} + \frac{1}{2} N^{-z} + \frac{N^{1-z}}{1-z}$$
$$- z \int_1^{N} \frac{\tilde{B}_1(u)}{u^{z+1}}\, du \quad .$$

It is natural to subtract these 2 formulae.

Get:

$$\zeta(z) \sim \sum_{1}^{N} k^{-z} = -\frac{1}{2} N^{-z} - \frac{N^{1-z}}{1-z} - z \int_N^\infty \frac{\tilde{B}_1(u)}{u^{z+1}}\, du,$$

where the final term is nicely analytic on $\{\mathrm{Re}(z) > 0\}$ thanks to $|\tilde{B}_1(u)| \leq 1/2$.

It follows that

$$(\star) \quad \zeta(z) = \sum_{1}^{N} n^{-z} - \frac{N^{1-z}}{1-z} - \frac{1}{2} N^{-z} - z \int_N^\infty \frac{\tilde{B}_1(u)}{u^{z+1}}\, du$$

on $\{\mathrm{Re}(z) > 0\} \sim \{1\}$. Compare Lec 6, 10 line 4.

In <u>numerical work</u> (evaluating $\zeta(z)$), one often uses the <u>counterpart</u> of $(\star)$ associated with <u>$\tilde{B}_{2R+1}(u)$</u> and the remainder term

$$(-1)\, z(z+1) \cdots (z+2R) \int_N^\infty \frac{\tilde{B}_{2R+1}(u)}{(2R+1)!} \frac{1}{u^{z+2R+1}}\, du.$$

Cf. Lec 9 pp. 18 + 19. One takes R and N appropriately large.

These comments hint that controlling the size of $\Xi(s)$ in the critical strip $\{0 < \mathrm{Re}(s) < 1\}$ comes down to doing the same for CERTAIN sums

$$\sum_{n=1}^{N} n^{-\sigma} n^{-it} \equiv \sum_{n=1}^{N} n^{-\sigma} e^{-it\ln n} .$$

The size of N will depend at least loosely on the magnitude of $|t|$. (See, e.g., p. 17.)

The issue of numerical calculation of $\Xi(s)$ deserves a separate lecture!! It will, however, play no role in the remaining lectures in this course.

In the present lecture, the goal is to simply obtain an important formula of Hardy and Littlewood growing out of 2 (*).

We need a preliminary.

THM

Let $f(x)$ be real and C^1 on $[a,b]$. Let $f'(x)$ be **monotonic** here. Assume, say, that $0 \leqq f'(x) \leqq \delta < 1$. Then:

$$\sum_{a<n\leqq b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)}dx + \frac{O(1)}{1-\delta},$$

with an absolute "implied" constant.

Proof

By inflating $O(1)$, wlog, a and b are integers and $b-a \geqq 100$. Indeed, we can also assume that $a=0$.

also Lec 8 14

By E-M version I (Lec 9 14 $R=0$), know:

$$\sum_{0\leqq n\leqq b} e^{2\pi i f(n)} = O(1) + O(1) + \int_0^b e^{2\pi i f(x)}dx$$

$$+ \int_0^b \left(x - [\![x]\!] - \tfrac{1}{2}\right) 2\pi i f' e^{2\pi i f} dx.$$

So,

$$\sum_{0<n\le b} e^{2\pi i f(n)} = O(1) + \int_0^b e^{2\pi i f(x)}\,dx$$

$$+\ 2\pi i \int_0^b \left(\sum_1^\infty \frac{\sin 2\pi n x}{-\pi n} \right) f' e^{2\pi i f}\,dx$$

purists may prefer an "m" here!

{ see Lec 9 p. 9 }

$$= O(1) + \int_0^b e^{2\pi i f(x)}\,dx$$

$$-\ 2i \sum_{n=1}^\infty \frac{1}{n} \int_0^b \sin(2\pi n x)\, f' e^{2\pi i f}\,dx$$

$$= O(1) + \int_0^b e^{2\pi i f(x)}\,dx$$

$$+ \sum_{n=1}^\infty \frac{1}{n} \int_0^b \left(e^{-2\pi i n x} - e^{2\pi i n x}\right) f' e^{2\pi i f}\,dx \,.$$

This last sum

$$= \sum_1^\infty \frac{1}{n}\left[\int_0^b e^{2\pi i (f(x) - n x)} f'\,dx - \int_0^b e^{2\pi i (f(x)+nx)} \cdot f'\,dx \right]$$

$$= \sum_{1}^{\infty} \frac{1}{n} \left(\frac{1}{2\pi i} \int_0^b \frac{f'(x)}{f'(x)-n} \, d e^{2\pi i [f(x) - nx]} \right.$$

$$\left. - \frac{1}{2\pi i} \int_0^b \frac{f'(x)}{f'(x)+n} \, d e^{2\pi i [f(x)+nx]} \right) .$$

We note herein that $n \neq 0$, f' is monotonic, and $0 \leqq f'(x) \leqq \delta < 1$.

Recall 2^{nd} mean value thm (Lec 22 5):

$$\int_A^B \underset{\text{monotonic}}{g(x)} \, d\underset{\alpha \text{ real and } C'[A,B]}{\alpha(x)} = g(A) \int_A^{\xi} d\alpha + g(B) \int_{\xi}^B d\alpha .$$

Complex α are treated via $\alpha = \alpha_1 + i\alpha_2$.

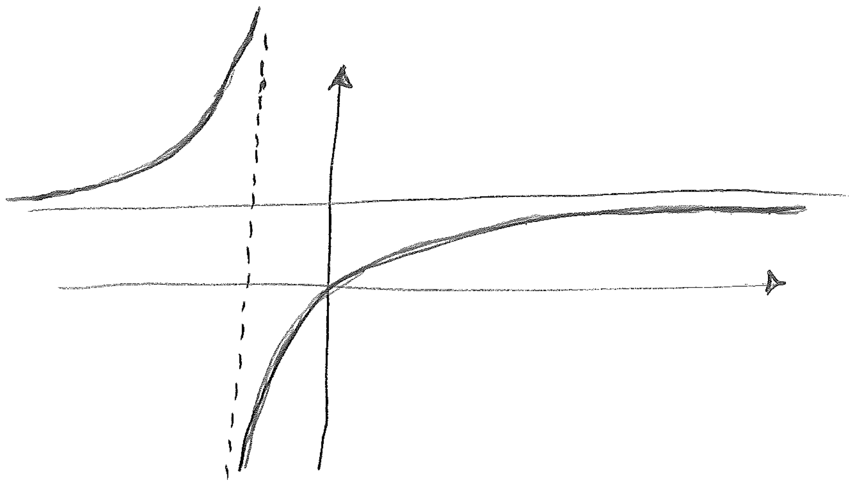
Also, notice that:

$$\left(\frac{u}{u+v}\right)' = \frac{(u+v)\cdot 1 - u\cdot 1}{(u+v)^2} = \frac{v}{(u+v)^2}$$

for $v \in \mathbb{Z} \sim \{0\}$.

This derivative has <u>fixed</u> <u>sign</u> for $u \neq -v$.

EG $\frac{u}{u+1}$



So, each $\dfrac{f'(x)}{f'(x) \pm n}$ is **monotonic** on $[0, b]$.

Look at:

$$\int_0^b \frac{f'(x)}{f'(x)+n} \, d e^{2\pi i (f(x)+nx)} \qquad (n \geq 1)$$

treated as $g_1 + i g_2$

and apply **2nd mean value thm** àla 6 bottom.

Get:

$\boxed{0 \leq f' \leq \delta}$

$$\int_0^b \frac{f'(x)}{f'(x)+n} \, d e^{2\pi i (f(x)+nx)}$$

$$= O\left(\frac{1}{n}\right).$$

Hence, on 6 top, we see that line 2 contributes

$$\sum_1^\infty \frac{1}{n} O\left(\frac{1}{n}\right) = O(1). //$$

We now look at

$$\int_0^b \frac{f'(x)}{f'(x)-n} \, d e^{2\pi i (f(x)-nx)} \qquad (n \geq 1)$$

treated as $\varphi_1 + i\varphi_2$

analogously. Since $0 \leq f'(x) \leq \delta < 1$,

$$n = 1 \Rightarrow O\left(\frac{1}{1-\delta}\right)$$

$$n \geq 2 \Rightarrow O\left(\frac{1}{n-1}\right)$$

$\Downarrow$

we see that on 6 top, in line 1, the collective contribution is:

$$O\left(\frac{1}{1-\delta}\right) + \sum_{n=2}^{\infty} \frac{1}{n} O\left(\frac{1}{n-1}\right)$$

$$= O\left(\frac{1}{1-\delta}\right). \quad //$$

Note how $n=1$ plays a special role in this portion of things.

Reviewing 5, we conclude that:

$$\sum_{0<n\leq b} e^{2\pi i f(n)} = O(1) + \int_0^b e^{2\pi i f(x)} dx$$

$$+ O(1) + O\left(\frac{1}{1-\delta}\right).$$

This proves p. 4 THM. ▮

Remark

One can obviously do $-\delta \leqq f'(x) \leqq 0$ in much the same way.

Still keeping f' monotonic, by splitting the original sum into 2 chunks, if need be, one can handle $-\delta \leqq f'(x) \leqq \delta$ as well.

Additional Remark

More general forms of p. 4 THM certainly suggest themselves. Cf., eg, Lec 22 5 + 9 bottom half. *

Theorems of this sort (with summations) arose in the work of van der Corput from 1921/22.

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* Also, Titchmarsh, Theory of $\zeta(s)$, near §4.10.

<u>THEOREM</u> (Hardy – Littlewood, Math. Zeit. 10 (1921).)

Given any $\sigma_0 \in (0, \frac{1}{10}]$, say. Given any $C > 1$.

Keep

$$\sigma_0 \leqq \sigma \leqq 2, \quad |t| \geqq 100.$$

Then:

$$\zeta(s) = \sum_{n \leqq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})$$

whenever $x > C\frac{|t|}{2\pi}$.

<u>Pf</u>

Note the formal similarity to ② (*).

Clearly, via a conjugation, WLOG $t \geqq 100$.

We immediately see by ② (*) that:

$$\zeta(s) = \sum_1^N n^{-s} - \frac{N^{1-s}}{1-s} + O(N^{-\sigma})$$

$$+ O(|s|) \int_N^\infty u^{-\sigma-1}\,du$$

$$= \sum_1^N n^{-s} - \frac{N^{1-s}}{1-s} + O(|s| N^{-\sigma})$$

$\{$since $\sigma_0 \leqq \sigma \leqq 2\}$.

Think of N as being giant and much greater than x. Let

$$A(v) = \sum_{x < n \leq v} n^{-it} = \sum_{x < n \leq v} e^{-2\pi i \left(\frac{t \ln n}{2\pi}\right)}.$$

Apply p. (4) + (10) line 2.

$$f(q) = -\frac{t \ln q}{2\pi} \qquad (q \geq x)$$

$$f'(q) = -\frac{t}{2\pi q} \qquad \text{monotonic}$$

$$0 \leq -f'(q) \leq \frac{t/2\pi}{x} < \frac{1}{c} < 1$$

So,

$$A(v) = \int_x^v e^{-2\pi i \left(\frac{t \ln q}{2\pi}\right)} dq + O(1)$$

$$= \int_x^v e^{-it \ln q}\, dq + O(1)$$

$$= \int_x^v q^{-it}\, dq + O(1)$$

$$= \frac{v^{1-it} - x^{1-it}}{1-it} + O(1).$$

But, now,

$$\sum_{x < n \leq N} n^{-\sigma - it} = \int_x^N v^{-\sigma}\, dA(v)$$

$$= v^{-\sigma} A(v) \Big]_x^N - \int_x^N A(v)(-\sigma) v^{-\sigma-1}\, dv$$

$$= \frac{A(N)}{N^{\sigma}} + \sigma \int_x^N \frac{A(v)}{v^{\sigma+1}}\, dv$$

$$= N^{-\sigma}\left[\frac{N^{1-it} - x^{1-it}}{1-it} + O(1)\right]$$

$$+ \sigma \int_x^N v^{-\sigma-1}\left[\frac{v^{1-it} - x^{1-it}}{1-it} + O(1)\right] dv$$

$$= \frac{N^{1-\sigma-it}}{1-it} - \frac{N^{-\sigma} x^{1-it}}{1-it} + O(N^{-\sigma})$$

$$+ \sigma \int_x^N \frac{v^{-\sigma-it}}{1-it}\, dv$$

$$- \sigma \int_x^N \frac{v^{-\sigma-1} x^{1-it}}{1-it}\, dv$$

$$+ O(1)(x^{-\sigma} - N^{-\sigma}) .$$

Remember that $N > x$.

Get:

$$\sum_{x<n\le N} n^{-s} = \frac{N^{1-\sigma-it}}{1-it} - \frac{N^{-\sigma}x^{1-it}}{1-it} + O(x^{-\sigma})$$

$$+ \frac{\sigma}{1-it}\int_x^N v^{-s}\,dv$$

$$- \frac{\sigma}{1-it}x^{1-it}\int_x^N v^{-\sigma-1}\,dv$$

$$= \frac{N^{1-\sigma-it}}{1-it} - \frac{N^{-\sigma}x^{1-it}}{1-it} + O(x^{-\sigma})$$

$$+ \frac{\sigma}{1-it}\left[\frac{N^{1-s}-x^{1-s}}{1-s}\right]$$

$$- \frac{\sigma}{1-it}x^{1-it}\left[\frac{N^{-\sigma}-x^{-\sigma}}{-\sigma}\right]$$

$$= \frac{N^{1-\sigma-it}}{1-it} - \underline{\underline{\frac{N^{-\sigma}x^{1-it}}{1-it}}} + O(x^{-\sigma})$$

$$+ \frac{\sigma}{1-it}\frac{1}{1-s}N^{1-s} - \frac{\sigma}{1-it}\frac{1}{1-s}x^{1-s}$$

$$+ \underline{\underline{\frac{x^{1-it}}{1-it}N^{-\sigma}}} - \frac{x^{1-it}}{1-it}x^{-\sigma}$$

{note the cancellation!}

$$= \frac{N^{1-s}}{1-it}\left[1+\frac{\sigma}{1-s}\right] + O(x^{-\sigma})$$

$$- \frac{\sigma}{1-it}\,\frac{1}{1-s}\,x^{1-s} - \frac{x^{1-it}}{1-it}\,x^{-\sigma}$$

$$= \frac{N^{1-s}}{1-it}\left[1+\frac{\sigma}{1-s}\right] + O(x^{-\sigma})$$

$$- \frac{x^{1-s}}{1-it}\left[\frac{\sigma}{1-s}+1\right]$$

$$\left\{ 1+\frac{\sigma}{1-s} = \frac{1-s+\sigma}{1-s} = \frac{1-it}{1-s} \right\}$$

$$= \frac{N^{1-s}}{1-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \ .$$

So, with our $N > x$, we get:

$$\boxed{\begin{aligned}\sum_{x<n\leqq N} n^{-s} &= \frac{N^{1-s}}{1-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \\ &= \int_x^N q^{-s}\,dq + O(x^{-\sigma})\end{aligned}} \ .$$

very natural term

Recall 11 bottom. Thus,

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \sum_{x < n \leq N} n^{-s}$$

$$- \frac{N^{1-s}}{1-s} + O(|s| N^{-\sigma})$$

by 15

$$= \sum_{n \leq x} n^{-s} + \underline{\underline{\frac{N^{1-s}}{1-s}}} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})$$

$$- \underline{\underline{\frac{N^{1-s}}{1-s}}} + O(|s| N^{-\sigma})$$

$$= \sum_{n \leq x} n^{-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})$$

$$+ O(|s| N^{-\sigma}) .$$

Now let $N \to \infty$ (to eliminate it).

Get:

$$\zeta(s) = \sum_{n \leq x} n^{-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}),$$

exactly as promised. ▮

Take $x > C\frac{t}{2\pi}$ with, say, $C \approx \pi$ and t big.

Apply p. 11 THM. Hence:

$$\zeta(\sigma+it) = \sum_{n \lesssim t} n^{-\sigma-it} - \frac{t^{1-s}}{1-s} + O(t^{-\sigma})$$

$$= \sum_{n \lesssim t} n^{-\sigma-it} - \frac{t^{1-\sigma} t^{-it}}{1-\sigma-it} + O(t^{-\sigma})$$

$$\Downarrow$$

$|\zeta(1+it)| \lesssim \ln t + O(1) + O(t^{-1})$ crudely;

$|\zeta(\frac{1}{2}+it)| \lesssim 2\sqrt{t} + O(1) + O(t^{-1/2})$ crudely.

Of course, by Lec 25 1 (h) [or Lec 24 16 line 4], we already know:

$$|\zeta(\tfrac{1}{2}+it)| = O(t^{\frac{1}{4}+\varepsilon}), \quad \text{each } \varepsilon.$$

This <u>hints</u> that p. 11 THM may be improved. <u>It</u> <u>can</u> be — but the argument is much harder.

$\boxed{\text{We only need p. 11 THM.}}$