Lecture 29

(Wed, 4 May)

We wish to prove

$$\psi(x) - x = \Omega_\pm(x^{\frac{1}{2}} \log\log\log x)$$

using the Ingham method from 1936, not the one in the book, pp. 92 – 100 (which follows Littlewood).

Ingham 1936 = Acta Arithmetica 1 (1936) 201 – 211

We use a variant in technique stressed by A. Selberg.

— — — — — —

(17)

Since $\psi(x) - x = \Omega_\pm(x^{\Theta - \delta})$ (Ing 90) à la Lec 21, we take $\Theta = \frac{1}{2}$ wlog. Ie, we assume RH.

Know:

Lec 13 p. (5)

(1) $\sum \frac{1}{|\rho|^2} < \infty$ for $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$

(2) $$\psi_1(x) = \frac{x^2}{2} - \frac{\zeta'(0)}{\zeta(0)}x + B - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{2k(2k-1)} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)}$$

$B = \frac{\zeta'(-1)}{\zeta(-1)} =$ unimportant, $x \geqq 1$

Lec 16 p. (15) Ing 73

3 For $x \geq 1+\delta_0$,

$$\psi^*(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} - \sum_{\rho} \frac{x^{\rho}}{\rho}$$

with some reasonable conditional convergence on the ρ-sum over compact subsets of $[1+\delta_0, \infty)$.

Also:

$$\frac{\zeta'(0)}{\zeta(0)} = \ln(2\pi) ;$$

Lec 18 35
Ing 77

$$\psi^*(x) \equiv \frac{\psi(x+0)+\psi(x-0)}{2} .$$

Let:

$$E_1(x) = -\frac{\zeta'(0)}{\zeta(0)} x + B - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{2k(2k-1)}$$

$$E(x) = -\frac{\zeta'(0)}{\zeta(0)} + \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} .$$

These fcns are clearly C^{∞} and we have

$$\boxed{E = E_1'} .$$

Obviously

$$\psi_1(x) - \frac{x^2}{2} - E_1(x) = -\sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} ;$$

$$\psi^*(x) - x - E(x) = -\sum_{\rho} \frac{x^{\rho}}{\rho} .$$

DEF (modified remainder terms)

$$P(x) = \psi^*(x) - x - E(x)$$

$$P_1(x) = \psi_1(x) - \frac{x^2}{2} - E_1(x)$$

Notice that $x + E(x)$ and $\frac{x^2}{2} + E_1(x)$ are C^∞, and that P and P_1 are piecewise C^1. In addition, P_1 is continuous.

FACT 1

For $x \geq 2$,

$$P_1(x) = b_1 + \int_2^x P(t)\,dt .$$

b_1 = some real constant.

Pf

$$\int_2^x P(t)\,dt = \int_2^x [\psi(t) - t - E(t)]\,dt \leftarrow \boxed{\text{by def of } \psi^* \text{ and baby integrals}}$$

$$= \int_2^x [\psi(t) - t - E_1'(t)]\,dt$$

$$= \text{constant} + \psi_1(x) - \frac{x^2}{2} - E_1(x)$$

$$\{ \psi_1(x) = \int_1^x \psi(v)\,dv , \quad \psi(v) = 0 \text{ for } v < 2 \} .$$

∎

By (2) bottom, we also have:

$$P_1(x) = -\sum_{\rho} \frac{x^{\frac{3}{2}+i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)}$$

$$\rho \equiv \tfrac{1}{2}+i\gamma, \text{ as usual} \quad .$$

The series is unif conv on $[2,\infty)$ compacta.

<u>FACT 2</u> (generalization of Dirichlet's "pigeon hole" principle) ← see Ing 94

Let $a_1, \ldots, a_N$ be real numbers. Let T_0 and $\delta_1, \ldots, \delta_N$ be positive. Then, there <u>exist</u> integers x_j and a number t_0 so that

$$|t_0 a_j - x_j| < \delta_j \quad \text{all } j \in [1,N]$$

$$T_0 \leqq t_0 \leqq T_0 \prod_{k=1}^{N} \left(1 + [\![\tfrac{1}{\delta_k}]\!]\right) \quad . \quad ★$$

The number t_0 can be taken to be a <u>multiple</u> of T_0.

$$\left\{ \text{Note:} \quad 1 + [\![\tfrac{1}{\delta_k}]\!] \leqq 1 + \frac{1}{\delta_k} \quad . \right\}$$

★ $[\![x]\!]$ can be replaced if desired by $[\![x]\!]' = \lim_{\varepsilon \to 0} [\![x-\varepsilon]\!]$. Useful for Ing 94.

PF

Classical pigeon hole principle:

$m+1$ things dumped into m boxes

$\Rightarrow$ some box contains $\underline{\geq 2}$ things.

Look at $[0,1)^N$. Partition this into

$$\underbrace{1+[\![\tfrac{1}{\delta_1}]\!]} \times \cdots \times \underbrace{1+[\![\tfrac{1}{\delta_N}]\!]} \leftarrow \text{total } m$$

subboxes (each semi-open; also disjoint). Note $\delta_j > 1 \Rightarrow 1+[\![\tfrac{1}{\delta_j}]\!] = 1 \Rightarrow$ no action in coordinate $\#j$.

Look at the $m+1$ points

$$(ta_1, \ldots, ta_N) \bmod 1 \quad \leftarrow \text{in } [0,1)^N$$

For $t = qT_0$, $0 \leq q \leq m$. Apply classical pigeon hole principle. Get obvious $0 \leq q' < q'' \leq m$, $t' < t''$,

$$|t''a_j - t'a_j - \text{integer}| < \frac{1}{1+[\![\tfrac{1}{\delta_j}]\!]},$$

each j. But $1+[\![u]\!] > u$ * when $u > 0$. Let $t_0 = t'' - t' = (q''-q')T_0$. This works. ∎

* For $[\![u]\!]'$, have $1+[\![u]\!]' \geq u$.

To continue, we now define

$$\boxed{F(v) = \alpha + \int_1^v \frac{P(e^u)}{\sqrt{e^u}}\,du}\ , \quad v \geq 1 ,$$

where α = a suitable constant (yet to be assigned).

Note

$$F(v) = \alpha + \int_e^{e^v} \frac{P(x)}{\sqrt{x}}\,\frac{dx}{x} \qquad \left\{ \begin{array}{l} x = e^u \\ u = \ln x \end{array} \right\}$$

$$= \alpha + \int_e^{e^v} x^{-\frac{3}{2}} P(x)\,dx$$

$$= \alpha + \int_e^{e^v} x^{-3/2}\,dP_1(x) \qquad \text{(R–S style)}$$

$$= \alpha + x^{-3/2} P_1(x) \Big]_e^{e^v} - \int_e^{e^v} P_1(x) \left(-\tfrac{3}{2}\right) x^{-\frac{5}{2}}\,dx$$

$$= \alpha + b_2 + e^{-\frac{3}{2}v} P_1(e^v) + \frac{3}{2} \int_e^{e^v} P_1(x) x^{-5/2}\,dx$$

we propose to plug in 4 top.

So, let's just look at:

$$x_0^{-3/2} P_1(x_0) + \frac{3}{2}\int_e^{x_0} P_1(x)\, x^{-5/2}\, dx$$

$$= \sum_{\gamma} \frac{(-1)}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)} \left[x_0^{-3/2} x_0^{\frac{3}{2}+i\gamma} + \frac{3}{2}\int_e^{x_0} x^{\frac{3}{2}+i\gamma}\, x^{-\frac{5}{2}}\, dx \right].$$

But,

$$\frac{3}{2}\int_e^{x_0} x^{\frac{3}{2}+i\gamma}\, x^{-\frac{5}{2}}\, dx = -\int_e^{x_0} x^{\frac{3}{2}+i\gamma}\, d(x^{-3/2})$$

$$= -\left[x^{\frac{3}{2}+i\gamma} x^{-\frac{3}{2}} \right]_e^{x_0} \quad \left(\text{by parts}\right)$$

$$+ \int_e^{x_0} x^{-\frac{3}{2}}\, d(x^{\frac{3}{2}+i\gamma})$$

$$= - x_0^{-3/2} x_0^{\frac{3}{2}+i\gamma}$$

$$+ e^{-3/2} e^{\frac{3}{2}+i\gamma}$$

$$+ (\tfrac{3}{2}+i\gamma)\int_e^{x_0} x^{-\frac{3}{2}}\, x^{\frac{1}{2}+i\gamma}\, dx$$

$$= -x^{-\frac{3}{2}} x^{\frac{3}{2}+i\gamma} + e^{i\gamma}$$

$$+ \left(\tfrac{3}{2}+i\gamma\right) \left[\frac{x^{i\gamma}}{i\gamma} \right]_e^x$$

$$\Downarrow$$

$$\frac{(-1)}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)} \text{ [big bracket on 7 top]}$$

$$= \frac{(-1)}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)} \left[e^{i\gamma} + \left(\tfrac{3}{2}+i\gamma\right)\left(\frac{x^{i\gamma}}{i\gamma} - \frac{e^{i\gamma}}{i\gamma}\right)\right]$$

$$= x^{i\gamma} \frac{(-1)}{(i\gamma)(\frac{1}{2}+i\gamma)} + \frac{e^{i\gamma}}{(i\gamma)(\frac{1}{2}+i\gamma)} - \frac{e^{i\gamma}}{(\frac{3}{2}+i\gamma)(\frac{1}{2}+i\gamma)}$$

$$\Downarrow$$

see 6 middle

$$F(v) = \alpha + b_3 - \sum_{\gamma} \frac{1}{(i\gamma)(\frac{1}{2}+i\gamma)} x^{i\gamma} \qquad x \equiv e^{v}$$

$$\Downarrow$$

it is now natural to

<u>declare</u> $\alpha = -b_3$

$$\Downarrow$$

$$\boxed{\begin{aligned} F(v) &= -b_3 + \int_1^v \frac{P(e^u)}{\sqrt{e^u}}\,du \\ F(v) &= -\sum_\gamma \frac{e^{i\gamma v}}{(i\gamma)(\frac{1}{2}+i\gamma)} \end{aligned}}$$

The sum over γ is unif conv for $v \in \mathbb{R}$.

[Remember $\gamma \in \mathbb{R}$.]

Recollection of baby Fourier analysis.

$$\tilde{f}(u) = \int_{-\infty}^{\infty} f(x)\, e^{-iux}\,dx$$

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(u)\, e^{iux}\,du$$

$$\int_{-\infty}^{\infty} \max(0, b-|x|)\, e^{-iux}\,dx = \left(\frac{\sin(\frac{u}{2}b)}{u/2}\right)^2$$

$$\max(0, b-|x|) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{u}{2}b}{\frac{u}{2}}\right)^2 e^{iux}\,du$$

$$\Downarrow$$

$$\max\left(0, 1-\frac{|x|}{2\pi}\right) = \int_{-\infty}^{\infty}\left(\frac{\sin\pi u}{\pi u}\right)^2 e^{iux}\,du$$

$$\max\left(0, 1-\frac{|u|}{2\pi}\right) = \int_{-\infty}^{\infty}\left(\frac{\sin\pi x}{\pi x}\right)^2 e^{iux}\,dx .$$

We let $\underline{k(x) = \left(\frac{\sin\pi x}{\pi x}\right)^2}$. Thus,

$$\tilde{k}(u) = \max\left(0, 1-\frac{|u|}{2\pi}\right) .$$

Crucial Facts:

$k(x) \geqq 0$, support of $\tilde{k} = [-2\pi, 2\pi]$

$k \in C^\infty(\mathbb{R})$, $\tilde{k} \geqq 0$,

$k, k' = O(x^{-2})$ for $|x| \geqq 1$.

$\{ k(x) = \text{Fejer kernel} \}$

WANT TO NOW CONSIDER:

$$\int_{t-A}^{t+A} \frac{P(e^v)}{\sqrt{e^v}}\, k[N(v-t)]\, dv$$

with $\begin{cases} A = \underline{\text{fixed}} \text{ positive integer}; \\ N = \text{positive integer (kept very large)}; \\ t = \text{real and very large}. \end{cases}$

Recall

$$F(v) = b_4 + \int_1^v \frac{P(e^u)}{\sqrt{e^u}}\, du \qquad \text{(9) box}$$

Need:

$$\int_{t-A}^{t+A} k[N(v-t)]\, dF(v) \qquad (R-5)$$

$$= k[N(v-t)]\, F(v)\Big]_{t-A}^{t+A}$$

$$- \int_{t-A}^{t+A} F(v)\, k'[N(v-t)]\, N\, dv$$

$$\left\{ \underline{\text{but } k(\pm NA) = 0} \right\}$$

$$= -N \int_{t-A}^{t+A} k'[N(v-t)]\, F(v)\, dv\,.$$

We can now substitute

$$F(v) = -\sum_{\gamma} \frac{e^{i\gamma v}}{(i\gamma)(\frac{1}{2}+i\gamma)} \qquad \text{à la } 9\,.$$

Do this manipulation <u>term-by-term</u>. Clearly there is no harm in taking $\gamma > 0$ and then getting the case $\gamma < 0$ by taking a conjugate.

Accordingly, with $\gamma > 0$, get:

$$\frac{N}{(i\gamma)(\frac{1}{2}+i\gamma)} \int_{t-A}^{t+A} k'[N(v-t)]\, e^{i\gamma v}\, dv$$

$$\left\{ w = N(v-t),\quad v = t + \frac{w}{N},\quad dv = \frac{dw}{N} \right\}$$

$$= \frac{N}{(i\gamma)(\frac{1}{2}+i\gamma)} \int_{-NA}^{NA} k'(w)\, e^{i\gamma(t+\frac{w}{N})}\, \frac{dw}{N}$$

$$= \frac{e^{i\gamma t}}{(i\gamma)(\frac{1}{2}+i\gamma)} \int_{-NA}^{NA} k'(w)\, e^{i\frac{\gamma}{N}w}\, dw$$

$O(w^{-2})$ for $|w|$ large

$$= \frac{e^{i\gamma t}}{(i\gamma)(\frac{1}{2}+i\gamma)} \left[\int_{-\infty}^{\infty} k'(w) e^{i\frac{\gamma}{N}w}\, dw + O\left(\frac{1}{NA}\right) \right].$$

Pause for a Second!!

$$\int_{-\infty}^{\infty} k'(w) e^{iqw}\, dw = \int_{-\infty}^{\infty} e^{iqw}\, dk(w)$$

$$= -\int_{-\infty}^{\infty} k(w)\, d_w(e^{iqw})$$

$$= -iq \int_{-\infty}^{\infty} k(w) e^{iqw}\, dw$$

$$= -iq \max\left(0, 1 - \frac{|q|}{2\pi}\right) \quad (10)$$

We therefore have:

$$(12) \text{ line } 1 = \sum_{\gamma > 0} \frac{e^{i\gamma t}}{(i\gamma)(\frac{1}{2}+i\gamma)} \left[-i\frac{\gamma}{N} \max\left(0, 1 - \frac{|\frac{\gamma}{N}|}{2\pi}\right) + O\left(\frac{1}{NA}\right) \right]$$

$$+ \text{ CONJUGATE}$$

$$= \sum_{0<\gamma<2\pi N} \frac{e^{i\gamma t}}{(i\gamma)(\frac{1}{2}+i\gamma)} \frac{(-i\gamma)}{N}\left(1-\frac{\gamma}{2\pi N}\right)$$

$$+ \sum_{\text{all } \gamma} \frac{1}{\gamma^2} O\left(\frac{1}{NA}\right)$$

$$+ \text{CONJUGATE}$$

$$= -\frac{1}{N} \sum_{0<\gamma<2\pi N} \frac{e^{i\gamma t}}{\frac{1}{2}+i\gamma}\left(1-\frac{\gamma}{2\pi N}\right)$$

$$+ \text{CONJUGATE}$$

$$+ O\left(\frac{1}{NA}\right) .$$

Here, note that

$$2\,\mathrm{Re}\left\{\frac{e^{i\phi}}{\frac{1}{2}+i\gamma}\right\} = 2\left\{\frac{\frac{1}{2}\cos\phi + \gamma\sin\phi}{\frac{1}{4}+\gamma^2}\right\}$$

$$= \frac{\cos\phi + (2\gamma)\sin\phi}{\frac{1}{4}+\gamma^2} .$$

So, we get

$$\underline{12 \text{ line } 1} = -\frac{1}{N}\sum_{0<\gamma<2\pi N} \frac{\cos(\gamma t) + 2\gamma\sin(\gamma t)}{\frac{1}{4}+\gamma^2}\left(1-\frac{\gamma}{2\pi N}\right)$$

$$+\ O\left(\frac{1}{NA}\right)$$

$$\Downarrow$$

$$\int_{t-A}^{t+A} \frac{P(e^v)}{\sqrt{e^v}}\, k\left[N(v-t)\right] dv$$

$$= O\left(\frac{1}{NA}\right)$$

$$-\frac{1}{N}\sum_{0<\gamma<2\pi N} \frac{\cos(\gamma t) + 2\gamma\sin(\gamma t)}{\frac{1}{4}+\gamma^2}\left(1-\frac{\gamma}{2\pi N}\right).$$

THE FINITENESS
OF THIS RANGE
IS CRUCIAL. 10 lines 5+7

This is Ingham's Trick.

Remember that "A" is fixed.

Also note that

$$\frac{1}{N} \sum_{0<\gamma<2\pi N} \frac{\cos(\gamma t)}{\frac{1}{4}+\gamma^2}\left(1-\frac{\gamma}{2\pi N}\right) = O\left(\frac{1}{N}\right).$$

And:

$$\frac{2\gamma}{\frac{1}{4}+\gamma^2} - \frac{2\gamma}{\gamma^2} = 2\gamma \cdot O(\gamma^{-4})$$
$$= O(\gamma^{-3})$$

$$\Downarrow$$

$$\frac{1}{N} \sum_{0<\gamma<2\pi N} \left[\frac{2\gamma}{\frac{1}{4}+\gamma^2} - \frac{2\gamma}{\gamma^2}\right] \sin(\gamma t)\left(1-\frac{\gamma}{2\pi N}\right)$$
$$= O\left(\frac{1}{N}\right).$$

$$\Downarrow$$

{ by recalling 11 }

$$\int_{t-A}^{t+A} \frac{P(e^v)}{\sqrt{e^v}}\, k[N(v-t)]\, dv$$

$$= O\left(\frac{1}{N}\right) - \frac{2}{N} \sum_{0<\gamma<2\pi N} \frac{\sin \gamma t}{\gamma}\left(1-\frac{\gamma}{2\pi N}\right)$$

$$= \Theta \frac{\mathcal{B}}{N} - \frac{2}{N} \sum_{0<\gamma<2\pi N} \frac{\sin \gamma t}{\gamma}\left(1-\frac{\gamma}{2\pi N}\right)$$

with $\mathcal{B} \gtrapprox 1$, $|\Theta| \lessapprox 1$.

NOTE THAT the last sum can be taken over $0<\gamma \lessapprox 2\pi N$, <u>if</u> desired.

Ingham <u>1936</u>	Here:
T	N
η	A
ω	t

$\rightsquigarrow$ $\boxed{\begin{array}{l} w = N(v-t) \\ v = t + \frac{w}{N} \end{array}}$

See Ingham p. 207 eq (11).

Ingham's statement is thus effectively:

$$\int_{-T\eta}^{T\eta} G\left(\omega + \frac{w}{T}\right) \mathcal{K}(w) \frac{dw}{T}$$

$$= O\left(\frac{1}{T}\right) - \frac{2}{T} \sum_{0<\gamma<2\pi T} \frac{\sin \gamma \omega}{\gamma}\left(1-\frac{\gamma}{2\pi T}\right).$$

As noted already on 1 (top), we plan to use a variant of Ingham's approach.

To continue, we now follow a direct approach with certain CHOICES. We'll hide the rough calculations which motivated these!!

Also, we will not seek an optimal constant in

$$\psi(x) - x = \Omega_{\pm}(x^{\frac{1}{2}} \log\log\log x)$$

on 1. That constant is conjectured to be arbitrarily large. [Ing; 2nd edition, p. xiv.]

Let $G =$ a sufficiently large constant. We hold G frozen.

$0 < \gamma_n \leq 2\pi N$

Apply 4 FACT 2 to get:

$$\left| t_0 \frac{\gamma_n}{2\pi} - (\text{integer}) \right| < \frac{\gamma_n}{2\pi G N}$$

$$T_0 \leq t_0 \leq T_0 \prod_{0 < \gamma_n \leq 2\pi N} \left(1 + \frac{2\pi G N}{\gamma_n}\right).$$

We'll select To in a few moments; it will be very big.

standard $|\Theta| \leqq 1$ meaning

Get:

$$t_0 \gamma_n = 2\pi(\text{integer}) + \Theta \frac{\gamma_n}{GN}$$

$$0 < \frac{\gamma_n}{GN} \leqq \frac{2\pi N}{GN} = \frac{2\pi}{G} < 10^{-6} \text{ (say)}.$$

$G > 2\pi(10^6)$

Put

$$\boxed{h = \frac{1000}{GN}}.$$

Let

$$\boxed{t_1 = t_0 + h}$$

àla Selberg

(similarly, at very end, consider $t_1 = t_0 - h$).

Clearly:

$$0 < h < 10^{-6} \quad (\text{since } N = \text{giant}).$$

$\boxed{0 < \gamma_n \leq 2\pi N}$

$$\gamma_n t_1 \equiv \gamma_n t_0 + \gamma_n h$$

$$= 2\pi(\text{integer}) + \underbrace{\Theta \frac{\gamma_n}{GN} + \frac{1000\,\gamma_n}{GN}}$$

but

$$\frac{999\,\gamma_n}{GN} \leq \underbrace{\Theta \frac{\gamma_n}{GN} + \frac{1000\,\gamma_n}{GN}} \leq \frac{1001\,\gamma_n}{GN} \leq \frac{2\pi(1001)}{G} < .002$$

$$\boxed{\text{and} \quad \frac{999\,\gamma_n}{GN} = \frac{999}{1000}\,\gamma_n(h)}$$

$$\Downarrow$$

$$\begin{bmatrix} \text{mod } 2\pi \text{ part} \\ \text{of } \gamma_n t_1 \end{bmatrix} \in \left[\frac{999}{1000}\gamma_n h\, ,\ .002\right)$$

$t_0 =$ NOT useful

$$\Downarrow$$

$$\sin(\gamma_n t_1) \geq \sin\left[\frac{999}{1000}\gamma_n h\right] > 0$$

KEY!!

$$\Downarrow$$

baby calc

$$\sin(\gamma_n t_1) \gtrapprox \frac{998}{1000}\gamma_n h > 0$$

$$\frac{\sin(\gamma_n t_1)}{\gamma_n} \gtrapprox \frac{998}{1000} h > 0$$

$$\sum_{0<\gamma_n \lessapprox 2\pi N} \frac{\sin(\gamma_n t_1)}{\gamma_n}\left(1-\frac{\gamma_n}{2\pi N}\right)$$

as on 17 top

$$\gtrapprox \sum_{0<\gamma_n \lessapprox 2\pi N} (.998) h \left(1-\frac{\gamma_n}{2\pi N}\right)$$

$$\gtrapprox \sum_{0<\gamma_n \lessapprox \pi N} (.49) h$$

$$\approx (.49) \frac{1000}{6N} \mathcal{N}(\pi N)$$

where $\mathcal{N}(v) = \#\{0<\gamma \lessapprox v, \underline{\text{with}} \text{ multiplicity}\}$

$$\approx \frac{v}{2\pi}\ln\frac{v}{2\pi} + O(v), \quad v \gtrapprox 2$$

{recall Lec 15 p. 29}

$$N = \text{giant}$$

$$\mathcal{N}(\pi N) = \frac{N}{2}\ln\frac{N}{2} + O(N)$$

$$\mathcal{N}(\pi N) = \frac{N}{2}\ln N + O(N)$$

also $\mathcal{N}(2\pi N) = N \ln N + O(N)$

$$\Downarrow$$

$$\sum_{0<\gamma_n \lesssim 2\pi N} \frac{\sin(\gamma_n t_1)}{\gamma_n}\left(1 - \frac{\gamma_n}{2\pi N}\right)$$

$$\gtrsim (.49)\,\frac{1000}{GN}\left[\frac{N}{2}\ln N + O(N)\right]$$

$$\gtrsim (240)\,\frac{1}{G}\ln N$$

{rather crudely!}

$$\Downarrow$$

$$\boxed{\text{(17) top at } t_1} \lesssim \frac{\beta}{N} - (480)\frac{\ln N}{GN} \lesssim -450\,\frac{\ln N}{GN}$$

for N suff large, G frozen, $\beta < \frac{\ln N}{G}$.

So, with our $t_0 \hookrightarrow t_1$ construction, it emerges that

$$\boxed{\int_{t_1 - A}^{t_1 + A} \frac{P(e^v)}{\sqrt{e^v}} k[N(v - t_1)]\, dv \leqq - \mathcal{C} \frac{\ln N}{N}}$$

holds for all N <u>suff. large</u> with $\mathcal{C} \simeq \frac{450}{G}$, <u>i.e.</u> some (fixed) appropriately small $\mathcal{C}$.

For convenience, we now just declare

$$T_0 = \prod_{0 < \gamma_n \leqq 2\pi N} \left(1 + \frac{2\pi G N}{\gamma_n}\right)$$

on p. 18 bottom. Recall too that $k(w) \geqq 0$.

We <u>seek</u> <u>to</u> transform the $\sim \mathcal{C} \frac{\ln N}{N}$ estimate into information about negative values of P.

$$T_0 \lesssim t_0 \lesssim T_0^2 \quad \text{on } 18 \text{ bot}$$

$$T_0 \lesssim \underline{t_1} \lesssim 2T_0^2$$

$$\frac{2\pi G N}{\gamma_n} \leq 1 + \frac{2\pi G N}{\gamma_n} \leq \frac{4\pi G N}{\gamma_n} \quad \text{trivially}$$

$$\prod_{0<\gamma_n \lesssim 2\pi N} \frac{2\pi G N}{\gamma_n} \lesssim T_0 \lesssim \prod_{0<\gamma_n \lesssim 2\pi N} \frac{4\pi G N}{\gamma_n}$$

$$(G)^{\mathcal{N}(2\pi N)} \prod_{0<\gamma_n \lesssim 2\pi N} \frac{2\pi N}{\gamma_n} \lesssim T_0 \lesssim (2G)^{\mathcal{N}(2\pi N)} \prod_{0<\gamma_n \lesssim 2\pi N} \quad (*)$$

---

positive

$$\log(T_0) = \left[N \log N + O(N)\right]\left[\ln G + \Theta \ln 2\right]$$

$$+ \sum_{0<\gamma_n \lesssim 2\pi N} \ln\left(\frac{2\pi N}{\gamma_n}\right)$$

---

best written as $\quad (\eta = \text{tiny})$

$$\int_{\eta}^{2\pi N} \log\left(\frac{2\pi N}{x}\right) d\mathcal{N}(x)$$

$$\boxed{\begin{array}{c}\xi(x) \neq 0 \\ \text{on } \mathbb{R}\end{array}}$$

$$\sum_{0<\gamma_n\leq 2\pi N} \ln\left(\frac{2\pi N}{\gamma_n}\right)$$

$$= 0 - 0 - \int_{\eta}^{2\pi N} \mathfrak{N}(x)\left(-\frac{1}{x}\right) dx$$ by parts

$$= \int_{\eta}^{2\pi N} \frac{\mathfrak{N}(x)}{x} dx$$

$$\int_{\eta}^{2\pi N} \frac{\mathfrak{N}(x)}{x} dx \sim \int_{2\pi}^{2\pi N} \frac{1}{2\pi} \ln\frac{x}{2\pi}\, dx$$

$$= \int_1^N \ln y \, dy$$

$$= N\ln N - N + 1$$

$$= N\ln N + O(N)$$

$$\boxed{\sum_{0<\gamma_n\leq 2\pi N} \ln\left(\frac{2\pi N}{\gamma_n}\right) = N\ln N + O(N)}$$

$$\Rightarrow \ln T_0 = N\log N \cdot (\ln G + \lambda)$$

with some $\lambda \in [1,2]$

G frozen + big

$$\Rightarrow \boxed{\ln\ln T_0 \sim \ln N} \quad \text{as } N \to \infty .$$

Needless to say, as $x \to \infty$,

$$\ln \ln x^\alpha \sim \ln \ln x \sim \ln \ln x^\beta$$

for any $0 < \alpha < 1 < \beta$. This has relevance for 24 line 1+2. EG $\ln \ln t_0 \sim \ln \ln t_1 \sim \ln \ln T_0$.

Looking at t_1 and 23 BOX, let

$$\left\| \; \mathfrak{M} = \inf_{[t_1 - A,\, t_1 + A]} \frac{P(e^v)}{\sqrt{e^v}} \; . \right\|$$

Get

$$\mathfrak{M} \int_{t_1 - A}^{t_1 + A} \underbrace{k[N(v - t_1)]}_{\text{non-neg}} \, dv \lessapprox -\mathcal{C} \frac{\ln N}{N}$$

$$\left\{ w = N(v - t_1), \quad v = t_1 + \frac{w}{N} \right\}$$

$$\mathfrak{M} \int_{-NA}^{NA} k(w) \frac{dw}{N} \lessapprox -\mathcal{C} \frac{\ln N}{N}$$

$$\left\{ \text{here } A \geqq 1, \; N \text{ giant}, \; \int_{-\infty}^{\infty} k(w)\, dw = 1 \; \text{cf. } ⑩ \text{ line } 3 \right\}$$

$$\Downarrow$$

$$\mathcal{M} \lessapprox -\mathcal{C} \ln N$$

$$\Downarrow$$

$$\mathcal{M} \lessapprox -\frac{99}{100}\mathcal{C} \ln\ln T_0$$ see 25 bottom

but

$0 < h < 10^{-6}$ 19

$A =$ pos. integer 11 (fixed)

$t_0 - 2A \lessapprox t_1 - A \lessapprox t_1 + A \lessapprox t_0 + 2A$ crudely

$\ln\ln\ln\,[e^{t_1 - A}, e^{t_1 + A}] \subseteq\approx \ln\ln\ln\,[e^{t_0 - 2A}, e^{t_0 + 2A}]$

$\ln\ln\ln\,[e^{v}\text{-range}]$

numerically asymptotic to

$\ln\ln\ln(e^{t_0})$

$= \ln\ln t_0$

---

by 26 top, get:

numerically asymptotic to

$$\boxed{\ln\ln T_0}$$

$\left\{\begin{array}{c}\text{but}\\ \text{see}\\ \text{line 2}\\ \text{above}\end{array}\right\} \Rightarrow$ Yes!

It follows that:

$$\frac{P(x)}{\sqrt{x}} = \Omega_-[\ln\ln\ln x]$$

holds with a constant of, say, $-\frac{98}{100}\,\mathcal{C}$.

One similarly establishes

$$\frac{P(x)}{\sqrt{x}} = \Omega_+[\ln\ln\ln x]$$

using $t_0 - h$ as t_1. See 19 second box!

On 2 + 3,

$$E(x) = O(1) \quad \text{for } x \geq 2 \quad \text{(obviously)}$$
$$P(x) = \psi^*(x) - x - E(x) \quad \text{by def}$$
$$P(x) = \psi(x) - x + O(\ln x).$$

Thus:

$$\psi(x) - x = \Omega_\pm(x^{\frac{1}{2}} \ln\ln\ln x). \qquad \boxed{\text{OK}}$$

We won't bother to explain the corresponding results for

$$\Pi(x) - li(x)$$

$\Pi(x)$ Lec 14 p. 7 8
Lec 21 p. 17

and, then, in regard to

$$\pi(x) - li(x) .$$

See Ingham (book) p. 103 theorem 35. One also recalls Ingham, p. 90 theorem 32! The issue of making things effective [e.g., by tampering with the kernel function $k(w)$], which was highlighted on p. 105, is quite interesting and was the subject of some famous work by S. Skewes (late 1930s – early 1950s).

> "The Skewes Number"
> (eg, see google.com)

Ingham, 2nd edition, page xiii (also viii) references this (work). The Selberg trick is also touched on there...

> See also the Hejhal–Odlyzko paper in the TURING centenary volume!!