Lecture 30 Synopsis

(Fri, 6 May 2016)

In this lecture — the last lecture — I thought that it would be fun to return to Euler and the PNT.

At the very end, to keep things more current, I (relented and) interjected a quick note about $S(T)$ [Lec 15, p. 26] which plays a role in ongoing computer tests of the Riemann Hypothesis.

EULER FIRST!

Recall

$$\mu(n) = \begin{cases} (-1)^{v}, & n = p_1 \cdots p_v, \text{ distinct } p_i \\ 0, & n \text{ not squarefree} \end{cases}$$

$[\mu(1) = 1]$. See Lec 17 p. 14. We have:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \operatorname{Re}(s) > 1$$

$$M(x) = \sum_{n \leq x} \mu(n), \quad x \geq 1, \; x \in \mathbb{R}.$$

Taking $s > 1$ and letting $s \to 1^+$, one would suspect (with Euler, 1748) that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 .$$

Of course, the "rub" here is that the <u>convergence</u> of the LHS is far from obvious!

NOTE. <u>IF</u> the LHS converges, its value must be 0. This follows immediately from Lec 21, p. 11 FACT 2(b) [with $\delta = \frac{\pi}{2}$]; see also Lec 25 p. 3 on "super Stolz".

In Lec 20 p. 28 $\boxed{D}$, I pointed out that it was known very early — using only ELEMENTARY techniques — that the following statements are equivalent:

$\boxed{\text{recall Lec 2 p. 2}}$

(i) $\psi(x) \sim x$ (i.e., the PNT);

(ii) $M(x) = o(x)$;

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$.

See Lec 20 pp. 33 – 40 for (i) $\Rightarrow$ (ii); then Lec 21 pp. 1 – 8 for (ii) ⇒ (i).

We now "finish" the job. ← And bring in Euler.

THM.

$$(i)(ii) \iff (iii).$$

I.E., Euler "already knew" the PNT.

Proof

Following Landau's, Handbuch der Primzahlen, 568 + 569, we set:

$$g(x) = \sum_{n \leqq x} \frac{\mu(n)}{n} \qquad (x \geqq 1)$$

$$f(x) = \sum_{n \leqq x} \frac{\mu(n) \ln n}{n} .$$

Recall that $|g(x)| < 1$ for $x \geqq 2$; Lec 20 p. 37.

ASSUME (iii): I.E., $g(x) = o(1)$.

Need to show $M(x) = o(x)$. Keep $x \geqq 2$. Get:

$$\sum_{n \leqq x} \mu(n) = 1 + \int_1^x v \, dg(v)$$

$$= 1 + \Big[v g(v)\Big]_1^x - \int_1^x g(v) \, dv$$

$$= 1 + x g(x) - 1 - \int_1^x g(v)\,dv$$

$$= x\,g(x) - \int_1^x g(v)\,dv \,.$$

But $|g(t)| \leqq \varepsilon$ for $t \geqq T_\varepsilon$. For $x \geqq T_\varepsilon$, we now get:

$$|M(x)| \leqq \varepsilon x + \int_1^{T_\varepsilon} |g(v)|\,dv + \int_{T_\varepsilon}^x \varepsilon\,dv$$

$$\leqq \varepsilon x + O(T_\varepsilon) + \varepsilon x$$

$$= 2\varepsilon x + O(T_\varepsilon)$$

$$\Downarrow$$

$$\limsup_{x \to \infty} \frac{|M(x)|}{x} \leqq 2\varepsilon$$

$$\Downarrow$$

$$M(x) = o(x) \,. \qquad \text{OK}$$

NEXT, ASSUME (i)(ii). <u>Must</u> <u>prove</u> $g(x) = o(1)$.

To achieve this, we require 2 lemmas.

Lemma 1

$$g(x) - \frac{f(x)}{\ln x} = O\left(\frac{1}{\ln x}\right).$$

$\boxed{x \geq 2}$

(So: $g(x) = o(1) \Leftrightarrow f(x) = o(\ln x)$.)

Pf

Recall that $\sum_{d|N} \mu(d) = \left\{ \begin{matrix} 1, & N=1 \\ 0, & N>1 \end{matrix} \right\}$ (Lec 20 p. 29).

Then that

$$1 = \sum_{n \leq x} \mu(n) [\![\tfrac{x}{n}]\!]$$

via $\sum_{nk \leq x} \mu(n)$ and the hyperbola method à la Lec 20 p. 37. This idea can be generalized using

$k = \frac{N}{n}$

$$1 = \sum_{nk \leq x} \frac{\mu(n)}{nk} \longleftarrow \boxed{\sum_{N \leq x} \frac{1}{N} \left\{ \sum_{n|N} \mu(n) \right\}}$$

$$\Downarrow$$

$$1 = \sum_{n \leq x} \frac{\mu(n)}{n} \left\{ \sum_{k \leq \frac{x}{n}} \frac{1}{k} \right\}$$

$$= \sum_{n \leq x} \frac{\mu(n)}{n} \left[\ln\left(\frac{x}{n}\right) + \gamma + O\left(\frac{n}{x}\right) \right]$$

Lec 18 p. 40 bottom

$$= \sum_{n \leq x} \frac{\mu(n)}{n} (\ln x - \ln n)$$

$$+ \gamma \sum_{n \leq x} \frac{\mu(n)}{n}$$

$$+ O(1) \sum_{n \leq x} \frac{1}{n} \frac{n}{x}$$

$$\Downarrow$$

$$1 = (\ln x) g(x) - \sum_{n \leq x} \frac{\mu(n) \ln n}{n} + \gamma g(x) + O(1)$$

$|g(x)| \leq 1$

$$\Downarrow$$

$$1 = (\ln x) g(x) - f(x) + O(1) \quad \text{by } 3$$

$$\Downarrow$$

$$(\ln x) g(x) - f(x) = O(1)$$

$$\Downarrow$$

$$g(x) - \frac{f(x)}{\ln x} = O\left(\frac{1}{\ln x}\right). \quad \blacksquare$$

<u>Must</u> <u>now</u> <u>seek</u> <u>to</u> <u>prove</u> <u>$f(x) = o(\ln x)$</u>.

Lemma 2

For $x \geqq 2$, we have:

$$\sum_{n \leqq x} \frac{1}{n} g\left(\frac{x}{n}\right) = 1 .$$

Pf

On 5 middle, we saw that

$$1 = \sum_{nk \leqq x} \frac{\mu(n)}{nk}$$

think hyperbolas

{sum the other direction!}

$$= \sum_{k \leqq x} \frac{1}{k} \left\{ \sum_{n \leqq \frac{x}{k}} \frac{\mu(n)}{n} \right\}$$

$$= \sum_{k \leqq x} \frac{1}{k} g\left(\frac{x}{k}\right) \quad \text{(by def of } g\text{)}.$$

▥

Recall that

$$-\mu(N)\ln N = \sum_{k\ell = N} \Lambda(k)\mu(\ell)$$

by Lec 20 p. 33. Accordingly

$$f(x) = \sum_{N \leq x} \frac{\mu(N)\ln N}{N} \qquad \text{à la } 3$$

$$= -\sum_{k\ell \leq x} \frac{\Lambda(k)\mu(\ell)}{k\ell}$$

{ think hyperbolas }

$$= -\sum_{k \leq x} \frac{\Lambda(k)}{k} \left\{ \sum_{\ell \leq \frac{x}{k}} \frac{\mu(\ell)}{\ell} \right\}$$

$$\boxed{f(x) = -\sum_{k \leq x} \frac{\Lambda(k)}{k}\, g\left(\frac{x}{k}\right)}$$

recall
$\Lambda(1) = 0$
$\Lambda(p^m) = \ln p, \; m \geq 1$.

Integration by parts with $\psi(v)$ is cumbersome since $g(*)$ is <u>not</u> continuous. <u>Avoid it!!</u>

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To prove $f(x) = o(\ln x)$ as $x \to \infty$, there is no loss of generality in taking $x =$ integer. See 3.

We have

$$\psi(v) = \sum_{n \leqq v} \Lambda(n)$$

and $\psi(v) = 0$, $v < 2$. Since we have assumed (i)(ii), it makes sense to write

$$\psi(v) = v + v\varepsilon(v), \quad v > 0$$

with

$$\lim_{v \to \infty} \varepsilon(v) = 0$$

and

$$\varepsilon(v) \equiv -1 \quad \text{for } 0 < v < 2 .$$

For convenience, declare:

$$\varepsilon(0) = 0 .$$

We still have $\psi(0) = 0 + 0\,\varepsilon(0)$.

CHOOSE M so that

$$|\varepsilon(v)| \leqq M, \quad \text{all } v \geqq 0 .$$

In 8 box, notice that (x = integer)

$$\sum_{k \leq x} \frac{\Lambda(k)}{k} g\left(\frac{x}{k}\right)$$

$$= \sum_{k \leq x} \frac{\psi(k) - \psi(k-1)}{k} g\left(\frac{x}{k}\right)$$

$$\{ \psi(k) = k + k\,\varepsilon(k) \}$$

$$= \sum_{1 \leq k \leq x} \frac{1 + k\,\varepsilon(k) - (k-1)\,\varepsilon(k-1)}{k} g\left(\frac{x}{k}\right)$$

$$= \sum_{k=1}^{x} \frac{1}{k} g\left(\frac{x}{k}\right)$$

$$+ \sum_{k=1}^{x} [\varepsilon(k) - \varepsilon(k-1)]\, g\left(\frac{x}{k}\right)$$

$$+ \sum_{k=1}^{x} \frac{\varepsilon(k-1)}{k} g\left(\frac{x}{k}\right)$$

$$= 1 \quad \leftarrow \text{by Lemma 2 on } 7$$

$$+ \sum_{k=1}^{x} \varepsilon(k)\, g\left(\frac{x}{k}\right) - \sum_{k=2}^{x} \varepsilon(k-1)\, g\left(\frac{x}{k}\right)$$

$\varepsilon(0) = 0$

$$+ \sum_{k=2}^{x} \frac{\varepsilon(k-1)}{k} g\left(\frac{x}{k}\right)$$

$$= 1 + \sum_{k=1}^{x} \varepsilon(k) g\left(\frac{x}{k}\right) - \sum_{\ell=1}^{x-1} \varepsilon(\ell) g\left(\frac{x}{\ell+1}\right)$$

$$+ \sum_{\ell=1}^{x-1} \frac{\varepsilon(\ell)}{\ell+1} g\left(\frac{x}{\ell+1}\right)$$

$$= 1 + \sum_{k=1}^{x} \varepsilon(k) g\left(\frac{x}{k}\right) - \sum_{\ell=1}^{x} \varepsilon(\ell) g\left(\frac{x}{\ell+1}\right)$$

$$+ \sum_{\ell=1}^{x} \frac{\varepsilon(\ell)}{\ell+1} g\left(\frac{x}{\ell+1}\right)$$

IF WE DECLARE
$g(v) \equiv 0$ for $v < 1$

$$= 1 + \sum_{k=1}^{x} \varepsilon(k) \left[g\left(\frac{x}{k}\right) - g\left(\frac{x}{k+1}\right) \right]$$

$$+ \sum_{\ell=1}^{x} \frac{\varepsilon(\ell)}{\ell+1} g\left(\frac{x}{\ell+1}\right)$$

$$(-f(x)) = 1 + \Sigma_1 + \Sigma_2 \text{, say.}$$

Fix any tiny $\varepsilon > 0$. Let $|\varepsilon(k)| \lessapprox \varepsilon$ for all $k \gtrapprox K_\varepsilon + 1$. Keep $x \gtrapprox K_\varepsilon + 100$.

Notice that:

$$|\Sigma_2| \leq \sum_{\ell=1}^{K_\varepsilon} \frac{M}{\ell+1} \cdot 1 \qquad \boxed{|g(v)| \leq 1}$$

$$+ \sum_{K_\varepsilon+1 \leq \ell \leq x} \frac{\varepsilon}{\ell+1} \cdot 1$$

$$\leq C_\varepsilon + \varepsilon \ln x + O(1)$$

$$\leq C'_\varepsilon + \varepsilon \ln x \, .$$

To estimate Σ_1, notice that

$$g\left(\frac{x}{k}\right) - g\left(\frac{x}{k+1}\right) = \sum_{\frac{x}{k+1} < n \leq \frac{x}{k}} \frac{\mu(n)}{n}$$

by p. 3 and 11 lines 5+6. Accordingly,

$$\left|g\left(\frac{x}{k}\right) - g\left(\frac{x}{k+1}\right)\right| \leq \left\{ \begin{array}{c} 2 \\ \\ \displaystyle\sum_{\left(\frac{x}{k+1}, \frac{x}{k}\right]} \frac{1}{n} \end{array} \right\}.$$

In the foregoing, $\mathbb{Z} \cap \left(\frac{x}{k+1}, \frac{x}{k}\right]$ may well be empty. Empty sums are zero.

Continue to keep $x \geq K_\varepsilon + 100$. By def,

$$\Sigma_1 = \sum_{k=1}^{x} \varepsilon(k) \left[g\left(\frac{x}{k}\right) - g\left(\frac{x}{k+1}\right) \right]. \quad (11)$$

Hence:

$$|\Sigma_1| \leq \sum_{1 \leq k \leq K_\varepsilon} \mathfrak{M} \cdot 2$$

$$+ \sum_{K_\varepsilon + 1 \leq k \leq x} \varepsilon \left(\sum_{\left(\frac{x}{k+1}, \frac{x}{k}\right]} \frac{1}{n} \right)$$

||| THE UNION OVER k WITH $\left(\frac{x}{k+1}, \frac{x}{k}\right]$ IS DISJOINT |||

$$\leq 2\mathfrak{M} K_\varepsilon + \varepsilon \sum_{n \leq \frac{x}{K_\varepsilon + 1}} \frac{1}{n}$$

$$\leqq 2\mathfrak{M} K_\varepsilon + \varepsilon\left(\ln x + O(1)\right)$$

$$\Downarrow$$

$$|\Sigma_1| \lessapprox C_\varepsilon'' + \varepsilon \ln x \, .$$

Upon reviewing 8 box, 10, (11,) 12 (middle), and line 2 above, we see that

$$|F(x)| \lessapprox 1 + C_\varepsilon''' + 2\varepsilon \ln x$$

for $x \gtrapprox K_\varepsilon + 100$. Since $\varepsilon > 0$ was arbitrary on 11, we get $f(x) = o(\ln x)$, as required on 6 bottom. By Lemma 1, on 5, we therefore have $g(x) = o(1)$, as needed.

This completes the proof of THM on 3. ∎

(I couldn't resist)

Finally, a very brief comment about $S(T)$!

vis à vis verifications of RH

We recall that:

$$S(T) = \frac{1}{\pi} \arg \zeta\left(\tfrac{1}{2} + iT\right)$$

àla Lec 15 26 (middle). Cf. also here Lec 15 pp. 22 - 25 and Lec 27 3 (FACT) - 6 (top).

When $T \neq$ all γ, we have

$$S(T) = -\frac{1}{\pi} \operatorname{Im} \int_{\frac{1}{2}}^{\infty} \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma .$$

If $T =$ some γ, one defines $S(T)$ by right continuity to preserve Lec 15 26 (lines 6+7).

One knows that:

$$\boxed{S(T) = O(\ln T)} .$$

THM

Define $\mathcal{L}og\, \zeta(s)$ in the usual <u>up and across</u> way beginning at point $A \in \mathbb{R}$, $A \gg 1$. Keep $T \geqq 2$, $T \neq$ all γ, and $-1 \leqq \sigma \leqq 2$. We then have:

$$\mathcal{L}og\, \zeta(s) = O(\ln T) + \sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \mathrm{Log}(s-\rho).$$

Here $s = \sigma + iT$ and Log = the <u>standard</u> principal value.

Pf

Review Lec 27 3 (bot) – 5 and Lec 15 22 (bot).

$$\mathcal{L}og\, \zeta(s) = 0 + \int_{\infty}^{\sigma} \frac{\zeta'}{\zeta}(u+iT)\,du$$

$$= \int_{\infty}^{2} O(2^{-u})\,du + \int_{2}^{\sigma} \frac{\zeta'}{\zeta}(u+iT)\,du$$

$$= O(1) + \int_{2}^{\sigma} \Big[O(\ln T) + \sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \frac{1}{u+iT-\rho} \Big] du$$

Lec 15 13 8

$$= O(1) + O(\ln T)$$

$$+ \sum_{\substack{\rho \\ |\gamma - T| \leq 1}} \left[\mathrm{Log}(s-\rho) - \mathrm{Log}(2+iT-\rho) \right]$$

$$= O(\ln T) + \sum_{\substack{\rho \\ |\gamma - T| \leq 1}} \mathrm{Log}(s-\rho)$$

$\{$ since $0 \leq \mathrm{Re}(\rho) \leq 1$ $\}$.

For $T \neq$ all γ, it is customary to define:

$$S_1(T) = -\frac{1}{\pi} \mathrm{Re} \int_{\frac{1}{2}}^{\infty} \left(\sigma - \frac{1}{2}\right) \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma .$$

For $T =$ some γ, use right continuity (cf. page 18 box).

It is easy to see that $S_1(T)$ is well-defined and nicely continuous insofar as T remains away from all γ. Lec 15, pp. 22 (bot) – 23 (top).

Notice here that:

$$S_1 = -\frac{1}{\pi} \operatorname{Re} \int_{\frac{1}{2}}^{\infty} (\sigma - \tfrac{1}{2})\, d \operatorname{Log} \Xi(\sigma + iT)$$

$$= -\frac{1}{\pi} \operatorname{Re} \left[0 - 0 - \int_{\frac{1}{2}}^{\infty} \operatorname{Log} \Xi(\sigma + iT)\, d\sigma \right]$$

$$\left\{ \begin{array}{l} \text{by integ by parts and} \\ \qquad \operatorname{Log} \Xi(s) = O(2^{-\sigma}), \quad \sigma \geqq 2 \end{array} \right\}$$

$$\Downarrow$$

$$\boxed{S_1(T) = \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \ln |\Xi(\sigma + iT)|\, d\sigma} \, .$$

This is for $T \neq$ all γ. By baby analysis and

$$\int_0^1 |\ln u|\, du < \infty ,$$

the box remains true for all $T \geqq 2$ (i.e., there is good continuous behavior).

Notice further (for $T \neq$ all γ) that:

$$S_1(T) = \frac{1}{\pi} \operatorname{Re} \int_{\frac{1}{2}}^{\infty} \mathcal{L}og\, \zeta(\sigma + iT)\, d\sigma \qquad \text{(18) line 3}$$

$O(2^{-\sigma}) \quad \sigma \geqq 2$

$$\Rightarrow \quad \underline{S_1'(T)} = \frac{1}{\pi} \operatorname{Re} \int_{\frac{1}{2}}^{\infty} \frac{\zeta'}{\zeta}(\sigma + iT)\, i\, d\sigma \qquad \left\{ \begin{matrix} \text{Leibnitz's} \\ \text{rule} \end{matrix} \right\}$$

$$= -\frac{1}{\pi} \operatorname{Im} \int_{\frac{1}{2}}^{\infty} \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma$$

$$= S(T) \quad \text{by (15) middle}$$

$\left\{ \begin{matrix} \text{with good uniform convergence} \\ \text{locally wrt } T \end{matrix} \right\}$

$$\Downarrow$$

$$\boxed{\begin{aligned} &S_1'(T) = S(T), \quad T \neq \text{all } \gamma \\ &S_1(T) = c_1 + \int_2^T S(u)\, du, \quad \text{all } T \geqq 2 \end{aligned}}$$

continuous wrt T

THM (very fundamental)

In the zero-counting formula

$$N(T) = \underbrace{\frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + O\left(\frac{1}{T}\right)}_{C^\infty} + S(T)$$

à la Lec 15 22 + 26, we have

$$S(T) = O(\ln T), \quad \int_2^T S(u)\,du = O(\ln T).$$

Pf.

Only the last assertion remains to be proved. WLOG $T \neq$ all γ. Apply 18 box. Get:

$$S_1(T) = \frac{1}{\pi}\int_2^\infty \ln|\zeta(\sigma+iT)|\,d\sigma + \frac{1}{\pi}\int_{\frac{1}{2}}^2 \ln|\zeta|\,d\sigma$$

$O(2^{-\sigma})$

$$= O(1) + \frac{1}{\pi}\int_{\frac{1}{2}}^2 \ln|\zeta(u+iT)|\,du$$

$$= O(1) + \frac{1}{\pi}\int_{\frac{1}{2}}^{2} \Big[O(\ln T) + \sum_{\substack{\rho \\ |\gamma - T| \leq 1}} \ln|u + iT - \rho| \Big] du$$

by p. 16 THM

observe too that we have

$$|u-\beta| \leq |(u-\beta) + i(T-\gamma)| \leq 3$$

$$= O(1) + O(\ln T) + O(\ln T)$$

$$= O(\ln T). \qquad \blacksquare$$

It is hardly surprising that the implied constants for $\ln T$ in p. 20 THM can be made explicit.

The relation $\int_2^T S(u)\,du = O(\ln T)$ qualitatively states that the <u>average</u> value of $S(u)$ is $\underline{\underline{0}}$.

These last 2 points are important. In the early 1950s, Alan Turing used these facts to develop a numerical criterion (now known as Turing's Law) by which the Riemann Hypothesis on interval $[T_1, T_2]$ can be checked simply by locating enough (that is to say, a requisite number of) sign-changes for the REAL-VALUED function

$$\xi\left(\tfrac{1}{2}+it\right) \quad \text{or, better,} \quad \frac{G\left(\frac{1}{2}+it\right)}{\left|G\left(\frac{1}{2}+it\right)\right|}\,\zeta\left(\tfrac{1}{2}+it\right)$$

in an interval slightly bigger than $[T_1, T_2]$. *

recall Lec 23, p. 3 lines 1-3 and (e); Lec 15, p. 26 (line -3)

OF COURSE, there is nothing to guarantee in advance that the requisite number will be found. One simply has to try!!!

The point here is 4-fold:

(A) if the requisite number is reached (by the machine), one is assured by Turing's theorem that all zeros are accounted for, and that there are none having $\mathrm{Re}(s) \neq \frac{1}{2}$;

(B) there is no need to check RH for $T < T_1$ first;

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* with special attention paid to the pattern near T_1, T_2

(C) there is no need to compute any $\mathfrak{I}$-values with $\mathrm{Re}(s) \neq \frac{1}{2}$;

(D) there is no need to "zero in" on the crossings in $[T_1, T_2]$ attached to the sign-changes detected by the machine.

To understand why Turing's Law is at least believable, simply pretend that one somehow knew that $S(t)$ was exactly zero in some short intervals centered at T_1 and T_2. See p. 20 THEOREM [the formula for $N(T)$] and ponder the logical consequences which ensue!

For further details about Turing and $\mathfrak{I}(s)$, see the paper of Hejhal and Odlyzko in the Turing Centenary volume "Alan Turing: His Work and Impact" published by Elsevier. The story is a VERY interesting one. ← with links to Lec 29 p. 29 and S. Skewes

Turing's Law is used in all modern computational work aimed at verifying the RH. When the approach is successful, the zeros of $\mathfrak{I}$ in the range $[T_1, T_2]$ are also known to be simple. Cf. (A) on 22.