PARTIAL DIARY ENTRY for
Lecture 6
(5 Feb 2016)

I. Move was discussed on infinite products, especially Weierstrass M-test. E.g., for
\[ \prod_{n=1}^{\infty} \cos \left( \frac{\pi}{n} \right). \]

II. \[ \prod_{n=1}^{\infty} (1 + b_n(z)) \] with \[ |b_n(z)| \leq \gamma < 1 \] on \( E \). We stressed that when \( \text{unif conv} \) holds, you get both
\[ \frac{p_N(z)}{p(z)} \to 1 \quad \text{AND} \quad p_N(z) \to p(z) \]
since \( c_1 < |p(z)| < c_2 \) on \( E \). Hence:
\[ b_n(z) \text{ continuous } \Rightarrow p(z) \text{ continuous on } E \]
\[ b_n(z) \text{ analytic } \Rightarrow p(z) \text{ analytic (in the usual "on compacta" sense associated with Weierstrass convergence thm)} \]
(III) Cauchy products. I remarked that

\[ A = \sum_{n=0}^{\infty} a_n, \text{ abs conv} \quad (a_n \in \mathbb{C}) \]

\[ B = \sum_{n=0}^{\infty} b_n, \text{ abs conv} \quad (b_n \in \mathbb{C}) \]

\[ \downarrow \]

\[ AB = \sum_{n=0}^{\infty} \left( \sum_{i+k=n} a_i b_k \right) = \sum_{n=0}^{\infty} c_n, \text{ abs conv} \quad (\text{Same proof}) \]

(as in \( \mathbb{R} \))

(IV) \( \text{Re}(z) > 1 \) say. Use (III). Take \( T \geq 2 \).

\[ \prod_{p \leq T} \frac{1}{1 - p^{-z}} = \prod_{p \leq T} \left\{ 1 + p^{-z} + p^{-2z} + \cdots \right\} \]

\[ = \sum_{n \geq 1} n^{-z} \]

\( n \) is factorizable into primes which are all \( \leq T \)

\( \text{includes all } n \leq T \)

\[ \text{obviously} \]

Notice too:

\[ |p^{-z} + p^{-2z} + \cdots| \leq p^{-x} + p^{-2x} + \cdots = \frac{p^{-x}}{1 - p^{-x}} = \frac{1}{p^x - 1} \]
\[ 1 - p^z + p^{2z} + \cdots \leq \frac{1}{2^x-1} \quad \text{for all } p > 0. \]

Hence, for \( x \geq 1 + \delta \), we have a good \( \mathcal{L} \) if

\[ \frac{1}{2^{1+\delta}-1} < \frac{1}{2^x-1}. \]

All of our earlier theorems about infinite products apply when we opt to let \( T \to \infty \). All is well.

We clearly get: (Euler)

\[ \prod_{p} \frac{1}{1-p^z} = \mathcal{L}(z) = \sum_{n=1}^{\infty} n^{-z} \]

with uniform absolute convergence on each closed half-plane \( \Re(z) \geq 1 + \delta \).

In particular, since LHS is nonzero (by def of conv infinite product), we get:

\[ \mathcal{L}(z) \neq 0 \quad \text{for } \Re(z) > 1. \]
I drew attention to Euler's identity

\[ \sum_{n=1}^{\infty} f(n) = \prod_p \left\{ 1 + f(p) + f(p^2) + f(p^3) + \cdots \right\} \]

in Ingham p. 16 — under the assumption that

\[ \sum_{1}^{\infty} |f(n)| < \infty \]

and \( f \) is multiplicative

\[ \begin{cases} f(1) = 1 \\ f(mn) = f(m)f(n) \text{ if } (m,n)=1 \end{cases} \]

(Read proof there!)

Defined a natural branch of \( \log J(z) \) on \( \Re(z) > 1 \) by writing

\[ \log J(z) = \sum_{n=2}^{\infty} \frac{A(n)}{\ln n} n^{-z} \]

This is NOT in general \( \log J(z) \)!

For \( z = x > 1 \), however, one readily checks

\[ \log J(z) = \ln J(x) \]

RECALL \( J(x) = \sum_{n} n^{-x} > 1 \).
Clarification: (regarding Log $J(x)$)

Recall $\circledast$ last line and $\boxed{3}$ lines 5–7. "All is well" because we are using Weierstrass $M$-test with

$$M_p = \frac{1}{p^{1+\delta} - 1} \quad \text{(For } x \geq 1 + \delta)$$

Our $\sum \log (1+b_p(x))$ for the "infinite product equivalence theorem" in Lecture 5 is

$$\sum_p \log (1-p^{-2\pi}) = \sum_p \frac{1}{\log p} n^{-2\pi}$$

This infinite series converges to some $S(z)$. The series is just

$$\sum_p \left\{ p^{-2\pi} + \frac{1}{3} p^{-3\pi} + \ldots \right\} \equiv \sum_n \frac{1(n)}{\ln n} n^{-2\pi} \quad \text{nice analytic fcn}$$

(with good abs conv). As in Lecture 5, we always have:

$$P(z) = \exp \left\{ S(z) \right\} \circledast$$

So, here, $\boxed{3}$ BOX,

$$J(z) = P(z) = \exp \left\{ S(z) \right\} \circledast$$

I.e., there is no question $S(z) = \text{some branch of log } J(z)$ on $\{ \text{Re}(z) > 1 \}$.

Clearly, by inspection, $S(x) > 0$ for $x > 1$. Hence, we do have:

$$S(x) \equiv \log J(x) = \text{Log } J(x) \circledast \quad (x > 1)$$
$\frac{J(z)}{J(\xi)} = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\xi}}, \quad \text{Re}(\xi) > 1$.

(by Weierstrass' conv thm for analytic funs.)

**Thm (Hadamard)**

For $x > 1$, $y \neq 0$,

$$|J(x)|^3 |J(x+iy)|^4 |J(x+2iy)| \leq 1.$$

**pf**

Take $\ln$. We want:

$$3\ln |J(x)| + 4\ln |J(x+iy)| + \ln |J(x+2iy)| \geq 0.$$

But,

$$\ln |J(z)| = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-x} \cos(y \ln n) \uparrow \Re\{n^{-x-iy \ln n}\}$$

by $\circ \{x > 1\}$.
Since, for any $\theta \in \mathbb{R}$,

$$3 + 4 \cos \theta + \cos(2\theta)$$

$$= 3 + 4 \cos \theta + 2 \cos^2 \theta - 1$$

$$= 2 + 4 \cos \theta + 2 \cos^2 \theta$$

$$= 2(1 + \cos \theta)^2 \geq 0,$$

and $\frac{A(n)}{\ln n} \geq 0$, a trivial substitution now gives what we claimed. \[\Box\]

Corollary (Hadamard's famous result)

$$\mathcal{S}(1 + iy) \neq 0 \text{ if } y \neq 0.$$ 

Pf

Suppose we had $\mathcal{S}(1 + iy) = 0$ at some $y \neq 0$. \[\downarrow\]
\[
\left( x - 1 \right)^3 \left| F(x) \right|^3 \left| \frac{F(x + iy)}{F(x - iy)} \right|^4 \left| F(x + 2iy) \right| \geq \frac{1}{x - 1}
\]

\[
\left\{ (x-1) \left| F(x) \right|^3 \left| \frac{F(x+iy) - F(1+iy)}{x-1} \right|^4 \left| F(x+2iy) \right| \right\} \left| \frac{1}{x - 1} \right| \leq 1
\]

let \( x \to 1^+ \)

\[
\left| \frac{1}{3} \left| F(1+iy) \right|^4 \left| F(1+2iy) \right| \right| \geq \infty \Rightarrow
\]

\( \sqrt[3]{\text{a contradiction}} \) since \( F(z) \) is nicely analytic for \( \text{Re}(z) > 0 \) except at \( z = 1 \)}. !!
Theorem (essentially like Ingham p. 27)

Let $0 < \delta \leq 1$. We then have:

(A) $|\zeta(x+iy)| \geq A \ln|y|$ for $x \geq 1$, $|y| \geq 2$

(B) $|\zeta'(x+iy)| \geq B \ln^2|y|$ for $x \geq 1$, $|y| \geq 2$

(C) $|\zeta(x+iy)| \geq \frac{c}{\delta(1-\delta)} |y|^{1-\delta}$ for $x \leq \delta$, $|y| \geq 2$.

Here $A, B, c$ are certain absolute constants.

Proof

\[ \sum_{n=1}^{N} n^{-2} = 1 + \frac{1-N^{1-\delta}}{1-2\delta} - \varepsilon \int_{1}^{N} \frac{\nu(t)}{t^{\delta+1}} dt \]

($\varepsilon \neq 1$) \hspace{1cm} Lec 5, p. 8 + 9

\[ \zeta(z) = 1 + \frac{1}{z-2} - \varepsilon \int_{1}^{\infty} \frac{\nu(t)}{t^{\delta+1}} dt \]

($\Re(z) > 1$) \hspace{1cm} Lec 5 p. 9

Then, we used this last formula to define $\zeta(z)$ for $\Re(z) > 0$. Lec 5 p. 9
By subtraction,

\[ f(z) - \sum_{n=1}^{N} n^{-z} = \frac{N^{1-z}}{z-1} - \pi \int_{N}^{\infty} \frac{\nu(t)}{t^{z+1}} dt. \]

This is a very useful TRICK!!

\[ f(z) = \sum_{n=1}^{N} n^{-z} + \frac{N^{1-z}}{z-1} - \pi \int_{N}^{\infty} \frac{\nu(t)}{t^{z+1}} dt. \]

\[ \text{Re}(z) > 0 \quad (z \neq 1) \]

We propose to begin with (C) [even though it looks to be the most complicated].

To prove (C), notice that it suffices to prove it for, say, \( |y| \leq 100 \).

In fact, for \( 2 \leq |y| \leq 100 \), we can just use our old VERY CRUDE Theorem 4 from Lec 5, page (2).

In this connection, recall too that

\[ |I(z)| \leq \frac{\text{const}}{\delta} + O(1) \leq \frac{\text{const}}{\delta} \]

for all \( \text{Re}(z) \geq 1+\delta \). (Also on p. (2).)
Use p. 10 line 4 above. 

\[ |J(x+iy)| \leq \sum_{n=1}^{N} n^{-\delta} + \frac{N^{1-\delta}}{|x|} + \frac{1}{2\pi} \int_{N}^{\infty} \frac{1}{t^{x+1}} \, dt \]

\[ |J(x+iy)| \leq \sum_{n=1}^{N} n^{-\delta} + \frac{N^{1-\delta}}{|y|} + (x+iy) \int_{N}^{\infty} \frac{dt}{t^{1+\delta}} \]

\[ \frac{N}{\sum_{n=1}^{N} n^{-\delta}} < 1 + \int_{1}^{N} u^{-\delta} \, du \quad \{ \text{by areas} \} \]

\[ \frac{N}{\sum_{n=1}^{N} n^{-\delta}} < 1 + \frac{N^{1-\delta}}{1-\delta} \]

\[ \frac{N}{\sum_{n=1}^{N} n^{-\delta}} < 1 + \frac{N^{1-\delta}}{1-\delta} < 2 \frac{N^{1-\delta}}{1-\delta} \]

\[ |J(x+iy)| \leq 2 \frac{N^{1-\delta}}{1-\delta} + \frac{N^{1-\delta}}{100} + (x+iy) \frac{N^{-\delta}}{\delta} \]

For \( x \geq 1+\delta \), we already know \( |J(x+iy)| \leq \frac{\text{const}}{\delta} \), hence (C) is certainly OK here [if \( \delta \) is taken sufficiently big].
For this reason, there is no harm in proceeding under the assumption

\[ |y| \geq 100, \ 5 \leq x \leq 1 + \delta \]

Get:

\[ |f(x+iy)| = 2 \frac{N^{1-\delta}}{1-\delta} + \frac{N^{1-\delta}}{100} + 2|y|\frac{N^{-\delta}}{\delta} \]

\[ \leq 3 \frac{N^{1-\delta}}{1-\delta} + 2|y|\frac{N^{-\delta}}{\delta} \]

\[ \leq 3N^{-\delta} \left[ \frac{N}{1-\delta} + \frac{|y|}{\delta} \right] \]

This estimate can admittedly be improved. But, a sloppy one is sufficient.

Also, recall \( f(x-iy) = \frac{1}{f(x+iy)} \). Hence, wlog, \( y \geq 100 \).

Let's try \( N = G \frac{y}{\delta} \), where \( 1 \leq G \leq 10 \), says always and we adjust it to make \( N \in \mathbb{Z} \).

(Note \( \frac{y}{\delta} \geq \frac{100}{\delta} \geq 100 \).)
Get:

\[ |f(x+iy)| = 3 \left( \frac{G y}{\delta} \right)^{-S} \left[ N + \frac{y}{\delta} \right] \frac{1}{1-\delta} \]

by (12)

\[ = 3 \left( \frac{G y}{\delta} \right)^{-S} \left[ \frac{G y}{\delta} + \frac{y}{\delta} \right] \frac{1}{1-\delta} \]

\[ = \frac{3}{1-\delta} G^{-S} y^{-S} \delta^S (G+1) \frac{y}{\delta} \]

\( \delta^S = e^{\delta \ln \delta} \)

is bad away from 0

and so for 0 \( \leq \delta \leq 1 \)

\[ \leq \frac{c_1}{1-\delta} G^{-S} y^{-S} \delta \frac{2G}{\delta} \quad \{ 1 \leq \delta \leq 10^3 \} \]

\[ \leq \frac{c_2}{(1-\delta)^S} y^{1-S}, \text{ as required} \]

This proves (C).

It is important to note \( \delta \in (0,1) \)

is arbitrary. It could even be taken

as a function of \( y \).
(A) is now a trivial consequence of (C).

Indeed, since \( f(z) \) is a nice analytic function for \( \Re(z) \geq 1, |y| \geq 2 \), there is nothing to do for \( \{ 1 \leq x \leq 2, 2 \leq |y| \leq 100 \} \). For \( \{ x \geq 2, 2 \leq |y| \leq 100 \} \) just use \( \theta \) last 3 lines; again nothing to do.

So, wlog, we can assume \( |y| \geq 100 \). Also \( y \geq 100 \).

Put \( \delta = 1 - \frac{1}{\ln y} \) in (C). Note \( \ln 100 = 4.605 \).

Hence \( 0.75 < \delta < 1 \). By (C), get (see \( \theta \)):

\[
|f(z + iy)| \leq 2Ce^{-y^{1-\delta}}, \quad x \geq 1 - \frac{1}{\ln y}
\]

\[
|f(z + iy)| \leq 2C(\ln y)^{\frac{1}{\ln y}}
\]

\[
|f(z + iy)| \leq 2Ce(\ln y) \leq 6Ce(\ln y).
\]

Now just specialize to \( x \geq 1 \). Done!

(\( \theta \) is "almost" as trivial once we recall Cauchy's inequality for \( |f'(z_0)| \).

**IE**

\[
f'(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2} \, dz
\]
\[ |f'(z_0)| \leq \frac{1}{2\pi} \frac{M(R)}{R} (2\pi R) \]

where \( M(R) \equiv \max \frac{|f|}{|z-z_0|=R} \)

\[ |f(z_0)| \leq \frac{M(R)}{R} \]

Here are the details for (B).

First, since \( f(z) \) is a nice analytic fun for \( \Re(z) = 1, |y| \leq 2 \), there is nothing to do for \( \{ z \in \mathbb{C}, 1 \leq |y| \leq 150 \} \). For \( \{ x \in \mathbb{R}, 2 \leq |y| \leq 150 \} \), apply Cauchy's inequality with \( R = 1/2 \) and (10) last 3 lines. Again nothing to do. \( \ast \)

So, wlog, take \( |y| \leq 100 \). We can also assume \( y \leq 100 \). We will use (C) with \( \delta \) similar to \( 1 - \frac{1}{\ln y} \).

\( \ast \) I do want 150 here, i.e., a slight overshoot over 100.
Take $\delta = 1 - \frac{2}{\ln y}$, where $0 < \delta < 1$. We'll choose $\lambda$ in a few moments. Note that we have $0.75 < \delta < 1$ by (14) line 8. Apply (C) on page 7.

Get:

\[ |S(x+iy)| \leq \frac{2e}{1-\delta} \frac{\lambda}{\ln y}, \quad \text{all } x \geq 1 - \frac{2}{\ln y}, \quad y \geq 100 \]

\[ |S(x+iy)| \leq \frac{2e}{\lambda} (\ln y) e^\lambda \]

\[ |S(x+iy)| \leq \frac{6e}{\lambda} (\ln y) \quad \text{for all } x \geq 1 - \frac{2}{\ln y}, \quad y \geq 100. \]

We want to rig things so we can select $y \geq 110$, then use $R = \frac{i}{10} \frac{2}{\ln y}$ (say) in Cauchy's inequality for a center $z_0$ along the segment $[1+iy, \infty+iy]$.

Note that $R \leq \frac{1}{10} \frac{1}{\ln y} \leq \frac{1}{16} < \frac{1}{2}$. \[ \ln 100 \approx 4.605. \]
As the circle slides along, its $y$-values clearly stay between $y-1$ and $y+1$.

Hence, obviously, $y \geq 100$.

But we must make certain that no matter what happens, we have $x \geq 1 - \frac{3}{10y}$ at all times on the circle.

**Baby Calculus Lemma**

Given $T \geq 110$. Keep $y \in [T-1, T+1]$.

Then:

$$\frac{4}{5} \leq \frac{\ln y}{\ln T} \leq \frac{5}{4}.$$

**PF**

$$\frac{\ln (T-1)}{\ln T} \leq \frac{\ln y}{\ln T} \leq \frac{\ln (T+1)}{\ln T}$$

But, by theorem of the mean:

$$\ln (T+1) - \ln (T) \approx \frac{1}{T} (1)$$

$$\ln (T) - \ln (T-1) \approx \frac{1}{T-1} (1)$$
\[ \ln(T+1) \leq \ln(T) + \frac{1}{T} \]
\[ \ln(T-1) \geq \ln(T) - \frac{1}{T-1} \]

\[ \Psi \]

\[ \ln(T+1) \geq \ln T + \frac{1}{110} < \ln T + \frac{1}{100} \]
\[ \ln(T-1) \leq \ln T - \frac{1}{109} > \ln T - \frac{1}{100} \]

\[ \frac{\ln T - \frac{1}{100}}{\ln T} \leq \frac{\ln y}{\ln T} \leq \frac{\ln T + \frac{1}{100}}{\ln T} \]

\[ \text{but } \ln T \leq \ln 100 = 4.605 \]

\[ 0.99 \leq \frac{\ln y}{\ln T} \leq 1.01 \quad \text{OK} \]

"the moving circle on (bottom) obviously"

\[ x \geq 1 - \frac{1}{10} \frac{2}{\ln y} \]

But \( \ln y = \omega \ln y \) with \( \frac{y}{5} \leq \omega \leq \frac{5}{4} \) by Calc Lemma.

So,

\[ x \geq 1 - \frac{1}{10} \frac{2}{\omega \ln y} \quad \text{on circle.} \]
Must make sure
\[
\frac{\lambda}{\ln y} > \frac{\lambda}{10w} \frac{1}{\ln y}
\]

i.e.,
\[
1 \geq \frac{1}{10w}
\]

Thus things are OK \(\lambda\) is irrelevant.

But
\[8 \leq 10w \leq 12.5\]

So just put \(\lambda = 1\).

Get:
\[
R = \frac{1}{10} \frac{1}{\ln y}
\]

By Cauchy's inequality (15), and (16) \((\text{middle})\), we find that:
\[
|\mathcal{F}(x+r,y)| \leq \frac{6e \mathcal{E} [\ln (y+1)]}{R} \leq \frac{12e \ln y}{R}
\]
\[
\leq 120 e (\ln y)^2
\]

for any \(x \in [1, \infty)\). Recalling (15) lines 5–9, we have thus proved (B).
Two Remarks

1. Take \( \lambda \in [11,14] \) say. Note \( \ln 10^6 = 13.815^+ \).

By mimicking (14) - (17), one easily sees that

\[
|f(x+iy)| = O(\ln y) \quad \text{for} \quad x \geq 1 - \frac{5}{\ln y}, \quad y \geq 10^6
\]

\[
|f'(x+iy)| = O(\ln^2 y) \quad \text{for same} \quad (x, y).
\]

One can take \( R = \frac{\lambda}{2\ln y} \). Key necessity is

\[
\frac{\lambda}{\ln y} \geq \frac{5}{\ln y} + \frac{\lambda}{2\ln y}
\]

or

\[
\frac{\lambda}{\ln y} \geq \frac{5}{\ln y} + \frac{\lambda}{2\ln y}
\]

or

\[
\lambda \left( 1 - \frac{1}{2\omega} \right) \geq \frac{5}{\omega}
\]

But, by (18), \( \omega = 1 \pm [0.01] \). \( \square \)

2. Why do we use \( \delta = 1 - \frac{\lambda}{\ln y} \)?

Answer: go to (13) 5 lines from bottom.
We wonder if $\delta$ is close to 1 and $y$ is very large, what is the smallest that

\[
\frac{y^{1-\delta}}{1-\delta}
\]

can be? This is a trivial calc problem. $\delta = 1-u$ \Rightarrow look at $\frac{yu}{u}$ \Rightarrow look at

\[f(u) = u\ln y - \ln u \quad 0 \leq u \leq 1\]

\[f' = \ln y - \frac{1}{u}\]

get $f' > 0 \iff u > \frac{1}{\ln y}$

So key $\delta$ is $1 - \frac{1}{\ln y}$, which gives $e^{\frac{1}{\ln y}}$.

The insertion of $\delta$ allows us to "move around" a bit.