# Lecture 7

(10 Feb)

## Abel Summation Lemma

Let $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_N \geq 0$.

Let $c_k \in \mathbb{C}$ and $\left|\sum_1^n c_j\right| \leq M$ for $1 \leq n \leq N$.

Then:

$$|\varepsilon_1 c_1 + \cdots + \varepsilon_N c_N| \leq M\varepsilon_1 .$$

Pf

$$\text{Sum} = \varepsilon_1 s_1 + \varepsilon_2 (s_2 - s_1) + \cdots + \varepsilon_N (s_N - s_{N-1})$$

$$= s_1(\varepsilon_1 - \varepsilon_2) + \cdots + s_{N-1}(\varepsilon_{N-1} - \varepsilon_N) + \varepsilon_N s_N$$

$$|\text{Sum}| \leq \underbrace{M(\varepsilon_1 - \varepsilon_2) + \cdots + M(\varepsilon_{N-1} - \varepsilon_N) + M\varepsilon_N}_{\text{equals } M\varepsilon_1} \quad \blacksquare$$

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2 immediate corollaries are:

**Thm** (Dirichlet Test for Unif Conv)

Let $1 \geq \varepsilon_1(\alpha) \geq \varepsilon_2(\alpha) \geq \cdots \geq 0$ and $\varepsilon_n(\alpha) \rightrightarrows 0$ for $\alpha \in E_1$. Let $\sum_1^\infty b_n(\beta)$ have unif bounded partial sums for $\beta \in E_2$. Then

$$\sum_{n=1}^\infty \varepsilon_n(\alpha) b_n(\beta)$$

conv unif on $E_1 \times E_2$.

<u>Pf</u>

Use uniform Cauchy criterion! $\blacksquare$

<u>Thm</u> (Abel's Test for Unif Conv)
$\alpha \in E_1$, $\beta \in E_2$ again. Let $1 \geq \varepsilon_1(\alpha) \geq \varepsilon_2(\alpha) \geq \cdots \geq 0$, <u>not</u> nec going to $0$. Let $\sum_1^\infty b_n(\beta)$ <u>conv</u> <u>unif</u> on $E_2$. Then

$$\sum_1^\infty \varepsilon_n(\alpha) b_n(\beta)$$

conv unif for $(\alpha, \beta) \in E_1 \times E_2$.

<u>Pf</u>

Use uniform Cauchy criterion! $\blacksquare$

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Use trivial geom series:

$$\left| \sum_{n=0}^{N} e^{2\pi i n \alpha} \right| \leq \frac{2}{2|\sin \pi \alpha|} = \frac{1}{|\sin \pi \alpha|}, \quad \alpha \notin \mathbb{Z}.$$

By Dirichlet's test, get

$$\sum_1^\infty \frac{1}{n} e^{2\pi i n \alpha}$$

conv <u>unif</u> for $\delta \leq \alpha \leq 1-\delta$. We connect this series with

$$-\operatorname{Log}(1-z).$$

$$-\mathrm{Log}(1-z) = z + \frac{z^2}{2} + \cdots, \quad |z|<1$$

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$z = re^{i\alpha}$ as usual

$$-\mathrm{Log}(1-re^{i\alpha}) = re^{i\alpha} + \frac{1}{2}r^2e^{2i\alpha} + \frac{1}{3}r^3e^{3i\alpha} + \cdots$$

$0 \leqq r < 1$ Wish to let $r \to 1$.

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2

Know $\sum_1^\infty \frac{1}{n} e^{2\pi i n\alpha}$ conv unif away from $\alpha \in \mathbb{Z}$, i.e. away from $z=1$. So, by Abel's test, get

$$\sum_{n=1}^\infty \frac{r^n}{n} e^{2\pi i n\alpha}$$

conv unif $0 \leqq r \leqq 1$, $\delta \leqq \alpha \leqq 1-\delta$ (say). We conclude therefore that

$$-\mathrm{Log}(1-e^{2\pi i\alpha}) = \sum_{n=1}^\infty \frac{e^{2\pi i n\alpha}}{n} \qquad \star$$

for $0<\alpha<1$.

One writes:

$$\begin{aligned} 1-e^{2\pi i\alpha} &= e^{\pi i\alpha}(e^{-\pi i\alpha} - e^{\pi i\alpha}) \\ &= e^{\pi i\alpha}(-2i\sin\pi\alpha) \\ &= 2\sin\pi\alpha \cdot e^{\pi i\alpha - i\pi/2}, \quad 0<\alpha<1. \end{aligned}$$

Conclude at once: (see 3 ✱)

$$-\ln(2\sin \pi\alpha) = \sum_{n=1}^{\infty} \frac{\cos(2\pi n\alpha)}{n}$$

$$\alpha - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin(2\pi n\alpha)}{\pi n},$$

with unif conv for $\delta \leqq \alpha \leqq 1-\delta$. These are two very basic Fourier series, especially #2.

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$$\alpha - [\![\alpha]\!] - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin(2\pi n\alpha)}{\pi n}, \quad \alpha \notin \mathbb{Z}$$

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We can now start moving very majorly toward the proof of PNT.

Recall from Lec 6:

$$|\zeta(x+iy)| \leqq A \ln|y|, \quad x \geqq 1,\ |y| \geqq 2$$

$$|\zeta'(x+iy)| \leqq B \ln^2|y|, \quad x \geqq 1,\ |y| \geqq 2$$

$$|\zeta(x+iy)| \leqq \frac{e}{\delta(1-\delta)} |y|^{1-\delta}, \quad x \geqq \delta,\ |y| \geqq 2,\ 0 < \delta < 1.$$

We also had:

$$\zeta(z) = \prod_p \frac{1}{1-p^{-z}} \qquad re(z)>1$$

$$Log\ \zeta(z) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-z}, \quad re(z)>1$$

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^z}, \quad re(z)>1$$

Hadamard

$$|\zeta(x)|^3 |\zeta(x+iy)|^4 |\zeta(x+2iy)| \geq 1, \quad x>1, \ y\neq 0.$$

I proved in Lec 6 that $\zeta(1+iy) \neq 0$.

Theorem (improvement over $\zeta(1+iy)\neq 0$)

$$\frac{1}{\zeta(x+iy)} = O(1)(\ln|y|)^7 \quad \text{for } x \geq 1, \ |y| \geq 3.$$

Pf

Very close to Ingham!

$x \geq 2$: $\left|\frac{1}{\zeta(z)}\right| = \left|\prod_p (1-p^{-z})\right| \leq \prod_p (1+p^{-x}) \leq \prod_p \frac{1}{1-p^{-x}}$

$$\left|\frac{1}{\zeta(z)}\right| \leq \zeta(x) \leq \zeta(2).$$

So, wlog, $1 \leq x \leq 2$. Also, wlog, $y \geq 3$.

Notice

$$[(x-1)\zeta(x)]^3 |\zeta(x+iy)|^4 |\zeta(x+2iy)| \geq (x-1)^3$$

use 4 bottom

$$\Rightarrow \quad |\zeta(x+iy)|^4 \geq \frac{(x-1)^3}{[(x-1)\zeta(x)]^3 A \ln(2y)}$$

trivial cont. fcn.

$$|\zeta(x+iy)|^4 \geq \frac{(x-1)^3}{A \ln y} \qquad \text{(not same } A\text{)}$$

$$|\zeta(x+iy)| \geq \frac{(x-1)^{3/4}}{A(\ln y)^{1/4}} \qquad \left\{ \begin{matrix} 1 \leq x \leq 2 \\ y \geq 3 \end{matrix} \right\}.$$

But $|\zeta'(x+iy)| = O(\ln^2 y)$, all $x \geq 1, y \geq 3$.

Take any $c \in (1,2)$, $x \in [1,2]$. Get:

$$\zeta(x+iy) - \zeta(c+iy) = \int_c^x \zeta'(u+iy)\,du$$

$$|\zeta(x+iy) - \zeta(c+iy)| \leq A|x-c| \ln^2 y \qquad \text{4 bottom}$$

$$|\zeta(x+iy)| \geq |\zeta(c+iy)| - A|x-c| \ln^2 y \; .$$

But,

$$|\zeta(c+iy)| \geq \frac{(c-1)^{3/4}}{A(\ln y)^{1/4}} \quad \text{by above.}$$

Get:

$$|\zeta(x+iy)| \gtrsim \frac{(c-1)^{3/4}}{A(\ln y)^{1/4}} - A|c-x|\ln^2 y \, .$$

If $x \gtrsim c > 1$, just use

$$|\zeta(x+iy)| \gtrsim \frac{(c-1)^{3/4}}{A(\ln y)^{1/4}} \, ,$$ ← ⑥ line 5

since it's better!! But, if $1 \lesssim x < c$, use

$$|\zeta(x+iy)| \gtrsim \frac{(c-1)^{3/4}}{A_1(\ln y)^{1/4}} - A_2|c-x|\ln^2 y$$

$$|\zeta(x+iy)| \gtrsim \frac{(c-1)^{3/4}}{A_1(\ln y)^{1/4}} - A_2(c-1)\ln^2 y$$

$$|\zeta(x+iy)| \gtrsim \frac{1}{A_1}\left[\frac{(c-1)^{3/4}}{(\ln y)^{1/4}} - A_3(c-1)\ln^2 y\right] .$$

Now have a trivial-type calculus problem to make bracket as <u>large</u> as possible.

Standard trick:

$$\frac{(c-1)^{3/4}}{(\ln y)^{1/4}} = \mathcal{C}\,(c-1)\ln^2 y \, , \quad \mathcal{C} = \text{adjustable}$$

$$\frac{1}{\mathcal{C}}\,\frac{1}{(\ln y)^{9/4}} = (c-1)^{1/4}$$

so we want

$$c - 1 = \frac{\sigma}{(\ln y)^9},$$

$\sigma$ to be adjusted.

We declare:

$$c = 1 + \frac{\sigma}{(\ln y)^9} \qquad (y \gtrsim 3)$$

and keep $\sigma$ small enough that $c \in (1,2)$.

Get:

$$|\zeta(x+iy)| \gtrsim \frac{1}{A_1}(c-1)^{3/4}\left[\frac{1}{(\ln y)^{1/4}} - A_3 (c-1)^{1/4} \ln^2 y\right] \text{ by } (7)$$

$$= \frac{1}{A_1}\frac{\sigma^{3/4}}{(\ln y)^{27/4}}\left[\frac{1}{(\ln y)^{1/4}} - A_3 \frac{\sigma^{1/4}}{(\ln y)^{9/4}} (\ln y)^{8/4}\right]$$

$$= \frac{1}{A_1}\frac{\sigma^{3/4}}{(\ln y)^7}\left[1 - A_3 \sigma^{1/4}\right].$$

Want $\sigma$ so <u>small</u> that $1 - A_3 \sigma^{1/4} \gtrsim \frac{1}{2}$.
(In addition to keeping $1 < c < 2$.) Get:

$$|\zeta(x+iy)| \gtrsim \frac{\text{constant}}{(\ln y)^7}, \quad 1 \le x < 1 + \frac{\sigma}{(\ln y)^9}.$$

(c)

For $x > 1 + \frac{G}{(\ln y)^9}$, we use [[ 7 line 4 ]]

$$|\zeta(x+iy)| \gtrsim \frac{(c-1)^{3/4}}{A(\ln y)^{1/4}} = \frac{\left(\frac{G}{\ln^9 y}\right)^{3/4}}{A(\ln y)^{1/4}}$$

$$= \frac{\text{constant}}{(\ln y)^7}.$$

So, in all cases,

$$|\zeta(x+iy)| \gtrsim \frac{\text{const}}{(\ln y)^7} \qquad \begin{matrix} x \in [1,2] \\ y \gtrsim 3 \end{matrix}.$$

So:

$$\frac{1}{|\zeta(z)|} \lesssim (\text{const})(\ln y)^7.$$

■

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One defines (following Riemann)

$$\psi_1(x) = \int_0^x \psi(v)\,dv \qquad \{\psi(v) = 0,\ v < 2\}$$

$$= \int_0^x \Big(\sum_{k \leq v} \Lambda(k)\Big)\,dv$$

$$= \sum_{k \leq x} \Lambda(k) \int_k^x dv = \sum_{k \leq x} \Lambda(k)(x-k).$$

We will also follow Riemann and begin writing

$$\boxed{s = \sigma + it}$$

instead of $z = x + iy$.

<u>Theorem</u> (Fund. Formula)

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left[ -\frac{\zeta'(s)}{\zeta(s)} \right] ds$$

for all $c > 1$, $x > 0$.

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Note that:

$$\left| \frac{x^{s+1}}{s(s+1)} \right| = \frac{x^{c+1}}{|s||s+1|} \leqq \frac{x^{c+1}}{|t|^2}.$$

There is no question that RHS converges!

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## Proof

We require a standard lemma from complex analysis.

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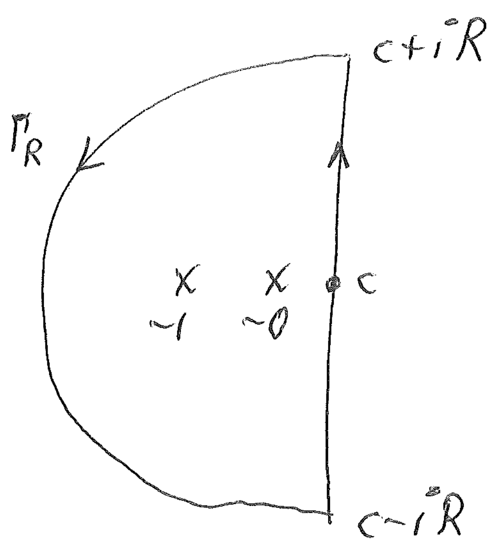
**Lemma**

Freeze $y$. Fix any $c > 0$. Then:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)}\,ds = \begin{cases} 0, & \text{if } y < 1 \\ 1 - y^{-1}, & \text{if } y \geq 1 \end{cases}.$$

**Pf**

$y \geq 1$ first.




Our Analytic fcn of $s$ is $\dfrac{y^s}{s(s+1)}$.

($y$ fixed)

Ingham $\swarrow$ p. 31 uses Cauchy residue thm on this shape. We prefer Cauchy integral theorem + Cauchy integral formula!!

$$\text{CIT} \Rightarrow \oint = \oint_{|s|=\varepsilon} + \oint_{|s+1|=\varepsilon}$$

But,

$$\frac{1}{2\pi i}\oint_{|s|=\varepsilon} \frac{1}{s}\,\frac{y^s}{s+1}\,ds = \frac{y^0}{0+1} = 1 \qquad \text{by CIF} \;;$$

$$\frac{1}{2\pi i}\oint_{|s+1|=\varepsilon} \frac{1}{s+1}\,\frac{y^s}{s}\,ds = \frac{y^{-1}}{-1} = -y^{-1} \qquad \text{by CIF} \,.$$

and

$$\left|\int_{\Gamma_R} \frac{y^s}{s(s+1)}\,ds\right| \lesssim \int_{\Gamma_R} \frac{y^c}{|s||s+1|}\,|ds| \quad \leftarrow \text{correct since } y \geqq 1$$

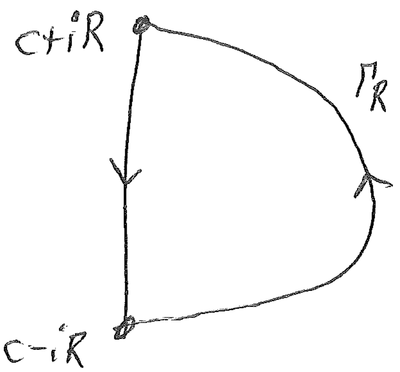
$$\lesssim (\text{const})\, y^c \frac{1}{R^2}\pi R$$

$$= O\left(\frac{y^c}{R}\right) \longrightarrow 0 \quad \text{as } R \to \infty$$

We immediately get:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)}\,ds + 0 = 1 - y^{-1} .$$

For $y < 1$, use shape:





$\left|\int_{\Gamma_R}\right| \to 0$ again since $y < 1$

■ (on the lemma)

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We are now ready for the Theorem.
Start on RHS (10). Freeze $c > 1$ and $x > 0$.

Must evaluate:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left[+\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right] ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{x^{c+1+it}}{(c+it)(c+1+it)} \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{c+it}}\, dt .$$

Want to integrate term-by-term.

Standard set-up applies.

$g(t)$ absolutely integrable on $\mathbb{R}$

$|f_n(t)| \leq M_n \quad \sum_1^\infty M_n < \infty$ for Weierstrass M-test

$$\Rightarrow \int_{-\infty}^{\infty} g(t) \sum_{n=1}^{\infty} f_n(t)\,dt = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(t) f_n(t)\,dt$$

(standard adv calculus.)

$$\left\{ \begin{array}{l} \text{NOTE THAT:} \\ \displaystyle\sum_1^\infty \int_{-\infty}^{\infty} |g(t)||f_n(t)|\,dt \\ \displaystyle\quad \leq \sum_1^\infty \int_{-\infty}^{\infty} |g(t)| M_n\,dt < \infty \end{array} \right\}$$

We immediately get:

$$\text{(13) bottom} = \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^1}{s(s+1)} (x/n)^s\,ds$$

$$= \sum_{n \leq x} \Lambda(n)\, x \left[1 - \frac{1}{x/n}\right] + 0 \quad \text{by Lemma}$$

$$= \sum_{n \leq x} \Lambda(n)[x - n] = \psi_1(x). \quad \blacksquare$$

Remark.

In Lec 8, I derived Thm 10 by elementary use of Fourier integrals. It is by no means essential to use complex variable. Riemann certainly knew this.

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Once having Riemann's fund formula, one seeks to move $\{\mathrm{Re}(s)=c\}$ over to the left.

THIS WILL USE COMPLEX VARIABLE!

I prefer to move the line to $\{\mathrm{Re}(s)=1\}$.

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I need one more little lemma.

See Ingham p. 31.

CLAIM: $y>0$ fixed, $c>0$ fixed

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s(s+1)(s+2)}\,ds=\begin{cases}0, & \text{if } y<1\\ \frac{1}{2}(1-y^{-1})^2, & \text{if } y\geq 1\end{cases}$$

Pf

Very similar to what we did on 11 - 13.

Omit $y<1$. For $y\geq 1$, notice that

$$\frac{1}{2\pi i}\oint_{|s|=\varepsilon} \frac{1}{s}\frac{y^s}{(s+1)(s+2)}\,ds = \frac{y^0}{1\cdot 2} = \frac{1}{2}$$

$$\frac{1}{2\pi i}\oint_{|s+1|=\varepsilon} \frac{1}{s+1}\frac{y^s}{(s)(s+2)}\,ds = \frac{y^{-1}}{(-1)(1)} = -y^{-1}$$

$$\frac{1}{2\pi i}\oint_{|s+2|=\varepsilon} \frac{1}{s+2}\frac{y^s}{s(s+1)}\,ds = \frac{y^{-2}}{(-2)(-1)} = \frac{1}{2}y^{-2}$$

$$\text{Sum} = \frac{1}{2}(1-y^{-1})^2 \qquad \text{OK}$$

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Keep $y \geqq 1$. Know: (any $\eta > 0$)

$$\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} \frac{y^s}{s(s+1)(s+2)}\,ds = \frac{1}{2}(1-y^{-1})^2$$

$$\xi = s+1$$

$$\frac{1}{2\pi i}\int_{\eta+1-i\infty}^{\eta+1+i\infty} \frac{y^{\xi-1}}{(\xi-1)\xi(\xi+1)}\,d\xi = \frac{1}{2}(1-y^{-1})^2$$

$$\underline{\underline{1+\eta > 1}}$$

The problem with moving $\{Re(s) = c\}$ in 10 directly to $Re(s) = 1$ is hitting the pole at $s = 1$. Must modify things slightly!

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near $s = 1$

$$\zeta(s) = (s-1)^{-1}\left[1 + c_1(s-1) + c_2(s-1)^2 + \cdots\right]$$

$$\zeta(s) = (s-1)^{-1}\phi(s) \quad \text{say}$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \frac{\phi'(s)}{\phi(s)} \quad \text{near } s = 1$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + [\text{analytic}]$$

$$\Downarrow$$

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \quad \text{is } \underline{\text{analytic}} \text{ for } Re(s) \overset{?}{\geq} 1$$

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Notice that: $(c > 1)$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{(s-1)s(s+1)}\,ds = \frac{1}{2}\left(1 - \frac{1}{x}\right)^2,$$

by 16 bottom for $x > 1$.

For $x > 1$, we thus have:

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]ds$$

$$\frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)}\left[\frac{1}{s-1}\right]ds$$

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right]ds.$$

Write

$$H(s) = -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \quad \text{for } \mathrm{Re}(s) \geqq 1.$$

For $|t| \geqq 3$, know that:

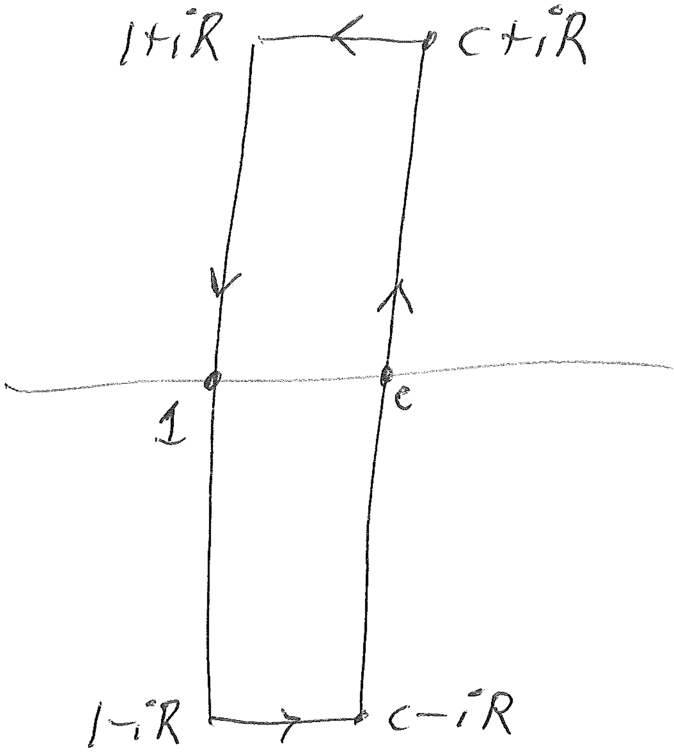
$$|H(s)| \lessapprox \text{constant} + O(1)\ln^2|t| \cdot \ln^7|t|$$

by 5 + 4 (bot). Ie,

$$|H(s)| \lessapprox O(1)\left(\ln|t|\right)^9$$

for $\sigma \geqq 1$, $|t| \geqq 3$.

Moving the contour in 18 line 4 is now trivial.




$x > 1$

$c > 1$

$R =$ giant

$$\left| \int_{\text{horizontal}} \frac{x^{s-1}}{s(s+1)} H(s)\,ds \right| \lesssim \int_1^c O(1)\, \frac{x^{c-1}}{R^2} (\ln R)^9\, d\sigma$$

$$\lesssim O(1)\,(c-1)\, x^{c-1} \frac{(\ln R)^9}{R^2}$$

$$\to 0 \text{ as } R \to \infty.$$

($x, c$ frozen)

Get:

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{x^{s-1}}{s(s+1)} H(s)\,ds.$$

Notice here that $|H(1+it)| \lesssim A(\ln|t|)^9$.
The integrand on RHS has abs value

$$\lesssim O(1)\frac{1}{t^2}(\ln|t|)^9. \qquad (|t| \gtrsim 3)$$

THEOREM (almost the PNT)

$$\lim_{x\to\infty} \frac{\psi_1(x)}{x^2} = \frac{1}{2}.$$

Proof

Must look at

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{x^{it}}{(1+it)(2+it)} H(1+it)\,dt$$

$$\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{H(1+it)}{(1+it)(2+it)} e^{it(\ln x)}\,dt.$$

$(\ln x \to +\infty)$

But, the Riemann-Lebesgue lemma tells us that

$$\int_{-\infty}^{\infty} \mathcal{H}(t)\, e^{it\lambda}\, dt \longrightarrow 0$$

as $\lambda \to \pm\infty$ for any piecewise continuous absolutely integrable $\left(\int_{-\infty}^{\infty} |\mathcal{H}|\, dt < \infty\right)$ $\mathcal{H}$.

Since $\dfrac{H(1+it)}{(1+it)(2+it)}$ is $C^{\infty}$ and $O(1)\dfrac{(\ln|t|)^{9}}{t^2}$ for $t \gtrsim 3$, we are done. $\blacksquare$

Proof of R-L lemma

Choose any $\varepsilon > 0$.

Choose $G$ so big: $\displaystyle\int_{|t|>G} |\mathcal{H}(t)|\,dt < \frac{\varepsilon}{3}$.

Find a piecewise constant fcn $s(t)$ on $[-G,G]$ so that

$$\int_{-G}^{G} |\mathcal{H}(t)-s(t)|\,dt < \frac{\varepsilon}{3}.$$

For $s(t)$, notice that on each "step",

$$\int_{\alpha_j}^{\beta_j} c_j e^{it\lambda}\,dt = c_j \frac{e^{i\lambda\beta_j}-e^{i\lambda\alpha_j}}{i\lambda}$$

$$[\text{abs value}] \lesssim 2|c_j|\frac{1}{\lambda}.$$

Hence:

$$\int_{-G}^{G} s(t)e^{it\lambda}\,dt = O\left(\frac{1}{\lambda}\right).$$

Writing

$$\int_{-\infty}^{\infty} \mathcal{H}(t)e^{it\lambda}\,dt = \int_{|t|>G} \mathcal{H}(t)e^{i\lambda t}\,dt + \int_{-G}^{G} [\mathcal{H}(t)-s(t)]e^{i\lambda t}\,dt$$

(continued)

$$+ \int_{-G}^{G} s(t) e^{i\lambda t}\, dt ,$$

we clearly get

$$\left| \int_{-\infty}^{\infty} \mathcal{H}(t) e^{it\lambda} dt \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

for all $|\lambda| > \lambda_\varepsilon$. Done! ∎

Note: if $\mathcal{H}$ is $\underline{C'}$, just do ($G$ fixed)

$$\int_{-G}^{G} \mathcal{H}(t) e^{i\lambda t} dt = \int_{-G}^{G} \mathcal{H}(t) d\left(\frac{e^{i\lambda t}}{i\lambda}\right)$$

$$= O\left(\frac{1}{\lambda}\right) - \int_{-G}^{G} \frac{e^{i\lambda t}}{i\lambda} \mathcal{H}'(t) dt$$

$$= O\left(\frac{1}{\lambda}\right)$$

to make a short-circuit. IE, R-L lemma is <u>TRIVIAL</u> if $\mathcal{H} \in C^1$.