Lecture 8
(12 Feb)

We know $\psi_1(x) \sim \frac{x^2}{2}$, where $\psi_1(x) = \int_0^x \psi(v)\,dv$.

Theorem (equiv to PNT)

$\psi(x) \sim x$ as $x \to \infty$.

Pf

Read Ingham p. 35 on your own. My method is closer to p. 64. Write $\psi_1(x) = \frac{x^2}{2} + R(x)$.

Keep $0 < h \leq \frac{x}{2}$ and x large. Obviously,

$$\frac{\psi_1(x+h) - \psi_1(x)}{h} = \frac{1}{h}\int_x^{x+h} \psi(v)\,dv \geq \psi(x)$$

$$\frac{\psi_1(x) - \psi_1(x-h)}{h} = \frac{1}{h}\int_{x-h}^{x} \psi(v)\,dv \leq \psi(x).$$

This gives

$$\psi(x) \leq \frac{\frac{(x+h)^2}{2} - \frac{x^2}{2} + R(x+h) - R(x)}{h}$$

$$\psi(x) \leq x + \frac{h}{2} + \frac{R(x+h) - R(x)}{h} \qquad ;$$

$$\psi(x) \gtrsim \frac{\frac{x^2}{2} - \frac{(x-h)^2}{2} + R(x) - R(x-h)}{h}$$

$$\psi(x) \gtrsim x - \frac{h}{2} + \frac{R(x) - R(x-h)}{h} .$$

Clearly:

$$-\frac{h}{2} - \frac{|R(x)| + |R(x-h)|}{h} \lesssim \psi(x) - x \lesssim \frac{h}{2} + \frac{|R(x+h)| + |R(x)|}{h} .$$

Suppose $|R(y)| \lesssim E(y)$ with some <u>explicit</u> <u>monotonic</u> increasing fcn E. Get:

$$-\frac{h}{2} - \frac{2E(x)}{h} \lesssim \psi(x) - x \lesssim \frac{h}{2} + \frac{2E(2x)}{h}$$

$$\Rightarrow \boxed{|\psi(x) - x| \lesssim \frac{h}{2} + \frac{2E(2x)}{h}} .$$

But, given $\varepsilon > 0$, we know $|R(y)| \lesssim \varepsilon y^2$ for all $y \gtrsim A_\varepsilon$. Keep $x \gtrsim \underline{2000 A_\varepsilon}$ so that $x - h \gtrsim \frac{x}{2} \gtrsim 1000 A_\varepsilon$.

We are free to take $E(y) = \mathcal{E} y^2$ in the ranges which are relevant so long as we make doubly certain $0 < h \leqq \frac{x}{2}$.

$$|\psi(x) - x| \lessapprox \frac{h}{2} + \frac{2E(2x)}{h}$$

$$|\psi(x) - x| \lessapprox \frac{h}{2} + \frac{8\mathcal{E}x^2}{h}$$

$$\left\{ \begin{array}{l} \text{wish to put } h = 4\sqrt{\mathcal{E}}\,x \\ \text{so just keep } \mathcal{E} < \frac{1}{100} \\ \text{and } x \text{ big} \end{array} \right\}$$

$$|\psi(x) - x| \lessapprox 2\sqrt{\mathcal{E}}\,x + 2\sqrt{\mathcal{E}}\,x$$

$$|\psi(x) - x| \lessapprox 4\sqrt{\mathcal{E}}\,x \quad , \quad \text{if } x \gtrapprox x_{\mathcal{E}} \equiv 2000 A_{\mathcal{E}}$$
(say)

Since $\mathcal{E}$ is arbitrary, we are done.

∎

From the earlier lectures (e.g. Lec 2, p. 2) we then get

$$\pi(x) \sim \frac{x}{\ln x} .$$

$\boxed{\text{PNT}}$

I remarked that, in Riemann's formula for $\psi_1(x)$, one would like to move $Re(s) = c$ over past $\sigma = \frac{1}{2}$.

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]ds$$

<u>IF</u> we expect the poles of $-\frac{\zeta'(s)}{\zeta(s)}$ to lie along $Re(s) = \frac{1}{2}$ (except for $s = 1$), it is reasonable [perhaps] for

$$\psi_1(x) \approx \frac{x^2}{2} + O(x^{3/2}).$$

$E(x) = Cx^{3/2}$ on p. 3 line 4 would lead to

$$|\psi(x) - x| \leq (\text{constant})\, x^{3/4}.$$

Riemann was aware of this. By being less sloppy with $R(x)$ on 1 + 2, <u>perhaps</u>

$$|\psi(x) - x| \leq (\text{constant})\, x^{1/2}$$

could be obtained. THIS IS ALL JUST VERY ROUGH, THOUGH.

I recalled that:

$$\int_a^b [f(x)g'(x) + f'(x)g(x)]\,dx = [f(x)g(x)]_a^b$$

holds for $f \in C'[a,b]$ and $g \in C[a,b]$ but only piecewise C'.

This could also be viewed à la Riemann-Stieltjes integration by parts, by declaring

$$\alpha(x) = g(a) + \int_a^x g'(v)\,dv\,.$$

Of course: $\alpha(x) \equiv g(x)$.

Another thing I remarked was how Riemann's fundamental formula for $\psi_1(x)$ was derivable by Fourier integrals.

"Fourier Integrals = Good."

Indeed,

R-S integral

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty x^{-s}\, d\psi(x) \quad , \quad Re(s) > 1$$

$$\left\{ \begin{array}{l} \psi(u) = O(u) \\ \text{Chebyshev} \end{array} \right\}$$

$$= \left[x^{-s}\psi(x)\right]_1^\infty - \int_1^\infty \psi(x)\, d(x^{-s})$$

$$= 0 - 0 - \int_1^\infty \psi(x)(-s)x^{-s-1}\, dx$$

$$= s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx \quad \Longrightarrow$$

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx \quad \left\{ \begin{array}{l} Re(s) > 1 \\ \text{Ingham p. 18} \end{array} \right\}.$$

But, $\psi_1(x) = \int_1^x \psi(v)\, dv$ for $x \geq 1$.

ψ_1 continuous piecewise C^1

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty x^{-s-1}\, d\left[\psi_1(x)\right]$$

$$= x^{-s-1}\psi_1(x)\Big]_1^\infty - \int_1^\infty \psi_1(x)\, d(x^{-s-1})$$

$$\{\psi_1(x) = O(x^2) \quad \text{Chebyshev}\}$$

$$= 0 - 0 + (s+1)\int_1^\infty \frac{\psi_1(x)}{x^{s+2}}\,dx$$

$$\Downarrow$$

$$-\frac{1}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty \psi_1(x)\,x^{-s-2}\,dx\,, \quad Re(s)>1$$

This is beginning to look like a Mellin transform.

Ingham p. 32 .

Recall Fourier inversion formula (heuristically).

$$\tilde{F}(p) = \int_{-\infty}^{\infty} F(v)\,e^{-ipv}\,dv$$ ← Fourier transform

$$\Rightarrow F(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(p)\,e^{ipv}\,dp \,.$$

This is very useful if

$$\mathfrak{M}(s) \equiv \int_0^\infty f(x)\,x^{-s}\,dx\,, \quad Re(s)>1$$

WITH $f(x) \equiv 0$ near $x = 0$, $|f(x)| \lesssim O(1)$.

Why? Because:

$$\mathfrak{M}(c+it) = \int_0^\infty f(x)\,x^{-c-it}\,dx \qquad \underline{c>1}$$

$$= \int_{-\infty}^{\infty} f(e^v)\,e^{v(-c-it)}\,e^v\,dv$$

$$\mathfrak{M}(c+it) = \int_{-\infty}^{\infty} \left[f(e^v) e^{-(c-1)v} \right] e^{-itv} \, dv$$

$\{ f(e^v) \equiv 0$ for v very negative $\}$

$$\Downarrow$$

$$f(e^v) e^{-(c-1)v} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathfrak{M}(c+it) e^{itv} \, dt$$

$$\Downarrow$$

$$f(e^v) e^v = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathfrak{M}(c+it) e^{cv} e^{itv} \, dt$$

$$f(e^v) e^v = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathfrak{M}(c+it) e^{(c+it)v} \, dt$$

$$f(e^v) e^v = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathfrak{M}(s) e^{sv} \, ds$$

$\{$ write $x = e^v$ $\}$

$$f(x) x = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathfrak{M}(s) x^s \, ds$$

$$\boxed{f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathfrak{M}(s) x^{s-1} \, ds} \; .$$

This is <u>essentially</u> what is called the Mellin inversion formula. $(s = 1 - \xi)$

Look at (7 top):

$$-\underbrace{\frac{1}{s(s+1)}\,\frac{\zeta'(s)}{\zeta(s)}}_{M(s)} = \int_1^{\infty} \underbrace{\frac{\psi_1(x)}{x^2}}_{f(x)\quad \{0 \text{ if } x<1\}} x^{-s}\,dx, \qquad Re(s) > 1$$

so we get, by 8 box,

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[-\frac{1}{s(s+1)}\,\frac{\zeta'(s)}{\zeta(s)}\right]x^{s-1}\,ds$$

which is <u>equivalent</u> to Riemann's

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[-\frac{1}{s(s+1)}\,\frac{\zeta'(s)}{\zeta(s)}\right]x^{s+1}\,ds.$$

THUS: you really do <u>not</u> need complex variable (analytic function theory) to derive Riemann's fund formula.

Riemann knew this!

FACT (very curious) Ingham 37

Suppose $\psi(x) \sim x$. Then we can see rather easily that $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}$. [Hence THIS fact is the essence of PNT!]

Pf

Suppose, e.g., that $\zeta(1+it_0) = 0$. Zero of multiplicity $m \geq 1$.

call this $\phi(s)$

$$f(s) = (s-s_0)^m \underbrace{[c_0 + c_1(s-s_0) + \cdots]}_{\phi(s)}$$

$$c_0 \neq 0$$

$$\frac{f'(s)}{f(s)} = \frac{m}{s-s_0} + \frac{\phi'(s)}{\phi(s)}$$

$$\frac{f'(s)}{f(s)} = \frac{m}{s-s_0} + O(1) \quad \text{near } s = s_0$$

recall Lec 7 p. 17

Get:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{m}{s-(1+it_0)} + O(1) \quad \text{near } 1+it_0.$$

But,

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx \qquad Re(s) > 1 \quad (6)$$

$$\frac{1}{s-1} = \int_1^\infty \frac{x}{x^{s+1}}\,dx \qquad Re(s) > 1$$

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = \int_1^\infty \frac{\psi(x)-x}{x^{s+1}}\,dx \,.$$

Assume $\varepsilon > 0$ is small. Get

$$|\psi(x) - x| < \varepsilon x \,, \quad x \geqq G_\varepsilon \,.$$

Hence:

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = \int_1^{G_\varepsilon} \frac{\psi(x)-x}{x^{s+1}}\,dx$$

$$+ \int_{G_\varepsilon}^\infty \frac{\psi(x)-x}{x^{s+1}}\,dx \,.$$

First integral on RHS is analytic for <u>all</u> $s \in \mathbb{C}$ (since G_ε = finite).

Let $s_0 = 1 + it_0$ and keep $Re(s) > 1$, $s \approx s_0$.

We have:

$$-\frac{1}{s}\left[\frac{m}{s-s_0} + O(1)\right] + O(1) = O(1) + \int_{G_\varepsilon}^{\infty} \frac{\psi(x)-x}{x^{s+1}}\,dx$$

$$-\frac{1}{s_0}\frac{m}{s-s_0} + O(1) = O(1) + \int_{G_\varepsilon}^{\infty} \frac{\psi(x)-x}{x^{s+1}}\,dx .$$

Take $s = \sigma + it_0$ and let $\sigma \to 1$. Get:

$$\frac{1}{|s_0|}\frac{m}{\sigma-1} + O(1) \lessapprox \int_{G_\varepsilon}^{\infty} \frac{\varepsilon x}{x^{\sigma+1}}\,dx$$

$$\frac{1}{|s_0|}\frac{m}{\sigma-1} + O(1) \lessapprox \varepsilon\int_{G_\varepsilon}^{\infty} x^{-\sigma}\,dx$$

$$\frac{1}{|s_0|}\frac{m}{\sigma-1} + O(1) \lessapprox \varepsilon\left[\frac{x^{1-\sigma}}{1-\sigma}\right]_{G_\varepsilon}^{\infty}$$

$$\frac{1}{|s_0|}\frac{m}{\sigma-1} + O(1) \lessapprox \varepsilon\,\frac{G_\varepsilon^{1-\sigma}}{\sigma-1} \qquad (\sigma>1)$$

$$\Rightarrow \quad \frac{m}{|s_0|} \lessapprox \varepsilon\, G_\varepsilon^{0} \quad \Rightarrow \quad \frac{|m|}{|s_0|} \leqq \varepsilon .$$

Hence $\frac{1}{|s_0|} \leq \varepsilon$. But ε was arbitrary!

Contradiction. $\blacksquare$

I remarked in the lecture that I would now pause* for about 2 lectures to fill in some background stuff on Bernoulli numbers, Euler–Maclaurin summation, and special values of $\zeta(s)$.

It's very pretty to work with this very <u>explicit</u> stuff!!!

* possibly a mistake

Thm (Euler – Maclaurin Sum Formula, version 1)

$f \in C^1[0,N] \Rightarrow$

$$\tfrac{1}{2}f(0) + f(1) + \cdots + f(N-1) + \tfrac{1}{2}f(N)$$
$$= \int_0^N f(x)\,dx + \int_0^N f'(x)\left(x - [\![x]\!] - \tfrac{1}{2}\right) dx .$$

Pf

Let $\beta(x) = x - [\![x]\!] - \frac{1}{2}$ for a few moments.
Note that $\beta(x)$ is the difference of 2 right continuous increasing fcns. By def,

$$[\![x]\!] = x - \tfrac{1}{2} - \beta(x) .$$

$$f(1) + \cdots + f(N) = \int_0^N f(x)\,d[\![x]\!] \quad \leftarrow \text{this is correct}$$
$$= \int_0^N f(x)\,d\left(x - \tfrac{1}{2} - \beta(x)\right)$$
$$= \int_0^N f(x)\,dx - \int_0^N f(x)\,d\beta(x)$$
$$= \int_0^N f(x)\,dx - [f\beta]_0^N + \int_0^N \beta(x) f'(x)\,dx$$

{by R-S parts}

$$= \int_0^N f\,dx - f(N)\beta(N) + f(0)\beta(0) + \int_0^N \beta f'\,dx$$
$$= \int_0^N f\,dx + \tfrac{1}{2}f(N) - \tfrac{1}{2}f(0) + \int_0^N \beta f'\,dx$$

$\Downarrow$

$$\tfrac{1}{2}f(0) + f(1) + \cdots + f(N-1) + \tfrac{1}{2}f(N) = \int_0^N f\,dx + \int_0^N \beta f'\,dx.$$

$\blacksquare$

We intend to use $\beta(x)$ *

$$x - [\![x]\!] - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{n\pi}, \quad x \notin \mathbb{Z}$$

repeatedly. We need a few facts.

* Recall that we got this equality via $-\mathrm{Log}(1-z)$. ← a nice way! Lec 7 p. 3

<u>Thus</u>

The partial sums $\displaystyle\sum_{n=1}^{N} \frac{\sin 2\pi n x}{\pi n}$ are <u>uniformly</u> <u>bounded</u> for all $x \in \mathbb{R}$.

<u>Pf</u>

Suffices to treat $\displaystyle\sum_{n=1}^{N} \frac{\sin(nt)}{n}$.

Use periodicity 2π and oddness wrt t.

Hence, wlog, $0 < t \lesssim \pi$.

Suffices to treat

$$\sigma_N(t) = \frac{t}{2} + \sum_{n=1}^{N} \frac{\sin nt}{n} .$$

But:

$$\sigma_N' = \frac{1}{2} + \sum_1^N \cos nt = \frac{1}{2}\left(\sum_{-N}^{N} e^{int}\right)$$

$$= \frac{1}{2} \frac{e^{-itN} - e^{it(N+1)}}{1 - e^{it}}$$

$$= \frac{1}{2} \frac{e^{-it(N+\frac{1}{2})} - e^{it(N+\frac{1}{2})}}{e^{-it/2} - e^{it/2}} = \frac{1}{2} \frac{\sin[(N+\frac{1}{2})t]}{\sin(t/2)} .$$

So,

$$\sigma_N(t) = \int_0^t \frac{\sin[(N+\frac{1}{2})v]}{2\sin(v/2)} \, dv .$$

Write:

$$\frac{1}{2\sin(v/2)} = \frac{1}{v} + h(v) , \quad 0 < v \leq \pi .$$

Obviously $h(v)$ is C^∞. The fcn $h(v)$ is also <u>analytic</u> near $v = 0$. Indeed,

$$\frac{1}{2\sin(\frac{v}{2})} - \frac{1}{v} = \frac{1}{2\left[\frac{v}{2} - \frac{1}{3!}\left(\frac{v}{2}\right)^3 + \cdots\right]} - \frac{1}{v}$$

$$= \frac{1}{v(1+b_2v^2+b_4v^4+\cdots)} - \frac{1}{v}$$

$$= \frac{1}{v}\left[1+A_2v^2+A_4v^4+\cdots\right] - \frac{1}{v}$$

$$= A_2v + A_4v^3 + \cdots \quad \text{near } v=0 \text{ in } \mathbb{C}.$$

So,

$$\sigma_N(t) = \int_0^t \left[\frac{1}{v} + h(v)\right] \sin\left(N+\tfrac{1}{2}\right)v\, dv$$

$$= \int_0^t \frac{\sin\left[\left(N+\frac{1}{2}\right)v\right]}{v}\, dv + \int_0^t h(v)\sin\left(N+\tfrac{1}{2}\right)v\, dv.$$

But,

$$\left|\int_0^t h(v)\sin\left(N+\tfrac{1}{2}\right)v\, dv\right| \leqq \int_0^\pi |h(v)|\, dv < \infty$$

and

$$\int_0^t \frac{\sin\left(N+\frac{1}{2}\right)v}{v}\, dv = \int_0^{\left(N+\frac{1}{2}\right)t} \frac{\sin q}{q}\, dq.$$

By baby calculus, however,

$$\left|\int_0^R \frac{\sin q}{q}\, dq\right| \leqq \text{constant}$$

for all $R \geqq 0$. Just look at the graph of $\frac{\sin q}{q}$ and consider <u>signed</u> area.

Or use:

$$\int_1^R \frac{\sin q}{q}\,dq = \int_1^R \frac{d(-\cos q)}{q}$$

$$= -\left[\frac{\cos q}{q}\right]_1^R + \int_1^R \cos q \, d\left(\frac{1}{q}\right)$$

$$= O(1) - \int_1^R \frac{\cos q}{q^2}\,dq$$

$$= O(1) + O(1).$$

One knows, in fact, that the improper integral $\int_0^\infty \frac{\sin q}{q}\,dq$ exists!

IN ANY EVENT, we clearly get (by 17)

$$|\sigma_N(t)| \leqq \text{some constant}$$

for all $0 < t \leqq \pi$. ▨

"Miracle #1" (by revisiting 16 – 18 with more <u>real</u> analysis)

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin q}{q}\, dq$$

Pf ← The standard proof in any Fourier series class.

On 16, we saw

$$\frac{1}{2} + \sum_{n=1}^{N} \cos nt = \frac{\sin[(N+\frac{1}{2})t]}{2\sin(t/2)}, \quad 0 < t \leq \pi.$$

For $t=0$, use a limit. Integrate over $[0,\pi]$. Get:

$$\frac{\pi}{2} = \int_0^\pi \frac{\sin[(N+\frac{1}{2})v]}{2\sin(v/2)}\, dv.$$

Use 16 bottom – 17 with $h(v)$. Get:

$$\frac{\pi}{2} = \int_0^\pi \underline{\underline{\left(\frac{1}{v} + h(v)\right)}} \sin[(N+\tfrac{1}{2})v]\, dv$$

$$\frac{\pi}{2} = \int_0^\pi \frac{\sin[(N+\frac{1}{2})v]}{v}\, dv + \int_0^\pi h(v)\sin[(N+\tfrac{1}{2})v]\, dv$$

C^∞ and analytic near $v=0$

$$\Downarrow$$

$$\frac{\pi}{2} \approx \int_0^{\pi(N+\frac{1}{2})} \frac{\sin q}{q}\, dq$$

$$+ \int_0^{\pi} h(v) \sin\left[(N+\tfrac{1}{2})v\right] dv .$$

Recall R-L lemma for C^1

$$\int_0^{\pi} h(v) e^{i\lambda v} dv = \int_0^{\pi} h(v)\, d\left[\frac{e^{i\lambda v}}{i\lambda}\right]$$

$$= h(v) \frac{e^{i\lambda v}}{i\lambda}\Big]_0^{\pi}$$

$$- \int_0^{\pi} \frac{e^{i\lambda v}}{i\lambda} h'(v)\, dv$$

$$= O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{\lambda}\right)$$

as in Lec 7 p. 23.

Let $N \to \infty$ and use R-L lemma.

Get:

$$\frac{\pi}{2} = \int_0^{\infty} \frac{\sin q}{q}\, dq + 0 .$$

OK!

Miracle #2

(by revisiting 16 – 18 with more real analysis)

I claim that p. 16 and 17 (middle) immediately imply

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin q}{q}\,dq$$

AND

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi n q)}{\pi n} = \frac{1}{2} - q + [\![q]\!] \; , \quad q \notin \mathbb{Z} .$$

"No need for $-\mathrm{Log}(1-z)$"

Pf

Use p. 16 for $0 < t \leq 2\pi - \delta$.

Notice that $h(v)$ is C^∞ on $(0, 2\pi - \delta]$ and analytic near $v = 0$.

We still have

$$\sigma_N(t) = \int_0^t \frac{\sin[(N+\frac{1}{2})v]}{2\sin(v/2)}\,dv \quad \longleftarrow 16$$

$$= \int_0^t \frac{\sin[(N+\frac{1}{2})v]}{v}\,dv$$

$$+ \int_0^t h(v)\sin[(N+\tfrac{1}{2})v]\,dv$$

àla 17 (middle).

Thus, for $0 < t \leq 2\pi - \delta$,

$$\frac{t}{2} + \sum_{1}^{N} \frac{\sin(nt)}{n} = \int_0^{(N+\frac{1}{2})t} \frac{\sin q}{q}\, dq$$

$$+ \int_0^t h(v) \sin\left[(N+\tfrac{1}{2})v\right] dv .$$

Freeze t temporarily and let $N \to \infty$.
Get:

$$\frac{t}{2} + \sum_{1}^{\infty} \frac{\sin(nt)}{n} = A + 0$$

cf. 20 middle
with minor change

where $A = \int_0^\infty \frac{\sin q}{q}\, dq$.

Thus:

$$\sum_{1}^{\infty} \frac{\sin(nt)}{n} = A - \frac{t}{2}, \quad \text{all} \quad 0 < t < 2\pi .$$

Plug in $t = \pi$; this forces A to be $\frac{\pi}{2}$.

Let $t = 2\pi q$, $0 < q < 1$, to get

$$\sum_{1}^{\infty} \frac{\sin(2\pi n q)}{n} = \frac{\pi}{2} - \pi q, \quad \text{all} \quad 0 < q < 1 .$$

Hence

$$\sum_{1}^{\infty} \frac{\sin(2\pi n q)}{\pi n} = \frac{1}{2} - q \quad , \quad 0 < q < 1 ,$$

and the rest is trivial by periodicity.

OK

NOTE:

On 22 line 4, if we keep t variable but inside $[\delta, 2\pi - \delta]$, this limit procedure is easily seen to be uniform wrt t as $N \to \infty$.

$t \geq \delta$ used in $\int_0^{(N+\frac{1}{2})t} \frac{\sin \xi}{\xi} d\xi$
$t \leq 2\pi - \delta$ used in $\int_0^t h(v) \sin\left[(N+\tfrac{1}{2}) v\right] dv$ Review 20 (middle) with obvious changes.