LECTURE 1: Introduction; very basic theorems and definitions; Chebyshev's theorem (part 1); statement of the prime number theorem.

LECTURE 2: $\pi(x) \log(x) \sim \psi(x) \sim \theta(x)$; Chebyshev's theorem (part 2); comments about improvements; list of dates concerning the PNT.

LECTURES 3 + 4: Review of some complex analysis, plus several high points involving Riemann-Stieltjes integrals.

LECTURE 5: More on Riemann-Stieltjes integrals; Abel's lemma; Abel's theorem on power series; complex logarithms; infinite products and their convergence properties; first results on the Riemann zeta function $\zeta(z)$ and its analytic continuation.

LECTURE 6: More on infinite products; Euler's identity; beginning estimates for $\zeta(z)$ in $\{\text{Re}(z) > 0\}$; nonvanishing of $\zeta(z)$ along $\{\text{Re}(z) = 1\}$.

LECTURE 7: The Abel summation lemma and related convergence tests; derivation of the Fourier series development of $x - [x] - \frac{1}{2}$; improved estimates for $\zeta(z)$ and $1/\zeta(z)$; Riemann's formula for $\psi_1(x)$; proof that $\psi_1(x) \sim x^2/2$.

LECTURE 8: Proof that $\psi(x) \sim x$, hence of the PNT; Fourier integral approach to $\psi_1(x)$; Euler-Maclaurin (version 1); boundedness of partial sums in the Fourier series of $x - [x] - \frac{1}{2}$.

LECTURES 9 + 10: Euler's formula for $\zeta(2k)$; the expansions of $\pi \cotn(\pi z)$ and $\sin(\pi z)$; Euler-Maclaurin (version 2); use of E-M to prove the analyticity of $\zeta(z)$ on $\mathbb{C} - \{1\}$; basic properties of $\Gamma(z)$, including Stirling's formula.

LECTURE 11: Review of Fourier series; Poisson summation formula; the Riemann $\xi$-function; the functional equation for $\xi$ and $\xi_0$; related estimates.

LECTURE 12: Review of some complex function theory, including Jensen's formula, the Hadamard-Borel-Caratheodory lemma, canonical products, and the Hadamard factorization theorem.

LECTURE 13: More function theory; application to $\xi(s)$; infinite number of zeros; Riemann's formula for $\zeta'(s)/\zeta(s)$; the classical zero-free region for $\zeta(s)$. 
LECTURE 14: Classical "big oh" estimates for \( \psi(x) - x \) and \( \pi(x) - \text{li}(x) \).

LECTURE 15: Preparations for deriving Riemann's explicit formula for \( \psi_1(x) \); the sliding partial fraction development of \( \zeta'(s)/\zeta(s) \); the function \( S(T) \) and its use in counting (à la Riemann) the number of zeros \( \rho \) up to height \( T \).

LECTURE 16: The explicit formula for \( \psi_1(x) \); relation to the prime number theorem; standard estimates for \( \psi(x) - x \) and \( \pi(x) - \text{li}(x) \) involving the "maximum abscissa" \( \Theta \).

LECTURES 17+18: The explicit formula for \( \psi(x) \); applications to the size of \( \psi(x) - x \).

LECTURES 19+20: The Perron summation formula with error term; the Möbius \( \mu \)-function; an estimate for the summatory function \( M(x) \); convergence of certain Dirichlet series involving \( \mu(n) \); the Möbius inversion formula; elementary equivalence of \( M(x) = o(x) \) and the prime number theorem.

LECTURE 21: Completion of the proof that PNT and \( M(x) = o(x) \) are equivalent; the classical Dirichlet divisor problem bound; review of some basic properties of generalized Dirichlet series and "Dirichlet integrals"; Landau's theorem on singular points; elementary \( \Omega_\pm \) estimates for \( \psi(x) - x \) and \( \pi(x) - \text{li}(x) \) referencing \( \Theta - \varepsilon \).

LECTURE 22: Preparations for Landau's proof of Hardy's theorem asserting an infinite number of zeros of \( \zeta(s) \) along the critical line \( \{ \text{Re}(s) = \frac{1}{2} \} \).

LECTURE 23: Landau's proof of the Hardy theorem. (Consult the Addendum in Lecture 24 for an improvement.)

LECTURE 24: Recollection of some complex function theory involving estimates of Phragmén-Lindelöf type; the Lindelöf \( \mu \)-function for \( \zeta(s) \); Littlewood's formula concerning the zero-counting function \( N(\sigma; T_1; T_2) \).

LECTURE 25: More on Lindelöf's function \( \mu(\sigma) \), also on generalized Dirichlet series; a quick rehash of Fourier transforms; development of a simple \( L_2 \) estimate for Dirichlet polynomials in the spirit of Hilbert's inequality.

LECTURE 26: Some van der Corput-type estimates of exponential sums; the 1921 Hardy-Littlewood theorem on approximating \( \zeta(s) \) by its partial sums.

LECTURE 27: The Bohr-Landau theorem that all but an infinitesimal proportion of the zeros of \( \zeta(s) \) are located within \( \{|\text{Re}(s) - \frac{1}{2}| < \varepsilon\} \).


LECTURE 29: J. E. Littlewood's 1914 \( \Omega_+ \) estimate for \( \psi(x) - x \) proved using a variant of Ingham's 1936 Fourier transform technique.
LECTURE 30: Connecting Euler's 1748 assertion that $\sum \frac{\mu(n)}{n} = 0$ to $M(x) = o(x)$ and the prime number theorem; some concluding remarks on $S(T)$ and Turing's Law (for checking the Riemann Hypothesis without ever leaving the line $\text{Re}(s) = \frac{1}{2}$).

ADDENDUM A: Pertaining to Lecture 30 ("an alternate ending").

ADDENDUM B: Pertaining to Lecture 28.

These notes will be a kind of loose/informal diary of what was discussed in the lecture. They are not a textbook, nor will they strive for "completeness".

Analytic Number Theory = that part of number theory wherein results are obtained by use of functions and appropriate analytic techniques.

Traditionally, the "analytic techniques" have come from complex analysis (i.e., the theory of analytic functions). But, this focus has gradually been widened.

Multiplicative A Number Theory (loosely speaking) deals with aspects/functions which are intimately tied to prime numbers and the unique factorization property of the positive integers.
The first 2 lectures will use elementary analysis combined with arithmetic to get some nontrivial information about primes.

I assume the unique factorization thm as being known; i.e.,

\[ n = p_1^{e_1} \cdots p_r^{e_r}, \]

uniquely with primes \( p_1 < \cdots < p_r \) and \( e_j \geq 1 \).

**Theorem 1 (Euclid)**

The number of primes is infinite.

**PF**

Assume not. Let the full list be \( p_1 < \cdots < p_m \).

Form \( N = p_1 \cdots p_m + 1 = \text{integer} \) Factor \( N \). Therefore, some \( p_e \) divides \( N \). But,

\[
\frac{N}{p_e} = p_1 \cdots \hat{p_e} \cdots p_m + \frac{1}{p_e} \quad \text{[1 means omit]}
\]

This is not an integer (since \( p_1 = 2 \)).

Contrad!
Def:

\[ \pi(x) = N[\frac{p}{\pi} \leq x] \quad \text{for} \quad x > 0. \]

By thm 1, \( \pi(x) \to \infty \) as \( x \to \infty \).

The idea of the proof of thm 1 can be used to prove \( \pi(x) \leq \ln \ln x \). See Hardy–Wright, thm 10. This is not very interesting.

Notation:

\[ f(t) = O[g(t)] \quad \text{as} \quad t \to \infty \]

means \( |f(t)| \leq M |g(t)| \) for \( t \geq t_0 \), some large \( t_0 \). But:

\[ f(t) = O[g(t)] \quad \text{for} \quad t \geq \alpha \]

means \( |f(t)| \leq M |g(t)| \) for all \( t \geq \alpha \).

In most situations, one can "get" the 2nd version merely by inflating \( M \) appropriately. This is why I wrote \( M \).
Def: 

\[ \Theta(x) = \sum_{\substack{p \leq x}} \ln p \] \[ \Psi(x) = \sum_{\substack{p^m \leq x}} \ln p \quad (x > 0). \]

Empty sums are defined to be 0; hence \[ \Theta(x) = \Psi(x) = 0 \] for \( x < 2 \).

Notation used by Chebyshev!

Obviously

\[ \Psi(x) = \Theta(x) + \Theta(x^{1/2}) + \Theta(x^{1/3}) + \cdots \]

where eventually \( \Theta(x^{1/N}) = 0 \).

Thm 2

\[ \Psi(x) = \sum_{\substack{p \leq x}} \left\lfloor \frac{\ln x}{\ln p} \right\rfloor \ln p \quad (x \geq 2). \]

For \( x < 2 \), both sides are 0.

Proof

Choose any prime \( p \leq x \). We seek its total contribution to the def. of \( \Psi(x) \).

It will contribute a \( \ln p \) so long as \( p^m \leq x \), i.e. \( m \leq \left\lfloor \frac{\ln x}{\ln p} \right\rfloor \), i.e. \( m \leq \left\lfloor \frac{\ln x}{\ln p} \right\rfloor \). Thus,
we collectively get \( \left\lceil \frac{\ln x}{\ln p} \right\rceil \ln p \). Now, just add over all relevant \( p \).

**Thm 3**

(a) \( \psi(x) = \Theta(x) + O \left[ x^{\frac{1}{2}} (\ln x)^2 \right] \), \( x \geq 2 \);

(b) we get \( \psi(x) = \Theta(x) + O(x^{\frac{1}{2}}) \) if we can somehow prove that \( \Theta(y) = O(y) \) for all \( y \gg 1 \).

**PF**

Need to estimate \( \sum_{n=2}^{\infty} \Theta \left( x^{\frac{1}{n}} \right) \). But \( \Theta \left( x^{\frac{1}{n}} \right) = 0 \) once \( x^{\frac{1}{n}} < 2 \), i.e. \( n > \frac{\ln x}{\ln 2} \). Get:

\[
\sum_{n=2}^{\infty} \Theta \left( x^{\frac{1}{n}} \right) = \sum_{n=2}^{L} \Theta \left( x^{\frac{1}{n}} \right) + L \geq \left\lceil \frac{\ln x}{\ln 2} \right\rceil + 10
\]

\[
11 L \left( L - 1 \right) \Theta \left( x^{\frac{1}{2}} \right) \leq L \sum_{p \leq x^{\frac{1}{2}}} \ln p
\]

\[
11 L \ln x \sum_{p \leq x^{\frac{1}{2}}} \frac{1}{p} \leq L \ln x \sum_{n \leq x^{\frac{1}{2}}} \frac{1}{n} = 0 \left[ x^{\frac{1}{2}} (\ln x)^2 \right] \quad \{ \text{very crudely} \}
\]

This proves (a). For (b), we need to look at:

\[
\sum_{n=3}^{\infty} \Theta \left( x^{\frac{1}{n}} \right) = \sum_{n=3}^{L} \Theta \left( x^{\frac{1}{n}} \right).
\]
A trivial modification shows that this sum is $O[x^{\frac{1}{3}} (\log x)^2]$. Hence,

$$
\psi(x) = \Theta(x) + \Theta(x^{\frac{1}{2}}) + O[x^{\frac{1}{3}} (\log x)^2] \\
= \Theta(x) + O(x^{\frac{1}{2}})
$$

by the hypothesis about $\Theta(y)$. \[\Box\]

Thm 4 (Legendre - early 1800s)

Given $m \geq 2$. We then have

$$
m! = \prod p^E_p, \quad E_p = \sum_{j=1}^{\Delta} \left\lfloor \frac{m}{p^j} \right\rfloor.
$$

Proof

We use "baby set theory" (i.e., classes).

Write $m! = \prod_{k=1}^{m} k$. The only primes that "go into" $k$ are clearly $\leq m$.

(temporarily)

With this being the case, fix any prime $p \leq m$.

Choose $L$ so that $p^L \leq m < p^{L+1}$.

Say that $k \in [1, m]$ is of "type $j$" when

$$
k = p^j (\text{integer relatively prime to } p).
$$

Here $j \geq 0$. THINK UNIQUE FACTORIZATION THM.
Let $\nu_j = \text{card} \{ k \in [1, m] : k \text{ is of type } j \}$. 

Notice that

\[
\begin{align*}
\nu_0 + \nu_1 + \nu_2 + \ldots + \nu_L &= m \quad \text{type } 0, \ldots, L \\
\nu_1 + \nu_2 + \ldots + \nu_L &= \left\lfloor \frac{m}{p} \right\rfloor \quad 1, \ldots, L \\
\nu_2 + \ldots + \nu_L &= \left\lfloor \frac{m}{p^2} \right\rfloor \quad 2, \ldots, L \\
& \vdots \\
\nu_L &= \left\lfloor \frac{m}{p^L} \right\rfloor \quad L
\end{align*}
\]

Then,

Delete the first row and add! Get:

\[
\nu_1 + 2\nu_2 + 3\nu_3 + \ldots + L\nu_L = \sum_{j=1}^{L} \left\lfloor \frac{m}{p^j} \right\rfloor.
\]

But, by construction, we clearly have

\[
E_p = \nu_1 + 2\nu_2 + \ldots + L\nu_L 
\]

(Just think about this a second!)

Now, just slap together all the $p^{E_p}$.}

Clearly any primes $> m$ get $E_p = 0$, exactly as they should.
Chebyshev became interested in quotients of various factorials which turn out to be integers.

E.g., the binomial coefficient \( \binom{2n}{n} = \frac{(2n)!}{n!n!} \).

He sought to use standard Stirling-type estimates to get information about the number of primes in certain intervals.

Clearly, thus it will be vital here.

We note:

**Baby Lemma**

Let \( x \) be a positive rational (say, \( \frac{m}{n} \) in lowest terms). The relation

\[
x = p_1^{A_1} \cdots p_r^{A_r} = p_1^{B_1} \cdots p_r^{B_r} \quad (p_1 < \cdots < p_r)
\]

holds with \( A_j \in \mathbb{Z} \) and \( B_j \in \mathbb{Z} \) only if \( A_j = B_j \).

If \( x \in \mathbb{Z}^+ \) (i.e., \( n=1 \)), each \( A_j \) is necessarily \( \geq 0 \).
\[ \text{pf} \]

Let \( G \) be giant. Notice that

\[ (p_1, \ldots, p_r)^G A_1 \ldots A_r = (p_1, \ldots, p_r)^G B_1 \ldots B_r = \text{integer}. \]

By unique factorization thm, \( G + A_j \cong G + B_j \).

The Baby Lemma says that unique factorization extends to \( \mathbb{A}^+ \) in an obvious way.

It also implies that the numbers \( h_n p_j \) are linearly independent over \( \mathbb{Q} \).

---

**Thm 5** (elementary integral calculus)

\[ m \geq 2 \quad \Rightarrow \quad \ln(m!) = m \ln m - m + O(\ln m). \]

**pf**

Suppose \( y = f(x) \) is continuous, non-neg, and increasing on \( 1 \leq x \leq N+1 \). By drawing a picture, clearly

\[ f(1) + \ldots + f(N) \leq \int_1^{N+1} f(t) dt \leq f(2) + \ldots + f(N+1) \]

\[ \Rightarrow \quad 0 \leq \int_1^{N+1} f(t) dt - \sum_{j=1}^{N} f(j) \leq f(N+1) - f(1). \]
Simply put \( f(t) = \ln t \) to get:

\[
0 \approx \int_1^{N+1} \ln t \, dt - \sum_{j=1}^{N} \ln j \leq \ln(N+1)
\]

\[
\frac{1}{x} \ln t \, dt = x \ln x - x + 1, \quad x = 1,2
\]

\[
0 \approx (N+1) \ln(N+1) - N - \ln(N!) \leq \ln(N+1)
\]

\[
(N+1) \ln(N+1) - N - \ln(N!) = \omega \ln(N+1)
\]

For some \( \omega \in [0,1] \)

\[
\ln(N!) = (N+1) \ln(N+1) - N - \omega \ln(N+1)
\]

\[
= N \ln(N+1) + (1 - \omega) \ln(N+1) - N
\]

\[
= N \ln(N+1) + O(1) + O(\ln N) - N
\]

\[
= N \ln N - N + O(\ln N)
\]
Let
\[ L(x) = \sum_{k \leq x} \ln k \quad \text{for } x > 0. \]

For \( x < 2 \), \( L(x) = 0 \). For \( x \geq 2 \), \( L(x) = \ln (\|x\|!) \).

**Thm 6**
\[ L(x) = x \ln x - x + O(\ln x) \quad \text{for } x \geq 2. \]

**Pf**
By inflating the constant, WLOG \( x \geq 100 \).
Let \( M \leq x < M+1 \). By Thm 5,
\[ L(x) = L(M) = M \ln M - M + O(\ln M). \]
The derivative of that - it is \( \ln \). Hence \( x \ln x - x \) differs from \( M \ln M - M \) by at most \( \ln(M+1) \). As such,
\[
\begin{align*}
L(x) &= x \ln x - x + O(\ln(M+1)) + O(\ln M) \\
&= x \ln x - x + O(\ln x).
\end{align*}
\]
Thm 7

Define (following von Mangoldt):

\[ \Lambda(n) = \begin{cases} \ln p & \text{if } n = p^j \ (j \geq 1) \\ 0 & \text{otherwise} \end{cases} \]

Here \( n \geq 1 \). We have:

(a) \( \Psi(x) = \sum_{n \leq x} \Lambda(n) \) \( \quad \sum \) \( \overrightarrow{\text{for } \log \left[ x \right]!} \)

(b) \( L(x) = \sum_{k \leq x} \left[ \frac{x}{k} \right] \Lambda(k) \)

(c) \( L(x) = \Psi(x) + \Psi \left( \frac{x}{2} \right) + \Psi \left( \frac{x}{3} \right) + \cdots \)

Proof

When \( x < 2 \), each of (a)(b)(c) just states \( 0 = 0 \).
So, WLOG, \( x \geq 2 \).

Assertion (a) is now a tautology. \( \square \)

For (b), write \( M \leq x < M + 1 \). By (ii)(top), obviously

\[ L(x) = L(M) = \log (M!) \]

By Thm 4, Legendre-style,

\[ L(M) = \sum_p E_p \ln p = \sum_p \left( \sum_{j=1}^{\alpha} \left[ \frac{M}{p^j} \right] \right) \ln p \]

\[ = \sum_{\text{all } k \leq M} \left[ \frac{M}{k} \right] \Lambda(k) \]
The key issue now boils down to:

if \( p^j \leq M \ (j \geq 1) \), why is \( \left\lfloor \frac{X}{p^j} \right\rfloor = \left\lfloor \frac{M}{p^j} \right\rfloor \)?

To verify this, let

\[ \ell = \left\lfloor \frac{M}{p^j} \right\rfloor \ (\geq 1) \]

We have

\[ \frac{M}{p^j} = \ell + \varphi \quad \text{with} \quad 0 \leq \varphi < 1 \]

Write \( M = \varphi(p^j) + \xi \), \( 0 \leq \xi \leq p^j - 1 \) à la Euclid.

Clearly,

\[ 0 \leq \varphi \leq \frac{p^j - 1}{p^j} \]

Also write \( X = \ell + \Theta \), \( 0 \leq \Theta < 1 \). We then have:

\[ \ell \leq \frac{X}{p^j} = \frac{\ell + \Theta}{p^j} = \frac{\ell}{p^j} + \frac{\Theta}{p^j} = \ell + \varphi + \frac{\Theta}{p^j} \]

\[ \leq \ell + \frac{p^j - 1}{p^j} + \frac{1}{p^j} = \ell + 1 \]

which gives \( \left\lfloor \frac{X}{p^j} \right\rfloor = \ell = \left\lfloor \frac{M}{p^j} \right\rfloor \), as desired.

Accordingly:

\[ L(x) = \sum_{k \leq x} \left\lfloor \frac{x}{k} \right\rfloor \Lambda(k) \quad \text{(OK)} \]
To prove (c), we start with \( \sum_{m=1}^{\infty} \psi \left( \frac{x}{m} \right) \). (cf. (a))

Take any given integer \( p^A \) with \( A \geq 1 \). We ask: how many times does \( \Lambda(p^A) \) assert "I am present" within \( \sum_{m=1}^{\infty} \psi \left( \frac{x}{m} \right) \)? This number will clearly be the largest \( \ell \) so that \( p^A \leq \frac{x}{\ell} \). In other words, \( \ell \leq \frac{x}{p^A} \) or \( \ell \leq \left\lfloor \frac{x}{p^A} \right\rfloor \). The collective contribution of \( p^A \) to \( \sum_{m=1}^{\infty} \psi \left( \frac{x}{m} \right) \) will therefore be \( \left\lfloor \frac{x}{p^A} \right\rfloor \Lambda(p^A) \).

In view of (b), it is now evident that \( \sum_{m=1}^{\infty} \psi \left( \frac{x}{m} \right) \) must reduce to \( L(x) \).  

---

Chebyshev played with \( \binom{2n}{n} \frac{(2n)!}{n!n!} \) and was therefore motivated to examine

\[ L(x) - 2L \left( \frac{x}{2} \right) \]

---

**N.B.** Theorem 7(b) applies to \( L(x) - 2L \left( \frac{x}{2} \right) \), but it is easier to use Thm 7(c).
Suppose for a moment that \( x \geq 4 \). A quick calculation with Thm 6 gives

\[
L(x) - 2L\left(\frac{x}{2}\right) = x (\ln 2) + O(\ln x).
\]

By inflating the constant à la \( \Theta \) (bottom), the same relation holds for \( x \geq 2 \).

On the other hand, by Thm 7(c),

\[
L(x) - 2L\left(\frac{x}{2}\right) = \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \psi\left(\frac{x}{4}\right) + \ldots
\]

(where, as usual, the terms are eventually 0).

---

We therefore have:

\[
x (\ln 2) + O(\ln x) = \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \psi\left(\frac{x}{4}\right) + \ldots
\]

for all \( x \geq 2 \).

---

This last relation can be manipulated.
Recall that $\psi(y)$ is non-neg and monotonically increasing. See Thm 7 (a).

Accordingly:

$$x (\ln 2) + O(\ln x) = \psi(x) - \left[ \psi\left(\frac{x}{2}\right) - \psi\left(\frac{x}{3}\right) \right] - \cdots$$

\begin{align*}
\forall \\
\psi(x) \geq x (\ln 2) + O(\ln x)
\end{align*}

At the same time:

$$x \ln 2 + O(\ln x) = \left[ \psi(x) - \psi\left(\frac{x}{2}\right) \right] + \left[ \psi\left(\frac{x}{2}\right) - \psi\left(\frac{x}{3}\right) \right] + \cdots$$

\begin{align*}
\forall \\
\psi(x) - \psi\left(\frac{x}{2}\right) \leq x \ln 2 + O(\ln x)
\end{align*}

In both instances, one keeps $x \geq 2$. 
Take $x \geq 1000$ (say) and make an iteration as follows:

$$
\psi(x) - \psi\left(\frac{x}{2}\right) \asymp x \ln 2 + B \ln x \quad \text{step 1}
$$

$$
\psi\left(\frac{x}{2}\right) - \psi\left(\frac{x}{4}\right) \asymp \frac{x}{2} \ln 2 + B \ln \frac{x}{2} \quad \text{step 2}
$$

$$
\vdots

$$

$$
\psi\left(\frac{x}{2^r}\right) - \psi\left(\frac{x}{2^{r+1}}\right) \asymp \frac{x}{2^r} \ln 2 + B \ln \frac{x}{2^r} \quad \text{step } r+1
$$

\[\begin{align*}
\text{take} \quad \frac{x}{2^r} & \in \left[4, 8\right] \text{ for safety} \\
\text{hence} \quad r & = \frac{\ln x - \ln 2}{\ln 2}, \quad \ln 4 \leq r \leq \ln 8
\end{align*}\]

ADD

\[\downarrow\]

$$
\psi(x) \geq O(1) \leq 2x \ln 2 + \frac{B}{\ln 2} \left(\ln x\right)^2
$$

$$
\downarrow
$$

$$
\psi(x) \leq x \left(\ln 4\right) + \frac{B}{\ln 2} \left(\ln x\right)^2
$$

$$
\downarrow
$$
\[ \psi(x) = x (\ln 4) + O[\ln^2 x]. \]

To include \( 2 \leq x < 1000 \), one can inflate the constant.

**Theorem A (Chebyshev \( \times 1850 \))**

For \( x \geq 2 \), we have:

\[ x (\ln 2) + O(\log x) \leq \psi(x) \leq x (\ln 4) + O(\log^2 x). \]

\[ x (\ln 2) + O(x^{1/2}) \leq \Theta(x) \leq x (\ln 4) + O(x^{1/2}). \]

**Proof**

The case of \( \psi(x) \) was just done. \( \checkmark \)

Since \( 0 \leq \Theta(y) \leq \psi(y) \), clearly \( \Theta(y) = O(y) \) for all \( y \gg 1 \). Recalling Thm 3(b), we have

\[ \Theta(x) = \psi(x) + O(x^{1/2}), \]

which produces the inequality for \( \Theta(x) \). \( \checkmark \)
Notice that
\[ \pi(x) \ln x = \sum_{\rho \leq x} \ln \rho \geq \sum_{\rho \leq x} \ln \rho = \Theta(x). \]

In light of this, theorem A assures us that
\[ \pi(x) \approx \frac{x}{(\ln 2 - \varepsilon) \ln x} \]

for \( x \geq x_0(\varepsilon) \). Here \( \varepsilon > 0 \) is arbitrary.
(p. 3 line 5 is trivial in comparison)

The celebrated Prime Number Theorem, which we seek to prove soon, states that
\[ \pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty. \]

We'll say more about \( \pi(x) \) in Lecture 2.
Our primary goal today is to improve Theorem A from Lecture 1. We also wish to address some preliminary stuff as well.

**Theorem 1**

\[ x \geq 2. \text{ We have} \]

\[ \frac{\Theta(x)}{\ln x} \leq \pi(x) \leq x^{1-\delta} + \frac{\Theta(x)}{\ln x} \frac{1}{1-\delta} \]

for any \( 0 < \delta < 1 \).

**PF**

\[ \pi(x) \ln x = \sum_{p \leq x} \ln x \approx \sum_{p \leq x} \ln p = \Theta(x) \text{ is obvious. On the other hand,} \]

\[ \Theta(x) - \Theta(x^{1-\delta}) = \sum_{x^{1-\delta} < p \leq x} \ln p \geq \ln(x^{1-\delta}) \left[ \pi(x) - \pi(x^{1-\delta}) \right] \]

\[ \pi(x) - \pi(x^{1-\delta}) \leq \frac{\Theta(x) - \Theta(x^{1-\delta})}{(1-\delta) \ln x} \]

\[ \pi(x) \approx \pi(x^{1-\delta}) + \frac{\Theta(x)}{(1-\delta) \ln x} \quad \{ \text{since } \Theta(y) \geq 0 \} \]

\[ \pi(x) \approx x^{1-\delta} + \frac{\Theta(x)}{(1-\delta) \ln x} \quad \{ \text{trivially: } \pi(y) \leq y \} \]

(OK)
Corollary 1

As \( x \to \infty \),

\[
\pi(x) \sim \frac{\Theta(x)}{\ln x} \sim \frac{\psi(x)}{\ln x}.
\]

Proof

By Lec 1 thm A, we know \( c_1 x < \Theta(x) < c_2 x \). By Lec 1 thm 3, we then get

\[
\frac{\psi(x)}{\Theta(x)} \rightarrow 1 \quad \text{as} \quad x \rightarrow \infty.
\]

We need to show \( \frac{\pi(x) \ln x}{\Theta(x)} \rightarrow 1 \) too. But here we can apply Thm 1 above to get:

\[
\Theta(x) \leq \pi(x) \ln x \leq x^{1-\delta} \ln x + \frac{\Theta(x)}{1-\delta}
\]

\[
1 \leq \frac{\pi(x) \ln x}{\Theta(x)} \leq x^{1-\delta} \ln x + \frac{1}{1-\delta}.
\]

Just take \( \delta \) smaller and smaller! Clearly

\[
\lim \sup_{x \to \infty} \frac{\pi(x) \ln x}{\Theta(x)} \leq \frac{1}{1-\delta}
\]

(again using \( c_1 x < \Theta(x) < c_2 x \)), so we are done. \( \square \)

This successively reducing the \( \delta \) seems a bit ugly. We can junk it.
Corollary 2

For \( x \geq x_0 \),

\[
\frac{\pi(x) \ln x}{\theta(x)} \leq 1 + \frac{O(1)}{\sqrt{\ln x}}.
\]

**Proof**

By inflating the constant in \( O(1) \), we can assume \( x \geq x_0 \) (suff. large) so that

\[
3 \frac{\ln \ln x}{\ln x} < \frac{1}{a}, \quad \text{say}.
\]

We propose to simply take \( \delta = 3 \frac{\ln \ln x}{\ln x} \) in Theorem 1 above. Note that

\[
\frac{1}{1 - u} < 1 + 2u \quad \text{for} \quad 0 < u < \frac{1}{a}.
\]

Plug in page 2 line 11 [which is just a rewrite of Thm 1]. Get:

\[
1 \leq \frac{\pi(x) \ln x}{\theta(x)} \leq \frac{1}{\theta(x)} x, e^{-\delta \ln x}, \ln x + 1 + 2 \delta
\]

\[
= \frac{1}{\theta(x)} x, \frac{\ln x}{(\ln x)^3} + 1 + 6 \frac{\ln \ln x}{\ln x}
\]

\[
= \frac{x}{\theta(x)} \frac{1}{(\ln x)^2} + 1 + \frac{6 \ln \ln x}{\ln x}.
\]

Remember that \( e_1 x < \theta(x) < e_2 x \) and \( x \geq x_0 \). The last expression is clearly \( \leq 1 + \frac{O(1)}{\sqrt{\ln x}} \).
**Theorem A' (Chebyshev ≈ 1850)**

For \( x \geq 2 \),

\[
x(ln2) + O(lnx) \leq \psi(x) \leq x(ln4) + O(ln^2 x)
\]

\[
x(ln2) + O(x^{1/2}) \leq \theta(x) \leq x(ln4) + O(x^{1/2})
\]

\[
[ln2 + o(1)] \frac{x}{lnx} \leq \pi(x) \leq [ln4 + o(1)] \frac{x}{lnx}
\]

Here \( o(1) \) means "bounded but tends to zero as \( x \to \infty \)."

**Proof**

For the first 2 lines, see Lec 1 Thm A.

The third line, pertaining to \( \pi(x) \), follows from corollary 1 or 2.

\[
ln2 = 0.693147^+
\]

\[
ln4 = 1.386294^+
\]

We would like to reduce the spread between these numbers!

Still using elementary techniques...
Chebyshev also played with the combination
\[
\frac{(30m)! \cdot m!}{(15m)! \cdot (10m)! \cdot (6m)!}.
\]

It is NOT obvious this is an integer!

Note that
\[
30 - 15 - 10 - 6 + 1 = 30 \left[ 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{30} \right] = 0.
\]

Correspondingly, let
\[
\Psi_0(x) = \ln(x) - \ln \left( \frac{x}{2} \right) - \ln \left( \frac{x}{3} \right) - \ln \left( \frac{x}{5} \right) + \ln \left( \frac{x}{30} \right)
\]
\[
\sum \left\{ \ln \left( \left\lfloor \frac{x}{n} \right\rfloor! \right) \right. = \sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor \Lambda(n) \right.,
\]
\[
= \sum_{n \leq x} \left( \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{2n} \right\rfloor - \left\lfloor \frac{x}{3n} \right\rfloor - \left\lfloor \frac{x}{5n} \right\rfloor + \left\lfloor \frac{x}{30n} \right\rfloor \right) \Lambda(n)
\]
\[
= \sum_{n \leq x} \sigma \left( \frac{x}{n} \right) \Lambda(n).
\]

\(\sigma(x)\) has interesting properties (as one soon discovers). First of all,
\[
\sigma(x) = \left\lfloor x \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor - \left\lfloor \frac{x}{3} \right\rfloor - \left\lfloor \frac{x}{5} \right\rfloor + \left\lfloor \frac{x}{30} \right\rfloor.
\]
Note that \([y] \text{ and } \sigma(y)\) are right continuous on \(R\). Also,

\[
\sigma(x+30) = \lfloor x \rfloor + 30 \\
- \lfloor \frac{x}{5} \rfloor - 15 \\
- \lfloor \frac{x}{3} \rfloor - 10 \\
- \lfloor \frac{x}{5} \rfloor - 6 \\
+ \left\lceil \frac{x}{30} \right\rceil + 1 = \sigma(x) .
\]

So, \(\sigma(x)\) is periodic 30. To appreciate \(\sigma\), one simply breaks down and computes it by hand or via a computer.

\([0,30) = [0,1) \cup [1,2) \cup [2,3) \cup \cdots \cup [29,30)\).

GET:

\[
\begin{array}{cccccc}
[0,1) & 0 & 10 & 0 & 20 & 0 \\
[1,2) & 1 & 11 & 1 & 21 & 0 \\
[2,3) & 1 & 12 & 0 & 22 & 0 \\
[3,4) & 1 & 13 & 1 & 23 & 1 \\
[4,5) & 1 & 14 & 1 & 24 & 0 \\
[5,6) & 1 & 15 & 0 & 25 & 0 \\
[6,7) & 0 & 16 & 0 & 26 & 0 \\
[7,8) & 1 & 17 & 1 & 27 & 0 \\
[8,9) & 1 & 18 & 0 & 28 & 0 \\
[9,10) & 1 & 19 & 1 & 29 & 1 \\
\end{array}
\]
We thus find that \( \sigma = 0 \) or 1 for all \( x \).

(It was not obvious a priori that, e.g., \( \sigma \geq 0 \).)

\[ \square \]

**Note**: Notice that the original factorial quotient on line 2 has logarithm \( \psi \sigma (30m) \). Since \( \sigma \in \{0, 1\} \), the formula on line 9 makes it clear that the original quotient is a positive integer.

Clearly,

\[ \psi \sigma (x) = \sum_{n \leq x} \sigma \left( \frac{x}{n} \right) \Lambda(n) \leq \sum_{n \leq x} \Lambda(n) = \psi(x). \]

Also:

\[ \sigma \left( \frac{x}{n} \right) = 1 \quad \text{for} \quad 1 \leq \frac{x}{n} < 6 \quad \text{is VERY convenient} \]

\[ \text{Let} \quad \sqrt[n]{\sigma \left( \frac{x}{n} \right)} = 1 \quad \text{for all} \quad \frac{x}{6} < n \leq x. \]

Notice that \( \sigma \left( \frac{x}{n} \right) = 1 \) in some other portions of \( n \leq x \) too. But, for now, we don't use this.
As a tautology,

\[ \psi_0(x) = \sum_{\frac{x}{6} < n \leq x} \sigma(\frac{x}{n}) \Lambda(n) + \sum_{n \leq \frac{x}{6}} \sigma(\frac{x}{n}) \Lambda(n) \]

\( \text{non-negative!} \)

\[ \downarrow \]

\[ \psi_0(x) = \sum_{\frac{x}{6} < n \leq x} \frac{1}{n} \Lambda(n) = \psi(x) - \psi(\frac{x}{6}) \]

\[ \downarrow \]

\[ \psi(x) \leq \psi_0(x) \leq \psi(\frac{x}{6}) + \psi_0(x). \]

So,

\[ \psi_0(x) \leq \psi(x) \leq \psi_0(x) + \psi(\frac{x}{6}) \].
Recall (Lec 1, Thm 6)

\[ L(y) = y \ln y - y + O(\ln y) \quad \text{"Stirling"} \]

for all \( y \geq 2 \).

We substitute into

\[ \psi_0(x) = L(x) - L(\frac{x}{2}) - L(\frac{x}{3}) - L(\frac{x}{5}) + L(\frac{x}{30}) \]

keeping \( x \approx 60 \) for safety. Get:

\[
x \ln x - x + O(\ln x)
\]
\[
- \frac{x}{2} \ln \left( \frac{x}{2} \right) + \frac{x}{2} + O(\ln x)
\]
\[
- \frac{x}{3} \ln \left( \frac{x}{3} \right) + \frac{x}{3} + O(\ln x)
\]
\[
- \frac{x}{5} \ln \left( \frac{x}{5} \right) + \frac{x}{5} + O(\ln x)
\]
\[
+ \frac{x}{30} \ln \left( \frac{x}{30} \right) - \frac{x}{30} + O(\ln x)
\]

\[
= \left( 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{30} \right) x \ln x
\]
\[
+ x \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 - \frac{1}{30} \right)
\]
\[
+ x \left( \frac{1}{2} \ln 2 + \frac{1}{3} \ln 3 + \frac{1}{5} \ln 5 - \frac{1}{30} \ln 30 \right)
\]
\[
+ O(\ln x)
\]
\[
\frac{1}{2} \ln 2 + \frac{1}{3} \ln 3 + \frac{1}{5} \ln 5 - \frac{1}{30} \ln 30
\]

\[= 0.9212920229^+\]

\[
\psi'(x) = (0.9212920229^+) x + O(\ln x)
\]

By \(\psi'(x) \leq \psi(x)\) on \(\{7\} + \{8\}\), we get:

\[
(0.921^+) x + O(\ln x) \leq \psi(x)
\]

By \(\psi(x) - \psi(x_0) \leq \psi'(x)\) on \(\{8\}\), we get:

\[
\psi(x) - \psi(x_0) \leq (0.921^+) x + B \ln x
\]

for some \(B\) whenever \(x \geq 60\). We can now iterate this to get:
\[ \psi(x) - \psi\left(\frac{x}{6}\right) \leq (0.921^+) \cdot x + B \cdot \ln x \quad \text{step 1} \]

\[ \psi\left(\frac{x}{6}\right) - \psi\left(\frac{x}{36}\right) \leq (0.921^+) \frac{x}{6} + B \cdot \ln \frac{x}{6} \quad \text{step 2} \]

\[ \vdots \]

\[ \psi\left(\frac{x}{6^r}\right) - \psi\left(\frac{x}{6^{r+1}}\right) \leq (0.921^+) \frac{x}{6^r} + B \cdot \ln \frac{x}{6^r} \quad \text{step } r+1 \]

\[
\begin{cases} 
\text{take } \frac{x}{6^r} \in [216, 1296] \text{ for safety} \\
\text{hence } r = \frac{\ln x - \ln 2}{\ln 6} \quad \ln 216 \leq r \leq \ln 1296 
\end{cases}
\]

ADD

\[ \psi \]

\[ \psi(x) + O(1) \geq \frac{6}{5} (0.921^+) \cdot x + B (r+1) \cdot \ln x \]

\[ \vdots \]

\[ \psi(x) \geq \frac{6}{5} (0.921^+) \cdot x + \frac{B}{\ln 6} (\ln x)^2. \]

Note:

\[ \frac{6}{5} (0.9212920229^+) = 1.05550427^+ \]
Theorem B (harder Chebyshev & 1850) (using \( \psi_0 \))

For \( x \geq 2 \), we have

\[
\begin{align*}
(0.921292^+ x + O(\ln x)) \leq \psi(x) & \leq (1.105550^+) x + O(\ln^3 x) \\
(0.921292^+ x + O(\sqrt{x})) \leq \Theta(x) & \leq (1.105550^+) x + O(\sqrt{x}) \\
\left[ 0.921292^+ + o(1) \right] \frac{x}{\ln x} \leq \pi(x) & \leq \left[ 1.105550^+ + o(1) \right] \frac{x}{\ln x}.
\end{align*}
\]

Here

\[ 1.105550^+ = \frac{6}{5} (0.921292^+) \]

Proof:

See (10) + (11) and inflate the \( \left( \text{implied} \right) \) constants \( (\text{as necessary}) \) for \( \psi(x) \). The rest is like

Theorem A'. \( \blacksquare \)
Addeudum (for completeness)

On (5), \( \psi_\sigma(x) = \sum_{n \leq x} \sigma(n) \Lambda(n) \). Let the value of \( \psi(y) \)
on \([k, k+1]\) be \( \sigma_k \). Here \( k \geq 0 \). Note that index \( k \geq 1 \)
corresponds to \( k \leq x/k < k+1 \) hence \( \frac{x}{k+1} < n \leq \frac{x}{k} \) in \( \psi_\sigma(x) \).

Hence:

\[
\psi_\sigma(x) = \sum_{k=1}^{\infty} \sigma_k \left\{ \sum_{\frac{x}{k+1} < n \leq \frac{x}{k}} \Lambda(n) \right\} = \sum_{k=1}^{\infty} \sigma_k \left[ \psi\left(\frac{x}{k}\right) - \psi\left(\frac{x}{k+1}\right) \right]
\]

\[
= \sigma_1 (\psi_1 - \psi_2) + \sigma_2 (\psi_2 - \psi_3) + \sigma_3 (\psi_3 - \psi_4) + \ldots
\]

\[\text{\{in shorthand\}}\]

\[
= \psi_1 \sigma_1 + \psi_2 (\sigma_2 - \sigma_1) + \psi_3 (\sigma_3 - \sigma_2) + \ldots
\]

\[\text{\{notice that \( \sigma_n - \sigma_{n-1} \) is periodic \( \equiv 0 \mod 2 \}}\]

\[
= \psi(x) + \psi\left(\frac{x}{2}\right)(0) + \psi\left(\frac{x}{3}\right)(0) + \psi\left(\frac{x}{4}\right)(0) + \psi\left(\frac{x}{5}\right)(0) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) + \psi\left(\frac{x}{8}\right)(0) + \psi\left(\frac{x}{9}\right)(0) + \psi\left(\frac{x}{10}\right)(-1)
\]

\[+ \psi\left(\frac{x}{11}\right) + \psi\left(\frac{x}{12}\right)(-1) + \psi\left(\frac{x}{13}\right) + \psi\left(\frac{x}{14}\right)(0) + \psi\left(\frac{x}{15}\right)(-1)
\]

\[+ \psi\left(\frac{x}{16}\right)(0) + \psi\left(\frac{x}{17}\right) + \psi\left(\frac{x}{18}\right)(-1) + \psi\left(\frac{x}{19}\right) + \psi\left(\frac{x}{20}\right)(-1)
\]

\[+ \psi\left(\frac{x}{21}\right)(0) + \psi\left(\frac{x}{22}\right)(0) + \psi\left(\frac{x}{23}\right) + \psi\left(\frac{x}{24}\right)(-1) + \psi\left(\frac{x}{25}\right)(0)
\]

\[+ \psi\left(\frac{x}{26}\right)(0) + \psi\left(\frac{x}{27}\right)(0) + \psi\left(\frac{x}{28}\right)(0) + \psi\left(\frac{x}{29}\right)(1) + \psi\left(\frac{x}{30}\right)(-1)
\]

\[+ \psi\left(\frac{x}{31}\right)(1) + \ldots\]

\[
\psi_\sigma(x) = (1) - (6) + (7) - (10) + (11) - (12) + (13) - (15)
\]

\[+ (17) - (18) + (19) - (20) + (23) - (24) + (39) - (30)
\]

\[+ (31) \pm \ldots\]

NOTE THE ALTERNATE SIGNS
Some Remarks.

It's not immediately clear what other combinations of $L(x^j)$ can be used — and how much of an improvement can be gained.

The standard reference is:


This reference is usually regarded as saying that if one already knows that the PNT is true, then there is in principle no obstruction to building better and better combinations that lead to "$1-\epsilon$ and $1+\epsilon$".

But, the assertion does not take into account the possibility of exploiting recursive relations and additional positive terms like \( f \) bottom line (and \( g \) top, for \( n \leq \frac{x}{6} \)).

J. J. Sylvester found improvements based on use of recursive relations. See:
Sylvester, Amer. J. Math. 4 (1881) 230-247

Also Mathews, Th. of Numbers, pp. 287-294 from 1892.

It seems fair to say the overall status of things is not as clear as one would like.

Incidentally, see: a classic!


For a tiny improvement in \( \frac{6}{5} (1.921292^+) \) based just on \( \Theta \) (bottom line) + \( \Theta \) (top), he got:

\[
\frac{171}{175} \cdot \frac{6}{5} (1.921292^+) = 1.080280^+
\]

We'll drop this stuff temporarily; it seems obvious that some essentially new idea would be needed to reach \( 1-\varepsilon, 1+\varepsilon \) via "elementary reasoning".
Corollary (related to Bertrand's Postulate)

There exists a positive \( c \) so that, for large \( x \),

\[
\sum_{x < p \leq 2x} \ln p \geq cx.
\]

**PF**

LHS \( = \theta(2x) - \theta(x) \).

By Thm B,

\[
\theta(2x) - \theta(x) \geq 2(0.921292^+)\times
\]

\[
- (1.105550^+) \times + O(\sqrt{x})
\]

\[
= \frac{4}{5}(0.921292^+)\times + O(\sqrt{x})
\]

\[
\approx (0.737)x + O(\sqrt{x}).
\]

Hence,

\[
\sum_{x < p \leq 2x} \frac{1}{p} \geq \frac{cx}{\log(2x)}.
\]
Theorem 2

For $x \geq 2$,

$$\sum_{p \leq x} \frac{\Lambda(n)}{p} = \ln x + O(1)$$

$$\sum_{n \leq x} \frac{A(n)}{n} = \ln x + O(1)$$

Proof

$$L(x) = \sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor A(n) \text{ for all } x \geq 2.$$  \{Lec 1, Thm 7\}

Hence,

$$x \ln x + O(x) = \sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor A(n)$$

by Lec 1 Thm 6. Temporarily write

$$\left\lfloor \frac{x}{n} \right\rfloor = \frac{x}{n} - \varphi(n) \quad 0 \leq \varphi(n) < 1.$$  

get

$$x \ln x + O(x) = \sum_{n \leq x} \frac{x}{n} A(n) - \sum_{n \leq x} \varphi(n) A(n)$$

This term is non-negative and $O(x)$ by Thm A'.
\[ x \ln x + O(x) = x \sum_{n \leq x} \frac{\Lambda(n)}{n} + O(x) \]

\[ \sum_{n \leq x} \frac{\Lambda(n)}{n} = \ln x + O(1) \]

**Notice however that**

\[ \sum_{p^2 \leq x} \frac{\ln p}{p^2} + \sum_{p^3 \leq x} \frac{\ln p}{p^3} + \ldots \]

\[ = \sum_{p \leq x} \frac{\ln p}{p^2} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots \right) \]

\[ = \sum_{p \leq x} \frac{\ln p}{p^2} \left( \frac{1}{1 - \frac{1}{p}} \right) = \sum_{p \leq x} \frac{\ln p}{p^2} < +\infty \]

**At once,**

\[ \sum_{p^2 \leq x} \frac{\ln p}{p^2} = \ln x + O(1) \quad \text{too} \]

\[ (\text{Thm 2} \sim \text{Mertens} \approx 1874) \]
**Theorem 3**

We have

\[
\liminf_{x \to \infty} \frac{\pi(x)/\ln x}{x} = \liminf_{x \to \infty} \frac{\psi(x)}{x} \leq 1
\]

\[
\limsup_{x \to \infty} \frac{\pi(x)/\ln x}{x} = \limsup_{x \to \infty} \frac{\psi(x)}{x} \geq 1.
\]

Hence, if \( \psi(x) \sim cx \), we must have \( c = 1 \).

**Proof**

By (2) corollary 1, need only treat \( \psi(x) \).

Recall

\[
L(x) = x \ln x + O(x) = \sum_{k \leq x} \psi\left(\frac{x}{k}\right)
\]

For \( x \geq 2 \), see lecture 7, hence 6 + 7.

Assume \( \liminf_{x \to \infty} \frac{\psi(x)}{x} = \varphi > 1 \). Hence \( \psi(y) \geq (1 + h)y \)

for all \( y \geq x_0 \). For large \( x \), we have

\[
x \ln x + O(x) = \sum_{1 \leq k \leq \frac{x}{x_0}} \psi\left(\frac{x}{k}\right) + \sum_{\frac{x}{x_0} < k \leq x} \psi\left(\frac{x}{k}\right)
\]

\[
\leq (1 + h) \sum_{1 \leq k \leq \frac{x}{x_0}} \frac{x}{k} + \sum_{\frac{x}{x_0} < k \leq x} O(1) \psi(x_0)
\]

\[
= (1 + h) x \sum_{k=1}^{x/x_0} \frac{1}{k} + O\left(\psi(x_0)\right) x
\]
The lim sup case is similar (again by contradiction).

As \( x \to x_0 \), we get \( l \leq 1+h \), an obvious contradiction.

\[ \limsup_{x \to x_0} (f(x) + \epsilon_0) = (1+h) \limsup_{x \to x_0} f(x) \]

[\( f(x) \)]
**Date Highlights**

- **Euler** suggests PNT \( \approx 1740 \sim 1760 \)
- The boy \( \rightarrow \) **Gauss** suggests PNT \( \approx 1790 \)
  - (counts in tables)
- **Legendre** looks like \( \frac{x}{\log x - c} \approx 1800 \)
- **Gauss letter to Encke** \( \int_2^x \frac{dt}{\ln t} \) \( \approx 1849 \)
- **Chebyshev** rigorous \( \frac{1}{\ln x} < \pi(x) < \frac{2}{\ln x} \approx 1850 \)
- **Riemann** \( 1859 \) introd. of complex variable (etc.)
- **Sylvester** elementary refinements \( \text{à la Chebyshev} \)
- **de la Valée-Poussin** \( \) proof of PNT \( \approx 1896 \)
- **Hadamard**
PARTIAL DIARY ENTRY FOR
Lectures 3 and 4
(27 Jan and 29 Jan)

+ SOME NEW STUFF

In lecture #3 and part of #4, we reviewed some key points in complex analysis (a subject regarded by many mathematicians as the most beautiful in mathematics, not only aesthetically but also vis à vis logical unity/coherence).

List of Some Definitions and Theorems

\( C^\infty(D) \) ← the usual

\( A^\infty(D) \) = the subset of \( C^\infty(D) \) consisting of those complex-valued \( f = u + iv \) for which we have a complex derivative

"A" for analytic

\[
A'(z_0) \equiv \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
\]

at each \( z_0 \in D \)

\( A(D) \) = the set of complex-valued \( f \) for which we have a complex derivative \( f'(z_0) \) at each \( z_0 \in D \) (no other assumptions about \( f = u + iv \))
Proved using \( \Delta x = \frac{A \bar{z} + \overline{A_z}}{2} \), \( \Delta y = \frac{A \bar{z} - \overline{A_z}}{2} \) that

\[
A^{\infty}(D) = \{ u + iv \in C^{\infty}(D) : u_x = v_y, u_y = -v_x \} \text{ (C-R eqs.)}
\]

Noted: \( A^{\infty}(D) \) is a subring of \( C^{\infty}(D) \) is also that quotients \( f_1/f_2 \), composites \( G[f(z)] \), and local inverses \{for \( w = f(z)\), \( f'(z_0) \neq 0 \}\) have good properties.

Also noted: \( f \in A^{\infty}(D) \Rightarrow f'(z) = u_x + iv_x = \frac{\partial f}{\partial x} \).

And that \( f' \in A^{\infty}(D) \) as well.

C-R equations are equivalent to

\[
\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} (f_x + if_y) = 0 \text{ on } D.
\]

Showed standard examples of \( f \in A^{\infty}(\text{suitable } D) \).

\( \mathbb{Z}^n \), rational fans of \( \mathbb{Z} \),

\( \exp(z) = e^x \cos y + i e^x \sin y \),

\( \log z = \ln |z| + i \arg(z) \) locally near some \( z_1 \neq 0 \).

In lec #4, did standard hand-waving about approximating "sensible" \( f \in C^{\infty}(\text{suitable } D) \) by polynomials in \((\bar{z}, \bar{z})\). Hence being able to explain \( A^{\infty}(D) \) to "man in the street".
Showed that
\[ F \in A^1(D) \Rightarrow F(z)dz = (u + iv)(dx + idy) \]
\[ = (udx - vdy) + i(vdx +udy) \]

is locally exact (i.e., closed). Accordingly, Green's Theorem can be brought to bear for suitable independence of path results for \( \int_Y f(z)dz \).

\[ \noindent \text{\underline{no need to belabor}} \]

Showed that
\[ F \in A^1(D) \Rightarrow F'(z)dz = d\Psi + id\Psi' \ (F = u + iv) \]
in the sense of standard differentials on RHS.

This produced
\[ \int_Y F'(z)dz = F(B) - F(A) \]

as the "fundamental theorem of complex integral calculus".

If \( h(z) \) is continuous on \( Y \), explained \( \int_Y h(z)dz \)

and why
\[ \int_Y h(z)dz \leq \int Y |h(z)|ds. \]
Proved the standard Cauchy-Goursat thm for \( f \in A(D) \), \( D \) = domain straddling closed rectangle \( R \). Got:
\[
\oint \frac{f(z)}{z-z_0} \, dz = 0 \quad (\approx 1900)
\]
Used bisection and nested interval/box thm.

Immediately went further to get the Cauchy integral formula
\[
f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-z_0} \, dz
\]
for \( z_0 \in \text{int}(R) \). Here \( f \in A(D) \).

Used Leibnitz's rule from adv calc to establish the fund thm that
\[
A(D) = A_0(D)
\]
on any domain \( D \).

Turned quickly to a host of classical theorems (in the "Cauchy theory").

(1) Cauchy Integral Thm
(2) Cauchy Integral Formula for \( f(z_0) \)
(3) Cauchy Integral Formula for $f^{(n)}(z_0)/n!$

(4) Max. Modulus Thm for $|f(z_0)|$, $z_0 \in D$ (gave the slick proof with CIF).

(5) Proved standard Cauchy-Taylor development

for $f \in A(D)$, $D = \{ |z - z_0| < R \}$: $f = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$.

(6) Given uniformly concvo $S(z) = \sum_{n=1}^{\infty} g_n(z)$ with $g_n$ continuous; remarked about $S(z)$ being continuous, and

$$\int_{\gamma} p(z) S(z) dz = \sum_{n=1}^{\infty} \int_{\gamma} p(z) g_n(z) dz$$

for sensible $p(z)$.

(7) Weierstrass M-test for uniform concvo.

(8) Used pathwise connectedness and a "marble" to prove that, when $f \in A(D)$, having $f \equiv 0$ on $\{ |z - z_0| < R \}$ implies $f \equiv 0$ on $D$.

(9) Stated Laurent series for $f \in A(D)$, $D = \{ |z - z_0| < R \}$.

(10) Mentioned in one microsecond the idea of isolated singularities. Never said the words 'removable singularity', pole of order $N$, essential singularity. 😞
(11) WANTED TO also stress the availability of Abel's lemma for an arbitrary power series $\sum_{n=0}^{\infty} a_n z^n$ which converges at $z \neq 0$.

(12) Likewise, WANTED TO develop the Cauchy residue theorem (CRT)

$$\oint_{\Gamma} f(z) dz = 2\pi i \sum_{j=1}^{m} \text{Res}(f, z_j)$$

after defining "Residue". Also wanted to do 2 quick examples.*

(13) Very quickly outlined the Weierstrass Convergence Theorem for $\sum_{n=1}^{\infty} f_n(z)$, $f_n \in A(D)$, and its proof (see 9 below).

(Time and Fortitude)

ran out on (10), (11), (12).

* Also missed the argument principle

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f'(z)}{f(z)} dz = N_0(f) = \frac{1}{2\pi} \text{Arg} f \text{ on } \Gamma.$$
Other topics reviewed:

Used nested interval/box thm, bisection, and a Cantor diagonal to prove the Bolzano-Weierstrass thm in $\mathbb{R}^2$. Similarly $\mathbb{R}^k$. Noted that same proof shows that any bdd+closed $E \subseteq \mathbb{R}^k$ is sequentially compact.

Discussed the Riemann integrability criterion for

$$\int_a^b f(x)\,d\phi(x) \uparrow \left\{ \begin{array}{l} \text{monotonic increasing on } [a,b] \\ \text{bdd} \end{array} \right.$$  

Using $U(P,f,r), L(P,f,r) \{ P = \text{partition} \}$ and

$$\int_a^b f(x)\,d\phi(x) = \lim_{\|P\| \to 0} \sum_{j=1}^{N} f(c_j^*) \Delta x_j.$$  

Using uniform continuity of $f \in C[a,b]$, proved that we actually have

$$\int_a^b f(x)\,d\phi(x) = \lim_{\|P\| \to 0} \sum_{j=1}^{N} f(c_j^*) \Delta x_j.$$  

whenever $f \in C[a,b]$.  \{ $\|P\| = \text{largest } \Delta x_j$ \}
For \( \int_a^b f(x) \, dx \) à la Riemann, remarked that

\[ f \text{ monotonic} \Rightarrow f \in \mathcal{R}[a,b] \quad \text{(easy)} \]

\[ f, g \in \mathcal{R}[a,b] \Rightarrow fg \in \mathcal{R}[a,b] \quad . \]

\[ \int_a^b F(x) \, dx = \int_a^c F(x) \, dx + \int_c^b F(x) \, dx \quad a < c < b \]

in a sensible way

Used summation by parts to derive the integration by parts formula:

\[ \int_a^b f(x) \, g'(x) \, dx = \left. [f(x) \, g(x)] \right|_a^b - \int_a^b g(x) \, f'(x) \, dx \]

valid whenever \( f \in C'[a,b] \) and \( g(x) \uparrow \) on \( [a, b] \).

This Riemann integral \text{ DOES exist!}
One gets an excellent review of the power of the structural properties of analytic functions by studying (developing) several results closely tied to the Weierstrass Convergence Theorem.

It is always striking when one appears to be getting something for nothing.

**Theorem** (the standard Weierstrass conv thm; similarly for multiply-connected $D$)

Let $D$ be a simply-connected domain bounded by a piecewise smooth Jordan curve $\Gamma$.

Let $\{s_n(z)\}_{n=1}^\infty$ be a sequence of analytic functions on $D$ which converges to a limit function $s(z)$ for each $z \in D$.

Assume that the convergence is **uniform** on every closed subset $K$ of $D$. Then:

(a) $s(z)$ must be analytic on $D$;

(b) we automatically have $s_n(z) \rightarrow s(z)$ **uniformly** on every closed $K \subseteq D$;

(c) similarly for $s_n^{(j)}(z) \rightarrow s^{(j)}(z)$, $j \geq 2$.

$s_n(z)$ could, for instance, be $\sum_{k=1}^n f_k(z)$ with $f_k \in A(D)$. 
Weierstrass' Theorem is very well-known (and important). I sketched the proof of it in Lec #4. See (20/4) - (20/4) below for the essential details.

In the pages that follow, we give a review of some techniques which, taken together, permit one to formulate 2 less well-known [and much more impressive] variants of Weierstrass' theorem.

Lemma (stated in \( \mathbb{R}^3 \), but valid for \( \mathbb{R}^k \), \( k \geq 2 \))

Let \( R \) be the box \([A_1, B_1] \times [A_2, B_2] \times [A_3, B_3]\). Let \( f \) be continuous on \( R \). Then:

\[
\mathcal{I}(x, y) \equiv \int_{A_3}^{B_3} f(x, y, z) \, dz
\]

is continuous on \( R = [A_1, B_1] \times [A_2, B_2] \), and

\[
\iint_R f(x, y, z) \, dV = \int_R \mathcal{I}(x, y) \, dA
\]

as in multi-variable calc.

\[PF\]

As indicated, this is just a form of Fubini's thm from elem calc. The continuity of \( \mathcal{I}(x, y) \) is a familiar fact which is part of that theorem (or should be!), and is a simple consequence of the uniform continuity of \( f \) on \( R \).
Lemma (stated in $\mathbb{R}^3$ but valid in $\mathbb{R}^k$, $k \geq 2$)

Let $R$ be the box $[A_1, B_1] \times [A_2, B_2] \times [A_3, B_3]$. Let $f(x, y, z)$ be continuous on $R$. In addition, let all partial derivatives of $f$ wrt $x, y$ also be continuous on $R$. Let

$$I(x, y) = \int_{A_3} f(x, y, z) \, dz.$$ 

Then $I(x, y)$ is $C^\infty$ on $[A_1, B_1] \times [A_2, B_2]$ and we have

$$\frac{\partial I}{\partial x} = \int_{A_3} \frac{\partial f}{\partial x}(x, y, z) \, dz,$$

$$\frac{\partial^2 I}{\partial x \partial y} = \int_{A_3} \frac{\partial^2 f}{\partial x \partial y}(x, y, z) \, dz,$$

etc etc.

PF

This is Leibnitz's rule from advanced calc stated in iterated form — and relying on the foregoing lemma with a general $g(x, y, z)$. The proof is standard adv calc.
Lemma

Let $F = u + iv$ be a $C^n$ function on $\Omega$, say, the open neighborhood $N = \{ |z - z_0| < \delta \}$. The Cauchy-Riemann equations for $u$ and $v$ on $N$ are equivalent to the relation

$$ F_x + if_y \equiv 0 $$

on $N$. Hence $F_x + if_y \equiv 0$ is the condition for $F$ to belong to $A^w(N)$ (i.e., $C^w + \text{analytic}$).

**Proof**

Trivial calculation gives

$$ F_x + if_y = u_x + i u_y + i (v_y + iv'_y) $$

$$ = (u_x - v_y) + i (v_y + v'_y) \quad \square $$

---

**Note:**

$$ \frac{\partial F}{\partial \bar{z}} = \frac{1}{2} (F_x + if_y) \quad \text{is the standard definition.} $$

**Observe:**

$$ \frac{\partial (\bar{z})}{\partial \bar{z}} = 1 \quad \text{and} \quad \frac{\partial (\bar{z})}{\partial z} = 0. $$
Lemma (about Cauchy-type integrals; similarly for multiply-connected \( D \))

Let \( D \) be a simply-connected domain bounded by a \underline{smooth} Jordan curve \( \Gamma \). (See fig.)

Let \( \sigma(w) \) be a \underline{piecewise continuous} complex-valued function on \( \Gamma \).

Let

\[
F(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\sigma(w)}{w-z} \, dw \quad (\text{DEF})
\]

For \( z \in D \). We then have:

(a) \( F(z) \in C^0(D) \)

(b) \( F(z) \in A^0(D) \) (i.e., \( C^0 \) and \underline{analytic})

(c) \[
F^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{\sigma(w)}{(w-z)^{n+1}} \, dw , \quad z \in D
\]

Proof

We can assume \underline{WLOG} that \( \Gamma \) is \underline{smooth} and that \( \sigma(w) \) is continuous on \( \Gamma \). Otherwise, in what follows, just split \( F(z) \) into a \underline{sum} of several natural "chunk integrals".
Convert \( \Gamma \) to a parametric equation \( w = w(t) \), \( a \leq t \leq b \), \( t \neq 0 \). Get:

\[
F(z) = \frac{1}{2\pi i} \int_a^b \frac{\sigma[w(t)]w'(t)}{w(t) - z} \, dt
\]

Write \( z = x + iy \). The integrand, as a func. of \((x, y, t)\), satisfies the hypotheses of the Leibnitz Lemma on page \( \text{(11)} \) so long as \( a_1 \leq x \leq b_1 \), \( a_2 \leq y \leq b_2 \) is some small rectangle inside \( D \).

Note that the numerator can be "pushed aside" since it has no dependence on \( z \). Also, the partial derivatives of any

\[
\frac{1}{W - x - iy}
\]

wrt \( x \) and \( y \) are 100% trivial calc. Of course \( \{(W - z)^{-1} \text{ lying in } \mathbb{A}^n \} \) is

\[
\frac{\partial}{\partial x} \left( \frac{1}{W - x - iy} \right) + i \frac{\partial}{\partial y} \left( \frac{1}{W - x - iy} \right)
\]

\[
= \frac{1}{(W - x - iy)^2} + i \frac{i}{(W - x - iy)^2} = 0
\]

(a) and (b) are immediate on each \( (a_1, b_1) \times (a_2, b_2) \) now.
hence on all $D$ by the lemmas on $10 + 11 + 12$

To get (c), since we know $F \in A^\omega(D)$, just use

$$F^{(n)}(x) = \left( \frac{\partial}{\partial x} \right)^n F(x)$$

and the trivial fact that

$$\left( \frac{\partial}{\partial x} \right)^n (qW - x - iy)^{-1} = n! (qW - x - iy)^{-n-1}$$

[In addition to Leibnitz' Lemma on $\Box$.]

\[\Box\]
Lemma (stated for $\mathbb{R}^2$, readily adapted to $\mathbb{R}^k$, $k \geq 1$)

Let $s_n(z) = s_n(x+iy)$ be a sequence of complex-valued functions on the closed rectangle $\mathbb{R} = [a_1, b_1] \times [a_2, b_2]$ OR, if you prefer, closed disk $\mathbb{D}$. Assume that $s_n(z) \to$ some function $s(z)$ pointwise for $z \in \mathbb{D}$. Assume further that, for some $M > 0$, we have

$$|s_n(z_1) - s_n(z_2)| \leq M |z_1 - z_2|$$

for all $n \geq 1$ and $z_1, z_2 \in \mathbb{D}$. (Uniform Lipschitz condition!) THEN: the convergence of $s_n(z)$ to $s(z)$ is automatically uniform on $\mathbb{D}$. 

PF

The procedure for this is standard. Choose any tiny $\varepsilon > 0$. Look at $\mathbb{R}$ and select a finite grid of points $\{P_1, \ldots, P_L\} \subseteq \mathbb{R}$ so that every point $z \in \mathbb{R}$ is located within $\frac{\varepsilon}{5M}$ units of some $P_k$. THIS IS CERTAINLY POSSIBLE!

Since $L$ is finite, we can select $N_\varepsilon$ so big

* The word "equiscontinuity" may come to mind here.
\[ |s_n(p_j^*) - s(p_j^*)| < \frac{\varepsilon}{10} \]

For all \( n \geq N_\varepsilon \) and all \( j \in \{1, 2\} \).

Get:

\[ |s_n(p_j^*) - s_m(p_j^*)| < \frac{\varepsilon}{5} \]

For all \( n \geq m \geq N_\varepsilon, j \in \{1, 2\} \).

Take any \( z \in \mathbb{R} \). Select \( p_e \) so that

\[ |z - p_e| \leq \frac{\varepsilon}{5M} \]

For \( n \geq m \geq N_\varepsilon \), notice that

\[ |s_n(z) - s_m(z)| \leq |s_n(z) - s_n(p_e)| + |s_n(p_e) - s_m(p_e)| + |s_m(p_e) - s_m(z)| \]

\[ \leq 9M \left( \frac{\varepsilon}{5M} \right) + \frac{\varepsilon}{5} + 9M \left( \frac{\varepsilon}{5M} \right) \]

\[ = \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} < \varepsilon \]

This is the standard **uniform Cauchy condition** for uniform conv of \( \{s_i(z)\}_{i=1}^\infty \) over \( \mathbb{R} \). Indeed, by letting \( n \rightarrow \infty \), we get

\[ |s(z) - s_m(z)| \leq \varepsilon \quad \text{anytime} \quad \{m \geq N_\varepsilon\} \}.
Theorem

Let $D$ be a simply-connected domain bounded by a piecewise smooth Jordan curve $\Gamma$. Let $\{s_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions on $D$ which converges pointwise to some $s(z)$ for each $z \in D$.

Assume that, for each closed subset $K$ of $D$, there exists a constant $M(K)$ so that

$$|s_n(z)| \leq M(K)$$

whenever $z \in K$ and $n \geq 1$. Then:

(a) $s(z)$ must be analytic on $D$;

(b) $s_n(z)$ converges UNIFORMLY to $s(z)$ on every closed subset of $D$;

(c) we have $s_n'(z) \rightarrow s'(z)$ UNIFORMLY on every closed subset of $D$;

(d) similarly for $s_n^{(j)}(z) \rightarrow s^{(j)}(z)$, $j \geq 2$.

Key Issues: where did all the UNIFORM conv. come from?
Pf

Not surprisingly, we rely on page 16 Lemma 1.

Every closed set \( K \subseteq D \) is automatically bounded. Hence \( K \) is \( bdd \) and closed; hence, sequentially compact, etc. Any such \( K \) will lie a positive distance from \( \mathbf{p} \). Likewise from a piecewise smooth Jordan curve \( \gamma \) in \( D \) "paralleling \( \mathbf{p} \)" extremely closely.

Let \( |s_n(z)| \leq M_0 \) for \( z \in \gamma, \ n \geq 1 \).

Consider now any small closed rectangle \( R \) situated "inside" \( \gamma \). Let \( h = \text{dist}(R, \gamma) \).

For \( z \in R \),
\[
S_n'(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{s_n(z)}{(z - \Xi)^2} \, dz
\]
by CIF for deriv.

\[
\Rightarrow \quad |S_n'(z)| \leq \frac{1}{2\pi} \int_{\gamma} \frac{M_0}{h^2} \, ds = \frac{1}{2\pi} \frac{M_0}{h^2} l(\gamma).
\]

Call this \( M \).
Connect any two points \( \xi_1 \) and \( \xi_2 \) of \( R \) by a line segment to get

\[
\text{SN}(\xi_2) - \text{SN}(\xi_1) = \int_{\xi_1}^{\xi_2} \text{SN}'(z) \, dz
\]

\( \text{linear} \)

\[
|\text{SN}(\xi_2) - \text{SN}(\xi_1)| = \int_{\xi_1}^{\xi_2} |\text{M}| \, ds = |\text{M}| |\xi_2 - \xi_1|.
\]

Because of this, Lemma on page 16 applies. Hence: \( \text{SN}(z) \to S(z) \) uniformly on \( R \).

Since \( R \) can be slid around, and \( y \) can always be pushed closer to \( \Pi \), we get that \( \text{SN}(z) \to S(z) \) uniformly on each closed subset \( K \) of \( D \) (It's best to fix \( K \) first, then select the Jordan curve \( \gamma \)). This proves (b).

At this point, the standard Weierstrass convergence theorem applies and, so, we get (a) + (c) + (d).

\( \text{OK} \)
A person wishing to keep matters completely self-contained would reason alternatively as follows.

Let $\Rightarrow$ signify uniform conv $\circ$

Refer to the picture on (19). Let $R_0$ be a closed rectangle just slightly bigger than $R_0$. Since we could have just as well used $R_0$ instead of $R_1$, we know that $\text{Jn}(z) \Rightarrow S(z)$ on $R_0$.

By unif conv, $S(z)$ is continuous on $R_0$.

For $z_1 \in R_1,$

$$\text{Jn}(z_1) = \frac{1}{2\pi i} \oint_{\partial R_0} \frac{\text{Jn}(w)}{w-z_1} \, dw \quad (n \geq 1).$$

By unif conv of $\text{Jn}(w)$ on $\partial R_0$ and use of test function $p(w) = \frac{1}{w-z_1}$, we get (via $n \to \infty$)

$$S(z_1) = \frac{1}{2\pi i} \oint_{\partial R_0} \frac{S(w)}{w-z_1} \, dw, \quad \text{each } z_1 \in R_0.$$

By FACT on $p$, (13), $S(z)$ must belong to $A^\infty$ on the interior of $R_0.$
Now just take the whole "picture frame" $\mathcal{F}(R, R_0)$ and slide it around slightly within the interior of $\gamma$. See picture on 19.

We get $s'(z) = \text{analytic on an open set containing } R_0$. We already know $s_u(z) \approx s(z)$ on $R_0$. Let $h = \text{dist}(R, \partial R_0)$. For $z \in R$,

$$s_u'(z) - s'(z) = \frac{1}{2\pi i} \oint_{\partial R} \frac{s_u(w) - s(w)}{(w-z)^2} \, dw$$

$\{ \text{by CIF for derivative} \}$

$$|s_u'(z) - s'(z)| \leq \frac{1}{2\pi} \oint_{\partial R} \frac{|s_u(w) - s(w)|}{|w-z|^2} \, |dw|$$

$$|s_u'(z) - s'(z)| \leq \frac{1}{2\pi h^2} \ell(\partial R_0) \max_{\partial R_0} |s_u(w) - s(w)|$$

But, choose $N_\varepsilon$ so $|s_u(w) - s(w)| < \varepsilon$ for all $n \leq N_\varepsilon$ and $w \in R_0$. Hence, $n \leq N_\varepsilon \implies$

$$|s_u'(z) - s'(z)| < \frac{\ell(\partial R_0) \varepsilon}{2\pi h^2} \leq \varepsilon \quad \text{for all } z \in R_0.$$

This proves $s_u'(z) \approx s'(z)$ on $R_0$. Similarly for $s_u''(z)$.
Since $R$ can be reduced in size and slid around, and $y$ can always be pushed closer to $\Gamma$, we immediately get (a), then (b)(c)(d).
The following version of the Weierstrass convexity theorem is still stronger and was proved by Vitali.

It shows how the structure of analytic functions can be exploited to induce a kind of "global rigidity".

Theorem (Vitali)

Let $D$ and $\Gamma$ be as on (18).

Let $\{s_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions on $D$ which satisfies the $M(K)$ hypothesis on (18) for each closed $K \subseteq D$.

Assume that $\lim_{n \to \infty} s_n(\xi_j)$ exists for each $j \leq J$ where the $\xi_j$ are distinct points of $D$ tending to a point (say $\xi_0$) of $D$.

Then:

(a) $\{s_n(z)\}_{n=1}^{\infty}$ automatically converges to a limit function $s(z)$ at each point of $\Gamma$.

(b) the convergence in (a) is uniform on each closed $K \subseteq D$. 
We recall a simple fact about complex numbers.

** Lemma **

Given sequence \( \{ w_n \}_{n=1}^{\infty} \) in \( \mathbb{C} \). This sequence converges to \( L \) as \( n \to \infty \) if and only if every subsequence \( \{ w_{n_j} \} \) admits a further subsequence which converges to \( L \).

** Proof of lemma **

The "only if" is obvious. For the "if", we use contradiction. Hence there must exist some bad \( \varepsilon_0 > 0 \) with no "\( N \varepsilon_0 \)". If we can find arbitrarily big \( n \) for which \( |w_n - L| \geq \varepsilon_0 \). Make a recursion construction to get \( n_j > n_j^* \geq 1 \) satisfying

\[
|w_{n_j} - L| \geq \varepsilon_0, \quad j \geq 1
\]

By hypothesis, there exists a (increasing) subseq \( \{ n_j \}_{j=1}^{\infty} \) for which

\[
\{ w_{n_j^*} \}_{j=1}^{\infty} \to L = L.
\]

But \( \exists \{ n_j \} \Rightarrow |w_k - L| \geq \varepsilon_0 \) for each \( k \in \mathbb{N} \).

Contradiction! 

We now turn to the proof of the THM.
The reasoning that follows is closely related to the Arzela-Ascoli theorem in real analysis (or point-set topology).


Choose any $\delta > 0$. By taking a grid of points on $D$, we can clearly find a finite set $E_\delta \subseteq D$ such that every point of $D$ lies within $\delta$ units of some point of $E_\delta$. The set

$$C = \bigcup_{k=1}^{\infty} E_{1/k}$$

is then countable and dense in $D$.

Let $Q_j = \lim_{n \to \infty} S_n(E_j)$ for each $j \geq 1$.

Also let $s$ be any increasing subsequence of $\{S_n\}_{n=1}^{\infty}$.

By combining hypothesis $H(K)$, the Bolzano-Weierstrass theorem, and the Cantor diagonal process, we can construct an increasing subsequence $S_1$ of $S$ such that

$$\lim_{n \to \infty} \{S_n(P) : n \in S_1\}$$

exists for each $P \in C$. 
At this juncture, we go back into the proof of p. 18 THM.

The key initial observation is this. Let $R$ be any closed rectangle situated within $\gamma$. See picture on 19. Since $C$ is dense in $D$, there exists a finite set of points $\{P_1, \ldots, P_L\} \subseteq R \cap C$ satisfying the $E/5M$-unit condition on 16 bottom. This assertion requires just a bit of care in handling points near $\partial R$; see also the very important page 20 (top 4 lines).

Keeping $n \in \mathbb{N}$, notice that lines 3-13 on 17 can now be re-used (since $P_j \in C$ and $\lim_n \{S_n(P_i) : n \in \mathbb{N}\}$ exists !!! $\Pi$).

We conclude that $\{S_n : n \in \mathbb{N}\}$ is uniformly Cauchy on $R$.

Hence $\{S_n : n \in \mathbb{N}\} \Rightarrow$ some $s(\xi)$ on each $R$.

By sliding $R$ as in the proof of p. 18 THM (see especially 20), we get that $\{S_n : n \in \mathbb{N}\} \Rightarrow s(\xi)$ on every closed $K \subseteq D$. 
Note that the points \( \{ \xi_j^* \}_{j=1}^\infty \) will lie in some fixed closed \( K \subseteq D \).

We can apply either the traditional or p. 18 strengthened Weierstrass conv theorem to \( \{ \sin \xi_1 \} \). The fn \( s(z) \) is thus analytic on \( D \).

Moreover, by substituting \( z = \xi_j^* \), we find that

\[
s(\xi_j^*) = a_j^* , \quad j \geq 1.
\]

For \( a_j^* \), recall 23.

Let \( \tilde{\xi} \) be any other increasing subseq of \( \{ \xi_n \}_{n=1}^\infty \).

Form \( \tilde{\xi} \) by analogy with \( \xi_1 \). See 23.

The limit function for \( \{ \sin \xi_1 \} \) will be \( \tilde{s}(z) \).

The function \( \tilde{s}(z) \) is again analytic on \( D \) and satisfies \( \tilde{s}(\xi_j^*) = a_j^* \).

The function \( H(z) \equiv s(z) - \tilde{s}(z) \) is analytic on \( D \) and vanishes at each \( \xi_j^* \). Hence also at \( \xi_\infty \).

If \( H(z) \neq 0 \) on \( D \), any zero at \( \xi_\infty \) would need to be isolated. This follows from the local Taylor expansion. Since \( \xi_j^* \to \xi_\infty \), we get an immediate violation. Hence: \( H(z) = 0 \) and \( s(z) = \tilde{s}(z) \) on \( D \).
CLAIM:
For each $r \in D$, \( \{s_n(r); n \geq 1\} \rightarrow s(r) \).

Pf of Claim
Just use the lemma on (22). We need only show that any increasing subseq \( \tilde{S} \) of \( \{s_n\}_{n=1}^{\infty} \) admits a subsequence \( \tilde{S}_0 \) such that
\[ \{s_n(r); n \in \tilde{S}_0\} \rightarrow s(r) \.
\] But \( \tilde{S}_1 \) works in the role of \( \tilde{S}_0 \) (since we just proved that \( \tilde{s}(z) = s(z) \)). OK on the Claim. \[ 18 \]

One is now exactly in the situation of \( p^* \). THM — and so we are done. \[ 18 \]
PARTIAL DIARY ENTRY for
Lecture 5
(3 Feb 2016)

We first went over a number of elementary facts and properties. The goal today was to begin the Riemann zeta function in earnest.

**Topic I**

*About Riemann-Stieltjes integrals.*

Showed that even if \( g(x) \) is right continuous and \( \mu \) on \([0,1]\) taking \( f \) to be piecewise continuous can lead to

\[
\int_0^1 f(x) \, d\mu(x) = 1 \quad \text{and} \quad \int_0^1 f(x) \, dx = 0.
\]

Discouraging! So, best to use R-5 for continuous \( f \) when possible.
Showed:

\[ F \in C[1, N] \Rightarrow \]
\[ \sum_{i=1}^{N} f_i d[x, y] = f(1) + \cdots + f(N) \]

\[ g \in C[\beta, N] \quad (0 < \beta < 1) \Rightarrow \]
\[ \sum_{i=1}^{N} g_i d[x, y] = g(1) + \cdots + g(N) \]

Hence, \( R - S \) has natural connection with sums.

\[ \text{Topic II} \]

\( \text{Abel's Lemma for power series} \)

Given \( \sum_{n=0}^{\infty} a_n z^n \) which converges at \( z \neq 0 \).

Then: \( |a_n| \leq \frac{M}{|z|^n} \) for some \( M \) and all \( n \geq 0 \).

Hence, the orig power series conv uniformly and absolutely on each closed disk \( \{ |z| \leq |z| - \delta \} \).

PF: Trivial.

And Weierstrass Conv Thm applies!! on \( |z| < |z| \).
Another well-known result of Abel.

**Thm (Abel)** \( S(z) \)

Let \( \sum_{n=0}^{\infty} a_n z^n \) converge at, say, \( z = 1 \) (to \( S \)).

Then:

\[
\sum_{n=0}^{\infty} a_n x^n
\]

converges uniformly on \([0,1]\). Hence \( \lim_{x \to 1^-} S(x) = S_0 \).

(Similarly along \( z = re^{i\theta} \).)

**Pf**

Uniform Cauchy estimate + Abel summation.

Must prove

\[
|S_N(x) - S_M(x)| < \varepsilon, \text{ all } N > M \geq M_\varepsilon.
\]

We know, of course,

\[
|a_{m+1} + \ldots + a_N| < \varepsilon \text{ for } N > M \geq M_\varepsilon.
\]

Claim that we can take \( M_\varepsilon = N_\varepsilon \). Put

\[
T_k = a_{m+1} + \ldots + a_k, \quad k \geq M+1.
\]
Get:

\[ a_{M+1} x^{M+1} + \ldots + a_N x^N \]

\[ = T_{M+1} x^{M+1} + (T_{M+2} - T_{M+1}) x^{M+2} + \ldots + (T_N - T_{N-1}) x^N \]

\[ = T_{M+1} (x^{M+1} - x^{M+2}) + \ldots + T_{N-1} (x^{N-1} - x^N) + T_N x^N \]

Know \( |T_k| < \epsilon \), \( k \geq M+1 \). Get:

\[ \text{Abs value} < \epsilon (x^{M+1} - x^{M+2}) + \ldots + \epsilon (x^{N-1} - x^N) + \epsilon x^N \]

\[ \leq \sum_{0 \leq j \leq k} \epsilon x^j \]

\[ \leq \epsilon x^{M+1} \leq \epsilon \]

Hence all is OK.  \[ \square \]

This proof can clearly be generalized to work in many other settings!
Topic IV

Traditional to define principal value of $\arg(w)$ by declaring $-\pi < \text{Arg}(w) < \pi$ and keeping $w$ off the negative real axis $(-\infty, 0]$. 

$$\text{Log}(w) = \ln|w| + i\text{Arg}(w)$$

Nice analytic fn on $C\setminus (-\infty, 0]$. 

$$\frac{d}{dw} \log w = \frac{1}{w} \quad \{\text{local inverses are analytic, etc}\}$$

So, $f(z) = \log(1+z)$ is analytic for $|z| < 1$. Cauchy - Taylor $\Rightarrow$

$$f(z) = \log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots, \quad |z| < 1$$

get uniform abs conv for $|z| \leq 1 - \delta$. 
Thm 1 (basic def of $\zeta(z)$)

A RIEMANN ZETA FCN.

We write

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{-\pi}{n^z} \quad \text{for } \Re(z) > 1.$$ 

The series converges uniformly and absolutely in every half-plane $\{\Re(z) \geq 1 + \delta\}$. Hence, $\zeta(z)$ is nicely analytic on $\{\Re(z) > 1\}$.

PF

Weierstrass M-test with $M_n = n^{-1-\delta}$. Also Weierstrass Conv Thm!

Thm 2

There exists a function $F(z)$ which is analytic on $\{\Re(z) > 0\} - \{1\}$ such that

$$F(z) = \zeta(z) \text{ whenever } \Re(z) > 1.$$ 

The function $F$ is unique (numerically). We call it the analytic continuation of $\zeta(z)$. One can see that, near $z=1$,

$$F(z) = \frac{1}{z-1} + [\text{something analytic}].$$

In, say, $|z-1| < \frac{1}{2}$.
Pf

Suppose there were two $F_1$ and $F_2$. The fcn $F_1 - F_2$ is analytic on $\{\text{Re}(z) > 0\} - S_1$ but $\equiv 0$ for $\text{Re}(z) > 1$. By properties of analytic fcn's, get $F_1 - F_2 \equiv 0$ everywhere.

Hence $F$ must be unique.

Must now find one $F_0$.

Take $R = N + \varepsilon$ for some tiny $\varepsilon > 0$. Keep $\text{Re}(z) > 1$.

$$1 + \int_1^R t^{-z} d[t^z] = 1 + 2^{-\varepsilon} + 3^{-\varepsilon} + \cdots + N^{-\varepsilon}.$$  

Notice (see 3) that nothing is lost if we simply take $\varepsilon = 0$ rather than let $\varepsilon \to 0$. (Make sure you understand this; this type of trick is used a lot!)

Write $t = \lceil t \rceil + r(t)$,

\[ 0 \leq r(t) < 1 \]

\[ \text{diff of two increasing right continuous fcn's} \]
Get:
\[
\sum_{n=1}^{N} n^{-\varepsilon} = 1 + \int_{1}^{N} t^{-\varepsilon} \, dt \\
= 1 + \int_{1}^{N} t^{-\varepsilon} \, d(t-r(t)) \\
= 1 + \int_{1}^{N} t^{-\varepsilon} \, dt - \int_{1}^{N} t^{-\varepsilon} \, dr(t)
\]
\(\overset{R-S \text{ integral and } t^{-\varepsilon} \text{ nicely } C^1 \text{ wrt } t}{\Rightarrow}\)

\[
\frac{d}{dv} e^{cv} = c e^{cv} \quad \text{for } v \in \mathbb{R}
\]

\[
\begin{align*}
\frac{d}{dt} t^c &= \frac{1}{t} \frac{d}{d(ln t)} t^c \\
&= \frac{1}{t} e^{c(ln t)} \\
&= \frac{1}{t} e^{c(ln t)} \cdot c = c t^{c-1} \\
&\text{for } c \in \mathbb{C} \text{ and } t > 0
\end{align*}
\]

\[
= 1 + \frac{N^{1-\varepsilon} - 1}{1-\varepsilon} - \left[ t^{-\varepsilon} r(t) \right]_{1}^{N} \\
+ \int_{1}^{N} r(t) (-\varepsilon) t^{-\varepsilon-1} \, dt \\
= 1 + \frac{1-N^{1-\varepsilon}}{\varepsilon-1} - 0 + 0 \\
- \varepsilon \int_{1}^{N} \frac{r(t)}{t^{\varepsilon+1}} \, dt
\]

\(r(1) = r(N) = 0\)
So, \( \text{Re}(\xi) > 1 \) gives (by taking \( N \to \infty \))

\[
\sum_{n=1}^{N} n^{-\xi} = 1 + \frac{1 - N^{1-\xi}}{\xi - 1} - \xi \int_{1}^{N} \frac{r(t)}{t^{\xi+1}} \, dt
\]

\( f(\xi) = 1 + \frac{1}{\xi - 1} - \xi \int_{1}^{\infty} \frac{r(t)}{t^{\xi+1}} \, dt \)

\[
\left\{ \begin{align*}
|t^\alpha| &= \left| e^{(\alpha+i\beta)\ln t} \right| = e^{\alpha \ln t} \\
&= t^\alpha = t^{\text{Re}(\xi)}, \quad \xi > 0
\end{align*} \right\}
\]

In the formulae above, note that:

- Formula #1 holds for any \( \xi \in \mathbb{C} \setminus \{1\} \);
- Formula #2 holds for \( \text{Re}(\xi) > 1 \);
- the integral in formula #2 is nicely absolutely and uniformly convergent so long as \( x \leq \delta > 0 \) (111)

\[ \text{HENCE analytic à la Weierstrass conv thm on Re}(\xi) > 0 \]
Note: there is a Weierstrass M-test for improper integrals \( \int_1^\infty f \) one should review it. (cf. any adv calc book.)

We can thus put

\[
F(z) = 1 + \frac{1}{z-1} - z \int_1^\infty \frac{v(t)}{t^{z+1}} \, dt
\]

For all \( \text{Re}(z) > 0 \) except \( z = 1 \). This works in Thm 2 on page 6.

**Thm 3**

We have \( |F(z)| \leq F(x) \) for all \( \text{Re}(z) > 1 \).

We also have

\[
|F(z) - 1| < 2^{-x} \left( 1 + \frac{2}{x-1} \right)
\]

for all \( \text{Re}(z) > 1 \). \( \exists \) Note that RHS is \( < 3 \cdot 2^{-x} \) whenever \( x > 2^{\cdot 5} \).
\[ \left| \sum_{n=1}^{\infty} n^{-x} \right| = x \sum_{n=1}^{\infty} \frac{1}{n^x} = \sum_{n=1}^{\infty} n^{-x} = f(x) \quad (\text{see } 9) \]

Also:

\[ \left| \sum_{n=2}^{\infty} n^{-x} \right| = x \sum_{n=2}^{\infty} \frac{1}{n^x} = \frac{2^{-x}}{1-x} + \int_{2}^{\infty} u^{-x} \, du \quad (x > 1) \]

\[ = \frac{2^{-x}}{1-x} + \left[ \frac{u^{1-x}}{1-x} \right]_{2}^{\infty} \]

\[ = \frac{2^{-x}}{1-x} + \frac{2^{1-x}}{x-1} \]

\[ = 2^{-x} \left\{ 1 + \frac{2}{x-1} \right\} \]

By Thm 2,

\[ f(x) = \frac{1}{x-1} + O(1) \]

as \( x \to 1^+ \).
Thm 4 (very crude)

Keep $|z-1| \geq \frac{1}{3}$ say. Take any $0 < \delta < 1$. We then have

$$|S(x+iy)| = O(1) \frac{1}{\delta} (1+|y|)$$

whenever $\delta \leq x \leq 1+\delta$. For $x \geq 1+\delta$, we have

$$|S(x+iy)| \leq O(1) \frac{1}{\delta} \cdot \begin{cases} \text{In } O(1), \text{ the implied constant} \text{ is absolute.} \\ \text{implied constant} \end{cases}$$

pf

Use (1)line 5. Keep $\delta \leq x \leq 1+\delta$. Get $\delta$

$$|S(x+iy)| \leq 1 + 3 + |z| \int_1^{\infty} \frac{1}{t^{x+1}} dt$$

$$\leq 4 + (|x|+|y|) \int_1^{\infty} \frac{1}{t^{x+1}} dt$$

$$\leq 4 + (2+|y|) \frac{1}{x} \quad (x > 0)$$

$$\leq 4(1+|y|) + 2(1+|y|) \frac{1}{x}$$

$$\leq (1+|y|) \left( 4 + \frac{2}{x} \right)$$

$$\leq (1+|y|) \left( \frac{4}{5} + \frac{2}{5} \right) \approx \frac{6}{5} (1+|y|)$$

For $x \geq 1+\delta$, simply use $|S(x)| \leq 5(x)$ (Thm 3) and

$$S(1+\delta) = \frac{1}{\delta} + O(1)$$.
We then paused to discuss infinite products, a topic which seems to have disappeared from UM's undergrad math curriculum!

We do not give a treatise, and we will deal with products of complex numbers using only real is much easier!!!

Like with \[ \sum_{n=1}^{\infty} a_n \] \( \text{conv/div} \) matters should focus on the tail end of the series (or product) NOT on the first 10100 terms.

Unless \( a_n \to 0 \), \[ \sum_{n=1}^{\infty} a_n \text{ is div}. \]

Unless \( a_n \to 1 \), \[ \prod_{n=1}^{\infty} a_n \text{ is (said to be) div}. \]

We therefore focus on products with the first 10100 terms erased and presuppose that \( a_n = 1 + b_n \) with \( |b_n| \leq \gamma < 1 \) for some \( \gamma \).
Given $a_n = 1 + b_n$ with $|b_n| \leq \lambda < 1$. We say

$$\prod_{n=1}^{\infty} a_n \text{ conv to } P$$

if

(a) $P \neq 0$

and

(b) $\frac{P_N}{P} \to 1$ as $N \to \infty$.

"Multiplicative style"

Here $P_N = a_1 \cdots a_N$.

If $a_n(z) = 1 + b_n(z)$, $|b_n(z)| \leq \lambda < 1$, $z \in E$, we say

$$\prod_{n=1}^{\infty} a_n(z) \text{ conv unif to } P(z)$$

if

$P(z) \neq 0$ and $\frac{P_N(z)}{P(z)} \to 1$ uniformly as $N \to \infty$. 
Def

Given $a_n = 1 + b_n$ as above. We say

$$
\prod_{n=1}^{\infty} a_n \quad \text{conv \ absolutely}
$$

when

$$
\prod_{n=1}^{\infty} (1 + |b_n|) \quad \text{converges}.
$$

\[ \text{N.B. see (24) below!} \]

NOT $|a_n|$

Lemma

Suppose $\prod_{n=1}^{\infty} a_n(z) \ \text{conv \ unif} \ \text{to} \ P(z) \ \text{on} \ E$.

Then, there exist $c_j > 0$ so that

$$
c_j < |P(z)| < c_2 \quad \text{on} \ E.
$$

PF

Choose $M$ so big that

$$
\left| \frac{P_n(z)}{P(z)} - 1 \right| < 10^{-6} \quad (z \in E)
$$

for all $N \geq M$. 

Get:

\[ \left| \frac{\nu}{\varphi} - 1 \right| < 10^{-6} \quad \left| \frac{\varphi}{\nu} - 1 \right| < 10^{-5} \]

\[ \Downarrow \]

\[ \frac{3}{4} < \left| \frac{\varphi}{\nu} \right| < \frac{5}{4} \quad \text{certainly} \]

\[ \Downarrow \]

\[ \frac{3}{4} |\nu| < |\varphi| < \frac{5}{4} |\nu| \]

\[ \Downarrow \]

\[ \frac{3}{4} (1-x)^M < |\nu| < \frac{5}{4} (1+x)^M \quad \Box \]

\[ \text{Corollary (important)} \]

Notation as above. Suppose \( \prod_{n=1}^{\infty} \psi_n(x) \) converges uniformly to \( \psi(x) \) on \( E \). We then also have

\[ \psi_n(x) \Rightarrow \psi(x) \text{ on } E. \]

\[ \uparrow \text{recall that this means UNIF CONV} \]
**Lemma**

Recall \( \log(w) \) and \( \text{Arg}(w) \) on (5).

Suppose that \( |w_1 - 1| < 1 \) and \( |w_2 - 1| < 1 \).

Then:

\[
w_1w_2 \notin (-\infty, 0]
\]

and

\[
\text{Arg}(w_1w_2) = \text{Arg}(w_1) + \text{Arg}(w_2).
\]

**pf**

Write \( w_j = R_j e^{i\theta_j} \). Clearly \( -\frac{\pi}{2} < \theta_j < \frac{\pi}{2} \) and \( R_j > 0 \). Hence

\[
w_1w_2 = R_1 R_2 e^{i(\theta_1 + \theta_2)}
\]

and

\[-\pi < \theta_1 + \theta_2 < \pi.\]

Done!

For \( |w_j - 1| < \frac{1}{100} \), baby Eng

\[
\Rightarrow \text{Arg}(w_1 \cdots w_{100}) = \sum_{j=1}^{100} \text{Arg}(w_j).
\]
**General Thm**

Let $E$ be some set which might possibly be just one point. Given $a_n(z) = 1 + bu(z)$, $|bu(z)| \leq \gamma < 1$, $z \in E$, as above.

We then have:

$$\prod_{n=1}^{\infty} a_n(z) \text{ conv unif to some } P(z) \text{ on } E$$

if and only if

$$\sum_{n=1}^{\infty} \log(1 + bu(z)) \text{ conv unif to some } S(z) \text{ on } E$$

And, if so,

$$P(z) = \exp \{ S(z) \}.$$

**Proof**

This thm is not just hand-waving trivia by "passing to logs". It is NOT true that

$$\log(\prod_{i}^{N} w_i) = \sum_{1}^{N} \log(w_i).$$

in general, even if $w_i \neq 1$.

$\uparrow$ think, eg, $w_i = e^{\frac{a_i}{N}}$
Suppose first that $\Sigma_N(z) \geq \Sigma(z)$, where
\[
\Sigma_N = \sum_{n=1}^{N} \log(1 + b_n(z))
\]

From uniform convexity (and $\lambda$), automatically $\sqrt{15(z)} < M$ for some $M$.

\[\uparrow\text{just imitate p. 15 Lemma}\]

We can now exponentiate freely.

\[p_N = \exp(\Sigma_N)\]
\[p = \exp(\Sigma) \quad \text{makes sense on } E\]

\[\frac{p_N}{p} \rightarrow 1 \quad \text{as } N \rightarrow \infty, \quad z \in E\]

Hence $\Pi_N(z)$ converges uniformly on $E$ and
\[p = \exp(\Sigma) \quad \text{<THIS MUCH IS TRIVIAL>}\]

The problem is with the converse!
Suppose now that $\prod_{n=1}^{\infty} a_n(z)$ converges uniformly to $p(z)$ on $E$. 

Choose $M$ so large that 

$$\left| \frac{p_N}{p} - 1 \right| < 10^{-6}$$

for all $N \geq M$, $z \in E$. Do some elementary algebra. Get 

$$\left| \frac{p_{N_2}}{p_{N_1}} - 1 \right| < 10^{-5} \quad \text{and} \quad \left| \frac{p}{p_{N_1}} - 1 \right| < 10^{-5}$$

for all $N_2 \geq N_1 \geq M$. See (16) above.

Let 

$$\theta = \sqrt{\frac{p_N}{p_M}}$$

$$\operatorname{Arg} \left[ (1 + b_{n+1}) \cdots (1 + b_N) \right] = \sum_{j=M+1}^{N} \operatorname{Arg}(1 + b_j) + 2\pi i t_N$$

for $N \geq M + 1$.

**Claim:** $t_N = 0$ for all $N \geq M + 1$.

**Proof of claim:**

Take $N_2 = N_1 + 1$ and $N_1 \geq M$. Get: 

$$\left| a_{N_2} - 1 \right| < 10^{-5}$$
\[ |a_L - 1| < 10^{-5} \text{ for all } L \geq M + 1.\]

Clearly \( t_{M+1} = 0 \) by def. 

Use induction. Suppose \( 0 = t_{M+1} = \cdots = t_N \).

Must prove \( t_{N+1} = 0 \).

Use Lemma on (17). Take:

\[
\begin{align*}
    w_1 &= (1 + b_{M+1}) \cdots (1 + b_N) \\
    &\leftarrow \frac{P_N}{P_M} \\
    w_2 &= 1 + b_{N+1} \\
    &\leftarrow \frac{P_{N+1}}{P_N}
\end{align*}
\]

Get:

\[
\begin{align*}
    \text{Arg}(w_1 w_2) &= \text{Arg}(w_1) + \text{Arg}(w_2) \quad \text{by } |w_j - 1| < 10^{-5} \\
    \text{OR} \\
    \text{Arg} \left[ (1 + b_{M+1}) \cdots (1 + b_{N+1}) \right] &= \sum_{j = M+1}^{N} \text{Arg}(1 + b_j) + 2\pi i(0) \\
    &\quad + \text{Arg}(1 + b_{N+1}) \\
    \text{hence } t_{N+1} &= 0.
\end{align*}
\]

\( \text{OK} \)
We have proved claim for our given \( M \).
But the same reasoning works with
\[ N_2 > N_1 \geq M \quad \text{and} \quad N_1 \text{ in place of } M. \]

If
\[
\text{Avg} \left[ (1+b_{N_1}) \cdots (1+b_{N_2}) \right] = \sum_{j=N_1+1}^{N_2} \text{Avg} \left[ 1+b_j^* \right] + O
\]

hence
\[
\log \left[ \frac{P_{N_2}(z)}{P_{N_1}(z)} \right] = \sum_{j=N_1+1}^{N_2} \log \left[ 1+b_j^*(z) \right]
\]

so long as \( N_2 > N_1 \geq M \)

This is the key equation! Since \( \frac{P_{N}(z)}{P(z)} \approx 1 \)
on \( \mathcal{E} \) and \( c_1 < |P(z)| < c_2 \) \((\circ)\), we get a multiplicative Cauchy condition
\[
\left| \frac{P_{N_2}(z)}{P_{N_1}(z)} - 1 \right| < \varepsilon \quad \text{anytime} \quad N_2 > N_1 \geq N_{\mathcal{E}}
\]
(and, wlog, \( N_{\mathcal{E}} \geq M \)).
This shows that there is a uniform Cauchy condition for
\[ \sum_{j=N_1}^{N_2} \log(1+b_j(x)), \quad \text{i.e.} \quad S_{N_2}(x) - S_{N_1}(x), \]
for \( x \in E \). \textbf{Hence}:
\[ \sum_{j=1}^{\infty} \log(1+b_j(x)) \text{ converges uniformly on } E \text{ to some } \Sigma(x). \]

By referring to (19), we again have \( P = \exp(\Sigma) \).
Done!

\begin{center}
\textbf{Important Remark:}
\end{center}

\textit{If you know } \( P(x) \),
\textit{note that you do NOT in general know } \( \Sigma(x) \) \textit{without further playing around with}
\[ \sum_{j=1}^{\infty} \log(1+b_j(x)). \]
\textit{Indeed: } for \( n \in \mathbb{Z} \)
\[ \exp \left[ \Sigma(x) + 2\pi i n \right] = P(x) \text{ too}. \]
\textit{I.e. which "branch" of } \log \textit{ } \text{ applies? You do NOT know this in general, even if } E = \{ \text{one point} \} \text{.}
Thm (Yes, this IS a theorem!)

Given \( a_n(z) = 1 + b_n(z), \ z \in E, \ |b_n(z)| \leq \lambda < 1 \)

as usual

If \( \prod_{n=1}^{\infty} (1 + b_n(z)) \) converges absolutely on \( E \),

then \( \prod_{n=1}^{\infty} (1 + b_n(z)) \) converges on \( E \).

[Remember \( E \) could be one point.]

PF

By hypothesis, we know \( \prod_{n=1}^{\infty} (1 + |b_n(z)|) \) converges at each \( z \in E \).

Apply (\( \Phi \)). Get \( \sum_{n=1}^{\infty} \ln(1 + |b_n|) \) converges on \( E \).

But, baby calculus \( \Rightarrow \)

\[
\frac{1}{2} t \leq \ln(1 + t) \leq t \quad \text{for} \quad 0 \leq t \leq 1.
\]

Hence \( \sum_{n=1}^{\infty} |b_n(z)| \) converges, each \( z \in E \).

But, baby analytic functions \( \Rightarrow \)

\[
a_n |w| \leq |\ln(1 + w)| \leq b_n |w| \quad \text{for} \quad |w| \leq \lambda < 1.
\]

\( 0 < a_n < b_n \rightarrow \infty \)
Accordingly,

$$\sum_{n=1}^{\infty} |\log(1+bn(z))| \text{ converges, each } z \in E.$$  

This absolute conv $\Rightarrow$ ordinary conv. of $\sum Log(1+bn)$.  

Now just apply (8) at each single point of $E$.  

---

EG use Taylor series or else

$$\log(1+w) = \int_{0}^{w} \frac{dx}{1+x}$$
Thm (Weierstrass M-test for products)

Given \( a_n(z) = 1 + b_n(z) \), \( |b_n(z)| \leq 1 \), \( z \in E \).

Assume that

\[ |b_n(z)| \leq M_n \quad \text{and} \quad \sum_{1}^{\infty} M_n < \infty \quad \text{on} \ E \]

Then:

\[ \prod_{1}^{\infty} \left( 1 + b_n(z) \right) \text{ conv unif on } E. \]

(In fact, so does \( \prod_{1}^{\infty} (1 + |b_n(z)|) \).)

PF

Apply (18), must show that \( \sum_{1}^{\infty} \log (1 + b_n(z)) \)

\[ \text{conv unif on } E. \text{ Recall } (24) \text{ last line! Get:} \]

\[ |\log (1 + b_n(z))| \leq b_0 |b_n(z)| \leq b_0 M_n. \]

A standard Weierstrass M-test now applies to \( \sum_{1}^{\infty} \log (1 + b_n(z)) \), hence \( \sum_{1}^{\infty} \log (1 + b_n(z)) \)

uniformly as needed. \( \Box \)

For the "in fact", just replace \( b_n(z) \) by \( |b_n(z)| \).

The same \( M_n \) still work. \( \Box \)
**Simple Exercise**

Given \( a_n(z) = 1 + b_n(z) \), \( |b_n(z)| \leq \lambda < 1 \) for \( z \in E \).

(a) \( \prod (1 + b_n) \) conv pointwise on \( E \) \( \iff \sum |b_n| \) does.

(b) \( \prod (1 + b_n) \) conv uniformly on \( E \) \( \iff \sum |b_n| \) does.

(c) \( \prod (1 + b_n) \) conv unif on \( E \) \( \Rightarrow \prod (1 + b_n) \) does too.

See \( \Box \). Note \( \log (1 + w) = \ln (1 + w) \sim |w| \) as \( w \to 0 \).

Also recall uniform Cauchy condition for unif conv.

---

**Mind-Twister Exercise** (otherwise known as)

Let \( E = \) one point. Keep \( |b_n| \leq 2 < 1 \). Put \( a_n = 1 + b_n \).

(a) Find \( \sum b_n \sim n \) so that \( \sum b_n \) conv, but \( \prod a_n \) div.

(b) Find \( \sum b_n \sim n \) so that \( \prod a_n \) conv, but \( \sum b_n \) div.
Eye-opening Exercise

This exercise really goes with lecture #6, but is placed here for convenience.

(A) Prove that \( \prod_{n=1}^{\infty} \cos(\frac{\pi}{2^n}) \) is unif and abs conv on every closed disk \( \{ |z| \leq R \} \), hence its value \( P(z) \) is some analytic fcn on \( \mathbb{C} \).

(B) [Main Problem!] Evaluate \( P(z) \) in simple terms.

Recall that:
\[
\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \\
\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}
\]
when \( z \in \mathbb{C} \). These trig fns are analytic on \( \mathbb{C} \). Standard identities are therefore true; eg \( \sin^2 z + \cos^2 z = 1 \).

Note too:
\( \sin(z) = 0 \iff z = n\pi \), etc.
I. Move was discussed on infinite products, especially Weierstrass M-test. E.g., for
\[ \prod_{n=1}^{\infty} \cos \left( \frac{x}{n} \right). \]

II. \[ \prod_{n=1}^{\infty} (1 + b_n(z)) \text{ with } |b_n(z)| \leq \lambda < 1 \text{ on } E. \] We stressed that when uniform conv holds, you get both
\[ \frac{p_N(z)}{p(z)} \to 1 \quad \text{AND} \quad p_N(z) \to p(z) \]

since \( c_1 < |p(z)| < c_2 \) on \( E \). Hence:
\[ b_n(z) \text{ continuous } \Rightarrow p(z) \text{ continuous on } \overline{E} \]
\[ b_n(z) \text{ analytic } \Rightarrow p(z) \text{ analytic (in the usual "on compacta" sense associated with Weierstrass convergence thm).} \]
Theorem \( \text{III} \): Cauchy products. I remarked that

\[
A = \sum_{n=0}^{\infty} a_n, \text{ abs conv} \quad (a_n \in \mathbb{C})
\]

\[
B = \sum_{n=0}^{\infty} b_n, \text{ abs conv} \quad (b_n \in \mathbb{C})
\]

\[
\downarrow
\]

\[
AB = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} a_j b_k \right) = \sum_{n=0}^{\infty} c_n, \text{ abs conv} \quad (\text{same proof})
\]

(\text{as in } \mathbb{R}, \text{ use } \text{III})

\[
\mathbb{R}(\mathbb{C}) > 1 \quad \text{say}, \quad \text{use } \text{III}, \text{ take } T \geq 2
\]

\[
\prod_{\frac{1}{p} \leq T} \frac{1}{1 - p^{-z}} = \prod_{\frac{1}{p} \leq T} \left\{ 1 + p^{-z} + p^{-2z} + \cdots \right\}
\]

\[
= \sum_{n \geq 1} n^{-z}
\]

\[
n \text{ is factorizable into primes which are all } \leq T
\]

\[
\text{includes all } n \leq T \text{ obviously}
\]

Notice too:

\[
\left| p^{-x} + p^{-2x} + \cdots \right| \leq p^{-x} + p^{-2x} + \cdots = \frac{p^{-x}}{1 - p^{-x}} = \frac{1}{p^{x-1}}
\]
\[ \prod_{p} \frac{1}{1-p^{-z}} = \prod_{p} \frac{1}{1-\frac{1}{p^z}} \leq \frac{1}{2^{x-1}} \quad \text{for all } p \quad \forall x \geq 1 + \delta \]

Hence, for \( x \geq 1 + \delta \), we have a good \( \mathcal{F} \) of
\[ \frac{1}{2^{x-1}} < 1 \quad \ast \]

All of our earlier theorems about infinite products apply — when we opt to let \( T \to \infty \). All is well.

We clearly get: (Euler)
\[ \prod_{p} \frac{1}{1-p^{-z}} = \prod_{p} \frac{1}{1-\frac{1}{p^z}} = \sum_{n=1}^{\infty} n^{-z} \]

with uniform absolute convergence on each closed half-plane \( \delta x \geq 1 + \delta \).

In particular, since LHS is nonzero (by def of conv infinite product), we get:
\[ \mathcal{F}(z) \neq 0 \quad \text{for } \text{Re}(z) > 1 \].
I drew attention to Euler's identity

$$\sum_{n=1}^{\infty} f(n) = \prod_p \left\{ 1 + f(p) + f(p^2) + f(p^3) + \ldots \right\}$$

in Ingham p.16 — under the assumption that

$$\sum_{n=1}^{\infty} |f(n)| < \infty$$

and $f$ is multiplicative

$$\begin{cases} f(1) = 1 \\ f(nm) = f(n)f(m) \text{ if } (m,n)=1 \end{cases}.$$

(Read proof there!)

I defined a natural branch of $\log S(z)$ on $\Re(z)>1$ by writing

$$\log S(z) = \sum_{n=2}^{\infty} \frac{A(n)}{\ln n} n^{-z}.$$

This is NOT in general $\log S(z)$!

For $z=x+iy$, however, one readily checks

$$\log S(z) = \ln S(x). \quad \text{RECALL } S(x) = \sum_{n=1}^{\infty} n^{-x} > 1.$$
Clarification: (regarding \( \log J(z) \))

Recall the last line and lines 5-7. "All is well" because we are using Weierstrass M-test with
\[
M_p = \frac{1}{p^{1+\delta}} \quad (\text{for } x \geq 1+\delta).
\]

Our \( \sum \log (1+b_p(z)) \) for the "infinite product equivalence theorem" in Lecture 5 is
\[
\sum_{p} \log (1 + b_p(z)) = \sum_{p} \log(1 - p^{-2z}).
\]
This infinite series converges to some \( \gamma(z) \).

The series is just
\[
\sum_{p} \left\{ p^{-z} + \frac{1}{2} p^{-2z} + \frac{1}{3} p^{-3z} + \cdots \right\}
\equiv \sum_{n=1}^{\infty} \frac{A(n)}{\log n} n^{-z}
\]
with good abs conv. As in Lecture 5, we always have:
\[
P(z) = \exp \left\{ \gamma(z) \right\}.
\]

So, here, we have:
\[
J(z) = P(z) = \exp \left\{ \gamma(z) \right\}.
\]

I.e., there is no question \( \gamma(z) = \text{some branch of } \log J(z) \) on \( \{ \text{Re}(z) > 1 \} \).

Clearly, by inspection, \( J(x) > 0 \) for \( x > 1 \).

Hence, we do have:
\[
\gamma(x) = \log J(x) = \log J(x).
\]
\[ \frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}, \quad \text{Re}(z) > 1. \]

(by Weierstrass' conv thm for analytic fns)

**Thm (Hadamard)**

For \( x > 1 \), \( y \neq 0 \)

\[ \left| \zeta(x)^3 \right| \left| \zeta(x+iy) \right|^4 \left| \zeta(x+2iy) \right| \leq 1. \]

**Pf**

Take \( \ln \phi \) \( \sqrt{\text{Want}} \)

\[ 3\ln |\zeta(x)| + 4\ln |\zeta(x+iy)| + \ln |\zeta(x+2iy)| \geq 0. \]

But,

\[ \ln |\zeta(z)| = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-x} \cos(y \ln n) \]

by \( \{ x > 1 \} \)
Since, for any $\theta \in \mathbb{R}$,
\[
3 + 4 \cos \theta + \cos(2\theta)
\]
\[
= 3 + 4 \cos \theta + 2 \cos^2 \theta - 1
\]
\[
= 2 + 4 \cos \theta + 2 \cos^2 \theta
\]
\[
= 2(1 + \cos \theta)^2 \geq 0,
\]
and \[
\frac{A(n)}{\ln n} \geq 0,
\]
a trivial substitution now gives what we claimed. \[\Box\]

Corollary (Hadamard famous result)

\[
S(1 + iy) \neq 0 \text{ if } y \neq 0.
\]

\underline{Pf}

Suppose we had $S(1 + iy) = 0$ at some $y \neq 0$. \[\downarrow\]
\[
\frac{(x-1)^3 \left| f(x) \right|^3 \left| f(x+i^2 y) \right|^4}{(x-1)^4} \geq \frac{1}{x-1}
\]

\[
\frac{\xi \left| f(x+i^2 y) \right|^3}{\left| f(x+i^2 y) - f(1+i^2 y) \right|} \left| f(x+2i^2 y) \right|^4 \geq \frac{1}{x-1}
\]

\text{Let } x \to 1^+

\[
1^3 \left| f(1+i^2 y) \right|^4 \left| f(1+2i^2 y) \right| \geq 0 \Rightarrow
\]

\[\sqrt[n]{\text{Contradiction}}\]

\[\text{since } \int f(z) \text{ is nicely analytic for } \Re(z) > 0 \text{ except at } z = 1 \text{ }^2 \]

\[\text{ Proof } \]

\[\text{ (Condition stated) }\]
Theorem (essentially like Ingham p.27)

Let \( 0 < \delta < 1 \). We then have:

(a) \( |S(x+iy)| \leq A \ln |y| \) for \( x \leq 1, |y| \geq 2 \)

(b) \( |S(x+iy)| \leq B \ln^2 |y| \) for \( x \leq 1, |y| \geq 2 \)

(c) \( |S(x+iy)| \leq \frac{C}{\delta(1-\delta)} |y|^{1-\delta} \) for \( x \leq \delta, |y| \geq 2 \)

Here \( A, B, C \) are certain absolute constants.

\[ \prod_{n=1}^{N} n^{-z} = 1 + \frac{1-N^{1-\delta}}{\delta - 1} - \frac{z}{\delta} \int_{1}^{\infty} \frac{r(t)}{t^{\delta+1}} \, dt \]  

\( (\delta \neq 1) \)  

Lec 5, p. 8 + 9

\[ S(z) = 1 + \frac{1}{\delta - 1} - \frac{z}{\delta} \int_{1}^{\infty} \frac{r(t)}{t^{\delta+1}} \, dt \]  

\( (\text{Re}(z) > 1) \)  

Lec 5, p. 9

Then, we used this last formula to define \( S(z) \) for \( \text{Re}(z) > 0 \). Lec 5, p. 9
By subtraction,

\[ f(z) - \sum_{n=1}^{N} n^{-z} = \sum_{n=1}^{N} \frac{1}{z} - \frac{N^{1-\frac{z}{z}}}{z-1} - \varepsilon \int_{N}^{\infty} \frac{v(t)}{t^{z+1}} \, dt. \]

This is a very useful TRICK!!

\[ f(z) = \sum_{n=1}^{N} n^{-z} + \frac{N^{1-\frac{z}{z}}}{z-1} - \varepsilon \int_{N}^{\infty} \frac{v(t)}{t^{z+1}} \, dt. \]

\[ \text{Re}(z) > 0 \quad (\varepsilon \neq 1) \]

We propose to begin with (C) [even though it looks to be the most complicated].

To prove (C), notice that it suffices to prove it for, say, \( |y| \geq 100 \).

In fact, for \( 2 \leq |y| \leq 100 \), we can just use our old VERY CRUDE Theorem 4 from Lec 5, page 12.

In this connection, recall too that

\[ |I(z)| \leq \frac{1}{5} + O(1) \leq \frac{\text{const}}{5} \]

for all \( \text{Re}(z) \geq 1 + 8 \). (Also on p. 12.)
Use p. 10 line 4 above.

\[ |s(x+iy)| \leq \sum_{n=1}^{N} n^{-x} + \frac{N^{1-x}}{|x|^2 - 1} + |z| \int_{N}^{\infty} \frac{1}{t^{x+1}} \, dt \]

\[ |s(x+iy)| \leq \sum_{n=1}^{N} n^{-\delta} + \frac{N^{1-\delta}}{|y|} + (x+1y) \int_{N}^{\infty} \frac{dt}{t^{1+\delta}} \]

\[ \sum_{n=1}^{N} n^{-\delta} < 1 + \int_{1}^{N} u^{-\delta} \, du \]\{ by areas \}

\[ \sum_{n=1}^{N} n^{-\delta} < 1 + \frac{u^{1-\delta}}{1-\delta} \bigg|_{1}^{N} \]

\[ \sum_{n=1}^{N} n^{-\delta} < 1 + \frac{N^{1-\delta}}{1-\delta} < 2 \frac{N^{\frac{1-\delta}{1-\delta}}} \]

\[ |s(x+iy)| \leq 2 \frac{N^{1-\delta}}{1-\delta} + \frac{N^{1-\delta}}{100} + (x+1y) \frac{N^{-\delta}}{\delta} \]

For \( x \geq 1+\delta \), we already know \( |s(x+iy)| \leq \frac{\text{const}}{\delta} \)

hence (C) is certainly OK here if \( \Omega \) is taken sufficiently big.
For this reason, there is no harm in proceeding under the assumption

\[ |y| \geq 100, \quad 5 \leq x \leq 1 + 5 \]

Get:

\[
|f(x+iy)| = 2 \frac{N^{-\delta}}{1-\delta} + \frac{N^{1-\delta}}{100} + 2 |y| \frac{N^{-\delta}}{\delta}
\]

\[ = 3 \frac{N^{1-\delta}}{1-\delta} + 2 |y| \frac{N^{-\delta}}{\delta}
\]

\[ = 3 N^{-\delta} \left[ \frac{N}{1-\delta} + \frac{|y|}{\delta} \right] \]

This estimate can admittedly be improved. But a sloppy one is sufficient.

Also, recall \( f(x-iy) = \overline{f(x+iy)} \). Hence, wlog, \( |y| \geq 100 \).

Let's try \( N = g \frac{y}{\delta} \) where \( 1 \leq g \leq 10 \), say.

And we adjust it to make \( N \in \mathbb{Z} \).

(Note \( \frac{y}{\delta} \geq \frac{100}{\delta} \geq 100 \)).
Get \( L(x + iy) \)

\[
L(x + iy) = 3 \left( \frac{y^2}{\delta} \right)^{-\delta} \left[ N + \frac{y}{\delta} \right] \frac{1}{1-\delta} \quad \text{by \ref{12}}
\]

\[
= 3 \left( \frac{y^2}{\delta} \right)^{-\delta} \left[ \frac{y^2}{\delta} + \frac{y}{\delta} \right] \frac{1}{1-\delta}
\]

\[
= \frac{3}{1-\delta} \delta^{-\delta} \frac{1}{\delta} \left( \delta + 1 \right) \frac{y}{\delta}
\]

\[
\delta^{-\delta} = e^{-\delta \ln \delta}
\]

is bad away from 0

and so for \( 0 \leq \delta \leq 1 \)

\[
\leq c_1 \frac{2\delta}{\delta} \frac{1}{1-\delta} \frac{2\delta}{\delta} \quad \{ 1 \leq \delta \leq 10 \}
\]

\[
\leq c_2 \frac{y^{1-\delta}}{(1-\delta)^{\delta}} \quad \text{as required}
\]

This proves \( (C) \).

It is important to note \( \delta \in (0,1) \)

is arbitrary. It could even be taken as a function of \( y \).
(A) is now a trivial consequence of (C).

Indeed, since \( S(z) \) is a nice analytic function for \( \Re(z) \geq 1, |y| \geq 2 \), there is nothing to do for \( \{ 1 \leq x \leq 2, 2 \leq |y| \leq 100 \} \). For \( \{ x \geq 2, 2 \leq |y| \leq 100 \} \), just use (10) last 3 lines; again nothing to do.

So, wlog, we can assume \( |y| \geq 100 \). Also \( y \geq 100 \).

Put \( \delta = 1 - \frac{1}{\ln y} \) in (C). Note \( \ln 100 = 4.605 \).

Hence \( 0.75 < \delta < 1 \). By (C), get (see (9)):

\[
|S(x+iy)| \leq 2e \frac{1}{1-\delta} y^{-\delta}, \quad x \geq 1 - \frac{1}{\ln y}
\]

\[
|S(x+iy)| \leq 2e (\ln y) y^{\frac{1}{\ln y}}
\]

\[
|S(x+iy)| \leq 2e (\ln y) \leq 6e (\ln y).
\]

Now just specialize to \( x = 1 \). Done!

(B) is "almost" as trivial once we recall Cauchy's inequality for \( |f'(z_0)| \). IE

\[
f'(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2} \, dz
\]
\[ \left| f'(z_0) \right| \leq \frac{1}{2\pi} \frac{M(R)}{R} (2\pi R) \]

where \( M(R) = \max_{|z-z_0|=R} |f| \)

\[
\left| f'(z_0) \right| \leq \frac{M(R)}{R}
\]

Here are the details for (B).

First, since \( f(z) \) is a nice analytic fun for \( \Re(z) = 1, |y| \leq 2 \), there is nothing to do for \( \{ 1 \leq x \leq 2, 2 \leq |y| \leq 150 \} \). For \( \{ x \leq 2, 2 \leq |y| \leq 150 \} \), apply Cauchy's inequality with \( R = \frac{1}{2} \) and (10) last 3 lines. Again nothing to do. ∗

So, wlog, take \( |y| \leq 100 \). We can also assume \( y \approx 100 \). We will use (C) with a δ similar to \( 1 - \frac{1}{\ln y} \).

∗ I do want 150 here, i.e., a slight overshoot over 100.
Take $\delta = 1 - \frac{2}{\ln y}$, where $0 < \delta \leq 1$. We'll choose $\lambda$ in a few moments. Note that we have $0.75 < \delta < 1$ by (14) line 8. Apply (C) on page 7.

Get:

\[
|S(x+iy)| \leq \frac{2e}{1-\delta} y^{\frac{\lambda}{\ln y}}, \quad \text{all } x \geq 1 - \frac{\lambda}{\ln y}, \quad y \geq 100
\]

\[
|S(x+iy)| \leq \frac{2e}{\lambda} (\ln y) e^{\lambda}
\]

\[
|S(x+iy)| \leq \frac{6e}{\lambda} (\ln y) \quad \text{for all } x \geq 1 - \frac{\lambda}{\ln y}, \quad y \geq 100.
\]

We want to RIG THINGS so we can select $y \geq 110$, then use $R = \frac{1}{10} \frac{\lambda}{\ln y}$ (say) in Cauchy's inequality for a center $z_0$ along the segment $[1+iy, \infty+iy)$.

Note that $R \leq \frac{1}{10} \frac{1}{\ln y} \leq \frac{1}{46} < \frac{1}{2}$. 

\[
\ln 100 = 4.6054
\]

\[
\begin{array}{c}
\text{R} \\
1+iy \\
2+iy \\
\infty+iy
\end{array}
\]
As the circle slides along, its y-values clearly stay between \( y-1 \) and \( y+1 \).

Hence, obviously, \( y \leq 100 \).

But we must make certain that no matter what happens, we have \( x \geq 1 - \frac{\theta}{\ln y} \) at all times on the circle.

**Baby Calculus Lemma**

Given \( T \geq 110 \). Keep \( y \in [T-1, T+1] \).

Then:

\[
\frac{y}{5} \leq \frac{\ln y}{\ln T} \leq \frac{5}{y}.
\]

**PF**

\[
\frac{\ln(T-1)}{\ln T} \leq \frac{\ln y}{\ln T} \leq \frac{\ln(T+1)}{\ln T}.
\]

But, by theorem of the mean:

\[
\ln(T+1) - \ln(T) \leq \frac{1}{T} (1)
\]

\[
\ln(T) - \ln(T-1) \leq \frac{1}{T-1} (1).
\]
So,

\[
\ln(T+1) \leq \ln(T) + \frac{1}{T} \\
\ln(T-1) \geq \ln(T) - \frac{1}{T-1}
\]

so,

\[
\ln(T+1) \leq \ln T + \frac{1}{110} < \ln T + \frac{1}{100} \\
\ln(T-1) \geq \ln T - \frac{1}{109} > \ln T - \frac{1}{100}
\]

so

\[
\frac{\ln T - \frac{1}{100}}{\ln T} \leq \frac{\ln y}{\ln T} \leq \frac{\ln T + \frac{1}{100}}{\ln T}
\]

\[
\text{(but)} \quad \ln T \geq \ln 100 = 4.605<3
\]

\[
\text{so} \quad 0.99 \leq \frac{\ln y}{\ln T} \leq 1.01 \quad \text{OK}
\]

\[
\text{(the) moving circle on (16 bottom), obviously}
\]

\[
x \geq 1 - \frac{1}{10} \frac{\lambda}{\ln y}
\]

But \(\ln y = \omega \ln y\) with \(\frac{y}{5} \leq \omega \leq \frac{5}{y}\) by Calc Lemma.

So,

\[
x \geq 1 - \frac{1}{10} \frac{\omega}{\ln y} \quad \text{on circle}
\]
Must make sure

\[ \frac{\lambda}{\ln y} \geq \frac{\lambda}{10} \cdot \frac{1}{\ln y} \]

i.e.,

\[ 1 \geq \frac{1}{10w} \]

Thus things are OK and \( \lambda \) is irrelevant.

But

\[ 8 \leq 10w \leq 12.5 \]

So: just put \( \lambda = 1 \).

\[ \text{Get: } R = \frac{1}{10} \frac{1}{\ln y} \]

By Cauchy's inequality (15) and (16) (middle), we find that:

\[ |\mathcal{F}(x+iY)| \leq \frac{6 \mathbb{E} [\ln(Y+1)]}{R} \leq \frac{120 \mathbb{E} \ln Y}{R} \]

\[ \ll 120 \mathbb{E} (\ln Y)^2 \]

for any \( x \in [-1, \infty) \). Recalling (15) lines 5-9, we have thus proved (B).
Two Remarks

1. Take \( \lambda \in [11, 17] \) say. Note \( \ln 10.6 = 13.815^+ \).

By mimicking \((14)-(17)\), one easily sees that

\[
|f(x+iy)| = O(\ln y) \quad \text{for} \quad x > 1 - \frac{5}{\ln y}, \quad y \geq 10.6
\]

\[
|f'(x+iy)| = O(\ln^2 y) \quad \text{for same} \ (x, y) \ .
\]

One can take \( R = \frac{\lambda}{2\ln y} \). Key necessity is

\[
\frac{\lambda}{\ln y} \geq \frac{5}{\ln y} + \frac{\lambda}{2\ln y}
\]

or

\[
\frac{\lambda}{\ln y} \geq \frac{5}{\ln y} + \frac{\lambda}{2(\ln y)}
\]

or

\[
\lambda \left(1 - \frac{1}{2\omega}\right) \geq \frac{5}{\omega} \ .
\]

But, by \((18)\) \( \omega = 1 \pm 0.01 \) \ . \ OK

2. Why do we use \( \delta = 1 - \frac{3}{\ln y} \)?

Answer: go to \((13)\) 5 lines from bottom \ .
We wonder if \( \delta \) is close to 1 and \( y \) is very large, what is the smallest that
\[
\frac{y^{1-\delta}}{1-\delta}
\]
can be? This is a trivial calc problem.\( \delta = 1-u \Rightarrow \text{look at } \frac{yu}{u} \Rightarrow \text{look at } \]
\[f(u) = u \ln y - \ln u, \quad 0 \leq u \leq 1\]
\[f' = \ln y - \frac{1}{u}\]
get \( f' > 0 \iff u > \frac{1}{\ln y} \).
So key \( \delta \) is \( 1 - \frac{1}{\ln y} \), which gives \( e^{\frac{1}{\ln y}} \).

The insertion of \( \delta \) allows us to "move around" a bit.
Lecture 7
(10 Feb)

Abel Summation Lemma

Let \( \sum \geq \sum_2 \geq \ldots \geq \sum_N \geq 0 \).

Let \( c_n \in C \) and \( | \sum_1^n c_n | \leq M \) for \( 1 \leq n \leq N \).

Then:

\[
| \sum_1^n + \cdots + \sum_N c_n | \leq M \varepsilon_1 + \varepsilon_N c_N
\]

Proof:

\[
\mathrm{Sum} = \sum_1^{n+1} + \sum_1^{n+1} \varepsilon_2 + \cdots + \sum_1^{n+1} \varepsilon_N (\sum_1^{n+1} - \sum_1^{n+1})
\]

\[
= \sum_1^{n+1} (\varepsilon_2 - \varepsilon_1) + \cdots + \sum_1^{n+1} (\varepsilon_n - \varepsilon_{n-1}) + \varepsilon_N \sum_1^{n+1}
\]

\[
| \mathrm{Sum} | \leq M (\varepsilon_2 - \varepsilon_1) + \cdots + M (\varepsilon_n - \varepsilon_{n-1}) + M \varepsilon_N
\]

\[ \varepsilon_1 \]

A immediate corollaries are:

**Theorem (Dirichlet Test for Uniform Convergence)**

Let \( 1 \geq \varepsilon_1 (\beta) \geq \varepsilon_2 (\beta) \geq \ldots \geq \geq 0 \) and \( \sum_{n=1}^{\infty} b_n (\beta) \)

for \( \beta \in E \). Let \( \sum_{n=1}^{\infty} b_n (\beta) \) have uniform bounded partial sums for \( \beta \in E \). Then

\[
\sum_{n=1}^{\infty} \varepsilon_n (\beta) b_n (\beta)
\]

conv. uniform on \( E \times E \).
Use uniform Cauchy criterion!

**Theorem (Abel's Test for Uniform Conv)**

Let \( a \in E_1 \), \( \beta \in E_2 \) again. Let \( 1 \leq \varepsilon_1(\alpha) \leq \varepsilon_2(\alpha) \leq \ldots \to 0 \), not nec going to 0. Let \( \sum_{\beta} b_n(\beta) \) conv unif on \( E_2 \). Then

\[
\sum_{\alpha} \varepsilon_n(\alpha) b_n(\beta)
\]

conv unif for \((\alpha, \beta) \in E_1 \times E_2 \).

**Proof**

Use uniform Cauchy criterion!

Use trivial geom series:

\[
| \sum_{n=0}^{N} e^{2\pi i n} \alpha | \leq \frac{2}{2|\sin\alpha|} = \frac{1}{|\sin\alpha|}, \quad \alpha \neq Z.
\]

By Dirichlet's test, get

\[
\sum_{\alpha} \frac{1}{\alpha} e^{2\pi i n} \alpha
\]

conv unif for \( \delta \geq \varepsilon \geq 1 - \delta \). We connect this series with

\[
- \log(1-Z).
\]
\[-\log(1-z) = z + \frac{z^2}{2} + \ldots, \quad |z| < 1\]

\[z = re^{i\alpha} \quad \text{as usual}\]

\[-\log(1-re^{i\alpha}) = re^{i\alpha} + \frac{1}{2} r^2 e^{2i\alpha} + \frac{1}{3} r^3 e^{3i\alpha} + \ldots\]

\[0 \leq r < 1 \quad \text{Wish to let } r \to 1\]

\[r \geq 0^\circ \quad \sum_n \frac{1}{n} e^{2\pi in^\alpha} \quad \text{coav unif away from}\]

\[\alpha \in \mathbb{Z}, \quad \text{i.e. away from } z = 1.\]

So, by Abels' test, get

\[\sum_n \frac{r^n}{n} e^{2\pi in^\alpha}\]

\[\text{coav unif } 0 \leq r \leq 1, \quad \delta \leq \alpha \leq 1 - \delta \quad (\text{say})\]

We conclude therefore that

\[-\log(1-e^{2i\alpha\theta}) = \sum_{n=1}^{\infty} \frac{e^{2\pi in\alpha\theta}}{n}\]

for \(0 < \alpha < 1\).

One writes:

\[1 - e^{2i\alpha\theta} = e^{i\theta}(e^{2i\alpha\theta} - e^{-i\theta})\]

\[= e^{i\theta}(-2i\sin\theta)\]

\[= 2\sin\theta e^{i\theta} e^{-i\theta/2}, \quad 0 < \alpha < 1.\]
Conclude at once: (see 3*)

\[-\ln(\alpha \sin \pi \alpha) = \sum_{n=1}^{\infty} \frac{\alpha}{n} \cos(2\pi n \alpha)\]

\[\alpha - \frac{1}{2} = \sum_{n=1}^{\infty} \frac{\alpha}{\pi n} \sin(2\pi n \alpha)\]

with unit conv for \(\Delta \equiv \frac{\pi}{2} \sim \Delta_0\). These are two very basic Fourier series, especially \#2.

\[\alpha - \left\lfloor \alpha \right\rfloor - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin(2\pi n \alpha)}{\pi n}, \quad \alpha \notin \mathbb{Z}\]

\[\left\lfloor \alpha \right\rfloor \sim \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin(2\pi n \alpha)}{\pi n}, \quad \alpha \notin \mathbb{Z}\]

(can)

We now start moving very majestically toward the proof of PNT.

Recall from Lec 6:

\[|\tilde{f}(x+iy)| \leq A \ln |y|, \quad x \geq 1, \quad |y| \geq 2\]

\[|\tilde{f}'(x+iy)| \leq B \ln^2 |y|, \quad x \geq 1, \quad |y| \geq 2\]

\[|\tilde{f}(x+iy)| \leq \frac{C}{\delta(1-\delta)} |y|^{-1-\delta}, \quad x \geq \delta, \quad |y| \geq 2, \quad 0 < \delta < 1\]
We also had:

\[
S(z) = \prod_{\rho} \frac{1}{1 - \rho^{-z}}, \quad \text{re}(z) > 1
\]

\[
\log S(z) = \sum_{n=2}^{\infty} \frac{A(n)}{\log n} n^{-z}, \quad \text{re}(z) > 1
\]

\[
\frac{S(z)}{S(1)} = -\sum_{n=2}^{\infty} \frac{A(n)}{n^z}, \quad \text{re}(z) > 1
\]

Hadamard:

\[
|S(x)|^2 |S(x+iy)|^2 \gtrless 1, \quad x > 1, \quad y \neq 0.
\]

I proved in Lec 6 that \( S(x+iy) \neq 0 \).

**Theorem (Improvement over \( S(1+iy) \neq 0 \))**

\[
\frac{1}{S(x+iy)} = O(\frac{\log|y|}{|y|})^7 \quad \text{for} \quad x \geq 1, \quad |y| \gtrsim 2.
\]

**Proof**

Very close to Ingham's

\[
x \geq 2 \Rightarrow \left| \frac{1}{S(z)} \right| = \left| \prod_{\rho} (1 - \rho^{-z}) \right| \leq \prod_{\rho} (1 + \rho^{-x}) \lesssim \prod_{\rho} \frac{1}{1 - \rho^{-x}}
\]

\[
\left| \frac{1}{S(z)} \right| \lesssim S(x) \lesssim S(2) \quad \ast
\]

So, wlog, \( 1 \lesssim x \lesssim 2 \). Also, wlog, \( y \gtrsim 3 \).
Notice

\[ \left[ (x-i) f(x) \right]^3 / f(x+i y) \overset{y}{\Rightarrow} f(x+2i y) \Rightarrow (x-i)^3 \]

\[ \Rightarrow \left| f(x+i y) \right|^4 \geq \frac{(x-i)^3}{(x-1)^3 \ln(2y)} \]

\[ \left| f(x+i y) \right|^4 \geq \frac{(x-i)^3}{A \ln y} \quad (not \ same \ A) \]

\[ \left| f(x+i y) \right| \geq \frac{(x-i)^{3/4}}{A(\ln y)^{1/4}} \quad \begin{cases} \frac{1}{\sqrt{x}} \leq 2 \\ \frac{1}{\ln y} \geq 3 \end{cases} \]

But \( \left| f(x+i y) \right| = O(\ln^2 y) \), all \( x \geq 1, y \geq 3 \).

Take any \( c \in (1,3) \), \( x \in [1,27] \). Get:

\( f(x+i y) - f(c+i y) = \int_c^x f(u+i y) \, du \)

\[ \left| f(x+i y) - f(c+i y) \right| \leq A/\sqrt{x-c/\ln^2 y} \]

\[ \left| f(x+i y) \right| \geq \left| f(c+i y) \right| - A/\sqrt{x-c/\ln^2 y} \]

But:

\[ \left| f(c+i y) \right| \geq \frac{(c-i)^{3/4}}{A(\ln y)^{1/4}} \text{ by above.} \]
Get:
\[ \int (x+y) \geq \frac{(c-1)^{3/4}}{A(\ln y)^{1/4}} - \frac{A}{c-x/\ln y} \]

If \( x \leq c > 1 \), just use
\[ \int (x+y) \geq \frac{(c-1)^{3/4}}{A(\ln y)^{1/4}} \]

since it's better. But if \( \frac{c-x}{\ln y} \), use
\[ \int (x+y) \geq \frac{(c-1)^{3/4}}{A_1(\ln y)^{1/4}} - \frac{A_1}{c-x/\ln y} \]
\[ \int (x+y) \geq \frac{(c-1)^{3/4}}{A_2(\ln y)^{1/4}} - A_2(c-1)\ln^2 y \]
\[ \int (x+y) \geq \frac{1}{A_1} \left[ \frac{(c-1)^{3/4}}{(\ln y)^{1/4}} - A_3(c-1)\ln^2 y \right] \]

Now have a trivial-type calculus problem to make bracket as large as possible.

Standard trick:
\[ \frac{(c-1)^{3/4}}{(\ln y)^{1/4}} = C(c-1)\ln^2 y, \quad C = \text{adjustable} \]
\[ \frac{1}{C} \frac{1}{(\ln y)^{1/4}} = (c-1)^{1/4} \]
so we want
\[ c - 1 = \frac{g}{(ln y)^9} \]

\( G \) to be adjusted.

We declare:
\[ c = 1 + \frac{g}{(ln y)^9} \quad (y \geq 3) \]

and keep \( G \) small enough that \( c \in (1/2) \).

Get:
\[ |S(k+iy)| \geq \frac{1}{A_1} (c-1)^{3/4} \left[ \frac{1}{(ln y)^{1/4}} - A_3 (c-1)^{1/4} (ln y)^{3/4} \right] \quad \text{by (7)} \]

\[ = \frac{1}{A_1} \frac{g^{3/4}}{(ln y)^{3/4}} \left[ \frac{1}{(ln y)^{1/4}} - A_3 \frac{g^{1/4}}{(ln y)^{3/4}} (ln y)^{3/4} \right] \]

\[ = \frac{1}{A_1} \frac{g^{3/4}}{(ln y)^3} \left[ 1 - A_3 g^{1/4} \right] \quad . \]

Want \( G \) so small that \( 1 - A_3 g^{1/4} \geq \frac{1}{2} \)
(in addition to keeping \( k << R \)).

Get:
\[ |S(k+iy)| \geq \frac{\text{constant}}{(ln y)^7}, \quad 15 < x < 1 + \frac{g}{(ln y)^9}. \]
For \( x > 1 + \frac{G}{(\ln y)^3} \) we use \([7]\) line 4.

\[
|J(x+iy)| \geq \frac{(c-1)^{3/4}}{A^2(\ln y)^{3/4}} = \frac{\left(\frac{G}{\ln y}\right)^{3/4}}{A(\ln y)^{1/4}}
\]

\[
\geq \frac{\text{constant}}{(\ln y)^{1/2}}.
\]

So, in all cases,

\[
|J(x+iy)| \geq \frac{\text{const}}{(\ln y)^{1/2}} \quad x \in [1, 2]
\]

\[
y > 3.
\]

So:

\[
\frac{1}{|J(z)|} \leq (\text{const})(\ln y)^{1/2}.
\]

One defines (Following Riemann)

\[
\psi_1(x) = \int_0^x \psi(v) \, dv \quad \{\psi(v) = 0, \; v < x\}
\]

\[
= \int_0^x \left( \sum_{k \leq x} A(k) \right) \, dv
\]

\[
= \sum_{k \leq x} A(k) \int_k^x \, dv = \sum_{k \leq x} A(k)(x-k).
\]
We will also follow Riemann and begin writing

\[ s = \sigma + it \]

instead of \( z = x + iy \).

**Theorem (Fund. Formula)**

\[
\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left[ -\frac{\zeta'(s)}{\zeta(s)} \right] ds
\]

for all \( c > 1, x > 0 \).

Note that:

\[
\left| \frac{x^{s+1}}{s(s+1)} \right| = \frac{x^{c+1}}{|s| |s+1|} \leq \frac{x^{c+1}}{|t|^2}.
\]

There is no question that RHS converges.
Proof

We require a standard lemma from complex analysis.

Lemma

Fix any \( c > 0 \). Then:

\[
\frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{y^s}{s(s+1)} \, ds = \begin{cases} 
0 & \text{if } y < 1 \\
1 - y^{-1} & \text{if } y \geq 1
\end{cases}
\]

PF

\( y \geq 1 \) first.

Our analytic function of \( s \)

is \( \frac{y^s}{s(s+1)} \) \( (y \text{ fixed}) \)
Ingham uses Cauchy residue thus on this shape. We prefer Cauchy integral theorem + Cauchy integral formula.

\[
\text{CIT} \Rightarrow \oint = \oint_\Gamma + \oint_\eta
\]

But,

\[
\frac{1}{2\pi i} \oint_\Gamma \frac{1}{s} \frac{y^s}{s+1} ds = \frac{y^0}{0+1} = 1 \quad \text{by CIF}
\]

\[
\frac{1}{2\pi i} \oint_\eta \frac{1}{s} \frac{y^s}{s+1} ds = \frac{y^{-1}}{-1} = -y^{-1} \quad \text{by CIF}
\]

and

\[
\left| \int\limits_{\Gamma} \frac{y^s}{s(s+1)} ds \right| \leq \int\limits_{\Gamma} \left| \frac{y^c}{|s||s+1|} ds \right| \leq (\text{const}) y^c \frac{1}{R^2} \pi R
\]

\[
\leq O \left( \frac{y^c}{R} \right) \to 0 \quad \text{as} \quad R \to \infty
\]
We immediately get:

\[
\frac{1}{2\pi i} \oint_{\Gamma} \frac{y}{s} ds = 0 \quad \text{for } y > 1.
\]

We are now ready for the theorem. Freeze \( c > 1 \) and \( x > 0 \).

\[\sum_{\text{on the imaginary axis}} \text{is again} \]

\[0 \rightarrow -1 \]

\[1 \]

\[0 \]
Want to integrate term-by-term.

Standard set-up applies:

\( g(t) \) absolutely integrable on \( \mathbb{R} \)

\[ |f_n(t)| \leq M_n \quad \sum_{n} M_n < \infty \quad \text{for Weierstrass M-test} \]

\[ \sum_{n=1}^{\infty} g(t) f_n(t) dt = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(t) f_n(t) dt \]

(Standard ADV calculus)

\[
\left\{ \begin{array}{l}
\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} |g(t)||f_n(t)| dt \\
\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} |g(t)| M_n dt < \infty
\end{array} \right. 
\]

We immediately get:

\[ \Omega \text{ bottom } = \sum_{n=1}^{\infty} \Lambda(n) \int_{-\infty}^{\infty} \frac{x}{s^{\frac{1}{2}}} \left\{ \frac{x}{\pi} \right\}^5 ds \]

\[ = \sum_{n=1}^{\infty} \Lambda(n) x \left[ 1 - \frac{1}{x/n} \right] \quad + \quad 0 \quad \text{by Lemma} \]

\[ = \sum_{n \leq x} \Lambda(n) \left[ x - n \right] = \psi_1(x). \]
Remarks

In Lec 8, I derived Thm 10 by elementary use of Fourier integrals. It is by no means essential to use complex variables. Riemann certainly knew this.

Once having Riemann's fund formula, one seeks to move \( \{ \text{Re}(s) = c \} \) over to the left. THIS WILL USE COMPLEX VARIABLE!

I prefer to move the line to \( \{ \text{Re}(s) = 1 \} \).

I need one more little lemma.

See Ingham p. 31.

CLAIM: \( y > 0 \) fixed, \( c > 0 \) fixed

\[
\frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} \frac{y^s}{s(s+1)} ds = \begin{cases} 0, & \text{if } y < 1 \\ \frac{1}{2} (1-y^2)^{1/2}, & \text{if } y \geq 1 \end{cases}
\]

Pf

Very similar to what we did on (1) \( \sim \) (3).

Omit \( y < 1 \). For \( y \geq 1 \), notice that
\[
\frac{1}{2\pi i} \oint \frac{1}{5} \frac{y^5}{(s+1)(s+2)} \, ds = \frac{y^0}{1 \cdot 2} = \frac{1}{2}
\]

\[
\frac{1}{2\pi i} \oint \frac{1}{5+1} \frac{y^5}{s(s+2)} \, ds = \frac{y^{-1}}{(-1)(1)} = -y^{-1}
\]

\[
\frac{1}{2\pi i} \oint \frac{1}{5+2} \frac{y^5}{s(s+1)} \, ds = \frac{y^{-2}}{(-2)(-1)} = \frac{1}{2} y^{-2}
\]

\[\sum = \frac{1}{2} (1-y^{-1})^2 \quad \text{(Ok)}\]

\[
\text{Keep } y \geq 1 \text{. Know: (any } \eta > 0) \]

\[
\frac{1}{2\pi i} \oint_{\gamma^{-1} \circ \epsilon} \frac{1}{s(s+1)(s+2)} \, ds = \frac{1}{2} (1-y^{-1})^2
\]

\[\bar{\epsilon} = s + 1\]

\[
\frac{1}{2\pi i} \oint_{\gamma^{-1} \circ \epsilon} \frac{y^{\bar{\epsilon}-1}}{(\bar{\epsilon}-1)\bar{\epsilon}(\bar{\epsilon}+1)} \, d\bar{\epsilon} = \frac{1}{2} (1-y^{-1})^2
\]

\[1 + \eta > 1\]
The problem with moving \( \{\text{Re}(s) = c\} \) in (10) directly to \( \text{Re}(s) = 1 \) is hitting the pole at \( s=1 \). Must modify things slightly.

\[ f(s) = (s-1)^{-1} \left[ 1 + c_1(s-1) + c_2(s-1)^2 + \cdots \right] \]

\[ f(s) = (s-1)^{-1} \phi(s) \quad \text{say} \]

\[ \frac{f'(s)}{f(s)} = -\frac{1}{s-1} + \frac{\phi'(s)}{\phi(s)} \quad \text{near } s=1 \]

\[ -\frac{f'(s)}{f(s)} = \frac{1}{s-1} + \frac{[\text{analytic}]}{\phi(s)} \]

\[ \forall \]

\[ -\frac{f(s)}{f(s)} = 1 \quad \text{is analytic for } \text{Re}(s) > 1 \]

\[ \text{Notice that } \quad (c > 1) \]

\[ \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} \frac{x^{s-1}}{(s-1)s(s+1)} ds = \frac{1}{2} \left( 1 - \frac{1}{x} \right)^2 \]

by (16) bottom for \( x > 1 \).
For \( x > 1 \), we thus have:

\[
\frac{\Psi_i(x)}{x^2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)} \left[ -\frac{s^{i\xi}}{s^{i\psi}} \right] ds
\]

\[
\frac{1}{2} (1 - \frac{1}{x})^2 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)} \left[ \frac{1}{s-1} \right] ds
\]

\[
\frac{\Psi_i(x)}{x^2} - \frac{1}{2} (1 - \frac{1}{x})^2 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)} \left[ -\frac{s^{i\xi}}{s^{i\psi}} - \frac{1}{s-1} \right] ds
\]

Write

\[
H(s) = -\frac{s^{i\xi}}{s^{i\psi}} \sim \frac{1}{s-1} \quad \text{for} \quad \text{Re} s \geq 1
\]

For \( |t| \geq 3 \), know that:

\[
|H(s)| \leq \text{constant} + O(1) \frac{\ln^2 |t|}{|t|} \cdot \ln^2 |t|
\]

by (5) + (4)(bot). I.e.,

\[
|H(s)| \leq O(1) \left( \ln^2 |t| \right)^9
\]

for \( \sigma \geq 1 \), \( |t| \geq 3 \).
Moving the contour in line 4 is now trivial.

\[ \int_{\text{horizontal}} \frac{x^{c-1}}{x^{5+1}} H(s) ds \approx \int_0^\infty O(1) \frac{x^{c-1}}{R^2} (\ln R)^g \, dx \]

\[ \approx O(1) (c-1) x^{c-1} \frac{(\ln R)^g}{R^2} \]

\[ \to 0 \text{ as } R \to \infty, \quad (x, c \text{ frozen}) \]
Geometrically,

\[
\sqrt{\frac{\psi(x)}{x^2}} - \frac{1}{\alpha} \left(1 - \frac{1}{x^2}\right) = \frac{1}{\alpha \pi i} \int_{1-i \infty}^{1+i \infty} \frac{x^{s-1}}{s(s+1)} H(s) \, ds.
\]

Notice here that \( |H(1\mp i \ell)| \leq A (\ln|\ell|)^{-1} \). The integrand on the right has abs value

\[
\leq O(1) \frac{1}{\ell^2} (\ln|\ell|)^{-9} , \quad (|\ell| \geq 3)
\]

**THEOREM** (almost the PNT)

\[
\lim_{x \to \infty} \frac{\psi(x)}{x^2} = \frac{1}{2}.
\]

**Proof**

Must look at

\[
\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{x^i t}{(1+i t)(2+i t)} H(1+i t) \, dt
\]

\[
= \frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{H(1+i t)}{e^{it(\ln x)}} \, dt \quad \text{as} \quad (\ln x \to +\infty)
\]
But, the Riemann–Lebesgue lemma tells us that

$$\lim_{t \to \infty} \int_{-\infty}^{\infty} H(t) e^{it\lambda} dt = 0$$

as $\lambda \to \pm \infty$ for any piecewise continuous absolutely integrable ($\int_{-\infty}^{\infty} |H| dt < \infty$) $H$.

Since $\frac{H(\lambda \pm i t)}{(\lambda \pm it)(2\pi \lambda)^{1/2}}$ is $C^\infty$ and $O(1) \frac{(\ln|\lambda|)^{1/2}}{t^2}$ for $t \geq 3$, we are done.
Proof of R-L lemma

Choose any $\epsilon > 0$.
Choose $G$ so big:

$$\int_{-G}^{G} |H(t)| dt < \frac{\epsilon}{3}.$$

Find a piecewise constant fn $s(t)$ on $[-G,G]$ so that

$$\int_{-G}^{G} |H(t) - s(t)| dt < \frac{\epsilon}{3}. $$

For $s(t)$, notice that on each "step"

$$\int_{t_j}^{t_{j+1}} e^{it\alpha} dt = j \frac{e^{i(j+\alpha)} - e^{i\alpha}}{i\alpha} $$

[abs value]$\leq 2|j|\frac{1}{\alpha}$.

Hence:

$$\int_{-G}^{G} s(t)e^{it\alpha} dt = O\left(\frac{1}{\alpha}\right).$$

Writing

$$\int_{-\infty}^{0} H(t)e^{it\alpha} dt = \int_{H| > G} H(t)e^{it\alpha} dt$$

$$+ \int_{G}^{\infty} [H(t) - s(t)]e^{it\alpha} dt$$
(continued)
\[ + \int_{-\infty}^{\infty} s(t) e^{i\omega t} dt \]

we clearly get

\[ \left| \int_{-\infty}^{\infty} \mathcal{H}(t) e^{i\omega t} dt \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \]

for all \( |\omega| > \lambda \). Done! \( \blacksquare \)

Note: if \( \mathcal{H} \) is \( \mathcal{C}^1 \), just do \( \langle G, \text{fixed} \rangle \)

\[ \int_{-\infty}^{\infty} \mathcal{H}(t) e^{i\omega t} dt = \int_{-\infty}^{\infty} \mathcal{H}(t) d\left( \frac{e^{i\omega t}}{i\omega} \right) \]

\[ = O\left( \frac{1}{\lambda} \right) - \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{i\omega} \mathcal{H}(t) dt \]

\[ = O\left( \frac{1}{\lambda} \right) \]

to make a short-circuit. \( \text{IE, R-L lemma is TRIVIAL if } \mathcal{H} \in \mathcal{C}^1. \)
We know \( \psi_1(x) \sim \frac{x^2}{2} \), where \( \psi_1(x) = \int_0^x \psi(v) \, dv \).

**Theorem (equiv to PNT)**

\( \psi(x) \sim x \) as \( x \to \infty \).

**PF**

Read Ingham p. 35 on your own. My method is closer to p. 64. Write \( \psi_1(x) = \frac{x^2}{2} + R(x) \).

Keep \( 0 < h < \frac{x}{2} \) and \( x \) large. Obviously,

\[
\frac{\psi_1(x+h) - \psi_1(x)}{h} = \frac{1}{h} \int_x^{x+h} \psi(v) \, dv \sim \psi(x)
\]

\[
\frac{\psi_1(x) - \psi_1(x-h)}{h} = \frac{1}{h} \int_{x-h}^{x} \psi(v) \, dv \sim \psi(x).
\]

This gives

\[
\psi(x) \sim \frac{(x+h)^2}{2} - \frac{x^2}{2} + R(x+h) - R(x)
\]

\[
\psi(x) \sim x + \frac{h}{2} + \frac{R(x+h) - R(x)}{h}
\]
\[
\psi(x) = \frac{x^2 - (x-h)^2}{h} + \frac{R(x) - R(x-h)}{h}
\]

Clearly:

\[
\psi(x) - x \leq \frac{h}{2} + \frac{|R(x+h) + R(x-h)|}{h}
\]

Suppose \( |R(y)| \leq E(y) \) with some explicit monotone increasing function \( E \). Get:

\[
-\frac{h}{2} \leq \frac{2E(x)}{h}
\]

\[
|\psi(x) - x| \leq \frac{h}{2} + \frac{2E(2x)}{h}
\]

But, given \( \varepsilon > 0 \), we know \( |R(y)| \leq \varepsilon y^2 \) for all \( y \leq \Delta \varepsilon \). Keep \( x \geq 2000 \Delta \varepsilon \) so that \( x-h \geq \frac{x}{a} \geq 1000 \Delta \varepsilon \).
We are free to take $E(y) \approx E y^2$ in the ranges which are relevant so long as we make doubly certain $0 < h \leq \frac{x}{2}$.

\[ |\psi(x) - x| \leq \frac{h}{2} + \frac{2E(2x)}{h} \]

\[ |\psi(x) - x| \leq \frac{h}{2} + \frac{8E x^2}{h} \]

\[
\begin{cases}
\text{wish to put } h = 4\sqrt{\varepsilon} x \\
\text{so just keep } \varepsilon < \frac{1}{100} \\
\text{and } x \text{ big}
\end{cases}
\]

\[ |\psi(x) - x| \leq 2\sqrt{\varepsilon} x + 2\sqrt{\varepsilon} x \]

\[ |\psi(x) - x| \leq 4\sqrt{\varepsilon} x \quad \text{if } x \geq x_\varepsilon \overset{\text{any}}{=} 2000 \varepsilon \]

Since $\varepsilon$ is arbitrary, we are done.

\[
\begin{align*}
\pi(x) &\sim \frac{x}{\ln x} \quad \text{PNT}
\end{align*}
\]

From the earlier lectures (e.g., Sec 2, p. 2) we then get $\pi(x) \sim \frac{x}{\ln x}$. 
I remarked that, in Riemann's formula for \( \Psi(x) \), one would like to move
Re(s) = \( \frac{1}{2} \) over past \( \sigma = \frac{1}{2} \).

\[
\Psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left[ \frac{\Gamma(s)}{s} \right] ds
\]

If we expect the poles of \( \frac{\Gamma(s)}{s} \) to lie along Re(s) = \( \frac{1}{2} \) (except for \( s = 1 \)), it is reasonable [perhaps] for

\[
\Psi(x) \sim \frac{x^2}{2} + O(x^{3/2}).
\]

\[E(x) = Cx^{3/2} \text{ on } p \circ \circ \text{ line } 4 \text{ would lead to}
\]

\[|\Psi(x) - x| \leq (\text{constant})x^{3/4}.
\]

Riemann was aware of this. By being less sloppy with R(x) on (1) + (2), perhaps

\[|\Psi(x) - x| \leq (\text{constant})x^{1/2}
\]

could be obtained. THIS IS ALL JUST VERY ROUGH, THOUGH.
I recalled that:

\[ \int_a^b [f(x)g'(x) + f'(x)g(x)] \, dx = [f(x)g(x)]_a^b \]

holds for \( f \in C'[a,b] \) and \( g \in C[a,b] \) but only piecewise \( C' \).

This could also be viewed as a Riemann–Stieltjes integration by parts, by declaring

\[ \alpha(x) = g(a) + \int_a^x g'(v) \, dv \]  

Of course: \( \alpha(x) \equiv g(x) \).

Another thing I remarked was how Riemann's fundamental formula for \( g'(x) \) was derivable by Fourier integrals.

"Fourier Integrals = Good."
Indeed,
\[- \frac{\mathcal{F}(s)}{\mathcal{F}(s)} = \int_{1}^{\infty} x^{-s} \psi(x) \, dx, \quad \text{Re}(s) > 1 \]
\[\{ \psi(u) = O(u) \} \]
\[\{ \text{Chebyshev} \} \]

\[= \left[ x^{-s} \psi(x) \right]_{1}^{\infty} - \int_{1}^{\infty} \psi(x) \, d(x^{-s-1}) \]

\[= 0 - 0 - \int_{1}^{\infty} \psi(x) (-s) x^{-s-1} \, dx \]

\[= -s \int_{1}^{\infty} \frac{\psi(x)}{x^{s+1}} \, dx \quad \Rightarrow \]

\[- \frac{1}{s} \frac{\mathcal{F}(s)}{\mathcal{F}(s)} = \int_{1}^{\infty} \frac{\psi(x)}{x^{s+1}} \, dx \quad \left\{ \begin{array}{l}
\text{Re}(s) > 1 \\
\text{Ingham, p. 18} \end{array} \right. \]

But, \( \psi(x) = \int_{1}^{x} \psi(v) \, dv \quad \text{for} \quad x \geq 1 \)

\[- \frac{1}{s} \frac{\mathcal{F}(s)}{\mathcal{F}(s)} = \int_{1}^{\infty} x^{-s-1} \, d \left[ \psi(x) \right] \]

\[= \left[ x^{-s-1} \psi(x) \right]_{1}^{\infty} - \int_{1}^{\infty} \psi(x) \, d(x^{-s-1}) \]

\[\{ \psi(x) = O(x^2) \} \quad \text{Chebyshev} \]
\[ = 0 - 0 + (s+1) \int_1^\infty \frac{\psi'(x)}{x^{s+2}} \, dx \]

\[ \therefore \frac{1}{s(s+1)} \frac{\pi(s)}{\pi(s)} = \int_1^\infty \psi'(x) x^{s-2} \, dx, \quad \text{Re}(s) > 1 \]

This is beginning to look like a Mellin transform.

Ingham p. 32

Recall Fourier inversion formula (heuristically).

\[ \tilde{F}(p) = \int_{-\infty}^{\infty} F(x) e^{-ixp} \, dx \quad \text{Fourier transform} \]

\[ \Rightarrow \quad F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{F}(p) e^{ixp} \, dp \]

This is very useful if

\[ QM(s) = \int_0^\infty f(x) x^{-s} \, dx, \quad \text{Re}(s) > 1 \]

WITH \( f(x) \equiv 0 \) near \( x = 0 \), \( \mid f(x) \mid \leq O(1) \)

Why? Because:

\[ QM(c + it) = \int_0^\infty f(x) x^{-c-it} \, dx \quad c > 1 \]

\[ = \int_{-\infty}^{\infty} f(e^v) e^{(c-it)v} \, dv \]
\[ M_c(t) = \int_{-\infty}^{\infty} \left[ f(e^v) e^{-(c-1)v} \right] e^{itv} dv \]

\[ \{ f(e^v) \equiv 0 \text{ for } v \text{ very negative} \} \]

\[ f(e^v) e^{-(c-1)v} = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_c(t) e^{itv} dt \]

\[ f(e^v) e^v = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_c(t) e^{cv} e^{itv} dt \]

\[ f(e^v) e^v = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_c(t) e^{(c-1)v} dt \]

\[ f(e^v) e^v = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_c(t) e^{sv} ds \]

\[ \{ \text{write } X = e^v \} \]

\[ f(x) x = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} M(s) x^s ds \]

\[ f(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} M(s) x^{s-1} ds \]

This is essentially what is called the Mellin inversion formula. \( s = 1-\xi \)
Look at \((\text{top})\):

\[- \frac{1}{s(s+1)} \frac{\mathcal{L}\{f(t)\}}{s} = \int_1^\infty \frac{y(x)}{x^2} x^{-s} \, dx, \quad \text{Re}(s) > 1\]

so we get, by \((\text{box})\),

\[\frac{y'(x)}{x^2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[ -\frac{1}{s(s+1)} \frac{\mathcal{L}\{f(t)\}}{s} \right] x^{-s} \, ds\]

which is equivalent to Riemann's

\[y'(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[ -\frac{1}{s(s+1)} \frac{\mathcal{L}\{f(t)\}}{s} \right] x^{s+1} \, ds\]

**Thus:** you really do not need complex variable (analytic function theory) to derive Riemann's fundamental formula.

Riemann knew this!
FACT (very curious) – Ingham 37

Suppose \( y(x) \sim x^0 \). Then we can see rather easily that \( y(1+it) \neq 0 \) for all \( t \in \mathbb{R} \). (Hence THIS is the essence of PNT!)

pf

Suppose e.g. that \( y(1+it_0) = 0 \). Zero of multiplicity \( m \geq 1 \).

\[ f(s) = (s-s_0)^m \left[ c_0 + c_1(s-s_0) + \cdots \right] \]

\( c_0 \neq 0 \)

\[ \frac{f'(s)}{f(s)} = \frac{m}{s-s_0} + \frac{\phi'(s)}{\phi(s)} \]

\[ \frac{f'(s)}{f(s)} = \frac{m}{s-s_0} + O(1) \text{ near } s = s_0 \]

Recall Lec 7 p. 17

Get:

\[ \frac{f'(s)}{f(s)} = \frac{m}{s - (1+it_0)} + O(1) \text{ near } 1+it_0. \]
But,
\[ -\frac{1}{s} \frac{s'(s)}{s(s)} = \int_1^\infty \frac{\psi(x)}{x^{s+1}} \, dx \quad \text{Re}(s) > 1 \] (6)

\[ \frac{1}{s-1} = \int_1^\infty \frac{x}{x^{s+1}} \, dx \quad \text{Re}(s) > 1 \]

\[ -\frac{1}{s} \frac{s'(s)}{s(s)} - \frac{1}{s-1} = \int_1^\infty \frac{\psi(x) - x}{x^{s+1}} \, dx \]

Assume \( \varepsilon > 0 \) is small. Get

\[ |\psi(x) - x| < \varepsilon x, \quad x \equiv \mathcal{G}_\varepsilon \]

Hence:

\[ -\frac{1}{s} \frac{s'(s)}{s(s)} - \frac{1}{s-1} = \int_1^{\mathcal{G}_\varepsilon} \frac{\psi(x) - x}{x^{s+1}} \, dx \]

\[ + \int_{\mathcal{G}_\varepsilon}^\infty \frac{\psi(x) - x}{x^{s+1}} \, dx \]

First integral on RHS is analytic for all \( s \in \mathbb{C} \) (since \( \mathcal{G}_\varepsilon = \text{finite} \))
Let \( s_0 = 1 + i \varepsilon \) and keep \( \text{Re}(s) > 1 \) \( 1 \leq s \leq s_0 \).

We have:

\[
- \frac{1}{s} \left[ \frac{m}{s - s_0} + O(1) \right] + O(1) = 0(1) + \int_{\mathcal{E}} \frac{\psi(x) - x}{x^{s+1}} \, dx
\]

\[
- \frac{1}{s_0} \frac{m}{s - s_0} + O(1) = 0(1) + \int_{\mathcal{E}} \frac{\psi(x) - x}{x^{s+1}} \, dx
\]

Take \( s = \sigma + i \varepsilon \) and let \( \varepsilon \to 0 + \). Get:

\[
\frac{1}{s_0} \frac{m}{\sigma - 1} + O(1) = \int_{\mathcal{E}} \frac{\varepsilon x}{x^{\sigma+1}} \, dx
\]

\[
\frac{1}{s_0} \frac{m}{\sigma - 1} + O(1) = \varepsilon \int_{\mathcal{E}} x^{-\sigma} \, dx
\]

\[
\frac{1}{s_0} \frac{m}{\sigma - 1} + O(1) = \varepsilon \left[ \frac{x^{1-\sigma}}{1-\sigma} \right]_{\mathcal{E}}^{\infty}
\]

\[
\frac{1}{s_0} \frac{m}{\sigma - 1} + O(1) \leq \varepsilon \frac{\mathcal{E}}{\sigma - 1} \quad (\sigma > 1)
\]

\[
\Rightarrow \quad \frac{m}{s_0} \leq \varepsilon \mathcal{E}^{-\sigma} \quad \Rightarrow \quad \frac{1/m}{s_0} \leq \varepsilon^{-\sigma}
\]
Hence \( \frac{1}{\log 2} \leq \varepsilon \). But \( \varepsilon \) was arbitrary.

Contradiction. \[ \square \]

I remarked in the lecture that I would now pause* for about 2 lectures to fill in some background stuff on Bernoulli numbers, Euler-Maclaurin summation, and special values of \( \zeta(s) \).

It's very pretty to work with this very explicit stuff! * *

* possibly a mistake
Thus (Euler–Maclaurin Sum Formula, version 1)

\[
F \in C^1[0, N] \Rightarrow \\
\frac{1}{2} f(0) + f(1) + \ldots + f(N-1) + \frac{1}{2} f(N) \\
= \int_0^N f(x) \, dx + \int_0^N f'(x) \left( x - \|x\| - \frac{1}{2} \right) \, dx.
\]

**PF**

Let \( \beta(x) \approx x - \|x\| - \frac{1}{2} \) for a few moments.

Note that \( \beta(x) \) is the difference of 2 right continuous increasing func. By def,

\[
\|x\| = x - \frac{1}{2} - \beta(x).
\]

\[
f(1) + \ldots + f(N) = \int_0^N f(x) \, dx \|x\| \quad \text{this is correct}
\]

\[
= \int_0^N f(x) \, dx \left( x - \frac{1}{2} - \beta(x) \right)
\]

\[
= \int_0^N f(x) \, dx - \int_0^N f(x) \, d\beta(x)
\]

\[
= \int_0^N f(x) \, dx - \left[ + \beta \right]_0^N + \int_0^N \beta(x) f'(x) \, dx
\]

\[
\text{\underline{\text{by R–S parts}}} \frac{3}{2}
\]

\[
= \int_0^N f(x) \, dx + \int_0^N \beta f'(x) \, dx
\]

\[
= \int_0^N f(x) \, dx + \frac{1}{2} f(N) - \frac{1}{2} f(0) + \int_0^N \beta f'(x) \, dx
\]
\[ \frac{1}{2} f(0) + f(1) + \ldots + f(N-1) + \frac{1}{2} f(N) = \int_0^N f(x) \, dx + \int_0^N \beta f(x) \, dx. \]

We intend to use \[ \beta(x) \]

\[ x - \left\lfloor x \right\rfloor - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin \pi n x}{n \pi}, \quad x \notin \mathbb{Z} \]

repeatedly. We need a few facts:

Recall that we got this equality via \[ -\log(1 - z) \] a nice way.

Thus

The partial sums \[ \sum_{n=1}^{N} \frac{\sin \pi n x}{n \pi} \]

are uniformly bounded for all \( x \in \mathbb{R} \).

\[ \text{Pf} \]

Suffices to treat \( \sum_{n=1}^{N} \frac{\sin(nt)}{n} \).

Use periodicity \( 2\pi \) and oddness \( \text{wrt } t \).

Hence, \( \log 1 \leq t \leq \pi \).
Suffices to treat

\[ \delta_N(t) = \frac{t}{2} + \sum_{N+1}^{N} \frac{\sin nt}{n} \]

But

\[ \delta_N = \frac{1}{2} + \sum_{N+1}^{N} \frac{\cos nt}{n} = \frac{1}{2} \left( \sum_{N}^{N} e^{int} \right) \]

\[ = \frac{1}{2} \frac{e^{-itN} - e^{it(N+1)}}{1 - e^{it}} \]

\[ = \frac{1}{2} \frac{e^{-it(N+\frac{1}{2})} - e^{it(N+\frac{1}{2})}}{e^{-it/2} - e^{it/2}} = \frac{1}{2} \frac{\sin \left[ (N+\frac{1}{2})t \right]}{\sin (t/2)} \]

So,

\[ \delta_N(t) = \int_{0}^{t} \frac{\sin \left[ (N+\frac{1}{2})\nu \right]}{2\sin(\nu/2)} d\nu \]

Write:

\[ \frac{1}{2\sin(\nu/2)} = \frac{1}{\nu} + h(\nu), \quad 0 < \nu \leq \pi \]

Obviously \( h(\nu) \) is \( C^\infty \). The function \( h(\nu) \) is also analytic near \( \nu = 0 \). Indeed,

\[ \frac{1}{2\sin(\nu/2)} \sqrt{\nu} = \frac{1}{2\left[ \frac{\nu}{2} - \frac{1}{3!} \left( \frac{\nu}{2} \right)^3 + \ldots \right]} \quad \sqrt{\nu} \]
\[ \frac{1}{\sqrt{1 + b_2 v^2 + b_4 v^4 + \cdots}} - \frac{1}{v} \]

\[ = \frac{1}{\sqrt{1 + A_2 v^2 + A_4 v^4 + \cdots}} - \frac{1}{v} \]

\[ = A_2 v + A_4 v^3 + \cdots \quad \text{near } v = 0 \text{ in } C_0. \]

So,

\[ \sigma_N(t) = \int_0^t \left[ \frac{1}{\sqrt{1 + h(v)}} \right] \sin \left( (N+\frac{1}{2})v \right) dv \]

\[ = \int_0^t \frac{\sin \left( (N+\frac{1}{2})v \right)}{v} dv + \int_0^t h(v) \sin \left( (N+\frac{1}{2})v \right) dv. \]

But,

\[ \left| \int_0^t h(v) \sin \left( (N+\frac{1}{2})v \right) dv \right| \leq \int_0^t |h(v)| dv < \infty \]

and

\[ \int_0^t \frac{\sin \left( (N+\frac{1}{2})v \right)}{v} dv = \int_0^{(N+\frac{1}{2})T} \frac{\sin \theta}{\theta} d\theta. \]

By baby calculus, however,

\[ \left| \int_0^R \frac{\sin \theta}{\theta} d\theta \right| \leq \text{constant} \]
for all \( R \geq 0 \). Just look at the graph of \( \frac{\sin \theta}{\theta} \) and consider signed area.

Or use:

\[
\int_1^R \frac{\sin \theta}{\theta} \, d\theta = \int_1^R \frac{d(-\cos \theta)}{\theta}
\]

\[
= -\left[ \frac{\cos \theta}{\theta} \right]_1^R + \int_1^R \cos \theta \, d\left( \frac{1}{\theta} \right)
\]

\[
= O(1) - \int_1^R \frac{\cos \theta}{\theta^2} \, d\theta
\]

\[
= O(1) + O(1)
\]

One knows, in fact, that the improper integral \( \int_0^\infty \frac{\sin \theta}{\theta} \, d\theta \) exists!

IN ANY EVENT, we clearly get (by (17))

\[
|\delta_N(t)| \leq \text{some constant}
\]

For all \( 0 < t \leq \pi \).
"Miracle #1" (with more real analysis)

\[ \frac{\pi}{2} \approx \int_0^\infty \frac{\sin \theta}{\theta} \, d\theta \]

pf ← The standard proof in any Fourier series class.

On (16), we saw

\[ \frac{1}{2} + \sum_{n=1}^N \cos nt = \frac{\sin \left( \left( N+\frac{1}{2} \right) t \right)}{2 \sin \left( \frac{t}{2} \right)} , \quad 0 < t \leq \pi. \]

For \( t = 0 \), use a limit. Integrate over \( [0, \pi] \). Get:

\[ \frac{\pi}{2} = \int_0^\pi \frac{\sin \left( \left( N+\frac{1}{2} \right) v \right)}{2 \sin \left( \frac{v}{2} \right)} \, dv \]

Use (16) bottom - (17) with \( h(v) \). Get:

\[ \frac{\pi}{2} = \int_0^\pi \left( \frac{1}{v} + h(v) \right) \sin \left( \left( N+\frac{1}{2} \right) v \right) \, dv \]

\[ \frac{\pi}{2} = \int_0^\pi \frac{\sin \left( \left( N+\frac{1}{2} \right) v \right)}{v} \, dv + \int_0^\pi h(v) \sin \left( \left( N+\frac{1}{2} \right) v \right) \, dv \]

\[ \downarrow \]

\( c^\infty \) and analytic near \( v = 0 \).
\[ \frac{\pi}{2} = \int_0^{\pi (N + \frac{1}{2})} \frac{\sin B}{B} dB + \int_0^{\pi} h(v) \sin [(N + \frac{1}{2})v] dv \]

Recall R-L lemma for

\[ \int_0^{\pi} h(v) e^{i\alpha v} dv = \int_0^{\pi} h(v) d\left[ \frac{e^{i\alpha v}}{i\alpha} \right] \]

\[ = h(v) \frac{e^{i\alpha v}}{i\alpha} \Bigg|_0^{\pi} \]

\[ - \int_0^{\pi} \frac{e^{i\alpha v}}{i\alpha} h'(v) dv \]

\[ \approx 0(\frac{1}{\alpha}) + 0(\frac{1}{\alpha}) \]

as in Lec 7 p. 23.

Let \( N \to \infty \) and use R-L lemma.

Get

\[ \frac{\pi}{2} = \int_0^{\infty} \frac{\sin B}{B} dB + 0 \quad \text{OK!} \]
Miracle #2

by revisiting \([16-18]\) (with more real analysis)

I claim that \(p. \, 16\) and \(17\) (middle)

immediately imply

\[
\frac{\pi}{2} = \int_0^\pi \frac{\sin \theta}{\theta} \, d\theta
\]

AND

\[
\sum_{n=1}^{\infty} \frac{\sin(2\pi n \theta)}{n} = \frac{1}{2} - \theta + \left\lfloor \theta \right\rfloor , \quad \theta \in \mathbb{Z}.
\]

"No need for \(-\log(1-\varepsilon)\)"

\[\text{Pf}\]

Use \(p. \, 16\) for \(0 < \varepsilon \leq 2\pi - \delta\).

Notice that \(h(\varepsilon)\) is \(C^\omega\) on \((0, 2\pi - \delta]\)

and analytic near \(\varepsilon = 0\).

We still have

\[
o_N(t) = \int_0^t \frac{\sin[(N+\frac{1}{2})\varepsilon]}{2\sin(\varepsilon/2)} \, d\varepsilon \quad \leftarrow 16
\]

\[
= \int_0^t \frac{\sin[(N+\frac{1}{2})\varepsilon]}{\sqrt{\varepsilon}} \, d\varepsilon
\]

\[
+ \int_0^t h(\varepsilon) \sin[(N+\frac{1}{2})\varepsilon] \, d\varepsilon
\]

\(\text{à la } 17\) (middle).
Thus, for $0 < t \leq 2\pi - \delta$,

$$\frac{t}{2} + \sum_{n=1}^{N} \frac{\sin(nt)}{n} = \int_{0}^{(N+\frac{1}{2})t} \frac{\sin \theta}{\theta} \, d\theta$$

$$+ \int_{0}^{\pi} h(\nu) \sin [(N+\frac{1}{2})\nu] \, d\nu.$$

Freeze $t$ temporarily and let $N \to \infty$.

Get:

$$\frac{t}{2} + \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} = A + 0$$

\[\text{\(\uparrow\) cf. (20) middle with minor change}\]

where $A = \int_{0}^{\infty} \frac{\sin \theta}{\theta} \, d\theta$.

Thus:

$$\sum_{n=1}^{\infty} \frac{\sin(nt)}{n} = A - \frac{t}{2}$$

all $0 < t < 2\pi$.

Plug in $t = \pi$; this forces $A$ to be $\frac{\pi}{2}$.

Let $t = 2\pi q$, $0 < q < 1$, to get

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi nq)}{n} = \frac{\pi}{2} - \pi q$$

all $0 < q < 1$. 

\[\text{(22)}\]
Hence

$$\sum_{n=1}^{\infty} \frac{\sin((2\pi n)q)}{\pi n} = \frac{1}{2} - q \quad \text{for} \quad 0 < q < 1$$

and the rest is trivial by periodicity.

NOTE:

On line 4, if we keep $t$ variable but inside $[\delta, 2\pi - \delta]$, this limit procedure is easily seen to be uniform wrt $t$ as $N \to \infty$.

<table>
<thead>
<tr>
<th>$t \leq \delta$ used in</th>
<th>$\int_{0}^{\frac{(N+\frac{1}{2})t}{\delta}} \frac{\sin b}{b} , db$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t \geq 2\pi - \delta$ used in</td>
<td>$\int_{0}^{t} h(v) \sin [(N+\frac{1}{2})v] , dv$</td>
</tr>
</tbody>
</table>

Review 20 (middle) with obvious changes.
\[ a_n(t) = \frac{t}{a} + \sum_{n=1}^{N} \frac{\sin nt}{n} \]

look at this on \( 0 < t \leq 2\pi - \delta \)

\[ a_n(t) = \int_{0}^{t} \frac{\sin[(N+\frac{t}{2})v]}{\sin(\frac{v}{2})} dv \]

\( \delta_n(t) \) \( \text{(Lec 8 p.16 etc)} \)

but \( \frac{1}{\sin \frac{v}{2}} = \frac{1}{\sqrt{\gamma(v)}} \) \( h(v) \) \( \text{on} \ [0, 2\pi - \delta] \)

and analytic near \( v = 0 \)

\[ \Rightarrow a_n(t) = \int_{0}^{t} \frac{\sin \left( \frac{x}{\gamma(v)} \right)}{\gamma(v)} dv + \int_{0}^{t} h(v) \sin \left( \frac{(N+\frac{t}{2})v}{\gamma(v)} \right) dv \]

Freeze \( \gamma \) \( t \to \infty \). Use \( \text{Im} \left[ \int_{0}^{t} h(v) e^{i\gamma(v)} dv \right] \)

\( = O\left( \frac{1}{A} \right) \). Get \[ a_n(t) \]

\[ \frac{t}{a} + \sum_{n=1}^{\infty} \frac{\sin nt}{n} = A + 0 \]

\( A \equiv \int_{0}^{\infty} \frac{\sin \gamma}{\gamma} dv \)

\[ \sum_{n=1}^{\infty} \frac{\sin nt}{n} = A - \frac{t}{a} \] all \( 0 < t < 2\pi \).

Plug in \( t = \pi \). Get \( A = \frac{\pi}{2} \).

Write \( t = 2\pi - \theta \), \( 0 < \theta < 1 \), get

\( \sum_{n=1}^{\infty} \frac{\sin(2\pi n\theta)}{n} = \pi - \pi \theta \Rightarrow \sum_{n=1}^{\infty} \frac{\sin(2\pi n\theta)}{-\pi n} = \pi - \frac{1}{2} \).
By the way, regarding this, note that this method of proof immediately yields uniform convergence for $\delta \leq t \leq 2\pi - \delta$.

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n, \quad 1/2 < 2\pi, \quad t \in \mathbb{C}$$

**Def of Bernoulli numbers**

<table>
<thead>
<tr>
<th>$k$</th>
<th>$B_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-1/2</td>
</tr>
<tr>
<td>2</td>
<td>1/6</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

**Easy Lemma**

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t}{e^t} \coth\left(\frac{t}{2}\right) = \text{even}$$

hence $B_2 = B_5 = B_7 = \ldots = 0$.

Put $t = 21\pi$. Get $0$.

$$i\pi \coth\left(\frac{i\pi}{2}\right) = 1 + \sum_{k=2}^{\infty} \frac{B_k}{k!} \left(\frac{2\pi i}{2}\right)^k$$

$k$ even

Thus

$$\pi \coth(\pi z) = 1 + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (-1)^n \frac{2^n}{(2n)}$$

and

$$\pi \coth(\pi w) = \frac{1}{w} \left[ \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} (-1)^n \frac{2^n}{(2n)} w^n \right], \quad |w| < 1.$$
The function \( \pi \cotn(x \pi) \) has familiar properties in complex analysis:

(A) periodic \( z \to z + 1 \)
(B) simple poles at \( z = n \), \( n \in \mathbb{Z} \)
(c) residue always 1
(D) \( \cotn \pi(x + iy) = -i + O(e^{-2\pi y}) \) \( y \to +\infty \)
(E) \( \cotn \pi(x + iy) = i + O(e^{-2\pi y}) \) \( y \to -\infty \)

Standard Cauchy residue theorem set-up with

\[
\frac{1}{2\pi i} \oint_{C_N} f(z) \cotn \pi z \, dz
\]

\( C_N = \) square with vertices \( (\pm (N+\frac{1}{2}), \pm (N+\frac{1}{2})) \).

\( f(z) = z^{-2m} \), \( m \geq 1 \). Use Thm on \( \Box \).

Take \( N \to \infty \). Get:

\[
I(2m) = -2^{2m-1} \pi^m \frac{(-1)^{m+1} B_{2m}}{(2m)!} \quad \text{(Euler)}
\]

In particular, note that \( B_{2m} = (-1)^{m+1} |B_{2m}| \)
since \( I(2m) > 0 \).
Another interesting \( f(z) \) is \( f = \frac{1}{\xi - z} \).

Think \( \Theta = \mathbb{C} - \mathbb{Z} \) and \( \xi \in \{ \Theta \text{ compacta} \} \).

By \( \text{CRT} \),

\[
-\pi \text{cm} \pi \xi + \sum_{1 \leq n \leq N} \frac{1}{\xi - n} = \frac{1}{2\pi i} \int_{\gamma} \frac{\pi \text{cm} \pi \xi}{\xi - z} \, dz.
\]

Write:

\[
f(z) = \frac{1}{z} - \left[ \frac{1}{\xi - z} + \frac{1}{\xi} \right] = \frac{1}{z} + \frac{\xi}{(z - \xi)z}
\]

\[
= -\frac{1}{z} + \frac{r(z)}{z}, \quad r(z) = O\left(\frac{1}{z^2}\right).
\]

Notice that:

\[
\int_{\gamma} h(z) \, dz = 0 \quad \text{for any even \& continuous} \quad h \quad \text{on} \quad \gamma.
\]

\( \gamma \) uses symmetry of \( \gamma_N \) and \( w = -\bar{z} = ze^{i\pi} \).

So, we still get:

\[
\int_{\gamma} \frac{\pi \text{cm} \pi \xi}{\xi - z} \, dz = O\left(\frac{1}{N}\right).
\]

So,

\[
\lim_{N \to \infty} \sum_{1 \leq n \leq N} \frac{1}{\xi - n} = \pi \text{cm} \pi \xi, \quad \xi \in \Theta.
\]
By reviewing the proof, we see the limit is uniform for $z \in \mathcal{D}$ compact.

**Tautology:**

$$\frac{1}{z} + \sum_{n \neq 0}^{N} \left( \frac{1}{z-n} + \frac{1}{n} \right) = \sum_{n}^{N} \frac{1}{z-n}.$$

**THM** $z \in \mathcal{D}$

(i) $\pi \text{ch} \pi z = \lim_{N \to \infty} \sum_{n}^{N} \frac{1}{z-n}$

(ii) $\pi \text{ch} \pi z = \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z-n} + \frac{1}{n} \right)$

$\pi \text{ch} \pi z$ uniform converges on $\mathcal{D}$ compact

---

Use thm on (2), get:

$$\pi \text{wch} \pi w = \sum_{m=0}^{\infty} \frac{B_{2m}}{(2m)!} (-1)^m w^{2m}.$$ $|w| < 1.$

But, now use the THM above:

$$\pi \text{ch} \pi z = \frac{1}{z} + \sum_{m \neq 0} \left( \frac{1}{z+m} - \frac{1}{m} \right).$$ Keep $|z| > 1.$

$$\frac{1}{m + z} = \frac{1}{m(1 + \frac{z}{m})} = \frac{1}{m} \left[ 1 - \frac{z}{m} + \frac{z^2}{m^2} + \ldots \right].$$

$$\pi \text{ch} \pi z = \frac{1}{z} + \sum_{m \neq 0} \left[ -\frac{z}{m} + \frac{z^2}{m^3} - \frac{z^3}{m^4} + \ldots \right].$$
\[
\pi \coth \pi z = \frac{1}{z} - 2 \sum_{n=1}^{\infty} \frac{\pi^2}{n^2} \frac{1}{z^n - 1^n}
\]

\[
\pi \coth \pi z = 1 - 2 \sum_{n=1}^{\infty} \frac{\pi^2}{n^2} \frac{1}{z^n - 1^n}, \quad |z| < 1
\]

\[
\Psi
\sim 2 \sum_{n=1}^{\infty} \frac{\pi^2}{n^2} \frac{1}{z^n - 1^n} = \frac{B_{2n}}{(2n)!} z^n (-1)^n
\]

\[
\Psi
\sim 2 \sum_{n=1}^{\infty} \frac{\pi^2}{n^2} \frac{1}{z^n - 1^n} = \frac{B_{2n}}{(2n)!} z^n (-1)^n 2^{2n - 1} \pi^{2n}
\]

\[
= 2 \text{nd proof of Euler's formula } \text{ NICE!}
\]

**Thm**

\[
\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}
\]

uniform convergence on compacta

**PF**

Differentiate THM on \(5\) by Weierstrass conv. thm.

Another nice trick. Take \(z_0\) and \(z\) in upper half-plane. Connect via \(y\).

\[
\gamma \to z
\]

\[
\gamma = \text{compact subset}
\]
\[
\frac{d}{dz} \log(\sin \pi z) = \pi \cot \pi z
\]

\[
\downarrow
\]

(with some branches)

\[
\log \left[ \sin \pi z \right] - \log \left[ \sin \pi z_0 \right] = \int \pi \cot \pi w \, dw
\]

\[
\frac{\sin \pi z}{\sin \pi z_0} = \exp \left[ \int \pi \cot \pi w \, dw \right]
\]

\[
\text{substitute THE E (10)}
\]

\[
= \exp \left[ o(1) + \sum_{n=1}^{N} \left( \log(z-n) - \log(z_0-n) \right) \right]
\]

\[
\approx 1 + o(1)
\]

\[
\frac{z-n}{z_0-n}
\]

\[
= \frac{z}{z_0} \prod_{n=1}^{N} \frac{1 - \frac{z}{k_n^2}}{1 - \frac{z_0}{k_n^2}}
\]

\[
\approx \left\{ \begin{array}{l}
\xi \in H, z_0 \in H \text{ think nonzero} \\
\text{infinite products etc etc}
\end{array} \right.
\]
\[
\frac{\sin n \pi z}{\pi z} = \left( \frac{\sin n \pi z_0}{n \pi z_0} \right) \frac{1}{\prod_{k=1}^{\infty} \left( 1 - \frac{z_0^2}{k^2} \right)}
\]

\[
\frac{\sin n \pi z}{\pi z} = e^{\sum_{k=1}^{\infty} \left( 1 - \frac{z^2}{k^2} \right)}
\]

First for \( \text{Im}(z) > 0 \) then for all \( z \in \mathbb{C} \) by analyticity of both sides.

Let \( z \to 0 \), get \( e = 1 \).

**THEM**
\[
\frac{\sin n \pi z}{\pi z} = \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{k^2} \right), \quad z \in \mathbb{C}.
\]

Known to Euler.
Baby Fact

Let $\beta(x) = x - \|x\| - \frac{1}{2}$ for $x \in \mathbb{R}$.

Let $F$ be Riemann integrable on $[a, b]$.

Then:

$$\int_a^b F(x) \beta(x) \, dx = \sum_{n=1}^\infty \int_a^b F(x) \frac{\sin 2\pi nx}{\pi n} \, dx.$$

Proof

$S_N(x) = \sum_{n=1}^N \frac{\sin 2\pi nx}{\pi n}.$

Know $S_N(x)$ is uniform for all $x \in \mathbb{N}$. Also,

$S_N(x) \to \beta(x)$ on compact subsets of $\mathbb{R}$ away from $\mathbb{Z}$.

Think $a < 1 < b$ (say) and

$$\left| \int_a^b F(x) \beta - S_N \, dx \right| = \left| \int_a^1 F(x) (\beta - S_N) \, dx \right| + \sum_{n=1}^{\lfloor \frac{1}{\delta} \rfloor} \int_{1-\delta}^{1+\delta} F(1) (\beta - S_N) \, dx$$

$$+ \int_{1-\delta}^b F(x) (\beta - S_N) \, dx$$

$$+ \int_{1+\delta}^b F(x) (\beta - S_N) \, dx$$

$\Rightarrow$ done!
Take e.g. \( f \in C_{[0,1]}^{2}\mathbb{R}+1 \). Know

\[
\frac{1}{2} f(0) + \frac{1}{a} f(1) = \int_0^1 f \, dx + \int_0^1 f' \beta(x) \, dx
\]

by E-M version I

\[
= \int_0^1 f \, dx + \int_0^1 f' \left( -\sum_{i=1}^{n} \frac{\sin(2\pi i x)}{\pi i} \right) \, dx
\]

\[
= \int_0^1 f \, dx + \sum_{i=1}^{n} \left( \frac{1}{\pi i} \right) \int_0^1 f' \sin(2\pi i x) \, dx
\]

by Baby Fact.

Now just look at EACH integral

\[
\int_0^1 f' \sin(2\pi i x) \, dx
\]

and repeatedly integrate by parts.

Step 1

\[
\int_0^1 f' \left( \frac{\cos(2\pi i x)}{-2\pi i} \right) \, dx
\]

\[
= \frac{f'}{-2\pi i} \left( \frac{\cos(2\pi i x)}{-2\pi i} \right) \bigg|_0^1 + \int_0^1 \frac{\cos(2\pi i x)}{2\pi i} f'' \, dx
\]

\[
= \frac{f'(1) - f'(0)}{-2\pi i} + \frac{1}{2\pi i} \int_0^1 \cos(2\pi i x) \cdot f'' \, dx
\]
Step 2

\[
\frac{f'(1) - f'(0)}{-2\pi n} + \frac{1}{2\pi n} \int_0^1 f''(x) \left( \frac{\sin(2\pi nx)}{2\pi n} \right) \, dx
\]

\[
= \frac{f'(1) - f'(0)}{-2\pi n} + 0 - 0 - \frac{1}{(2\pi n)^2} \int_0^1 \sin(2\pi nx) \cdot f'''(x) \, dx
\]

\[
= \frac{f'(1) - f'(0)}{-2\pi n} - \frac{1}{(2\pi n)^2} \int_0^1 f'''(x) \sin(2\pi nx) \, dx
\]

Clearly a recursion has begun.

\[
\int_0^1 f'(x) \sin(2\pi nx) \, dx = \sum_{k=1}^R \frac{(-1)^k}{(2\pi n)^{2k-1}} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]
\]

\[
+ \frac{(-1)^R}{(2\pi n)^{2R}} \int_0^1 f^{(2R+1)}(x) \sin(2\pi nx) \, dx
\]

So, on line 5

\[
-\frac{2}{2\pi n} \int_0^1 f'(x) \sin(2\pi nx) \, dx
\]

\[
= \sum_{k=1}^R \frac{2(-1)^{k+1}}{(2\pi n)^{2k}} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]
\]

\[
+ \frac{2(-1)^{R+1}}{(2\pi n)^{2R+1}} \int_0^1 f^{(2R+1)}(x) \sin(2\pi nx) \, dx
\]
Theorem (E-M version #2, prelim form)

Let $f \in C^{2R+1}[0,N]$. Then:

$$
\sum_{n=0}^{N} f(n) = \frac{1}{2} f(0) + \frac{1}{2} f(N) + \int_{0}^{N} f(x) \, dx
$$

$$
+ \sum_{k=1}^{R} 2\frac{(-1)^{k+1}}{(2\pi)^{2k}} f(2k) \left[ f^{(2k)}(N) - f^{(2k)}(0) \right]
$$

$$
+ 2\frac{(-1)^{R+1}}{2\pi} \int_{0}^{N} f^{(2R+1)}(x) \left[ \sum_{l=1}^{\infty} \frac{\sin 2\pi nx}{(2\pi l)^{2R+1}} \right] \, dx
$$

(in the above)

and we actually have

$$
\frac{2\frac{(-1)^{k+1}}{(2\pi)^{2k}}}{(2\pi)^{2k}} f(2k) \equiv \frac{B_{2k}}{(2k)!}.
$$

Proof

Write

$$
\frac{1}{2} f(0) + f(1) + \cdots + f(N-1) + \frac{1}{2} f(N) = \sum_{j=0}^{N-1} \frac{1}{2} \left[ f(j) + f(j+1) \right]
$$

then use (top) + (bottom). The final observation about $B_{2k}$ follows from Euler's formula for $I(2k)$.
Lec 10 begins here.

Think about \( \int_{C_N} f(z) \frac{e^{iz}}{1 - e^{2\pi iz}} \, dz \). Entertain yourself.

\[ f(2k) \to 1 \text{ as } k \to \infty \quad \text{so:} \quad I \sim \frac{(2\pi)^{2k}}{2} \frac{B_{2k}}{(2k)!} \]

\[ I \equiv \frac{(2\pi)^{2k}}{(2k)!} B_{2k} \]

\[ B_2 = \frac{1}{6} \quad B_{10} = \frac{5}{66} \quad \text{there are long tables!} \]
\[ B_4 \approx -\frac{1}{30} \quad B_{12} \approx -\frac{691}{7930} \]
\[ B_6 = \frac{1}{42} \quad B_{14} \approx \frac{7}{6} \]
\[ B_8 = \frac{1}{30} \quad B_{16} \approx -\frac{3617}{510} \]

\( (B_1 = -\frac{1}{2}) \) \( \odot \)

Bernoulli polynomials on \( 0 < x < 1 \)?

(underline) Bernoulli polynomials (underline) on \( 0 < x < 1 \)?

\[ B_n(x) = \text{degree } n, \text{ leading coefficient } 1 \]
\[ B_n(0) = B_n \]

\[ \frac{e^{tx} - 1}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \quad |t| < 2\pi \]

\[ B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6} \]

Let \( \tilde{B}_n(x) \) be the period 1 periodic extension of \( B_n(x) \) to \( \mathbb{R} - \mathbb{Z} \).
Not hard to check:

\[ \frac{\tilde{B}_{2R}(x)}{(2R)!} = 2^{(-1)} R^{R+1} \sum_{j=0}^N \frac{\cos(j\pi nx)}{(2\pi j)^{2R}} \]

\[ \frac{\tilde{B}_{2R+1}(x)}{(2R+1)!} = 2^{(-1)} R^{R+1} \sum_{j=0}^N \frac{\sin(j\pi nx)}{(2\pi j)^{2R+1}} \]

\[ \Leftarrow \text{compare line 60} \]

Note that \( B_0(x) \) \( a la R = 0 \) certainly fits!

**THEOREM (E-M version 2)**

\[ f \in C^{2R+1}(0, N], f \text{ complex OK. } \]

\[ \sum_{j=0}^N f(j^2) = \frac{1}{2} f(0) + \frac{1}{2} f(N) + \int_0^N f(x) \, dx \]

\[ + \sum_{k=1}^R \frac{\tilde{B}_{2k}(x)}{(2k)!} \left[ f((k-1)(N)) - f((2k-1)(0)) \right] \]

\[ + \text{Remainder}_R \], \text{ where} \]

\[ \text{Rem}_R = \int_0^N f((2R+1)(x)) \frac{\tilde{B}_{2R+1}(x)}{(2R+1)!} \, dx \]

\[ - \int_0^N f((2R)(x)) \frac{\tilde{B}_{2R}(x)}{(2R)!} \, dx \] via easy integ by parts

If

Use (12) THM the formulae at top of this page.
Cor 1

\[
|\text{Rem}_R| = \frac{|B_{2R}|}{(2R)!} \int_0^N |f(2R)(x)| \, dx
\]

Proof:
Plug into 2nd form of Rem R on (14).

\[
|\text{Rem}_R| = \int_0^N |f(2R)(x)| \, dx \geq 2 \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^{2R}} \, dx
\]

\[
= \frac{2}{(2\pi)^{2R}} \int_0^N |f(2R)(x)| \, dx
\]

\[
= \frac{|B_{2R}|}{(2R)!} \int_0^N |f(2R)(x)| \, dx \quad \text{by (1) box.}
\]

Cor 2 (very useful in numerical work)

\[
\text{Rem}_R = 2(-1)^{R+2} \int_0^N f(2R+2)(x) \sum_{n=1}^{\infty} \frac{1 - \cos 2\pi nx}{(2\pi n)^{2R+2}} \, dx
\]

Proof:

\[
\text{Rem}_R = \int_0^N f(2R+1)(x) \left[ 2(-1)^{R+1} \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{(2\pi n)^{2R+1}} \right] \, dx
\]

\[
= \sum_n \frac{2(-1)^{R+1}}{(2\pi n)^{2R+1}} \int_0^N f(2R+1)(x) \left( \frac{1 - \cos 2\pi nx}{2\pi n} \right) \, dx
\]

\[
= \sum_n \frac{2(-1)^{R+1}}{(2\pi n)^{2R+1}} \left[ 0 - 0 - \int_0^N \frac{1 - \cos 2\pi nx}{2\pi n} f(2R+1)(x) \, dx \right]
\]
\[ \sum \frac{2(-1)^{R+2}}{(2\pi n)^{2R+2}} \int_0^N (1 - \cos 2\pi nx) f^{(2R+2)}(x) \, dx \]

\[ 2(-1)^{R+2} \int_0^N \sum_{n=1}^{\infty} \frac{1 - \cos 2\pi nx}{(2\pi n)^{2R+2}} f^{(2R+2)}(x) \, dx \]

**Cor 3** (very commonly used)

If complex, we always have:

\[ \left| \text{Rem}_R \right| \leq 2 \frac{|B_{2R+2}|}{(2R+2)!} \int_0^N f^{(2R+2)}(x) \, dx \]

**PF**

Use Cor 2.

\[ \left| \text{Rem}_R \right| \leq 2 \int_0^N \left| f^{(2R+2)}(x) \right| \sum_{n=1}^{\infty} \frac{2}{(2\pi n)^{2R+2}} \, dx \]

\[ = 2 \cdot \frac{2}{(2\pi)^{2R+2}} \int_0^N \left| f^{(2R+2)}(x) \right| \, dx \]

\{ apply box \}

\[ = 2 \cdot \frac{|B_{2R+2}|}{(2R+2)!} \int_0^N \left| f^{(2R+2)}(x) \right| \, dx \]
For the sake of clarity, notice too that

\[ R_{n+1} = \frac{B_{n+2}}{(2n+2)!} \left[ f^{(n)}(N) - f^{(n+1)}(0) \right] + R_{n+1} \]

Cor 3 can thus be obtained equally well by Cor 1: indeed,

\[ |R_{n+1}| = \frac{|B_{n+2}|}{(2n+2)!} \int_0^N |f^{(n)}(x)| \, dx \]

\[ + \frac{|B_{n+2}|}{(2n+2)!} \int_0^N f^{(n+2)}(x) \, dx \quad \text{Cor 1} \]

\[ \leq 2 \frac{|B_{n+2}|}{(2n+2)!} \int_0^N f^{(n+2)}(x) \, dx \]
THM (Corollary of E-M à la Euler)

\( J(z) \) is analytic on each half-plane \( \{ \text{Re}(z) > -A \} \) except for a simple pole of residue 1 at \( z = 1 \). Hence,

\[ J(z) \sim \frac{1}{z-1} \quad \text{is analytic on } \mathbb{C}. \]

PF

E-M

\[ f(t) = (1 + t)^{-z} \quad \text{keep } \text{Re}(z) > 1 \text{ at first} \]

\[ f^{(j)}(t) = (-1)^j \frac{d^j}{dz^j} [(z+1) \cdots (z+j-1)(1+t)^{-z-j}] \]

\[ \sum_{k=1}^{N+1} k^{-z} = \frac{1}{z} + \frac{1}{2} (1 + N)^{-z} + \int_0^N (1 + t)^{-z} \, dt \]

\[
+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)!} \left[ (-1)^k (z+1) \cdots (z+2k-2)(1+N)^{-z-(2k-1)} \right. \\
\left. + \frac{z(z+1) \cdots (z+2k-2) 0}{(2k+1)} \right] \\
+ \int_0^N \frac{B_{2k+1}(t)}{(2k+1)!} (-1)^k (z+1) \cdots (z+2k) (1+t)^{-z-2k-1} \, dt
\]
Let \( N \to \infty \).

\[
\tilde{f}(z) = \frac{1}{2} + \int_0^\infty \frac{dz}{1+t}^z dt + \sum_{k=1}^R \frac{B_{2k}}{(2k)!} z(z+1) \cdots (z+2k-2) \cdot 1 \\
+ \int_0^\infty \frac{(-1)^z}{(1+t)^{z+2R+1}} \frac{B_{2R+1}(t)}{(2R+1)!} dt
\]

\[
\tilde{f}(z) = \frac{1}{2} + \frac{1}{z+1} \\
+ \sum_{k=1}^R \frac{B_{2k}}{(2k)!} z(z+1) \cdots (z+2k-2) \\
+ (-1)^z z(z+1) \cdots (z+2R) \int_0^\infty \frac{1}{(1+t)^{z+2R+1}} \frac{B_{2R+1}(t)}{(2R+1)!} dt
\]

Remark with \( R \geq 1 \)

\begin{itemize}
  \item note \( R = 0 \) is OK too;
  \item recall Lec 5 p. 17 line 3
  \item and \( v(t) = \frac{1}{2} + \beta(t) \)
\end{itemize}

The integral containing \( \frac{B_{2R+1}(t)}{(2R+1)!} \) is clearly nicely convergent, hence analytic, for compact
subsets of \( \{ \text{Re}(z) > -2R \} \).

At once,
\[
\mathcal{S}(z) = \frac{1}{z-1}
\]
is analytic on each \( \{ \text{Re}(z) > -2R \} \) and we are done!

---

**Examples (Numerology!)**

Recall \( B_k \) on p. 13.

\( R \geq 3 \) say \( \Rightarrow \) get

\[
\mathcal{S}(z) = \frac{1}{2} + \frac{1}{z-1} + \frac{B_2}{2!} z + \frac{B_4}{4!} \frac{z(z+1)(z+2)}{}
\]
\[
+ \frac{B_6}{6!} \frac{z(z+1) \ldots (z+4)}{}
\]
\[
+ (-1)^{1} \frac{z(z+1) \ldots (z+6)}{6!} \int_0^\infty \frac{1}{(1+t)^{z+7}} \cdot \frac{\tilde{B}_7(t)}{7!} dt
\]
\[
\text{Re}(z) > -6
\]

\[
\mathcal{S}(0) = -\frac{1}{2}
\]

\[
\mathcal{S}(-1) = \frac{1}{2} - \frac{1}{2} + \frac{B_2}{2!} (-1) + 0 = -\frac{1}{12}
\]

\[
\mathcal{S}(-2) = \frac{1}{2} - \frac{1}{3} + \frac{B_2}{2!} (-2) + 0 = \frac{1}{6} + \frac{1}{12} (-2) = 0
\]
\[ f(-4) = \frac{1}{2} - \frac{1}{5} + \frac{B_2}{2!} (-4) + \frac{B_4}{4!} (-4)(-3)(-2) + 0 \]

\[ = \frac{1}{2} - \frac{1}{5} + \frac{1}{12} (-4) + \left(-\frac{1}{30}\right) \frac{(-1)^{4/4} - 1}{4!} + 0 \]

\[ = \frac{3}{10} - \frac{1}{3} + \frac{1}{30} = \frac{9 - 10 + 1}{30} = 0 \]

One conjectures \( f(-2k) = 0, \ k \geq 1 \).

Euler proved this (playing with the B2k).

Of course he used only \( f(x) \)

and his "natural formulae".
\[ \Gamma(z) \text{, more accurately } \Gamma \left( \frac{z}{2} \right) \text{, is part of the} \]

"modern Riemann zeta fun". It's the

Archimedean part:

\[
\pi^{-z/2} \Gamma \left( \frac{z}{2} \right) \Gamma(z)
\]

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \text{Re}(z) > 0 \]

\[ \Gamma(k) = (k-1)! \quad k \geq 1 \]

By Weierstrass M-test, the improper integral

is uniformly convergent on \( \{ \text{Re}(z) > 0 \} \) - compacta.

\[ |t^{z-1}| = t^{x-1} \]

Hence \( \Gamma(z) \) analytic on \( \text{Re}(z) > 0 \).

Easy integ by parts:

\[ \text{Re}(z) > 0 \implies \Gamma'(z+1) = z \Gamma(z) \]

Hence

\[ \Gamma(z+R) = z(z+1) \cdots (z+R-1) \Gamma(z) \quad R \geq 1 \]

\[ \implies \Gamma(z) = \frac{\Gamma(z+R)}{z(z+1) \cdots (z+R-1)} \]

But, RHS is analytic for \( \{ \text{Re}(z) > -R \} \), except at \( 0, 1, 2, \ldots, -R \).
Hence, \( f(z) \) is analytic on
\[ \mathbb{C} - \{0, -1, -2, \ldots \} \, \mathbb{F} \).

Easy to check:
\[ z = -k \quad \text{is a simple pole} \]
\[ \text{Res} = \frac{(-1)^k}{k!} \, \mathbb{F} \]

**Thm (Euler)**

\[ f(z) = \lim_{N \to \infty} \frac{N! \, N^z}{(z+1) \cdots (z+N)} \]

with uniform convergence on \( \{\text{Re}(z) > 0\} \), compacta.

**PF**

Keep \( z \in K \) say \[ \mathbb{F} \]

\[ e^{-t} = \lim_{n \to \infty} (1 - \frac{t}{n})^n, \quad t > 0 \quad \text{(baby calc)} \]

We hope to approximate \( f(z) \) by
\[ \int_0^\infty e^{-t} (1 - \frac{t}{n})^n \, dt \]
\[ \exp \left[ (z-1) \ln t \right] \]
Let \((1-u)^{\frac{1}{N}} = \nu\), \(0 < \nu < 1\).

So \(t = \frac{1}{N}(1-\nu)\), \(\frac{1}{N} < t < 1\).

\[
\int_0^t e^{-s} (1-s)^{\frac{1}{N}} \; ds = \frac{1}{N} \sum_{n=0}^{\infty} \frac{(t^N)^n}{(1+n)^{\frac{1}{N}}}
\]

We will look at \(\int_0^1 e^{-s} (1-s)^{\frac{1}{N}} \; ds\) which equals \(N\int_0^1 (1-u)^{\frac{1}{N}} \; du\).

Must prove \(\lim_{N \to \infty} N\int_0^1 (1-u)^{\frac{1}{N}} \; du = 0\).

\[
\lim_{N \to \infty} N\int_0^1 (1-u)^{\frac{1}{N}} \; du = \int_0^1 e^{-s} \; ds = 1
\]

\[
1 = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\frac{1}{N}}}
\]

\[
N\int_0^1 (1-u)^{\frac{1}{N}} \; du = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\frac{1}{N}}}
\]

\[
\int_0^1 (1-u)^{\frac{1}{N}} \; du = \frac{1}{N+1}
\]

\[
\int_0^1 (1-u)^{\frac{1}{N}} \; du = \frac{1}{N+1}
\]

\[
\int_0^1 (1-u)^{\frac{1}{N}} \; du = \frac{1}{N+1}
\]
For one moment, let $t > 0$ and keep $0 < t < n$.

\[
\left(1 - \frac{t}{n}\right)^n = e^{n \log \left(1 - \frac{t}{n}\right)}
\]

\[
= e^{n \left[-\frac{t}{n} - \frac{1}{2} \frac{t^2}{n^2} - \cdots\right]}
\]

\[
= e^{-t} e^{-\frac{1}{2} \frac{t^2}{n}} e^{-\frac{1}{3} \frac{t^3}{n^2}} \cdots
\]

\[
0 < \left(1 - \frac{t}{n}\right)^n < \left(1 - \frac{t}{n+1}\right)^{n+1} < \cdots < e^{-t}
\]

We thus see that

\[
0 \leq \left(1 - \frac{t}{N}\right)^N \uparrow \text{ to } e^{-t}
\]

as $N \to \infty$. In addition, by the expansion above, we have uniform convergence as $N \to \infty$ on any $0 \leq t \leq \Delta$, $\Delta$ big.

The issue with \( \int_0^\infty t^{z-1} \left(1 - \frac{t}{N}\right)^N dt \) is now a simple manipulation with the dominated convergence for $z \in \mathbb{C}$. 
Just write

\[ \int_0^\infty t^{\alpha-1} \left( 1 - \frac{t}{N} \right)^N \, dt \]

\[ = \int_0^\delta + \int_\delta^T t^{\alpha-1} \left( 1 - \frac{t}{N} \right)^N \, dt + \int_T^\infty \]

\[ \left\{ \begin{array}{l}
\text{then adjust } \delta \text{ and } T > 1 \\
\text{appropriately}
\end{array} \right. \]

\[ \alpha = \inf \text{Re}(K), \quad \beta = \sup \text{Re}(K) \]

\[ |\int_0^\delta| \leq \int_0^\delta t^{\alpha-1} e^{-t} \, dt < \frac{\varepsilon}{100} \]

\[ |\int_\delta^T| \leq \int_\delta^T t^{\beta-1} e^{-t} \, dt < \frac{\varepsilon}{100} \]

\[ \text{etc etc} \]

Thus

\[ f(z) \neq 0 \quad \text{for } \text{Re}(z) > 0. \]

If

\[ f(x) > 0 \quad \Rightarrow \quad f'(z) \neq 0, \]

Notice that

\[ \frac{N! \, N^z}{\Gamma(z+1) \cdots \Gamma(z+N)} \neq 0 \quad \text{on } \text{Re}(z) > 0. \]
By Hurwitz's thm in analytic fns, the limit is either \( \equiv 0 \) or \( \text{never zero} \). So, we are done thanks to Thm 23.

**Cor**

\[ \Gamma'(z) \neq 0 \quad \text{for } z \in \mathbb{C}. \]

**PF**

- \( z = -k \) is no problem! \((k \geq 0)\)
- \( \Gamma'(z) = 0 \) for some \( \Re(z_0) \leq 0 \), \( z_0 \in \mathbb{C} \).
- But then, by box \( \Rightarrow \)

\[ \Gamma(z + m) = z_0 (z_0 + 1) \cdots (z_0 + m - 1) \Gamma(z_0) = 0 \]

For all \( m \) large \( \Rightarrow \) Contradicts Thm 26.

Hence, \( \frac{1}{\Gamma(z)} \) is analytic on \( \mathbb{C} \) with zeros at \( z = 0, 1, -2, \ldots \), each of multiplicity 1.
Let $K$ be any compact subset of 
\[ \mathbb{C} - \{0, -1, -2, \ldots\} \].

Choose $m$ so big that
\[ m + \inf \text{ Re}(K) \geq 1 \).

The relation
\[ \Gamma(z) = \frac{\Gamma(z+m)}{z(z+1) \cdots (z+m-1)} \]
holds first for $\text{Re}(z) > 0$, then all $z$ by analytic continuation.

One can apply Thm (23) to $\Gamma(z+m)$ for $z \in K$.

**Thm**
\[ \Gamma(z) = \lim_{N \to \infty} \frac{N! \ N^z}{z(z+1) \cdots (z+N)} \]
for $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$ with uniform conv on compacta.

**PF**

Exercise — using the procedure suggested.
Lemma

Let $a_n(z)$ be analytic on $|z-z_0| < 2h$.

Let $a_n(z) \text{ conv unif on } |z-z_0| = h$.

Then $a_n(z) \text{ conv unif on } |z-z_0| \leq h$ too.

Proof

Apply max mod principle to $a_m(z) - a_n(z)$, $m > n$.

Get unif Cauchy condition for $|z-z_0| \leq h$.

Thus

$$\frac{1}{f(z)} = \lim_{N \to \infty} \frac{z(z+1) \cdots (z+N)}{N! \ z^N}$$

on $C$.

with unif conv on compacta.

Proof

Combine Thm 28 with Lemma.
Thus (all well-known)

Let \( D = \mathbb{C} - \{0, -1, -2, \ldots\} \).

(a) \( \Gamma(z) \neq 0 \) on \( D \), \( \frac{1}{\Gamma(z)} \) entire, simple zeros at \( z = 0, -1, -2, \ldots \).

(b) \[
\frac{\Gamma'(z)}{\Gamma(z)} = -z - \frac{1}{z} = \sum_{k=1}^{\infty} \left( \frac{1}{z+k} - \frac{1}{k} \right)
\]
with unif conv on \( D \) compacta.

(c) \( \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z} \)

(d) \( \Gamma(\frac{1}{2}z) = 2^{\frac{1}{2}z-1} \pi^{-\frac{1}{2}} \Gamma(z) \Gamma(z + \frac{1}{2}) \)

(e) \( \eta = \lim_{N \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \ln N \right) = -\Gamma'(1) \)

**Proof**

**Exercise.**

Some pointers. In (b), by analytic continuation it suffices to work on a compact subset of \( \text{Re}(z) > 0 \) containing 1. Use thm (28).

\[
\frac{f_n(z)}{\Gamma(z)} \to \Gamma(z) \quad \Rightarrow \quad \frac{f_n'(z)}{\Gamma(z)} \to \frac{\Gamma'(z)}{\Gamma(z)}
\]

so

\[
\frac{f_n'(1)}{f_n(1)} \to \frac{\Gamma'(1)}{\Gamma(1)}
\]
Study

\[ \frac{f_n'(z)}{f_n(z)} = \frac{f_n'(1)}{f_n(1)} \quad \text{as} \quad n \to \infty. \]

Get (b) and (e).

For (c), take \( \Im(z) > 0 \) wlog. Use theorem (28)

and recall

\[ \frac{\sin \pi z}{\pi z} = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right). \]

In (d), use theorem (28) again with obvious factorings to get

\[ \Gamma(2z) = A \Gamma(z) \Gamma(z + \frac{1}{2}) 2^z z. \]

Evaluate \( A \) via the knowledge that \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \)

\[ \text{must prove} \]

\[ \Gamma(z)\Gamma(-z) = \frac{\pi}{-z\sin(\pi z)} \]
Theorem (very well-known)

\[
\frac{1}{f(x)} = \frac{1}{\prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right)^{\frac{x}{k}}} \\
\text{with uniform convergence on compacta.}
\]

Proof

Work on \( K = \{ |x| \leq R \} \), \( R \) large.

\[
\frac{1}{f(x)} = \lim_{N \to \infty} \frac{\prod_{k=1}^{N} (x+k)}{N^{x} \prod_{k=1}^{N} k}
\]

\[
= \lim_{N \to \infty} \frac{\prod_{k=1}^{N} (1 + \frac{x}{k})}{e^{\frac{x}{k} \ln N}}
\]

\[
\left\{ \begin{array}{l}
\sum_{k=1}^{N} \frac{1}{k} = \ln N + \gamma + O_N, \\
\quad E_N \to 0
\end{array} \right.
\]

\[
\ln N = \sum_{k=1}^{N} \frac{1}{k} - \gamma - E_N
\]

\[
= \lim_{N \to \infty} \frac{\prod_{k=1}^{N} (1 + \frac{x}{k})}{e^{\frac{x}{k} \ln N}}
\]

\[
= \lim_{N \to \infty} e^{\frac{x}{k} \ln N} \prod_{k=1}^{N} \left(1 + \frac{x}{k}\right) e^{-\frac{x}{k}}
\]

\[
= \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right) e^{-\frac{x}{k}}
\]
Note:

\[
\begin{aligned}
\text{for } k \geq 100 \text{ say, } \frac{1}{k} &\approx \frac{1}{100} \\
\text{but } (1+w) e^{-w} &\approx 1 + O(w^2) \quad \text{for } |w| \leq \frac{1}{100} \\
(1+\frac{z}{k}) e^{-\frac{z}{k}} &\approx 1 + O\left(\frac{R^2}{k^2}\right)
\end{aligned}
\]

\[\psi\]

Weierstrass $M$-test for product is fine.

We now return to (28) and wish to apply E-M version 2 to $\log \Gamma(z)$. We wish to do this for $z \in D = \mathbb{C} - \{-0, -1, -2, \ldots\}$, keeping $z$ on some compact subset of $D$ initially.

Clearly, we need:

\[\ln N! + \varepsilon \ln N - \sum_{j=0}^{N} \log (z+j)\]

and for the logs, we should try to use $\log$.\]
For ease of calculations, focus on

\[ T_N(z) = \sum_{j=0}^{N} \log(z+j) - \sum_{k=1}^{N} \ln k - z \ln N \]

\[ \text{Can use E-M} \]

\[ \sum_{j=0}^{N-1} \log(1+j) \]

The calculation is easy, but boring. We just give a few steps.

\[ f(t) = \log(t+z) \quad z = \text{held fixed} \]

\[ f'(t) = (t+z)^{-1} \]

\[ f''(t) = (-1)(t+z)^{-2} \]

\[ f^{(j)}(t) = (-1)^j (j-1)! (t+z)^{-j} \]

\[ \nabla \]
\[ \sum_{j=0}^{N} \log(\varepsilon + j) = \frac{1}{2} \log \varepsilon + \frac{1}{2} \log(\varepsilon + N) \]

\[ + \int_{0}^{N} \log(\varepsilon + t) \, dt \]

\[ + \sum_{k=1}^{R} \frac{\beta_{2k}}{(2k)!} \left[ (2k-1)! (\varepsilon + N)^{-(2k-1)} - (2k-2)! (\varepsilon)^{-(2k-1)} \right] \]

\[ + \int_{0}^{N} \frac{\tilde{B}_{2k+1}(t)}{(2k+1)!} \left( \frac{\varepsilon}{t + \varepsilon} \right)^{-2k-1} \, dt \]

\[ \left\{ \int_{0}^{N} \log(\varepsilon + t) \, dt = (\varepsilon + N) \log(\varepsilon + N) - \varepsilon \log \varepsilon - N \right\} \]

\[ \sum_{j=0}^{N} \log(\varepsilon + j) = \frac{1}{2} \log \varepsilon + (\varepsilon + N + \frac{1}{2}) \log(\varepsilon + N) \]

\[ - \varepsilon \log \varepsilon - N \]

\[ + \sum_{k=1}^{R} \frac{\beta_{2k}}{(2k)(2k-1)} \left[ (N+\varepsilon)^{-(2k+1)} - (\varepsilon)^{-(2k+1)} \right] \]

\[ + \int_{0}^{N} \frac{\tilde{B}_{2k+1}(t)}{2k+1} \left( \frac{\varepsilon}{t + \varepsilon} \right)^{-2k-1} \, dt \]

To get \( \sum_{k=1}^{N} \log k \) just take \( \varepsilon = 1 \) and \( N \to N-1 \)

\( \psi \)
\[
\sum_{k=1}^{N} \ln k = \left( N + \frac{1}{2} \right) \log N - (N-1)
\]
\[
\quad + \sum_{l} \frac{R}{(2k)(2k-1)} \left[ N^{-2k+1} - 1 \right]
\]
\[
\quad + \int_{0}^{N-1} \frac{B_{2R+1}(t)}{2R+1} \left( t+1 \right)^{-2R-1} \, dt
\]

So, by flipping all signs in the above, get:

\[
T_{N}(\varepsilon) = \left( \varepsilon + N + \frac{1}{2} \right) \left\{ \log (\varepsilon+N) - \log (N) \right\}
\]
\[
\quad - \left( \varepsilon - \frac{1}{2} \right) \log \varepsilon - 1
\]
\[
\quad + \sum_{l} \frac{R}{(2k)(2k-1)} \left[ 1 - \varepsilon^{-2k+1} + \varepsilon^{-2k+1} \right]
\]
\[
\quad + \frac{1}{2R+1} \int_{0}^{N} \frac{B_{2R+1}(t)}{(t+\varepsilon)^{-2R-1}} \, dt
\]
\[
\quad - \frac{1}{2R+1} \int_{0}^{N-1} \frac{B_{2R+1}(t)}{(t+1)^{-2R-1}} \, dt
\]

\( \left\{ \begin{array}{l}
\text{use } \log (\varepsilon+N) - \log (N) = \log \left( \frac{\varepsilon}{N} + 1 \right) \\
\varepsilon \in D
\end{array} \right\} \)
\[ T_N(z) = (z + N + \frac{1}{2}) \log \left( 1 + \frac{z}{N} \right) \]

\[ \approx (z - \frac{1}{2}) \log z - 1 \]

\[ + \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} \left[ \ldots \right] \]

\[ + \frac{1}{2R+1} \int_{0}^{N} \tilde{B}_{2R+1}(t) \left( t + z \right)^{-2R-1} dt \]

\[ - \frac{1}{2R+1} \int_{0}^{N-1} \tilde{B}_{2R+1}(t) \left( t+1 \right)^{-2R-1} dt \]

Remember that \( |\tilde{B}_{2R+1}(t)| \leq some \, \mathcal{O}_R, \, all \, t \in \mathbb{R}. \)

Now let \( N \to \infty. \)

\[ (z + \frac{1}{2}) \log \left( 1 + \frac{z}{N} \right) \to 0 \]

\[ N \log \left( 1 + \frac{z}{N} \right) \to z \]

Conclude:

\[ T_N(z) \to z - (z - \frac{1}{2}) \log z - 1 \]

\[ + \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} \left[ \ldots \right] \]

\[ + \frac{1}{2R+1} \int_{0}^{\infty} \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}} dt \]

\[ - \frac{1}{2R+1} \int_{0}^{\infty} \frac{\tilde{B}_{2R+1}(t)}{(t+1)^{2R+1}} dt \]
Recall \((33)\) bottom. Deduce:

\[
- \log \Gamma(z) = z - \left( z - \frac{1}{2} \right) \log z - \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{2k+1} + \frac{1}{2R+1} \int_{0}^{\infty} \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}} dt + C_R
\]

where \(C_R = \text{some appropriate real constant}\).

Thus, on \(D\) \(D = \mathbb{C} - \{0, 1, 2, \ldots, 3\}\):

\[
\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z - \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{2k+1} + \frac{1}{2R+1} \int_{0}^{\infty} \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}} dt + E_R
\]

\(E_R = \text{suitable real constant}\).
The preceding relation is an identity. Note that RHS is real if \( \varepsilon = x > 0 \).

Hence the branch of \( \log \Gamma(z) \) under discussion is the one that reduces to \( \ln \Gamma(x) \) for \( \varepsilon = x > 0 \).

Also, note:

\[
\left| \frac{1}{2R+1} \int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+x)^{2R+1}} \, dt \right|
\]

\[
\approx \frac{1}{2R+1} \int_0^\infty \frac{\Theta_R}{(t+x)^{2R+1}} \, dt
\]

\[
\approx \frac{1}{(2R+1)(2R)} \Theta_R x^{-2R}
\]

\[
= O\left( \frac{1}{x^{2R}} \right), \text{ each } R.
\]

So,

\[
\ln \Gamma(x) = (x - \frac{1}{2}) \ln x - x + \sum_{r=1}^{R} \frac{B_{2k} x^{-2k+1}}{(2k)(2k-1)}
\]

\[
+ E_R + O(x^{-2R}) \quad \text{as } x \to +\infty.
\]
At once, by comparing for different $R$'s,

\[ E_1 = E_2 = E_3 = \ldots = E_R = \ldots \]

Call the common value $E$.

An easy substitution into

\[ \Gamma(2x) = 2^{2x-1} \pi^{-1/2} \Gamma(x) \Gamma(x + \frac{1}{2}) \]

\[ \downarrow \]

\[ \ln \Gamma(2x) = (2x-1) \ln 2 - \frac{1}{2} \ln \pi \]

\[ + \ln \Gamma(x) + \ln \Gamma(x + \frac{1}{2}) \]

gives (for $x > 1$)

\[ (2x - \frac{1}{2}) \ln(2x) - 2x + O(\frac{1}{x}) + E \]

\[ = (2x - 1) \ln 2 - \frac{1}{2} \ln \pi \]

\[ + (x - \frac{1}{2}) \ln x - x + O(\frac{1}{x}) + E \]

\[ + (x) \ln(x + \frac{1}{2}) - (x + \frac{1}{2}) + O(\frac{1}{x}) + E \]

\[ \downarrow \]
\[-\frac{1}{2} \ln 2 + O\left(\frac{1}{x}\right) + E\]

\[= -\ln 2 - \frac{1}{2} \ln \pi + 2E + O\left(\frac{1}{x}\right)\]

\[\frac{1}{2} \ln 3 + \frac{1}{2} \ln \pi + O\left(\frac{1}{x}\right) = E\]

\[\Rightarrow E = \frac{1}{2} \ln (2\pi).\]

On the bottom, we therefore get:

\[\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \ln (2\pi)\]

\[+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1}\]

\[- \frac{1}{z} \int_{0}^{\infty} \frac{B_{2R+1}(t)}{(t+z)^{2R+1}} \, dt\]

As an identity on $D$. 
Theorem (Stirling — corollary of E-M)

Keep \( z \in \mathbb{D} \), where \( \mathbb{D} = \{ z \in \mathbb{C} \mid 0, 1, 2, \ldots \} \).

We have

\[
\log \Gamma(z) = \left(1 - \frac{1}{z} \right) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} \frac{z^{-2k+1}}{z} + \sum_{k=1}^{R} \frac{B_{2k+1}}{(2k+1)!} \frac{1}{(z+1)(z+2)\cdots(z+k)}
\]

where the branch of \( \log \Gamma(z) \) reduces to \( \ln \pi(x) \) when \( z = x > 0 \).

\[\text{Famous} \]

Proof

As above.

Corollary (Stirling's asymptotic formula)

Fix any \( R \geq 1 \) and \( \delta > 0 \). Then:

\[
\log \Gamma(z) = \left(1 - \frac{1}{z} \right) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} \frac{z^{-2k+1}}{z} + O_{R, \delta} \left( \frac{1}{|z|^{2R+1}} \right)
\]

as \( z \to \infty \) in \( |\arg(z)| \leq \pi - \delta \).
A simple absolute value estimate of

\[ -\frac{1}{2R+1} \int_0^{\infty} \frac{\tilde{B}_{2R+1}(t)}{(t+\varepsilon)^{2R+1}} \, dt \]

does not work. One uses a minor trick instead. Namely:

\[ \text{Rem}_R = \frac{B_{2R+2}}{(2R+2)(2R+1)} \varepsilon^{-2R-1} \]

\[ -\frac{1}{2R+3} \int_0^{\infty} \frac{\tilde{B}_{2R+3}(t)}{(t+\varepsilon)^{2R+3}} \, dt \]

Notice that:

\[ |\text{last term}| \leq \frac{B_{R+1}}{2R+3} \int_0^{\infty} \frac{dt}{(t+\varepsilon)^{2R+3}} \]

\[ \varepsilon = \rho e^{i\omega}, \quad |\omega| \leq \pi - \delta, \quad \delta \]

\[ \rho \text{ large} \]

\[ \text{put } t = \rho \nu \]
\[
= \frac{\mathcal{R} + 1}{2R + 3} \int_0^{\beta} \frac{d\nu}{\nu^{2R+3}} \left| \frac{1}{\nu + e^{i\omega}} \right|^{2R+3}
\]

\[
= \frac{\mathcal{R} + 1}{2R + 3} \nu^{-2R-2} \int_0^{\beta} \frac{d\nu}{\nu^{2R+3}} \left| \frac{1}{\nu + e^{i\omega}} \right|^{2R+3}
\]

These integrals are
\[
O(1) \quad \text{for} \quad |\omega| \leq \pi - \delta
\]

\[
|\text{Re} w_\mathcal{R}| \ll \frac{O_{\mathcal{R}}(\nu^{-2R-1}) + O_{\mathcal{R}}(1)|\nu|^{-2R-2}}{|\nu|^{-2R-1}}
\]

\[
= \frac{O_{\mathcal{R}}(1)}{|\nu|^{-2R-1}}.
\]
Remark 1

Refer to (14) line 4 and (14) Thm (E-M vers 2). Though it may be tempting to allow \( R = 0 \) on page 42, note that

\[
\sum_{k=0}^{\infty} \frac{\beta_k(t)}{t + \varepsilon} dt
\]

is not absolutely convergent. For this reason, allowing \( R = 0 \) in Thm (42) is usually avoided. In the Corollary, it is of course OK, since

\[
\frac{B_2}{2^1 \varepsilon^1} + O_5\left(\frac{1}{|z|^3}\right) = O_5\left(\frac{1}{|z|}\right).
\]

Remark 2 (classical Stirling) \( R = 0 \) OK HERE.

By use of (15) Cor 2 (nontrivial), it is possible to show

\[
\log N! = (N + \frac{1}{2}) \ln N - N + \frac{1}{2} \ln (2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} N^{2k-1} + REM_R
\]

\[
REM_R = \frac{B_{2R+2}}{(2R+2)(2R+1)} N^{2R-1} \quad \text{with} \quad 0 < \gamma < 1.
\]
I began with a quick development of basic Fourier series — in a nonstandard way, i.e., via E-M summation.

\[ \langle f, g \rangle = \int_a^b f(x) g(x) \, dx \]

On \([0,1]\) or any \([a_j + \epsilon, b_j + \epsilon]\):

\[ \langle \varphi_m, \varphi_n \rangle = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \]

For

\[ \varphi_n(x) = e^{2\pi inx}, \quad n \in \mathbb{Z}. \]

So, we have the usual idea of trying to write \( f \) "most of the time" as \( \sum_{n=1}^\infty c_n \varphi_n \), \( c_n = \langle f, \varphi_n \rangle \).

**Lemma**

\( f \in C[0,N] \). Assume \( f \) is only piecewise \( \leq \) \( c_1 \).

Then we still have

\[ \frac{1}{2} f(0) + f(1) + \ldots + f(N-1) + \frac{1}{2} f(N) = \int_0^N f \, dx \]

\[ + \int_0^N f \beta(x) \, dx. \]

\[ \beta(x) = x - \lfloor x \rfloor - \frac{1}{2}. \]
**Proof**

Begin as before

\[
\begin{align*}
\sum_{i=0}^{N} f(x) &= \int_{0}^{N} f(x) \, dx \\
&= \int_{0}^{N} f(x) \, dx \left( x - \frac{1}{2} \right) - f(x) \\
&= \int_{0}^{N} f(x) \, dx - \int_{0}^{N} f(x) \, d\theta(x) \\
&= \int_{0}^{N} f(x) \, dx - \int_{0}^{N} f(x) \, d\theta(x).
\end{align*}
\]

Split \( \int_{0}^{N} f(x) \, dx \) into chunks corresponding to corners of \( F_0 \).

Then do the integration by parts and recombine. Ambiguous \( f' \) at a finite \( n \) of corners does not affect \( \int_{0}^{N} \theta f'(x) \, dx \).

\( \Rightarrow \) All is fine.

---

Take \( N=1 \). Assume \( f \in C[0,1] \), piecewise \( C^1 \).

Hence, by Lemma

\[
\frac{1}{2} f(0) + \frac{1}{2} f(1) = \int_{0}^{1} f(x) \, dx + \int_{0}^{1} f'(x) \left( - \sum_{n=1}^{N} \frac{\sin \frac{2\pi n x}{\Gamma}}{\pi n} \right) \, dx
\]

\[
= \int_{0}^{1} f(x) \, dx + \sum_{n=1}^{N} \int_{0}^{1} f' \left( -\sin \frac{2\pi n x}{\pi n} \right) \, dx,
\]

the 2nd line by Lec 9, p. 9, Baby Fact.
Note that the error term after $N$ is

$$\pm \int_0^1 f'(\beta - \lambda N) \, dx$$

i.e.

$$\text{ABS VALUE} \leq M \int_0^1 |\beta - \lambda N| \, dx$$

$$M \equiv \sup_{[0,1]} |f'|$$

The $|\beta - \lambda N|$ integral is an absolute expression, say $\omega_N$, and $\omega_N \to 0$ so

$$|\text{Error}| \leq M \omega_N$$

Note too that

$$\left( n \geq 1 \right)$$

$$\int_0^1 f' \frac{\sin 2\pi n x}{-\pi n} \, dx = \frac{1}{2\pi i n} \int_0^1 f'(e^{-2\pi inx} - e^{2\pi inx}) \, dx$$

Write the last expr. as

$$\frac{1}{2\pi i n} \int_0^1 f' e^{-2\pi inx} \, dx + \frac{1}{2\pi i (-n)} \int_0^1 f' e^{-2\pi i(-n)x} \, dx$$
But,

\[ \frac{1}{2\pi in} \int_0^1 f(x)e^{-2\pi inx} \, dx \]

\[ = \frac{1}{2\pi in} \int_0^1 e^{-2\pi inx} \, dx \] (standard parts)

\[ = \frac{1}{2\pi in} \left[ e^{-2\pi inx} \right]_0^1 \]

\[ = \frac{1}{2\pi in} \left( e^{-2\pi in} - 1 \right) = 0 \]

\[ = \frac{f(1) - f(0)}{2\pi in} + \int_0^1 f'(x)e^{-2\pi inx} \, dx . \]

Similarly for \(-n\). Now add! Get:

\[(\text{term } n) + (\text{term } -n) \equiv C_n + C_{-n} \]

where

\[ C_k = \int_0^1 f(x)e^{-2\pi ikx} \, dx . \]

So,

\[ \frac{1}{2} f(0) + \frac{1}{2} f(1) = C_0 + \sum_{n=1}^{\infty} \frac{c_n}{\pi n} \quad \left( \text{as } n \rightarrow \infty \right) \]
\[ \frac{1}{2} f(0) + \frac{i}{2} f(1) = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n, \]

any \( f \in C[0,1], \) piecewise \( C^1 \).

\[ \text{Example} \]

AHA! This is really a Fourier series!

\[ \text{i.e., } \lim_{N \to \infty} \sum_{n=-N}^{N} c_n e^{-i \pi n} = 0. \]

The proof was just basic E-M, version I, and

\[ x - \lfloor x \rfloor = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n \pi x)}{n}, \quad x \in \mathbb{Z}. \]

\[ \text{NOTE THAT} \quad \text{error term for } |x| > N \text{ is} \]

\[ \leq M \omega N. \]
Initial Thm

Given \( f \in C[0,1], \) piecewise \( C^1. \)

Let \( c_k = \int_0^1 f(x) e^{-2\pi ikx} \, dx = \langle f, \phi_k \rangle. \)

Then:
\[
\frac{1}{2} f(0) + \frac{1}{2} f(1) = \lim_{N \to \infty} \sum_{-N}^{N} c_k \overline{e^{2\pi ik0}}
\]
\[|\text{Error}| \leq M \|f\|_\infty.\]

pf
As above. \( \square \)

Thm

Let \( f \in C(\mathbb{R}), \) periodic \( 1 \) is piecewise \( C^1. \)

Then:
\[
\sum_{-N}^{N} c_k e^{2\pi ikx} \rightarrow f(x) \quad \text{on} \quad \mathbb{R}.
\]

pf
Fix any \( x_0 \in \mathbb{R}. \) Consider \( g(x) = f(x + x_0) \) on \( [0,1] \) in previous Thm. Note
\[
c_k(g) = \int_0^1 g(x) e^{-2\pi ikx} \, dx = \int_0^1 f(x + x_0) e^{-2\pi ikx} \, dx
\]
\[
\left\{ \begin{array}{l}
y = x + x_0 \\\{ y \in [0,1] \}
\end{array} \right.
\]
\[
= \int_{x_0}^{x_0+1} f(y) e^{-2\pi ikx} e^{2\pi ikx_0} \, dy
\]
\[ \sum_{n=-N}^{N} c_k(f) e^{2 \pi i k x} \]

So

\[ f(x_0) = \lim_{N \to \infty} \sum_{n=-N}^{N} c_k(f) e^{2 \pi i k x_0} \]

\[ |\text{Error}| \leq M \frac{1}{N} \quad M = \sup_{\mathbb{R}} |f'| \]

QED.

The next theorem is a commonly used augmentation of theorem 6 (bottom).

**Theorem**

Let \( f \) belong to \( \mathcal{C}^2(\mathbb{R}) \) and be periodic 1. We then have

\[ |c_k| \leq \frac{1}{(2\pi k)^2} \int_{-1/2}^{1/2} |f''| dx \quad k \neq 0 \]

This ensures that, on the bottom, \( \sum_{k} c_k 2 \pi i k x \) converges uniformly and absolutely to \( f(x) \) on \( \mathbb{R} \).
Pf
Simply integrate by parts twice:
\[ c_k = \frac{1}{(2\pi i k)^2} \int_0^1 f(e^{-2\pi i k x}) dx, \quad k \neq 0. \]

The following is our main assertion in this approach to FS based on E-M.

**THEOREM** (standard Fourier series thus in undergrad analysis)

Let \( f \) be given on \( TR \) and be periodic \( 2\pi \).
Assume \( f \) is piecewise \( C^1 \). (See pictures)
Let \( c_k = \int_0^1 f e^{-2\pi i k x} dx \) and
\[ \text{FS}(f) \equiv \sum_{-\infty}^{\infty} c_k e^{2\pi i k x} \]
as a formal sum.

We then have \( \lim_{N \to \infty} \left( \sum_{-N}^{N} c_k e^{2\pi i k x} \right) \to f(x) \) as \( N \to \infty \)
away from the discontinuities of \( f \). At the points of discontinuity, we have
\[ \sum_{-N}^{N} c_k e^{2\pi i k x} \to \frac{1}{2} \left[ f(x+0) + f(x-0) \right]. \]
Here \( f(x+0) \), \( f(x-0) \) are the one-sided limits.

(cont'd)
In addition, the partial sums \( \sum_{-N}^{N} c_k e^{2\pi i k x} \) will be uniformly bounded on \( \mathbb{R} \).

**Proof**

The thm is certainly correct if \( f \) has no discontinuities on \( \mathbb{R} \). See (6) bottom.

We now do a trick. (using \( \beta \))

**Baby Lemma**

Let \( H(x) \equiv \lim_{N \to \infty} \sum_{-N}^{N} a_k e^{2\pi i k x} \), where the limit exists pointwise on all of \( \mathbb{R} \). Assume that the partial sums \( \sum_{-N}^{N} \) are uniformly bounded on \( \mathbb{R} \). Finally, assume that the partial sums \( \sum_{-N}^{N} \) converge uniformly away from \( \{c_1, \ldots, c_m \} \bmod \mathbb{Z} \) (in finite). Then:

(A) \( H(x) \) is Riemann integrable on \( [0, 1] \).

(B) \( a_k = \int_{0}^{1} H(x) e^{-2\pi i k x} dx \), each \( k \in \mathbb{Z} \).

No! (B) is not a tautology!
Pf of Lemma

The discontinuities of \( H \) are contained in \( \{ c_1, \ldots, c_m \} \mod \mathbb{Z} \) by the unit convolution \( H(x) \) is also bounded by the unit boundedness of

\[
S_N(x) = \sum_{-N}^{N} a_k e^{2\pi i k x}.
\]

By baby calculus, \( H \) is Riemann integrable on any finite \([a, b] \). Hence \([0, 1]\).

As we saw earlier, baby analysis \( \Rightarrow \)

\[
\int_{0}^{1} |H(x) - S_N(x)| \, dx \to 0 \quad \text{as} \quad N \to \infty.
\]

See Lec 9 p. 9.

By that same idea, we have:

\[
\int_{0}^{1} e^{-2\pi i m x} S_N(x) \, dx \to \int_{0}^{1} e^{-2\pi i m x} H(x) \, dx
\]

for each \( m \in \mathbb{Z} \). But LHS = \( am + O \) \( \frac{1}{N} \) for large \( N \).

Hence,

\[
am = \int_{0}^{1} H(x) e^{-2\pi i m x} \, dx.
\]
Before continuing, observe that \( \beta \) \((x) \geq 1\)

\[
\frac{e^\frac{2\pi i}{n}lx}{-2\pi i(x)} + \frac{e^{-\frac{2\pi i}{n}lx}}{-2\pi i(-l)} = -\frac{1}{2\pi i l} (e^{\frac{2\pi i}{n}lx} - e^{-\frac{2\pi i}{n}lx}) = \frac{\sin(2\pi l x)}{-\pi l}.
\]

Also write

\[
\tilde{\beta}(y) = \begin{cases} 0, & y \in \mathbb{Z} \\ \beta(y), & y \notin \mathbb{Z} \end{cases}.
\]

We already know that

\[
\tilde{\tilde{\beta}}(x) = \sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{\pi m} = \sum_{n \neq 0} \frac{1}{2\pi i n} e^{\frac{2\pi i}{n}inx},
\]

all \( x \in \mathbb{R} \). Unit conv away from \( \mathbb{Z} \)'s partial sums unit bounded. Similarly

\[
\tilde{\beta}(x-c) = \sum_{n \neq 0} \frac{e^{-2\pi i nx}}{R_{\min}} e^{\frac{2\pi i}{n}inx},
\]

all \( x \in \mathbb{R} \). By Baby Lemma on \( \tilde{\beta} \) automatically

\[
\int \tilde{\beta}(x) e^{-2\pi i nx} dx = \begin{cases} 0 & x = 0 \\ -\frac{1}{2\pi i n} & n \neq 0 \end{cases}
\]
\[
\int_0^1 \tilde{\beta}(x-c) e^{-2\pi i n x} dx = \begin{cases} 
0, & n = 0 \\
\frac{-e^{-2\pi i n c}}{2\pi i n}, & n \neq 0
\end{cases}
\]

\[THUS:\]
\[
F_S \left[ \tilde{\beta}(x) \right] = \sum_{n \neq 0} \frac{1}{2\pi i n} e^{-2\pi i n x}
\]
\[
F_S \left[ \tilde{\beta}(x-c) \right] = \sum_{n \neq 0} \frac{e^{-2\pi i n c}}{2\pi i n} e^{2\pi i n x}
\]

Obviously, the "\(n\)" can be removed from \(\beta\).

[These Fourier series can of course be checked directly, but we prefer the slick approach.]

We now return to the proof of p. \(\Xi\) THM.

Let \(f(x)\) have nontrivial discontinuities at points \(c_1, \ldots, c_m \mod \mathbb{Z}\). Let the "right-left" jump be \(J_p\). Saying \(J_p = 0\) means \(f(c_p + 0) = f(c_p - 0)\) but \(f(c_p) \neq f(c_p + 0)\).

Recall that
\[
\beta(0+) - \beta(0-) = -\frac{1}{2} - \frac{1}{2} = -1
\]
Define:

\[ q(x) = f(x) + \sum_{i=1}^{m} J_i \beta(x - c_i), \quad x \in \mathbb{R}. \]

For \( g \) is very interesting! It is obviously periodic 1. Also, it is obviously piecewise \( C^1 \). It may have discontinuities, but these lie in \( \{c_1, \ldots, c_m\} \mod 2 \).

Note however that

\[ q(c_i + 0) - q(c_i - 0) = J_i - J_i + 0 = 0 \quad \text{each} \quad 1 \leq i \leq m. \]

The points \( c_i \) are thus "removable discontinuities" if \( q \) is redefined correctly at these points.

Apply p. 6 bottom THM to this modified \( q \). We conclude that \( FS(q) \) converges uniformly over \( \mathbb{R} \) to \( \frac{1}{2} \left[ q(x + 0) + q(x - 0) \right] \). The partial sums are automatically uniformly bounded on \( \mathbb{R} \).

By linearity, however, as series,

\[ FS(f) = FS(g) - \sum_{i=1}^{m} J_i \cdot FS[\beta(x - c_i)]. \]
At once, the partial sums of $FS(f)$ are unif bounded on $\mathbb{R}$ (by the corresponding fact for $\beta$).

Also, $FS(f)$ conv uniformly away from the $\{c_i\}$ mod $\mathbb{Z}$ (by the corr fact for $\beta$).

At points $x \not\equiv c_i, \ldots, c_m$ mod $\mathbb{Z}$, clearly $FS(f)$ converges to

$$g(x) = \sum_{i=1}^{m} J_i \beta(x - c_i) = F(x).$$

(Big surprise!!)

At $c_i$, $FS(f)$ converges to

$$\frac{1}{2} [g(c_i^+0) + g(c_i^-0)] = 0 - \sum_{\lambda \neq i} J_\lambda \beta(c_i - c_\lambda).$$

But:

$$g(c_i^+0) = f(c_i^+0) + J_i(-\frac{1}{2}) + \sum_{\lambda \neq i} J_\lambda \beta(c_i - c_\lambda)$$

$$g(c_i^-0) = f(c_i^-0) + J_i(\frac{1}{2}) + \sum_{\lambda \neq i} J_\lambda \beta(c_i - c_\lambda)$$

$$\frac{g(c_i^+0) + g(c_i^-0)}{2} = \frac{f(c_i^+0) + f(c_i^-0)}{2} + \sum_{\lambda \neq i} J_\lambda \beta(c_i - c_\lambda)$$
Thus, all is now proved.

\( L(f) \) conv to \( f(x) + \int_{-\infty}^{0} f(x) dx \) (against big surprise!)

\( Y \)

\( F(f) \) conv to \( f(x) + \int_{-\infty}^{0} f(x) dx \) (against big surprise!)

\( Y \)}
We get:

\[ \int_0^1 f(x) \overline{S_N(x)} \, dx \to \int_0^1 f(x) f(x) \, dx \quad (N \to \infty) \]

but

\[
\text{LHS} = \int_0^1 f(x) \left( \sum_{-N}^N c_k e^{2\pi i k x} \right) \, dx
\]

\[
= \sum_{-N}^N c_k \overline{c_k} = \sum_{-N}^N |c_k|^2.
\]

The Fourier theory so far has been a kind of \( L_0 \times L_1 \) theory. In traditional real analysis courses, one investigates to see if an \( L_2 \times L_2 \) theory might be better (or more natural).

We will not bother to pursue the latter beyond 2 quick remarks.

Very use of completing the square on integrals like

\[ \int_0^1 |f(x) - S_N(x)|^2 \, dx, \quad \int_0^1 |f(x) - \sum_{-N}^N A_k e^{2\pi i k x}|^2 \, dx \]

for a general piecewise continuous, periodic \( \mathbb{1}_{\frac{1}{2}} f(x) \)

leads to
\[
\sum_{-N}^{N} |c_k|^2 \leq \int_{0}^{1} |f(x)|^2 \, dx \quad \text{(each N)}
\]

\[
\sum_{-\infty}^{\infty} |c_k|^2 \leq \int_{0}^{1} |f(x)|^2 \, dx
\]

(i.e., Bessel's inequality)

Here \(c_k = \int_{0}^{1} e^{-2\pi i k x} \, dx\).

(Actually, equality holds — but this is a harder theorem. One uses \(15\) Thm and "approximates" \(f\) by piecewise \(C^1\) functions.) SEE ANY STANDARD BOOK ON F.S.

Our second remark is a theorem.

\textbf{THM} (slight strengthening of \(p.6\) bottom)

Let \(f \in C(\mathbb{R})\), periodic 1, and be piecewise \(C^1\).

Let \(c_k = \int_{0}^{1} e^{-2\pi i k x} \, dx\). The Fourier series

\[
\sum_{-\infty}^{\infty} c_k e^{2\pi i k x}
\]

then converges \underline{uniformly} to \(f(x)\) on \(\mathbb{R}\).

AND we also have

\[
\sum_{-\infty}^{\infty} |c_k| < \infty.
\]

[\(\text{i.e., have nice abs conv.}\)]
\textbf{pf}

Take $k \neq 0$. By standard integ by parts,

$$c_k = \frac{1}{2\pi i k} \int_0^1 f'(x) e^{-2\pi i k x} dx.$$  \hfill (4)

Again, \textbf{NOTE THAT RHS is not affected by a few ambiguities in $f'$}. Write the foregoing as

$$c_k = \frac{1}{2\pi i k} \langle c_k (f') \rangle$$

and recall \textbf{Box (Bessel's ineq)}. By \textbf{Cauchy-Schwarz inequality},

$$\sum_{k=1}^{\infty} |c_k| = \frac{1}{2\pi} \sum_{h=1}^{\infty} \frac{1}{k} |c_k (f')|$$

$$\leq \frac{1}{2\pi} \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \sqrt{\sum_{k=1}^{\infty} |c_k (f')|^2} < + \infty.$$

Similarly for $k < 0$. \hfill \square
Next topic: Poisson summation formula.

THM

Given \( \varphi \in C^2(\mathbb{R}) \) such that, say

\[
|\varphi(x)|, |\varphi'(x)|, |\varphi''(x)| \quad \text{all} = O\left(\frac{1}{1+|x|}\right)^2.
\]

Let

\[
\hat{\varphi}(p) = \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i px} \, dx \quad (p \in \mathbb{R}).
\]

Let

\[
F(x) = \sum_{n=-\infty}^{\infty} \varphi(x+n) \quad (x \in \mathbb{R}).
\]

We then have

\[
F(x) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(k) e^{2\pi i kx} \quad \text{Poisson summation formula}
\]

for \( x \in \mathbb{R} \), with absolute and uniform conv on both sides over every interval \([-A,A]\).

Proof:

The series \( \sum_{n=-\infty}^{\infty} \varphi^{(j)}(x+n) \), \( 0 \leq j \leq 2 \), are clearly conv both abs and uniformly on every \([-A,A]\).

As such, we immediately get \( F \in C^2(\mathbb{R}) \). It is also apparent that \( F(x+1) = F(x) \).

Apply Thm (bottom). Get.
\[ F(x) = \sum_{n=-\infty}^{\infty} A_k e^{2\pi i k x} \quad \text{nicely} \]

\[ A_k = \int_0^1 F(x) e^{-2\pi i k x} \, dx \]

But:

\[ A_k = \int_0^1 \left( \sum_{n=-\infty}^{\infty} \varphi(x+n) \right) e^{-2\pi i k x} \, dx \]

\[ = \sum_{n=-\infty}^{\infty} \int_0^1 \varphi(x+n) e^{-2\pi i k x} \, dx \quad \text{by unit conv} \]

\[ = \sum_{n=-\infty}^{\infty} \int_{n}^{n+1} \varphi(y) e^{-2\pi i k y} \, dy = \int_{\mathbb{R}} \varphi(y) e^{-2\pi i k y} \, dy \]

\[ = \varphi(k) \quad \square \]

**Example**

\[ \varphi(x) = e^{-ax^2}, \quad a > 0 \]

\[ \int_{-\infty}^{\infty} e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}} \quad \Rightarrow \]

\[ \int_{-\infty}^{\infty} e^{-ax^2} e^{-2\pi i k x} \, dx = \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2 k^2}{a}} \]

(by elementary contour shift).

Hence, by Poisson summation formula.
\[ \sum_{n=-\infty}^{\infty} e^{-a(x+n)^2} = \sqrt{\frac{\pi}{a}} \sum_{k=-\infty}^{\infty} e^{-n^2\pi^2 k^2} e^{2\pi i k x}. \]

**Special Case:**

\[ \sum_{n=-\infty}^{\infty} e^{-\beta n^2} = \sqrt{\frac{1}{\beta}} \sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{\beta}}, \quad (\beta > 0). \]

The famous \( \Theta \) relation of Jacobi:

\[ \theta(\beta) = \frac{1}{\sqrt{\beta}} \theta\left(\frac{1}{\beta}\right). \]

We are now ready to derive (following Riemann) a slick formula for \( \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \).

**Easily check:**

\[ \Gamma\left(\frac{s}{2}\right) = \int_{0}^{\infty} x^{\frac{s}{2}} e^{-x} \frac{dx}{x}, \quad \text{Re}(s) > 1 \quad \text{say}. \]

\[ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_{0}^{\infty} y^{\frac{s}{2}} e^{-\pi n^2 y} \frac{dy}{y}, \quad \text{for } y > 0 \]

\[ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_{0}^{\infty} y^{\frac{s}{2}} \left[ \sum_{n=1}^{\infty} e^{-\pi n^2 y} \right] \frac{dy}{y}. \]

A nice positive term.

Clearly \( O(1/\sqrt{y}) \) for \( y \to 0^+ \) by \( \Theta \)-relation.
Note: the foregoing integral is nicely convergent near \( y = 0 \) because

\[
\int_0^1 \frac{\frac{\sigma}{2}}{\sqrt{y}} \frac{dy}{y} < \infty \quad \text{for} \quad \sigma > 1
\]

Write

\[
\psi(y) = \sum_{n=1}^{\infty} e^{-\pi n^2 y} \quad \text{and} \quad \Theta(y) = 2\psi(y) + 1.
\]

So:

\[
\psi(y) + \frac{1}{\sigma} = \frac{1}{\sqrt{y}} \left[ \psi(y) + \frac{1}{\sigma} \right] \quad \text{for} \quad y > 0
\]

\[
\psi(y) = -\frac{1}{\sigma} + \frac{1}{\sigma} y^{-\frac{1}{2}} + y^{-\frac{1}{2}} \psi(y)
\]

Get \( \Theta \):

\[
\frac{\sigma}{2} \Gamma\left(\frac{\sigma}{2}\right) \Theta(y) = \int_0^1 y^\frac{\sigma}{2} \psi(y) \frac{dy}{y} + \int_1^{\infty} y^\frac{\sigma}{2} \psi(y) \frac{dy}{y}
\]

Put \( y = \frac{1}{\sqrt{y}} \) here

\( \$\) now grind! \( \$\)
\[ \Gamma(s) = \int_1^\infty \sqrt{\frac{x}{2}} \left[ -\frac{1}{2} + \frac{1}{2} x^{1/2} + x^{1/2} \psi(x) \right] \frac{dx}{x} \]

\[ = -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty \sqrt{\frac{x}{2}} \psi(x) \frac{dx}{x} \]

\[ + \int_1^\infty y^{s/2} \psi(y) \frac{dy}{y} \]

\[ = -\left( \frac{1}{s} + \frac{1}{1-s} \right) + \int_1^\infty \left( y^{s/2} + y^{1/2} \right) \psi(y) \frac{dy}{y} \]

\[ = -\frac{1}{s(1-s)} + \int_1^\infty \left( y^{s/2} + y^{1/2} \right) \psi(y) \frac{dy}{y} \]

So, for \( \Re(s) > 1 \),

\[ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \Gamma(s) = \frac{1}{s(1-s)} + \int_1^\infty \left( y^{s/2} + y^{1/2} \right) \psi(y) \frac{dy}{y} \]

\[ = O(e^{-\pi y}) \quad \text{as} \quad y \to +\infty \]

The integral is analytic for all \( s \in \mathbb{C} \).

The \( \frac{1}{s(s-1)} \) is trivially analytic on \( \mathbb{C} \).
Theorem (Functional Equation)

\[ \Xi(s) \equiv \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) L(s) \] is analytic on 
\( \mathbb{C} \setminus \{0,1\} \) and satisfies
\[ \Xi(s) = \Xi(1-s). \]

We also have for \( \Xi \) :
- \( s = 1 \) simple poles, residue 1
- \( s = 0 \) simple poles, residue -1

\[ \Xi(s) = \frac{1}{(1+h)h} + O(1) \]

\[ = \frac{1}{h} + O(1) \]

And, similarly, with \( s = h \)
\[ \Xi(h) = \frac{1}{h(h-1)} + O(1) = -\frac{1}{h} + O(1). \]
**Cor 1**

\[ \xi_0(s) = s(s-1) \xi(s) = s(s-1) \pi^{-3/2} \Gamma\left(\frac{3}{2}\right) \Gamma(s) \] is an entire fcn which satisfies

\[ \xi_0(s) = \xi_0(1-s), \quad \xi_0(1) = 1. \]

**Pf**

\[ \xi_0(s) = 1 + s(s-1) \int_1^\infty \left( y - \frac{1}{2} + \frac{3}{4} \right) \psi(y) \frac{dy}{y} \quad \text{by (23).} \]

**Cor 2**

\[ \zeta(-ak) = 0 \quad \text{for} \quad k \geq 1 \quad (\text{simple zero}). \]

**Pf**

\[ \xi(x) = \pi^{-1/2} \Gamma\left(\frac{x}{2}\right) \xi(x) > 0 \quad \text{for} \quad x > 1. \quad \text{But} \quad \xi(x) = \xi(1-x). \]

Hence \( \xi(x) > 0 \) for \( x < 0 \). Let \( x \to -2k \).

Since \( \Gamma\left(\frac{x}{2}\right) \to \Gamma(-k) \) simple poles get \( \xi(x) \to 0 \) \( \text{al a} \)

**simple zero.**

Lemma

\[(1 - x^{1-k}) \sum_{k=1}^{\infty} (\alpha^{k+1} - \alpha^k) = \sum_{k=1}^{\infty} (\alpha^{k+1} - \alpha^k)\]

For \(\text{Re}(s) > 1\) and the RHS is actually analytic for \(\{\text{Re}(s) > 0\}\).

**Proof**

If \(\text{Re}(s) > 1\), then

\[
I(s) = \sum_{k=1}^{\infty} (\alpha^{k-1})^{-s} + \sum_{k=1}^{\infty} (\alpha^k)^{-s} \quad \text{trivially}
\]

\[
q^{-s} I(s) = q^{-s} \sum_{m=1}^{\infty} (\alpha^m)^{-s}
\]

The difference is

\[
\sum_{k=1}^{\infty} ((\alpha^{k-1})^{-s} - (\alpha^k)^{-s}) \quad \text{nice abs conv}
\]

Keep \(s \in K\) where \(K\) is a compact subset of \(\{\text{Re}(s) > 0\}\). Observe that:

\[
(\alpha^{k-1})^{-s} - (\alpha^k)^{-s} = (\alpha^k)^{-s} \left[ (1 - \frac{1}{k})^{-s} - (\alpha^k)^{-s} \right]
\]

\[
\left\{ (1 + u)^{-s} = 1 - s u + O_K(1) u^2 \right\} \quad \text{for } u = \frac{3}{4}
\]

\[
(\alpha^{k-1})^{-s} - (\alpha^k)^{-s} = (\alpha^k)^{-s} \left[ \frac{s}{\alpha^k} + O(1) \frac{1}{k^2} \right]
\]

\[
= O(1) k^{-s-1} \quad \text{for } s \in K.
\]
Corollary

In the sense of analytic continuation,

\[ \xi(x) \neq 0 \text{ for } x \in \mathbb{R} \]
\[ \xi_0(x) \neq 0 \text{ for } x \in \mathbb{R} \]
\[ \delta(x) < 0 \text{ for } 0 < x < 1. \]

Proof

That \( \delta(x) < 0 \) on \( 0 < x < 1 \) is obvious from (26). Hence \( \xi(x) \neq 0 \) on \( 0 < x < 1 \). The points \( x = 0, 1 \) are poles and take care of themselves.

For \( x > 1 \) and \( x < 0 \), we have \( \xi(x) > 0 \) à la (25). Since \( \xi_0(x) = x(x-1) \xi(x) \), the assertions for \( \xi_0 \) are immediate. \( \square \)
We wish to bound the size of
\[ F(z) \equiv z(z-1) \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \Gamma(z) \]
(roughly) using Stirling. \( F(z) = F(1-z) \) and basic properties of \( \Gamma(z) \).

Because of \( F(z) = F(1-z) \), we can clearly restrict to \( \Re(z) \geq \frac{1}{2} \).

We had
\[
|f(x+iy)| \leq \frac{C}{\delta(1-\delta)} |y|^{1-\delta} \left\{ \begin{array}{l}
x \geq \delta \\
y \geq 2 \end{array} \right. 
\]
any \( 0 < \delta < 1 \) \( \text{ Lec 6 page 9} \) \( \text{ EG } \delta = \frac{1}{2} \).

Also, we had
\[
|f(z) - 1| < \frac{3}{4} \quad \text{for } \Re(z) > 2
\]
by Lec 5 page 10.
\[ F(z) = z(z-1) \pi^{-\frac{x}{2}} \Gamma\left(\frac{x}{2}\right) \Gamma\left(\frac{x}{2}\right) \]

\[ |F(z)| = |z| |z-1| \pi^{-\frac{x}{2}} |\Gamma\left(\frac{x}{2}\right)| \Gamma\left(\frac{x}{2}\right) | \]

\[ |F(z)| \approx |z|^3 \left[ 1 + O\left(\frac{1}{r}\right) \right] \pi^{-\frac{x}{2}} |\Gamma\left(\frac{x}{2}\right)| \Gamma\left(\frac{x}{2}\right) | \]

Know:
\[ |\Gamma(z)| < \frac{3}{4} \text{ for } x > 2 \]
\[ |\Gamma(x+iy)| \approx O(1/y^{1/2}) \]
\[ x \approx \frac{1}{2}, \quad |y| \approx 2 \]

Also:
\[ \ln |\Gamma\left(\frac{z}{2}\right)| = \text{Re}\left\{ \text{Log} \Gamma\left(\frac{z}{2}\right) \right\} \]
\[ \textit{Stirling} \quad \text{p. 412} \]
\[ \log \pi \left( \frac{\pi}{x} \right) = \left( \frac{\pi}{x} - \frac{1}{2} \right) \log \left( \frac{\pi}{2} \right) - \frac{\pi}{2} + \frac{1}{2} \ln(2\pi) + O \left( \frac{1}{x} \right) \]

\[ \{ \text{For } r \text{ say, } |z| = r \text{, large, } \text{Arg } z \leq \frac{3}{4} \pi \} \]

As in Ingham 56–57, we get

\[ \ln \left( \Gamma \left( \frac{\pi}{2} \right) \right) \approx \frac{\pi}{2} \ln \pi + A_1 r \]

\[ \{ \text{For } r \text{ say, } |z| = r \text{, } \text{Arg } z \leq \frac{3}{4} \pi \} \]

\[ \downarrow \]

\[ \ln |F(re^{i\theta})| \approx \frac{\pi}{2} \ln r + A_2 r \]

\[ \text{For } |z| = r, \text{ } \Re(z) \approx \frac{r}{2} \]

then, using \( F(z) = F(1-z) \), similarly for \( \Re(z) \approx \frac{1}{2} \).
Also, looking at $\Theta = 0$,

$$F(r) = R^2 \left[ 1 + O\left( \frac{1}{R} \right) \right] \pi^{-\frac{R}{2}} \Gamma\left( \frac{R}{2} \right) J(r)$$

$$\geq (\text{constant}) \ R^2 \ \pi^{-\frac{R}{2}} \ \Gamma\left( \frac{R}{2} \right)$$

\[\begin{align*}
\ln \Gamma\left( \frac{R}{2} \right) &\sim \left( \frac{R}{2} - \frac{1}{2} \right) \ln \left( \frac{R}{2} \right) - \frac{R}{2} \\
\downarrow \\
\ln F(r) &\geq \frac{r}{2} \ln r - A_3 \ r \quad (r \ \text{large})
\end{align*}\]

\[\text{THM} \]

Let $F(z) = z(z-1) \pi^{-z/2} \Gamma\left( \frac{z}{2} \right) J(z)$. Let

$$M(r) = \max_{|z| = r} |F(z)| \quad (r \ \text{large})$$

Then:

$$\ln M(r) \sim \frac{r}{2} \ln r \quad$$
Proof

As above.

For any entire func \( g(z) \), \( g \not= 0 \), we write

\[
M(r) = \max_{|z| = r} |g(z)|.
\]

Then put:

\[
\rho = \inf \{ \omega : \frac{\ln M(r)}{r} \leq \omega, \text{all large } r \}
\]

\[
\tau = \inf \{ \beta : \frac{\ln M(r)}{r} \leq \beta r^\rho, \text{all large } r \}
\]

Herein \( \omega \geq 0 \) and \( \beta \geq 0 \). Empty braces mean \( \inf = +\infty \). We call:

\[
\rho = \text{ORDER of } g(z)
\]

\[
\tau = \text{TYPE of } g(z)
\]

For our \( F(z) \), clearly \( \rho = 1 \) and \( \tau = +\infty \).
Lecture 12  
(26 Feb)

2 Notes

Lec 10 p. 42 Stirling (Corollary).

We also have:

Thm (Stirling)

\[ \frac{f'(z)}{f(z)} = \log z - \frac{1}{2z} + \sum_{k=1}^{R} \left( -\frac{B_{2k}}{2k} \right) z^{-2k} + O_{R} \left( \frac{1}{|w|^{2R+1}} \right) \]

as \( z \to \infty \) in \( |\text{Arg } z| \leq \pi - \delta \).

\[ \text{PF} \]

Call the \( \bar{\Omega}_{R+1} \) integral term in 42 Thm \( V(z) \).

Note \( v(z) \) is nicely analytic and \( r(z) = O(z^{-2R-1}) \)

by the Cor on 42. But:

\[ r'(z) = \frac{1}{2 \pi i} \oint_{|w-z|=1} \frac{r(w)}{(w-z)^2} dw \]

Just use \( |\text{Arg } z| \leq \pi - 2 \delta \) in place of \( |\text{Arg } z| \leq \pi - \delta \).

Done. \( \square \)

About \( \Gamma' (z) \neq 0 \), Lec 10 p. 26. One can avoid Hurwitz's Thm.

Thm

Let \( f_{n}(z) \to f(z) \) on \( \{ z \in \mathbb{C} : |z| < R \} \), compacta. We assume \( f_{n}, f \) are analytic. Let \( f_{n}(z) \neq 0 \) for all \( z \) and \( f(z) \neq 0 \). Then \( f(z) \neq 0 \).
Proof

The zeros of $f$ are isolated. Hence finite $n$ on each $\mathbb{D}$. Find $R_n \uparrow R$ so $f(R_n e^{i\theta}) \neq 0$.

Fix any $N$. Find $m, M > 0$ so $m \leq |f(R_n e^{i\theta})| \leq M$. By uniform convexity,

$$\frac{m}{2} \leq |f_n(z)| \leq 2M, \quad n \geq N.$$

Apply maximum modulus principle to $f_n$ and $1/f_n$. Get

$$|f_n(z)| \leq 2M, \quad n \geq N,$$

$$\left|\frac{1}{f_n(z)}\right| \leq \frac{2}{m}$$

on $|z| \leq R_N$. So,

$$\frac{m}{2} \leq |f_n| \leq 2M.$$

Let $N \to \infty$ to get $\frac{m}{2} \leq |f| \leq 2M$ on $|z| \leq R$.

Now let $N \to \infty$. Done.

END OF NOTES

We then turned to the issue of the entire function

$$\zeta_0(s) = s(s-1)\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \xi(s)$$

and trying to get a product expansion over the zeros of $\zeta_0$, trying to get a "Hadamard factorization" of $\zeta_0$ to justify Riemann's
unproved assertion [from 1859]

Standard lemmas.

Lemma 1

\[ D \text{ a simply-connected domain} \]
Let \( f = u + iv \) be analytic on \( D \).
Then \( u \) is harmonic on \( D \) (i.e., \( C^2 \) and \( u_{xx} + u_{yy} = 0 \)).

Conversely, given real-valued \( u \) harmonic on \( D \), we can cook up \( V \), harmonic on \( D \), so \( F = u + iV \) is analytic on \( D \).

\[ \overline{\text{Cor}} \]
Every harmonic \( u \) on \( D \) is actually \( C^\infty \).

Lemma 2 (mean-value property)

Let \( u \) be harmonic on \( D \) (as above).
Let \( |z - z_0| \leq R \) be contained in \( D \).
Then
\[ u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + R e^{i\theta}) \, d\theta \]

Lemma 3

\( D \) as above. Let \( g \) be analytic on \( D \) and \( g(z) \neq 0 \). We can always find an analytic function \( \phi(z) \) on \( D \) such that \( \exp(\phi) = g \).

[\( \phi \) is unique up to \( +2\pi i k \)]
Theorem (Jensen's formula) Lemma 4

Let $D$ as above and let $\{1/|z| \leq R\} \subseteq D$. Let $f$ be analytic on $D$, $f \neq 0$ on $1/|z| = R$, $f(0) \neq 0$.

Then:

$$\ln|f(0)| + \sum_{j \neq 1} \ln \frac{R}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \ln|f(Re^{i\theta})|d\theta.$$ 

Here $a_1, \ldots, a_m$ are the zeros of $f$ in $0 < |z| < R$ listed with multiplicity.

Proof.

Wlog $D = \{1/|z| < R + \varepsilon\}$, $\varepsilon$ tiny.

Wlog $f \neq 0$ on $\{RE/|z| = R + \varepsilon\}$. Form analytic function

$$F(z) = f(z)^m \prod_{j \neq 1} \frac{R^2 - a_j \cdot z}{R(z - a_j)}.$$ 

Get $|F| = |f|^m$ on $1/|z| = R$, $F(z) \neq 0$ on $1/|z| < R + \varepsilon$.

Apply Lemma 3 to get $\log F(z)$. By Lemma 2 + 1,

$$\ln|F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \ln|F(Re^{i\theta})|d\theta.$$ 

Done.

If $f(0) = 0$, people usually just pass to $\frac{f(z)}{z^N}$. 
Theorem (Lemma 5)

Let \( f \) be entire of order \( \leq \rho \) \(( \rho \neq 0)\).
Then, counting with multiplicity,

\[
n(r) = N[\text{zeros of } f \text{ in } |z| \leq r] = O(r^{\rho + \varepsilon})
\]

for all \( r \) large. Here \( \varepsilon > 0 \).

Proof:

If \( f(0) = 0 \Rightarrow \) pass to \( g = \frac{f(z)}{z^N} \). \( g \) is still entire and has order \( \leq \rho \).

WLOG \( f(0) = 1 \). Know \( \ln M(R; f) \leq R^{\rho + \varepsilon} \) large \( R \).

Perturb \( R \) slightly to make \( f(Re^{i\theta}) \neq 0 \).

Apply Lemma 4 (Jensen):

\[
0 + \sum_{j=1}^{m} \ln \frac{R}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(Re^{i\theta})|d\theta \leq R^{\rho + \varepsilon}.
\]

Hence:

\[
n\left( \frac{R}{\alpha} \right) \ln 2 \leq R^{\rho + \varepsilon}
\]

\[
\Rightarrow n(r) = O(r^{\rho + \varepsilon}) \text{ for all large } r \; .
\]
KEY THM (Lemma 6, Hadamard/Borel/Caratheodory)

Let $D$ as above. $f$ analytic on $D$. Suppose $|z-z_0| \leq R^2 \subseteq D$. Let $f = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ on the closed disk.

Assume further that

$$\Re f(z) \leq M$$

on the closed disk. Then:

(A) $|c_n| \leq \frac{2}{R^n} (M-\Re c_0), \quad n \geq 1$

(B) $|f(z) - f(z_0)| \leq \frac{2R}{R-1} \left\{ \frac{M-\Re c_0}{1} \right\} |z-z_0| \leq r \leq R$

(C) $\left| \frac{f(z)}{k!} \right| \leq \frac{2R}{(R-r)^{k+1}} \left\{ \frac{M-\Re c_0}{k!} \right\}, \quad k = 1, 2, \ldots$

PF

See Ingham 50-51. There is a famous trick in this proof (starts 50 bottom).

Lemma 7

Suppose we have an entire fn $f$ such that $f(0) \neq 0$. Let its zeros (listed with multiplicity) be $\{a_j\}$. WLOG $|a_j| \leq |a_{j+1}|$. Assume that we know $\|u(r) = O(r^\beta) \text{ for large } r$. Then:

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} \gamma < \infty \quad \text{for each } \gamma > \beta$$
\textbf{Pf}

Take \( \delta \) tiny, look at \( \int_0^r r^{-y} \, dn(r) \) and integrate by parts.

\[ \sum \frac{1}{|a_n|^p + \varepsilon} < \infty, \text{ each } \varepsilon > 0. \]

\textbf{Corollary}

Let \( f \) be entire and \( f(0) \neq 0 \). Then:

\[ \sum \frac{1}{|a_n|^{\frac{p}{2}+1}} < \infty. \]

\textbf{Pf}

Lemma 5 + 7.

Thus, we always have (for \( f(z) \) entire)

\[ \sum \frac{1}{|a_n|^{\frac{p}{2}+1}} < \infty. \]

We let

\[ p = \frac{\|f\|}{\|f'\|} \]

when we play with a given \( f_0 \).

Do not confuse \( p \) with a prime.
When \( p = \text{non-neg integers} \) following Weierstrass it is customary to define:

\[
E(u, j, p) = \begin{cases} 
 1 - u, & p = 0 \\
 1 - u \exp \left[ u + \frac{u^2}{2} + \cdots + \frac{u^p}{p} \right], & p \geq 1
\end{cases}
\]

Note that \( E(z, j, p) \) is entire.

Take \( |u| \leq \lambda < 1 \). With some branch of \( \log \)

\[
\log E(u, j, p) = \log (1 - u) + u + \cdots + \frac{u^p}{p}
\]

\[
= -\sum_{n=1}^{\infty} \frac{u^n}{n} + u + \cdots + \frac{u^p}{p}
\]

\[
= -\sum_{n=p+1}^{\infty} \frac{u^n}{n}
\]

Clearly,

\[
|\log E(u, j, p)| \leq \frac{|u|^{p+1}}{1 - |u|} \quad (p = 0 \quad \text{OK too}).
\]

Hence:

\[
\ln |E(u, j, p)| \leq \frac{|u|^{p+1}}{1 - |u|}, \quad |u| \leq \lambda < 1.
\]
Given \( p \geq 0 \). Also given \( a_n \in C - \{0\} \), \( a_n \to \infty \), and

\[
\sum_{n} \frac{1}{|a_n|^{p+1}} < \infty.
\]

We call

\[
\prod_{n=1}^{\infty} E\left( \frac{z}{a_n} \cdot p \right)
\]

a CANONICAL PRODUCT of genus \( p \).

**THM**

In the above, the canonical product of genus \( p \) converges uniformly + absolutely on \( C - \text{compacta} \). Hence it is an entire function with zeros exactly at \( \{a_n\} \).

**PF**

We use our standard reduction to the "\( \sum_{k=1}^{\infty} \log(1+b_k(z)) \) theorem".

Take \( K = \{ |z| \leq R \} \). Restrict attention to \( |a_n| > 1000R \). Hence, in products each term has

\[
|\frac{z}{a_n}| < \frac{1}{1000} \text{ for } z \in K.
\]
Get:

$$\left| \log E\left( \frac{z}{an} ; p \right) \right| \leq \left| \frac{\frac{\tilde{z}}{an}}{1 - \frac{1}{1000}} \right|$$

$$= (1.01) \left| \frac{\tilde{z}}{an} \right|^{p+1}$$

$$= (1.01) \left( \frac{1}{1000} \right)^{p+1}$$

$$\ll 0.002$$

Therefore, the "log" is actually \( \log \).

And:

$$\left| \log E\left( \frac{z}{an} ; p \right) \right| \approx 0.002$$

$$\left| E\left( \frac{z}{an} ; p \right) - 1 \right| \approx 0.01$$

i.e. "\( |bn(z)| \approx 0.01" \ (on \ K) ."

Must look at

$$\sum_n \left| \log (1+bn(z)) \right|$$

on \( K \). This sum will be

$$\ll \sum_n (1.01) \left( \frac{R}{|an|} \right)^{p+1} \ \{ \text{by the above} \}$$

for all \( z \in K \). \( \Rightarrow \) all is OK. \( \square \)
When we study canonical products, it is helpful to conceptualize them as

\[ \prod E(\frac{\pi}{an+jp}) \equiv \prod_{|an| \leq 1000R} E(\frac{\pi}{an+jp}) \]

\[ \prod_{|an| > 1000R} E(\frac{\pi}{an+jp}) \]

over \( \{IZ1 \leq R \} \).

\[ \triangleleft \text{This part is NONZERO} \]

**Theorem (preliminary factorization)**

Let \( f \) be entire, let the order be \( \rho < \infty \). Put \( \rho = \|f\|_{\infty} \), let the \textit{zeros} of \( f \) in \( \mathbb{C} - \{0\} \) be \( \{an\} \). [This set could be empty.]

We then have:

\[ f(z) = z^N \exp[\phi(z)] \prod_{an \neq 0} E(\frac{\pi}{an+jp}) \]

where \( \phi \) is some entire function and where the product (if infinite) is \textit{absolutely uniformly} convex on \( \mathbb{C} \) compact.
Pf

Pass first to \( g(z) \equiv \frac{f(z)}{z^N} \), as usual.

The fun \( g \) is entire, order \( p \), \( g(0) \neq 0 \).

Now review (7) and form

\[
h(z) = \frac{g(z)}{\prod_{n \neq 0} E(\frac{z}{a_n}, p)}
\]

See (1) top. Get \( h(z) \neq 0 \) for all \( z \in \mathbb{C} \).

By Lemma 3 applied to \( h(z) \), we can write \( h = \exp(\phi(z)) \). Done. \( \square \)

Hadamard realized, in studying Riemann's work, that he needed some way of controlling \( \phi(z) \) using only information about \( \Re \phi(z) \).

This is what led to p. 6 Key Thm!
Hadamard's Factorization Theorem \( \sim 1893 \)

Given the situation of \( p \geq 1 \) then we have that \( \phi(z) \) must be a polynomial of degree \( \leq p \).

(Recall that \( p = \| p \| \).)

In the case of \( \Xi_0(s) \), we had \( p = 1 \) and type \( \tau = \infty \). So, here,

\[
\Xi_0(s) = e^{As+B} \prod E(\frac{5}{\alpha_n}; 1).
\]

(proof)

The proof of the HFT either follows an approach of Landau or else one based on the so-called Poisson–Jensen formula (a very common identity used in Nevanlinna theory II). The proof is function theory, not number theory.
The Landau approach is remarkable for its simplicity. See:

Landau, Vorlesungen "über Zahlentheorie",
Satz 423 (from 1927)


INGHAM, pages 54(bottom) - 55(bottom)
is compressed, but follows LANDAU.

I presented the details of this in Lecture #12 and the first half of Lecture #13. I will omit them here!
Lecture 13
(2 Mar 2016)

I began by finishing the proof of the Hadamard Factorization theorem.

I then turned to some simple function-theoretic facts and some corollaries of HFT.

**Simple Facts**

1. \( f \) order \( \rho \), \( g \) order \( \rho' \) \( \Rightarrow \) \( f + g \) order \( \rho \)
2. \( f \) order \( \rho_1 \), \( g \) order \( \rho_2 \) \( \Rightarrow \) \( fg \) order \( \leq \max \{ \rho_1, \rho_2 \} \)
3. \( p(z) \neq 0 \) polynomial, \( f \) order \( \rho \) \( \Rightarrow \) \( p(z)f(z) \) order \( \rho \) too
4. \( f \) order \( \rho \). Zeros at \( \{ a_1, \ldots, a_m \} \). Then \( g(z) = \frac{f(z)}{(z-a_1) \cdots (z-a_m)} \) order \( \rho \)
5. Let \( f \) have order \( \rho \). Let \( z_0 \in \mathbb{C} \). Then: \( h(z) = f(z + z_0) \) has order \( \rho \).

\{ Must fiddle and use max mod principle? \}
VI. Let \( f \) be entire, even, and order \( \rho \).

Then
\[
g(z) = f(z^{1/2}) \quad \text{has order} \quad \rho/2.
\]

---

**Cor 1 to HFT**

Let \( f \) be entire, order \( \rho \), \( \rho \not\in \mathbb{Z} \).
Let \( a \in \mathbb{C} \). Then \( f(z) = a \) has infinitely many roots.

**Proof**

\[ \rho \not\in \mathbb{Z} \implies \rho > 0. \]
Let \( g = f(z) - a \) has order \( \rho \). Suppose \( g(z) \) has only finitely many zeros \( \rho(g) = \rho, \rho = \lfloor \rho \rfloor \).

Apply HFT.

\[
g(z) = z^{-\rho} e^{\phi(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, \rho\right), \quad \| \phi \| < \infty
\]

But \( \rho < \rho < \rho + 1 \). RHS has order at most \( \rho \) since \( \deg \phi \leq \rho \). (Recall def of \( E \).) Let \( \rho < \rho \).

Contradiction. \( \square \)
Cor 2 to HFT  (a form of the baby Picard thm)

Let \( f \) be entire, order \( \rho > 0 \), \( \rho \in \mathbb{Z} \).

Then \( f(z) \) assumes every \( a \in \mathbb{C} \) with AT MOST ONE exception.

\[ P \]
Suppose 2 exceptions: \( f \neq \alpha, f \neq \beta \). Write

\[ g(z) = \frac{f(z) - \alpha}{\beta - \alpha} \]

Order \( \rho \) again. And \( g \neq 0, 1 \).

Apply HFT. Get \( g = \exp(\phi) \), \( \phi = \text{polynomial} \),

degree \( \leq \rho \). HERE \( \rho = \rho' \).

But, order of \( g(z) \) is \( \rho = \rho' \), so \( \deg \phi = \rho' \).

Since \( \rho' \geq 1 \), we can solve \( \phi(z) = 2k\pi i \) for any integer \( k \). Thus, we get many points \( z_k \) where \( g(z_k) = 1 \). Contrad!  

\[
\text{WE NOW GO TO } \xi_0(z) \]

\( \xi_0 \)
**THM**

Recall $\Xi_0(s) = 5(s-1) \pi^{-s/2} \Gamma(s/2) \Gamma(1/2)$. Also Lec 11 p. 25 and 31. We have $\Xi_0(s) = \Xi_0(1-s)$, $\Xi_0(1) = 1$. Also order $p = 1$ and type $r = \infty$.

$\Xi_0(s)$ has **infinitely many zeros**; these lie exclusively in $\{0 < \text{Re}(s) < 1/2\}$.

**PF**

Let $f(z) = \Xi_0(\frac{1}{2} + iz)$, so $F$ has order 1 AND $f(z)$ IS **EVEN**. Form $g(z) = f(z^{1/2})$, which has order $1/2$. Page 2 top.

By Cor 1, $g$ has **infinitely many zeros**. Hence so does $f$, and hence $\Xi_0$.

By Lec 11 p. 27, $\Xi_0 \neq 0$ along the real axis.

For $\text{Re}(s) > 1$, we know $5(s-1) \neq 0$, $\pi^{-s/2} \neq 0$,

$\Gamma(s/2) \neq 0$, $\Gamma(s) = \pi^{1-s} / \Gamma(1-s) \neq 0$.

The same is true for $s = 1 + it$, $t \text{ real} \neq 0$.

(Recall $5(1 + it) \neq 0$ was proved Lec 6 p. 7.)
Hence, \( \xi_0(s) \neq 0 \) for all \( \Re(s) \geq 1 \).

By \( \xi_0(s) = \xi_0(1-s) \), get same for \( \Re(s) \leq 0 \).

So, all zeros lie in \( 0 < \Re(s) < 1 \). \( \square \)

\( \xi_0(1) = \xi_0(0) = 1 \), \( \xi_0(s) = \xi_0(1-s) \).

HFT now implies

\[
\xi_0(s) = e^{A + Bs} \prod_{\rho} \left(1 - \frac{s}{\rho} \right) e^{\frac{s}{\rho}}
\]

\( 0 < \Re(\rho) < 1 \).

It might be better to use a letter other than \( \rho \) (to avoid confusion with order \( \rho \)).

But "everyone" uses \( \rho \) for these zeros, Riemann, Landau, etc etc.

Order = 1, \( \|\text{Order}\| + 1 = 2 \) for \( \xi_0 \).

So,

\[
\sum_{\rho} \frac{1}{|\rho|^2} < \infty.
\]
We had $\Xi_0(x) \neq 0$ for $x \in \mathbb{R}$, so

$\Im(\rho) \neq 0$.

Recall that:

$$\Xi_0(s) = 1 + s(s-1) \int_1^\infty \left[ y^{\frac{1-s}{2}} + y^{\frac{s-1}{2}} \right] \psi(y) \frac{dy}{y}$$

\[ = \sum_{n=1}^{\infty} e^{-\pi n^2 y} \]

*a la Lec 11, p. 25*. Clearly

$$\Xi_0(s) = \Xi_0(\overline{s})$$

for all $s \in \mathbb{C}$.

Hence any zeros occur in conjugate pairs. The *canonical product* on (6) is thus real-valued for $s \in \mathbb{R}$. So is $\Xi_0(s)$. As such, we conclude $B$ must be real.

Letting $s \to 0$ on (6) middle, we get $e^A = 1$, so WLOG $A = 0$. 
With some branches,

\[ \log \xi_0(s) = A + B s + \sum_{\rho} \log \left\{ \frac{1}{\rho} e^{\frac{s}{\rho}} \right\}. \]

Keep \( s \) on some \( \mathcal{C} \) at first!

Following Riemann, get:

\[ \frac{\frac{\xi'}{\xi}(s)}{\xi_0(s)} = B + \sum_{\rho} \left\{ \frac{1}{s - \rho} + \frac{1}{\rho} \right\} \]

with nice convergence on \( \mathcal{C} \)-compacta away from the \( \rho \)'s.

(Weierstrass conv theorem)

But, \( \odot \) line 2:

\[ \log \xi_0(s) = \log s + \log (s-1) - \frac{s}{2} \ln \pi + \log \Gamma\left(\frac{s}{2}\right) + \log S(s). \]

\[ \frac{\xi_0(s)}{\xi_0(s)} = s \frac{\log s}{\log (s-1)} - \frac{s}{2} \ln \pi + \frac{1}{2} \frac{\Gamma'(\frac{s}{2})}{\Gamma\left(\frac{s}{2}\right)} + \frac{S(s)}{S(s)}. \]
\[ \Gamma'(z+1) = z \Gamma(z) \]

\[ \frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{1}{z} + \frac{\Gamma'(z)}{\Gamma(z)} \]

\[ \frac{\xi_0'(s)}{\xi_0(s)} = \frac{1}{s-1} - \frac{1}{2} \ln \pi + \frac{1}{2} \frac{\pi}{\pi} \left( \frac{s}{2} + 1 \right) + \frac{\xi_0'(s)}{\xi_0(s)} \]

On the bottom, if desired, one could also substitute

\[ \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{z+n} \right) \]

From Lec 10 p. 3D, we'll skip this for now.

Combine line 5 with line 3 above. Get:

\[ \frac{\xi_0'(s)}{\xi_0(s)} = \frac{1}{s-1} + \left( B + \frac{1}{2} \ln \pi \right) - \frac{1}{2} \frac{\pi}{\pi} \left( \frac{s}{2} + 1 \right) + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right) \]
Thm (Riemann)

Away from the set \( \{1\} \cup \{\rho \} \cup \{ -2k \}_{k=1}^{\infty} \)
we have

\[
\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + (B + \frac{1}{2} \ln \pi) - \frac{1}{2} \frac{\pi'}{\pi} \frac{s}{2} + 1 + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right) .
\]

Pf

As above. \( \square \)

---

\[3 + 4 \cos \theta + \cos 2\theta = 2(1 + \cos \theta)^2 \geq 0.\]

\[-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} n^{-it}\]

\[\Re \left[ -\frac{\zeta'(s)}{\zeta(s)} \right] = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \cos (t \ln n) = \Re (s) > 1.\]
Fact
\[
\sigma > 1, \ t \in \mathbb{R}, \ t \neq 0
\]
\[
\operatorname{Re} \left[ 3 \frac{s}{s^2} + 4 \frac{s}{s^2} (s+it) + \frac{s}{s^2} (s+2it) \right] \leq 0.
\]

Proof
As above.

Abbreviate Thm on \( \rho \) as
\[
\frac{\xi(s)}{\zeta(s)} = -g(s) + f(s)
\]
\[
\sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right)
\]

For a few moments. Keep \( \sigma > 1 \).

We immediately get:
\[
\operatorname{Re} \left[ 3f(\sigma) + 4f(\sigma+it) + f(\sigma+2it) \right]
\]
\[
\leq \operatorname{Re} \left[ 3g(\sigma) + 4g(\sigma+it) + g(\sigma+2it) \right],
\]
\( t \neq 0 \) real, \( \sigma > 1 \).
Theorem ( classical zero-free region )

There exists an absolute constant \( a > 0 \) so that

\[ f(s) \neq 0 \quad \text{on} \quad \{ \sigma > 1 - \frac{a}{\log(1 + t)} \} \]

Proof

Essentially following INGHAM 59–60.

WLOG \( t > 0 \).

We can always re-adjust "a" to take care of bounded \( t \). So, WLOG, we need only look at the case of \( t > 6 \), 6 = giant.

We play with \( 10 \) bottom for \( 1 < \sigma < 3 \),

\( t > 6 \). Remember that

\[ g(s) = \frac{1}{s - 1} - b + \frac{1}{2} \rho \left( \frac{s}{2} + 1 \right) \]

for some real constant \( b \). Hence \( g(s) \) is rather explicit.
Recall Stirling's for $\frac{n!}{n^n}$; Lec 12 p. 10.

We clearly get

$$g(s) = O(\ln t) \quad 1 < \text{Re}(s) < 3$$

anytime $\text{Im}(s) \geq 100$, say.

Of course, for $1 < \sigma < 3$,

$$g(\sigma) = \frac{1}{\sigma - 1} + O(1)$$

On (D) bottom, we get:

$$\text{Re} \left[ 3f(\sigma) + 4f(\sigma + it) + f(\sigma + 2it) \right]$$

$$\leq \frac{3}{\sigma - 1} + A \ln t \quad \left\{ \begin{array}{l} 1 < \sigma < 3 \\ t > 6 \end{array} \right.$$.

Here:

$$f(s) = \sum_{\rho} \left\{ \frac{1}{s - \rho} + \frac{1}{\rho} \right\} \quad \leftarrow 9 \text{ line 5}$$

For $\sigma > 1$ and $\rho = \beta + iy$, note that

$$\text{Re} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) = \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - y)^2} + \frac{\beta}{\beta^2 + y^2} \geq 0$$

...
Consider now any zero $\rho_0 = \beta_0 + i\gamma_0$ of $\Xi_0$
with $\gamma_0 > G$.

Apply (12) lines 8 + 9 keeping (12) bottom in mind.

Get:

$$4 \frac{\frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (t - \gamma_0)^2}}{\gamma} = \frac{3}{\sigma - 1} + A \log(\tau)$$

all $1 < \sigma < 3$, $\tau > G$.

---

Notice that

$LHS \leq \frac{4}{\sigma - \beta_0}$

trivially for ANY $\sigma > 1$. For $\sigma = 3$ get

$LHS \leq \frac{4}{3 - \beta_0} \leq 2$.

---

By revising $A$, we can thus say

$$4 \frac{\frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (t - \gamma_0)^2}}{\gamma} \leq \frac{3}{\sigma - 1} + A \ln(\tau)$$

for ANY $\sigma > 1$ and $\tau > G$. (And any $\rho_0$ with $\gamma_0 > G$)
Put $t = \gamma_0$ to see that
\[
\frac{4}{\sigma - \beta_0} \leq \frac{3}{\sigma - 1} + \lambda \ln \gamma_0
\]
for all $\sigma > 1$.

Let $\sigma \to 1$ to see that $\beta_0 = 1$ is impossible (a fact we already know).

One expects that $\sigma - \beta_0$ and $\sigma - 1$ of comparable size will be most illuminating. Take:
\[
\sigma = 1 + \lambda(1 - \beta_0), \quad \lambda > 0
\]
\[
\sigma - \beta_0 = \sigma - 1 + 1 - \beta_0 = (\lambda + 1)(1 - \beta_0)
\]
\[
\downarrow
\]
\[
\frac{4}{(1 + \lambda)(1 - \beta_0)} - \frac{3}{\lambda(1 - \beta_0)} \leq \lambda \ln \gamma_0
\]
for any $\lambda > 0$.

For large $\lambda$, obviously $\frac{4}{1 + \lambda} - \frac{3}{\lambda} > 0$ since $4 > 3$. 
\( a = 4 \) gives

\[
\frac{0.80 - 0.75}{1 - \beta_0} \leq \frac{\ln \gamma_0}{A}
\]

\[
1 - \beta_0 \geq \frac{0.05}{A (\ln \gamma_0)}.
\]

Hence,

\[
\beta_0 \leq 1 - \frac{0.05}{A (\ln \gamma_0)}.
\]

This is sufficient to prove Thm. on p. 11.
Lecture 14 Synopsis
(4 Mar 2016)

The aim in this lecture was to develop standard estimates for $\psi(x) - x$ and $\pi(x) - li(x)$ based on given zero-free regions for $I(s)$. Ingham 60–67.

I began by recalling the Hadamard/Bovey/Caratheodory lemma from Ingham 50.

I then turned to the development of Ingham’s general estimate on $\frac{I'(s)}{I(s)}$ for a given zero-free region $0 > 1 - \eta(\delta l)$. See Ingham 60–62.

\[ 0 < \eta(t) \leq \frac{1}{2} \]
\[ \eta(t) \text{ decreasing on } [0, \infty) \text{, } C^1 \]
\[ \eta'(t) \to 0 \text{ as } t \to \infty \]
\[ \eta(t) \geq \frac{1}{6 \ln t} \text{ for } t \text{ large } (G = \text{big constant}) \]

\[ \frac{I'(s)}{I(s)} = O(\ln^2 |t|) \text{ on } 0 > 1 - \eta(\delta l) \]

For $|t|$ large and any $0 < \gamma < 1$.\]
One can then use \((c > 1)\)

\[
\psi_1(x) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{x^{s+1}}{s(s+1)} \left[ -\frac{\psi'(s)}{\psi(s)} \right] ds
\]

\((x \geq 1)\) and begin to do contour shifts over to the left, beyond \(s = 1\), using the Cauchy Residue Theorem. Here one wants to move the path of integration over to

\[
\sigma = 1 - \alpha \gamma(1/\alpha)
\]

for a fixed \(0 < \alpha < 1\). \{due to bottom\}

Ingham 62(bottom) - 63 gets

\[
\psi_1(x) = \frac{x^2}{2} + O(x^2 e^{-\alpha \omega(x)})
\]

\[
\omega(x) \equiv \min_{t \geq e} \left[ \eta(t) \ln x + \ln t \right]
\]

\(\uparrow\) I prefer "e" over Ingham's "1"

The introduction of \(\omega(x)\) is slightly "slick".

Classical estimates simply did "each \(\eta(t)\)"
separately, using whatever technique was natural.

Concerning \( w(x) \), I noted:

**Lemma**

Keep \( x \geq 1 \). Then:

(a) \( w(x) \) strictly \( \uparrow \)

(b) \( \ln x - w(x) \) strictly \( \uparrow \)

(c) \( 1 < w(x) < 1 + \ln x \) \( \quad x > 1 \).

**Proof**

let \( 1 \leq x_2 < x_1 \). Let \( w(x_2) \) "occur" for \( t_2 \). \( w(x_1) \) "occur" for \( t_1 \). Get:

\[
w(x_1) \leq \eta(t_2) \ln x_1 + \ln t_2 \quad \text{a priori}
\]

\[
= \eta(t_2) \ln x_2 + \ln t_2 + \eta(t_2) \left[ \ln x_1 - \ln x_2 \right]
\]

\[
\leq w(x_2) + \frac{1}{\alpha} \left( \ln x_1 - \ln x_2 \right) \quad \text{see p. 0}
\]

\[
< w(x_2) + \ln x_1 - \ln x_2
\]

\[
\Rightarrow w(x_1) - \ln x_1 < w(x_2) - \ln x_2
\]

\[
\Rightarrow \ln x_1 - w(x_1) > \ln x_2 - w(x_2) \quad \text{i.e. (b)}.
\]
Similarly:
\[ \omega(x_2) \leq \eta(t_1) \ln x_2 + \ln t_1 \quad \text{a priori} \]
\[ < \eta(t_1) \ln x_1 + \ln t_1 = \omega(x_1) \]
\[ \Rightarrow \omega(x) \text{ strictly } \uparrow \quad \text{i.e., (a).} \]

And,
\[ \omega(1) = 1 \quad \text{by def} \]
\[ \text{but } \omega(x) \uparrow \text{ strictly, so } \omega(x) > 1 \quad (x > 1) \]
\[ \text{and, } \ln x - \omega(x) \uparrow \text{ strictly, so } \ln x - \omega(x) > -1 \quad (x > 1) \]

IE, for \( x > 1 \),
\[ 1 < \omega(x) < \ln x + 1 \quad \text{This is (c).} \]

Theorem

Given \( \eta(1/t) \) as above. For large \( x \), we have
\[ \psi(x) = x + O \left[ x e^{-\frac{\alpha}{2} \omega(x)} \right] \]
\[ \pi(x) = \text{li}(x) + O \left[ x e^{-\frac{\alpha}{2} \omega(x)} \right] . \]
Here \( 0 < \alpha < 1 \).
Essentially like Ingham 64-65.

Keep \( x \geq 1000 \) say. Take \( 0 < h < \frac{x}{2} \). Know

\[
\frac{1}{h} \int_{x-h}^{x} \psi(u)du \leq \psi(x) \leq \frac{1}{h} \int_{x}^{x+h} \psi(u)du
\]

\[
\frac{\psi(x) - \psi(x-h)}{h} \leq \psi(x) \leq \frac{\psi(x+h) - \psi(x)}{h}
\]

\[
\begin{cases}
\text{here } x-h > \frac{x}{2} \geq 500 \\
\text{ } \\
x+h < \frac{3}{2}x
\end{cases}
\]

Let's look at upper side first

\[
\psi(x) \leq \frac{1}{h} \left[ \frac{(x+h)^2 - x^2}{2} + o\left[ (x+h)^2 e^{-\omega(x+h)} \right] \
+ o\left[ x^2 e^{-\omega(x)} \right] \right]
\]

\[
\begin{cases}
\omega(u) \text{ strictly } \uparrow \\
\end{cases}
\]

\[
\psi(x) \leq x + \frac{h}{2} + \frac{o\left[ x^2 e^{-\omega(x)} \right]}{h} + o\left[ x^2 e^{-\omega(x)} \right]
\]

\[
\Rightarrow \psi(x) \leq x + \frac{h}{2} + \frac{1}{h} o\left[ x^2 e^{-\omega(x)} \right]
\]
Next, do lower side; get

\[ \psi(x) \geq x - \frac{h}{2} - \frac{1}{h} \left[ O(x^2 e^{-\alpha \omega(x)}) + O(x e^{-\alpha \omega(x)}) \right] \]

\[ x - h > \frac{x}{2}, \text{ and} \]
\[ \omega(u) \text{ is strictly increasing.} \]

\[ \begin{cases} 
\ln u - \omega(u) \uparrow \text{strictly} \Rightarrow \\
\ln x - \omega(x) > \ln \frac{x}{2} - \omega(\frac{x}{2}) \\
\omega(\frac{x}{2}) > \omega(x) - \ln 2 \\
e^{-\omega(\frac{x}{2})} < \frac{1}{2} e^{-\omega(x)} < \frac{1}{2} e^{-\omega(x)}
\end{cases} \]

\[ \Rightarrow \psi(x) \geq x - \frac{h}{2} - \frac{1}{h} \left[ x^2 e^{-\alpha \omega(x)} \right] \]

So,

\[ \psi(x) = x + O(h) + O\left[ \frac{1}{h} x^2 e^{-\alpha \omega(x)} \right] \]

Put

\[ h = \frac{x}{3} e^{-\frac{x}{2} \omega(x)}, \quad \text{say.} \]

(cf.)

(3) Lemma (c)
Get:
\[ \psi(x) = x + 0 \left[ x e^{-\frac{x}{2} \omega(x)} \right] \]

Recall Lec 1 p.4 middle • Then define:
\[ \Pi(x) = \sum_{2 \leq n \leq x} \frac{\Lambda(n)}{\ln n} \quad (x \geq 2) \]
\[ = \sum_{p^m \leq x} \frac{1}{m} \]
\[ = \pi(x) + \frac{1}{2} \pi(x^{\frac{1}{2}}) + \frac{1}{3} \pi(x^{\frac{1}{3}}) + \cdots \]

Ingham p.16

Note that
\[ x^{-\frac{1}{2}} < 2 \quad \text{for} \quad M = \left\lfloor \frac{\ln x}{\ln 2} \right\rfloor + 10 \]

Get:
\[ \Pi(x) = \int_c^x \frac{1}{\ln t} \, d\psi(t) \quad 1 < c < 2 \]
\[ = \frac{\psi(t)}{\ln t} \bigg|_c^x - \int_c^x \psi(t) \, d \left( \frac{1}{\ln t} \right) \]
\[
\psi(x) = \frac{\psi(x)}{\ln x} - O - \int_c^x \frac{\psi(t) \psi(t)^{-1}}{(\ln t)^2} \frac{1}{t} \, dt
\]

\[
= \frac{\psi(x)}{\ln x} + \int_c^x \frac{\psi(t)}{t(\ln t)^2} \, dt
\]

Let \( c \to 2 \) to get

\[
\Pi(x) = \frac{\psi(x)}{\ln x} + \int_2^x \frac{\psi(t)}{t(\ln t)^2} \, dt
\]

Of course, we also have

\[
\ln^*(x) = \int_2^x 1 \frac{1}{\ln u} \, du \quad \text{by def (Ingham p.3)}
\]

\[
= \frac{x}{\ln x} - \frac{2}{\ln 2} - \int_2^x u \, d\left(\frac{1}{\ln u}\right)
\]

\[
= \frac{x}{\ln x} - \frac{2}{\ln 2} + \int_2^x \frac{u}{u(\ln u)^2} \, du
\]

So,

\[
\Pi(x) - \ln^*(x) = \frac{\psi(x)}{\ln x} - x + \frac{2}{\ln 2} + \int_2^x \frac{\psi(t) - t}{t(\ln t)^2} \, dt
\]

which is a very useful identity, clearly.
We get:

\[ \beta \equiv \frac{x}{2} \quad 0 < \beta < \frac{1}{2} \]

\[ |T(x) - 1| \leq O(1) + \frac{0 \left[ xe^{-\beta \omega(x)} \right]}{\ln x} \]

\[ + \int_{2}^{x} \frac{0 \left[ te^{\beta \omega(t)} \right]}{t (\ln t)^2} \, dt \]

\{ the implied constant needs inflation for small \( t \) \}

\[ \leq 0 \left[ xe^{-\beta \omega(x)} \right] + O(1) \]

\[ + O(1) \int_{2}^{x} e^{-\beta \omega(t)} \, dt \]

\[ \begin{align*}
&\left\{ \begin{array}{l}
\omega(t) < 1 + \ln t \quad \text{p.}\, (3) \\
x e^{-\beta \omega(x)} \geq x e^{-\beta(1+\ln x)} \\
\quad = e^{-\beta} x^{1-\beta}
\end{array} \right. \end{align*} \]

\[ \leq 0 \left[ xe^{-\beta \omega(x)} \right] + O(1) \int_{2}^{x} e^{-\beta \omega(t)} \, dt \]
\( \{ \text{but } \ln u - w(u) \not\to \text{ strictly } p \} \)

\[
\leq 0 \left[ x e^{-\beta w(x)} \right] + \int_{2}^{x} O(1) e^{\beta (\ln t - w(t))} \frac{dt}{t^\beta}
\]

\[
\leq 0 \left[ x e^{-\beta w(x)} \right] + \int_{2}^{x} O(1) e^{\beta (\ln t - w(t))} \frac{dt}{t^\beta}
\]

\[
\leq 0 \left[ x e^{-\beta w(x)} \right] + O(1) x^\beta e^{-\beta w(x)} \left[ \frac{t^{1-\beta}}{1-\beta} \right]_{2}^{x}
\]

\[
\leq 0 \left[ x e^{-\beta w(x)} \right] + O(1) x^\beta e^{-\beta w(x)} \frac{x^{1-\beta}}{1-\beta}
\]

\[
\leq 0 \left[ x e^{-\beta w(x)} \right] .
\]

So,

\[
\Pi(x) - \Pi'(x) = 0 \left[ x e^{-\frac{3}{2} \beta w(x)} \right] .
\]

But

\[
\Pi(x) - \pi(x) = \sum_{m=2}^{\infty} \frac{1}{m} \pi(x/\mu_m) \quad \text{see } 7
\]

\[
= 0 \left[ \frac{x^{1/2}}{\ln x} \right] + O \left[ M x^{1/3} \right]
\]

\[
M = \frac{\ln x}{\mu_1^2} + 10
\]

\[
= 0 \left[ \frac{x^{1/2}}{\ln x} \right] .
\]
Hence, for large $x$,

$$
\pi(x) - \lambda(x) = 0 \left[ x e^{-\frac{x}{2} \omega(x)} \right]
$$

\[\text{noting } \theta \text{ 2 lines from bottom}\]
So, by Lec 13 p. 11 and p. 4. Then above,

\[
\psi(x) = x + O\left[xe^{-c\sqrt{\ln x}}\right]
\]
\[
\pi(x) = \psi(x) + O\left[xe^{-c\sqrt{\ln x}}\right]
\]

for suitably small \( c > 0 \).

The estimates in the box are the famous classical estimates of de la Vallée Poussin \( \sim 1899 \).

---

**Example II**

Assume the Riemann Hypothesis, i.e., \( \text{Re}(\rho) = \frac{1}{2} \) for all zeros of \( \xi_{\theta}(s) \).

Here \( \eta(t) = \frac{1}{2} \).

\[
\omega(x) = \min_{t \leq e} \left\{ \frac{1}{2} \ln x + \ln t \frac{3}{2} \right\} = \frac{1}{2} \ln x + 1 \triangleq 2
\]
In this situation, we get

\[ \psi(x) = x + O\left[ x e^{-\frac{\alpha}{2} \ln x} \right] \]

\[ \approx x + O\left[ x^{1 - \frac{\alpha}{4}} \right] \quad \alpha = 1 - 4\varepsilon \]

\[ \approx x + O\left[ x^{\frac{3}{4} + \varepsilon} \right] \quad \varepsilon = \frac{1}{4} (1 - \alpha) \]

\[ \pi(x) = \text{li}(x) + O\left[ x^{\frac{3}{4} + \varepsilon} \right] \]

WE EXPECT AN EXPONENT MORE LIKE \( \frac{1}{2} + \varepsilon \) \, NOT \( \frac{3}{4} + \varepsilon \) \, \text{(Under RH)} \]

To fix this, we must use a more refined technique. The idea on page 2 top is too crude! Not enough structure!!

Riemann recognized this fact. I.E., a need for a more explicit formula for \( \psi_1(x) \).
Before starting, I noted a simple lemma having relevance to \( f_0(z) \).

**Lemma**

Let \( f(z) \) be entire, with order \( \rho \in [1,2) \).

Let \( \{a_n\} \) be the nonzero zeros of \( f \) (listed with multiplicity). We must then have

\[
\sum_{n} \frac{1}{|a_n|} = +\infty
\]

if either

(a) \( 1 < \rho < 2 \)

(b) \( \rho = 1 \) but type \( \gamma = +\infty \).

**Pf**

Apply Hadamard factorization. \( \rho = \sum p_j \gamma_j = 1 \).

\[
f(z) = \sum_{\gamma} e^{Q(z)} \prod_{n \in \gamma} E\left( \frac{\pi}{|a_n|}; 1 \right) \]

\[ \deg Q \leq 1 \]

\[(\text{Lec 12}) \quad (p, 13)\]
Know $\sum \frac{1}{\lambda_n^2} < \infty$ by Lec 12 p.7.

Assume that $\sum \frac{1}{\lambda_n} < \infty$. Take $R$ large.

Observe that, for $|z| = R$,

\[ |f(re^{i\theta})| = R^g \left| e^{Az} + g \right|, \quad \prod_{n} \left| 1 - \frac{|z|}{\lambda_n} \right| \left| e^{\frac{z}{\lambda_n}} \right| \]

\[ \leq R^g e^{O(R)} \prod_{n} \left( 1 + \frac{|z|}{\lambda_n} \right) e^{\frac{|z|}{\lambda_n}} \]

\[ \leq R^g e^{O(R)} \prod_{n} e^{\frac{|z|}{\lambda_n}} e^{\frac{|z|}{\lambda_n}} \]

\[ \leq R^g e^{O(R)} \prod_{n} e^{2R \frac{|z|}{\lambda_n}} \]

\[ = R^g e^{O(R)} e^{2R \left( \sum \frac{1}{\lambda_n} \right)} \]

\[ \leq e^{O(R)} \quad \Rightarrow \]

\[ M(R; f) \leq e^{O(R)} \]

\[ \ln M(R; f) \leq O(R) \]
This is a contradiction to both (a) and (b).

We conclude that

\[ \sum \frac{1}{\tan f} = +\infty \]

whenever (a) or (b) holds.

Since \( \xi_0(z) \) had \( \rho = \frac{1}{2}, r = \infty \), we conclude at once that

\[ \sum \frac{1}{\tan f} = +\infty \]

whereupon \( \xi_0(z) \) must have infinitely many zeros (each lying in \( 0 < x < 1 \)).

There is no need to use the \( \rho = \frac{1}{2} \) trick following from \( \xi_0(z + \frac{1}{2}) = \text{EVEN} \).

One might also recall that, for \( 0 < \rho \leq 1 \), an entire \( f \) has infinitely many zeros by Lec 13, p. 2.
I then remarked that I. M. Vinogradov showed (with methods of trigonometric sums) that

\[ f(s) \neq 0 \quad \text{for} \quad \sigma = 1 - c \left( \ln t \right)^{-2/3} \]

\( t \) large. [As noted in Ingham, 2nd edition, p. xii], there is some question about this, so only \( (\ln t)^{-2/3} (\ln \ln t)^{-1/3} \) is properly justified.

Taking \( \eta(t) = \frac{1}{G(\ln t)^{2/3}}(t \geq e) \), Lec 14 leads to

\[ \omega(x) = \min_{t \leq e} \left\{ \frac{\ln x/G}{(\ln t)^{2/3}} + \ln t \right\} \]

i.e., trivial case for \( u \geq 1 \) on the Fcn.

\[ q(u) = \frac{\ln x/G}{u^{2/3}} + u \quad \text{(compare Lec 14, (ii))} \]

\( \Rightarrow \) min for \( u = (\text{const}) (\ln x)^{3/5} \) if \( x \) is large

\( \Rightarrow \omega(x) = (\text{const}) (\ln x)^{3/5} \)
\[ y(x) \sim x = \mathcal{O}\left[ xe^{-c(lnx)^{3/5}} \right] \]
\[ \pi(x) \sim 1/x = \mathcal{O}\left[ xe^{-c(lnx)^{3/5}} \right] \]

with suitable \( c > 0 \). Compare Lec 14. 

The box needs a slight adjustment if only
\[ \sigma \geq 1 - \left( \frac{\ln t}{\ln \ln t} \right)^{2/3} \left( \ln \ln t \right)^{-1/3} \]
zero-free
holds rigorously.

The "3/5" is about where things still lie in 2016 unconditionally. On this point, note that:
\[ X = X^* x^{-\frac{1}{2}} = x e^{-\frac{1}{2} (\ln x)^{2}} \]

There is obviously "some" distance yet to go!!!

\[ \frac{1}{2} \rightarrow \frac{3}{5} \rightarrow 1 \]
\( \sim 1900 \quad \sim 1958 \quad \text{when??} \)
The rest of the lecture was devoted to some important preparations for getting an explicit formula for \( \psi_1(x) \).

Know

\[
\begin{align*}
\Xi_0(s) &= e^{B_5} \prod_{\rho} (1 - \frac{s}{\rho} e^{\pi \rho})^{-1/2} \quad B \in \mathbb{R} \\
\Xi_0(s) &= \pi^{(s-1)/2} \Gamma(s/2) \tilde{J}(s)
\end{align*}
\]

\[
\frac{\Gamma'(s)}{\Gamma(s)} = \log z - \frac{1}{2z} - \sum_{k=1}^{R} \frac{B_{2k}}{2k} z^{-2k} + O_R \delta (1) z^{-2R-1}
\]

\[|\arg z| \leq \frac{\pi}{2} - \delta\]

\[
\frac{\tilde{J}'(s)}{\tilde{J}(s)} = -\frac{1}{s} - \frac{1}{s-1} + \left( B + \frac{1}{2} \log \pi \right) - \frac{1}{2} \frac{\pi'}{\pi} \left( \frac{s}{2} \right) + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right)
\]

\[\text{Im}(\rho) \neq 0, \quad 0 < \text{Re}(\rho) < 1, \quad \rho = \beta + i\gamma\]
by virtue of

\text{Lec 12 ① Stirling } \frac{1}{\sqrt{n}}

\text{Lec 13 ④ - ⑦}

and \( \frac{\Gamma(n')}{\Gamma(n)} (1 + \varepsilon) = \frac{1}{\varepsilon} + \frac{\Gamma(n')}{\Gamma(n)} \).

We also have:

\[
\frac{\xi_0'(s)}{\xi_0(s)} = B + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right)
\]

\[
\frac{\xi'(s)}{\xi(s)} = -\left[ \frac{1}{s} + \frac{1}{s-1} \right] + B + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right)
\]

\( \{ \xi(s) \equiv \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \} \)

\[
\sum \frac{1}{|\rho|^a} < \infty \quad \sum \frac{1}{\gamma^2} < \infty
\]

by \text{Lec 13 ⑦, ⑤ (bottom)}, and \( \text{Im}(\rho) = \gamma \neq 0 \).
For large $t$,
\[ N[\rho; |y-t| \leq 1] = O(\ln t). \]

**pf**

We follow von Mangoldt's method.

\[ \frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{A(n)}{n^s} \quad \text{Re}(s) > 1 \]

\[ \left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq \sum_{n=2}^{\infty} \frac{A(n)}{n^\sigma} \quad \text{Re}(s) > 1 \]

Keep $s = \sigma + it$, $t \geq \text{big } G$.

Keep $1 < \sigma \leq 10$, say.

Apply Riemann's formula for $\frac{\zeta'(s)}{\zeta(s)}$ on $\Gamma$ bottom.

Take real part only !!!

Get $\sqrt{\text{Stirling}}$

\[ O(1) = O(1) + O(\ln t) + \sum_{\rho} \left\{ \text{Re} \left( \frac{1}{s-\rho} \right) + \text{Re} \left( \frac{1}{\overline{s-\rho}} \right) \right\} \]

\[ O(\ln t) = \sum_{\text{all } \rho} \frac{\sigma-\rho}{(\sigma-\rho)^2 + (t-y)^2} + \sum_{\text{all } \rho} \frac{\overline{\rho}}{\rho^2 + t^2} \]
but \( \sigma - \beta > 0 \) since \( 0 < \beta < 1 \)

\[ O(\ln t) = \sum_{\text{all } \rho} \frac{2 - \beta}{(2 - \beta)^2 + (t - \rho)^2} \]

\[ \sum_{\text{all } \rho} \frac{1}{1 + (t - \rho)^2} = O(\ln t). \]

Restrict box to terms with \( |\gamma - t| \leq 1 \).

Thus,

\[ N[|\gamma - t| \leq 1] = O(\ln t), \]

as promised. \( \{ \text{Valid for } t \geq 2 \text{ by inflation of constant}\} \)

**Corollary**

\[ N[\rho : 0 < \gamma \leq T] = O(T \ln T), \quad T \geq 2. \]
\[ \frac{f'(s)}{f(s)} = O\left(\frac{1}{t}\right) + O(\text{ln} t) + \sum_{\rho} \left(\frac{1}{s + it - \rho} + \frac{1}{\rho}\right) \]

\[ \frac{f'(3+it)}{f(3+it)} = O\left(\frac{1}{t}\right) + O(\text{ln} t) + \sum_{\rho} \left(\frac{1}{3+it - \rho} + \frac{1}{\rho}\right) \]
\[
\frac{\mathfrak{I}'(s)}{\mathfrak{I}(s)} + O(1) = O\left(\frac{1}{t}\right) + O(\ln t)
\]
\[
+ \sum_{\rho} \left( \frac{1}{\sigma+i\varepsilon-\rho} - \frac{1}{3+i\varepsilon-\rho} \right)
\]

\[
\frac{\mathfrak{I}'(s)}{\mathfrak{I}(s)} = O(\ln t) + \sum_{|\gamma-t| > 1} \frac{3-\sigma}{(\sigma+i\varepsilon-\rho)(3+i\varepsilon-\rho)}
\]
\[
+ \sum_{|\gamma-t| \leq 1} \left[ \frac{1}{\sigma+i\varepsilon-\rho} - \frac{1}{3+i\varepsilon-\rho} \right]
\]

\[
\begin{aligned}
\left\{ \begin{array}{l}
\text{but }, \text{for } |\gamma-t| \leq 1 \\
\end{array} \right.
\end{aligned}
\]

\[
\left| \frac{1}{3+i\varepsilon-\rho} \right| = \frac{1}{|3-\beta + i(t-\nu)|} = \frac{1}{3-\beta}
\]

\[
= \frac{1}{2}
\]

\[
\frac{\mathfrak{I}'(s)}{\mathfrak{I}(s)} = O(\ln t) + \sum_{|\gamma-t| > 1} \frac{3-\sigma}{(\sigma+i\varepsilon-\rho)(3+i\varepsilon-\rho)}
\]
\[
+ \sum_{|\gamma-t| \leq 1} \frac{1}{\sigma+i\varepsilon-\rho}
\]
but, for $|\gamma-t| > 1$, $-1 \leq \sigma \leq 2$,

\[
\left| \frac{3-\sigma}{(\sigma-\beta+i(t-\gamma))(\sigma-\beta+i(t-\gamma))} \right| \leq \frac{4}{|t-\gamma|^2}
\]

while, by (9) box,

\[
\sum_{\text{all } \rho} \frac{1}{1+(t-\gamma)^2} = O(ln t)
\]

\[
\frac{F(s)}{F(s)} = O(ln t) + \sum_{\rho} \frac{1}{s-\rho}.
\]

One remarks here that the "1" can be replaced (if convenient) by any positive constant. Just review the earlier steps!
Theorem (Very Important and Basic)

Let \(-1 \leq \sigma \leq 2\) and \(t\) be large, with \(t \neq \) all \(\gamma\). We then have

\[
\frac{\bar{\Xi}(s)}{\Xi(s)} = O(\ln t) + \sum_{\rho, |\gamma - t| \leq 1} \frac{1}{s - \rho}
\]

For \(s = \sigma + it\). The "1" can be replaced by any positive constant (as convenient).

\[\text{Pf.} \]
As above. \[\blacksquare\]

On p. 17, recall that \(\bar{\Xi}(s) = \pi^{s/2} \Gamma(\frac{s}{2}) J(s)\) had \(\Xi(s) = \Xi(1-s)\) and simple poles at \(s = 1\) and \(s = 0\).

\[\uparrow\text{Lec 11, p. 24-25}\]

Recall too:

\(\bar{\Xi}(s) = (s-q)^M \phi(s), \quad \phi(s) = \text{power series in } (s-q)\)

with \(\phi(q) \neq 0\)

\[\Rightarrow \frac{\bar{\Xi}'(s)}{\bar{\Xi}(s)} = \frac{M}{s-q} + \text{[analytic near } s=q]\]


And, remember that

$$\Xi(x) \neq 0, \quad x \in \mathbb{R}$$

We propose to try to count the zeros of $\Xi(s)$ — following Riemann.

Let $T \geq 2$, say $T \neq \text{all } y$.
Let $A > 1$.

Draw the rectangle

$$R(A, T) = [1-A, A] \times [-T, T]$$
Put
\[ N(T) = N\left[ \rho \circ 0 < y \leq T \right] . \]

By (13) bottom and CRT, we get
\[ \frac{1}{2\pi i} \oint_{AR(A, T)} \frac{\xi'(s)}{\xi(s)} \, ds = 2N(T) - 2 \]
from \( s = 0, 1 \)
simple poles of \( \xi \)

Note:
\[ \frac{1}{2\pi i} \int_{\text{left}} \frac{\xi'(s)}{\xi(s)} \, ds = \frac{1}{2\pi i} \int_{\text{u-image of left}} \frac{\xi'(1-u)}{\xi(u)(1-u)} (-du) \]
\[ = -\frac{1}{2\pi i} \int_{\text{u-image of left}} \frac{\xi'(1-u)}{(1-u)} \, du \]
\[
\begin{align*}
\{ & \text{easily check } u \text{- image of "left"} \\
& \text{is exactly "right" (in the correct direction)} \\
& \quad = -\frac{1}{2\pi i} \int_{\text{right}} \frac{\xi'(u)}{\xi(u)} du \quad \tag{16}
\end{align*}
\]

But,
\[
\xi(z) = \xi(1-z)
\]
\[
\Rightarrow \log \xi(z) = \log \xi(1-z)
\]
\[
\Rightarrow \frac{\xi'(z)}{\xi(z)} = -\frac{\xi'(1-z)}{\xi(1-z)}
\]

So,
\[
\frac{1}{2\pi i} \int_{\text{left}} \frac{\xi'(s)}{\xi(s)} ds = \frac{1}{2\pi i} \int_{\text{right}} \frac{\xi'(u)}{\xi(u)} du \quad \tag{17} \text{line 4}
\]

Hence,
\[
2N(r) - 2 = \frac{2}{2\pi i} \int_{\text{right}} \frac{\xi'(s)}{\xi(s)} ds
\]
\[ N(t) = 1 + \frac{1}{2\pi i} \int_{\text{right}} \frac{\xi'(s)}{\xi(s)} \, ds. \]

Notice that \( A > 1 \) has not been specified yet in \( R(A, T) \).

Put \( \xi(s) = G(s) \zeta(s) \), \( G(s) = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \).

\[ \Rightarrow \frac{\xi'(s)}{\xi(s)} = \frac{G'(s)}{G(s)} + \frac{\zeta'(s)}{\zeta(s)} \]

The function \( G(s) \) is analytic on \( \text{Re}(s) > 0 \) and not zero.

By CIT,

\[ \frac{1}{2\pi i} \int_{\text{right}} \frac{G'(s)}{G(s)} \, ds = \frac{1}{2\pi i} \int_{\frac{1}{2} + iT}^{1 + iT} \frac{G'(s)}{G(s)} \, ds. \]

So,

\[ N(t) = 1 + \frac{1}{2\pi i} \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} \frac{G'(s)}{G(s)} \, ds + \frac{1}{2\pi i} \int_{\text{right}} \frac{\zeta'(s)}{\zeta(s)} \, ds. \]
To handle the $G'$ integral, recall our use of the uniquely determined branch $\log \Gamma(z)$ near $p = \frac{1}{2}$ of Lec 10. We had $\log \Gamma(z) = \log \Gamma(x) = \ln \Gamma(x)$ for $x > 0$.

We can therefore unambiguously declare
\[
\log G(s) = -\frac{s}{2} \ln \pi + \log \Gamma\left(\frac{s}{2}\right)
\]
for $\text{Re}(s) > 0$. At once:
\[
\frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} G'(s) ds
\]
\[
= \frac{1}{2\pi i} \left[ \log G(s) \right]_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}
\]
\[
\begin{cases}
G(s) = \frac{G(s)}{G(s)} \\
\log G(s) = \log G(s) \\
\log G(u) = \ln |G(u)| + i \text{Arg } G(u)
\end{cases}
\]
\[0 > 0\]
\[
\frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\zeta'}{\zeta}(s) \, ds
\]

\[
= \frac{1}{2\pi i} \, 2i \arg \zeta \left( \frac{1}{2} + iT \right) \quad \text{or middle}
\]

\[
= \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + iT \right) \quad \text{see (18)}
\]

\[
= \frac{1}{\pi} \left[ -\frac{T}{2} \ln \pi + \arg \Gamma \left( \frac{1}{2} + iT \right) \right].
\]

So far, then, we have:

\[
N(T) = 1 + \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + iT \right)
\]

\[
+ \frac{1}{2\pi i} \int_{\text{right}} \frac{\zeta'(s)}{\zeta(s)} \, ds.
\]

This box clearly holds for any \( T > 0 \), \( T \neq \text{all } y \). There was nothing used about \( T \geq 2 \) yet.
We now PAUSE to apply Stirling to

\[ \frac{1}{\pi} \left[ -\frac{T}{2} \ln \pi + \text{Arg} \, \Gamma \left( \frac{1}{4} + i \frac{T}{2} \right) \right] . \]

Here we keep \( T \approx 2 \) and imagine \( T \approx 6 \), if necessary (along the way).

\[ \text{Arg} \, \Gamma \left( \frac{1}{4} + i \frac{T}{2} \right) = \text{Im} \, \log \, \Gamma \left( \frac{1}{4} + i \frac{T}{2} \right) \]

\[ = \text{Im} \left[ \left( \frac{1}{4} + i \frac{T}{2} - \frac{1}{2} \right) \log \left( \frac{1}{4} + i \frac{T}{2} \right) \right. \]
\[ - \left( \frac{1}{4} + i \frac{T}{2} \right) \]
\[ + \ln \sqrt{2\pi} + O \left( \frac{1}{T} \right) \left\{ \text{Lec 10 p. 42} \right\} \]

\[ = \text{Im} \left[ \left( -\frac{1}{4} + i \frac{T}{2} \right) \left\{ \log \left( \frac{T}{2\pi} \right) \left( 1 + \frac{1}{2iT} \right) \right\} \right. \]
\[ - \frac{1}{4} - i \frac{T}{2} + \ln \sqrt{2\pi} + O \left( \frac{1}{T} \right) \left\{ \text{Lec 10 p. 42} \right\} \]
\[
\left\{ \log \left(1 + \frac{1}{2iT} \right) \right\} = \frac{1}{2iT} + O(T^{-2})
\]

\[
= \frac{T}{2\pi} \ln \frac{T}{2} - \frac{\pi}{8} + O(T^{-1})
\]

\[
= \frac{T}{2\pi} \ln \left( \frac{T}{2e} \right) + O(T^{-1})
\]

This is valid for \( T \geq 6 \), then by constant inflation for \( T \geq 2 \). Compare Ingham 69 line 60.

Get:

\[
\frac{1}{\pi} \text{Arg} \, G \left( \frac{1}{2} + iT \right)
\]

\[
= \frac{1}{\pi} \left[ -\frac{T}{2} \ln \pi + \text{Arg} \, \Gamma \left( \frac{1}{2} + iT \right) \right]
\]

\[
= -\frac{T}{2\pi} \ln \pi + \left( -\frac{1}{8} \right) + \frac{T}{2\pi} \ln \frac{T}{2e} + O(T^{-1})
\]

\[
= \frac{T}{2\pi} \ln \left( \frac{T}{2\pi e} \right) - \frac{1}{8} + O(T^{-1})
\]

\( T \geq 2 \).
On \( 19 \), for \( T \geq 2 \) (\( T \neq \text{all } \gamma \)), we therefore have

\[
N(T) = \frac{T}{2\pi} \ln \left( \frac{T}{2\pi e} \right) + \frac{T}{8} + O(T^{-1})
\]

\[+
\frac{1}{2\pi i} \int_{\text{right}} \frac{f'(s)}{f(s)} \, ds\]

To address \( \frac{1}{2\pi i} \int_{\text{right}} \frac{f'(s)}{f(s)} \, ds \), we proceed in 2 ways.

First, recall that:

\[
|f(z) - 1| < 3 \cdot 2^{-x} \quad , \quad x \geq 2
\]

\{ \text{Lec 5, p. 10} \}

\[
\log f(z) = \sum_{n=2}^{\infty} \frac{A(n)}{\ln n} n^{-z} \quad , \quad x > 1
\]

\{ \text{Lec 6, p. 4 + 3} \}

\[
\log f(z) = O(2^{-x}) \quad , \quad x \text{ large}
\]
\[
\frac{f'(z)}{f(z)} = - \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^z}, \quad x > 1
\]
\{ \text{Lec 6, p. 6} \}

\[
\frac{f'(z)}{f(z)} = O(2^{-x}), \quad x \text{ large}
\]

We can now freeze \( T \) and let \( A \to \infty \) in \( R(A, T) \) to get

\[
\frac{1}{2\pi i} \int_{\text{right}} \frac{f'(s)}{f(s)} ds = \frac{1}{2\pi i} \int_{\frac{1}{2}}^{\infty} \frac{f'(u-iT)}{f(u)} du
\]

\[
+ \frac{1}{2\pi i} \int_{\alpha}^{1/2} \frac{f'(u+iT)}{f(u)} du
\]

\[
+ 0 \quad \{2^{-A} \to 0\}
\]

\[
\begin{cases}
\text{but } \frac{f'(u-iT)}{f(u)} = \frac{f'(u+iT)}{f(u)} \\
\text{and }
\end{cases}
\]

\[
= \frac{1}{2\pi i} \int_{\alpha}^{1/2} \left[ \frac{f'(u+iT)}{f(u)} - \frac{f'(u+iT)}{f(u)} \right] du
\]

\[
= \frac{1}{2\pi i} \int_{\alpha}^{1/2} 2i \Im \frac{f'}{f}(u+iT) du
\]
\[ \int_{\text{right}} \frac{1}{2\pi i} \int_{\frac{1}{2}}^{0} \text{Im} \left( \frac{s'}{s} \right) ds = -\frac{1}{\pi} \text{Im} \int_{\frac{1}{2}}^{0} \frac{s'}{s} \left( u+iT \right) du. \]

Thus,

\[ \int_{\text{right}} \frac{1}{2\pi i} \int_{\frac{1}{2}}^{0} \text{Im} \left( \frac{s'}{s} \right) ds = -\frac{1}{\pi} \text{Im} \int_{\frac{1}{2}}^{0} \frac{s'}{s} \left( u+iT \right) du. \]

The second way is more basic. One starts with \( \log J(s) \) on \( \Re(s) > 1 \) (see (22)) and forms an analytic continuation along the line segments \( \left[ \frac{1}{2} + iT, A+iT \right] \) and \( \left[ \frac{1}{2} - iT, A-iT \right] \) in an obvious way (starting at \( A \pm iT \)).

This is legal since \( T \neq \) all \( y \).

No zeros of \( J(s) \) will be hit.

This branch of \( \log J(s) \) clearly satisfies

\[ \log J(s) = \log J(s). \]

[Analytic \( f(z) \) with \( f(x) \in \mathbb{R} \Rightarrow f(\bar{z}) = \overline{f(z)} \).]
One typically says \( \log S(s) \) has been found by continuing "up" from \( s_0 = A \), then "across" to \( s \) from \( A \pm iT \).

\[
\begin{align*}
\sigma + iT \quad \downarrow \quad \left( A > 1 \right) \\
\uparrow \\
1 \\
s_0 = A
\end{align*}
\]

With this convention,

\[
\frac{1}{2\pi i} \int_{\text{right}} \frac{f'}{f} (s) \, ds = \frac{1}{2\pi i} \int_{\text{right}} \, d \left[ \log f(s) \right]
\]

\[
= \frac{1}{2\pi i} \left[ \log f\left( \frac{1}{2} + iT \right) \right. \\
\left. - \log f\left( \frac{1}{2} - iT \right) \right]
\]

\[
= \frac{1}{2\pi i} \Im \left[ \log f\left( \frac{1}{2} + iT \right) \right] \Rightarrow
\]

\[
\frac{1}{2\pi i} \int_{\text{right}} \frac{f'}{f} (s) \, ds = \frac{1}{\pi} \arg f\left( \frac{1}{2} + iT \right)
\]

in an obvious "up and across" sense.
Let $T > 0$, $T \neq \text{any } \gamma$.

Let

$$ \xi(s) = G(s) \zeta(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s). $$

We then have:

$$ N(T) = N[\rho : 0 < \gamma \leq T] $$

$$ = 1 + \frac{1}{\pi} \text{Arg } G\left(\frac{1}{2} + i\gamma\right) + S(T) $$

wherein

$$ S(T) = \frac{1}{\pi} \text{Arg } \zeta\left(\frac{1}{2} + i\gamma\right) \quad \text{"up and across"} $$

$$ = -\frac{1}{\pi} \text{Im} \int_{\frac{1}{2}}^{\infty} \frac{\zeta'(\sigma + i\gamma) d\sigma}{\zeta(\sigma + i\gamma)} $$

and $\text{Arg } G\left(\frac{1}{2} + i\gamma\right)$ is defined a la Stirling.

For $T \geq 2$, we have:

$$ \frac{1}{\pi} \text{Arg } G\left(\frac{1}{2} + i\gamma\right) = \frac{T}{2\pi} \ln\left(\frac{T}{2\pi e}\right) - \frac{1}{8} + O\left(\frac{1}{T}\right). $$

$C^\infty \text{fcn of } T$
THM (Very Important and Basic)

Introduce \( s(T) \) for \( T > 0 \), \( T \neq 0 \) as on p. 26. We then have

\[
\frac{s'(s)}{s(s)} = -\sum_{n=2}^{\infty} \frac{A(n)}{n^s}, \quad \Re(s) > 1
\]

\[
= O(2^{-\sigma}), \quad \sigma \geq 2.
\]

PF

Apply Thm on 13. Remember that

\[
\frac{s'(s)}{s(s)} = -\sum_{n=2}^{\infty} \frac{A(n)}{n^s}, \quad \Re(s) > 1
\]

\[
= O(2^{-\sigma}), \quad \sigma \geq 2.
\]

See 23. Get:

\[
S(T) = -\frac{1}{\pi} \text{Im} \int_{\frac{1}{2}}^{2} \frac{s'}{s}(\sigma + iT) \, d\sigma
\]

\[
-\frac{1}{\pi} \text{Im} \int_{0}^{\infty} \frac{s'}{s}(\sigma + iT) \, d\sigma
\]
\[ = -\frac{1}{\pi} \text{Im} \int_{1/2}^{2} \left[ O(\ln T) + \sum_{1/2 - iT \leq \rho \leq 1/2 + iT} \frac{1}{s - \rho} \right] ds \]

\[ + O(1) \int_{1/2}^{2} 2^{-s} ds \]

\[ \{ \text{here } s = \sigma + iT, \quad \rho = \beta + i\gamma \} \]

\[ = O(\ln T) \]

\[ -\frac{1}{\pi} \text{Im} \sum_{1/2 - iT \leq \rho \leq 1/2 + iT} \left( \int_{1/2}^{2} \frac{1}{s - \rho} ds \right) \]

\[ + O(1) \]

\[ \left\{ \begin{array}{l}
\text{but } \int_{1/2 + iT}^{2 + iT} \frac{1}{s - \rho} ds \quad \text{for all } \gamma \\
\text{and } \log (2 + iT - \rho) - \log (1/2 + iT - \rho) \\
\Rightarrow \text{imaginary part has absolute value } \leq \pi \end{array} \right\} \]

\[ = O(\ln T) + O(1) \sum_{1/2 - iT \leq \rho \leq 1/2 + iT} \frac{1}{\gamma} + O(1) \]

\[ = O(\ln T) \quad \text{by } \nabla HIS \]
THM (stated by Riemann)

\[ N(T) = N\left[ \rho : 0 < \gamma \leq T \right] \quad \text{definition} \]

\[ = \frac{T}{2\pi} \ln \left( \frac{T}{2\pi e} \right) + O(\ln T) \]

\text{for all } T \geq 2 \quad \left\{ \text{Ingham p. 68 thm 25} \right\}

Proof

For \( T \neq \text{all } \gamma \), just combine \( \circ \) + \( \bullet \).

If \( T = \text{some } \gamma \), just use the right continuity of \( N(t) \) as a counting function. \( \square \)

For later use, notice that:

\[
\frac{d}{dt} \left( \frac{t}{2\pi} \ln \left( \frac{t}{2\pi e} \right) \right) = \frac{1}{2\pi} \frac{d}{du} \left( u \ln \left( \frac{u}{e} \right) \right) \quad u = \frac{t}{2\pi} \\
= \frac{1}{2\pi} \ln u \\
= \frac{1}{2\pi} \ln \left( \frac{t}{2\pi} \right) \quad t = 2\pi u
\]
We seek to use
\[
\varphi_1(x) = \frac{i}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left[ -\frac{\Gamma'(s)}{\Gamma(s)} \right] \, ds
\]
\((c > 1, x < 1)\)
to get an explicit formula for \(\varphi_1(x)\).

We shall use an appropriate rectangle
\[ R = [-\Delta, c] \times [-T, T] \]
and let \(T \to \infty, \Delta \to 0 \).

\underline{Lemma}
\[
\begin{align*}
(a) \quad & \sum_{0 < \gamma \leq T} \frac{1}{\gamma} = O(\ln^2 T) \quad \text{for } T \to \infty \\
(b) \quad & \sum_{\gamma > 1} \frac{1}{\gamma^2} = O\left(\frac{\ln T}{T}\right)
\end{align*}
\]

**Proof**
We know \(N(t) = \frac{t}{2\pi} \ln \left(\frac{t}{2\pi e}\right) + O(\ln t)\), \(t \to \infty\),
by Lec 15 p.29. Recall that \(N(t)\) is right continuous.

In both (a) and (b), WLOG \(T \geq 1000\).
Write \( N(t) = \frac{4}{\pi} \ln \left( \frac{4}{\pi(t)} \right) + R(t) \)

For (a), get \( \varepsilon \) \[
\sum_{0 < y \leq T} \frac{1}{y^2} = O(1) + \int_2^T \frac{1}{t^2} dN(t)

= O(1) + \int_2^T \frac{1}{t^2} \left\{ \frac{1}{2\pi} \ln \frac{4}{\pi(t)} \right\} \, dt \leq \text{Lec 15, eq. 29} \]

= O(1) + \int_2^T \frac{1}{t^2} \, dt

+ \frac{R(T)}{T} - \frac{R(2)}{2} - \int_2^T \frac{R(t)}{t^2} \, dt \leq R(t) = O(\ln t)

= O(1) + O(\ln T) \int_2^T \frac{1}{t^2} \, dt

+ \frac{R(T)}{T} - \frac{R(2)}{2} - \int_2^T \frac{R(t)}{t^2} \, dt

= O(\ln^2 T) + \frac{O(\ln T)}{T} + O(1)

+ O(1) \int_2^T \frac{\ln t}{t^2} \, dt

= O(\ln^2 T) + O(\ln T) \cdot \int_2^T \frac{1}{t^2} \, dt

= O(\ln^2 T) \quad \text{OK}

For (b),

\[
\sum_{y > T} \frac{1}{y^2} = \int_T^\infty \frac{1}{t^2} \, dN(t) \quad \{\text{this is correct even if } T = \text{some } y\}
\[= \int_T^\infty \frac{1}{t^2} \left\{ \frac{1}{2\pi} \ln \frac{t}{2\pi} \right\} \, dt \]

\[+ \int_T^\infty \frac{1}{t^2} \, dR(t) \]

\[= O(1) \int_{T/2\pi}^\infty \frac{\ln u}{u^2} \, du \]

\[+ \frac{R(t)}{t^3} \bigg|_T^\infty - \int_T^\infty R(t) (-2) \frac{1}{t^3} \, dt \]

\[= O(1) \int_{\infty}^{\infty} \ln u \, du \left( \frac{1}{u} \right) \]

\[= O(1) \int_T^{\infty} \frac{\ln t}{t^3} \, dt \]

\[+ O\left( \frac{\ln T}{T^2} \right) + O(1) \int_T^\infty \frac{\ln t}{t^3} \, dt \]

\[= O(1) \int_T^\infty \frac{\ln t}{t^3} \, dt \]

\[+ O\left( \frac{\ln T}{T^3} \right) + O(1) \int_T^\infty \frac{\ln t}{t^3} \, dt \]

\[= O(1) \frac{\ln T}{T} + O(1) \int_T^\infty \frac{\ln t}{t^3} \, dt \]

\[= O(1) \frac{\ln T}{T} + O(1) \int_T^\infty \frac{\ln t}{t^3} \, dt \]

\[= O(1) \frac{\ln T}{T} + O(1) \frac{\ln T}{T^2} = O(1) \frac{\ln T}{T}. \]
Lemma

For $m \geq 2$, we can always find some $T_m \in (m, m+1)$ so that

$$\left| \frac{s(s+iT_m)}{s(s+iT_m)} \right| \leq A_1 \ln^2 T_m \quad \text{for } -1 \leq s \leq 2.$$

Here $A_1$ is a suitable absolute constant.

**PF**

When $m \geq 1000$. ($\ln 1000 = 6.90^+$)

By Lec 15 Thm p.8, see also p.29, we know:

$$N[m-2 \leq \gamma \leq m+2] = O(\ln m).$$

Write this as

$$N[m-2 \leq \gamma \leq m+2] \leq B \ln m.$$

When $B \geq 1$. Divide $(m, m+1]$ into $2 \ll \ln m$ equal left-open subintervals. Some interval must therefore contain NO $\gamma$. Let $T_m = \text{midpoint of this subinterval}$. By construction,
\[ |y - Tm| \geq \frac{1}{4 \sigma 2 \| Blm \|} \geq \frac{1}{8 \sigma \ln \| m \|} \]

For all \( y \), apply Lec 15, p. 13 (the partial fraction thm). With \( z = Tm \), we clearly get

\[ \frac{1}{5} (e^{i \sigma Tm}) = O(\ln Tm) + O(\ln m) \sim O(\ln m) \]

\[ = O(\ln^2 Tm) \]

for \( -1 \leq \sigma \leq 2 \). \( \Box \)

**Lemma**

Consider the domain

\[ \mathcal{E}_0 = \left\{ \Re(s) < -1 \right\} - \bigcup_{k=1}^{\infty} \left\{ 1 + 2k \frac{\pi \sigma}{2} \right\} \]

We have

\[ \left| \frac{S'(s)}{S(s)} \right| \leq A_2 \ln (|s| + 10) \]

for \( s \in \mathcal{E}_0 \). Here \( A_2 \) = suitable absolute constant.
Recall the functional equation of \( \xi(s) \), \( \Gamma(s) \).

Get:

\[
\Gamma(s) = \frac{\prod_{r=1}^{s - 1} \Gamma\left(\frac{r}{2}\right) \Gamma\left(\frac{s - r}{2}\right)}{\prod_{r=1}^{s - 1} \Gamma\left(\frac{r}{2}\right)} \Gamma\left(\frac{s}{2}\right)
\]

\[= \pi^{s - \frac{1}{2}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s - 1}{2}\right)
\]

\[
\therefore \Gamma(s) = \frac{\pi^{s - \frac{1}{2}} \Gamma\left(\frac{s - 1}{2}\right) \Gamma\left(\frac{s + 1}{2}\right)}{\pi^{\frac{1}{2}} 2^{1-s} \Gamma(s)}
\]

\[
\Gamma(s) = \pi^{s - 1} 2^{s - 1} \frac{\Gamma\left(\frac{s - 1}{2}\right) \Gamma\left(\frac{s + 1}{2}\right)}{\Gamma(s)}
\]

\[
\therefore \Gamma(s) = \frac{\pi^{s - 1} 2^{s - 1} \Gamma\left(\frac{s - 1}{2}\right) \Gamma\left(\frac{s + 1}{2}\right)}{\sin \pi \left(\frac{s - 1}{2}\right)}
\]

\[
\Gamma(s) = \frac{\pi^{s - 1} 2^{s - 1} \Gamma\left(\frac{s - 1}{2}\right) \Gamma\left(\frac{s + 1}{2}\right)}{\cos \frac{\pi s}{2}}
\]

\[
\Rightarrow \Gamma(1-s) = \pi^{-s} 2^{-s} \Gamma(s) \cos \frac{\pi s}{2} \cdot \Gamma(s)
\]
Here \( s \) = any generic value in \( \mathbb{C} \). Take logarithmic derivatives to get

\[
- \frac{f'(1-s)}{f(s)} = - \ln 2\pi - \frac{\pi}{2} \tan \frac{\pi s}{2} + \frac{f'(s)}{f(s)} + \frac{s'(s)}{s(s)}
\]

Flip \( s \rightarrow 1-s \)

\[
- \frac{f'(s)}{f(s)} = - \ln 2\pi - \frac{\pi}{2} \coth \frac{\pi s}{2} + \frac{f'(1-s)}{f(1-s)} + \frac{1}{f(1-s)}
\]

\[
\frac{f'(s)}{f(s)} = \ln 2\pi + \frac{\pi}{2} \coth \frac{\pi s}{2} - \frac{f'(1-s)}{f(1-s)} - \frac{1}{f(1-s)}
\]
Recall that \( \pi \text{ch} n/\pi \) is periodic \( z \to z + 1 \),

\[
\pi \text{ch} n/\pi = \lim_{N \to \infty} \sum_{-N}^{N} \frac{1}{\pi \text{ch} n/\pi - n} \]

and

\[
|\text{ch} n/\pi - i| = O(e^{-2\pi y}) \quad \text{for } y \geq 1.
\]

Similarly

\[
|\text{ch} n/\pi - i| = O(e^{-2\pi y}) \quad \text{for } y \leq -1.
\]

See Lec 9, p. 10, (a), (d), THM.

For \( s \in \mathbb{E}_0 \), p. 7, 2nd box gives \( s \)

\[
\frac{f'(s)}{f(s)} = O(1) + O(1) + O(1) \left| \frac{n'}{n} \right| (1 - s)
\]

\[
\left| \frac{f'(z)}{f(z)} \right| \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{nx} \quad \text{as } x \to 2
\]

\[
= O(1) + O(1) \left| \log (1 - s) + O(1) \right|
\]

Stirling, Lec 12, p. 10
\begin{align*}
&= O(1) + O(1) \ln |1 - s| \\
&\leq O(1) + O(1) \ln (|s|+10) \\
&\leq O(1) \ln (|s|+10),
\end{align*}

as was to be proved.

For our rectangle $R$ on $\mathbb{C}$ we take
\begin{align*}
&c = 2 \\
&\Delta = 2m+1, \text{ } m \text{ big} \\
&T = T_m.
\end{align*}

We know that
\[ \psi_1(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^{s+1}}{s(s+1)} \left[ -\frac{\Gamma(s)}{\Gamma(s+1)} \right] ds. \]

Here $x \geq 1$. Notice that
\[ \left| \int_{a+iT_m}^{a+i\infty} \frac{x^{s+1}}{s(s+1)} \left[ -\frac{\Gamma(s)}{\Gamma(s+1)} \right] ds \right| \leq \int_{T_m}^{\infty} \frac{x^3}{t^2} O(1) dt. \]
Similarly, for \( \int_{2-iT_m}^{2+iT_m} \). Thus,

\[
\psi_1(x) + O\left(\frac{x^3}{T_m}\right) = \frac{i}{2\pi i} \int_{2-iT_m}^{2+iT_m} \frac{e^{sx}}{s^3} ds
\]

By Cauchy Residue Theorem, we have

\[
\frac{1}{2\pi i} \oint_{\partial R} \frac{x^{s+1}}{s(s+1)} \left[ -\frac{s'}{s(s+1)} \right] ds
\]

\[
= \text{Res} \text{ at } 1 + \text{Res} \text{ at } 0 + \text{Res} \text{ at } -1
\]

\[
+ \sum_{k=1}^{m} \text{Res} \text{ at } -2k + \sum_{|y|<T_m} \text{Res} \text{ at } \rho
\]
\[ \frac{1}{2\pi i} \oint_{\partial R} \frac{x^{s+1}}{s(s+1)} \left[ -\frac{f'(s)}{f(s)} \right] ds \]

\[ = \frac{x^2}{2} + x^1 \left[ -\frac{f'(0)}{f(0)} \right] + x^0 \left[ -\frac{f'(1)}{f(-1)} \right] \]

\[ + \sum_{k=1}^{m} (-1)^{k+1} \frac{x^{1-2k}}{(2k)(2k-1)} \]

\[ + \sum_{|\gamma| \leq T_m} (-1)^{\gamma+1} \frac{x^{\rho+1}}{\rho(\rho+1)} \]

\[ \rho = \beta + iT \]

\[ \text{as usual} \]

\[ 0 < \beta < 1 \]

\[ \text{Note that:} \]

\[ \text{LHS} = \psi(x) + O\left(\frac{x^3}{T_m}\right) \]

\[ + \frac{1}{2\pi i} \int_{\text{horiz}} e^{st} \left( \frac{d}{dt} \right) \left( e^{\gamma t} \right) \bigg|_{t=T_m} dt \]

\[ + \frac{1}{2\pi i} \int_{\text{vert}} e^{\gamma t} \left( \frac{d}{d\gamma} \right) \left( e^{st} \right) \bigg|_{\gamma = -2m-1} d\gamma \]

\[ + \frac{1}{2\pi i} \int_{\text{horiz}} e^{st} \left( \frac{d}{dt} \right) \left( e^{\gamma t} \right) \bigg|_{t=-T_m} dt \]

See (D) bottom.
Apply (4) + (5) to \([\text{horiz, } t = T_m] \). Get:

\[
\int_{\text{horiz}}^{\text{t} = T_m} = O(1) \int_{-2m^{-1}}^{-1} \frac{x^{1+\sigma}}{T_m^2} \ln m \, dx
\]

\[
+ O(1) \int_{-1}^{2} \frac{x^{1+\sigma}}{T_m^2} \ln^2 m \, dx
\]

\[
\left\{ \begin{array}{l}
x^{\sigma+1} = x^{\sigma+1} \text{ and } x \approx 1 \end{array} \right. \]

\[
= O(1) \left( \frac{1}{m^2} (\ln m) O(m) \right)
\]

\[
+ O(1) \frac{x^3}{m^2} \ln^2 m
\]

\[
= O(1) \frac{\ln m}{m} + O(1) x^3 \frac{\ln^2 m}{m^2}
\]

Similarly for \([\text{horiz, } t = -T_m] \).
Apply (5) to $[\text{vertical}]$, $\sigma = -2n-1$. Get:

$$
\int_{\sigma = -2n-1} = O(1) \int_{-Tm}^{Tm} \frac{x^{1+(-2n-1)}}{m^2} \ln m \, dt
$$

$$
\approx O(1) \int_{-Tm}^{Tm} \frac{1}{m^2} \ln m \, dt
$$

$$
= O(1) \frac{\ln m}{m}
$$

We conclude that on (11) bottom:

$$
\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{x^{s+1}}{s(s+1)} \left[ \frac{\Gamma(s)}{\Gamma(s)} \right] \, ds
$$

$$
= \psi(x) + O\left(\frac{x^3}{Tm}\right)
$$

$$
+ O(1) x^3 \left(\frac{\ln m}{m}\right)^2 + O(1) \frac{\ln m}{m}
.$$
Combining (11) top with (13) bottom, we get

\[ \psi_1(x) + O(1) \frac{\ln m}{m} + O\left(\frac{x^3}{m}\right) \]

\[ = \frac{x^2}{2} + Ax + B \]

\[ + \sum_{k=1}^{m} (-1)^k \frac{x^{1-2k}}{(2k)(2k-1)} \]

\[ + \sum_{|y| < TM} (-1)^y \frac{x^{\rho+1}}{y^{\rho+1}} \]

\[ \begin{cases} \text{but } \sum \frac{1}{|y|^{\rho+2}} < \infty \end{cases} \]

\[ \downarrow \quad \text{LET } m \to \infty \]

\[ \psi_1(x) = \frac{x^2}{2} + Ax + B - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{(2k)(2k-1)} \]

\[ - \sum_{\text{all } \rho} \frac{x^{\rho+1}}{\rho^{\rho+1}} \]

For each \( x \geq 1 \), both series \( \psi_1 \) converge.
Remark.

One definitely wants to keep $x \geq 1$.
Indeed, for $0 < x < 1$, i.e., $\frac{1}{x} > 1$, we notice that

$$\sum_{k=1}^{\infty} \frac{x^{-2k}}{(2k)(2k-1)} = + \infty \cdot$$

**THM** (Riemann's explicit formula for $\psi_1(x)$)

For each $x \geq 1$, we have

$$\psi_1(x) = \frac{x^2}{2} + Ax + B - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{(2k)(2k-1)} - \sum_{\text{all } \rho} \frac{x^{\rho+1}}{\rho (\rho+1)}$$

wherein $A = -\frac{f'(0)}{f(0)}$, $B = \frac{f'(1)}{f(1)}$.

**PF**

As above. See (14) bottom.

[Ingham 73]
Important Procedural Remark.

To keep things completely clear logically, notice that our proof of \( p.15 \) THM technically only relied on \( 0 \leq \beta \leq 1 \). If we did not need to know \( I(\text{for} y) \neq 0 \).

To verify this, observe that certain "expungements" can be safely made:

- Lec 6 pp. 6-8 Hadamard (note that pp. 9-21 do not use 6-8)
- Lec 7 pp. 5-9 (middle), 15-23 using \( \psi_j(z) \)
  (NOTE Lec 7 pp. 9 (foot) - 14 on \( \psi_j(x) \) is OK)
- Lec 8 pp. 1-4, 10-13 (top) on \( \psi_j(x) + \text{PNT} \)
  (NOTE Lec 9+10 is E-M1, no \( \psi_j(z) \).
  (NOTE Lec 11 got functional eq for \( I(z) \), never needed \( \psi_j(z) \).

In Lec 13, \( p.4 \) THM, state only that \( 0 \leq \text{Re}(\beta) \leq 1 \). Expunge \( 0 < \text{Re}(\beta) < 1 \).
Also on \( p.5 \) in connection with HFTs.
Expunge Lec 13 pp. 9 (bot) - 15 \( \sqrt{\text{all of Lec 14}} \) (related to zero-free regions).
With these expungements, a quick review shows that Lec 15 goes thru perfectly well — knowing only that $0 \leq \beta \leq 1$ and $\text{Im}(\rho) \neq 0$.

Pages 1 - 15 above are then recovered without difficulty.

This being said, we now get a "new" proof of the PNT as follows:

- Develop the explicit formula for $\psi(x)$ as on p.15. (Note that this requires the CRT.)
- Do the Hadamard trick to get $f(it^g) \neq 0$. See Lec 6, pp. 6 - 8.
- Use the functional equation of $\xi_0(z)$ to get $0 < \beta < 1$. See Lec 11, pp. 24 - 25; also 27.
- Choose $R$ so big that $\sum_{|\rho| > R} < \epsilon$.
- Exploit the explicit formula to get

$$\limsup_{x \to \infty} \left| \psi(x) \frac{1}{x^2} - \frac{1}{2} \right| \leq O + O + O + \limsup_{x \to \infty} \left| \sum_{\rho \neq \frac{x^{\beta-1}}{\rho(x+1)} \right|$$

(continued)
\[ \lim_{x \to \infty} \left( \sum_{|y| \leq R} \frac{x^{\beta - 1}}{|y|^{\beta + 1}} + \sum_{|y| > R} \frac{x^{\beta - 1}}{|y|^{\beta + 1}} \right) \leq \frac{1}{|p|^2} \]

\[ \delta \text{ but } |p + 1| \geq 1/p \] since \( \beta \geq -\frac{1}{2} \)

\[ \lim_{x \to \infty} \sum_{|y| > R} \frac{1}{|y|^2} \leq \delta \]

\[ < \varepsilon \]

Hence \( \psi_1(x) \sim \frac{x^3}{2} \) and we can repeat.

Lec 8 pp. 1–3.

This proof corresponds to Ingham 82 (middle).

Loosely Put:

Explicit Formula for \( \psi_1(x) \) plus \( \zeta(1+iy) \neq 0 \), \( y \in \mathbb{R} \), immediately implies the PNT.
It is now customary to define

\[ \Theta = \sup \{ \text{Re}(\rho) \}^2. \]

The Riemann Hypothesis is equivalent to stating that \( \Theta = \frac{1}{2} \). Obviously \( \frac{1}{2} \leq \Theta \leq 1 \).

**THM**

\[ \psi_1(x) = \frac{x^2}{2} + O(x^{\Theta+1}) \text{ for large } x. \]

**PF**

Obvious from p. 15 Thm since \( \sum_{\rho} \frac{1}{|\rho|^2} < \infty \).

**THM (Very Basic and Interesting)**

\[ \psi(x) = x + O(x^{\Theta \ln^2 x}) \]

\[ \pi(x) = \psi(x) + O(x^{\Theta \ln x}). \]

Here \( \psi(x) \equiv \int_2^x \frac{dt}{\ln t} \).
Corollary

Assume RH. Then:

\[ \Psi(x) = x + O\left( x^{\frac{1}{2} + \varepsilon} \ln^2 x \right) \]

\[ \pi(x) = \Pi(x) + O\left( x^{\frac{1}{2} + \varepsilon} \ln x \right) \]

Proof of Theorem

Know:

\[ \Psi'_1(x) = \frac{x^2}{2} + Ax + B + E(x) - \sum_{\rho \in \rho \cdot (\rho + 1)} \]

\[ E(x) = -\sum_{k=1}^{\infty} \frac{x^{1-2k}}{(2k)(2k-1)} \]

by \( \rho \cdot (\rho + 1) \)

Note that \( E(x) = b_1 x^{-1} + b_3 x^{-3} + b_5 x^{-5} + \cdots \) is a nice power series in \( x^{-1} \).

Also know:

\[ \frac{\Psi'_1(x) - \Psi'_1(x-h)}{h} \leq \Psi(x) \leq \frac{\Psi'_1(x+h) - \Psi'_1(x)}{h} \]

(x large)

For all \( 1 \leq h \leq \frac{x}{2} \) (say).
Look at upper part of the inequality.

\[
\frac{(x+h)^2 - x^2}{h} = x + \frac{h}{2}
\]

\[
\frac{A(x+h) + B - A - B}{h} = A
\]

\[
\frac{E(x+h) - E(x)}{h} = E\left(x + \frac{h}{2}\right), \quad 0 < h < h
\]

\[
= O(x^{-2}) \quad \text{by Taylor series}
\]

\[
\left| \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h^{\rho}(\rho+1)} \right| \leq \frac{(x+h)^{\Theta+1} + x^{\Theta+1}}{h \gamma^2}
\]

\[
\leq \text{(constant)} \frac{x^{\Theta+1}}{h \gamma^2}
\]

\{very crude\}
less crudely,

\[ \left| \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h \rho (\rho+1)} \right| = \frac{1}{h} \left| \int_{x}^{x+h} u^\rho \, du \right| \]

\[
\begin{align*}
\{ \text{no ambiguity: } & u^s \equiv \exp\{ s \ln u \} \} \\
u > 1
\end{align*}
\]

\[
\leq \frac{1}{h} \frac{1}{|\rho|} \int_{x}^{x+h} u^\Theta \, du
\]

\[
\leq \frac{1}{h} \frac{1}{|\rho|} (x+h)^\Theta
\]

\[
\leq \frac{(\text{constant}) x^\Theta}{|\gamma|}
\]

Hence,

\[
\left| \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h \rho (\rho+1)} \right| \leq (\text{const}) \min \left[ \frac{x^{\Theta+1}}{h \gamma^2}, \frac{x^\Theta}{|\gamma|} \right]
\]
\[ \leq (\text{const}) \frac{\Theta x^\Theta}{|y|} \min \left( \frac{x}{h|y|}, 1 \right) \]

\[ = (\text{const}) \frac{\Theta x^\Theta}{|y|} \left\{ \begin{array}{ll}
1 & \text{if } |y| < \frac{x}{h} \\
\frac{x}{h|y|} & \text{if } |y| > \frac{x}{h}
\end{array} \right\} \]

We thus get:

\[ \psi(x) \leq x + \frac{h}{2} + A + O(x^{-2}) \]

\[ + O(1) \sum_{1|y| < \frac{x}{h}} \frac{x^\Theta}{1|y|} \]

\[ + O(1) \sum_{1|y| > \frac{x}{h}} \frac{x^{\Theta+1}}{h|y|^2} \]

The lower part of (20) but will give similar if simply replace \( x + \frac{h}{2} \) by \( x - \frac{h}{2} \).
\[
\Psi(x) = x + O(h) + O(1) + O(x^{-q}) \\
+ O(1) \sum_{|y| < \frac{x}{h}} \frac{x^\Theta}{1+y} \\
+ O(1) \sum_{|y| > \frac{x}{h}} \frac{x^{\Theta+1}}{h y^2}.
\]

Here \( 1 \leq h \leq \frac{x}{2} \) and \( p \cdot 1 \) LEMMA applies.

\[
\Psi(x) = x + O(h) + O(1) x^\Theta \ln^2 \left(\frac{x}{h} \right) \\
+ O(1) x^{\Theta+1} \frac{\ln \left(\frac{x}{h} \right)}{1+y} \]

\[
\approx x + O(h) + O(1) x^\Theta \ln^2 \left(\frac{x}{h} \right)
\]

\[
\approx x + O(h) + O(1) x^\Theta \ln^2 \left(\frac{x}{h} \right)
\]

We get \( \Psi(x) = x + O(x^\Theta \ln^2 x) \) with \( h = 1 \).
Do calculus problems for safety.

Let $h = \frac{x}{t}$, $2 \leq t \leq x$

\[ h + x^{\theta} \ln^3 \left( \frac{x}{h} \right) = \frac{x}{t} + x^{\theta} \ln^3 (t) \]

\[ \Theta = 1 \Rightarrow x \left[ \frac{1}{t} + \ln^3 t \right] \Rightarrow \text{(const)} x \]

Minimum at $t = 2$

$\Theta < 1 \ (x \ large) \quad x^{\theta} \left[ \frac{x^{1-\Theta}}{t} + \ln^3 t \right]$

Derivative of bracket:

\[ -\frac{x^{1-\Theta}}{t^2} + \frac{2\ln t}{t} < 0 \]

iff

\[ \frac{2\ln t}{t} < \frac{x^{1-\Theta}}{t^2} \]

iff

\[ 2t \ln t < x^{1-\Theta} \]

\[ \Rightarrow \ t_{\text{critical}} \sim \frac{\frac{1}{2} x^{1-\Theta}}{(1-\Theta) \ln x} \]

\[ \Rightarrow \ \text{bracket min is} \ \approx (1-\Theta)^{\frac{3}{2}} \ln^2 x \]

\[ \Rightarrow \ \text{overall} \ (\text{const}) x^{\theta} \ln^3 x. \]
We now continue via

\[ \prod(x) \sim \log x + \int_2^x \frac{\psi(t) - t}{t \ln t^2} \, dt + O(1) \]

\[ \prod(x) = \pi(x) + \sum_{n=2}^{\infty} \frac{1}{n} \pi \left( x^{1/n} \right) \]

À la Lec 14 pp. \( \Theta \) + \( \Omega \) + \( \Omega \) (bottom)

\[ \pi(x) \sim \log x + \int_2^x \frac{\psi(t) - t}{t \ln t^2} \, dt + O(1) \frac{x^{1/2}}{\ln x} \]

\[ |\pi(x) - \log x| \leq O \left( \frac{x^{1/2}}{\ln x} \right) + O(1) \frac{x^{1/2}}{\ln x} + O(1) \int_2^x \frac{t \ln^2 t}{t \ln t^2} \, dt \]
\[ \ll 0(1) x^\Theta \ln x \\
\quad + O(1) \int_2^x t^{\Theta-1} dt \\
= 0(1) x^\Theta \ln x + O(1) \frac{1}{\Theta} x^\Theta \\
= 0(1) x^\Theta \ln x \]
2 HW problems

1. Prove rigorously that, for large $x$,
the number of primes in $(1, x]$ 

exceeds that in $(x, 2x]$. 

compare Leq 2 p. 20

2. Regarding Legendre and Ingham p. 2 (bottom).
Prove that there is exactly one constant $C$ 
such that 

$$\left| \pi(x) - \frac{x}{\ln x - C} \right| = O\left( \frac{x}{\ln^3 x} \right)$$

and that value is 1.
Before proceeding to the explicit formula for $\Psi(x)$, we treat an important connection between $\Theta$ (lec 16, p. 14) and the PNT. 

Let $\Theta' = \inf \{ \omega > 0 : \Psi(x) - x = O(x^\omega), \ x \geq 2 \}$.

\[ \begin{align*}
\text{Thm} \\
\Theta' = \Theta.
\end{align*} \]

\[ \begin{align*}
\text{Ingham 84}
\end{align*} \]

\[ \begin{align*}
\text{Pf} \\
\text{By lec 16 p. 14 2nd Thm,} \quad \Theta' \leq \Theta.
\end{align*} \]

Suppose now that $\Psi(x) = x + O(x^\omega), \ x \geq 2$. \[ \omega > 0 \]

For $\Re(s) > 1$, we immediately check

\[ -\frac{\frac{d}{ds} \zeta(s)}{\zeta(s)} = \int_1^\infty u^{-s} \psi(u) \, du = \int_1^\infty \frac{\psi(u) - u}{u^{s+1}} \, du \]

\[ \frac{s}{s-1} = s \int_1^\infty u \cdot u^{-s-1} \, du \]

\[ \frac{s}{s-1} = s \int_1^\infty u \cdot u^{-s-1} \, du \]

\[ -\frac{\frac{d}{ds} \zeta(s)}{\zeta(s)} - \frac{1}{s-1} - 1 = s \int_1^\infty \frac{\psi(u) - u}{u^{s+1}} \, du \]

\[ \text{see lec 8 p. 11} \]
The RHS is analytic wrt \( s \) for \( \text{Re}(s) > \omega \).

Note that \( \frac{\psi'(s)}{\psi(s)} + \frac{1}{s-1} \) has a removable singularity at \( s = 1 \). We thus find that

\[
\frac{\psi'(s)}{\psi(s)} + \frac{1}{s-1} = \text{analytic for Re}(s) > \omega.
\]

Clearly, for any \( \rho \) with \( E_0(\rho) = 0 \), we must then get \( \text{Re}(\rho) \leq \omega \). Hence \( \Theta \leq \omega \). Hence \( \Theta \leq \Theta' \).

In the near future, we will improve the theorem on p. 1.

One interprets p. 1 THM as saying that \( \Theta \) controls the size of the remainder term in \( \psi(x) - x \) or \( \Pi(x) - \zeta^{-1}(x) \). See here Lec 14 p. 8 BOX and 10 bottom. Also Lec 16 p. 26 - 27.

All of this will be improved/sharpened soon.

* Recall Lec 13, p. 4 THM. Also see: Lec 8, p. 10.
The discussion that I gave of the explicit formula for $\psi(x)$ can be seen as something having 2 basic stages:

(A) an initial "fleshing it out" in the style of Landau's *Vorlesungen über Z...*

(B) tightening that up - and strengthening it.

I follow the same procedure in these notes, but make some slight changes to streamline things.

We had

$$\psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{\Gamma(s+1)} \left( -\frac{t}{x} \right) ds$$

*à la* Lec 7. Here $c > 1$, $x > 0$. By a purely *formal* differentiation wrt $x$, one expects that

$$\psi(x)$$ is associated with

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s}}{s} \left( -\frac{t}{x} \right) ds$$

(basically)
Danger exists here since for each \( x \),

\[
\int_{-i\infty}^{c+i\infty} \frac{x^c}{|s|} |ds| = +\infty \quad \text{Ie absolute convergence fails!}
\]

By Lec 7, we expect to study

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s} \, ds
\]

for \( y > 0 \).

**Fundamental Lemma (Perron + others)**

Keep \( 0 < y < \infty \), \( 0 < c \leq 2 \), \( T \geq 3 \).

Let

\[
\eta(y) = \begin{cases} 
1, & y > 1 \\
\frac{1}{2}, & y = 1 \\
0, & 0 < y < 1
\end{cases}
\]

We then have

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s} \, ds = \eta(y) + \text{Remainder}
\]

where
One also has, if one is willing to be quite crude,

\[ |\text{Remainder}| \lesssim \begin{cases} \frac{y^c}{\pi T |\ln y|}, & y \neq 1 \\ \frac{c}{\pi T}, & y = 1 \end{cases} \]

for \( 0 < y < 1 \) with an "implied" constant which depends solely on \( A \).

**Proof**

Take \( 0 < y < 1 \) first. Look at \( \frac{1}{2\pi i} \int_{\partial R} \frac{y^s}{s} \, ds \) on

\[
\begin{array}{c|c}
\text{c+iT} & \Delta+iT \\
\hline
\text{c-iT} & \Delta-iT
\end{array}
\]

with \( T \) frozen and let \( \Delta \to \infty \). Note that

\( y^\Delta \to 0 \) along \( [\Delta-iT, \Delta+iT] \). By the Cauchy Integral Theorem, we get

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} = \frac{1}{2\pi i} \int_{c-iT}^{\infty-iT} + \frac{1}{2\pi i} \int_{\infty+iT}^{c+iT}.
\]
At once,

\[ | L H S | \sim \frac{2}{2\pi} \int e^{-\sigma |\ln y|} d\sigma \quad (s = \sigma \pm iT) \]

\[ = \frac{1}{\pi T} \int e^{-\sigma |\ln y|} d\sigma \]

\[ = \frac{1}{\pi T} \frac{e^{-c |\ln y|}}{|\ln y|} = \frac{1}{\pi T} \frac{y^c}{|\ln y|} \]

Take \( I y < \infty \) next. Here we use \( \frac{y^{s}}{s} \) and

\[ \begin{array}{c}
-\Delta + iT \\
R \\
-\Delta - iT
\end{array} \]

Notice that \( \text{Res} \left\{ \frac{y^{s}}{s} | s = 0 \right\} = 1 \) (simple pole).

Freeze \( T \) and let \( \Delta \to \infty \) again. By the Cauchy Residue Theorem (or Cauchy Integral Formula), get:

\[ \frac{1}{2\pi i} \int_{c-iT}^{c+iT} = 1 - \frac{1}{2\pi i} \int_{-\infty-iT}^{c-iT} - \frac{1}{2\pi i} \int_{c+iT}^{\infty+iT} \]

\[ = 1 - \frac{1}{2\pi i} \int_{-\infty-iT}^{c-iT} + \frac{1}{2\pi i} \int_{c+iT}^{\infty+iT} \]

\[ = 1 - \frac{1}{2\pi i} \int_{-\infty-iT}^{c-iT} + \frac{1}{2\pi i} \int_{c+iT}^{\infty+iT} \]
At once

\[
\text{\textbf{Remainder}} = \frac{2}{\alpha_0} \int_{-\infty}^{c} \frac{y^0}{|s|} \, ds \quad (s = s \pm i\tau)
\]

\[
\text{\textbf{Remainder}} = \frac{1}{\pi T} \left[ \int_{-\infty}^{c} e^{\sigma(\ln y)} \, d\sigma \right] \quad (y > 1)
\]

\[
= \frac{1}{\pi T} \left. \frac{e^{\sigma(\ln y)}}{\ln y} \right|_{c}^{0} = \frac{y^c}{\pi T (\ln y)}
\]

For \( y = 1 \) we notice that \( (0 < c \leq 2) \)

\[
\frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} \frac{1}{s} \, ds = \frac{1}{2\pi i} \left[ \text{Log} s \right]_{c-i\tau}^{c+i\tau}
\]

\[
= \frac{1}{2\pi i} \left[ \text{Log} (c+i\tau) - \text{Log} (c-i\tau) \right]
\]

\[
= \frac{1}{\pi} \text{Arg} (c+i\tau)
\]

\[
= \frac{1}{\pi} \arctan \frac{T}{c}
\]

\[
= \frac{1}{\pi} \left[ \int_{0}^{\infty} \frac{dz}{1+z^2} - \int_{T/c}^{\infty} \frac{dz}{1+z^2} \right]
\]

\[
= \frac{1}{2} - \frac{1}{\pi} \int_{T/c}^{\infty} \frac{dz}{1+z^2}
\]

\[
\hspace{2cm} \Psi
\]

\[
\text{\textbf{Remainder}} < \frac{1}{\pi} \left( \frac{1}{T/c} \right) = \frac{c}{\pi T}
\]
To conclude we assume that $0 < y < A$ for some $A \geq 2$ (wlog).

For $0 < y < 1$, look at

$$\text{radius } R = \sqrt{c^2 + \tau^2}$$

and get

$$\frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} = \frac{1}{2\pi i} \int_{\mathcal{C}_R} \frac{y^s}{s} \, ds$$

by CIT

\[ \Rightarrow \]

$$|\text{Remainder}| \leq \frac{1}{2\pi} \int_{\mathcal{C}_R} \frac{|y^s|}{|s|} |ds|$$

\[ \leq \frac{1}{2\pi} \cdot \frac{y^c}{R} \cdot (\pi R) \]

\[ \leq \frac{1}{2} \]

since $0 < c \leq 2$ (and $0 < y < 1$).
Next, consider $1 \leq y < A$. Use

$$R = \sqrt{c^2 + R^2}$$

and get

$$\frac{1}{2\pi i} \int_{c-iR}^{c+iR} = 1 + \frac{1}{2\pi i} \int_{C_R} \frac{y^s}{s} \, ds$$

$$\Rightarrow \eta(1) = \frac{1}{2}!$$

$$|\text{Remainder}| \leq \frac{1}{2} + \frac{1}{2\pi} \int_{C_R} \frac{y^s}{s!} \, ds$$

$$\leq \frac{1}{2} + \frac{1}{2\pi} \frac{y^c}{R} \left(2\pi R \right)$$

$$\leq \frac{1}{2} + \frac{1}{2} y^c$$

but $0 \leq c \leq 2$ and $1 \leq y < A \Rightarrow \frac{1}{2} + \frac{1}{2} A^2$.
Corollary of lemma

\[ \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s} \, ds = \gamma(y) = \begin{cases} 
1, & y > 1 \\
\frac{1}{2}, & y = 1 \\
0, & y < 1 
\end{cases} \]

for \( 0 < y < \infty \), \( 0 < c \leq 2 \).

Guided by p. 3 bottom, we now turn our attention to

\[ \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \left[ \frac{1}{s} - \frac{1/3}{s^{1/3}} \right] \, ds \]

under the hypothesis that

\[ x \geq 1 + \delta_0 \quad \text{AND} \quad 1 < c \leq 2 \]

Here \( \delta_0 \) is some small positive constant (e.g., \( 1/2 \)) which we fix once and for all.

N.B. Landau, Vorlesungen über Z... likes \( c = 2 \) and arbitrary \( x > 1 \).
Since we plan to use p.14 Fud Lemmas, we also insist that
\[ T \geq 3. \]

Conceptually, in phase (A) on p.3, it is simplest to declare \( c = 2 \) [for instance] like Landau does, and then stay with that.

To facilitate making improvements, we prefer however [in contrast to what was done in the lectures] to keep \( c \) free for the time being.
Let $\mathcal{F}$ be the set $\{p^m : p = \text{prime}, m = \text{pos. integer}\}$.

Obviously $\mathcal{F} \subseteq \mathbb{Z} \cap [2, \infty)$.

Let $\| u \|' = \begin{cases} |u| & \text{if } u \neq 0 \\ \infty & \text{if } u = 0 \end{cases}$. Here $u \in \mathbb{R}$.

For $x \geq 1 + \delta_0$, let $\xi(x) = \min_{\lambda \in \mathcal{F}} |x - \lambda|$. Also write

$$\langle x \rangle = \min \{ \frac{1}{100}, \| \xi(x) \|' \}$$.

Notice that $\langle x \rangle = \frac{1}{100}$ unless $\| \xi(x) \|' < \frac{1}{100}$, which would mean that $x \notin \mathcal{F}$ but $x$ lies LESS THAN $\frac{1}{100}$ units from $\mathcal{F}$.

In particular, we see that:

$$\langle x \rangle = \frac{1}{100} \quad \text{anytime } x \in \mathbb{Z}$$.

In all cases, obviously:

$$0 < \langle x \rangle \leq \frac{1}{100}$$.
Lemma 2

Keep $x \geq 1 + \delta_0$, $1 < c \leq 2$, $T \geq 3$.

Let

$$
\psi^*(x) = \frac{\psi(x+0) + \psi(x-0)}{2}.
$$

We then have:

$$(i) \quad \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \left(-\frac{f(s)}{f(1)}\right) ds = \psi^*(x) + O\left[ \frac{x \ln x}{T(c-\Re s)} \right] + O\left( \frac{\ln x}{T(c-\Re s)} \right) + O\left( \frac{\ln x}{T} \right).$$

$$(ii) \quad \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \left(-\frac{f(s)}{f(1)}\right) ds = \psi^*(x) + O\left[ \frac{x \ln x}{T(c-\Re s)} \right] + O\left( \ln x \right).$$

The implied constants will depend on \( \delta_0 \).
Proof

We use p.\textnumero 4 \textbf{LEMMA}. Get:

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \left( \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \right) ds = \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{(X/n)^s}{s} ds
\]

\[
= \sum_{n=1}^{\infty} \Lambda(n) \left\{ \begin{array}{ll}
\frac{\lambda_2}{\pi T} \frac{(X/n)^s}{1} & \text{if } n \neq X \\
\frac{2\lambda_2}{\pi T} & \text{if } n = X \\
\mathcal{O}(1) & \text{if } \frac{x}{A} < n \leq \infty
\end{array} \right. \]

Here \(|\lambda| \leq 1\). Clearly

\[
\sum_{n=1}^{\infty} \Lambda(n) \eta \left( \frac{x}{n} \right) = \Psi^*(x) \quad .
\]

We first look at (\textit{i}). This will use chunks \#1 and \#2 in the big bracket.\]
Make the obvious

\[ 1 \leq n \leq \|x\| - 1 \]

\[ n = \|x\| \quad \Leftarrow \quad n_1 \]

\[ n = \|x\| + 1 \quad \Leftarrow \quad n_2 \]

\[ n \geq \|x\| + 2 \]

Splitting.

\[ \sum_{n=\|x\|+2}^{\infty} \Lambda(n) \frac{n}{\pi T} \frac{(x/n)^c}{\ln x - \ln n} \]

\[ = \sum_{n=\|x\|+2}^{\|x\|+4} + \sum_{n=\|x\|+5}^{\infty} \]

Must bound the expressions in absolute value.

\[ n \geq \|x\| + 5 \implies n \geq 2(\|x\|+1) > 2x \]

So, \( \ln(n/x) > \ln 2 \). Get:

\[ \leq \frac{x^c}{T} \sum_{n=\|x\|+5}^{\|x\|+4} \frac{\Lambda(n)}{n^c} \frac{1}{\ln 2} \]

\[ = O \left( \frac{x^c}{T} \right) \left[ \frac{1}{c-1} + O(1) \right] = O \left( \frac{x^c}{T} \right) \frac{1}{c-1} \]
$$\frac{f(x)}{\log x} = \sum_{n \leq x} \frac{\Lambda(n)}{\log n} \sim \frac{1}{\log x} + O(1)$$ \quad \text{for} \quad 1 < x \leq 2$$

Next, \quad \lfloor x \rfloor + 2 \leq n \leq 2 \lfloor x \rfloor + 4. \quad \text{Get}:

$$\eta \sim \frac{1}{T} \sum_{\lfloor x \rfloor + 2}^{2 \lfloor x \rfloor + 4} \Lambda(n) \left( \frac{x}{n} \right)^c \frac{1}{\log n - \log x}$$

$$\eta \sim \frac{1}{T} O(\ln(10x)) \sum_{\lfloor x \rfloor + 2}^{2 \lfloor x \rfloor + 4} \frac{1}{\log n - \log x}$$

apply mean value theorem to \( \log n - \log x \),

get final sum

$$\eta \sim O(1) \sum_{\lfloor x \rfloor + 2}^{2 \lfloor x \rfloor + 4} \frac{1}{x \log(n-x)}$$

$$\leq O(1) x \cdot O(1) \ln(10x)$$ \quad \text{note here that this estimate is trivial if, say,} \quad x \leq 1000$$

$$\leq \frac{x}{T} O(1) (\ln 10x)^2 \leq \frac{x}{T} O(1) \ln^2 x$$ \quad \text{for} \quad x \geq 1 + \delta_0$$

\text{But:} \quad \frac{x}{\log x} \sim (\text{const}) x \ln x \quad \text{for} \quad x \geq 1 + \delta_0.
This follows by elem calculus; the "const" will depend on \( \delta_0 \).

\[
\begin{align*}
\left\{ \begin{array}{l}
\text{For } x > e, \text{ the fcn } \frac{x^v}{v} \text{ on } 0 < v \leq 1 \\
\text{has its min at } v = \frac{1}{\ln x} \\
\end{array} \right.
\end{align*}
\]

Accordingly: for \( x \geq 1 + \delta_0 \)

\[
\frac{x^c \ln x}{T(c-1)} \approx (\text{const}) \frac{x \ln^2 x}{T}.
\]

Hence,

\[
\sum_{x \leq 1 + \delta_0} \frac{1}{T} A(n) \left( \frac{x}{n} \right)^c \frac{1}{\ln n - \ln x} \approx O(1) \frac{x^c \ln x}{T(c-1)}.
\]

The "implied" constant will depend on \( \delta_0 \).

---

Pause for a moment!

In Lec 17 with \( c = 2 \), what I remarked following Landau was that:

(proceeding somewhat crudely... )
\[
\sum_{\pi \times \pi + 2} \frac{1}{n} L(n) \left( \frac{x}{n} \right)^2 \frac{1}{\ln n - \ln x}
\]

= \( O \left( \frac{x^2}{T} \right) \) + \( \sum_{\pi \times \pi + 2} \frac{1}{n} L(n) \left( \frac{x}{n} \right)^2 \frac{1}{\ln n - \ln x} \)

= \( O \left( \frac{x^2}{T} \right) \) + \( \frac{x^2}{T} \sum_{\pi \times \pi + 2} \frac{L(n)}{n^2} \frac{1}{(\text{const}) \frac{1}{x}(n-x)} \)

= \( O \left( \frac{x^2}{T} \right) \) + \( \frac{x^2}{T} \sum_{\pi \times \pi + 2} \frac{\ln(10x)}{x^2} \frac{\text{const}}{n-x} \)

\leq \( O \left( \frac{x^2}{T} \right) \) + (\text{const}) \frac{x^2}{T} \ln(10x) \cdot O(1)

= \( O(1) \frac{x^2 \ln x}{T} \) \quad \text{for } x \geq 1 + 50

END of PAUSE. \{ \text{See Landau, Vorlesungen, proof of } \}

\text{Satz 451 near (564) } \}

---

\text{Step } \( \text{II } \circ \)

\[
\sum_{n=1}^{\pi \times \pi - 1} L(n) \frac{1}{n^2} \left( \frac{x}{n} \right)^2 \frac{1}{(\ln x - \ln n)}
\]

= \sum_{n < \frac{1}{T} \pi \times \pi} + \sum_{\frac{1}{T} \pi \times \pi \leq n \leq \pi \times \pi - 1}

Expect things to be similar to step \( \text{I } \circ \).
\[ \sum_{n \leq \frac{T}{2}} A(n) \frac{1}{T} \frac{(x/n)^s}{(\ln x/n)^s} \]

\[ \leq \sum_{n \leq \frac{T}{2}} A(n) \frac{x^n}{n^c} \frac{1}{\ln 2} \]

\[ = O\left(\frac{x^n}{T^{c-1} \ln 2}\right) \]

Also,

\[ \sum_{\frac{T}{2} \leq x \leq T} A(n) \frac{1}{T} \frac{(x/n)^s}{(\ln x/n)^s} \]

\[ \leq \frac{O(\ln(10x))}{T} \left(\text{const.}\right) \sum_{\frac{T}{2} \leq x \leq T} \frac{x}{x-n} \]

\[ \leq O(1) \frac{x \ln(10x)}{T} \cdot \ln(10x) \]

\[ \sim \frac{x}{T} O(1) \ln^2 x \quad \text{for} \quad x \geq 1 + 5 \delta \]

Line 6 on (17) is still valid, so we get
\[ \sum \frac{1}{T} \lambda(n) \left( \frac{x}{n} \right)^c \frac{1}{\ln x - \ln n} \]

\[ \approx \frac{x^c}{T} \frac{1}{\ln x} \quad \text{as} \quad n \to \infty \]

Recall \( n_1 = \lfloor x \rfloor, \quad n_2 = \lfloor x \rfloor + 1 \) on the top.

By (14), we must still check that:

\[ \sum_{n \in \{n_1, n_2\}} \lambda(n) \left[ \frac{1}{T} \left( \frac{x/n}{\ln x - \ln n} \right)^c \right] \quad n \neq x \]

\[ \sum_{n \in \{n_1, n_2\}} \lambda(n) \left[ \frac{1}{T} \frac{x^c}{\ln x} \right] \quad n = x \]

\[ = O \left[ \frac{x^c}{T(c-1)} \right] + O \left[ \frac{x^c}{T\langle x \rangle} \right]. \]

**Step III A**

Suppose first that \( x = \) integer \((\geq 2)\).

Here \( \langle x \rangle = \frac{1}{100} \) by (12), \( n_1 = x, \) \( n_2 = x+1 \), and

\[ \text{LHS} = \lambda(n_1) \frac{1}{T} + \lambda(n_2) \frac{1}{T} \left( \frac{x/n_2}{\ln x - \ln n_2} \right)^c \]
\[
\begin{align*}
&= \frac{O(\ln x)}{T} + \frac{O(\ln (x+1))}{T} \frac{1}{\ln (x+1) - \ln x} \\
&= \frac{O(\ln x)}{T} + \frac{O[ x \ln (x+1) ]}{T} \quad \{ \text{by mean value thm} \}
\end{align*}
\]
which is subsumed by both \( \frac{x \ln x}{T} \) and \( \frac{x \ln x}{T} \).

\[
\text{Step III B}
\]
Suppose next that \( x \neq \text{integer} \) (\( x \geq 1 + \delta_0 \)).

For each \( j \), notice that (in LHS on (20)):

\[
\begin{align*}
\Lambda(n_j^*) \frac{1}{T} \left( \frac{x}{n_j^*} \right)^2 \frac{1}{|\ln x - \ln n_j^*|} \\
&= \Lambda(n_j^*) \frac{O(1)}{T} \frac{x}{|x - n_j^*|} \quad \{ \text{by mean value thm} \}
\end{align*}
\]
If \( |x - n_j^*| \geq \frac{1}{100} \), the foregoing bound is

\[\text{ie, TERM}\]
\[ O(\ln x) \frac{x}{T} \langle 100 \rangle = O \left[ \frac{x/\ln x}{T} \langle x \rangle \right] \text{.} \]

This is obviously subsumed by \( O \left[ \frac{x/\ln x}{T} \langle x \rangle \right] \).

\[ \langle x \rangle \leq \frac{1}{100} \text{ by (12)} \]

On the other hand, suppose \(|x - n_j^o| < \frac{1}{100}\). The term on (21) bottom is either

\[ 0 \]

or else \( O(\ln x) \frac{O(1)}{T} \frac{x}{\langle x \rangle} \)

by recalling the def of \( \langle x \rangle \) again. Here, then, we again get something subsumed by \( O \left[ \frac{x/\ln x}{T} \langle x \rangle \right] \).

Both cases in the "either" can occur so the \( O \left[ \frac{x/\ln x}{T} \langle x \rangle \right] \) is essentially sharp.

Bottom line:

\[ \begin{cases} \text{for } x \neq \text{integer} \\ \text{LHS on (30)} \end{cases} = O(1) \frac{x/\ln x}{T} \langle x \rangle \text{.} \]

\[ \text{(OK) III B} \]
By the OK's, we get: in \((\text{II})\)

\[
\sum_{n \in \{m_1, m_2, \ldots\}} A(n) \left[ \frac{\frac{1}{T} \left( \frac{\gamma n}{\ln x - \ln n} \right)^{\frac{1}{T}}}{\ln x - \ln n} \right]_{n \neq x} + \left[ \frac{1}{T} \right]_{n = x} = O(1) \frac{x \ln x}{T(x)} \cdot \]

All told, chunks #1 and #2 in the big bracket on \((\text{IIV})\) lead to an error term [à la \((\text{I})\), \((\text{IV})\), \((\text{V})\), \((\text{VII})\), and line 3 above] of

\[
O(1) \frac{x \ln x}{T(x)} + O(1) \frac{x \ln x}{T(e - 1)} \quad \text{for } x \geq 1 + \delta_0, 
\]

where the "implied constants" depend on \(\delta_0\).

Assertion (\(i^*\)) on p. \((\text{I})\) is thus proved.

One expects that \((\text{ii}^*\) will be VERY similar — with use of chunk #3 in the big bracket on \((\text{IIV})\) at an appropriate point.

Note that steps \((\text{I})\) and \((\text{II})\) are OK as \(= \rangle i^*\).
We need only fiddle with step (III). See (20) and (14) bracket.

\[ n_1 = \lceil x \rceil, \quad n_2 = \lceil x \rceil + 1 \]

\( \left\{ \begin{array}{l} n_j^\ast \text{ is relevant} \\ \text{only if } \lambda(n_j^\ast) \neq 0 \end{array} \right. \)

Want \( \frac{x}{A} < n_j^\ast < \infty \)

\[ A=1 \text{ OK for } j=2 \]

for \( j=1 \), \( n_1 \) is relevant only if \( \|x\| \geq 2 \)

whereupon \( A=2 \) is adequate

\[ \sum_{n \in \{n_1, n_2\}} \lambda(n) \left[ O_2(1) \text{ wlog} \right] \]

\( \leq O(1) \ln(x+1) \leq O(1) \ln x \)

\[ \text{for } x \geq 1 + \delta_0. \]

Page (13) assertion (ii') follows at once.

In Lec 17, following Landau (\( c=2 \)), we got

\[ O\left(\frac{x^2}{\ln x}\right) + O(\ln x) \]

for assertion (ii').
To continue, consider \( \{x, c, T\} \) as on \( \square + \square \) top.
Assume that
\[
T = \text{all } y \neq \gamma
\]
(rectangle)
Form \( \chi \) as on \( p \cdot 6 \) with \( A = 2m+1 \). Treat \( \{x, c, T\} \) as frozen for a few moments. By letting \( m \to \infty \), we immediately get
\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \left( \frac{\zeta(s)}{\zeta(3)} \right) ds + \frac{1}{2\pi i} \int_{c+iT}^{-a_0+iT} \]
\[
+ \frac{1}{2\pi i} \int_{-a_0-iT}^{c-iT} = \sum \text{Res}
\]
\[
= x - \frac{T(0)}{T(0)} - \sum_{k=1}^{\infty} \frac{x^{-2k}}{(-2k)} - \sum_{|y| \leq T} \frac{x^y}{y}
\]
thanks to
\[
\frac{T'(s)}{T(s)} = O(1) \ln (|s| + 10) \quad \text{for } s \in \mathcal{E}.
\]
a fact proved in Lec 16 pp. 5 - 9.

Unfreeze \( x, c, T \)!
The horizontal integrals have \( s = \sigma \pm iT \) and are absolutely convergent (since \( x > 1 \)).

To estimate them, it suffices to look at

\[
\left| \frac{1}{2\pi i} \int_{c+iT}^{\infty+iT} \frac{x^s}{s} \left(-\frac{\Gamma'(s)}{\Gamma(s)}\right) ds \right|.
\]

For \( -1 \leq \sigma \leq 2 \), we know that

\[
\frac{\Gamma'(s)}{\Gamma(s)} = O(\ln T) + \sum_{|\gamma-T| = 1} \frac{1}{s-\gamma}
\]

when \( s = \sigma + iT \). See Lec 15 p. 13. One inflates the implied constant in \( O(\ln T) \) to handle moderate \( T \).

The portion of the horizontal integral arising from \( (-\infty+iT, -1+iT] \) is clearly

\[
O(1) \int_{-\infty}^{-1} \frac{x^\sigma}{|s|} O(\ln |s|) ds \quad \text{by } \frac{\Gamma'(s)}{\Gamma(s)} \text{ estimate}
\]

\[
\leq O(1) \frac{\ln T}{T} \int_{-\infty}^{-1} x^\sigma d\sigma
\]

\[
\leq O(1) \frac{\ln T}{T} \frac{x^{-1}}{\ln x} \leq O(1) \frac{\ln T}{T} x^{-1} \quad \{ x \geq 1 + \delta_0 \}.
\]
To treat the stretch \([\frac{-1+iT}{c+iT}]\), we must look at

\[
\left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s} \left[ O(\ln T) + \sum_{|\gamma-n| \leq 1} \frac{1}{s-n} \right] ds \right|
\]

wherein \(s = \sigma + iT\). We stress that the bracket is a continuous function of \(\sigma\).

The portion with \(O(\ln T)\) clearly gives:

\[
O(\ln T) \int_{-1}^{c} \frac{x^\sigma}{T} d\sigma
\]

\[
= O\left(\frac{\ln T}{T}\right) \int_{-1}^{c} x^\sigma d\sigma
\]

\[
= O\left(\frac{\ln T}{T}\right) \int_{-\infty}^{c} x^\sigma d\sigma
\]

\[
= O\left(\frac{\ln T}{T}\right) \frac{x^c}{\ln x}
\]

\[
= O\left(\frac{\ln T}{T}\right) x^c \quad \{ x \geq 1 + \delta_0 \}\]

Compare 26 bottom.
To handle the rest, we look at

\[ \sum_{\rho} \left| \frac{1}{2\pi i} \int_{-1+ iT}^{c+ iT} \frac{x^5}{\zeta(s-\rho)} \, ds \right| \]

where \(|\gamma-\rho| \leq 1\)

using the Cauchy integral theorem on each integral separately.

We will not seek the best possible estimate (especially for \( c \) very close to 1), just something that leads to an explicit formula for \( Y(x) \) of a quality comparable to the best that is currently known for practical use.

Remember that \(1 < c \leq 2\). Also that \( \rho = \beta + iy \) has \(0 \leq \beta \leq 1\). We propose to deform \([-1 + iT, c + iT]\).

A rectangle of height \( \frac{1}{2}(c+1) \) can also be used.
$T < y \leq T+1 \Rightarrow$ make similar $C$, but go down. 

For $s \in C$ notice first that:

$$|s-\rho| \geq |s-(\beta+iT)|.$$ 

Elem geometry shows that the sliding circle

$$\frac{1}{2} |s-(\gamma+iT)| = \frac{1}{a} (c-1)^2$$

never intersects $C$ for $0 \leq \gamma \leq 1$. [At $\gamma = 0$, $\frac{c-1}{a} < 1$.]

Accordingly, $s \in C \Rightarrow |s-\rho| = \frac{1}{2} (c-1)$ for $1 \leq c \leq 2$. 

For EACH $\rho$, get:

$$\left| \frac{-1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s(s-\rho)} ds \right|$$

$$= \left| \frac{1}{2\pi} \int_C \frac{x^s}{s(s-\rho)} ds \right| \quad (\text{by CIT})$$

\[\n\]

\[\n\]

\[\n\]
Since \( N[\rho : |y-T| \leq 1] = O(\ln T) \) by Lec 15 p. 8, we now get:

\[
\left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s} \left[ \sum_{1 \leq |y-T| \leq 1} \frac{1}{s-\rho} \right] ds \right| \leq O(\ln T) \frac{1}{T^{\frac{c}{2}}} \cdot
\]

By (27) below, we get:

\[
\left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s} \left[ O(\ln T) + \sum_{1 \leq |y-T| \leq 1} \frac{1}{s-\rho} \right] ds \right| = O\left(\frac{\ln T}{T}\right) \frac{x^c}{c-1} \cdot
\]

By (26), we thus see that:

\[
\frac{1}{2\pi i} \int_{-\infty + iT}^{\infty + iT} \frac{x^s}{s} \left( -\frac{\xi'(s)}{\xi(s)} \right) ds = O\left(\frac{\ln T}{T}\right) \frac{x^c}{c-1} \cdot
\]

For \( 1 < c \leq 2 \), \( x \geq 1 + \delta_0 \), \( T \geq 3 \), \( T \neq \text{all } y \).
Referring to (25) middle and (13), we find that:

\[ \psi^*(x) + O\left(\frac{x^c \ln x}{T^{c-1}}\right) + \left\{ \begin{array}{c}
O(1) \ln x \\
O\left(\frac{x \ln x}{T^{c-1}}\right)
\end{array} \right\} \]

\[ + O\left(\ln \frac{1}{T}\right) \frac{x^c}{c-1} \quad \text{by (30)} \]

\[ = x - \frac{f'(0)}{f(0)} + \frac{1}{\alpha} \sum_{k=1}^{\infty} \frac{x^{-2k}}{k} - \sum_{\|y\| \leq T} \frac{x}{\gamma^p} \]

for \( x \geq 1 + \delta_0, \quad T \geq 3, \quad T \neq \text{all } y, \quad 1 < c \leq 2. \)

Notice that, for any small \( h \in (0,1) \),

\[ \left| \sum_{|y-t| \leq h} \frac{x^p}{\gamma^p} \right| \leq O\left(\ln \frac{1}{T}\right) \frac{x}{t} \leq O\left(\ln \frac{1}{T}\right) x \]

anytime \( t \geq \frac{5}{2} \). Similarly for \( |y+t| \leq h \).

Here \( x \geq 1 + \delta_0 \), as usual.
Holding $\{x, c\}$ fixed for a moment, we can now apply right continuity in $T$ to address cases on top where $T \geq 3$, but $T = \text{some } \gamma$.

\[ \psi^*(x) = x - \frac{\gamma(0)}{\gamma(0)} - \frac{1}{\alpha} \ln (1 - x^{-2}) + \sum_{1 \leq i \leq T} x^\rho \]

\[ + O\left( \frac{\ln T}{T} \right) \frac{x^c}{c-1} \]

\[ + O\left( \frac{x^c \ln x}{T(c-1)} \right) + \left\{ O(\ln x) \right\} \]

\[ + O\left[ \frac{x \ln x}{T(c-1)} \right] \]

for $x \geq 1 + \delta_0$, $1 < c \leq 2$.

\[ \text{NB} \]

Compare: Ingham, p. 77, wherein $c = 2$, AND Landau, Vorlesungen, Satz 452 (wherein $c = 2$). Note how our formula is better than both, via taking $c = 3/2$. 
To optimize $32$ middle, one basically wants to minimize

$$\frac{X}{c}$$

where $b = a$ tiny constant.

Here $x \gtrsim 1+\delta_0$, and we need to have $1 < c \leq 2$.

See (17) top. Since $c$ is free in $32$ middle, this choice of $c$ is completely legal.

For $x \gtrsim 1+\delta_0$, $T \gtrsim 3$, we thus get:

$$\psi^*(x) = x - \frac{\tau(0)}{\delta(0)} - \frac{1}{2} \ln(1-x^2) - \sum_{|\lambda| \leq T} \frac{x^\lambda}{\lambda}$$

$$+ O\left(\frac{x \ln T \cdot \ln x}{T}\right) + O\left(\frac{x \ln^2 x}{T}\right)$$

$$+ O(\ln x) \min \left\{ 1, \frac{x}{T\langle x \rangle} \right\}.$$ 

The implied constants will depend [solely] on $\delta_0.$
The "trivial terms"

\[- \frac{f(0)}{f(0)} = \frac{1}{2} \ln(1-x^{-2}) \quad \text{on (33)}\]

define an obvious power series in $x^{-2}$ and are often simply replaced by $O(1)$. Insofar as that is done, once any given remainder term on both drops below $-\Delta x^{-6}$, say, (with $|\Delta| \leq 1$) that term takes on a patently secondary role vis-à-vis $x$, especially if $x \to \infty$.

That being said, we now observe that:

(A) For $x \geq 1 + \delta_0$ and $T \geq 3x^{10}$ (say),

\[
\frac{x \ln T \ln x}{T} + \frac{x \ln^3 x}{T} \leq \frac{x(\ln T + \ln x)^2}{T} = O(x^{-8})
\]

(B) For $x \geq 1 + \delta_0$ and $3 \leq T \leq 3x^{10}$,

\[
c(\delta_0) \frac{x}{T} (\ln T + \ln x)^2 \leq \frac{x}{T} (\ln T \ln x + \ln^2 x) \leq \frac{x}{T} (\ln T + \ln x)^2
\]

This final chunk scales like $<1.144 \frac{x \ln^2 x}{T}$ if $x \leq 3$. 
where \( c(\delta_0) \) is some appropriately tiny positive constant.

**Theorem (Standard statement of Explicit Formula for \( \Psi(x) \))**

Let \( x \approx 1 + \delta_0 \), \( T \geq 3 \). Define \( \mathcal{I} \) and \( \langle x \rangle \) as on (12). We then have:

\[
\Psi^*(x) = x - \sum_{|\lambda| \leq T} \frac{x^\lambda}{\lambda} - \frac{\mathcal{I}(0)}{\mathcal{I}(0)} - \frac{1}{2} \ln (1 - x^{-2})
\]

\[
+ O\left[ \frac{x}{T} (\ln T + \ln x)^2 \right]
\]

\[
+ O(\ln x) \min \left\{ 1, \frac{x}{T \langle x \rangle} \right\}
\]

wherein

\[
\Psi^*(x) = \frac{\Psi(x+0) + \Psi(x-0)}{2}
\]

The implied constants will depend on \([\text{at most}] \delta_0\). In addition, one has:

\[
\frac{\mathcal{I}(0)}{\mathcal{I}(0)} = \ln (2\pi)
\]
Proof
See (33) (bottom) and then the obvious relation

\[
\frac{x \ln T \ln x}{T} + \frac{x \ln^2 x}{T} \leq \frac{x (\ln T + \ln x)^2}{T}
\]

used on (34). This proves the formula for \( \psi^*(x) \). OK

We'll verify \( \frac{s(0)}{s(0)} = \ln(2\pi) \) in a theorem stated several pages below. See (41).

No B. The formula on (35) middle can be found many places; e.g., in Davenport, *Multiplicative Number Theory*, 2nd ed., p. 109 (10.10) OR Prachar, *Primzahl-verteilung*, Satz 4.5 on pp. 231-2.
We define:

\[ \sum_{\rho} \frac{x^\rho}{\rho} \equiv \lim_{T \to \infty} \sum_{1 \leq \gamma \leq T} \frac{x^\rho}{\rho}. \]

Recall (31) bottom concerning slight "slippiness" in \( T \).

**Corollary 1.**

For each \( x \geq 1 + 50 \), we have

\[ \psi^*(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \ln(1 - x^{-2}) \]

In this regard, we also have (in an obvious sense)

\[ \sum_{|\gamma| > T} \frac{x^\rho}{\rho} = O \left[ \frac{x}{T} \left( \ln T + \ln x \right)^2 \right] + O(\ln x) \min \{ \frac{1}{T} + \alpha, \frac{x}{T + x} \} \]

If desired, the 1st term on the RHS can be replaced by

\[ O \left[ \frac{x}{T} \left( \ln T + \ln x + \ln^2 x \right) \right] \]

**Pf.**

Straightforward. See (33) bottom for last assertion.

---

* As already hinted in the two boxes on (28), the term \( \frac{x \ln T \cdot \ln x}{T} \) can in fact be improved slightly. This will not affect the estimates for \( \psi(x) - x \) though. See (34). Also p. (39).
Corollary 2.

Let \([x_1, x_2]\) be any closed interval in \([1 + \delta_0, \infty)\).

(a) If \([x_1, x_2] \cap \mathbb{E} = \emptyset\), then \(\sum \frac{x^\rho}{\lambda^\rho}\) converges uniformly on \([x_1, x_2]\) as a symmetric limit in \(\mathbb{T}\).

(b) In every instance, the partial sums \(\sum_{|\lambda| \leq T} \frac{x^\rho}{\lambda^\rho}\) are uniformly bounded on \([x_1, x_2]\) for all \(T \geq 3\).

Pf.

For (a), use corollary 1.

For (b), rearrange (35) middle and use the "1" in the minimum.

Thm (recall Lec 16, p. 19)

The explicit formula for \(\psi(x)\) immediately gives

\[
\psi(x) = x + O(x^{\Theta \ln^2 x}) \quad \text{for} \quad x \geq 2
\]

Pf.

By Lec 16, p. 11, assertion (a), we have:

\[
\sum_{|\lambda| \leq T} \frac{1}{|\lambda|} = O(\ln^2 T), \quad T \geq 3.
\]
We stress that the foregoing bound is essentially sharp due to

\[
\int_{3}^{T} \frac{1}{t} \, d\left( \frac{1}{2\pi} \ln \frac{t}{2\pi e} \right) = \int_{3}^{T} \frac{1}{t} \left[ \frac{1}{2\pi} \ln \frac{t}{2\pi e} \right] \, dt \quad \text{Lec 15 p. 27}
\]

\[
\{ t = 2\pi u \}
\]

\[
= \frac{1}{2\pi} \int_{3/2\pi}^{T/2\pi} \frac{\ln u}{u} \, du
\]

\[
\sim \frac{1}{4\pi} \left( \ln \frac{T}{2\pi} \right)^{2} \sim \frac{1}{4\pi} \left( \ln T \right)^{2}.
\]

Apply (35) with, say, \( x \leq 100 \) and \( T = x^{2} \).

Get:

\[
| \psi^{*}(x) - x | \leq \sum_{|y| \leq x^{2}} \frac{x^{\Theta}}{|y|} + O(1) + O\left[ \frac{x \ln^{2} x}{x^{2}} \right] + O(\ln x)
\]

\[
\leq O(1) x^{\Theta} \ln^{2} x.
\]

But \( \Psi(x) = \psi^{*}(x) + O(\ln x) \). Hence,

\[
| \Psi(x) - x | \leq O(1) x^{\Theta} \ln^{2} x
\]

as promised. \( \Box \)
We now PAUSE for some elementary facts (better late than never) related to Lec 5 and Lec 16, p. 7.

**THM**

Let $\gamma$ = the Euler constant. Near $x = 1$, we then have:

$$f(x) = \frac{1}{x-1} + \gamma + O(1|x-1|).$$

**Proof**

Recall Lec 5, pp. 8 - 10 with $r(x) = t - [t]$. The function $G(x) = f(x) - \frac{1}{x-1}$ is analytic on $\{x > 0\}$.

$$G(x) = 1 - x \int_1^\infty \frac{r(t)}{t^2+1} \, dt$$

Apply Lec 5, p. 8 but take $x = 1$. Get:

$$\sum_{n=1}^N \frac{1}{n} = 1 + \ln N - \int_1^N \frac{r(t)}{t^2} \, dt$$

$$\therefore$$

$$\gamma = 1 - \int_1^\infty \frac{r(t)}{t^2} \, dt.$$
Accordingly,

\[ G(1) = 1 - \int_1^\infty \frac{\gamma(t)}{t^2} \, dt = \gamma. \]

By Taylor series,

\[ G(x) = \sum_{k=0}^{\infty} \frac{\xi(k+1)}{k!} (x-1)^k \]

\[ = \gamma + O(x-1) \quad \text{near } x=1, \]

and we are done. \( \Box \)

**Theorem**

\[ f'(0) = -\frac{1}{a} \ln(2\pi) \quad \frac{f''(0)}{f'(0)} = \ln(2\pi). \]

**Proof**

Know \( f(0) = -\frac{1}{a} \) by Lec 9, pp. 18 + 20.

Now have:

\[ f(s) = \frac{1}{s-1} \left[ 1 + \gamma(s-1) + O((s-1)^3) \right] \]

by \( \Box \). Accordingly:

\[ \log f(s) = -\log(s-1) + \gamma(s-1) + O(s-1)^2 \]
\[ \frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s-1} + \gamma + O(s-1) \]

This sharpens Lec 7, p. 17. We can now apply the functional equation

\[ \frac{\Gamma'(z)}{\Gamma(z)} = \ln 2\pi + \frac{\pi}{2} \text{ctn} \frac{\pi z}{2} - \frac{\pi'}{\pi} (1-z) - \frac{1}{z} (1-z) \]

from Lec 16, p. 7. Recall

\[ \Gamma(1) = 1, \quad \Gamma'(1) = -\gamma \]

by Lec 10, p. 30 assertion (e) [and p. 22]. Take \( z \to 0 \) to get

\[ \frac{\Gamma'(0)}{\Gamma(0)} = \ln 2\pi - \frac{\Gamma'(1)}{\Gamma(1)} \]

\[ + \lim_{z \to 0} \left[ \frac{1}{z} + O(z) + \frac{1}{z} - \gamma + O(z) \right] \]

\[ = \ln 2\pi + \gamma - \gamma + O = \ln 2\pi. \]

Multiply by \( \Gamma(0) \) to get \( \Gamma'(0) = -\frac{1}{2} \ln (2\pi) \). \[ \text{END OF PAUSE} \]
We closed Lec 18 with a statement of and very brief sketch-of-the-proof for the so-called PERRON SUMMATION FORMULA associated with a general Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and \( \sum an \quad (x \geq 10) \).

The formula is based on p. 4 Lemma [above] and its verification parallels pp. 14 - 24.  

Due to the (already excessive!) length of these notes for Lec 17 + 18, we postpone this "Perron matter" until the notes for Lec 19.

\( \sim \)
The development in Lectures 17-18, pp. 47-54, is highly suggestive and capable of a major generalization.

We touched on this already in Lec 18 at the very end. See Lec 18 p. 43.

This brought us to the Perron summation formula with error term.

We'll sketch this formula emphasizing the similarity to Lec 17-18, pp. 47-54.

We begin with a slight revision of Lec 17's p. 4 lemma. Compare Ingham p. 75.
FACT

In Lec 17, p. 4, Lemma 7 things can be broadened to have

\[ 0 < y < \infty, \quad 0 < c \leq \frac{3}{2}, \quad T \geq 3 \]

giving

\[ |\text{Remainder}| \leq \begin{cases} 
O(1) \frac{y^c}{T|\ln y|}, & y \neq 1 \\
O(1) \frac{1}{T} & y = 1 \\
O(1) y^c & \text{any } y
\end{cases} \]

The "implied" constant in the \( O(1) \) is absolute. One also has [the somewhat cruder]

\[ |\text{Rem}| \leq O(1) \frac{y^c}{1 + T|\ln y|} \]

Proof

Simply review pp. 5-9 and modify several lines. For the final [cruder] assertion, divide into \( T|\ln y| > 1 \) and \( T|\ln y| \leq 1 \).
Recall Lec 17 p. 12. We generalize this!

Let \( \{a_n \}_{n=1}^{\infty} \) be given. Assume that \( a_n \neq 0 \) infinitely often as \( n \to \infty \).

Let \( \mathcal{F} \) be any subset of \( \mathbb{Z}^+ \) which includes the set \( \{ n : a_n \neq 0 \} \).

Define \( \| u \| = \{ \| u \|, u \neq 0 \} \) for \( u \in \mathbb{R} \).

For \( x \equiv \frac{3}{2} \mod 1 \), let \( \xi(x) = \min_{\xi \in \mathcal{F}} |x - \xi| \). Also write

\[
\langle x \rangle = \min \left\{ \frac{1}{100}, \| \xi(x) \| \right\}.
\]

Notice that \( \langle x \rangle = \frac{1}{100} \) unless \( \| \xi(x) \| < \frac{1}{100} \), which would mean \( x \notin \mathcal{F} \), but lies \text{LESS THAN} \( \frac{1}{100} \) units from \( \mathcal{F} \).

We clearly have:

\[
\langle x \rangle = \frac{1}{100} \quad \text{anytime} \quad x \in \mathbb{Z} \quad (x \equiv \frac{3}{2})
\]

\[
0 < \langle x \rangle \leq \frac{1}{100} \quad \text{always}.
\]
THEOREM (Perron summation formula with error term)

Given a Dirichlet series

\[ F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \]

which is absolutely convergent on \( \{ \Re(s) > 1 \} \). Assume that

\[ \sum_{n=1}^{\infty} \frac{|a_n|}{n^\alpha} = O\left( \frac{1}{(\alpha-1)^q} \right) \text{ for } 1 < \alpha \leq 2. \]

Here \( \alpha \geq 0 \). Assume further that \( |a_n| \leq \Phi(n) \), where \( \Phi(v) \) is some continuous, nonnegative, monotonic increasing function on \( \{ 1 \leq v < \infty \} \).

Consider \( \{ c, x, T \} \) such that

\[ 1 < c \leq 2, \quad x \geq 10, \quad T \geq 3. \]

Taking \( \sigma = \Re(s) \) and \( \sigma + c > 1 \), we then have the following relations insofar as \( -9 \leq \sigma \leq 10 \) (say):

(1) in a style reminiscent of the explicit formula

\[ \sum_{n < x} a_n n^{-s} + \begin{cases} 0 & x \notin \mathbb{Z} \\ \frac{i}{2} a_x x^{-s}, & x \in \mathbb{Z} \end{cases} \]

\[ = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s+w) \frac{x^w}{w} dw + O\left[ \frac{x^c}{T(\sigma+c-1)^{\alpha'}} \right] \]

\[ + O\left[ \frac{\Phi(2x) x^{-\sigma} \ln x}{T} \right] + \text{ (see next page) } \]
\[ + O(1) \frac{\Phi(2x)}{x_0} \min \left\{ 1, \frac{x}{T\langle x \rangle} \right\} \]

[see (3) for \( \langle x \rangle \)]

(2) in a style closer to that of an \textit{a priori} bound,

\[
\sum_{n < x} a_n n^{-s} + \left\{ \begin{array}{ll}
O, & x \notin \mathbb{Z} \\
\frac{1}{2} a_n x^{-s}, & x \in \mathbb{Z}
\end{array} \right\}
\]

\[
= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} \, dw
\]

\[+ O(1) \sum_{n=1}^{\infty} \frac{|a_n| n^{-s} \left( \frac{x}{n} \right)^{C}}{1 + \tau |\ln \frac{x}{n}|} \]

In (1), the "implied" constants are absolute apart from a mild dependence on \( \alpha \) and the implied constant associated with \( O[(w-1)^{-\gamma}] \).

In (2), the implied constant is absolute.

\textbf{Proof}

It will be convenient to let \( N = \) the integer nearest to \( x \) (with \( x = k + \frac{1}{2} \) \( \Rightarrow N = k \)).
We look first at:

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} \, dw
\]

\[
= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \sum_{n=1}^{\infty} \frac{a_n}{n^{s+w}} \frac{x^w}{w} \, dw
\]

\[
= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \left( \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{(x/n)^w}{w} \, dw \right)
\]

in the setting of p. 2 FACT. Think

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^s} \left\{ \eta \left( \frac{x}{n} \right) + REM, \right\}
\]

where REM, takes various formats.

Assertion (2) follows immediately.

For (1), we need to adapt Lec 17 pp. 14-24. We merely sketch this.

First of all, corresponding to Lec 17 p. 15 top,
we split things according to 

\[ 1 \leq n \leq N - 1 \]
\[ n = N \]
\[ N + 1 \leq n < \infty \]

Note that, by hypothesis, \( N - \frac{1}{2} < x \leq N + \frac{1}{2} \).

(When \( x = k + \frac{1}{2} \), \( N = k \) and all is OK.)

\[
\begin{align*}
N - \frac{1}{2} & \quad N \quad N + \frac{1}{2} \\
N & \quad N & \quad N + \frac{1}{2}
\end{align*}
\]

For \( 1 \leq n < \frac{x}{2} \), one easily checks

\[
\sum_{1 \leq n < \frac{x}{2}} \frac{a_n}{n^s} \{ \text{REM}_n \} = O \left[ \frac{x^c}{T(\sigma+c-1)^s} \right].
\]

No. \(-9 \leq \sigma \leq 10, 1 < c \leq 2 \Rightarrow -8 < \sigma + c \leq 12\) a priori.

All is OK if \( 1 < \sigma + c \leq 2 \). For \( 2 < \sigma + c \leq 12 \), one easily adjusts the implied constant in \( O[(\omega-1)-c] \) to accommodate \( 2 < \omega \leq 12 \). This clearly involves \( \delta \).

Inflation by \( 11^\delta \) is sufficient.

For \( 2x < n < \infty \), we also get

\[
O \left[ \frac{x^c}{T(\sigma+c-1)^s} \right].
\]
For $\frac{x}{2} \leq n \leq N-1$, we write $n = N - r$ and insert from Lec 17 p. 19 bottom. We get

$$\sum_{\frac{x}{2} \leq n \leq N-1} \frac{a_n}{n^s} \left\{ \text{REM}_0 \right\}$$

$$= O(1) \frac{\Phi(N)}{x^\sigma} \frac{x/\ln x}{T} \left\{ \text{recall } x \geq 10, \right\} \left\{ -9 \leq \sigma \leq 10 \right\}$$

$$= O(1) \frac{\Phi(2x)}{x^\sigma} \frac{x/\ln x}{T}$$

via trivial insertion of absolute values.

Not surprisingly, noting Lec 17 p. 16 middle, we find that

$$\sum_{N+1 \leq n \leq 2x} \frac{a_n}{n^s} \left\{ \text{REM}_0 \right\}$$

$$= O(1) \frac{\Phi(2x)}{x^\sigma} \frac{x/\ln x}{T}$$

as well.
It remains to discuss

\[ \frac{|an|}{N^0} \mid \text{REM} \mid \quad \{ n = N \} \quad \text{if } n \geq 10 \quad \exists \quad N \geq 10 \]

If \( an = 0 \), we get 0 which is subsuend by anything. Suppose therefore that \( an \neq 0 \).

Hence \( N \in \mathbb{N} \).

\[ \frac{1}{N^0} \mid \text{REM} \mid = O(1) \frac{\Phi(N)}{N^0} \frac{1}{T} \frac{1}{|\ln x - \ln N|} \]

\[ \leq O(1) \frac{\Phi(N)}{x^0} \frac{1}{T} \frac{1}{x |x - N|} \]

\[ \leq O(1) \frac{\Phi(N)}{x^0} \frac{x}{T} \]

\[ = O(1) \frac{\Phi(2x)}{x^0} \frac{x}{T} \]
\[ = O(1) \Phi(2x) x^{1-\sigma} \] \( \frac{1}{T} \)

This is subsumed by \( O(1) \Phi(2x) x^{1-\sigma} \frac{\ln x}{T} \) without further ado.

\[ \text{OK} \]

Cf. page 8 above; also 4 bottom.

To finish up, we therefore take \( |x-N| < \frac{1}{100} \) w.l.o.g. We still have \( q_N \neq 0 \) and \( N \in \mathcal{F} \).

Must consider 2 cases.

\underline{Case I} \quad N = x.

Here \( \langle x \rangle = \frac{1}{100} \) (cf. 3) and

\[ \frac{|N|}{N^\sigma} \mathsf{REM}(1) = O(1) \frac{|N|}{N^\sigma} \min \left\{ \frac{1}{3}, \frac{1}{T} \right\} \text{ by } 2 \]

\[ = O(1) \frac{\Phi(2x)}{x^{\sigma}} \min \left\{ \frac{1}{3}, \frac{x}{T \langle x \rangle} \right\} \].

Of course, \( T \leq 3 \). In any event, this last expression is safely subsumed by

\[ O(1) \frac{\Phi(2x)}{x^{\sigma}} \min \left\{ \frac{1}{3}, \frac{x}{T \langle x \rangle} \right\} \] \( \text{OK} \)
Though not really necessary, we remark that:

\[
\frac{|\lambda_n|}{N_0^{\sigma}} |\text{REM}| = O(1) \frac{\Phi(2x)}{x^{\sigma}} \frac{1}{T} = O(1) \frac{\Phi(2x)}{T} x^{1-\sigma} \ln x
\]

i.e., matters are also subsumed by the term

\[
O(1) \frac{\Phi(2x)}{T} x^{1-\sigma} \ln x
\]

à la § above (cf. also § 4).

**Case II** \(N \neq x\) but \(|x-N| < \frac{1}{100}\).

By § 2,

\[
\frac{|\lambda_n|}{N^{\sigma}} |\text{REM}| = O(1) \frac{|\lambda_n|}{N^{\sigma}} \min \left\{ \frac{(x/N)^c}{T/(\ln x - \ln N)} \right\}
\]

\(1 < c \leq 2, -9 \leq \sigma \leq 10, x \geq 10, T \geq 3\)

\[
= O(1) \frac{|\lambda_n|}{N^{\sigma}} \left( \frac{x}{N} \right)^c \min \left\{ 1, \frac{1}{T/(x-N)} \right\}
\]

\[
= O(1) \frac{|\lambda_n|}{x^{\sigma}} \min \left\{ 1, \frac{x}{T(\langle x \rangle)} \right\}
\]

\(\langle x \rangle = |x-N|\) here; see § 3.
By combining the three OK's [on 10 here], we deduce that:

\[
\frac{|\ln N|}{N^\sigma} \leq O(1) \frac{\Phi(2x)}{\ln x} \left( \frac{x}{T(x)} \right)^{1-\sigma} + O(1) \frac{\Phi(2x)}{x^\sigma} \min\left\{ \frac{1}{T(x)}, \frac{x}{T(x)} \right\}
\]

This completes the proof of (1) on 14.

Page 4 THM is clearly little more than a pedestrian revamp of Lec 17 plus pp. 4-24.

Note that:

\[
\sum_{n<x} a_n n^{-s} + \left\{ \frac{1}{2} a_n x^{-s}, x \in \mathbb{Z} \right\} = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{-i T}^{i T} f(s+w) \frac{x^w}{w} \, dw
\]
pointwise w.r.t. $x$. This is the original version of the Perron summation formula.

On the matter of assertion (2), we leave it as an easy exercise to obtain

$$|\text{remainder term}| = O \left[ \frac{X^c}{T(\sigma+c-1)^c} \right]$$

$$+ O(1) \Phi(\beta x)^{1/2} - \sigma \frac{x \ln \left[ \min (T_j x^{1/2}) \right]}{T}$$

$$+ O(1) \Phi(\beta x)^{-\sigma} \min \left\{ \frac{1}{T} \Phi_0(x) \right\}$$

where

$$\Phi_0(x) = \min \left\{ \frac{1}{100} , \Phi(x) \right\}$$

and $\Phi(x) = \min_{\lambda \in \mathbb{Z}} |x - \lambda|$ as on (3). Of course,

$$\min (T_j x^{1/2}) \leq \min (T_j x) \leq \min (T_j x^{1/2})$$

$$\Downarrow$$

$$\frac{1}{2} \ln \left[ \min (T_j x) \right] \leq \ln \left[ \min (T_j x^{1/2}) \right] \leq \ln \left[ \min (T_j x) \right]$$

This allows line 9 to be cleaned up slightly.

* As suggested, treat $T < x^{1/2}$ and $T \geq x^{1/2}$ separately. Note that $\Phi_0(x)$ can be 0.
For $T > x^{1/2}$ assertion (1) is better than lines 5-7 on (13). Cf. $\langle x \rangle$ versus $\xi_0(x)$. For $T < x^{1/2}$ lines 5-7 will sometimes give the better result. E.g. take $x \notin \mathbb{Z}$ and $T = \exp \left[ \frac{(\ln x)^5}{5} \right]$, $\delta$ tiny.

NEW TOPIC.

Let

$$\mu(n) = \begin{cases} 0, & \text{if } n \text{ is NOT squarefree} \\ (-1)^v, & \text{if } n = p_1^{a_1} \cdots p_v^{a_v} \text{ (distinct primes)} \end{cases}.$$  

Of course, $\sqrt{n} = 0$ for $n = 1$. It is completely standard by Euler's identity (see Ingham 16 + Lec 6 p.4) to verify that

$$\frac{1}{s(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

For Re($s$) > 1. It is also standard to verify that

$$f(s)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$

wherein $d(n) = N \{ k \geq 1 : k|n \}$, Thus $d(6) = 4$ via $\{1,2,3,6\}$. Elementary number theory.
gives
\[ d(n) = (1+E_1) \cdots (1+E_r) \quad \text{for } n = \prod_{i=1}^{r} p_{E_i}^{\varepsilon_{E_i}}. \]
Here \( p_1 < p_2 < \cdots < p_r \) are primes. An easy analysis shows
\[ d(n) = O(n^\varepsilon) \]
for every \( \varepsilon > 0 \). Indeed:
\[ \sum_{\substack{\ell \\ \ell \neq \ell' \neq \ell''}} E_\ell + \cdots + E_r \leq (1.5) \ln n \]
\[ 1 + m \leq (p^{\varepsilon})^m \quad \text{for all } p \geq 2^{1/\varepsilon} \quad (m \geq 0) \]
\[ \prod_{j=1}^{r} (1+E_j^*) = \prod_{\substack{j < 2^{1/\varepsilon} \quad p_j^* \geq 2^{1/\varepsilon}}} (1+E_j) \]
\[ \leq \prod_{p_j < 2^{1/\varepsilon}} (1+E_j^*) \cdot \prod_{p_j \geq 2^{1/\varepsilon}} (1+E_j^*) \]
\[ \leq \prod_{p_j < 2^{1/\varepsilon}} (1+E_j^*) \cdot \prod_{p_j \geq 2^{1/\varepsilon}} (1+E_j^*) \cdot (3 \ln n) \cdot n^\varepsilon \]
\[ \leq O(n^{2\varepsilon}) \quad \{ \text{each } \varepsilon \} \]

For real \( y \geq 1 \), we define
\[ M(y) = \sum_{n \leq y} \mu(n) \]
\[ T(y) = \sum_{n \leq y} d(n) \]
The flip-flopping function \( \mu(n) \) has average 0 in the sense that

\[
\lim_{x \to \infty} \frac{M(x)}{x} = 0.
\]

In fact,

\[
M(x) \leq O \left( x e^{-c(\ln x)^{4/10}} \right)
\]

with some suitably small \( c > 0 \).

**Pf**

We use (Perron) p. 4 version (1) with \( x = m + \frac{1}{a} \), \( m \) large. We take

\[
a_n = \mu(n), \quad \psi(v) = 1, \quad s = 0, \quad \sigma = 0.
\]

We note that

\[
\frac{1}{s(s)} = O \left[ \ln^7 t \right] \quad \text{for} \quad \sigma \geq 1 - \lambda (\ln t).
\]

Whenever \( \lambda \) is appropriately small and \( t \geq 3 \).

In checking this, wlog \( t = \) giant. We remember that...
\[ \frac{1}{s(\sigma + it)} = O(\ln^q t), \quad \sigma \geq 1, \quad t \geq 3 \quad \text{Lec 7 p. 6} \]

\[ |I(\sigma + it)| \geq A(\ln t)^{-7} \quad \text{here} \]

\[ |I(\sigma + it)| \leq \frac{C}{\delta(1 - \delta)} \left| t \right|^{1 - \delta} \quad \sigma \geq \delta, \quad t \geq 2 \quad \text{Lec 6 p. 7} \]

\[ |I(\sigma + it)| \leq A_2 \ln \left| t \right|, \quad \sigma \geq 1 - \frac{5}{\ln \left| t \right|}, \quad t \geq t_0 \quad \text{Lec 6} (14) (20) \]

\[ |I(\sigma + it)| \leq A_2 \ln^2 \left| t \right|, \quad \sigma \geq 1 - \frac{5}{\ln \left| t \right|}, \quad t \geq t_0 \quad \text{Lec 6} (20) \]

and

\[ s_1 = 1 + it, \quad s_2 = \sigma_2 + it, \quad \frac{1}{2} < \sigma_2 < 1 \quad \Rightarrow \]

\[ |I(s_2)| \geq |I(s_1)| - |I(s_2) - I(s_1)| \]

\[ \geq |I(s_1)| - \int_{s_2}^{s_1} |I'(w)| dw \quad \{ w = \sigma + it \} \]

Get: \((t \geq t_0)\)

\[ |I(\sigma_2 + it)| \geq A(\ln t)^{-7} - \int_{\sigma_2}^{1} A_2 \ln^2 t \, d\sigma \]

\[ \geq A(\ln t)^{-7} - A_2 \ln^2 t \left( 1 - \sigma_2 \right) \]

**KEEP** \( \sigma_2 > 1 - \frac{2}{(\ln t)^q} \quad \text{tiny} \)

\[ \Downarrow \]

\[ \geq (A - \frac{\lambda A_3}{2})(\ln t)^{-7} \geq \frac{1}{2} A(\ln t)^{-7} \]
In other words,

\[ \frac{1}{\zeta(\sigma + it)} = O(\ln^7 t), \quad \sigma \geq 1 - \frac{2}{\ln t}, \quad t \geq 3 \]

for some tiny \( \lambda \).

By p. 49 (1), with \( \mathfrak{N} = \mathbb{Z}^+ \) and \( c \in (1, 2) \),

\[ M(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{\zeta(w)} \frac{x^w}{w} dw \]

\[ + O \left[ \frac{x^c}{T(c-1)} \right] + O \left[ \frac{x}{T} \right] + O(1) \min \left\{ 1, \frac{x}{T^{\frac{1}{2}}} \right\} \]

To optimize \( O \left[ \frac{x^c}{T(c-1)} \right] \) in regard to \( c \), we put

\[ c = 1 + \frac{1}{\ln x} \]

Since \( x = \ln x + \frac{1}{2} \) is big, this is legal. Get:

\[ M(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{\zeta(w)} \frac{x^w}{w} dw \]

\[ + O \left[ \frac{x}{T} \right] + O \left[ \frac{x \ln x}{T} \right] \]

\[ + O(1) \frac{x}{T} \]
\[ M(x) \approx \frac{1}{2\pi i} \int_{c-iT}^{c+iT} + O \left[ \frac{x}{T} (\ln x) \right]. \]

We now deform \([c-iT, c+iT]\) remembering pp. 17, 18 top. Of course, \(1/\sin w\) has a simple zero at \(w = 1\) (in the sense of a removable singularity).

\[ 1 - \frac{2}{(\pi T)^2} i T \]

\[ \text{use a rectangle!} \]

\[ 1 - \frac{2}{(\pi T)^2} i T \]

By either reducing \(\lambda\) or inflating \(T\), we can hypothesize that \(1/\sin w\) is nicely analytic on this closed rectangle. By 18 top, the issue occurs only for \(|\text{Im} w| \leq 3\).
By the Cauchy Integral Theorem looking at \( \frac{1}{\gamma(w)} \frac{w^n}{w} \), we get:

\[
|M(n)| \leq O(1) (\ln T)^{\frac{7}{8}} \left(1 - \frac{A}{(\ln T)^{\theta}}\right) \int_{3}^{T} \frac{1}{V} \, dV 
+ O(1) |c - 1\rangle + \frac{A}{(\ln T)^{\theta}} \int_{0}^{3} (\ln T) \frac{x^{c}}{T} \lla 19 \rra_{\text{top}} 
+ O \left[ \frac{x \ln x}{T} \right]
\]

\[
= O(1) (\ln T)^{\frac{7}{8}} 1 - \frac{A}{(\ln T)^{\theta}}
+ O(1) \frac{1}{\ln x} \frac{x (\ln T)^{\theta}}{T} \lla c = 1 + \frac{1}{\ln x} \rra
+ O(1) \frac{A}{(\ln T)^{2}} \frac{x}{T} + O \left[ \frac{x \ln x}{T} \right]
\]

\[
= O(1) x \cdot (\ln T)^{\frac{7}{8}} x - \frac{A}{(\ln T)^{\theta}}
+ O(1) x \cdot \frac{(\ln T)^{\theta}}{T \ln x}
+ O(1) x \cdot \frac{A}{T (\ln T)^{2}}
+ O(1) x \cdot \frac{\ln x}{T}
\]
\[ = O(x) \int (\ln T) \left[ x^{-2} (\ln T)^8 + \frac{(\ln T)^7}{T \ln x} + \frac{1}{T (\ln T)^2} + \frac{1}{T} \right] \]

Continue to keep \( x \) large, also keep \( T \geq e^{10} \).

Since \( \lambda \) is tiny, obviously

\[ \frac{\ln x}{T} \geq \frac{\lambda}{T} \geq \frac{\lambda}{T (\ln T)^2} \]

We can therefore erase the term \( \frac{\lambda}{T (\ln T)^2} \).

**ASSUME NOW THAT** \( T \geq x \).

At once:

\[ |M(x)| \leq O(x) (\ln x)^8 \cdot \frac{\lambda}{(\ln T)^9} \]

\[ + O(x) (\ln x)^6 \cdot \frac{1}{T} \]

\[ + O(x) (\ln x) \cdot \frac{1}{T} \]
\[ x = 0(x) (\ln x)^8 \left[ x^{-2 \left( \frac{2}{(\ln T)^9} \right)} + \frac{1}{T} \right] \]

\[ = 0(x) (\ln x)^8 \left[ e^{-\frac{2\ln x}{(\ln T)^9}} + e^{-\ln T} \right] . \]

But,

\[ e^{\min\{A,B\}} \leq e^{-A} + e^{-B} \leq 2e^{\min\{A,B\}} \]

For \( A > 0, B > 0 \).

So,

\[ |M(x)| = 0(x) (\ln x)^8 e^{-\min\left\{\frac{\ln x}{(\ln T)^9}, \ln T\right\}} . \]

Optimize by graphing "\( \frac{\ln x}{v} \) versus \( v \)" and thus setting

\[ \frac{\ln x}{(\ln T)^9} = \ln T \]

\( \text{I.E.,} \)

\[ \ln T = \left( \frac{\ln x}{10} \right) \]

\( \text{I.E.,} \)

\[ T = \exp\left[ \left( \frac{\ln x}{10} \right) \right] . \]

\( \text{clearly} \)

\[ T \leq x \]
Hence:

\[ |M(x)| = O(x) (\ln x)^8 e^{-\lambda \ln x} \]
\[ = O(x) e^{\frac{8 \ln \ln x}{\lambda}} e^{-\frac{1}{10} (\ln x)^{\frac{1}{10}}} \]
\[ = O(x) e^{-\frac{1}{2} (\lambda \ln x)^{\frac{1}{10}}} \quad \text{for large } x. \]

Corollary

For \( x \geq 3 \) and any big \( \Delta \), we have

\[ M(x) = O\left( \frac{x}{\ln^{\Delta} x} \right). \]

PF

\[ \Delta \ln \ln x < \frac{1}{2} (\lambda \ln x)^{\frac{1}{10}} \text{ once } x \text{ is big enough.} \]
We now give some easy corollaries of p. 23 Corollary.

**Proposition 1**

The series \( \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \) converges at each point of \( \{ \text{Re}(s) = 1 \} \). The convergence is uniform on compact subsets.

**Pf**

Consider a general point \( s_0 = 1 + it \) \( (t \in \mathbb{R}) \).

Know \( M(x) = 1 \) for \( 1 \leq x < 2 \). Keep \( U \) large and in \( \mathbb{Z}^+ \).

\[
\sum_{n=1}^{U} \frac{\mu(n)}{n^{s_0}} = 1 + \int_{1}^{U} x^{-s_0} dM(x)
\]

\[
= 1 + \left[ x^{-s_0} M(x) \right]_{1}^{U} - \int_{1}^{U} M(x) d(x^{-s_0})
\]

\[
= U^{-s_0} M(U) + \sum_{n=1}^{U} \frac{M(x)}{x^{s_0+1}}
\]

\[
= O(1) \frac{U^{-1}}{\ln^n U} + \sum_{n=1}^{U} \frac{M(x)}{x^{2+it}}
\]

But \( \int_{3}^{\infty} \frac{M(x)}{x^{2+it}} dx \) is nicely majorized by \( \int_{3}^{\infty} \frac{O(1)}{x(\ln x)^4} dx \)
A moment's thought now gives the 2 statements in the proposition.

Notice that we get

\[
\sum_{i=1}^{\infty} \frac{\mu(n)}{n^{s_0}} = s_0 \int_1^{\infty} \frac{M(x)}{x^{s_0+1}} \, dx.
\]

**Prop 2**

In connection with \( \sum_{i=1}^{\infty} \frac{\mu(n)}{n^s} \), we have uniform convex on compact subsets on \( \Re(s) \geq 1/2 \).

**Pf**

Easy modification of Prop 1.

Again,

\[
\sum_{i=1}^{\infty} \frac{\mu(n)}{n^s} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}} \, dx = \frac{1}{\zeta(s)},
\]

this time for \( \Re(s) \geq 1 \).

**Prop 3**

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0. \quad (\text{Euler, 1748})
\]

(not rigorously)
Let $s \to 1$ in the 2nd box and use unif conv.

**Prop 4**

Let $l \geq 1$. The series

$$\sum_{n=1}^{\infty} \frac{\mu(n)(\ln n)^{l}}{n^s}$$

conv unif on compact subsets of $\frac{1}{2} \text{Re}(s) \geq 1$.

**pf**

Imitate Prop 1 + 2 with $\Delta \geq l+4$ in (23) Corollary.

EG

$$\sum_{n=1}^{U} \frac{\mu(n)(\ln n)^{l}}{n^{s_0}} = \int_{1}^{U} \frac{(\ln x)^{l}}{x^{s_0}} dM(x)$$

$$= \left[ M(x) \frac{(\ln x)^{l}}{x^{s_0}} \right]_{1}^{U}$$

$$- \int_{1}^{U} M(x) d\left[ \frac{(\ln x)^{l}}{x^{s_0}} \right]$$

$$= \text{etc.}$$

Using the Weierstrass conv thm (for analytic func.), notice that

$$- \frac{f'(s)}{f(s)^2} = - \sum_{n=1}^{\infty} \frac{\mu(n)\ln n}{n^s}, \quad \text{Re}(s) > 1.$$

(Clean up by erasing the minus signs.)
By virtue of the unit conv in Prop 4, we immediately get

$$\frac{\zeta'(1+it)}{\zeta(1+it)^2} = \sum_{n=1}^{\infty} \frac{\mu(n) \ln n}{n^{1+it}}, \quad t \neq 0.$$  

Letting $t \to 0$ gives

$$\sum_{n=1}^{\infty} \frac{\mu(n) \ln n}{n} = -1.$$

Indeed:

$$\zeta(s) = \frac{1}{s-1} \left[ 1 + \gamma(s-1) + O(s-1)^2 \right]$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + O(s-1)$$

$$\frac{\zeta'(1+w)}{\zeta(1+w)} = -\frac{1}{w} + \gamma + O(w) \quad w \to 0$$

$$\frac{1}{\zeta(1+mw)} = w \left[ 1 - \gamma w + O(w^2) \right] \quad w \to 0$$

$$\frac{\zeta'(1+w)}{\zeta(1+w)^2} = (-1) \left[ 1 - 2\gamma w + O(w^2) \right] \quad w \to 0.$$  

OK
These facts using Corollary are nice and are of interest because of several facts (very old ones) which we will not prove fully at this time.

A] The Riemann Hypothesis is equivalent to the statement that \( M(x) = O(x^{\frac{1}{2}+\varepsilon}) \).

B] The RH is equivalent to the statement that \( \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \) converges for all \( \{\text{Re}(s) > \frac{1}{2}\} \).

C] By elementary techniques (no use of complex variable), one can show

\[
M(x) = o(x) \quad \text{is equiv to } \psi(x) \sim x.
\]

D] By elementary techniques, one can show that the following are equivalent

\((i)\) \( \psi(x) \sim x \);
\((ii)\) \( M(x) = o(x) \);
\((iii)\) \( \sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 \).

so Euler "knew" the PNT

And, supplementing this, that box \( \Rightarrow (iii)(ii)(i) \).
In Lec 20, I proposed to look at $\lfloor C \rfloor$.

There is a certain amount of "fun" in doing so. Plus being instructive!

It is convenient to first review some preliminaries involving elementary number theory and $\mu(n)$. We do so quickly.

Given $f(n)$ for $n \in \mathbb{Z}^+$. Recall that $f$ is multiplicative when

$$f(mn) = f(m)f(n) \quad \text{if} \quad (m,n) = 1.$$

To avoid trivialities, we assume $f(1) = 1$.

Review proposition

\textbf{Prop R1}

$$\sum_{d|n} \sigma(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}.$$ 

\textbf{Pf}

$$1 = \frac{1}{s} \Gamma(s), \quad \text{Think of} \quad s \text{ real}, \quad s > 1.$$
\[ l = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \cdot \sum_{l=1}^{\infty} \frac{1}{l^s} \]

By absolute conv of both series, get:

\[ \begin{cases} 1, & n=1 \\ 0, & n>1 \end{cases} \sum_{k \mid n} \mu(k) \cdot 1 = \sum_{k \mid n} \mu(k) \cdot \frac{1}{k \cdot n} \]

(ABS CONV ensures that an infinite series can be summed/rearranged in any order.)

\textbf{Prop R2} \ (\textit{Möbius inversion})

Let \( f \) be any sfn defined on \( \mathbb{Z}^+ \). Let

\[ g(n) = \sum_{d \mid n} f(d) \cdot \frac{n}{d} \]

Then:

\[ f(n) = \sum_{d \mid n} \mu(d) \cdot \frac{n}{d} \]

\textbf{PF}

For an elementary proof, see R1 and any basic

\[ \sum_{k=1}^{\infty} c_n x^{-s} = \left( \sum_{k=1}^{\infty} a_k k^{-s} \right) \left( \sum_{l=1}^{\infty} b_l l^{-s} \right) \iff c_n = \sum_{k \mid n} a_k b_{n/k} \]
book on number theory. E.g. Hardy and Wright.

The "slick informal" proof goes like so:

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^s} = f(s) \sum_{l} \frac{f(l)}{l^s} \quad s > \sigma (G = \text{giant})$$

$$\Rightarrow \quad \frac{1}{f(s)} \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \sum_{l} \frac{f(l)}{l^s}$$

$$\Rightarrow \quad f(n) = \sum_{d | n} \mu(d) g\left(\frac{n}{d}\right) \quad n \text{ frozen}$$

Prop R3 (converse of Möbius inversion)

If $g$ on $\mathbb{Z}^+$, Assume $f(n) = \sum \mu(d) g\left(\frac{n}{d}\right)$. Then:

$$g(n) = \sum_{d | n} f(d)$$

Proof

Look in, e.g., Hardy and Wright (using Prop R1). Slick/informal proof:

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{1}{f(s)} \sum_{k=1}^{\infty} \frac{g(k)}{k^s} \quad s > \sigma \text{ giant}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = f(s) \sum_{l} \frac{f(l)}{l^s}$$

$$\Rightarrow \quad g(n) = \sum_{d | n} 1 \cdot f(k) = \sum_{k | n} f(k)$$
Prop $R4$

Let $f$ be multiplicative \( \text{on } \mathbb{Z}^+ \). Then, so is
\[
g(n) = \sum_{d|n} f(d).
\]

**Pf**

\( g(1) = f(1) = 1 \) OK.

Given \( n_j \geq 2 \) for \( j = 1, 2 \),

Suppose \( n_1 n_2 \) has \( (n_1, n_2) = 1 \).

Use elem numb th.

\[
g(n_1 n_2) = \sum_{d|m} f(d) = \sum_{d|n_1} f(d) \sum_{d|n_2} f(d).
\]

Since \( d \) are uniquely \( d_1 d_2 \) with \( d_1 | n_1 \), \( d_2 | n_2 \),

\[
= \sum_{d_1 | n_1} f(d_1) \sum_{d_2 | n_2} f(d_2) = g(n_1) g(n_2).
\]

The case \( n_1 = 1 \), \( n_2 \geq 2 \) is trivial.

\( n_2 = 1 \), \( n_1 \geq 2 \).
OK, then. To prove \( C \), there are 2 halves:

\[
\psi(x) \sim x \implies M(x) = o(x) \quad \text{and} \\
M(x) = o(x) \implies \psi(x) \sim x.
\]

We begin with \( \psi(x) \sim x \implies M(x) = o(x) \).

\[\text{FACT}\]

\[
\sum_{n \leq x} \mu(n) \ln \frac{x}{n} = M(x) \ln x + \sum_{k \leq x} A(k) \mu(k).
\]

Here \( x \in \mathbb{R} \), \( x \geq 2 \).

\[\text{Pf}\]

\[
\begin{align*}
\left( \frac{1}{J} \right)' &= -\frac{1}{J^3} \frac{d}{ds} J^s \\
&= \left( -\frac{t(s)}{J(s)} \right) \frac{1}{J(s)} \\
&\implies \\
-\sum_{n=1}^{\infty} \frac{\mu(n) \ln n}{n^s} &= \left[ \sum_{k=1}^{\infty} \frac{A(k)}{k^s} \right] \left[ \sum_{s=1}^{\infty} \frac{\mu(s)}{s^s} \right]
\end{align*}
\]

\[
-\mu(n) \ln n = \sum_{\substack{k \cdot l = n \\ell \leq n}} A(k) \mu(k) \ell.
\]
\[
\sum_{n \leq x} \tau(n) \ln \frac{x}{n} = M(x) \ln x + \sum_{n \leq x} \sum_{k \leq x} \Lambda(k) \mu(k) \mu(l) \\
\{ \text{view in first quadrant of uv-plane, and under hyperbola } v = \frac{x}{u} \}
\]
FACT

\[ x \in \mathbb{R}, \ x \geq 2. \] Then:

\[
\sum_{n \leq x} \ln \frac{x}{n} = x + O(\ln x).
\]

pf

By Lec 1, \( \ln \lfloor x \rfloor! = x \ln x - x + O(\ln x) \).

See Lec 1 p. 11, Thm 6. But, in the above:

\[
\begin{align*}
\text{LHS} &= \lfloor x \rfloor \ln x - \ln \lfloor x \rfloor! \\
&= \lfloor x \rfloor \ln x - x \ln x + x + O(\ln x) \\
&= (\lfloor x \rfloor - x) \ln x + x + O(\ln x) \\
&= O(\ln x) + x + O(\ln x) \\
&= x + O(\ln x).
\end{align*}
\]

\[ \square \]
FACT

\[ M(x) \ln x = O(x) - \sum_{\ell \leq x} \mu(\ell) \psi(\frac{x}{\ell}) \quad x \in \mathbb{R} \]
\[ x \geq 2 \]

PF

use (33) Fact. Note there that:

\[ \left| \sum_{n \leq x} \mu(n) \ln \frac{x}{n} \right| \leq \sum_{n \leq x} \ln \frac{x}{n} = O(x) \]

by (35). Get:

\[ M(x) \ln x = O(x) - \sum_{\ell \leq x} \Lambda(\ell) \mu(\ell) \]

\[ \left\{ \begin{array}{l}
\text{view the hyperbola region in first quadrant (34)} \\
\text{given any } \ell \leq x, \text{ note that } k \text{ then must range in } [1, \frac{x}{\ell}] \\
\end{array} \right. \]

\[ M(x) \ln x = O(x) - \sum_{\ell \leq x} \mu(\ell) \left( \sum_{k \leq x/\ell} \Lambda(k) \right) \]

\[ = O(x) - \sum_{\ell \leq x} \mu(\ell) \psi(\frac{x}{\ell}) \quad \Box \]
\textbf{FACT}

\[ \left| \sum_{m \leq x} \frac{\mu(m)}{m} \right| < 1 \quad \text{for all } x \in \mathbb{R}^+ \quad x \geq 2. \]

\textbf{PF}

\textit{whoosh } x \in \mathbb{Z} \Rightarrow x \neq 1 \text{ gives } \text{Sum} = 1 \text{.}

\[ x = 2 \text{ OK } \Rightarrow \text{Sum } = \frac{1}{2} \text{. So, wlog, } x \geq 3. \]

\textbf{Note:}

\[ \sum_{m \leq x} \mu(m) \left\lfloor \frac{x}{m} \right\rfloor = \sum_{m \leq x} \mu(m) \cdot 1 \]

\text{as in the proof of hyperbola region}

\[ \begin{aligned}
\text{\{ write } N = \prod_{k \leq x} \mu(k) \text{ for each } N \text{, note how } u
\text{ ranges over the divisors of } N \} \text{ and } k \equiv \frac{N}{u} \text{.}
\end{aligned} \]

\[ uv = N \]

\[ \sum_{N \leq x} \left\{ \sum_{m \mid N} \mu(m) \right\} \]

\[ = \sum_{N \leq x} \left\{ \begin{cases} 1, & N = 1 \\ 0, & N > 1 \end{cases} \right\} \leftarrow \text{RI} \]

\[ = 1. \quad \therefore \]

\textit{Accordingly,}
\[
\sum_{m=1}^{X} \mu(m) \left( \frac{x}{m} - \psi\left( \frac{x}{m} \right) \right) = 1
\]

\[
\begin{cases}
\varepsilon(t) = t - \lfloor t \rfloor
\end{cases}
\]

\[
x \sum_{m=1}^{X} \frac{\mu(m)}{m} = 1 + \sum_{l=1}^{X} \mu(l) \psi\left( \frac{x}{l} \right)
\]

\[
\begin{cases}
m = 1 \Rightarrow \mu(1) \psi(x) = 0 \\
m = x \Rightarrow \mu(x) \psi(1) = 0 \\
\text{in general, } 0 \leq \psi\left( \frac{x}{m} \right) < 1
\end{cases}
\]

\[
x \left| \sum_{l=1}^{X} \frac{\mu(l)}{l} \right| \leq 1 + (x-2) = x-1 < x
\]

\[
\sum_{m=1}^{X} \frac{\mu(m)}{m}
\]

\[
\leq 1 \text{ for all } x \geq 2.
\]

Recall (36) to get:

\[
M(x) \ln x = O(x) - \sum_{l \leq x} \mu(l) \psi\left( \frac{x}{l} \right) \quad x \in \mathbb{R}
\]

\[
\begin{cases}
\text{use (37)}
\end{cases}
\]

\[
= O(x) + \sum_{l \leq x} \mu(l) \left( \frac{x}{l} - \psi\left( \frac{x}{l} \right) \right)
\]

Very Slick
FACT

\[ \psi(x) \sim x \quad \text{as} \quad x \to \infty \quad \Rightarrow \\
M(x) = o(x). \]

pf

Use \( \text{(38)} \) below.

Choose any tiny \( \varepsilon > 0 \). Let \( |\psi(y) - y| < \varepsilon y \) for all \( y \geq A = 5 \), say. (\( A \) depends on \( \varepsilon \).)

Assume \( x \geq 1000 A \). Get:

\[ |M(x)| \ln x \leq O(x) + \sum_{l \leq \frac{x}{A}} |\mu(l)| \left| \frac{x}{l} - \psi\left(\frac{x}{l}\right)\right| \]

\[ + \sum_{\frac{x}{A} < l \leq x} |\mu(l)| \left| \frac{x}{l} - \psi\left(\frac{x}{l}\right)\right| \]

since \( \psi(x) \sim x \) as \( x \to \infty \), one knows trivially that

\[ |\psi(y)| \leq M y \]

for all \( y \geq 1 \) with some \( M \)

also recall Chebyshev estimate for \( \psi \) from Lec 1 p. 18.
\[ T = O(x) + \sum_{\frac{x}{A} \leq l \leq x} \varepsilon \frac{x}{l} \]

\[ + (1 + M) \sum_{\frac{x}{A} \leq l \leq x} \frac{x}{l} \]

\[ = O(x) + (1) E x \left[ \ln \frac{x}{A} + O(1) \right] \]

\[ + O(1) x \left[ \ln \frac{x}{x/2} + O(1) \right] \]

\[ \downarrow \]

\[ |M(x)| \ln x \leq O(x) + O(x) + E x \ln x \]

\[ + O(x) \]

\[ \downarrow \]

\[ |M(x)| \leq O \left( \frac{x}{\ln x} \right) + E x \cdot \text{yes!} \]

Hence \( M(x) = o(x) \) as \( x \to \infty \).

We'll treat \( M(x) = o(x) \Rightarrow \psi(x) \sim x \)

in the next set of notes.

(p. 33)

Recall

"40 pages is long enough!"

for a synopsis.
Lecture 21 Synopsis
(6 April)

We must now prove that

\[ M(x) = o(x) \implies \psi(x) \sim x \]

by elementary methods. See Lec 20 p. 33.

We need to make a detour into the Dirichlet divisor problem:

\[ T(x) = \sum_{n \leq x} d(n) = \sum_{(k, d) \text{ lattice point}} 1 \]

Trivially:

\[ T(x) = \sum_{m \leq x} \left\lfloor \frac{x}{m} \right\rfloor \]

\[ = \sum_{m \leq x} \frac{x}{m} - \sum_{m \leq x} r\left(\frac{x}{m}\right) \]

\[ = x \ln x + O(x) + O(x) \]

\[ = x \ln x + O(x) \]
One checks that
\[ \text{card} \{ (m_1, m_2) : m_1, m_2 \leq \frac{x}{2} \} \]

\[ = \text{card} \{ m_1 \leq \left\lfloor \frac{x}{2} \right\rfloor, \ m_2 \leq \left\lfloor \frac{x}{2} \right\rfloor \} \]

\[ + \text{card} \{ m_1 \leq \left\lfloor \frac{x}{2} \right\rfloor, \ m_2 > \left\lfloor \frac{x}{2} \right\rfloor, \ m_1, m_2 \leq \frac{x}{2} \}
\]

\[ + \text{card} \{ m_1 > \left\lfloor \frac{x}{2} \right\rfloor \ \text{and} \ m_1, m_2 \leq \frac{x}{2} \right\rfloor \]

\[ \downarrow \]

but \[ m_2 \leq \frac{x}{m_1} \] and \[ m_1 > \left\lfloor \frac{x}{2} \right\rfloor \]

\[ \Rightarrow m_1 > \frac{x}{2} \Rightarrow \ m_2 \leq \left\lfloor \frac{x}{2} \right\rfloor \]

\[ \Rightarrow m_2 \leq \left\lfloor \frac{x}{2} \right\rfloor \] automatically

\[ \Rightarrow (m_1 \leftrightarrow m_2) \]

\[ = A + B + B \quad (\text{in obvious notation}) \]

\[ = A (A + B) - A \]

\[ = 2 \sum_{m_1 \leq \left\lfloor \frac{x}{2} \right\rfloor} \left\lfloor \frac{x}{m_1} \right\rfloor - \left\lfloor \sqrt{x} \right\rfloor^2 \]

\[ = 2 \sum_{x \leq \frac{k}{2}} \frac{x}{k} - 2 \sum_{k \leq \left\lfloor \sqrt{x} \right\rfloor} r \left( \frac{x}{k} \right) - x + O(\sqrt{x}) \]

\[ = 2x \left[ \ln \left\lfloor \sqrt{x} \right\rfloor + y + O \left( \frac{1}{\sqrt{x}} \right) \right] + O(\sqrt{x}) - x \]

\[ = 2x \ln \left\lfloor \sqrt{x} \right\rfloor + 2yx + O(\sqrt{x}) - x \]
\[
\begin{align*}
\ln(\sqrt{x} - \theta) &= \ln \sqrt{x} (1 - \frac{\theta}{\sqrt{x}}) \\
&= \frac{1}{2} \ln x + O(\frac{1}{\sqrt{x}}) \\
&= x \ln x + O(\sqrt{x}) + (2y-1) x \cdot
\end{align*}
\]

Thus \((\text{Dirichlet})\)

For \(x \geq 1\),

\[T(x) = \sum_{n \leq x} d(n) = x \ln x + (2y-1) x + O(\sqrt{x}) \]

\textbf{PF}

As above.

\textit{Dirichlet divisor problem}

\(\alpha = \frac{1}{4} + \varepsilon \quad ??

\textbf{Fact}

Let \(A(x) = \sum_{n \leq x} (\ln n - d(n) + 2y)\), \(x \geq 1\).

Then:

\[A(x) = O(\sqrt{x}) \cdot \]

\textbf{PF}

WLOG \(x = \text{integer}\). Just use THM and \(\ln x\). \[\square\]
Note that:

\[ I = I^2 \cdot \frac{1}{I} \]

\[ 1 = \sum_{k \cdot l = n} d(k) \mu(l) \]

\[ -\frac{I'}{I} = (\sim I') \cdot \frac{1}{I} \]

\[ \Lambda(n) = \sum_{k \cdot l = n} (\ln k) \mu(l) \]

and, as before,

\[ \left( \frac{I}{I^2} \right)' = -\frac{I'}{I^2} = \left( -\frac{I'}{I} \right) \cdot \frac{1}{I} \]

\[ -\mu(n) \ln n = \sum_{k \cdot l = n} \Lambda(k) \mu(l) \]

Fact:

\[ \psi(x) - x + 2y = \sum_{kk \leq x} m(k) \left\{ \ln k - d(k) + 2y \right\} \]

\[ x \in \mathbb{Z}^+ \]
Proof

\[ \sum_{n \leq x} \Lambda(n) = \sum_{n \leq x} \left( \sum_{k \mid n} (\ln k) \mu(k) \right) \]

by (4)

\[ = \sum_{(k, \ell)} \mu(\ell) / \ell \ln k \]

\[ = \sum_{\ell k \leq x} \mu(\ell) / \ell \ln k \]


\[ -\sum_{n \leq x} \frac{1}{n} = -\sum_{n \leq x} \left( \sum_{k \mid n} d(k) \mu(k) \right) \]

by (4)

\[ = -\sum_{\ell k \leq x} \mu(\ell) d(k) \]


\[ 2 \gamma \sum_{\ell k \leq x} \mu(\ell) = 2 \gamma \sum_{N \leq x} \mu(\ell) \]

with \[ N = \ell k \]

\[ \text{Lec 20} \]

\[ \text{(hyperbola)} \]

\[ = 2 \gamma \sum_{N \leq x} \sum_{\ell \mid N} \mu(\ell) \]

\[ = 2 \gamma \sum_{N \leq x} \left\{ \begin{array}{ll} 1, & N = 1 \\ 0, & N > 1 \end{array} \right\} \]

\[ = 2 \gamma \cdot \]

Add together. ALL IS FINE! \[ \blacksquare \]
**FACT**

\[ M(x) = o(x) \implies \psi(x) \sim x \]

**Proof**

Keep \( x \in \mathbb{Z}^+ \). Choose any large integer \( G \),
\[ 100 \leq G \leq x \] Recall (4) bottom.

\[
\psi(x) \sim x + 2y = \sum_{\ell k \leq x} \mu(\ell) \left\{ \ln k - \frac{d(k)}{k} \right\} + 2y \\
+ \sum_{\ell k \leq x} \mu(\ell) \left\{ \ln k - \frac{d(k)}{k} \right\} + 2y
\]

For the \( k \leq G \) piece, note:

\[
\sum_{\ell k \leq x} = \sum_{k=1}^{G} \sum_{1 \leq \ell \leq \frac{x}{k}} \mu(\ell) \left\{ \ln k - \frac{d(k)}{k} \right\} + 2y
\]

\[
= \sum_{k=1}^{G} \left( \ln k - \frac{d(k)}{k} + 2y \right) M\left( \frac{x}{k} \right)
\]

\( G \) is held fixed. As \( x \to \infty \), by hypothesis,
the RHS = \( o(x) \).
Next, for $k > G$, notice that:

$$lk \leq x \Rightarrow 1 \leq l \leq \frac{x}{G} \quad \text{a priori}$$

For each $l \in \left[ \frac{x}{x}, \frac{x}{G} \right]$, we look at

$$G < k \leq \frac{x}{k}$$

$$\sum_{lk \leq x \atop k > G} = \sum_{1 \leq l < \frac{x}{G}} \sum_{k \in \left( \frac{x}{k}, \frac{x}{G} \right]} \mu(l) \left( \ln k - d(lk) + 2y \right)$$

$$= \sum_{1 \leq l < \frac{x}{G}} \mu(l) \left[ \Delta \left( \frac{x}{k} \right) - \Delta \left( \frac{x}{G} \right) \right]$$

$$\uparrow \text{see } 3 \text{ bottom}$$

$$= \sum_{1 \leq l < \frac{x}{G}} \mu(l) \left[ O(1) \sqrt{\frac{x}{k}} + O(1) \sqrt{G} \right]$$

$$\{ |\mu(l)| \leq 1 \}$$

$$= O(1) \sum_{l < \frac{x}{G}} \sqrt{\frac{x}{k}} + O(1) \frac{x}{G} \sqrt{G}$$
\[ 0(1) x^{1/2}\left\{ 2\sqrt{G} + o(1) \right\} + O(1) \frac{x}{\sqrt{G}} \]

\[ \text{for } x \geq 1 \]

\[ = O(1) \frac{x}{\sqrt{G}} + O(x^{1/2}) \]

\[ \text{Hence: } \text{middle} \]

\[ |\psi(x) - x + 2y| \leq o(x) + O(1) \frac{x}{\sqrt{G}} \quad \text{as } x \rightarrow \infty. \]

By moving \( G \) upward in successive jumps, we get:

\[ \psi(x) - x = o(x), \quad \text{i.e. } \psi(x) \sim x. \]

\[ \text{ok} \]
Remark:

Recall Perron's formula: \( x \geq 10, \ x \in \mathbb{Z} \)

\[
\sum_{n<x} a_n w^{-s} = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} f(s+w) \frac{x^w}{w} dw
\]

\( s \rightarrow \sigma \)

Lec 19 p. 40 etc.

\[
5 = 0 \Rightarrow \\
\sum_{n<x} d(n) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} f(w) x^w \frac{d}{dw} dw
\]

An easy manipulation gives:

\[
\text{Res} \left[ f(w) x^w \frac{d}{dw}; w = 1 \right] = x/nx + (2y-1)x
\]

BIG SURPRISE!!! (See Thm on 3).

* Lec 18 p. 40

note the \( y \)
In the remainder of Lec 21, I pretty much followed Ingham 86-89 (middle), 90-91 (top).

There's no point repeating that discussion here except for a couple of special items that I plan to refer to later.

Definitions:

Dirichlet Series
\[\sum_{n=1}^{\infty} \frac{a_n}{n^s}\]

Generalized Dirichlet Series
\[\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}\]

with \(0 < \lambda_1 < \lambda_2 < \cdots \to \infty\)

[usually \(\lambda_1 = 1\)]

one often puts \(\lambda_n = e^{\mu_n}\)

\[\sum an e^{-\lambda_n x}\]

"Dirichlet integral"
\[\int_1^{\infty} \frac{a(x)}{x^s} dx\]

Here \(a(x)\) = a piecewise continuous fcn on \([1, \infty)\) with a discrete set of (possible) jump points which run off to \(+\infty\). Also

\[\int_1^{\infty} = \lim_{R \to \infty} \int_1^{R} \quad (R = \text{real})\]
Fact 1: (for Dirichlet series)

Let \( \sum_{n=1}^{\infty} a_n n^{-s} \) conv at \( s_0 \). Then:

(a) the series conv absolutely on \( \{ \text{Re}(s) > \text{Re}(s_0) + 1 \} \)

(b) there is uniform conv on any half-plane \( \{ \text{Re}(s) \geq \text{Re}(s_0) + 1 + \delta \} \).

Proof (sketch)

WLOG \( s_0 = 0 \).

(a) Let \( T = \sum_{n=1}^{\infty} a_n \lambda_n \) \( \lambda_n \leq x \)

is a right continuous step function. \( \lim_{x \to \infty} F(x) = T \).

Fact 2: (for generalized Dirichlet series)

Given \( \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \) with, say, \( \lambda_1 = 1 \). Assume that we have convergence at \( s_0 \). Then:

(a) we have pointwise conv on \( \{ \text{Re}(s) > \text{Re}(s_0) + \epsilon \} \)

(b) we have uniform conv on sectors of the form \( \{ |\arg(s-s_0)| \leq \frac{\pi}{\alpha} - \delta \} \).
Fix any $s$ with $\text{Re}(s) > 0$. Integrate by parts in the standard way.

$$
\lim_{N \to \infty} \left\{ a_1 + \int_1^N x^{-s} dF(x) \right\} = \lim_{N \to \infty} \left\{ a_1 + [x^{-s} F(x)]_1^N + s \int_1^N F(x) x^{-s-1} dx \right\}
$$

$$
= \lim_{N \to \infty} \left\{ N^{-s} F(N) + s \int_1^N F(x) x^{-s-1} dx \right\}
$$

$$
= s \int_1^\infty \frac{F(x)}{x^{s+1}} \, dx \quad \text{integ is absolutely conv}
$$

(b) For uniform conv, by modifying $a_1$, WLOG $T = 0$. Note that $N^{-s} F(N)$ [above] tends uniformly to 0 on $\{ \text{Re}(s) \geq 0 \}$. The issue is how fast

$$
\int_1^N \frac{F(x)}{x^{s+1}} \, dx \to s \int_1^\infty \frac{F(x)}{x^{s+1}} \, dx
$$

on $\{ |\text{Arg}(s)| \leq \frac{\pi}{2} - \delta \}$, $s \neq 0$. Write $\alpha = \frac{\pi}{2} - \delta$.

Assume that $|F(x)| < \varepsilon$ for $x \geq N_\varepsilon$ ($N_\varepsilon \in \mathbb{Z}^+$). We know $\lim_{x \to \infty} F(x) = 0$ since $T = 0$. 
Get:

\[ |\text{relevant remainder}| = |s| \left| \int_{N}^{b} \frac{F(x)}{x^{\rho \cos \theta + 1}} \, dx \right| \]

\{ keep \: N = N_e \}

\[ \leq |s| \int_{N}^{b} \frac{\varepsilon}{x^{\rho \cos \theta + 1}} \, dx \]

\{ we \: write \: s = re^{i\theta}, \: r > 0, \: |\theta| \leq \gamma \}

\[ = \rho \varepsilon \int_{N}^{b} x^{\rho \cos \theta - 1} \, dx \]

\[ = \rho \varepsilon \frac{N^{\rho \cos \theta}}{\rho \cos \theta} \]

\[ \leq \frac{\varepsilon}{\cos \theta} N^{-\rho \cos \theta} \]

\{ \cos \theta \: decreases \: on \: [0, \gamma] \}

\[ \leq \frac{\varepsilon}{\cos \gamma} \cdot \frac{1}{N^{\rho \cos \theta}} \]

By recalibrating \( \varepsilon \) for our given \( \gamma = \frac{\pi}{2} - \delta \),

we are done.
Fact 3 (for generalized D.o.S.)

In Fact 2, the associated function $f(s)$ is analytic on $\{\Re(s) > \Re(s_0)\}$ with uniform convergence on compacta. Hence,

$$f^{(k)}(s) = \sum_{n=1}^{\infty} \frac{a_n (-\lambda_n \lambda_n)^k}{\lambda_n^n}$$

with uniform convergence on compacta too.

Fact 4 (for generalized D.o.S.)

Every $\sum a_n \lambda_n^{-s}$ with, say, $\lambda_1 = 1$ has an abscissa of convergence $\sigma_c \in [-\infty, +\infty]$ so that

$$\sum_{n=1}^{\infty} a_n \lambda_n^{-s} \text{ converges} \begin{cases} \text{on} & \{\Re(s) > \sigma_c\} \\ \text{diverges} & \{\Re(s) < \sigma_c\} \end{cases}$$

No assertion about $\Re(s) = \sigma_c$. $\square$
FACTS 2-4 have easy analogs for Dirichlet integrals

\[ f(s) = \int_1^\infty \frac{a(x)}{x^s} \, dx \]

\[ s_0 = 0. \text{ Define } F(x) = \int_1^x a(y) \, dy. \text{ Note that } F(x) \text{ is continuous and piecewise } C^1 \text{ on } [1, \infty). \]

Also:

\[ \int_1^R x^{-s} a(x) \, dx = \int_1^R x^{-s} \, dF(x) \quad (R > 1) \]

\[ f(s) = s \int_1^\infty \frac{F(x)}{x^{s+1}} \, dx \text{ etc etc} \]

insofar as \( \Re(s) > 0 \).

**IMPORTANT NOTE:** (Lec 11 p. 26)

The standard example

\[ f(s) = (1 - 2^{1-s}) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \]

has \( \sigma_c = 0 \) [via \( s = \epsilon \)] yet no singularity anywhere along \( \Re(s) = 0 \).

Not like a Taylor series.
**USEFUL REMARK** *(not stated in lec)*

\[ f(s) = \text{generalized D. S. or Dirichlet integral} \]

Assume \( \sigma_c \neq +\infty \). Take any \( A > \sigma_c \).

Then:

\[ |f(\sigma + it)| = O(1 + |t|) \]

on \( \{ \Re(s) \geq A \} \).

---

**PF**

Do a trivial translation to position things so that the point \( s_0 = 0 \) is a point of convergence. Look at

\[ \sum_{1}^{\infty} \frac{F(x)}{x^{\sigma+1}} \]

on \( \textcircled{12} + \textcircled{15} \). Keep \( \Re(s) \geq A > 0 \). Obviously

previous expression

\[ \leq |s| \int_{1}^{\infty} \frac{|F(x)|}{x^{\sigma+1}} \, dx \]

\[ \leq |s| \int_{1}^{\infty} \frac{e}{x^{\sigma+1}} \, dx \]

\[ = |s| \frac{e}{\sigma} \leq |s| \frac{e}{A} \quad \text{OK} \]
Landau's Thm

\[ f(s) = \text{generalized } \cos \text{ or } \text{Dirichlet integral}. \]
Assume \( \sigma_c \in \mathbb{R} \). Assume also that the \( a_n \) or \( a(x) \) are real and eventually of fixed sign. Then, as an analytic function (cf. 14), \( f(s) \) must have a true singularity at the point \( s = \sigma_c \).

Proof
Famous. As in Ingham 88-89.

THM

Introduce \( \Theta = \sup \text{Re}(\rho) \) for \( I \) as usual. Then:
\[ \psi(x) - x = \sum_{\pm} (x^{\Theta - \varepsilon}) \]
\[ \Pi(x) - li(x) = \sum_{\pm} (x^{\Theta - \varepsilon}) \]
for each tiny \( \varepsilon > 0 \) (as \( x \to +\infty \)). Here
\[ \Pi(x) = \sum_{n \leq x} \Lambda(n) = \pi(x) + \frac{1}{2} \pi(x^{1/2}) + \ldots \]
The "implied constant" in each of the foregoing can be taken arbitrarily large.

Also

\[ \ln^*(x) = \int_2^x \frac{dt}{\ln t} \]

\[ \ln^*(x) = \int_2^x \frac{dt}{\ln t} . \]

Proof

Ingham 90-91 top.

\[ \text{plus baby calculus} \]

This uses

\[ S(x) \neq 0 \text{ on } 0 < x < 1. \]

Lec 11 p. 27

STANDARD DEFINITION.

Recall:

\[ h(x) \text{ real} \]

\[ g(x) > 0 \]

\[ h(x) = \sum [g(x)] \leftrightarrow \]

\[ h(x) > c g(x) \text{ infinitely often as } x \to +\infty \]

for some constant \( c > 0 \). Similarly

for \( h(x) = \sum [g(x)] \) and \( \sum [g(x)] \).
We will basically not repeat Ingham 91 (middle) - 92 (top) here. The reasoning in the book is quite clear. We get:

\[ \limsup_{x \to \infty} \frac{\psi(x) - x}{x^{\frac{3}{2}}} \geq \frac{\gamma_1}{\frac{1}{2} + i\gamma_1} \]

\[ \liminf_{x \to \infty} \frac{\psi(x) - x}{x^{\frac{3}{2}}} \leq -\frac{\gamma_1}{\frac{1}{2} + i\gamma_1} \]

If RH is true \( \left[ \frac{1}{2} + i\gamma_1 = \text{first zero on critical line}, m_1 = \text{its multiplicity} \right] \), if RH is false \( \Theta > \frac{1}{2} \) and the aforementioned \( \limsup, \liminf \) are +\infty and -\infty; see Lec 21 p. 17.

(Actually)

Littlewood proved in 1914 that a much better result can be obtained if RH holds. See Ingham p. 100. If time permits, we will discuss his result later.
We turn now to a proof of Hardy's theorem that $T(s)$ has infinitely many zeros along $\Re(s) = \frac{1}{2}$. (1914)

We follow the approach of Landau in 1927 in his Vorlesungen.

We begin with some calculus lemmas.

**Fact 1**

Let $f \in C[a, b]$ and $\varphi(x)$ be monotonic on $[a, b]$ (either increasing or decreasing). Then

$$
\int_a^b f(x) \varphi(x) \, dx = f(b) \varphi(b) - f(a) \varphi(a) - \int_a^b \varphi(x) f'(x) \, dx,
$$

the "$dx$" integral existing as a nice Riemann integral.

**PF**

See Lec 3 p. 8 bottom; also Lec 3 middle - 8 middle.
Fact 2 (1st mean-value thm)

Let \( q \in C[a, b] \) (and real). Let \( q(x) \uparrow \) on \([a, b]\). Then

\[
\int_a^b q(x) \, d\sigma(x) = q(\xi) \int_a^b \, d\sigma(x)
\]

for some \( \xi \in [a, b] \).

Proof

If \( q(b) = q(a) \), matter is trivial.

If \( q(b) > q(a) \), take \( m = \min \{g\} \), \( M = \max \{g\} \) on \([a, b]\). Note

\[
LHS \in \left[ m(q(b) - q(a)), M(q(b) - q(a)) \right]
\]

So, we can write \( LHS = C \, [q(b) - q(a)] \) for a unique \( C \in [m, M] \). Apply intermediate value thm to \( g \). Get \( g(\xi) = C \).

Fact 3A (rudimentary 2nd mean value thm)

Let \( f \) be monotonic on \([a, b]\). Let \( q \) be real and in \( C'[a, b] \). Then there exists \( \xi \in [a, b] \) so that

\[
\int_a^b f(x) \, d\sigma(x) = f(a) \int_a^\xi \, d\sigma(x) + f(b) \int_\xi^b \, d\sigma(x)
\]

where \( d\sigma(x) = q'(x) \, dx \).
The ideas in Lec 3 (7-8) assure us that

\[
(R-5) \int_a^b H(x) \, dq(x) = (R) \int_a^b H(x) q'(x) \, dx
\]

holds whenever \( H \) is either continuous or monotonic. To be strictly correct, one writes

\[
q = q_1 - q_2
\]

with \( q_j \in C^1 \) and \( q_j \uparrow q \).

To prove the relation stated in Fact 3A, we flip \( f \) to \(-f\) if needed and declare \( f \uparrow \) wlog. By Fact 1 \( (\text{i.e., integ by parts}) \)

\[
\int_a^b f(x) \, dq(x) = f(b) q(b) - f(a) q(a) - \int_a^b q(x) \, df(x) \quad \text{if } f \text{ increasing}
\]

By Fact 2,

\[
\int_a^b q(x) \, df(x) = q(\xi) \int_a^b df = q(\xi) [f(b) - f(a)]
\]

Hence:
\[ \int_a^b f(x) \, dx = \int_a^b f(a) \, dx + \int_a^b f(b) \, dx \]

Fact 3B (2nd Mean-Value Theorem)

Let \( f \) be monotonic \( \uparrow \) on \([a, b]\), let \( \theta \in \mathbb{R} \), \( \theta \in C^1([a, b]) \), and \( \theta \) real. Let \( A = f(a^+) \), \( B = f(b^-) \).

We can then find \( \xi \in [a, b] \) so that

\[ \int_a^b f(x) \, dx = A \int_a^\xi f(x) \, dx + B \int_\xi^b f(x) \, dx. \]

Similarly for \( f \downarrow \) on \([a, b]\).

\[ \text{pf} \]

Let \( f_0(x) = \begin{cases} a, & x = a \\ f(x), & a < x < b \\ b, & x = b \end{cases} \) . Note that \( f_0 \uparrow \).

Apply Fact 2A to get

\[ \int_a^b f_0(x) \, dx = A \int_a^\xi f(x) \, dx + B \int_\xi^b f(x) \, dx. \]

But,
\[ \int_{a}^{b} y_0(x)dx = \int_{a}^{b} y_0(x) y'(x)dx = \int_{a}^{b} f(x) y'(x)dx = \int_{a}^{b} f(x) dx \]

by (**) top 3

**NOTE:** For \( f \downarrow \), one takes
\[ A \geq f(a^+), \quad B \leq f(b^-) \]

---

**Lemma I**

Let \( F \in C^1[a, b] \) and real. Assume that \( F'(x) \) is *monotonic* on \([a, b]\). Assume further that
\[ F'(x) \geq m > 0 \quad \text{or} \quad F'(x) \leq -m < 0 \]
for all \( x \in [a, b] \). Then:
\[ \left| \int_{a}^{b} e^{iF(x)} dx \right| \leq \frac{4}{m} \]
\textbf{Pf}

\underline{WLOG \ F(x) \ is \ monotonic \ \uparrow \ (simply \ flip \ F \ to \ -F \ if \ need \ be)}.

Suppose first that \( F'(x) = m > 0 \).

\[
\int_a^b \cos F \, dx = \int_a^b \frac{1}{F'} \left( \cos F \circ F' \right) \, dx
= \int_a^b \frac{1}{F'} \, d(\sin F)
\]

\[
\begin{cases}
\frac{1}{F'} \text{ is monotonic } \downarrow \text{ and positive}
\end{cases}
\]

Recall \textbf{Fact 3B)}.

\[
\begin{array}{c}
\Delta \ \frac{1}{F'(x)} \\
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Similarly:

\[ \left| \int_a^b \sin F \, dx \right| \leq \frac{2}{m} \]

Now suppose \( F'(x) \leq -m < 0 \) on \([a, b]\):

\[ \int_a^b \cos F \, dx = \int_a^b \frac{1}{F'} \, d(sin F) \quad \text{as before} \]

\[
\begin{cases} 
F' \text{ is negative, monotonic up} \\
\frac{1}{F'} \text{ is negative, monotonic down}
\end{cases}
\]

Fact 3B

\[
\begin{align*}
A &= 0 \\
B &= \frac{1}{F'(b)}
\end{align*}
\]

\[ \int_a^b \frac{1}{F'} \, d(sin F) = 0 \int_a^b \, d(sin F) + \frac{1}{F'(b)} \int_a^b \, d(sin F) \]

\[ \Rightarrow \left| \int_a^b \cos F \, dx \right| \leq \frac{2}{|F'(b)|} \leq \frac{2}{m} \]
Similarly \\

\[ \left| \int_a^b \sin F \, dx \right| \leq \frac{2}{\mu} \]

\[ \square \]

**Lemma II**

Given \( F, G \) real, \( G(x) > 0 \). \( F, G \) in \( C^1[a, b] \).

Assume \( \frac{F'(x)}{G(x)} \) is monotonically on \([a, b]\).

Assume further that

\[ \frac{F'(x)}{G(x)} \geq m > 0 \text{ OR } \frac{F'(x)}{G(x)} \leq -m < 0 \]

for all \( x \in [a, b] \). Then:

\[ \left| \int_a^b G(x) e^{iF(x)} \, dx \right| \leq \frac{4\mu}{m} \]

**Proof**

Flip \( F \) to \(-F\) if need be so take \( \frac{F'}{G} \) monotonically on \([a, b]\) WLOG.

Suppose first that \( \frac{F'(x)}{G(x)} \geq m > 0 \).

\[ \int_a^b G(x) \cos F(x) \, dx = \int_a^b \frac{G(x)}{F'(x)} \, d(sin F) \]
\[ \sqrt{G \cos F \, dx} \leq \frac{2}{m} \quad \text{pos} \]

Then:
\[ \int_{a}^{b} G \sin F \, dx \leq \frac{2}{m} \quad \text{pos} \]

Suppose next that \( \frac{F'(x)}{G(x)} \leq -m < 0 \).

\[ \sqrt{G \cos F \, dx} \leq \frac{2}{m} \quad \text{neg} \]

Then:
\[ \int_{a}^{b} G \sin F \, dx \leq \frac{2}{m} \quad \text{neg} \]
Lemma III

\[ F \text{ real, } F \in C^2[a,b]. \ F''(x) \geq r > 0 \ \text{OR} \]
\[ F''(x) \leq -r < 0 \ \text{for all} \ x \in [a,b]. \ \text{Then} \]

\[ \left| \int_a^b e^{iF(x)} \, dx \right| \leq \frac{8}{\sqrt{r}}. \]

Pf

Note that \( \left| \int_a^b e^{iF(x)} \, dx \right| \leq (b-a) \) trivially.

Hence, wlog \( \frac{8}{\sqrt{r}} < b-a. \)

Case 1

\[ F'(x) > 0 \ \text{on} \ a < x < b. \]

Know \( F''(x) \neq 0 \ \text{on} \ [a,b] \) and \( |F''(x)| \geq r > 0. \)

Hence \( F'(x) \) is \textbf{monotonic} on \( a \leq x \leq b. \)

Suppose, e.g., \( F'(x) \) is decreasing.

Clearly

\[ F'(x) - F'(b) \geq r \delta \ \text{for} \ a \leq x \leq b - \delta. \]

We assume here \( \delta < b-a. \) Hence, \( F'(x) \geq r \delta. \)
Write

\[ I = \int_a^b e^{iF} \, dx = \int_a^{b-\delta} e^{iF} \, dx + \int_{b-\delta}^b e^{iF} \, dx \]

to get

\[ |I| \leq \left| \int_a^{b-\delta} e^{iF} \, dx \right| + \delta \]

\{ \text{apply Lemma 1 \textsuperscript{1}} \}

\[ |I| \leq \frac{4}{r^{\delta}} + \delta \quad . \]

We propose to take \( \delta = \frac{2}{\sqrt{r}} \). We have

\[ b - a > \frac{8}{\sqrt{r}} > \frac{2}{\sqrt{r}} = \delta \]

as needed. \( \delta \) so for (ii) bottom

\[ |I| \leq \frac{4}{\sqrt{r}} \quad . \quad \text{(OK)} \]

Similarly for \( F'(x) \) increasing.

Case 2

\( F'(x) < 0 \) on \( a < x < b \).

Here, just flip \( F \) to \( -F \) and use Case 1. \( \text{(OK)} \)
For all other cases, we must then have $F'(c) = 0$ for some $c \in (a, b)$.

We know $F''(x) \neq 0$ on $[a, b]$ and $|F''(x)| \geq r > 0$.

Hence $F'(x)$ is strictly monotonic on $[a, b]$.

The point $c$ is therefore unique.

**Case 3**

$F'(x) > 0$ on $[a, c)$, $F'(x) < 0$ on $(c, b]$.

We maintain that

$$\int_a^c e^{iF(x)} dx \leq \frac{y}{\sqrt{r}} \quad \text{and} \quad \int_c^b e^{iF(x)} dx \leq \frac{y}{\sqrt{r}}.$$

Take $y$, say $[c, b]$. If $b - c \geq \frac{y}{\sqrt{r}}$, we are fine. Suppose therefore

$$b - c > \frac{y}{\sqrt{r}}.$$

Know $F'(c) - F'(x) \geq r(x - c)$ for $c < x \leq b$.

Hence

$$F'(c) - F'(x) \geq r \delta \quad \text{for} \quad c + \delta \leq x \leq b.$$

We assume here $\delta < b - c$. Get $F'(x) \leq -r \delta$.

See (11) bottom.
Get:

\[ \left| \int_{c}^{b} e^{F} dx \right| = \left| \int_{c}^{c+\delta} e^{F} dx \right| + \left| \int_{c+\delta}^{b} e^{F} dx \right| \]

\[ \leq \delta + \frac{Y}{r} \delta \] by Lemma 1.0

Propose to put \( \delta = \frac{2}{\sqrt{Y}} \) to get \( \sqrt{\frac{Y}{r}} \).

We need:

\[ \frac{2}{\sqrt{Y}} < b - c \quad \text{but} \quad b - c > \frac{Y}{\sqrt{r}} \]

(by hypothesis). Hence:

\[ \left| \int_{c}^{b} e^{F} dx \right| \leq \frac{Y}{\sqrt{r}} \]

(by hypothesis). Hence:

The case \([a, c]\) is similar, of course.

Get:

\[ \left| \int_{a}^{b} e^{F} dx \right| \leq \frac{Y}{\sqrt{r}} + \frac{Y}{\sqrt{r}} = \frac{8}{\sqrt{r}} \]

Case \(4\): \( F'(x) < 0 \) on \([a, c]\), \( F'(x) > 0 \) on \((c, b]\).

Here, just flip \( F \) to \(-F\) and use Case 3.
Lemma IV

\[ F \text{ real, } G > 0 \text{, } F \in C^2, G \in C^1 \text{ on } [a, b] \circ \]

Assume \( F''(x) \geq r > 0 \) or \( F''(x) \leq -r < 0 \) for all \( x \circ \)

Assume also that \( \frac{F'}{G} \) is monotonic on \([a, b]\) and \( |G(x)| \leq M \). Then:

\[ \int_a^b G(x)e^{iF(x)}dx \leq \frac{8M}{\sqrt{r}} \circ \]

Proof

Imitate proof of Lemma III. \( \text{WLOG, } \frac{8}{\sqrt{r}} < b - a \).

Use Lemma II. Etc.

EG case 1 \( \implies M \delta + \frac{4}{(r \delta M)} = M \left[ \delta + \frac{4}{r \delta} \right] \Rightarrow \text{etc.} \)

In case 3, p. 13 middle, refer to:

\[ \frac{4M}{\sqrt{r}} \text{ and } \frac{4M}{\sqrt{r}} \]

\( \Box \)
We use a series of Facts (in the writing style of Landau) to establish Hardy's theorem that

\[ N_{\text{critical}}(T) \to \infty \text{ as } T \to \infty. \]

Here \( N_{\text{critical}}(T) = N\left\{ \rho : \text{Re}(\rho) = \frac{1}{2}, 0 < \text{Im}(\rho) \leq T \right\}. \)

**Fact 1**

\[ T \geq 2, \quad \forall \in \mathbb{R}. \quad \text{Then} \]

\[ \left| \int_{T}^{2T} t^{-\frac{1}{8}} e^{\frac{i}{2} \left( t \ln t + 4 \tau \right)} \, dt \right| = O(T^{5/8}) \]

with an implied constant which is absolute.

(No dependence on \( \epsilon \))

**PF**

Lec 22, p. 15 Lemma IV.

\( g(t) = t^{1/8} \quad \Rightarrow \quad 2F(t) = t \ln t + 4 \tau \)

\( M = (2T)^{1/8} \quad \Rightarrow \quad 2F'(t) = 1 + \ln t + \frac{4}{t} \)

\( 2F''(t) = \frac{1}{t} \quad \Rightarrow \quad r = \frac{1}{4T} \quad \text{for} \quad [T, 2T] \)
\[
\frac{F'(t)}{G(t)} = \frac{1}{2} \frac{1 + A + \ln t}{t^{1/8}}
\]
\[
\frac{d}{dt} \left( \frac{A + \ln t}{t^{1/8}} \right) = \frac{t^{1/8} (t^{-1}) - (A + \ln t) \frac{1}{8} t^{-7/8}}{t^{1/4}}
\]
\[
= \frac{t^{-7/8}}{t^{1/4}} \left[ 1 - \frac{A + \ln t}{8} \right]
\]

Critical pt \iff \( 8 = A + \ln t \) (etc)

so \( \frac{F'(t)}{G(t)} \) has AT MOST ONE crit pt

on \([T, 2T]\)

\[
\frac{M}{\sqrt{r}} = (\text{constant}) T^{5/8}
\]

Apply Lemma IV from Lec 22 either once or twice \( \square \)

\textbf{NOTE:}

Analogous fact holds for \([T, T + H]\), any \( H \in [1, T] \).
Recall

\[ \xi(s) = G(s) \Gamma(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \]

\[ \xi(s) = \xi(1-s) \]

`a la Lec II eq 24 + 27.`

\[ f(s) = e^{-\frac{\pi}{4} \left(s-\frac{1}{2}\right)^2} \xi(s) \]

for \( \text{Im}(s) \geq 1 \).

---

**Fact 2**

(a) \( f(s) \) is analytic on \( \{ \text{Im}(s) \geq 1 \} \)

(b) \( |f(\sigma + it)| = e^{\frac{\pi t^2}{4}} |f(\sigma + it)| \)

(c) \( |f(1-\sigma + it)| = |f(\sigma + it)| \)

(d) \( f\left(\frac{1}{2} + it\right) \) is real for \( t \in [1, \infty) \)

(e) \( \xi\left(\frac{1}{2} + it\right) \) is real for \( t \in \mathbb{R} \).

**Proof**

Easy. \( \blacksquare \)
Fact 3

Given any $-\infty < \sigma_1 < \sigma_2 < \infty$. We then have

$$| \Gamma(\sigma+it) | = \sqrt{2\pi} |t|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2} |t|} \left( 1 + O\left( \frac{1}{|t|} \right) \right)$$

uniformly on $\frac{\sigma_j}{2} \leq \sigma \leq \sigma_2$, $|t| \geq \frac{1}{10}$. \\

Pf.

Standard corollary of Stirling's formula for $\log \Gamma(\sigma+it)$, Lec 10 around 42.

Recall that:

$$| \Gamma(\sigma+it) | \leq \frac{C}{\delta(1-\delta)} t^{-\delta} \quad \sigma \geq \delta, \quad t \geq 3$$

$$| \Gamma(\sigma+it) | = O(\ln t) \quad \sigma \geq 1 - \frac{e}{\ln t}, \quad t \geq 3$$

$$| \Gamma(\sigma+it) | = O(\ln^2 t) \quad \sigma \geq 1 - \frac{e}{\ln t}, \quad t \geq 3$$

$$\log \Gamma(\sigma) = \Theta(1) \text{ for } \sigma \approx 1 + \epsilon$$

Here $0 < \delta < 1$, $c = \text{small}$, $0 < \epsilon < \frac{1}{2}$. See Lec 6 9 20 4.
Fact 4

On \( \{ -\frac{1}{4} \leq \sigma \leq \frac{5}{4}, \ t \geq 1 \} \) we have

\[ |f(s)| = O(t^{\sigma/2}) \]

**CRUDE BOUND**

**Proof (\text{\textbullet})**

By Fact 2(c), wlog \( \frac{1}{2} \leq \sigma \leq \frac{5}{4} \). Apply \( p \circ 4 \) bottom with \( \delta = \frac{1}{2} \). Get:

\[ |f(s+i\tau)| \leq c e^{\frac{\pi}{4} t} \left| \Gamma \left( \frac{\sigma}{2} + \frac{i\tau}{2} \right) \right| |J(s+i\tau)| \]

\[ \leq c e^{\frac{\pi}{4} t} \left| \Gamma \left( \frac{\sigma}{2} + \frac{i\tau}{2} \right) \right| |f(s+i\tau)| \]

\[ \leq c e^{\frac{\pi}{4} t} \left( \frac{t}{\sigma} \right)^{\frac{\sigma}{2} - \frac{1}{2}} e^{-\frac{\pi}{4} t^2} |f(s+i\tau)| \]

\[ \leq c \left( \frac{t}{\sigma} \right)^{\frac{\sigma}{2} - \frac{1}{2}} t^Q \]

where "c" can change from line to line and

\[ Q = \begin{cases} \frac{1}{2}, & 0 \leq 1 \\ \frac{1}{100}, & 0 > 1 \end{cases} \]

The extreme exponents are \( \frac{1}{2} \) and \( \frac{1}{8} + \frac{1}{100} \), so we are done.
Fact 5

For $\sigma = \frac{5}{4}$ and $-\frac{1}{4}$, we have

$$|f(\sigma + it)| = O(t^{1/8})$$

in Fact 4.

\[ \text{pf} \]

Just review the proof and recall $|f(s)| \leq f(\sigma)$ whenever $\sigma > 1$. Get

$$|f(\frac{5}{4} + it)| = O(t^{1/8}).$$

Treat $\sigma = -1/4$ via Fact 2(c).

Fact 6

On $\frac{1}{4} \leq \sigma \leq \frac{5}{4}$, $t \geq 1$, we actually have

$$|f(\sigma + it)| = O(t^{1/8})$$

for any $\sigma$.

\[ \text{pf} \]

This is an immediate consequence of Facts 4 + 5 when the Phragmén–Lindelöf principle for
(general) analytic functions is applied. To avoid interruptions, we prove P-L in Lec 24.

Fact 7

For \( t \geq \frac{1}{10} \) and some \( \beta \in \mathbb{C} \) with \( |\beta| = \sqrt{2}\pi \)
we have

\[
\Gamma\left(\frac{5}{8} + it\right) = \beta e^{-\frac{\pi}{4}t} \frac{1}{8} e^{it\ln\left(\frac{t}{2}\right)} \left[ 1 + O\left(\frac{1}{t}\right) \right].
\]

PF

Kindergarten calculation with Stirling's formulas
see Lec 10 around 42.

In what follows, we plan to compare

\[
\int_T^{2T} \left| f\left(\frac{1}{2} + it\right) \right| dt \quad \text{with} \quad \left| \int_T^{2T} f\left(\frac{1}{2} + it\right) dt \right|.
\]

Also, similarly for \([T, T + H], \, 1 \leq H \leq T\).

in Lec 24
Fact 8

\[ \int_{\frac{1}{2} + i(2T)}^{\frac{1}{2} + i(T)} f(z) \, dz = iT + O(T^{1/2}) \]

\[ \text{Pf} \]

Cauchy integral theorem \Rightarrow use new path

\[ \text{Recall } \delta \text{ bottom } \delta = \frac{1}{2} \text{. Get} \]

\[ \left| \int f(z) \, dz \right| = O(T^{1/2}) \text{.} \]

Along \( \sigma = 2 \), we get the revised integral

\[ \int_{2 + iT}^{2 + i(2T)} \left[ 1 + \sum_{n=2}^{\infty} \frac{-2-ist}{n^2} \right] \, dt \]
\[\begin{align*}
\theta (2T-1) & \quad + \sum_{n=2}^{\infty} n^{-2} \int_T^{2T} \frac{e^{-it\ln n}}{-it\ln n} dt \\
& \quad + 0(1) \sum_{n=2}^{\infty} \frac{1}{n^2 \ln n} \\
& \quad + 0(1) . \quad \text{Adding things, we are done.}
\end{align*}\]

Note that \([T, T+H]\) gives \(iH + O(T^{-1/2})\) insofar as \(1 \leq H \leq T\).

**Fact 9**

For large \(T\), one has

\[
\int_T^{2T} |I(\frac{1}{2}+it)| dt > \frac{1}{2} T.
\]

\(\text{Pf}\)

Trivial corollary of Fact 8.
Thm

\[ N_{\text{critical}}(T) \to \infty \quad \text{as} \quad T \to \infty. \]

In fact,

\[ N_{\text{crit}}(T) \approx c \ln T \quad \text{for } T \text{ large}. \]

PF

We study \( f(s) \) on \( \left[ \frac{1}{2} + \frac{5}{4} \right] \times \left[ T, 2T \right] \). Know

\[ f(s) = e^{-\frac{\pi}{4} \left( T - \frac{1}{2} \right)} e^{\frac{\pi t}{4}} \xi(s). \]  

Apply CFT to

\[ \int_{T}^{2T} f\left( \frac{1}{2} + it \right) dt = \int_{\frac{T}{2} + i(2T)}^{\frac{5}{4} + i(2T)} f(s) ds \]

\[ = \] 

\[ \frac{1}{2} + i(2T) \]

\[ \frac{5}{4} + i(2T) \]
By Fact 6,

\[ \int \sum_{r} f(r) ds = O(\gamma^{1/8}) \cdot \prod_{\text{horizontal}} \]

For the \((\frac{5}{4})\) contribution, note that

\[ \int_{r} f(\frac{5}{4}+it) \, dt \]

\[ \int_{r} f(\frac{5}{4}+it) \, ds \]

has

\[ f(\frac{5}{4}+it) = e^{-\frac{3\pi i}{16}} e^{\frac{\pi t}{4}} \prod_{\text{vertical}} \Gamma\left(\frac{5}{8}+\frac{it}{2}\right) \Gamma\left(\frac{5}{4}+it\right) \]

\[ = e^{\frac{\pi t}{4}} e^{-\frac{it}{2} \ln \pi} \prod_{\text{vertical}} \Gamma\left(\frac{5}{8}+\frac{it}{2}\right) \Gamma\left(\frac{5}{4}+it\right) \]

\[ = e^{\frac{\pi t}{4}} e^{-\frac{it}{2} \ln \pi} \beta, e^{\frac{\pi t}{4}} e^{\frac{it}{2} \ln(\frac{t}{2\pi})} \]

[1 + O(\frac{1}{t})] \cdot \Gamma\left(\frac{5}{4}+it\right) \]

See Fact 7.

Complex and nonzero changes from line to line.
\[ = Ct \frac{1}{8} e^{\frac{\pi}{2} \ln \left( \frac{e}{2\pi i} \right)} \cdot [1 + O(\varepsilon)] \]

\[ \cdot J \left( \frac{5}{4} + it \right) \]

\[ = Ct \frac{1}{8} e^{\frac{\pi}{2} \ln \left( \frac{e}{2\pi i} \right)} \cdot J \left( \frac{5}{4} + it \right) \]

\[ + O(t^{-1/8}) \cdot O \left( \frac{1}{t} \right) \cdot \exp \left[ O(1) \right] \]

\[ \text{bottom} \]

\[ \mathcal{F} = Ct \frac{1}{8} e^{\frac{\pi}{2} \ln \left( \frac{e}{2\pi i} \right)} \left\{ \sum_{n=1}^{\infty} n^{-\frac{5}{4} - it} \right\} \]

\[ + O(t^{-7/8}) \]

\[ \text{Of course,} \]

\[ \int_{\frac{5}{4} + it}^{\frac{5}{4} + it} \frac{O(t^{-7/8})}{dt} = O(T^{1/8}). \]

We need to focus on

\[ \frac{1}{T} \sum_{n=1}^{\infty} n^{-\frac{5}{4} - it} \ln n \cdot \frac{1}{8} e^{\frac{\pi}{2} \ln \left( \frac{e}{2i} \right)} \]

\[ \cdot e^{\frac{\pi}{2} \ln \left( \frac{e}{2i} \right)} \]
\[
\begin{align*}
&= C \sum_{n \geq 1} n^{-\frac{5}{4}} \int_{T}^{\infty} \frac{1}{N} e^{\frac{t}{2} \ln \left( \frac{t}{2\pi n^{2}} \right)} dt \\
&= C \sum_{n \geq 1} n^{-\frac{5}{4}} O\left(T^{-5/8}\right) \quad \text{by } \text{Fact 1} \\
&= O\left(T^{-5/8}\right).
\end{align*}
\]

It follows, by (11) + (12) + the above, that

\[
\int_{\frac{5}{4} + iT}^{\frac{5}{4} + iT} f(s) ds = O\left(T^{-5/8}\right).
\]

By (10) bottom + (11) top, we finally get:

\[
\int_{-2}^{2} f\left(\frac{1}{2} + it\right) dt = O\left(T^{1/8}\right) + O\left(T^{-5/8}\right) \\
= O\left(T^{-5/8}\right).
\]

\underline{Remark}.

Landau uses Fact 6 for (11) line 2, i.e., \( \int_{\text{horiz}} f(s) ds \).

Exploitation of the weaker Fact 4 produces \( O(T^{1/2}) \).

This is sufficient since

\[ O\left(T^{-5/8}\right) + O\left(T^{1/8}\right) + O(T^{1/2}) = O\left(T^{-5/8}\right). \]

Phragmén-Lindelöf can thus be avoided.
\[ \int_{-T}^{T} f\left(\frac{1}{2} + it\right) dt = O(T^{5/8}) \]

with main contribution due to \((13)\) lines 1-3 and Fact 1.

On the other hand, by Fact 2, on \((3)\),

\[ |f\left(\frac{1}{2} + it\right)| = e^{\frac{\pi t}{4}} |\xi\left(\frac{1}{2} + it\right)| \]

\[ = e^{\frac{\pi t}{4}} \left| \pi^{-\frac{1}{2}} \left(\frac{1}{2} + it\right)^{-\frac{1}{2}} \xi\left(\frac{1}{2} + it\right) \right| \]

\[ \geq c e^{\frac{\pi t}{4}} \sqrt{2\pi} t^{-\frac{1}{2}} e^{-\frac{\pi t}{8}} \left[ 1 + O\left(\frac{1}{t}\right) \right] |\xi\left(\frac{1}{2} + it\right)| \]

\[ \geq c t^{-1/4} |\xi\left(\frac{1}{2} + it\right)| \]

for \( t \) large. Hence,

\[ \int_{-T}^{T} \left| f\left(\frac{1}{2} + it\right) \right| dt \geq c T^{-1/4} \int_{-T}^{T} \left| \xi\left(\frac{1}{2} + it\right) \right| dt \]

\[ \geq c T^{-1/4} (T/2) \quad \text{Fact 9 (9)} \]

\[ \geq c T^{3/4} \]
Accordingly, for each large $T$, 

$$\left| \int_T^{2T} f(\frac{1}{2}+it) \, dt \right| < \frac{1}{2} \int_T^{2T} |f(\frac{1}{2}+it)| \, dt.$$ 

As such, there must be some point in $(T, 2T]$ where the real-valued continuous function $f(\frac{1}{2}+it)$ undergoes a change of sign.

---

Remember that $f(s)$ is nicely analytic à la local Taylor series!

---

In other words: $(T, 2T]$ contains at least one odd order zero of $f(\frac{1}{2}+it)$.

By (3) top, hence likewise for $f(\frac{1}{2}+it)$.

By studying the cases $T = 2^k$, we clearly get 

$$N_{\text{crit}}(T) \to \infty$$

and, indeed, 

$$N_{\text{crit}}(T) \asymp c \ln T \quad (\text{all large } T).$$
A moment's thought about Proposition 15 shows that we have actually proved:

\[ \# \left\{ \text{distinct } \beta : \Re(\beta) = \frac{1}{2}, \text{ } 0 < \Im(\beta) \leq T \right\} \geq c \ln T. \]

Some further refinements were left for discussion in Lec 24.

[End of Lec 23]
Function theory — centered on max mod principle,
Phragmén–Lindelöf principle, Lindelöf mu-function,
Littlewood’s formula for $\int_0^\beta N(\theta; T_1, T_2) \, d\theta$.

**Thm (Max Mod Principle)**

Let $D$ be a bounded domain in $\mathbb{C}$, let $F$ be analytic on $D$. Let

$$\lim_{z \to \zeta} \sup_{\mathbb{D}} |F(z)| \leq M \quad \text{all } \zeta \in \partial D.$$

Then:

$$|F(z)| \leq M \quad \text{all } z \in D.$$

If equality ever holds, then $F(z) \equiv GM e^{i\theta}$

for some $\theta \in \mathbb{R}$.

**Pf**

As in function theory, with standard use of

$$F(z_0) = \frac{1}{2\pi} \int_0^{2\pi} F(z_0 + re^{i\phi}) \, d\phi$$

for $0 < r < \text{dist}(z_0, \partial D)$.
Thu (Phragmén–Lindelöf)

Let \( D \) be a \underline{simply-connected} domain.

Let \( f \) be analytic on \( D \). Let \( |f| \leq C \) for some big constant \( C \). Let

\[
\lim_{z \to \infty} |f(z)| \leq M, \quad \text{all } z \in \partial D \setminus \{a_1, \ldots, a_m\}.
\]

Then:

\[
|f(z)| \leq M, \quad \text{all } z \in D.
\]

\textbf{PF}

\( m = 1 \). \( z - a_1 \neq 0 \) on \( D \). Construct a single-valued branch \( \log(z-a_1) \). Also \( (z-a_1)^\varepsilon \).

Let \( f_{\varepsilon} = f \circ \left( \frac{z-a_1}{\varepsilon} \right) \). Here \( \varepsilon = \text{Adiam}(D) \).

Note \( \lim_{z \to a_1} |f_{\varepsilon}| = 0 \). And \( \lim_{z \to \infty} |f_{\varepsilon}| \leq M \cdot 1 \).

Hence \( |f_{\varepsilon}(z)| \leq M \). Fix any \( z \in D \). Get

\[
|f(z)| \leq M \left| \frac{\varepsilon}{z-a_1} \right|^{\varepsilon}.
\]

Let \( \varepsilon \to 0 \). \hfill \square

\begin{itemize}
  \item Simply-connected \( D \)
  \item Taken for maximal simplicity in the proof.
\end{itemize}
Counterexample if no $G$ exists.

$$F(z) = \exp\left(\frac{i}{z}\right), \quad D = \{ |z| < 1, y > 0 \}$$

$$M = e^1, \quad a_1 = \{ 0 \}$$

$$\frac{1}{y} \to \infty \text{ as } y \to 0^+$$

\[\text{Fact}\]

**Fact**

Given $E = \{ a < x < b, y > 0 \}$, let $F$ be analytic on $E$. Let $|F(z)| \leq M$.

Let

$$\lim_{z \to \alpha} |F(z)| \leq M, \quad \forall \alpha \in \partial E \cup C.$$

Then $|F(z)| \leq M$ on $E$.

**Proof**

Apply $p \circ \varphi$ after passing to a change of variable $\varphi = \frac{1}{x + c}$, with $c$ big enough to have $c + a > 0$. The new domain $E_{\varphi}$ is bounded. ($c = \text{real!!}$)
Fact

Let \( E = \left\{ -\frac{\pi}{2} < x < \frac{\pi}{2}, \ y > 0 \right\} \). The function \( w = \sin z \) maps \( E \) in a 1-1 way onto \( \{ \text{Im}(w) > 0 \} \). \( E \) corresponds to \( \mathbb{R} \) in a nice fashion.

Proof

Look at the formula

\[
\sin(x+iy) = \sin x \cdot \cosh y + i \cos x \cdot \sinh y
\]

Use standard fun theory.

Note that:

\[
F(z) = e^{-i \sin(z)} \quad (z \in E)
\]

has \( |F(z)| > 1 \), although \( \lim_{z \to \xi} |F(z)| = 1 \),

each \( z \in \mathbb{R} \setminus \mathbb{C} \). Also, for fixed \( x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \), we have:

\[
|F(x+iy)| = e^{\cos x \cdot \sinh y}
\]

and \( \cos x \cdot \sinh y \sim \cos x \cdot \frac{1}{2} e^y \quad (y \to \infty) \)
Theorem (compare Ingham p. 95) 

Let \( E = \{ q_1 < x < q_2, y > 0 \} \). Let \( F \) be analytic on \( E \); let

\[
\lim_{z \to \xi} |F(z)| \leq M, \quad \text{all } \xi \in \partial E \setminus C_j.
\]

\[|F(x+iy)| \leq C \exp[to be filled] \left[ e^{cy} \right], \quad \text{some } C_j \text{,}
\]

some \( 0 < c < \frac{\pi}{q_2 - q_1} \).

Then:

\[|F(z)| \leq M \text{ on } E.\]

Proof:

wlog \( q_1 = -\frac{\pi}{2}, q_2 = \frac{\pi}{2} \). Take \( c < b < 1 \).

Study

\[F_\xi(z) = F(z) e^{i \xi \sin(bx)} \text{ on } E.\]

By formula for \( \sin(x+iy) \) on \( \xi \), get

\[|F_\xi(z)| = |F(z)| e^{-E \cos(bx) \sinh(by)} \]

\[\lim_{z \to \xi} |F_\xi(z)| \leq M \cdot 1\]
but

\[ e^{by} - E(\cos \frac{b}{x}) \sinh(by) \to -\infty \]

as \( y \to +\infty \)

\[ \forall \]

\[ |F_{\varepsilon}| \to 0 \quad \text{as} \quad y \to \infty \]

\[ \forall \]

\[ |F_{\varepsilon}| \leq \text{some } G \quad \text{on } E \]

and

\[ \lim_{x \to x} |F_{\varepsilon}| \leq M, \quad \text{all } \xi \in \partial E \cup \mathcal{C}. \]

Apply (3). Get \( |F_{\varepsilon}(x)| \leq M \quad \text{on } E \]

so

\[ |F(\varepsilon)| \leq M e^{\varepsilon \cos(bx) \sinh(by)}, \quad \text{each } \varepsilon. \]

Let \( \varepsilon \to 0^+ \). Get

\[ |F(z)| \leq M. \]
Corollary

\[ E = \delta_{x_1 < x < x_2, \ y > 0} \]  \ F \ \text{analytic}

on \ E. \ \text{Let} \ \lim_{\Gamma_0} |F(z)| = O(e^{\xi y})

some \ \text{graint} \ \delta. \ \text{Let} \ \lim_{z \to \xi} |F(z)| \ \text{be bdd} \ a/a

Then:

\[ |F(z)| = \max\left\{ c_1, c_2, c_3 \right\} \ \text{on} \ E. \]
**Thm (convexity thm)**

Given \( E = \{ a < x < b, y > y_0 \} \) with \( y_0 > 0 \). Let \( F \) be analytic on \( E \) and have

\[
|F(x+iy)| = O(e^{Gy})
\]

\( G \approx \text{giant} \).

Let \( \lim_{z \to E} |F(z)| \) be bounded a la sizes

We can then find a constant \( M \) depending in an explicit way on

\[
\{ E, A, B, \max\{c_1, c_2, c_3\} \}
\]

such that

\[
|F(x+iy)| \leq M y^A \left( \frac{b-x}{b-a} \right) + B \left( \frac{x-a}{b-a} \right)
\]
\[
\begin{align*}
\textbf{pf (sketch)} & \quad \text{wlog } c_1 = c_2 = c_3 = 1 \quad \text{and } a = 0, b = 1. \\
\text{Introduce (on } \mathbb{E}) & \quad \log(-i\mathbb{E}) = \log \mathbb{E} - i \frac{\pi}{2} \\
\text{Look at} & \quad g(\mathbb{E}) = \exp \left[ (A(1-\mathbb{E}) + B \mathbb{E}) \log(-i\mathbb{E}) \right]. \\
\text{Write, for } 0 < x < 1, y > y_0 & \quad \\
\log(y-i\mathbb{E}) & = \ln y + \log(1 - i \frac{x}{y}) \\
& = \ln y - i \frac{x}{y} + O\left(\frac{1}{y^2}\right). \\
\text{Get } [0 < x < 1, y > y_0]: & \quad \\
|g(x+i\mathbb{E})| & = y^{A(1-x) + Bx} \exp\left[O(1)\right] \\
& \text{depends on } A, B, E. \\
\text{Form} & \quad \\
H(\mathbb{E}) & = \frac{F(\mathbb{E})}{g(\mathbb{E})} \text{ on } \mathbb{E}.
\end{align*}
\]
\[
\lim_{n \to \infty} |H(z)| \leq B, \quad \forall z \in \mathbb{C}
\]

while

\[
|H(x+iy)| \leq \frac{O(1)e^{\gamma y}}{y^{A(1-x)+Bx} \exp\left[O(1)\right]} \leq O(1)e^{2\gamma y} \quad \text{on} \quad E
\]

\[
|H(z)| \leq B, \quad \forall z \in E
\]

\[
|F(z)| \leq B|g(z)|
\]

\[
|F(z)| \leq y^{A(1-x)+Bx} \quad \text{on} \quad E_0.
\]

Lindelöf property, \(\mu\)-function \(\mu\) positive

Let \(F(z)\) be analytic on

\[
E_0 = \{\alpha < x < \beta, \quad y > y_0 \}
\]

Assume:

\[
|F(x+iy)| \leq O(e^{\gamma y}) \quad \text{on} \quad E_0.
\]

(some giant \(G_0\))
We define \( \mu(x) \) for \( -1 < x < 1 \):

\[
\mu(x) = \inf \left\{ \omega : |F(x+i\omega)| = O(y^\omega) \right\}
\]

Here we allow \( \mu(x) = \pm \infty \) in an obvious sense.

Tautologically, for each \( x \),

\[
\mu(x) = \lim_{y \to \infty} \frac{\ln |F(x+i\omega)|}{\ln y}
\]

---

**Note:**

\( \alpha = -1 \), \( \beta = 1 \), \( y_0 = 1 \)

\[
F(z) = e^{-iz^2}
\]

\[
|F(z)| = e^{2xy}
\]

\[
|F(x+i\omega)| \leq e^{2y} \text{ on } E_0
\]

\[
\mu(x) = \begin{cases} 
-\infty & -1 < x < 0 \\
0 & x = 0 \\
+\infty & 0 < x < 1
\end{cases}
\]
Fact

Suppose that \( \mu(x) < +\infty \) for all \( x \in (a, b) \).
If \( \mu(x_0) = -\infty \) for some \( x_0 \in (a, b) \), we must then have \( \mu(x) \equiv -\infty \) on \( (a, b) \).

PF

Simply apply p. \( \Theta \) THM with appropriate \( a, b, A, B \) and let one of \( A \) or \( B \) tend incrementally to \(-\infty \). 

Thm (convexity of \( \mu \))

Given \( F \) on \( E_0 \) as above.
Assume that \( -\infty < \mu(x) < +\infty \) for each \( x \in (a, b) \). The fn \( \mu(x) \) is then convex on \( (a, b) \); i.e.,
\[
\mu\left(\left(1-t\right)x_1 + tx_2\right) \leq \left(1-t\right)\mu(x_1) + t\mu(x_2)
\]
for \( t \in [0,1] \) and \( x_1 < x_2 \) in \( (a, b) \).

PF

Easy consequence of p. \( \Theta \) THM.
It is a standard thm of basic analysis that every \( (\text{finite}) \) convex fn \( \phi(x) \) on \((a, b)\) is automatically continuous.

Let \( F(x) = \int \phi(z) \, dz \).

The Euler–Maclaurin development (in the style of Euler) given in Lec 7 p. 19, immediately shows that \( \mu(x) < +\infty \) for every \( x \in \mathbb{R} \).

(Cf. Lec 5 pp. 10 (line 5) + 12 (thm)
for \( x > 0 \).

Recall \( \log \Gamma(z) \) in Lec 6, pp. 3 + 4, in connection with

\[
\Gamma(z) = \prod_{p} \frac{1}{1 - p^{-z}}, \quad \text{Re}(z) > 1.
\]

Clearly:

\[ \log \Gamma(z) = \mathcal{O}_\varepsilon(1) \quad \text{for} \quad x > 1 + \varepsilon. \]

Hence:

\[-A_\varepsilon \leq \ln |\phi(x + iy)| \lesssim A_\varepsilon \quad \text{here}.\]
Fact

Given \( f(x) \). We have \(-\infty < \mu(x) < +\infty\) for all \( x \in \mathbb{R} \). In fact \( \mu(x) = 0 \) for all \( x > 1 \).

\( \text{Pf} \)

Obvious by p. 13 and the Fact on 12.

Now exploit \( \Xi(s) = \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s) \) and \( \Xi(s) = \Xi(1-s) \) à la Lec 11 p. 14.

Recall:

\[
|\Gamma(s+it)| = \sqrt{2\pi} |t|^{\frac{s}{2}} e^{-\frac{\pi}{2} |t|} (1 + O\left(\frac{1}{|t|}\right))
\]

for any \( \sigma_1 < \sigma < \sigma_2 \) and \( |t| \to 1 \). See Lec 23 p. 4 Fact 3 also Lec 10 p. 12 for Stirling.
Get:

\[ \pi^{-\frac{5}{2}} \left| \frac{\Gamma\left(\frac{\sigma + it}{2}\right)}{\Gamma\left(\frac{\sigma - it}{2}\right)} \right| \]

\[ = \pi^{-\frac{1-\sigma}{2}} \left| \frac{\Gamma\left(\frac{1-\sigma - it}{2}\right)}{\Gamma\left(\frac{1-\sigma + it}{2}\right)} \right| \]

\[ \downarrow \]

\[ \{ \text{by (14) line } -4 \} \]

\[ \pi^{-\frac{\sigma}{2}} \sqrt{2\pi} \left( \frac{\pi}{2} \right)^{\frac{\sigma}{2} - \frac{1}{2}} e^{-\frac{\pi}{4} t^2} \left| \mathcal{L}(\sigma + it) \right| \]

\[ \sim \pi^{-\frac{1-\sigma}{2}} \sqrt{\frac{\pi}{2}} \left( \frac{\pi}{2} \right)^{\frac{1-\sigma}{2} - \frac{1}{2}} e^{-\frac{\pi}{4} t^2} \left| \mathcal{L}(1-\sigma - it) \right| \]

\[ \{ \text{compare Lec 23 p. 5} \} \]

\[ \left| \mathcal{L}(\sigma + it) \right| \sim C(\sigma) t^{\frac{1-\sigma}{2}} \left| \mathcal{L}(1-\sigma - it) \right| \]

\[ \text{as } t \to +\infty. \]

\[ \text{THM} \]

For \( F(s) = \mathcal{L}(s) \), we have

\[ \mu(\sigma) = \mu(1-\sigma) + \frac{1}{2} \sim \sigma \]

\[ \text{Pf} \]

As above.
By \((13)\) (top), \((14)\) (top), \((15)\) THM, we get:

\[
\mu(\sigma) = \begin{cases} 
0, & \sigma \geq 1 \\
\frac{1}{\sigma - \frac{1}{2}}, & \sigma < 0 \\
\frac{1}{\sigma - \frac{1}{2}}, & \sigma \approx 0
\end{cases}
\]

Application of p. \((12)\) THM then gives:

\[
\mu(\sigma) \approx \frac{1}{2} - \frac{\sigma}{2} \quad \text{for} \quad 0 < \sigma < 1.
\]

The exact value of \(\mu(\sigma)\) at any given \(\sigma \in (0, 1)\) remains a mystery.

Lindelöf has conjectured that

\[
\mu(\sigma) = \begin{cases} 
\frac{1}{\sigma - \frac{1}{2}}, & 0 < \sigma < \frac{1}{2} \\
0, & \frac{1}{2} \approx \sigma - 1
\end{cases}
\]

It is known that the Riemann Hypothesis (Lec 16, p. \((17)\) \(\Theta = \frac{1}{2}\)) implies Lindelöf's
Fact

Lindelof's conjecture is equivalent to proving that $\mu(\frac{1}{2}) = 0$.

Pf

Clearly $\text{Lindelof} \Rightarrow \mu(\frac{1}{2}) = 0$.

Now suppose $\mu(\frac{1}{2}) = 0$. By convexity, $\mu(x) = 0$ when $0 < x < 1$, we get $\mu \leq 0$ on $[\frac{1}{2}, 1]$.

If we had $\mu(x_0) < 0$ for some $x_0 \in (\frac{1}{2}, 1]$, application of (2) again would give $\mu(x_0) < 0$, $\mu(\frac{1}{2}) = 0$ $\Rightarrow$ $\mu(\frac{3}{2}) < 0$.

Contrad. Hence $\mu = 0$ on $[\frac{1}{2}, 1]$.

By p. 15 THM, get $\mu = \frac{1}{2} - \sigma$ on $[0, \frac{1}{2}]$.

Hence all is OK. \qed
The best that is currently known is that $\mu(\frac{1}{2})$ is at most a specific fraction somewhat less than $\frac{1}{6}$.

It has sometimes been claimed that $\mu(\frac{1}{2}) \approx \frac{1}{8}$, but this has never panned out [i.e., proven to be correct]. The conventional wisdom is that achieving even this would be a "major advance."
Now we turn to Littlewood's formula.

Let \((\sigma, \beta) \times (T_1, T_2)\) be a given rectangle. We'll call it \(R\). Let \(f(s)\) be analytic on \(RUER\). Let \(f(\beta+it) \neq 0\). Also let

\[f(\sigma+iT_1) \neq 0, \quad f(\sigma+iT_2) \neq 0.\]

We are completely happy if \(f\) vanishes at some points of \(\sigma = \varphi \alpha\) \(\{t \neq T_1, T_2\}\).

Begin by defining a single-valued branch of \(\log f(s)\) on a narrow open set containing \(\sigma = \beta, T_1 \leq t \leq T_2\). For \(t\)-values not matching the ordinate of a zero of \(f(s)\) on \(RUER\), define \(\phi(s) = \log f(s)\) by horizontal analytic continuation starting with \(\log f(s)\). Compare Lec 15 p. 23.

Once that is done, then use continuity from above \(\text{write}\) to take care of the ordinates of \(f\)-zeros. \(\{\text{Note that this makes good sense even for } \sigma = \varphi \alpha \}\).
THM (Littlewood)

Given $R$, $f$ as above. Let

$$N(u^*, T_1, T_2) = \# \text{ of zeros of } f(s) \text{ on } RU \in \mathbb{R}$$

having abscissa $> u$ (and counted WITH multiplicity).

We then have:

$$-\frac{1}{2\pi i} \oint_{\partial \mathbb{R}} \phi(s) ds = \sum_{j=1}^{N} [\text{Re}(\rho_j^*) - \alpha]$$

$$= \int_{\alpha}^{\beta} N(s^*, T_1, T_2) ds$$

using an obvious $\rho_j^*$ notation for the zeros of $f$.

Proof

Make the connected open set $R'$ by drawing

in an obvious way. The $x$'s corr to $\rho_j^*$.

* Note that $N(u^*, T_1, T_2)$ is right continuous.
Write
\[ \Phi(s) = \Phi_0(s)(s-\rho_1) \cdots (s-\rho_N) \]
\[ \{ \Phi_0(s) \text{ analytic and nonzero} \} \text{ on } \Re \sigma \subset \Re \}

The branch \( \text{Log } \Phi_0(s) \) is uniquely defined on \( \Re \sigma \subset \Re \) once it is "started" on \( \sigma = \beta \).

Let us agree that the standard principal value \( \text{Log } z \) has \( -\pi < \text{Arg}(z) \leq \pi \). Then:
\[ \text{Log } (-q) = \lim_{\varepsilon \to 0^+} \text{Log } (-q+i\varepsilon) \]
for every \( q > 0 \).

There is no loss of generality in presupposing that
\[ \Phi(s) \]
\[ \text{Log } \Phi(s) = \text{Log } \Phi_0(s) + \sum_{j=1}^{N} \text{Log } (s-\rho_j) \]
first along \( \sigma = \beta \), then throughout \( \Re \).

Naturally, along \( \Re \), one must be more careful [utilizing, e.g., the continuity from above idea].

\[ \text{Also in } \text{Log } z \]
Take just one zero $p_j$ and drop the $j^\circ$.

For simplicity, take $\alpha < \text{Re}(\beta) < \beta$. The case $\text{Re}(\beta) = \alpha$ is an easy adaptation.

\[
\int_{C_1} + \int_{\gamma_1} + \int_{\Gamma_a} + \int_{\gamma_2} + \int_{\log(s - \beta)} ds = 0
\]

(by CIT)

\[
\int_{C_2} + \int_{\gamma_3} + \int_{\Gamma_b} + \int_{\gamma_4} + \int_{\log(s - \beta)} ds = 0
\]

\[
\left| \int_{\Gamma_a} \log(s - \beta) ds \right| \leq \int_{\Gamma_a} \left[ \ln \frac{1}{\varepsilon} + 2\pi \right] ds
\]

\[
= O(\varepsilon \ln \frac{1}{\varepsilon}) \rightarrow 0
\]

\[
\left| \int_{\Gamma_b} \log(s - \beta) ds \right| = O(\varepsilon \ln \frac{1}{\varepsilon}) \rightarrow 0
\]

Similarly.
Obviously \\
\[
\int_{\gamma_2} + \int_{\gamma_3} = 0 \quad (\text{Arg}(s-\rho) = 0).
\]

But \\
\[
\int_{\gamma_1} \log(s-\rho) \, ds = \int_{\gamma_1} \frac{\text{Re}(\rho)-\varepsilon}{\ln |s-\rho| + i\pi} \, ds
\]
\[
\int_{\gamma_3} \log(s-\rho) \, ds = -\int_{\gamma_3} \frac{\text{Re}(\rho)-\varepsilon}{\ln |s-\rho| - i\pi} \, ds
\]

\[\Rightarrow \int_{\gamma_1} + \int_{\gamma_3} = 2\pi i \left[ \text{Re}(\rho) - \varepsilon \right] + O(\varepsilon).
\]

Hence, collectively we get \\
\[
\int_{\gamma_1} + \int_{\gamma_3} + 2\pi i \left[ \text{Re}(\rho) - \varepsilon \right] = o(1)
\]
\[\Rightarrow \int_{\gamma_2} \log(s-\rho) \, ds = -2\pi i \left[ \text{Re}(\rho) - \varepsilon \right].
\]

This will hold for each \(\rho_j\).
Of course, by CIT,

\[ \oint \log f_0(s) \, ds = 0. \]

Adding produces:

\[ \oint \log f(s) \, ds \]

\[ = -2\pi i \sum_{j=1}^{N} \left[ \text{Re}(\lambda_j^*) - \eta_j \right] \]

OR

\[ \frac{1}{2\pi i} \oint \phi(s) \, ds = \sum_{j=1}^{N} \left[ \text{Re}(\lambda_j^*) - \eta_j \right] \]

OK

If one writes

\[ N(\sigma_j, T_1, T_2) = \sum_{\text{each } \lambda_j^*} N_{\lambda_j^*}(\sigma_j, T_1, T_2) \]

in an obvious way, we clearly get
\[ \int_\alpha^\beta N(\omega; T_1, T_2) d\omega = \sum_j \left[ \text{Re}(\beta_j) - \gamma \right] . \]

Here, of course, one can suppress any \( \beta_j \) having \( \text{Re}(\beta_j) = \gamma \).

**Corollary (Littlewood)**

\[ 2\pi \int_\alpha^\beta N(\omega; T_1, T_2) d\omega = \int_{T_1}^{T_2} \text{Ln} f(t+it) dt - \int_{T_1}^{T_2} \text{Ln} f(\beta+it) dt \]
\[ - \int_\alpha^\beta \text{arg} f(\omega+it) d\omega + \int_\alpha^\beta \text{arg} f(\omega+i\tau) d\omega \]

wherein \( \text{arg} f \) comes from \( \text{Log} f(s) \) \( \ominus \).

**PF**

Use \( \ominus \) and take the appropriate real part. \( \ominus \)

\[ \text{Re} \left[ i \int_\alpha^\beta f(\omega) d\omega \right] \]
Addendum
(a remark about Lec 23)

I commented that the technique of Lec 23 actually gives \( \leq \) (const.) \( T^\omega \) zeros on the critical line for some small \( \omega \). I claim that \( \omega = 1/8 \) works.

More precisely, I claim that:

\[
\# \{\text{on-line\} zeros with } U < y \leq U^2 \}
\leq (\text{small constant}) \ U^{1/8}
\]

once \( U \) is large enough.

Let \( H \) be any number in \([T^{1/100},T]\). Keep \( T \) large. Note that Lec 23, Fact 1, holds equally well for

\[
\int_T^{T+H} f(x+it) \, dt
\]

Lec 23 Facts 2-7 require no change. On \( \Omega \) (bottom) of Lec 23, look at

\[
\int_T^{T+H} f'(\frac{x}{2}+it) \, dt \quad \text{vs.} \quad \left| \int_T^{T+H} f(\frac{x}{2}+it) \, dt \right|
\]

Analog of Fact 8 is

\[
\int_{\frac{x}{2}+it}^{\frac{x}{2}+(T+H)} J(s) \, ds = iH + O(T^{1/2})
\]

\[\text{note role of } n=1\]
See Lec 23 p. 9 middle. The analog of

Fact 9 is:

$$\int_T^{T+H} \left| f\left(\frac{x}{2}+it\right) \right| \, dt > \frac{1}{2} H \quad \text{once } T \text{ is large enough.}$$

On Lec 23 pp. 10 - 11, use \([\frac{1}{2}, \frac{5}{4}] \times [T, T+H]\).

On \(12\), get

$$\int_{\frac{5}{4} - iT}^{\frac{5}{4} + i(T+H)} O(T^{-7/8}) \, dt = O(HT^{-7/8})$$

$$= O(T^{7/8}) \quad \text{since } H \leq T.$$ On \(13\) line 3, get \(O(T^{-5/8})\).

Again. Hence, on \(14\) top,

$$\int_T^{T+H} f\left(\frac{x}{2} + it\right) \, dt = O(T^{7/8}).$$

On \(14\) (bottom), we get

$$\int_T^{T+H} \left| f\left(\frac{x}{2} + it\right) \right| \, dt \leq c_T^{-1/4} \int_T^{T+H} \left| f\left(\frac{x}{2} + it\right) \right| \, dt$$

$$\leq T^{-1/4} (H/2) \quad \text{in } c_2 HT^{-1/4}.$$
Observe however that
\[ T^{5/8} \leq c_2 H^{-1/4} \]
any time
\[ H \geq \frac{1}{c_2} T^{7/8} \cdot \]

This suggests keeping
\[ H \geq G T^{7/8} \]
for some \( \text{giant constant } G \). Doing so clearly produces
\[ \left| \int_{T}^{T+H} f(\frac{1}{2} + it) dt \right| < \frac{1}{2} \int_{T}^{T+H} |f(\frac{1}{2} + it)| dt \]

once \( T \) is large enough.

Hence, under (\( \ast \)), we find at least one true change of sign for \( f(\frac{1}{2} + it) \) in
\( (T, T+H) \). See Lec 23 \( \text{(lines 3-5)} \).

\( \text{(now)} \)

All this being said, let \( U \) be large and take:
\[ H = G(2U)^{7/8} . \]
Let 
\[ U_n = U + nH, \quad 0 \leq n \leq \left\lfloor \frac{y}{H} \right\rfloor. \]

Look at the disjoint intervals
\[ (U_{n-1}, U_n], \quad (n \geq 1). \]

We clearly get at least \( \left\lfloor \frac{y}{H} \right\rfloor \) true changes of sign of \( f(\frac{1}{2} + it) \) [hence, distinct zeros] on \( (U, 2U] \). This number clearly exceeds
\[ \left\lfloor \frac{y}{H} \right\rfloor \]
(small constant) \( \frac{y}{8} \).

A review of this proof shows that a similar estimate holds for a wider class of Dirichlet series
\[ \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad (a_1 \neq 0) \]

having functional equation similar to that of \( \zeta(s) \). The total number of zeros will still be \( \sim (\text{constant}) \ln T \). And the existence of an Euler product will NOT be required.
Going back to $J(s)$, I also noted that with a much harder proof, A. Selberg proved

\[ N_{\text{crit}}(T) > (\text{tiny constant}) T \ln T \]

(1942)

In the early 1970s, N. Levinson used a different [but related] approach to get

\[ > \frac{1}{3} \left( \frac{T}{2\pi} \ln \frac{T}{2\pi e} \right) \]

Conrey pushed this to

\[ > 40 \% \left( \frac{T}{2\pi} \ln \frac{T}{2\pi e} \right) \]

\[ \star \] Hardy and Littlewood reached $> eT$ in 1921.
Lecture 25 Synopsis
(Wed, 20 Apr)

The lecture covered a variety of topics.

First, regarding Lindelöf's μ-function for \( I(s) \). Cf.

Lec 24 p. 15 ff.

Thus

Consider \( f(s) \approx I(s) \) for \( \text{Im}(s) \geq 1 \), say.

(a) \( \mu(\sigma) + (\sigma - \frac{1}{2}) = \mu(1 - \sigma) \)
(b) \( \mu(\sigma) = 0 \) for \( \sigma > 1 \)
(c) \( \mu(\sigma) = 0 \) for \( \sigma < 0 \)
(d) \( \mu(\sigma) \) is convex on every \( [a, b] \)
(e) \( \mu(\sigma) \) is continuous on \( \mathbb{R} \)
(f) \( \mu(\sigma) \geq 0 \)
(g) \( \mu(\sigma) \) is monotonic decreasing
(h) \( \mu \left( \frac{1}{3} \right) \leq \frac{1}{4} \)
(i) Lindelöf's conjecture is true \( \iff \mu \left( \frac{1}{3} \right) = 0 \).

PF

(a) Lec 24 15.
(b) Lec 24 12 bot + 14.
(c) Combine (a) + (b). See Lec 24 16.
(d) Lec 24 12.
(e) \( \text{Lec 24 (13) top} \).

(f) Know \( \mu(\sigma) = 0 \) if \( \sigma > 1 \). Hence \( \mu(\sigma) = 0 \) if \( \sigma \geq 1 \).

Suppose \( \mu(\sigma_1) \leq 0 \) with some \( \sigma_1 < 1 \). Take \( \sigma_2 = \frac{3}{2} \) and apply convexity over \([0_1, 0_2 \leq 1] \).

Set \( \mu(\frac{3}{2}) < 0 \). Contradition!

(g) Know \( \mu(\sigma) \geq 0 \). And \( \mu(\sigma) = 0 \) if \( \sigma \geq 1 \).

Suppose \( \sigma_1 < \sigma_2 \) has \( 0 \leq \mu(\sigma_1) < \mu(\sigma_2) \).

So, \( \sigma_2 < 1 \). Look at convexity over \([0_1, 2 \leq 1] \).

This violates convexity (at \( \sigma_2 \)).

(h) \( \text{Lec 24 p.16 line 4, put } \sigma = \frac{1}{2} \).

(i) \( \text{Lec 24 p.17} \).

Recall \textbf{Lindelöf's Conjecture}:

\[
\mu(\sigma) = \begin{cases}
0 & \frac{1}{2} \leq \sigma < \infty \\
\frac{1}{2} - \sigma & -\infty < \sigma \leq \frac{1}{2}
\end{cases}
\]

It is known that \( \text{RH} \Rightarrow \text{Lindelöf Conjecture} \).
I briefly discussed the following theorem.

**Theorem**

Let \( f(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \) be a given generalized Dirichlet series with \( 1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty \).

Suppose the series converges at \( s_0 \in \mathbb{C} \).

Then:

(a) the series converges uniformly on every 
   Stolz angle
   \[ \{ \arg(s-s_0) \leq \frac{\pi}{2} - \delta \} \]

(b) the series converges uniformly on every 
   "super" Stolz angle
   \[ \{ |t-t_0| \leq e^{M(t-t_0)} - 1 \} \]
   \( (M > 0) \).

The proof (omitted here) is an interesting exercise. Of course, (a) is known already by Lec 21, p. 11 Fact 2. Concerning (b),
I simply remark: just study Lec 21, p. 13, line 7 when (wlog) $s_0 = 0$. For $0 > A$ big \[ \text{[but frozen]} \] notice that:

\[
\sigma < e^{M\sigma} \quad (M \geq 1 \text{ wlog})
\]

\[
|t| < e^{M\sigma} - 1 \leq e^{M\sigma}
\]

\[
|s| < 2 e^{M\sigma} \quad \text{a priori}
\]

\[
\downarrow
\]

get a $\frac{E \cdot 2}{A} e^{-\sigma (\ln N - M)}$ term!

Needless to say, by a minor expungement and insertion (of a new "$\lambda$"), we can actually allow ANY $\lambda_1$ in the above. Thus, we do not need $\lambda_1 = 1$ only $\lambda_1 > 0$.

Because of (3) Thus, Stolz angles or "super" Stolz angles are natural vehicles on which to discuss, e.g., identity theorems of the sort $f_1(\xi_k) = f_2(\xi_k)$, all $k \geq 1 \Rightarrow a_{m_1} = a_{n_2}$. 
3rd topic

We did a quick review of basic Fourier transforms and related analysis.

\[ \hat{f}(p) \equiv \int_{-\infty}^{\infty} f(x) e^{-2\pi i px} \, dx \quad p \in \mathbb{R} \]

\[ \int_{-\infty}^{\infty} |f(x)| \, dx < \infty , \text{ if piecewise } C^1 \text{ basically} \]

\[ \frac{f(x+0) + f(x-0)}{2} = \int_{-\infty}^{\infty} \hat{f}(p) e^{2\pi i px} \, dp \]

RHS \equiv \lim_{R \to \infty} \int_{-R}^{R} \hat{f}(p) e^{2\pi i px} \, dp

\[ \hat{f}(u) \equiv \int_{-\infty}^{\infty} f(x) e^{-iux} \, dx \]

\[ \frac{f(x+0) + f(x-0)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(u) e^{iux} \, du \]
For "nice" functions \( f, g \) (real or complex) on \( \mathbb{R} \), we define the convolution

\[
H(x) = \int_{-\infty}^{\infty} f(t) g(x-t) \, dt
\]

\( H(x) \) is a reasonable function, due to

\[
|H(x)| \leq \int_{-\infty}^{\infty} |f(t)| |g(x-t)| \, dt
\]

\[
\int_{-\infty}^{\infty} |H(x)| \, dx \leq \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |f(t)| |g(x-t)| \, dt \right) \, dx
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t)| |g(x-t)| \, dx \, dt
\]

\( \leq \int_{-\infty}^{\infty} |f(t)| \left( \int_{-\infty}^{\infty} |g(x)| \, dx \right) \, dt
\]

\[
\leq \int_{-\infty}^{\infty} |f(t)| \left( \int_{-\infty}^{\infty} |g(x)| \, dx \right) \, dt
\]

Often, \( f \) and \( g \) are initially kept in the Schwartz class \( \mathcal{S} \).
One easily checks that $H(x)$ is continuous and bounded if either $f$ or $g$ is known to be bounded. This is true in all "sensible" cases.

What is also checked quite easily by Fubini and the key fact that

$$e^{i\theta} e^{i\phi} = e^{i(\theta + \phi)} \quad \theta, \phi \in \mathbb{R}$$

is the relation from Fourier transform theory

$$\hat{H}(p) = \hat{f}(p) \hat{g}(p)$$

(also $\hat{H}(u) = \hat{f}(u) \hat{g}(u)$).

This is why the convolution $H = f \ast g$ is so useful!

---

Another useful property goes as follows. Assume $f, g, \hat{f}, \hat{g}$ are all "nice". Then observe that:
\[ \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx \]

\[ = \int_{-\infty}^{\infty} f(x) \left[ \int_{-\infty}^{\infty} \overline{\hat{g}(p)} e^{2\pi i px} \, dp \right] \, dx \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \overline{\hat{g}(p)} e^{-2\pi i px} \, dx \, dp \]

(by Fubini)

\[ = \int_{-\infty}^{\infty} \overline{\hat{g}(p)} \left[ \int_{-\infty}^{\infty} f(x) e^{-2\pi i px} \, dx \right] \, dp \]

\[ = \int_{-\infty}^{\infty} f_{1}(p) \overline{f_{2}(p)} \, dp \]

Thus,

\[ \int_{-\infty}^{\infty} (f_{1}(x) - f_{2}(x))^2 \, dx = \int_{-\infty}^{\infty} \left| \hat{f}_{1}(p) - \hat{f}_{2}(p) \right|^2 \, dp \]

for nice \( f_{j} \). In particular:

\[ \int_{-\infty}^{\infty} |f_{1}(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}_{1}(p)|^2 \, dp \]

This is the Plancherel formula.
Let $\chi_{E}(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$.

One easily checks:

\[
\chi_{[-c, c]}(x) = 2 \frac{\sin cu}{u} \quad \text{and} \quad \max(0, b - |x|) = 2 \frac{1 - \cos bu}{u^2} \approx \frac{4 \sin^2 \left( \frac{b}{2} u \right)}{u^2}.
\]

If $u = 2\pi p^2$,

\[
\chi_{[-c, c]}(x) = \frac{\sin 2\pi pc}{\pi p} \quad \text{and} \quad \max(0, b - |x|) = \frac{1 - \cos 2\pi pb}{2 \pi^2 p^2} = \frac{\sin^2 (\pi pb)}{\pi^2 p^2}.
\]

It will be convenient to consider the convolution

\[
T(x) \equiv \frac{1}{2 \eta} \chi_{[-\eta, \eta]}(x) \ast \chi_{[-A-\eta, A+\eta]}(x).
\]
Theorem

If $A > 0$, $\eta > 0$, $T(x)$ as above. Then:

1. $T(x)$ is the trapezoid

   \[
   \begin{array}{c}
   -A-2\eta \quad -A \\
   \downarrow \quad \quad \quad 1 \\
   A \quad A+2\eta
   \end{array}
   \]

2. \[
   \sim T(x) = \frac{\cos(Au) - \cos((A+2\eta)u)}{\eta u^2}
   \]

3. \[
   \sim T(x) = 2 \frac{\sin(\eta u)}{\eta u} \frac{\sin((A+\eta)u)}{u}
   \]

Proof

By (7) + (9)

\[
\sim T(x) = \frac{1}{2\eta} \quad 2 \frac{\sin\eta u}{u} \quad 2 \frac{\sin(A+\eta)u}{u}
\]
\[
\cos(\theta - \phi) - \cos(\theta + \phi) = 2 \sin \theta \sin \phi
\]

\[
\downarrow
\]

\[
\cos(Au) - \cos((A+2\eta)u) = 2 \sin((A+\eta)u) \sin(\eta u)
\]

\[
= 2 \sin(\eta u) \sin((A+\eta)u),
\]

so (3) is OK. Of course,

so (2) is OK too.

For (1), define \( f(x) \) to be the trapezoid on \( (0, A-1) \):

\[
2\eta f(x) =
\]

\[
\max(0, A-1x) =
\]

(shown)
\[ 2\eta \phi_b(x) = \max(0, A-x) = \max(0, A+2\eta - 1) \]

\[ \Rightarrow \phi_b(x) = \frac{\cos(Au) - \cos((A+2\eta)u)}{u^2} \]
By (2) on p. 10,

\[ q(x) = T(x) \quad (\text{all } u \in \mathbb{R}) \]

Apply 5 last line to this situation. Get

\[ q(x) = T(x), \quad \text{each } x \in \mathbb{R} \]

since \( q \) and \( T \) are continuous on \( \mathbb{R} \). (6) also the \( f^* \) counterpart of 8 line - 4. In any case, (1) is now true.
THM (an important estimate for Dirichlet polynomials). 

We have

\[
\int_{x}^{x+H} \left| \sum_{j=1}^{N} b_j e^{-i\lambda_j t} \right|^2 dt = H \sum_{j=1}^{N} |b_j|^2 + \frac{O(1)}{\delta} \sum_{j=1}^{N} |b_j^0|^2
\]

anytime

\[
0 < \lambda_1 < \lambda_2 < \ldots < \lambda_N
\]

\[
|\lambda_k - \lambda_j| \geq \frac{\delta}{N}, \quad \text{all } k \neq j
\]

\[
b_j^0 \in \mathbb{C}, \quad \varphi \in \mathbb{R}, \quad H > 0
\]

The "implied" constant in \( O(1) \) is absolute; it can be taken to be \( \frac{4\pi}{\sqrt{3}} \).
Pf

Some easy wlog's (giving same implied constant in $O(1)$).

First one: $\eta = -\frac{H}{2}$.
Second one: $H = 2$.

Must prove:

$$\int_{-1}^{1} \left| \sum_{j=1}^{N} b_j e^{-i\lambda_j t} \right|^2 dt = 2 \sum_{j \neq 1}^{N} |b_j|^2 + O(1) \sum_{j=1}^{N} |b_j|^2.$$

Take $T(t)$ on $\mathbb{C}$ with $A=1$, $\eta = \eta_0$. Let

$$\mathcal{F} = \int_{-1}^{1} \left| \sum_{j=1}^{N} b_j e^{-i\lambda_j t} \right|^2 dt.$$

Clearly

$$\mathcal{F} \leq \int_{\mathbb{R}} \left| \sum_{j=1}^{N} b_j e^{-i\lambda_j t} \right|^2 T(t) dt$$

$$\mathcal{F} \leq \sum_{j, k} |b_j^* b_k| \int_{\mathbb{R}} T(t) e^{-i(\lambda_j - \lambda_k) t} dt.$$

{put $d_{jk} = \lambda_j - \lambda_k$}
\[ F = \sum_{j, k} b_j^* b_k \left\{ \frac{\sin(\eta d_{jk}^* \gamma)}{d_{jk}^* \gamma} \cdot \frac{\sin[(1 + \eta)d_{jk}^* \gamma]}{d_{jk}^* \gamma} \right\} \]

by (10)(3)

\[ F = \sum_{j=1}^{N} |b_j|^2 \left\{ \frac{2}{\eta} \eta (1 + \eta) \right\} \]

\[ + \frac{2}{\eta} \sum_{j \neq k} b_j^* b_k \frac{\sin(\eta d_{jk}^* \gamma)}{d_{jk}^* \gamma} \frac{\sin[(1 + \eta)d_{jk}^* \gamma]}{d_{jk}^* \gamma} \]

\[ = (2 + 2\eta) \sum_{j=1}^{N} |b_j|^2 \]

\[ + \frac{2}{\eta} \sum_{j \neq k} \Re(b_j^* b_k) \]

\[ \left\{ \begin{array}{l}
\text{but } 2 \Re(b_j^* b_k) \leq |b_j|^2 + |b_k|^2 \end{array} \right\} \]

\[ = (2 + 2\eta) \sum_{j=1}^{N} |b_j|^2 + \frac{1}{\eta} \sum_{j \neq k} \frac{|b_j|^2 + |b_k|^2}{d_{jk}^* \gamma} \]

\[ = (2 + 2\eta) \sum_{j=1}^{N} |b_j|^2 + \frac{2}{\eta} \sum_{j \neq k} \frac{|b_j|^2}{d_{jk}^* \gamma} \]
\[ \begin{align*}
\text{(17)} & \quad \sum_{j=1}^{N} |b_j|^2 \\
\quad & \quad + \frac{2}{\eta} \sum_{j=1}^{N} |b_j|^2 \left( 2 \sum_{m=1}^{\infty} \frac{1}{m^2 \delta^2} \right) \\
\quad & \quad \text{\{ somewhat crudely \}} \\
\quad & \quad \text{\{ via } |\lambda_j|^2 \cdot \eta_0 | \geq \delta > 0 \text{ \} } \\
\quad & \quad \text{\{ for } j \neq k \text{ \} } \\
\quad & \quad = (2 + 2\eta) \sum_{j=1}^{N} |b_j|^2 + \frac{4\eta}{\eta\delta^2} \sum_{j=1}^{N} |b_j|^2 \frac{\pi^2}{6} \\
\quad & \quad \text{\{ let } D = \frac{\pi}{\sqrt{3}} \text{ \} } \\
\quad & \quad = (2 + 2\eta) \sum_{j=1}^{N} |b_j|^2 + \frac{2D^2}{\eta\delta^2} \sum_{j=1}^{N} |b_j|^2 \\
\quad & \quad = (2 + 2\eta + \frac{2D^2}{\eta\delta^2}) \sum_{j=1}^{N} |b_j|^2. \\
\text{To minimize RHS, take} \\
\eta = \frac{D}{\delta}.
\end{align*} \]
Get:
\[ f \leq (2 + \frac{4D}{5}) \sum |b_j|^2 \]
or
\[ \int_{-1}^{1} \left| \sum_{j=1}^{N} b_j e^{-i\omega_j t} \right|^2 dt \leq 2 \sum_{j=1}^{N} |b_j|^2 + \frac{4\pi/\sqrt{3}}{5} \sum_{j=1}^{N} |b_j|^2. \]

This, of course, is the upper bound positioned on page 14.

The remainder of the proof was done in Lec 26, but we include it here!

The lower bound for \( f \) is similar but slightly harder. Must prove:
\[ f \geq (2 - \frac{4D}{5}) \sum_{j=1}^{N} |b_j|^2. \]
If $S \leq 2D = \frac{2\pi}{\sqrt{3}}$, matters are trivial.

So, wlog $S > 2D$. Hence, $\frac{1}{2} > \frac{D}{S}$.

We consider $T(t)$ on $[0, \infty)$ with $A = 1 - 2\eta$, $0 < \eta < \frac{1}{2}$, and observe that $A + 2\eta = 1$. Here

$$F \geq \int_{-\infty}^{\infty} T(t) \left| \sum_{i} b_{j} e^{-i\eta_{j}t} \right|^{2} dt$$

$$= \sum_{j,k} b_{j}^{*} b_{k} \int_{-\infty}^{\infty} T(t) e^{-i(\eta_{j}-\eta_{k})t} dt$$

$$= \sum_{j,k} b_{j}^{*} b_{k} 2 \frac{\sin(\eta d_{jk})}{\eta d_{jk}} \frac{\sin((1-\eta)d_{jk})}{d_{jk}}$$

$$= \sum_{j=i}^{N} |b_{j}|^{2} (2-2\eta)$$

$$+ \sum_{j \neq k} \text{Re}(b_{j}^{*} b_{k}) 2 \frac{\sin \eta d_{jk}}{\eta d_{jk}} \frac{\sin((1-\eta)d_{jk})}{d_{jk}}$$

$\xi$ as on (16)$$

= (2-2\eta) \sum_{i} |b_{i}|^{2} - \sum_{j \neq k} \left( |b_{j}|^{2} + |b_{k}|^{2} \right) \frac{1}{\eta d_{jk}} \frac{1}{d_{jk}}$$

$$= (2-2\eta) \sum_{i} |b_{i}|^{2} - \frac{2}{\eta} \sum_{j} |b_{j}|^{2} \left( \sum_{k \neq j} \frac{1}{d_{jk}^{2}} \right)$$

$\xi$ as on (16) bottoms.
\[
\begin{align*}
\Rightarrow (2-2\eta) \sum_{j=1}^{N} |b_j|^2 - \frac{2}{\eta} \sum_{j=1}^{N} |b_j|^2 \frac{1}{\delta^4} 2 \left( \frac{\pi^2}{\delta} \right) \\
\text{cf. (17) middle } \{ D = \frac{\pi}{\sqrt{3}} \} \\
= \sum_{j=1}^{N} |b_j|^2 \left( 2 - 2\eta - \frac{2D^2}{\eta \delta^2} \right) \\
\end{align*}
\]

Take \( \eta = \frac{D}{\delta} \). Since \( \frac{1}{\delta^2} > \frac{D}{\delta} \), \( \eta \) is admissible. Get

\[
J \geq 2 \sum_{j=1}^{N} |b_j|^2 - \frac{4D}{\delta} \sum_{j=1}^{N} |b_j|^2, \]

with \( 4D = \frac{4\pi}{\sqrt{3}} \) parallel to (18) lines 4 - 5. This is the lower bound promised. \[\square\]

Let \( C \) be the constant in the \( O(1) \) on (14). The theorem on (14) is very closely related to the generalized Hilbert inequality:

\[
(\star) \quad \left| \sum_{j \neq k} \frac{z_j \overline{z}_k}{\lambda_j - \lambda_k} \right| \leq \frac{\varepsilon/2}{\delta} \sum_{j=1}^{N} |z_j|^2 \quad (z_j \in C). 
\]

One readily checks that \((\star) \Rightarrow \text{thm on (14)} \). Selberg has noted \([\text{in a very slick proof}]\) that the thm on (14) \( \Rightarrow (\star) \).
By choosing majorant/minorant functions more sophisticated than trapezoids, one finds that the best $C$ is $2\pi$.

Note:

\[ \frac{4\pi}{\sqrt{3}} = 2\pi \left( 1.1547^* \right) \]

which is not too bad!
Recall E-M version 2 from Lec 9, pp. 13 and 14.

Taking $R = 0$ led to

$$\widetilde{B}_1(x) = -2 \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{2\pi n} = x - \|x\| - \frac{1}{2}$$

for $x \notin \mathbb{Z}$. See Lec 9, pp. 13 bottom, 14 top.

In Lec 9, p. 19, we saw that

$$\tilde{f}(\varepsilon) = \frac{1}{2} + \frac{1}{\varepsilon - 1} - \varepsilon \int_1^{\infty} \frac{\tilde{B}_1(u)}{u^{\varepsilon + 1}} \, du \quad \text{if } x > 1.$$ 

In fact, the derivation on 18 bottom, with $N \to N - 1$ and $u = 1 + t$, produced:

$$\sum_{k=1}^{N} k^{-\varepsilon} = \frac{1}{2} + \frac{1}{\varepsilon - 1} + \frac{1}{2} N^{-\varepsilon} + \frac{N^{1-\varepsilon}}{1-\varepsilon} - \varepsilon \int_1^{N} \frac{\tilde{B}_1(u)}{u^{\varepsilon + 1}} \, du\,.$$ 

It is natural to subtract these formulae.
Get:
\[ I(\varepsilon) \sim \sum_{k=1}^{N} k^{-\varepsilon} \approx -\frac{1}{\varepsilon} N^{-\varepsilon} - \frac{N^{1-\varepsilon}}{1-\varepsilon} - \varepsilon \int_{N}^{\infty} \frac{\widetilde{B}_{1}(u)}{u^{\varepsilon+1}} \, du \]

where the final term is nicely analytic on \( \{ \text{Re}(\varepsilon) > 0 \} \) thanks to \( |\widetilde{B}_{1}(u)| \lesssim \frac{1}{u^{1/2}} \).

It follows that
\[ I(\varepsilon) = \sum_{k=1}^{N} k^{-\varepsilon} - \frac{N^{1-\varepsilon}}{1-\varepsilon} - \frac{1}{\varepsilon} N^{-\varepsilon} - \varepsilon \int_{N}^{\infty} \frac{\widetilde{B}_{1}(u)}{u^{\varepsilon+1}} \, du \]
on \( \{ \text{Re}(\varepsilon) > 0 \} \sim \{ 1 \} \). Compare Lec 6) (10) line 4.

In numerical work (evaluating \( I(\varepsilon) \)) one often uses the counterpart of (\#) associated with \( \widetilde{B}_{2R+1}(u) \) and the remainder term
\[ (-1)^{\varepsilon+1} \cdots (\varepsilon+2R) \int_{N}^{\infty} \frac{\widetilde{B}_{2R+1}(u)}{(2R+1)!} \frac{1}{u^{\varepsilon+2R+1}} \, du \]
(\#) Lec 9 pp. (18) + (19). One takes \( R \) and \( N \) appropriately large.
These comments hint that controlling the size of $I(s)$ in the critical strip $0 < \Re(s) < 1$ comes down to doing the same for certain sums:

$$\sum_{n=1}^{N} n^{-\sigma} e^{-\nu \ln n} = \sum_{n=1}^{N} n^{-\sigma} e^{-\nu \ln n}$$

The size of $N$ will depend at least loosely on the magnitude of $|\nu|$ (see, e.g., Chapter 17).

The issue of numerical calculation of $I(s)$ deserves a separate lecture. It will, however, play no role in the remaining lectures in this course.

In the present lecture, the goal is to simply obtain an important formula of Hardy and Littlewood growing out of \( \mathbb{R}(\mathbb{A}) \).
We need a preliminary.

**Theorem**

Let \( f(x) \) be real and \( C^1 \) on \([a,b]\). Let \( f'(x) \) be monotonic here. Assume, says that \( 0 \leq f'(x) \leq \delta < 1 \). Then:

\[
\sum_{a < n \leq b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)} \, dx + \frac{O(1)}{1-\delta},
\]

with an absolute "implied" constant.

**Proof**

By inflating \( O(1) \), wlog \( a \) and \( b \) are integers and \( b-a \geq 100 \). Indeed, we can also assume that \( a = 0 \).

By E-M version I (Lec 9 (14) \( R = 0 \)), know:

\[
\sum_{0 \leq n \leq b} e^{2\pi i f(n)} = O(1) + O(1) + \int_0^b e^{2\pi i f(x)} \, dx + \int_0^b (x - \|x\| - \frac{1}{2}) e^{2\pi i f(x)} \, dx.
\]
\[ \sum_{0 \leq n \leq b} e^{2\pi i f(n)} = O(1) + \int_0^b \sum_{0 \leq n \leq b} d\epsilon_n \sin(2\pi \epsilon_n x) \, dx \]

\[ + \pi \sum_{n=1}^b \left( \frac{\sin(2\pi nx)}{2\pi n} \right) f e^{2\pi i f(x)} \, dx \]

\[ \text{see Lec 9 p. 9} \]

\[ = O(1) + \int_0^b e^{2\pi i f(x)} \, dx \]

\[ - \frac{\pi}{2} \sum_{n=1}^b \frac{1}{n} \int_0^b \sin(2\pi nx) \, f e^{2\pi i f(x)} \, dx \]

\[ = O(1) + \int_0^b e^{2\pi i f(x)} \, dx \]

\[ + \sum_{n=1}^b \frac{1}{n} \int_0^b (e^{2\pi i nx} - e^{-2\pi i nx}) \, f e^{2\pi i f(x)} \, dx \]

This last sum

\[ = \sum_{1}^{\infty} \frac{1}{n} \left[ \int_0^b e^{2\pi i f(x) - nx} \, f e^{2\pi i f(x) + nx} \, dx - \int_0^b e^{2\pi i f(x) + nx} \, f e^{2\pi i f(x) - nx} \, dx \right] \]
\[ \sum_{i=1}^{\infty} \frac{1}{n} \left( \frac{1}{2\pi i} \int_{0}^{b} \frac{f(x)}{f'(x)} \, d\xi \right) e^{2\pi i \cdot [f(x) - nx]} 
- \frac{1}{2\pi i} \int_{0}^{b} \frac{f'(x)}{f(x) + n} \, d\xi e^{2\pi i \cdot [f(x) + nx]} \right). \]

We note herein that \( n \neq 0 \), \( f' \) is monotonic, and \( 0 \leq f'(x) \leq \delta < 1 \).

Recall 2nd mean value theorem (lec 20 5):

\[ \int_{A}^{B} g(x) \, dx = g(A) \int_{A}^{E} \, dx + g(B) \int_{E}^{B} \, dx \]

\[ \text{monotonic} \quad \text{a real and } C[ A, B ] \]

Complex are treated via \( \eta = \eta_r + i\eta_x \).

Also, notice that \( \psi \).
\[
\left( \frac{u}{u+v} \right)' = \frac{(u+v)^{-1} - u^{-1}}{(u+v)^2} = \frac{v}{(u+v)^2}
\]

for \( v \in \mathbb{Z} - \{0\} \).

This derivative has fixed sign for \( u \neq -v \).

Example: \( \frac{u}{u+1} \)

So, each \( \frac{f'(x)}{f'(x) \pm n} \) is monotonic on \([0,b]\).

Look at:

\[
\int_0^b \frac{f'(x)}{f'(x) + n} \, de^{2\pi i (f(x) + nx)} \quad (n \geq 1)
\]

\( e^{2\pi i (f(x) + nx)} \) treated as \( q_1 + i q_2 \)

and apply 2nd mean value thm à la @ bottom.
Get:

\[
\int_0^b \frac{f'(x)}{f(x)+n}\, dx = e^{2\pi i (f(x)+nx)}
\]

\[= O\left(\frac{1}{n}\right) \quad .\]

Hence, on top, we see that line 2 contributes

\[
\sum_{i=1}^\infty \frac{1}{n} O\left(\frac{1}{n}\right) = O(1) \quad .\]

We now look at

\[
\int_0^b \frac{f'(x)}{f(x)-n}\, dx = e^{2\pi i (f(x)-nx)} \quad (n \geq 1)
\]

\[
\text{treated as } g_1 + i\cdot g_2
\]

analogously. Since \(0 \leq f(x) \leq 5 < 1\),

\[n = 1 \quad \Rightarrow \quad O\left(\frac{1}{1-5}\right)\]

\[n \geq 2 \quad \Rightarrow \quad O\left(\frac{1}{n-1}\right)\]
we see that on \( O \) top, in line 1, the collective contribution is:

\[
O \left( \frac{1}{1-\delta} \right) + \sum_{n=2}^{\infty} \frac{1}{n} O \left( \frac{1}{n-1} \right)
\]

\[
= O \left( \frac{1}{1-\delta} \right) .
\]

Note how \( n=1 \) plays a special role in this portion of things.

Reviewing \( (5) \), we conclude that:

\[
\sum_{0 < n \leq b} e^{2\pi i f(n)} = O(1) + \int_0^b e^{2\pi i f(x)} \, dx
\]

\[
+ O(1) + O \left( \frac{1}{1-\delta} \right) .
\]

This proves \( p \circ \theta \) THM. \[\square\]
Remark

One can obviously do $-\delta \leq f'(x) \leq \delta$ in much the same way.

Still keeping $f'$ monotonic, by splitting the original sum into 2 chunks, if need be, one can handle $-\delta \leq f'(x) \leq \delta$ as well.

Additional Remark

More general forms of p. 4 THM certainly suggest themselves. (cf., e.g., lec 22 5+6 bottom half.)

Theorems of this sort (with summations) arose in work of van der Corput from 1921/22.

*Also, Titchmarsh, Theory of $J(x)$, near §4010.
THEOREM (Hardy-Littlewood, Math. Zeits. 10 (1921), 9) 

Given any \( \sigma_0 \in (0, \frac{1}{10}] \), say. Given any \( C > 10 \) 
Keep \( \sigma_0 \leq \sigma \leq \sigma_0 \), \( |t| \leq 100 \).

Then:
\[
J(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})
\]

whenever \( x > C \frac{|t|}{2\pi} \).

\[\text{PF} \overset{\text{the formal similarity to}}{\Rightarrow} \Box (\ast) .\]

Clearly, via a conjugation, \( \Box (\ast) \) \( \forall C \geq 100 \).

We immediately see by \( \Box (\ast) \) that:
\[
J(s) = \sum_{n=1}^{N} n^{-s} - \frac{N^{1-s}}{1-s} + O(N^{-\sigma})
\]

\[+ O(|s|) \int_{N}^{\infty} u^{-\sigma-1} du \]

\[= \sum_{n=1}^{N} n^{-s} - \frac{N^{1-s}}{1-s} + O(|s|N^{-\sigma}) \]

\[\text{since } \sigma_0 \leq \sigma \leq \sigma_0 \].
Think of $N$ as being giant and much greater than $x$. Let

$$A(v) = \sum_{x \leq n \leq v} n^{-it} = \sum_{x \leq n \leq v} e^{-2\pi i \left( \frac{x \ln n}{2\pi} \right)}.$$ 

Apply p. (4)+10 line 2.

$$f(q) = -\frac{t \ln q}{2\pi}$$ \hspace{1cm} (q \geq x)

$$f'(q) = -\frac{t}{2\pi q} \quad \text{monotonic}$$

$$0 \leq -f'(q) \leq \frac{t/2\pi}{x} < \frac{1}{x} < 1$$

So,

$$A(v) = \int_{x}^{v} e^{-2\pi i \left( \frac{t \ln q}{2\pi} \right)} dq + O(1)$$

$$= \int_{x}^{v} e^{-it \ln q} dq + O(1)$$

$$= \int_{x}^{v} q^{-it} dq + O(1)$$

$$= \int_{x}^{v} \frac{q^{-it}}{1-\frac{it}{x}} dq + O(1)$$

$$= \frac{v^{-it} - x^{-it}}{1-\frac{it}{x}} + O(1).$$
But now,

\[
\sum_{n < x \leq N} n^{-\sigma - \frac{it}{\sqrt{\sigma}}} = \int_{x}^{N} n^{-\sigma} \, dA(n)
\]

\[
= \int_{x}^{N} A(n)^{(-\sigma)} \frac{\sqrt{\sigma}}{\sqrt{\sigma+1}} \, d\nu
\]

\[
= A(N) \frac{1}{N^{\sigma}} + \sigma \int_{x}^{N} \frac{A(n)}{\sqrt{\sigma+1}} \, d\nu
\]

\[
= N^{-\sigma} \left[ -\sigma \frac{n^{-\sigma - \frac{it}{\sqrt{\sigma}}}}{1-\frac{i \nu}{\sqrt{\sigma}}} + O(1) \right]
\]

\[
+ \sigma \int_{x}^{N} N^{-\sigma - 1} \frac{n^{-\sigma - \frac{it}{\sqrt{\sigma}}}}{1-\frac{i \nu}{\sqrt{\sigma}}} + O(1) \, d\nu
\]

\[
= \frac{N^{-\sigma - \frac{it}{\sqrt{\sigma}}}}{1-\frac{i \nu}{\sqrt{\sigma}}} - \frac{N^{-\sigma - \frac{it}{\sqrt{\sigma}}}}{1-\frac{i \nu}{\sqrt{\sigma}}} + O(N^{-\sigma})
\]

\[
+ \sigma \int_{x}^{N} \frac{n^{-\sigma - \frac{it}{\sqrt{\sigma}}}}{1-\frac{i \nu}{\sqrt{\sigma}}} \, d\nu
\]

\[
- \sigma \int_{x}^{N} \frac{n^{-\sigma - \frac{it}{\sqrt{\sigma}}}}{1-\frac{i \nu}{\sqrt{\sigma}}} \, d\nu
\]

\[
+ O(1) \left( x^{-\sigma} - N^{-\sigma} \right)
\]

Remember that \(N > x\).
Get $\Sigma$:

$$\sum_{x<n \leq N} n^{-s} = \frac{N^{1-\sigma-i\delta}}{1-i\delta} - \frac{N \chi}{1-i\delta} + O(x^{-\sigma})$$

$$+ \frac{-\sigma}{1-i\delta} \int_{x}^{N} v^{-s} dv$$

$$- \frac{-\sigma}{1-i\delta} x^{1-i\delta} \int_{x}^{N} v^{-\sigma-1} dv$$

$$= \frac{N^{1-\sigma-i\delta}}{1-i\delta} - \frac{N \chi}{1-i\delta} + O(x^{-\sigma})$$

$$+ \frac{-\sigma}{1-i\delta} \left[ \frac{N^{1-s} - x^{1-s}}{1-s} \right]$$

$$- \frac{-\sigma}{1-i\delta} x^{1-i\delta} \left[ \frac{N^{-\sigma} - x^{-\sigma}}{-\sigma} \right]$$

$$= \frac{N^{1-\sigma-i\delta}}{1-i\delta} - \frac{N \chi}{1-i\delta} + O(x^{-\sigma})$$

$$+ \frac{\sigma}{1-i\delta} \frac{1}{1-s} N^{1-s} - \frac{\sigma}{1-i\delta} \frac{1}{1-s} x^{1-s}$$

$$+ \frac{x^{1-i\delta} - \sigma}{1-i\delta} N - \frac{x^{1-i\delta} \chi^{-\sigma}}{1-i\delta}$$

\[\text{Note the cancellation!}\]
\[
\frac{N^{1-s}}{1-1+ \frac{\sigma}{1-s}} \left[ 1 + \frac{\sigma}{1-s} \right] + O(x^{-\sigma})
\]

\[
- \frac{\sigma}{1-1+ \frac{\sigma}{1-s}} x^{1-s} - \frac{x^{1-\sigma}}{1-1+ \frac{\sigma}{1-s}} x
\]

\[
= \frac{N^{1-s}}{1-1+ \frac{\sigma}{1-s}} \left[ 1 + \frac{\sigma}{1-s} \right] + O(x^{-\sigma})
\]

\[
- x^{1-s} \left[ \frac{\sigma}{1-s} + 1 \right]
\]

\[
\{ 1 + \frac{\sigma}{1-s} = \frac{1-\sigma+\sigma}{1-s} = \frac{1-1+ \frac{\sigma}{1-s}}{1-s} \}
\]

\[
= \frac{N^{1-s}}{1-1+ \frac{\sigma}{1-s}} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})
\]

So, with our \( N > x \), we get:

\[
\sum_{x < n \leq N} n^{-s} = \frac{N^{1-s}}{1-1+ \frac{\sigma}{1-s}} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})
\]

\[
= \int_{x}^{N} \frac{1}{q^s} dq + O(x^{-\sigma})
\]

very natural term
Recall \( \iota(\sigma) = \sum_{n \leq x} \frac{1}{n^s} + \sum_{n \geq N} \frac{x^{1-s}}{n} \). Thus,

\[
\iota(\sigma) = \sum_{n \leq x} \frac{1}{n^s} + \frac{N^{1-s}}{1-s} + O\left(1/x \right),
\]

by (15).

\[
= \sum_{n \leq x} \frac{1}{n^s} + \frac{N^{1-s}}{1-s} - \frac{x^{1-s}}{1-s} + O\left(1/x \right),
\]

by (15).

\[
= \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O\left(1/x \right) + O\left(1/x \right).
\]

Now let \( N \to \infty \) (to eliminate it).

Get:

\[
\iota(\sigma) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O\left(1/x \right),
\]

exactly as promised. \( \square \)
Take $x > C \frac{t}{2 \pi}$ with, say, $C = \pi$ and $t$ big. Apply $p \circ \text{THM}$. Hence:

$$f(s+i\varepsilon) = \sum_{n \leq x} \frac{\sigma - i\varepsilon}{t - \sigma - i\varepsilon} + O(t^{-\sigma})$$

$$= \sum_{n \leq x} \frac{n^{-\sigma - i\varepsilon}}{t - \sigma - i\varepsilon} + O(t^{-\sigma})$$

\[ \downarrow \]

$$\left| f(s+i\varepsilon) \right| \leq \text{ln} t + O(1) + O(t^{-1}) \text{ crudely}.$$

$$\left| f\left(\frac{1}{2}+i\varepsilon\right) \right| \leq 2\sqrt{t} + O(1) + O(t^{-1/2}) \text{ crudely}.$$ 

Of course, by Lec 25 (1) (h) [or Lec 24 (16) line 4], we already know:

$$\left| f\left(\frac{1}{2}+i\varepsilon\right) \right| = O(t^{\frac{1}{4}+\varepsilon}), \text{ each } \varepsilon > 0.$$ 

This hints that $p \circ \text{THM}$ may be improved. It can be — but the argument is much harder. 

We only need $p \circ \text{THM}$.
Lecture 27 Synopsis
(Wed, 27 Apr)

We seek to develop the famous thm of Bohr–Landau about zeros of \( f(s) \) to the right of \( \text{Re}(s) = \frac{1}{2} \).

Recall, from Lec 24, (19 20), that we had

\[
N(u; T_1, T_2) = \# \text{ of zeros of } f(s) \text{ on } RV \mathbb{R}
\]

having abscissa > u (and counted with multiplicity)

wherein \( R = (\alpha, \beta) \times (T_1, T_2) \) ; \( f(s) \) is analytic on \( RV \mathbb{R} \) ; \( f(\beta + it) \neq 0 \), \( f(\sigma + iT_1) \neq 0 \), \( f(\sigma + iT_2) \neq 0 \).

One defines \( \phi(s) = \text{Log } f(s) \) via horizontal analytic continuation starting at \( \sigma = \beta \) insofar as \( t \neq \text{ ordinate of a zero of } F \) otherwise by an obvious right continuity from above.

In this framework, we get Littlewood's formula

\[
-\frac{1}{2\pi i} \oint_{\mathcal{R}} \phi(s) ds = \int_{T_1}^{T_2} N(u; T_1, T_2) du
\]

and then the simplified version in Lec 24, (25).
The Bohr–Landau theorem will arise by specializing \( f(s) \) to be \( f(s) \) in the foregoing — and playing with appropriate \( \alpha \) and \( \beta \).

Not-too-surprisingly, matters will need to be looked at along the way with the aid of some of our previously obtained estimates.

THEOREM (basic Form of Bohr–Landau thm)

Consider \( f(s) \). For \( T \geq 2 \), say, \( \ln \) not the ordinate of a zero of \( f(s) \), we have:

\[
N\left( \frac{1}{2} + \varepsilon \ ; \ h, T \right) = O \left( T \right)
\]

for each \( \varepsilon > 0 \). Here \( h \) is a tiny number such that \( f \neq 0 \) on \( \left\{ \text{Re}(s) \geq 0 \right\} \), \( 0 \leq \text{Im}(s) \leq h \), \( \pi \ln(2) \).

See Lec 11, p. 87 and Lec 13, pp. 6 (top) + 7 (top).

We stress here that, in toto, we have

\[
N(0; h, T) \sim \frac{T}{2 \pi} \ln \left( \frac{T}{2 \pi e} \right)
\]

by Lec 15, pp. 29 + 30 (box). As such, by the functional equation of \( f \), only \( 0 \% \) of the complex zeros of \( f \) can lie outside \( |\text{Re}(s) - \frac{1}{2}| \leq \frac{\varepsilon}{2} \).
The relevant "PREVIOUSLY OBTAINED" estimates referenced on \( \sqrt{2} \) are those found on:

- page 8 of Lec 15 (a priori upper bound)
- page 13 of Lec 15 (the partial fraction thing)
- page 14 of Lec 25 (L2 estimate)
- page 11 of Lec 26 (Hardy-Littlewood estimate).

In connection with the first two — from Lec 15 — we note the following:

**FACT**

Keep \( T \geq 2 \), say, and not the ordinate of a zero of \( \zeta \). Use the standard up and across definition of \( \log |f(s)| \) beginning at some point \( A \in \mathbb{R} \), \( A \gg 1 \). Then:

\[
\text{Arg} \ J(\sigma + it) = O(\ln T),
\]

for all \(-1 \leq \sigma \leq 2\).
We know $|\log J(s)| = O(1)$, $\sigma \geq 1 + \gamma$.

Use

$$N[|y-t| \leq 1] = O(ln t)$$

$$\frac{J'(s)}{J(s)} = O(ln t) + \sum_{\rho} \frac{1}{s-\rho} \chi_{-1 \leq \sigma \leq 2}$$

$$|y-t| \leq 1$$

$$\sigma + iT$$

$$A + iT$$

$$\text{Arg } J(\sigma + iT) = \text{Im } \int_{A}^{\sigma} \frac{J'(u + iT)}{J(u)} \text{d}u + \text{Im } \log J(A + iT)$$

(this is $O(e^{-A})$

by the Dirichlet series for $\log J$

\{ we can let $A \to \infty \}
\[ \text{Arg } \zeta(s + iT) = - \text{Im} \int_{0}^{\infty} \frac{1}{\zeta'} (u + iT) \, du \]

\[ = - \text{Im} \int_{0}^{2} \frac{1}{\zeta'} (u + iT) \, du - \text{Im} \int_{2}^{\infty} \frac{1}{\zeta'} (u + iT) \, du \]

\[ = \text{O}(2^{-\delta}) \text{ by the Dirichlet series for } \zeta = \frac{1}{\zeta'} \]

\[ = - \text{Im} \int_{0}^{2} \left\{ \text{O} \left( \ln T \right) + \sum_{\rho} \frac{1}{u + iT - \rho} \right\} \, du \]

\[ + \text{O}(1) \]

\[ = \text{O} \left( \ln T \right) \]

\[ - \sum_{\rho} \left\{ \text{Arg } (\zeta'(a + iT - \rho)) - \text{Arg } (\zeta'(s + iT - \rho)) \right\} \]

\[ + \text{O}(1) \]

Arg = ordinary principal value
\[= O(\ln T) + O(\ln T) + O(1) \quad \text{(6)}\]

\[= O(\ln T)\]

\[\xi \text{ much like Lec 15, p. 28} \quad \text{5} \quad \text{of 5}\]

By Littlewood's identity, Lec 24, p. 25, know:

\[\frac{1}{2} \leq \delta < \beta \leq 2\]

\[\downarrow\]

\[2\pi \int_{\delta}^{\beta} N(\sigma; T_1, T_2) d\sigma\]

\[= \int_{T_1}^{T_2} \ln |\zeta(\sigma + it)| dt\]

\[= \int_{T_1}^{T_2} \ln |\zeta(\beta + it)| dt\]

\[+ \mathcal{O}\left( (\beta - \delta) \ln T_2 \right) \quad \leftarrow \text{by FACT on 3}\]

at least for \(T_0\) which are not the ordinates of \(\zeta\)-zeros. We'll keep \(T_2 > T_1 \geq 2\). Cf. also h on page 2.
Must focus on

\[ \sum_{T_1}^{T_2} \ln |I(\sigma + it)| \, dt \]

(see 15 middle)

(since \( \beta \) will be taken \( \geq \frac{3}{2} \) later).

We propose to look first at

\[ \sum_{T/2}^{T} |I(\sigma + it)|^2 \, dt \]

with \( T \geq 1000 \) and \( \frac{1}{2} \leq \sigma \leq 2 \).

Use H-L estimate from Lec 26 (11), \( \sigma_0 = \frac{1}{10} \).

Take \( C = \pi \). We get:

\[ I(\sigma + it) = \sum_{n \leq T} \omega^{-\sigma - it} - \frac{T^{-1-\sigma-it}}{1-\sigma-it} + O(T^{-\delta}) \]

for \( t \in \left[ \frac{1}{2}T, T \right] \).
So,

$$f(\sigma + it) = \sum_{n \leq T} n^{-\sigma} \cdot e^{-i\tau n} + \frac{O(T^{-\sigma})}{T} + O(T^{-\sigma})$$

$$= \sum_{n \leq T} n^{-\sigma} e^{-i\tau \ln n} + O(T^{-\sigma})$$

for \( \frac{1}{2} \leq \sigma \leq T \). We'll write this as

$$f(\sigma + it) = \Sigma + R$$

Hence,

$$|f(\sigma + it)|^2 = (\Sigma + R) \bar{(\Sigma + R)}$$

$$= \Sigma \bar{\Sigma} + 2 \text{Re}(R \bar{\Sigma}) + |R|^2$$

$$= \Sigma \bar{\Sigma} + 0(1) T^{-\sigma} |\Sigma| + 0(1) T^{-2\sigma}$$

for \( t \in [\frac{1}{2}T, T] \) and \( \frac{1}{2} \leq \sigma \leq 2 \).
Put:

\[ \Sigma = \sum_{n \leq T} n^{-\sigma} e^{-i\lambda_n t} \quad \text{with} \]

\[ \lambda_n = \ln n \]

and then use Lec 25 p. 04 (the L2 estimate).

Here, of course, \( n > m \Rightarrow \)

\[ \lambda_n - \lambda_m = \ln n - \ln m \]

\[ = \frac{1}{\tilde{n}} (n - m) \quad \text{where} \quad \tilde{n} \in (m, n) \]

\[ \geq \frac{1}{T} \]

We can apply Lec 25 04 with \( \delta = \frac{1}{T} \). So,

\[ \int_{T/2}^{T} |\Sigma|^2 dt = \frac{T}{2} \sum_{n \leq T} \tilde{n}^{-2\sigma} \]

\[ + \frac{O(1)}{1/T} \sum_{n \leq T} n^{-2\sigma} \]

{\text{this may be improvable, but we prefer to stay with a crude bound.} \}
\[ \int_{T/2}^{T} |\Sigma|^2 \, dt = O(T) \sum_{n \leq T} n^{-2\sigma} . \]

Suppose now that \( \sigma > \frac{1}{2} \). In that case, we go further and get
\[
\int_{T/2}^{T} |\Sigma|^2 \, dt = O(T) \int (2\sigma) \\
= O(T) \frac{1}{2\sigma - 1} .
\]

At the same time, (see \( \Box \) bottom)
\[
\int_{T/2}^{T} T^{-\sigma} |\Sigma| \, dt \leq \left\{ \int_{T/2}^{T} T^{-2\sigma} \, dt \right\}^{1/2} \\
\cdot \left\{ \int_{T/2}^{T} |\Sigma|^2 \, dt \right\}^{1/2} .
\]

\( \leq \text{by Cauchy-Schwarz} \)
\[
\leq \sqrt{T^{1-2\sigma}} \sqrt{\frac{O(T)}{2\sigma - 1}} \\
= O(1) \frac{T^{1-\sigma}}{\sqrt{2\sigma - 1}} = O(1) \frac{T^{1-\sigma}}{2\sigma - 1} .
\]
Referring to (8) bottom again, we get
\[
\int_{T/2}^{T} |f(\sigma+it)|^2 \, dt = \frac{O(T)}{2\sigma-1} + \frac{O(T^{1-\sigma})}{2\sigma-1} + O(T^{1-2\sigma})
\]
for \( \frac{1}{2} < \sigma \leq 1 \). Accordingly (via \( 2\sigma \geq 1 \)):
\[
\int_{T/2}^{T} |f(\sigma+it)|^2 \, dt \leq \frac{O(1)}{\sigma - \frac{1}{2}} T
\]
for \( \frac{1}{2} < \sigma \leq 2 \). The case \( \sigma = \frac{1}{2} \) is also OK, but [obviously] not very informative.

Suppose NEXT that we only know \( \sigma \geq \frac{1}{2} \). Observe that (10) line 1 is still usable. Being totally crude, we can say:
\[
\sum_{n \leq T} n^{-2\sigma} \leq \sum_{n \leq T} \frac{1}{n} = \zeta \cdot \ln T.
\]
Hence,

\[
\int_{T/2}^{T} |\Sigma|^{2} \, dt = O(T \ln T)
\]

\(\Delta\) can then mimic \(\heartsuit\) bot + \(\spadesuit\) top \(\heartsuit\)

\[
\int_{T/2}^{T} |f(\sigma + it)|^{3} \, dt = O(T \ln T)
\]

\[
+ O(1) T^{1-\sigma} \sqrt{\ln T}
\]

\[
+ O(T^{1-2\sigma})
\]

\[
\int_{T/2}^{T} |f(\sigma + it)|^{2} \, dt \leq O(1) T \ln T
\]

\(\heartsuit\) for \(\frac{1}{2} \leq \sigma \leq 2\).

We have obtained the above box, and \(\heartsuit\) box, assuming \(T \geq 1000\) (see \(\heartsuit\)).
THEOREM (standard a priori estimate)

For $T \geq 3$ and $\sigma \in [\frac{1}{2}, 2]$, we have:

$$\int_2^T |I(\sigma + it)|^2 \, dt = O(T) \min \left\{ \ln T, \frac{1}{\sigma - \frac{1}{2}} \right\}.$$ 

**Proof.**

Matters are obvious for $3 \leq T \leq 10^6$. In fact, here,

$$\min \left\{ \ln T, \frac{1}{\sigma - \frac{1}{2}} \right\} = \min \left\{ \ln 3, \frac{1}{3/2} \right\} = \frac{\ln 3}{3}$$

and it suffices to adjust the implied constant in $O(T)$.

For $T > 10^6$, choose $\lambda$ so that

$$\frac{T}{2^{\lambda+1}} \leq 1000 < \frac{T}{2^\lambda}.$$ 

Apply (1) box and (12) box to $\frac{T}{2^k}$ for $k \in [0, \lambda]$. Add. Get:
\[ \int_{1000}^{T} |I(\sigma+it)|^2 \, dt = O(T) \min \left\{ \frac{1}{\sigma - \frac{1}{2}} \right\} \]

To replace 1000 by 2, repeat the observation used for 3 \( \leq T \leq 10^6 \).

With more work, one can prove that:

\[ \int_{2}^{T} |I(\sigma+it)|^2 \, dt \sim T \, I(\sigma), \quad \sigma > \frac{1}{2} \]

and

\[ \int_{2}^{T} |I(\frac{1}{2}+it)|^2 \, dt \sim T \, \ln T \]

See Titchmarsh, Theory of \( \zeta(s) \), first few sections in chapter 7. We won't need these more precise results.

---
On p. 6 bottom, fix $T_1 \in [2, 2.5]$ to be well away from the ordinate of any $\rho$-zero. Since numerical work shows that

$$\rho_1 = \frac{1}{2} + i \left[ 14.134725 + \right]$$

one can declare $T_1 = 2$. We prefer not to use this, however.

Keep $T = T_2 \geq 3$ and then take

$$\frac{1}{2} < \varpi \leq 1 \quad \text{and} \quad \beta = 2 \ (\text{say}).$$

One gets:

$$2\pi \int_{\varpi}^{2} N(\sigma, T_1, T) \, d\sigma = \int_{T_1}^{T} \frac{\ln |\Gamma(\sigma + it)|}{|t|} \, dt$$

$$- \int_{T_1}^{T} \ln |\Gamma(2 + it)| \, dt$$

$$+ O(\ln T).$$
**BABY LEMMA**

(a) For \( x \in \mathbb{R} \), \( e^x \geq 1 + x \).

(b) For \( y \geq 0 \), \( \ln y \leq \frac{1}{2} (y^2 - 1) \).

**PF**

We give 2 proofs of (a). The first notes that \( g(x) = e^x \) is concave upward since \( g'' > 0 \). Hence \( g(x) \) sits on or above the tangent line at each point \( x_0 \). Take \( x_0 = 0 \). Get: \( g(x) \geq 1 + x \) by inspection.

The 2nd proof is more boring. For \( x > 0 \), apply mean value theorem to get:

\[
e^x - 1 = e^\tilde{x}(x - 0) = e^{\tilde{x} \cdot x}, \quad 0 < \tilde{x} < x
\]

\[
e^x - 1 \geq e^0 \cdot x = x \quad \text{(OK)}
\]

For \( x < 0 \), apply mean value theorem to get:

\[
1 - e^x = e^{\tilde{x}}(0 - x) = (-x)e^{\tilde{x}}, \quad x < \tilde{x} < 0
\]

\[
1 - e^x \leq (-x)e^0
\]

\[
e^x - 1 \geq x \quad \text{(OK)}
\]

In (b), wlog \( y > 0 \). Put \( y = e^u \) with \( u \in \mathbb{R} \).
Must check that
\[ u \leq \frac{1}{2} (e^{2u} - 1) \]
\[ 2u \leq e^{2u} - 1 \]
\[ e^{2u} \geq 1 + 2u \]

But this is obvious by (a).

By Baby Lemma, then,
\[ \int_{T_1}^{T} \ln |I(\alpha + it)| \, dt \leq \frac{1}{2} \int_{T_1}^{T} (|S(\alpha + it)|^2 - 1) \, dt \]
\[ \leq \frac{1}{2} \int_{T_1}^{T} |S(\alpha + it)|^2 \, dt \]
\[ \leq \frac{1}{2} \int_{T}^{T} |I(\alpha + it)|^2 \, dt \]
\[ \leq \frac{O(T)}{\gamma - \frac{1}{2}} \text{ by (3).} \]
In addition,

\[
\log J(2+it) = \sum_{n=2}^{\infty} \frac{A(n)}{\ln n} n^{-2-it}
\]

\[
\int_{T_1}^{T} \ln \left| J(2+it) \right| dt = \text{Re} \int_{T_1}^{T} \log J(2+it) \, dt
\]

\[
= \text{Re} \int_{T_1}^{T} \sum_{n=2}^{\infty} \frac{A(n)}{\ln n} n^{-2-it} \, dt
\]

\[
= O(1) \sum_{n=2}^{\infty} \frac{A(n)}{\ln n} n^{-2} \left[ \frac{e^{-it \ln n}}{-it \ln n} \right]_{T_1}^{T}
\]

\[
= O(1) \sum_{n=2}^{\infty} \frac{A(n)}{\ln n} n^{-2} \frac{1}{\ln n}
\]

\[
= O(1)
\]

Page 15 bottom then gives

\[
2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{N(\sigma; T_1, T)}{\sigma - \frac{1}{2}} \, d\sigma \leq \frac{O(T)}{\sigma - \frac{1}{2}} + O(1) + O(\ln T)
\]

\[
\Rightarrow \quad \frac{O(T)}{\sigma - \frac{1}{2}}
\]

\[\text{here } \sigma \in \left[ \frac{1}{2}, 1 \right]\]
Write \( \alpha = \frac{1}{2} + 2\omega \), \( 0 < \omega \leq \frac{1}{4} \). Notice that

\[
N\left(\frac{1}{2} + 2\omega i, T, \bar{T}, T\right) \leq \frac{1}{\omega} \int_{\frac{1}{2} + \omega}^{\infty} N(\sigma^i, T, \bar{T}) \ d\sigma
\]

since \( N(\sigma^i, T, \bar{T}) \) is monotonic decreasing in \( \sigma \).

Thus:

\[
N\left(\frac{1}{2} + 2\omega i, T, \bar{T}, T\right) \leq \frac{1}{\omega} \int_{\frac{1}{2} + \omega}^{\infty} N(\sigma^i, T, \bar{T}) \ d\sigma
\]

\[
\leq \frac{1}{\omega} \frac{O(T)}{\omega} = \frac{H}{(2\omega)^2} \ O(T).
\]

In other words,

\[
N(\alpha, T, \bar{T}, T) \leq \frac{O(T)}{(\alpha - \frac{1}{2})^2}, \quad \frac{1}{2} < \alpha \leq 1.
\]

This proves the THM on p. 2.

Because our use of (16)(b) was so crude, one suspects that the foregoing box can be improved in the \( \alpha \)-aspect.  

\[ \text{cf. also 18 bottom} \]
This is indeed correct. We claim, in fact, that

$$\int_{T_1}^{T} \ln|I(x + it)| \, dt = O(T) \ln \left( \frac{1}{x - \frac{1}{2}} \right).$$

On (19), this leads to

$$N(x; T_1, T) \leq O(T) \frac{1}{\sqrt{x - \frac{1}{2}}} \ln \left( \frac{1}{x - \frac{1}{2}} \right).$$

There are 2 approaches to prove line 2.

Method I

By (18) (bottom), \( x \in \left( \frac{1}{2}, \frac{3}{5} \right) \).

Under this hypothesis, we have \( r \equiv \frac{1}{x - \frac{1}{2}} \geq 10 \).

Notice that \( \ln y \leq \frac{1}{2A} (y^{2A} - 1) \) for \( 0 < y \leq 1 \) \([\text{and } y \geq 0] \). See (16). We'll keep \( \lambda < 1 \). Get:
\[
\int_{T_1}^{T} \ln |f(t)| \, dt = \frac{1}{2\lambda} \int_{T_1}^{T} (15^{2\lambda} - 1) \, dt
\]

\[
\leq \frac{1}{2\lambda} \int_{T_1}^{T} 15^{2\lambda} \, dt
\]

\[
\leq \frac{1}{2\lambda} \left( \int_{T_1}^{T} 15^{\frac{r}{2}} \, dt \right)^{\lambda} \left( \int_{T_1}^{T} 1 \, dt \right)^{1-\lambda}
\]

\[
\boxed{\text{by Hölder's inequality}}
\]

\[
p = \frac{r}{2}, \quad q = \frac{1}{1-\lambda}
\]

\[
= \frac{1}{2\lambda} \left( \frac{O(T)}{q^{\frac{1}{2}}} \right)^{\lambda} T^{1-\lambda}
\]

\[
= \frac{c}{\lambda} \left( \frac{1}{q^\frac{1}{2}} \right)^{\lambda} T
\]

\[
= \frac{c}{\lambda} r^\lambda T
\]

Put \( \lambda = \frac{1}{\ln r} \). This is admissible since \( r \geq 10 \).

Get:

\[
\int_{T_1}^{T} \ln |f(t) + i t| \, dt \leq C (\ln r) e T
\]

\[
\leq O(T) \ln \left( \frac{1}{r^{-\frac{1}{2}}} \right)
\]

\[\text{OK}\]
Method II

We use Jensen's inequality common in probability and measure theory.

To recall it, let $\Phi(v)$ be non-negative and convex on $\mathbb{R}$ (hence $\hat{\Phi}$ is automatically continuous.) Let $Y$ be an extended real-valued random variable on a probability space $(\mathcal{X}, \mu)$. Assume that $E(Y)$ exists (i.e., that $Y \in L_1(\mu)$). Then:

$$\Phi(E(Y)) \leq E(\Phi(Y)).$$

One simply approximates $E(Y)$ by an obvious Riemann sum and uses

$$\left\{ \begin{array}{l}
\Phi \left( \sum_{j=1}^{N} t_j v_j' \right) \leq \sum_{j=1}^{N} t_j \Phi(v_j') \\
\text{for } t_j \in [0,1] \text{ with } t_1 + \cdots + t_N = 1
\end{array} \right\}.
$$

Put $\Phi(v) = \exp(v)$ to get

$$\exp \left( \frac{1}{H} \int_0^H f(t) \, dt \right) \leq \frac{1}{H} \int_0^H \exp \left[ f(t) \right] \, dt$$

anytime $f \in L_1([0, H])$. 

\[22\]
A trivial specialization gives:

\[
\exp \left( \frac{1}{T-T_1} \int_{T_1}^{T} 2 \ln |I(x+it)| \, dt \right)
\]

\[
\leq \frac{1}{T-T_1} \int_{T_1}^{T} \left| I(x+it) \right|^2 \, dt
\]

\[
\leq \frac{O(T)}{\gamma - \frac{i}{2}} \quad \text{by (13)}
\]

\[
\leq \frac{O(1)}{\gamma - \frac{i}{2}}
\]

Accordingly:

\[
\frac{1}{T-T_1} \int_{T_1}^{T} 2 \ln |I(x+it)| \, dt
\]

\[
\leq \ln \left( \frac{B}{\gamma - \frac{i}{2}} \right) \quad \{ \text{some } B \geq 1 \}
\]

\[
\downarrow
\]

\[
\int_{T_1}^{T} \ln |I(x+it)| \, dt \leq O(T) \ln \left( \frac{1}{\gamma - \frac{i}{2}} \right)
\]

for \( \gamma \in (\frac{1}{2}, 1] \). \( \text{OK} \)
Incidentally, observe how the THM gets applied in both methods $I + II$. Switching to $\ln T$ in place of $\frac{1}{q - \frac{1}{2}}$ produces the following:

**Method I**

$$\frac{c}{\lambda} (\ln T)^\gamma T \quad \text{on } 21 \text{ middle}$$

$$\Rightarrow \text{take } \lambda = \frac{1}{\ln T} \quad (T \text{ giant})$$

$$\Rightarrow \left[ \ln |I(s + it)| \right]_T \, dt = O(T \ln \ln T)$$

uniformly for $\gamma \in \left[\frac{1}{2}, 1\right]$.

**Method II**

$$\ll O(1) \ln T \quad \text{on } 23 \text{ middle}$$

$$\Rightarrow \frac{1}{T - T_1} \left[ \ln |I(s + it)| \right]_T \, dt \leq \ln (\Theta \ln T)$$

$$\Rightarrow \left[ \ln |I(s + it)| \right]_T \, dt = O(T \ln \ln T)$$

uniformly for $\gamma \in \left[\frac{1}{2}, 1\right]$.

**Hence:**

$$\left[ \ln |I(s + it)| \right]_T \, dt = O(T) \ln \left[ \min \left( \ln T, \frac{1}{a - \frac{1}{2}} \right) \right].$$
Closing Remark

The estimate

\[ N\left(\frac{1}{2} + \varepsilon; T_1, T\right) = O(T) \frac{1}{\varepsilon} \ln \left(\frac{1}{\varepsilon}\right) \]

mentioned on p. 20 (line 4) was obtained by Littlewood in 1924. \textit{Proc. Camb. Phil. Soc. 22 (1924)}

It is possible to expunge the term \( \ln \frac{1}{\varepsilon} \).

This was shown by A. Selberg around 1942 with the aid of some fundamentally new ideas. By use of a so-called \underline{mollifier method},

Selberg was able to demonstrate that

\[ \int_{\frac{1}{2}}^{2} N(\sigma; T_1, T) \, d\sigma = O(T) \cdot \]

\underline{Compare \(15\) (bottom) + \(18\) (middle)}.

See also Titchmarsh, \textit{Theory of \(\zeta(s)\)}, around § 9.24.
Lecture 28
(Fri, Apr 29)


**Known Facts**

1. \( J(s) = \sum_{n=1}^{\infty} n^{-s} \) analytic \( \text{re}(s) > 1 \)

2. \( J(s) = \prod_{p} \frac{1}{1-p^{-s}} \) nice convergence \( \text{re}(s) > 1 \) Lec 6 \( \triangleright \)

3. \( J(s) = \frac{1}{s-1} \) analytic \( \text{re}(s) > 0 \) Lec 5 \( \triangleright \)

4. \( J(s) = \frac{1}{s-1} + \gamma + O(s^{-1}) \) near \( s = 1 \) Lec 17 \( \triangleright \)

5. \( \log J(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln(n)} n^{-s} \) \( \text{re}(s) > 1 \) Lec 6 \( \triangleright \)

6. \( -\frac{J'(s)}{J(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \) \( \text{re}(s) > 1 \) Lec 6 \( \triangleright \)

7. \( J(s) \neq 0 \) on \( \text{Re}(s) \geq 1 \)

Newman likes

\[ \phi(s) = \sum_{p} \frac{\ln p}{p^s}, \; \text{re}(s) > 1 \]

Obviously,

\[ -\frac{J'(s)}{J(s)} = \phi(s) + \sum \frac{\ln p}{p^{2s}} + \sum \frac{\ln p}{p^{3s}} + \cdots \]

and the underlined term is analytic on \( \text{Re}(s) > \frac{1}{2} \).
Indeed,

\[ \sum_{n=2}^{\infty} \frac{p^{-n \sigma}}{\ln p} = (\ln p) \frac{p^{-2 \sigma}}{1 - p^{-\sigma}} \]

and

\[ (\ln p) \frac{p^{-2 \sigma}}{1 - p^{-\sigma}} \quad \sigma \geq 1 + \varepsilon \]

e tc e tc .

Let \( E(s) \) mean a fun analytic on \( \Re(s) > \frac{1}{2} \).

Not necessarily the same one each time...

\[ \frac{-I(s)}{S(s)} = \phi(s) + E(s) \]

\textbf{FACT}

Write

\[ I(s) \approx (s-1)^{-\gamma} \left[ 1 + \gamma(s-1) + O(s-1)^{2} \right] \quad \text{near} \quad s=1. \]

Take logarithmic derivative. Get

\[ \frac{I'(s)}{S(s)} = \frac{-1}{s-1} + \left( \gamma + O(s-1) \right) \]

\[ \frac{I'(s)}{S(s)} \approx \frac{-1}{s-1} + \gamma + O(s-1) \quad \text{near} \quad s=1. \]

\textbf{FACT 2}

see Lec 17 p. 42
Recall

\[ \psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \ln p \quad (x > 1) \]

\[ = \theta(x) + \theta(x^{\frac{1}{2}}) + \theta(x^{\frac{1}{3}}) + \ldots \]

with

\[ \theta(x) = \sum_{p \leq x} \ln p \]

See Lec 1 [4][12].

---

**Additional Known Fact For \( x \geq 2 \)**

\[ c_1 x \leq \psi(x) \leq c_2 x \quad c_3 x \leq \theta(x) \leq c_4 x \quad (c_j > 0) \]

\[ \psi(x) = \theta(x) + O(x^{\frac{1}{2}}) \text{ "Chebyshev"} \]

See Lec 1 [4][5][16][18].

---

\[ \psi(x) = \theta(x) + R(x) \]

\[ R(x) = O(x^{\frac{1}{2}}) \quad x \geq 1 \]

FACT 3
Recall

\( \text{Re}(s) > 1 \quad \Rightarrow \quad (\psi(x) = 0, \ x < 2) \)

\[-\frac{f'(s)}{f(s)} = \int_{1}^{\infty} x^{-s} \psi(x) \, dx \]

\[= \left[ x^{-s} \psi(x) \right]_{1}^{\infty} - \int_{1}^{\infty} x^{-s} \, d(\psi(x)) \]

\[= 0 + s \int_{1}^{\infty} \frac{\psi(x)}{x^{s+1}} \, dx \]

\[-\frac{1}{s} \frac{f'(s)}{f(s)} = \int_{1}^{\infty} \frac{\psi(x)}{x^{s+1}} \, dx \]

and \( \frac{1}{s-1} = \int_{1}^{\infty} \frac{x}{x^{s+1}} \, dx \)

\[-\frac{1}{s} \frac{f'(s)}{f(s)} - \frac{1}{s-1} = \int_{1}^{\infty} \frac{\psi(x) - x}{x^{s+1}} \, dx \]

\( \text{Re}(s) > 1 \quad \Rightarrow \)

\[\text{FACT 4} \]

See Ingham 18(17), 91(8) and Lec 8 (11).
But,

$$\psi(x) - x = \Theta(x) - x + R(x) \quad \text{see (3), Fact 3}$$

and

$$\int_1^\infty \frac{R(x)}{x^{s+1}} \, dx = \text{analytic on } \text{Re}(s) > \frac{1}{2}$$

(since \( R(x) = O(\sqrt{x}) \)).

So,

$$-\frac{1}{s} \frac{\theta'(s)}{\theta(s)} - \frac{1}{s-1} = \int_1^\infty \frac{\Theta(x) - x}{x^{s+1}} \, dx + E(s)$$

\( \text{à la (2)} \)

**Fact 5**

$$\int_1^\infty \frac{\Theta(x) - x}{x^{s+1}} \, dx = -\frac{1}{s} \frac{\theta'(s)}{\theta(s)} - \frac{1}{s-1} + E(s)$$

on \( \text{Re}(s) > 1 \) and we get:

(a) LHS has a meromorphic continuation to \( \text{Re}(s) > \frac{1}{2} \)

(b) LHS has no pole at \( s = 1 \)

(c) LHS has no poles on \( \{ \text{Re}(s) \geq 1 \} \).
(a) is obvious by (1).
(b) is easy by (bottom) so (c) then follows by (1).

Next:
\[
\int_1^\infty \frac{\theta(x) - x}{x^s} \frac{dx}{x} = \int_0^\infty \frac{\theta(e^v) - e^v}{e^{sv}} dv
\]
\[
\begin{cases}
x \approx e^v \\
v = \ln x
\end{cases}
\]
shift \quad s \rightarrow s + 1
\Rightarrow \quad \text{get } \int_0^\infty \frac{\theta(e^v) - e^v}{e^{sv}} dv
\]
\[
= \int_0^\infty e^{-sv} \left[ \frac{\theta(e^v)}{e^v} - 1 \right] dv.
\]

\underline{FACT 6}
\[
\int_0^\infty e^{-sv} \left[ \frac{\theta(e^v)}{e^v} - 1 \right] dv
\]
(a) is analytic on \( \text{Re}(s) \geq 0 \)
(b) is meromorphic on \( \text{Re}(s) > -\frac{1}{2} \)
(c) secretly has poles at \( s - 1 \), where
\[
\xi_0(s) = s(s-1)\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \xi(s)
\]
\( \text{à la Lec 13 (4)5} \).
On the other hand, following Newman

$$\phi(s) = \int_1^\infty u^{-s} \theta(u) \, \text{d}u, \quad \text{Re}(s) > 1 \quad (\theta(u) = 0 \quad \text{for} \quad u < 2)$$

$$= u^{-s} \theta(u) \left[ \int_1^u - \int_1^\infty \theta(u) \text{d}u \right]$$

$$= s \int_1^\infty \frac{\theta(u)}{u^{s+1}} \text{d}u$$

$$= s \int_0^\infty e^{-sv} \theta(e^v) \text{d}v$$

but

$$\frac{s}{s-1} = s \int_0^\infty e^{-sv} e^v \text{d}v$$

$$\Rightarrow$$

$$\phi(s) - \frac{s}{s-1} = s \int_0^\infty e^{-sv} \left[ \theta(e^v) - e^v \right] \text{d}v$$

$$\phi(s) - \frac{1}{s-1} = \int_0^\infty e^{-sv} \left[ \theta(e^v) - e^v \right] \text{d}v \quad \text{Re}(s) > 1$$

$$\frac{\phi(s+1)}{s+1} - \frac{1}{s} = \int_0^\infty e^{-sv} \left[ \frac{\theta(e^v)}{e^v} - 1 \right] \text{d}v \quad \text{Re}(s) > 0$$

FACT 7

same fcn
as in Fact 6
**Definition**

\[ g(s) = \frac{\theta(s+1)}{s+1} - \frac{1}{s} \quad i \]

\[ f(v) = \frac{\theta(e^v)}{e^v} - 1 \quad (v \geq 0). \]

By (7), Fact 7, know

\[ g(s) = \int_0^\infty e^{-sv} f(v) \, dv, \quad \text{Re}(s) > 0 \]

in the style of a Laplace transform.

Fact 6 allows us to better understand \( g \).

**Theorem**

\( g \) and \( f \) as above. Then:

(i) \( g(s) = \int_0^\infty e^{-sv} f(v) \, dv, \quad \text{Re}(s) > 0 \)

(ii) \( f(v) \) is bounded and piecewise \( C^\infty \)

(iii) \( g(s) \) is meromorphic on \( \text{Re}(s) > -\frac{1}{2} \)

**BUT** HAS NO POLES on \( \text{Re}(s) \geq 0 \).
Proof
(i) as above. (ii) by Chebyshev on \( \mathbb{S} \).
(iii) see Fact 6. \( \square \)

NEWMAN'S GENERAL THM

\[ \text{Let } f(n) \text{ be ANY} \text{ bounded, piecewise continuous function on } [0, \infty). \text{ Let} \]
\[ g(s) = \int_0^\infty e^{-sv} f(n) \, dv, \quad \text{Re}(s) > 0. \]

**ASSUME THAT** \( g \) **extends to a single-valued analytic function** on a connected open set slightly bigger than \( \text{Re}(s) \geq 0 \) (Call it \( g \) again.) **Then:**
\[ \int_0^\infty f(n) \, dv \text{ exists and equals } g(0). \]

\( \text{Pf} \)
\[ \int_0^\infty f(n) \, dv \text{ means } \lim_{T \to \infty} \int_0^T f(n) \, dv. \]

To convey the function theory flavor, switch to \( z = x + iy \) instead of \( s \).

Let
\[ g_T(z) = \int_0^T e^{-zv} f(n) \, dv, \quad T > 0. \]
The fn $g_T(z)$ is entire for each $T$. Simply view as a standard limit of Riemann sums. Recall Lec 3 $\lim_{|T| \to \infty} g_T(0) = g(0)$. 

Must prove:

$$\lim_{T \to \infty} g_T(0) = g(0).$$

Take $R$ giant and freeze it! 

Select a tiny $\delta > 0$ (depending on $R$) so that $g(z)$ is nicely analytic on

$$\{ |s| \leq R \} \cap \{ x \geq -\delta \}.$$

(That means on a slightly bigger open set.) 

\[ C = \text{heavy path} \]

\[ C_+ = \text{portion with } x > 0 \]

\[ C_- = \text{portion with } x < 0 \]
Apply Cauchy integral formula to

\[ \int \left[ g(z) - g_T(z) \right] e^{zT} \left( 1 + \frac{z^2}{R^2} \right) \frac{dz}{z} \]

KEY

Newman's Trick

Get:

\[ g(0) - g_T(0) = \frac{1}{2\pi i} \oint_C \left[ g - g_T \right] e^{zT} \left( 1 + \frac{z^2}{R^2} \right) \frac{dz}{z} \]

We estimate RHS in several steps:

First over \( C^+ \):

Let \( B = \sup_{v \leq 0} |f(v)| \)

On \( C^+ \):

\[ |g(z) - g_T(z)| = \left| \int_T^\infty e^{-zv} f(v) dv \right| \]

\[ \leq B \int_T^\infty e^{-zv} dv \]

\[ = B \frac{e^{-zT}}{z} \]
\[ |e^{zT}| = e^{xT} \]

\[ |1 + \frac{z^2}{R^2}| = |1 + \frac{\bar{z}^2}{\bar{z}z}| = |1 + \bar{z}/z| \]

\[ = \frac{|\bar{z} + z|}{R} \]

\[ = \frac{2|x|}{R} = \frac{2x}{R} \]

\[ |g - g_T| \cdot |e^{zT}| \cdot |1 + \frac{z^2}{R^2}| \leq \frac{2B}{R} \]

\[ \left| \frac{1}{2\pi i} \int_{C_T} (g - g_T) e^{zT} (1 + \frac{z^2}{R^2}) dz \right| \]

\[ \leq \frac{1}{2\pi} \int_{C_T} \frac{2B}{R} \cdot \frac{|dz|}{R} \]

\[ \leq \frac{1}{2\pi} \frac{2B}{R} \pi R = \frac{B}{R} \]
For \( C^{-} \), we write:

\[
I_1 = \frac{1}{2\pi i} \int_C g(z) e^{zT} \left( 1 + \frac{z^2}{R^2} \right) \frac{dz}{z}
\]

\[
I_2 = \frac{1}{2\pi i} \int_C g_T(z) e^{zT} \left( 1 + \frac{z^2}{R^2} \right) \frac{dz}{z}
\]

\( g_T \) is entire \( 10 \).

For \( I_2 \), by deformation of contour, note that

\[
I_2 = \frac{1}{2\pi i} \int_{iR}^{R} \left[ \cdots \right] \frac{dz}{z}
\]

\( (\text{left half}) \left\{ \begin{array}{c}
\mid z \mid = R
\end{array} \right. \) \( \left\{ \begin{array}{c}
\text{see } 10
\end{array} \right. \)

here, on \( \mid z \mid = R \), have

\[
\left| g_T(z) \right| \leq \int_0^T e^{-\nu x} |f(\nu)| d\nu \quad 9 \text{ bot}
\]

\[
\begin{align*}
\text{for } x < 0 & \\
& = B \int_0^T e^{\nu |x|} d\nu \\
& = B \frac{e^{T|x|} - 1}{|x|} \leq B \frac{e^{T|x|}}{|x|}
\end{align*}
\]
\[ |e^{x^2T}| = e^{x^2} = e^{-|x|^2T} \]

\[ \left| 1 + \frac{z^2}{R^2} \right| = \left| 1 + \frac{z^2}{z^2} \right| = \left| 1 + \frac{z}{z} \right| \]

\[ = \frac{2|x|}{R} \]

take product to get

\[ |\ldots| \leq B \frac{e^{T|x|}}{|x|} e^{-|x|^2T} \frac{2|x|}{R} = \frac{2B}{R} \]

\[
\left| \frac{1}{2\pi i} \int_{|z|=R} \frac{[\ldots]}{z} \, dz \right| \leq \frac{1}{2\pi} \frac{2B}{R} \pi = \frac{B}{R}.
\]

Must now do \( I_1 \). We'll do Newman's method first and then note an alternate reasoning.
$R$ is frozen, as is $\gamma$. (10)

Look at the integrand

$$e^{\pi T} g(x) \left( 1 + \frac{x^2}{R^2} \right) \frac{1}{x} \, dx$$

on curve $C_-$. Each chunk

$$\left\{ g(x), \ 1 + \frac{x^2}{R^2}, \ \frac{1}{x} \right\}$$

is bounded by something, so is $e^{\pi T}$;

$$\left| e^{\pi T} \right| = e^{\pi T} \leq e^0 = 1.$$

Switch now to a parametric representation of $C_-$, say $x = x(\theta)$, $0 \leq \theta \leq \pi$.

Get new integral

$$\int_0^\pi \beta(\theta) e^{\pi T} x(\theta) d\theta$$

$\uparrow$ continuous & bounded
Can now apply an elementary bounded convergence theorem for Riemann integrals, since

\[ |e^{\alpha(T)}| = e^{\alpha(T)} , \quad \alpha \in [0,1] \]

\[ \leq 1 \]

AND

\[ \lim_{T \to \infty} e^{\alpha(T)} = 0 \quad \text{pointwise on } 0 < \alpha < 1 \]

In fact, this last limit is \underline{uniform} on each \([\epsilon, 1-\epsilon]\). Get:

\[ \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\gamma} g(z) \left( 1 + \frac{z^2}{R^2} \right) e^{\alpha(T)} \frac{dz}{z} = 0 \]

A highly suggestive \underline{alternate} approach to \(I_1\) goes as follows.
Take \( h > 0 \) microscopic. Make a new path \( C(h) \) a little circle of radius \( h \) centered at \( 0 \), a little circle of radius \( h \) centered at \( \infty \).

By the extended (limit) form of the CIF, we have

\[
g(0) - g_T(0) = \frac{1}{2\pi i} \int_{C_T} (g - g_T) e^{zT} \left( 1 + \frac{z^2}{R^2} \right) \frac{dz}{z}
\]

\[
+ \frac{1}{2\pi i} \int_{C(-h)} (g - g_T) e^{zT} \left( 1 + \frac{z^2}{R^2} \right) \frac{dz}{z}
\]

(see (9))

anytime \( g \) is ONLY known to be continuous on \( \{ x \neq 0 \} \) and analytic near \( z = 0 \).

The \( \mathbb{R} \) part of the \( C(-h) \) integral again gives \( \Theta \frac{B}{R} \), \( |\Theta| \leq 1 \). See (13) (14).
For the $I_1$ portion, use $C_-(h)$ as given:

$$\frac{i}{2\pi i} \int_{iR}^{iR} g(z) e^{\frac{r}{z}} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \quad \leftarrow I_{11}$$

$$+ \frac{i}{2\pi i} \int_{|z|=\frac{R}{h} \text{ left}} g(z) e^{\frac{r}{z}} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \quad \leftarrow I_{12}$$

$$+ \frac{i}{2\pi i} \int_{-iR}^{iR} g(z) e^{\frac{r}{z}} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \quad \leftarrow I_{13}$$

Note:

$$I_{11} = (\text{const}) \int_{iR}^{R} g(iy) e^{\frac{i}{y} \left(1 - \frac{y^2}{R^2}\right)} \frac{1}{y} dy$$

$$= o(1) \quad \text{by Riemann-Lebesgue lemma}$$

$$\{ h, R \text{ fixed} \}$$

$$I_{13} = o(1) \quad \text{similarly}$$

$$I_{12} = o(1) \quad \text{by a mimic of (15)(hot) + (16)}$$

$$\{ h > 0 \text{ fixed} \}$$
So,

\[ I_1 = o(1) \]

\[ \text{End of Alternate Approach} \]

Remember \( R = \text{giant, but fixed} \).

Get:

\[
\limsup_{T \to \infty} \left| \frac{1}{2\pi i} \int_C (g - g_T) e^{zT} \left( 1 + \frac{z^2}{R^2} \right) \frac{dz}{z} \right|
\]

\[ = \limsup_{T \to \infty} \left| f(0) - g_T(0) \right| \]

\[ \leq \frac{B}{R} + \frac{B}{R} + 0 \quad \text{by } 12, 14, 16 \text{ or line } 2 \text{ above} \]

\[ = \frac{2B}{R} \]

Since \( R \) is arbitrary, deduce that

\[
\limsup_{T \to \infty} \left| g(0) - g_T(0) \right| = 0
\]
Corollary

\[ \int_{1}^{\infty} \frac{\theta(x) - x}{x^2} \, dx \text{ is convergent.} \]

**pf**

Recall \( \mathbb{7} \) (bottom) + \( \mathbb{8} \). Then apply Newman’s general thm. Get

\[ \int_{0}^{\infty} \left[ \frac{\theta(e^v)}{e^v} - 1 \right] \, dv \text{ converges} \]

\( \begin{cases} x = e^v, \quad v = \ln x \end{cases} \)

\[ \int_{1}^{\infty} \frac{\theta(x) - x}{x} \, dx \text{ converges.} \]

\( \Box \)
FACT

Suppose \( H(x) \) is piecewise continuous on \([1, \infty)\) and \( H(x) \uparrow \). Suppose

\[
\int_1^\infty \frac{H(x) - x}{x^2} \, dx
\]

converges as an improper integral. Then

\( H(x) \sim x \) as \( x \to \infty \).

PROOF

Suppose \( H(x) \geq \lambda x \) frequently as \( x \to \infty \) for some \( \lambda > 1 \). Notice that, \( \forall \) such \( x \),

\[
H(u) \geq \lambda x \text{ on } [x, \lambda x] \quad (H \uparrow)
\]

\( H(u) - u \geq \lambda x - u \) here

\[
\int_x^{\lambda x} \frac{H(u) - u}{u^2} \, du \geq \int_x^{\lambda x} \frac{\lambda x - u}{u^2} \, du
\]

\[
\Rightarrow \int_x^{\lambda x} \frac{\lambda x - u}{u^2} \, du
\]

\[
\Rightarrow \int_1^{\lambda x} \frac{\lambda x - xw}{x^2 w^2} \, (x \, dw)
\]

\[
\Rightarrow \int_1^{\lambda x} \frac{\lambda - w}{w^2} \, dw > 0
\]
This violates
\[ \left| \int_{y_1}^{y_2} \frac{H(u) - u}{u^2} \, du \right| < \varepsilon \]
for all \( y_2 \geq y_1 \geq y \varepsilon \).

Now let \( H(x) \leq \eta x \) frequently as \( x \to \infty \) for some \( \eta < 1 \). Look at such \( x \).

\[ H(u) \leq \eta x \text{ on } [\eta x, x] \]
(\( \eta \downarrow \))

\[ H(u) - u \leq \eta x - u \text{ here} \]

\[ \int_{\eta x}^{x} \frac{H(u) - u}{u^2} \, du = \int_{\eta x}^{x} \frac{\eta x - u}{u^2} \, du \]
(put \( u = xw \))

\[ = \int_{\eta}^{1} \frac{\eta x - xw}{x^2 w^2} (x \, dw) \]

\[ = \int_{\eta}^{1} \frac{\eta - w}{w^2} \, dw \]

\[ = - \int_{\eta}^{1} \frac{w - \eta}{w^2} \, dw < 0 \]
This violates the \( Y_1, Y_2 \) condition above.

So,

\[
H(x) \sim x \circ \Theta(x) \sim x .
\]

\underline{Corollary (PNT)}

\[
\Theta(x) \sim x .
\]

\underline{PF}

Combine \((20) + (21)\).

\underline{REMARKS}

\[1\] Clearly, a very nice proof! ☺ (actually)

\[2\] It is reasonable to conjecture Newman began with (4) box, the FACT on (21), and Landau, Gött. Nachr. 1932 [attached below].

\[3\] Various extensions of the THM on (9) have been made based on the idea of (17), (18), (19) top.
We'll return to page 9 THM a bit later in a comment about lecture 30.
Über Dirichletsche Reihen.

Von

Edmund Landau.


Durch Weiterführung der N. Wienerischen Methode bewiesen Herr Heilbron and ich 1) den

Satz: Es gibt zwei für \( \lambda > 0 \) definierte positive Funktionen \( P_1(\lambda) \) und \( P_2(\lambda) \) mit

\[
\lim_{\lambda \to \infty} P_1(\lambda) = \lim_{\lambda \to \infty} P_2(\lambda) = 1
\]

und folgender Eigenschaft.

Die Dirichletsche Reihe

\[
f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad a_n \geq 0,
\]

konvergiert für \( \sigma > 1 \).

(Trivialerweise ist also,

\[
e^{-y} \sum_{\lambda_n \leq y} a_n = H(y) \quad \text{für} \quad y \geq 0
\]

gesetzt 2),

\[
f(s) = s \int_{-\infty}^{\infty} H(y) e^{-\sigma y} \, dy \quad \text{für} \quad \sigma > 1.
\]

Für \( |t| \leq 2\lambda \), \( \sigma = 1 + \varepsilon, \varepsilon > 0 \), sei bei \( \varepsilon \to 0 \) gleichmäßig in \( t \)

\[
h_\varepsilon(t) = f(s) - \frac{1}{s - 1} \to h(t).
\]

Dann ist

\[
P_1(\lambda) \geq \lim_{y \to \infty} H(y) \geq \lim_{y \to \infty} H(y) \geq P_2(\lambda).
\]

1) Bemerkungen zur vorstehenden Arbeit von Herrn Bochner (Mathematische Zeitschrift, im Druck). Wegen aller historischen Bemerkungen verweise ich auf diese Arbeit.

2) Für \( y_2 \geq y_1 \) ist also \( H(y_2) \geq H(y_1) e^{y_1 - y_2} \).

\[\text{Landau, Collected Works vol. 9}\]
Ich teile hier einen noch mehrfach vereinfachten Beweis unseres Satzes mit, bei dem ich o. B. d. A. \( \lambda_i = 0 \) annehmen darf.

**Hilfsatz:**
\[
\lim_{y \to \infty} \int_{-\infty}^{y} H\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2} \, dv = \int_{-\infty}^{\infty} \frac{\sin^2 v}{v^2} \, dv = \pi. \quad (3)
\]

**Beweis:** Für \( \varepsilon > 0 \) ist
\[
\frac{1}{2} \int_{-2\lambda}^{2\lambda} e^{\varepsilon t} \left(1 - \frac{|t|}{2\lambda}\right) \frac{h(t) - 1}{1 + \varepsilon + t^2} \, dt
\]
\[
= \frac{1}{2} \int_{0}^{\infty} (H(u) - 1) e^{-\varepsilon u} \, du \int_{-2\lambda}^{2\lambda} \left(1 - \frac{|t|}{2\lambda}\right) e^{\varepsilon(t-u)} \, dt
\]
\[
= \int_{0}^{\infty} H(u) e^{-\varepsilon u} \frac{\sin^2 \lambda (y-u)}{\lambda(y-u)^2} \, du - \int_{0}^{\infty} e^{-\varepsilon u} \frac{\sin^2 \lambda (y-u)}{\lambda(y-u)^2} \, du.
\]
\[
\varepsilon \to 0 \text{ ist links und im Subtrahendus rechts, also im Minuendus rechts unter dem Integralzeichen ausführbar.}
\]
\[
\frac{1}{2} \int_{-2\lambda}^{2\lambda} e^{\varepsilon t} \left(1 - \frac{|t|}{2\lambda}\right) \frac{h(t) - 1}{1 + t^2} \, dt
\]
\[
= \int_{0}^{\infty} H(u) \frac{\sin^2 \lambda (y-u)}{\lambda(y-u)^2} \, du - \int_{0}^{\infty} \frac{\sin^2 \lambda (y-u)}{\lambda(y-u)^2} \, du
\]
\[
= \int_{-\infty}^{\lambda y} H\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2} \, dv - \int_{-\infty}^{\lambda y} \frac{\sin^2 v}{v^2} \, dv.
\]

Bei \( y \to \infty \) strebt das Integral links als Fourierkonstante gegen 0, das letzte Integral gegen \( \pi \).

**Beweis des Satzes:** 1) Mit \( a = \sqrt{\lambda} \) ist
\[
\pi \geq \lim_{y \to \infty} \int_{-a}^{a} H\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2} \, dv \geq \lim_{y \to \infty} \int_{-a}^{a} H\left(y - \frac{a}{\lambda}\right) e^{-\frac{2\pi}{\lambda} \frac{\sin^2 v}{v^2}} \, dv,
\]
\[
\lim_{y \to \infty} H(y) \leq \pi e^{-\frac{2\pi}{\lambda}} \int_{-a}^{a} \frac{\sin^2 v}{v^2} \, dv = P_1(\lambda) \to 1.
\]

3) Gebraucht wird nur \( \pi > 0 \), nicht \( \pi = \text{Ludolfsche Zahl} \).
Über Dirichletsche Reihen.

2) Nach 1) ist \( H(y) \) beschränkt und zuletzt \( < 2 P_1(\lambda) \). Mit
\[
\beta = \frac{4}{\pi} P_1(\lambda) + \sqrt{\lambda}
\]
ist also
\[
\pi = \lim_{y \to \infty} \int_{-\infty}^{\infty} H\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2} dv
\]
\[
\leq 2 P_1(\lambda) \int_{-\infty}^{b} \frac{dv}{v^2} + \lim_{y \to \infty} \int_{-b}^{b} H\left(y + \frac{b}{\lambda}\right) e^{-\frac{b^2}{\lambda}} \frac{\sin^2 v}{v^2} dv + 2 P_1(\lambda) \int_{b}^{\infty} \frac{dv}{v^2},
\]
\[
\lim_{y \to \infty} H(y) \geq e^{-\frac{2b^2}{\lambda}} \left( 1 - \frac{4P_1(\lambda)}{\pi b} \right) = P_1(\lambda) + 1.
\]
We wish to prove

\[ \psi(x) - x = \sum_{\pm} \left( x^{\frac{1}{2}} \log \log \log x \right) \]

using the Ingham method from 1936, not the one in the book, pp. 92-100 (which follows Littlewood).

Ingham 1936 = Acta Arithmetica 1 (1936) 201-211

We use a variant in technique stressed by A. Selberg.

---

Since \( \psi(x) - x = \sum_{\pm} \left( x^{\Theta - \delta} \right) \) \( (\text{Ing 90}) \) à la Lec 21, we take \( \Theta = \frac{1}{2} \) wlog. If we assume RH,

Know:

1. \[ \sum \frac{1}{|n|^2} < \infty \quad \text{for } \xi(s) = \frac{\zeta(s/2)}{\zeta(s)} \xi(s) \] \( \text{Lec 13 p. 5} \)

2. \[ \psi_1(x) = \frac{x^2}{2} - \frac{\zeta'(0)}{\zeta(0)} x + B - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{2k(2k-1)} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} \]

   \[ B = \frac{\zeta'(-1)}{\zeta(-1)} \approx \text{unimportant} \quad x \geq 1 \] \( \text{Lec 16 p. 57} \)

\[ \text{Ing 73} \]
For $x \geq 1 + \delta_0$,

$$
\psi^*(x) = x - \frac{f'(0)}{f(0)} + \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} - \sum_{\rho} \frac{x^\rho}{\rho}
$$

with some reasonable conditional convergence on the $\rho$-sum over compact subsets of $[1+\delta_0, \infty)$.

Also:

$$
\frac{f'(0)}{f(0)} = \ln(2\pi) \quad 0
$$

$$
\psi^*(x) = \frac{\psi(x+0) + \psi(x-0)}{2}
$$

Let:

$$
E_1(x) = -\frac{f'(0)}{f(0)} x + B - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{2k(2k-1)}
$$

$$
E(x) = -\frac{f'(0)}{f(0)} + \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k}
$$

These fens are clearly $C^\infty$ and we have

$$
E = E_1
$$

Obviously

$$
\psi_1(x) - \frac{x^2}{2} - E_1(x) = -\sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)}
$$

$$
\psi^*(x) - x - E(x) = -\sum_{\rho} \frac{x^\rho}{\rho}
$$
DEF \ (modified \ remainder \ terms) \n
\[ p(x) = \psi^*(x) - x - E(x) \]
\[ p_1(x) = \psi_1(x) - \frac{x^2}{2} - E_1(x) \]

Notice that \( x + E(x) \) and \( \frac{x^2}{2} + E_1(x) \) are \( C^1 \) and that \( p \) and \( p_1 \) are piecewise \( C^1 \). In addition, \( p_1 \) is continuous.

FACT 1

For \( x \in \mathbb{R} \),

\[ p_1(x) = b_1 + \int_2^x p(t) \, dt \]

\( b_1 = \) some real constant.

PF

\[ \int_2^x p(t) \, dt = \int_2^x [\psi(t) - t - E(t)] \, dt \]

by def of \( \psi^* \)

and bary integrals

= \[ \int_2^x [\psi(t) - t - E_1(t)] \, dt \]

= constant + \( \psi_1(x) - \frac{x^2}{2} - E_1(x) \)

\( \begin{cases} \psi_1(x) = \int_1^x \psi(v) \, dv, & \psi(v) = 0 \text{ for } v < 2 \end{cases} \)
By (2) bottom, we also have:

\[ p_1(x) = - \sum_{\rho} \frac{x^{\frac{3}{2}+iy}}{(\frac{1}{2}+iy)(\frac{3}{2}+iy)} \]

\[ \rho = \frac{1}{2} + iy \] as usual.

The series is uniformly convergent on \([a, \infty)\) compacta.

**FACT 2** (generalization of Dirichlet’s “pigeon hole” principle) \(\leftarrow\) see Ing. 94

Let \(a_1, \ldots, a_N\) be real numbers. Let \(T_0\) and \(\delta_1, \ldots, \delta_N\) be positive. Then there exist integers \(x_j^*\) and a number \(t_0\) so that

\[ |t_0a_j^* - x_j^*| < \delta_j \] all \(j \in [1, N]\)

\[ T_0 \leq t_0 \leq T_0 \prod_{k=1}^{N} \left(1 + \frac{1}{\delta_k} \right) \]

The number \(t_0\) can be taken to be a multiple of \(T_0\).

Note:

\( \left\{ \begin{array}{c}
1 + \left\| \frac{1}{\delta_k} \right\| \leq 1 + \frac{1}{\delta_k} \\
\end{array} \right\} \)

\( \left\{ \right\} \) can be replaced if desired by \( \|x\| = \lim_{\epsilon \to 0} \|x - \epsilon x\| \). Useful for Ing. 94.
Classical pigeon hole principle:

\[ m + 1 \text{ things dumped into } m \text{ boxes} \]

\[ \Rightarrow \text{ some box contains } \geq 2 \text{ things} \]

Look at \([0,1)^N\). Partition this into

\[
\left( 1 + \left\lceil \frac{t}{s_j} \right\rceil \right) \times \ldots \times \left( 1 + \left\lceil \frac{t}{s_N} \right\rceil \right)
\]

subboxes (each semi-open \(s_j\) also disjoint). Note

\[ s_j > 1 \Rightarrow 1 + \left\lceil \frac{t}{s_j} \right\rceil = 1 \Rightarrow \text{ no action in coordinate } \#j. \]

Look at the \(m+1\) points

\[ (\tau_1, \ldots, \tau_N) \text{ mod } 1 \in [0,1)^N \]

For \(t = q \cdot T_0 \Rightarrow 0 \leq q \leq m\). Apply classical pigeon hole principle. Get obvious \(0 \leq q' < q'' \leq m\), \(t' < t''\)

\[
\left| t_{q''} - t_{q'} \right| \text{ integer } \leq \frac{1}{1 + \left\lceil \frac{1}{s_j} \right\rceil}
\]

each \(j\). But \(1 + \left\lceil \|u\| \right\rceil > u^* \) when \(u > 0\). Let

\[ t_0 = t'' - t' = (q'' - q')T_0 \text{. This works.} \]

\[ \left( \text{For } \|u\|, \text{ have } 1 + \|u\| \geq u^* \right) \]
To continue, we now define

$$F(v) = \alpha + \int_1^v \frac{p(e^u)}{\sqrt{e^u}} \, du$$

where $\alpha$ is a suitable constant (yet to be assigned).

Note

$$F(v) = \alpha + \int_e^v \frac{p(x)}{\sqrt{x}} \, \frac{dx}{x} \quad \left\{ \begin{array}{l}
X = e^u \\
u = \ln x
\end{array} \right\}$$

$$= \alpha + \int_e^v x^{-\frac{3}{2}} p(x) \, dx$$

$$= \alpha + \int_e^v x^{-\frac{3}{2}} \, dp_1(x) \quad \left( R-S \text{ style} \right)$$

$$= \alpha + x^{-\frac{3}{2}} p_1(x) \int_e^v - \int_e^v p_1(x) \left( -\frac{3}{2} \right) x^{-\frac{5}{2}} \, dx$$

$$= \alpha + b_2 + e^{-\frac{3}{2}v} p_1(e^v) + \frac{3}{2} \int_e^v p_1(x) x^{-5/2} \, dx$$

we propose to plug in $0$ at top

So, let's just look at $0$. 
\[
\begin{aligned}
&\frac{-3}{2} p_1(\rho) + \frac{3}{2} \int e^{-\frac{3}{2} + i \gamma} d\rho

= \sum_{\gamma} \frac{(-1)}{(\frac{1}{2} + i \gamma)(\frac{3}{2} + i \gamma)} \left[ \frac{-3}{2} \rho \frac{3}{2} + i \gamma \rho \right. \\
&+ \frac{3}{2} \int e^{-\frac{3}{2} + i \gamma} d\rho \left. \right]

\text{But}

\frac{3}{2} \int e^{-\frac{3}{2} + i \gamma} d\rho \frac{-\frac{3}{2}}{2} = -\int e^{-\frac{3}{2} + i \gamma} d\left(\rho^{-\frac{3}{2}}\right)

= -\left[ e^{-\frac{3}{2} + i \gamma} \rho^{-\frac{3}{2}} \right]_e^{(\text{parts})}

+ \int e^{-\frac{3}{2} + i \gamma} d\left(\rho^{-\frac{3}{2}}\right)

= -\rho^{-\frac{3}{2} + i \gamma}

+ e^{-\frac{3}{2} + i \gamma}

+ (\frac{3}{2} + i \gamma) \int e^{-\frac{3}{2} + i \gamma} d\rho
\end{aligned}
\]
\[= -\chi^3 \frac{3}{2} \chi^\frac{3}{2} + e^{i\gamma} \]
\[+ \left( \frac{3}{2} + i\gamma \right) \left[ \frac{\chi^i\gamma}{i\gamma} \right] e^{i\gamma} \]
\[\downarrow\]
\[\frac{(-1)}{(\frac{1}{2} + i\gamma)(\frac{3}{2} + i\gamma)} \left[ \text{big bracket on top} \right] \]
\[= \frac{(-1)}{(\frac{1}{2} + i\gamma)(\frac{3}{2} + i\gamma)} \left[ e^{i\gamma} + \left( \frac{3}{2} + i\gamma \right) \left( \frac{\chi^i\gamma}{i\gamma} - \frac{e^{i\gamma}}{i\gamma} \right) \right] \]
\[= \chi^i\gamma \frac{(-1)}{(i\gamma)(\frac{1}{2} + i\gamma)} + \frac{e^{i\gamma}}{(i\gamma)(\frac{1}{2} + i\gamma)} \]
\[\downarrow\]
\[\text{see } \bigcirc \text{ middle} \]
\[\downarrow\]
\[F(v) = \chi + b_3 - \sum_{\gamma} \frac{1}{(i\gamma)(\frac{1}{2} + i\gamma)} \chi^i\gamma \]
it is now natural to
\[ \overline{\text{declare}} \quad \nu = -b_3 \]
\[ \downarrow \]
\[ F(v) = -b_3 + \int_1^\nu \frac{\varphi(e^u)}{\sqrt{e^u}} \, du \]
\[ F(v) = -\sum_{\gamma} \frac{e^{iyv}}{(iv)(\frac{1}{2} + iv)} \]

The sum over \( \gamma \) is uniformly convergent for \( v \in \mathbb{R} \).

[Remember \( \gamma \in \mathbb{R} \).]

---

Recollection of baby Fourier analysis:

\[ \tilde{f}(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} \, dx \]

\[ \frac{F(x+o) + F(x-o)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(u) e^{iux} \, du \]

\[ \int_{-\infty}^{\infty} \max(0, b-1|x|) e^{-iux} \, dx = \left( \frac{\sin \left( \frac{u}{2} b \right)}{u/2} \right)^2 \]
\[
\max (0, b - |x|) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\sin \frac{\mu}{2} b}{\frac{\mu}{2}} \right)^2 e^{i\mu x} d\mu
\]

\[
\max (0, 1 - \frac{|x|}{2\pi}) = \int_{-\infty}^{\infty} \left( \frac{\sin \pi x}{\pi x} \right)^2 e^{i\pi x} d\mu
\]

\[
\max (0, 1 - \frac{|x|}{2\pi}) = \int_{-\infty}^{\infty} \left( \frac{\sin \pi x}{\pi x} \right)^2 e^{i\pi x} dx
\]

We let \( k(x) = \left( \frac{\sin \pi x}{\pi x} \right)^2 \). Thus,

\[
\hat{k}(u) = \max (0, 1 - \frac{|u|}{2\pi})
\]

Crucial Facts:

\( k(x) \geq 0 \) \quad \text{support of} \ k = [-2\pi, 2\pi]

\( k \in C^\infty(\mathbb{R}) \), \( \hat{k} \geq 0 \)

\( k, k' = O(x^{-2}) \) for \( |x| \geq 1 \).

\( \{ k(x) = \text{Fejer kernel} \} \)
Want to now consider:
\[
\int_{t-A}^{t+A} \frac{P(e^v)}{\sqrt{e^v}} k[N(v-t)] dv
\]

with
\[
\begin{cases}
A = \text{fixed positive integer}\; \delta \\
N = \text{positive integer (kept very large)}; \\
t = \text{real and very large}.
\end{cases}
\]

Recall
\[
F(v) = b v + \int_{1}^{v} \frac{P(e^u)}{\sqrt{e^u}} du \quad \text{box}
\]

Need:
\[
\int_{t-A}^{t+A} k[N(v-t)] dF(v)
= k[N(v-t)] F(v)]_{t-A}^{t+A} - \int_{t-A}^{t+A} F(v) k'[N(v-t)] N dv
\]
\[
\begin{cases}
\text{but } k(\pm NA) = 0
\end{cases}
\]
\[ -N \int_{t-A}^{t+A} k' \left[ N(N-t) \right] F(v) \, dv. \]  

We can now substitute

\[ F(v) = -\sum_{\gamma} \frac{e^{i\gamma v}}{(i\gamma)(\frac{1}{2}+i\gamma)} \]  

Do this manipulation term-by-term. Clearly there is no harm in taking \( \gamma > 0 \) and then getting the case \( \gamma < 0 \) by taking a conjugate.

Accordingly, with \( \gamma > 0 \), get:

\[ \frac{N}{(i\gamma)(\frac{1}{2}+i\gamma)} \int_{t-A}^{t+A} k' \left[ N(N-t) \right] e^{i\gamma v} \, dv \]

\[ \begin{cases} w = N(N-t), & v = t + \frac{w}{N}, & dv = \frac{dw}{N} \end{cases} \]

\[ = \frac{N}{(i\gamma)(\frac{1}{2}+i\gamma)} \int_{-NA}^{NA} k'(w) e^{i\gamma \left( t + \frac{w}{N} \right)} \, \frac{dw}{N} \]

\[ = \frac{e^{i\gamma t}}{(i\gamma)(\frac{1}{2}+i\gamma)} \int_{-NA}^{NA} k'(w) e^{i\frac{N}{w} w} \, dw \]

\[ O(w^{-2}) \text{ for } |w| \text{ large} \]
$$= \frac{e^{i\gamma t}}{(i\gamma)(\frac{1}{2} + i\gamma)} \left[ \int_{-\infty}^{\infty} k'(w) e^{i\frac{\gamma w}{N}} dw + O\left(\frac{1}{NA}\right) \right]. \tag{13}$$

Pause for a Second!

$$\int_{-\infty}^{\infty} k'(w) e^{i\frac{\gamma w}{N}} dw = \int_{-\infty}^{\infty} e^{i\frac{\gamma w}{N}} dk(w)$$
$$= -\int_{-\infty}^{\infty} k'(w) d_{w}(e^{i\frac{\gamma w}{N}})$$
$$= -i\frac{\gamma}{N} \int_{-\infty}^{\infty} k'(w) e^{i\frac{\gamma w}{N}} dw$$
$$= -i\frac{\gamma}{N} \max \left(0, 1 - \frac{1}{2\pi} \right) \tag{10}$$

We therefore have:

$$\sum_{\gamma > 0} \frac{e^{i\gamma t}}{(i\gamma)(\frac{1}{2} + i\gamma)} \left[ -i\frac{\gamma}{N} \max \left(0, 1 - \frac{1}{2\pi} \right) + O\left(\frac{1}{NA}\right) \right] + \text{CONJUGATE} \tag{12}$$
\[
= \sum_{0 < y < 2\pi N} \frac{e^{i\gamma t}}{i\gamma(\frac{1}{2} + i\gamma)} \left( \frac{-i\gamma}{N} \right) \left( 1 - \frac{y}{2\pi N} \right) \\
+ \sum_{\text{all } y} \frac{1}{y^2} O\left( \frac{1}{NA} \right) \\
+ \text{CONJUGATE} \\
= -\frac{1}{N} \sum_{0 < y < 2\pi N} \frac{e^{i\gamma t}}{\frac{1}{2} + i\gamma} \left( 1 - \frac{y}{2\pi N} \right) \\
+ \text{CONJUGATE} \\
+ O\left( \frac{1}{NA} \right)
\]

Here, note that

\[2 \text{Re}\left\{ \frac{e^{i\phi}}{\frac{1}{2} + i\gamma} \right\} = 2 \left\{ \frac{\frac{1}{2} \cos \phi + y \sin \phi}{\frac{1}{4} + y^2} \right\} \]

\[= \frac{\cos \phi + (2\gamma) \sin \phi}{\frac{1}{4} + y^2} \]

\[= \frac{\cos \phi + \cos(2\gamma) \sin \phi}{\frac{1}{4} + y^2} \]

\[= \frac{\cos \phi + (2\gamma) \sin \phi}{\frac{1}{4} + y^2} \]

\[= \frac{\cos \phi + (2\gamma) \sin \phi}{\frac{1}{4} + y^2} \]

\[= \frac{\cos \phi + (2\gamma) \sin \phi}{\frac{1}{4} + y^2} \]

\[= \frac{\cos \phi + (2\gamma) \sin \phi}{\frac{1}{4} + y^2} \]
So, we get

\[ \frac{1}{N} \sum_{0 < y < 2\pi N} \frac{\cos(yt) + 2\gamma \sin(yt)}{\frac{1}{4} + y^2} \left( 1 - \frac{y}{2\pi N} \right) + O\left( \frac{1}{NA} \right) \]

\[ \downarrow \]

\[ \int_{t-A}^{t+A} \frac{P(\nu)}{\sqrt{\nu^2}} k \left[ N(\nu - t) \right] d\nu \]

\[ = O\left( \frac{1}{NA} \right) \]

\[ - \frac{1}{N} \sum_{0 < y < 2\pi N} \frac{\cos(yt) + 2\gamma \sin(yt)}{\frac{1}{4} + y^2} \left( 1 - \frac{y}{2\pi N} \right) \]

\[ \text{THE FINITENESS OF THIS RANGE IS CRUCIAL.} \]

\[ \text{This is Ingham's Trick.} \]
Remember that "A" is fixed.

Also note that

\[
\frac{1}{N} \sum_{0 < y < 2\pi N} \frac{\cos(yt)}{\frac{i}{4} + y^2} \left(1 - \frac{y}{2\pi N}\right) = O\left(\frac{1}{N}\right).
\]

And:

\[
\frac{2y}{\frac{i}{4} + y^2} - \frac{2y}{y^2} = 2y \cdot O(y^{-4})
\]

\[
= O(y^{-3})
\]

\[
\downarrow
\]

\[
\frac{1}{N} \sum_{0 < y < 2\pi N} \left[ \frac{2y}{\frac{i}{4} + y^2} - \frac{2y}{y^2} \right] \sin(yt) \left(1 - \frac{y}{2\pi N}\right)
\]

\[
= O\left(\frac{1}{N}\right).
\]

\[
\downarrow
\]
\[ \int_{t-A}^{t+A} \frac{P(e^v)}{\sqrt{v}} k[N(v-t)] dv \]

\[ = O\left( \frac{1}{N} \right) - \frac{2}{N} \sum_{0 \leq \gamma < 2\pi N} \frac{\sin \gamma t}{\gamma} \left( 1 - \frac{\gamma}{2\pi N} \right) \]

\[ = \Theta \frac{O}{N} - \frac{2}{N} \sum_{0 \leq \gamma < 2\pi N} \frac{\sin \gamma t}{\gamma} \left( 1 - \frac{\gamma}{2\pi N} \right) \]

with \( \Theta \geq 1 \), \( |\Theta| \leq 1 \)

\[ \text{NOTE THAT the last sum can be taken over } 0 < \gamma < 2\pi N, \text{ if desired} \]

---

**Ingham 1936**

\[ T \]
\[ \eta \]
\[ \omega \]

**Here:**

\[ N \]
\[ A \]
\[ t \]

\[ w = N(v-t) \]
\[ v = t + \frac{w}{N} \]

See Ingham p. 207 eq (11).

Ingham's statement is thus effectively:

\[ \int_{-\eta}^{T\eta} G(\omega + \frac{w}{T}) \mathcal{K}(w) \frac{dw}{T} \]

\[ = O\left( \frac{1}{T} \right) - \frac{2}{T} \sum_{0 \leq \gamma < 2\pi T} \frac{\sin \gamma \omega}{\gamma} \left( 1 - \frac{\gamma}{2\pi T} \right) \]
As noted already on (top), we plan to use a variant of Ingham’s approach.

To continue, we now follow a direct approach with certain CHOICES. We’ll hide the rough calculations which motivated these!!

Also, we will not seek an optimal constant in

\[ \psi(x) - x = \Omega_\pm (x^{\frac{1}{2}} \log \log \log x) \]

on (1). That constant is conjectured to be arbitrarily large. [Ing, 2nd edition, p. xiv.]

Let \( G \) = a sufficiently large constant. We hold \( G \) frozen.

Apply \( \Theta \) FACT 2 to get:

\[ |t_0 \frac{\gamma_n}{2\pi} - \text{(integer)}| < \frac{\gamma_n}{2\pi G N} \]

\[ T_0 \leq t_0 \leq T_0 \prod_{0<\gamma_n \leq 2\pi N} \left( 1 + \frac{3\pi G N}{\gamma_n} \right) \]
We'll select $t_0$ in a few moments; it will be very big.

Get:

$$t_0 + \gamma_n = 2\pi (\text{integer}) + \Theta \frac{\gamma_n}{G_N}$$

$$0 < \frac{\gamma_n}{G_N} \Rightarrow \frac{2\pi N}{G_N} = \frac{2\pi}{G} < 10^{-6} \quad \text{(say)}.$$ 

$$G > 2\pi (10^6)$$

Put

$$h = \frac{1000}{G N}$$

Let

$$t_1 = t_0 + h \quad \text{à la Selberg}$$

(similarly, at\* end, consider $t_1 = t_0 - h$).

Clearly:

$$0 < h < 10^{-6} \quad (\text{since } N=\text{giant}).$$
\[ y_{nt_1} = y_{nt_0} + y_n h \]
\[ = 2\pi (\text{integer}) + \Theta \frac{y_n}{GN} + \frac{1000 y_n}{GN} \]

but

\[ \frac{999 y_n}{GN} \leq \Theta \frac{y_n}{GN} + \frac{1000 y_n}{GN} \leq \frac{1001 y_n}{GN} \leq \frac{2\pi (1001)}{G} < .002 \]

and

\[ \frac{999 y_n}{GN} = \frac{999}{1000} y_n (h) \]

\[ \left[ \text{mod } 2\pi \text{ part of } y_{nt_1} \right] \in \left[ \frac{999}{1000} y_n h > .002 \right) \]

(to = NOT useful)

\[ \sin(y_{nt_1}) \geq \sin\left[ \frac{999}{1000} y_n h \right] > 0 \]

KEY!!
$$\sin(y_{n+1}) \geq \frac{998}{1000} y_n h > 0$$

$$\frac{\sin(y_{n+1})}{y_n} \geq \frac{998}{1000} h > 0$$

$$\sum_{0 < y_n \leq 2\pi N} \frac{\sin(y_{n+1})}{y_n} \left(1 - \frac{y_n}{2\pi N}\right) \sim$$

as on 17 top

$$\geq \sum_{0 < y_n \leq 2\pi N} (0.998) h \left(1 - \frac{y_n}{2\pi N}\right)$$

$$= \sum_{0 < y_n \leq \pi N} (0.49) h$$

$$= (0.49) \frac{1000}{6\pi} \mathcal{N}(\pi N)$$

where $\mathcal{N}(v) = \#\{0 < y \leq v, \text{ with multiplicity } \}$

$$= \frac{v}{2\pi} \ln \frac{v}{2\pi} + O(N), \quad v \geq 2$$

\{ recall Lec 15 p. 29 \}
\[ N = \text{giant} \]
\[ \mathcal{N}(\pi N) = \frac{N}{2} \ln \frac{N}{2} + O(N) \]
\[ \mathcal{N}(\pi N) = \frac{N}{2} \ln N + O(N) \]
\[ \text{also}\ \mathcal{N}(2\pi N) = N \ln N + O(N) \]

\[ \sum_{0 < \gamma_n < 1} \frac{\sin(\gamma_n t_1)}{\gamma_n} \left(1 - \frac{\gamma_n}{2\pi N}\right) \]

\[ \geq 0.49 \left( \frac{1000}{G} \right) \left[ \frac{N}{2} \ln N + O(N) \right] \]

\[ \geq (240) \frac{1}{G} \ln N \]

\[ \text{rather crudely!} \]

\[ \text{at } t_1 \]

\[ B < \frac{\beta}{N} - (480) \frac{\ln N}{GN} \approx -450 \frac{\ln N}{GN} \]

\[ \text{for } N \text{ suff large, } G \text{ frozen, } B < \frac{\ln N}{G} \]
So, with our $t_0 \rightarrow t_1$ construction, it emerges that

$$\int_{t_1^- A}^{t_1 + A} \frac{P(e^v)}{\sqrt{\pi \sigma^v}} \exp \left[ N(v - t_1) \right] dv \leq - C \frac{\ln N}{N}$$

holds for all $N$ sufficiently large, with $C = \frac{450}{\sigma}$, i.e., some (fixed) appropriately small $C$.

For convenience, we now just declare

$$T_0 = \prod_{0 < y_n \leq 2\pi / N} \left(1 + \frac{2\pi\sigma N}{y_n} \right)$$

on p. 18 bottom. Recall too that $k(w) \geq 0$.

We seek to transform the $- C \frac{\ln N}{N}$ estimate into information about negative values of $P \sigma$. 
\[ T_0 \leq t_0 \leq T_0^2 \quad \text{on } \mathbb{R}^{+} \text{} \]

\[ T_0 \leq t_1 \leq 2T_0^2 \]

\[ \frac{2\pi G N}{\gamma_n} \leq 1 + \frac{2\pi G N}{\gamma_n} \leq \frac{4\pi G N}{\gamma_n} \quad \text{trivially} \]

\[ \prod_{0 < \gamma_n \leq 2\pi N} \frac{2\pi G N}{\gamma_n} \leq T_0 \leq T_0 \leq \prod_{0 < \gamma_n \leq 2\pi N} \frac{4\pi G N}{\gamma_n} \]

\[ (G) \prod_{0 < \gamma_n \leq 2\pi N} \frac{2\pi N}{\gamma_n} \leq T_0 \leq (G) \prod_{0 < \gamma_n \leq 2\pi N} \frac{9\pi N}{\gamma_n} \]

\[ \log(T_0) = \left[ N \log N + O(N) \right] \left[ \ln G + \Theta \ln 2 \right] \]

\[ + \sum_{0 < \gamma_n \leq 2\pi N} \ln \left( \frac{2\pi N}{\gamma_n} \right) \]

best written as \((\eta = \text{tiny})\)

\[ \int_\eta^{2\pi N} \log \left( \frac{2\pi N}{x} \right) dN(x) \]

\[ \xi(x) = 0 \quad \text{on } \mathbb{R} \]
\[
\sum_{0 < y_n \leq 2\pi N} \ln \left( \frac{2\pi N}{y_n} \right) \\
= 0 - 0 - \int_{\eta}^{2\pi N} g_2(x) \left( -\frac{i}{x} \right) \, dx \quad \text{by parts} \\
= \int_{\eta}^{2\pi N} \frac{g_1(x)}{x} \, dx \\
\int_{\eta}^{2\pi N} \frac{g_1(x)}{x} \, dx \sim \int_{\eta}^{2\pi N} \frac{1}{2\pi} \ln \frac{x}{2\pi} \, dx \\
= \int_{1}^{N} \ln y \, dy \\
= N \ln N - N + 1 \\
= N \ln N + O(N)
\]

\[
\sum_{0 < y_n \leq 2\pi N} \ln \left( \frac{2\pi N}{y_n} \right) = N \ln N + O(N)
\]

\[\Rightarrow \ln T_0 = N \log N \cdot (\ln G + \lambda) \text{ with some } \lambda \in [1, 2] \]

\[\Rightarrow \ln \ln T_0 \sim \ln N \text{ as } N \to \infty \]
Needless to say, as $x \to \infty$,

$$\ln \ln x^r \sim \ln \ln x \sim \ln \ln x^\beta$$

for any $0 < r < 1 < \beta$. This has relevance for line 1+2. E.g., $\ln \ln t_0 \sim \ln \ln t_1 \sim \ln \ln t_0$.

Looking at $t_1$ and BOX, let

$$M = \inf_{[t_1-A, t_1+A]} \frac{P(v)}{\sqrt{v}}$$

Get

$$g M \int_{t_1-A}^{t_1+A} k[N(v-t_1)] dv \leq -c \frac{\ln N}{N}$$

\{ $w = N(v-t_1), \; v = t_1 + \frac{w}{N}$ \}

$$g M \int_{-NA}^{NA} k(w) \frac{dw}{N} \leq -c \frac{\ln N}{N}$$

\{ here $A \geq 1$, N giant, \}$\int_{-\infty}^{\infty} k(w) dw = 1$ \} (cf. line 3)
\[ M \leq -C \ln N \]

\[ M \approx -\frac{99}{100} \ln \ln T_0 \quad \text{see 25 bottom} \]

but

\[ 0 < h < 10^{-6} \quad 19 \]

\[ A = \text{pos. integer} \quad 11 \quad \text{(fixed)} \]

\[ t_0 - 2A \leq t_1 - A \leq t_1 + A \leq t_0 + 2A \quad \text{crudely} \]

\[ \ln \ln \ln \left[ e^{t_1 - A} \right] \leq \ln \ln \ln \left[ e^{t_0 + 2A} \right] \]

\[ \ln \ln \ln \left[ e^{t_1 - A} \right] \quad \text{numerically asymptotic to} \]

\[ \ln \ln \ln \left[ e^{t_0} \right] \]

\[ = \ln \ln T_0 \]

by 26 top, get \quad \text{numerically asymptotic to}

\[ \ln \ln T_0 \]

\[ \{ \text{but see line 2 above} \} \Rightarrow \{ \text{yes!} \} \]
It follows that:

\[ \frac{P(x)}{\sqrt{x}} = N - \left\lfloor \ln \ln \ln x \right\rfloor \]

holds with a constant of, say, \(-\frac{98}{100}\).

One similarly establishes

\[ \frac{P(x)}{\sqrt{x}} = N + \left\lfloor \ln \ln \ln x \right\rfloor \]

using \( t_0 = h \) as \( t_1 \). See (9) second box!

On \((2 + 3)\),

\[ E(x) = O(1) \quad \text{for} \quad x \geq 2 \quad \text{(obviously)} \]

\[ P(x) = \Psi^*(x) - x - E(x) \quad \text{by def} \]

\[ P(x) = \Psi(x) - x + o(\ln x) \]

Thus:

\[ \Psi(x) - x = N_{\pm} \left( x^{\frac{1}{3}} \ln \ln \ln x \right) \]

OK
We won't bother to explain the corresponding results for 

\[ \Pi(x) - \Pi^*(x) \]

and, then, in regard to 

\[ \pi(x) - \pi^*(x) \]

See Ingham (book) p. 103 theorem 35. One also recalls Ingham, p. 90 theorem 32! The issue of making things effective [e.g., by tampering with the kernel function \( k(w) \)], which was highlighted on p. 105, is quite interesting and was the subject of some famous work by S. Skewes (late 1930s - early 1950s).

"The Skewes Number"
(e.g., see google.com)

Ingham, 2nd edition, page xiii references this. The Selberg trick is also touched on there...

See also the Hejhal-Odlyzko paper in the Turing centenary volume!!
In this lecture — the last lecture — I thought that it would be fun to return to Euler and the PNT.

At the very end, to keep things more current, I interjected a quick note about $S(T)$ [Lec 15, p. 26] which plays a role in ongoing computer tests of the Riemann Hypothesis.

**Euler First!**

Recall

$$
\mu(n) = \begin{cases} 
(-1)^r & n = p_1 \cdots p_r \text{, distinct } p_j \\
0 & n \text{ not squarefree}
\end{cases}
$$

$[\mu(1) = 1]$. See Lec 17, p. 14. We have:

$$
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \text{Re}(s) > 1
$$

$$
M(x) = \sum_{n \leq x} \mu(n) \quad x \geq 1, \ x \in \mathbb{R}
$$
Taking $s > 1$ and letting $s \to 1^+$, one would suspect (with Euler, 1748) that
\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0.
\]
Of course, the "rub" here is that the convergence of the LHS is far from obvious!

\textbf{NOTE.} If the LHS converges, its value must be 0. This follows immediately from Lec 21, p. 11, FACT 2 (b) \{with $s = \frac{\pi}{2}$\}; see also Lec 25, p. 3, on "super Stolz".

In Lec 20, p. 28, I pointed out that it was known very early being only ELEMENTARY techniques --- that the following statements are equivalent:

(i) $\Psi(x) \sim x$ \{i.e., the PNT\}
(ii) $M(x) = o(x)$
(iii) $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$

See Lec 20, pp. 33 - 40 for (i) $\Rightarrow$ (ii); then Lec 21
We now "finish" the job.

\[ \text{THM.} \]

\( (i)(ii) \iff (iii) \)

\[ \text{Proof} \]

Following Landau's Handbuch der Primzahlen, we set:

\[ g(x) = \sum_{n \leq x} \frac{\mu(n)}{n} \quad \text{if} \quad x \geq 1 \]

\[ f(x) = \sum_{n \leq x} \frac{\mu(n) \ln n}{n} \]

Recall that \( |g(x)| < 1 \) for \( x \geq 2 \); Lec 20 p. 37.

Assume \( (iii) \): Euler \( g(x) = o(1) \).

Need to show \( M(x) = o(x) \). Keep \( x \geq 2 \). Get:

\[ \sum_{n \leq x} \mu(n) = 1 + \int_1^x v d g(v) \]

\[ = 1 + [v g(v)]_1^x - \int_1^x g(v) d v \]
\[
\begin{align*}
\quad = & \quad 1 + xg(x) - i - \int_1^x g(v) \, dv \\
\quad = & \quad xg(x) - \int_1^x g(v) \, dv \\
\quad But \quad |f(t)| \leq \varepsilon \quad for \quad t \geq T_\varepsilon. \quad For \quad x \geq T_\varepsilon, \quad we \quad now \quad get:\n\quad |M(x)| \leq && \varepsilon x + \int_1^T_\varepsilon |g(v)| \, dv + \int_{T_\varepsilon}^x \varepsilon \, dv \\
\quad \leq && \varepsilon x + O(T_\varepsilon) + \varepsilon x \\
\quad = && 2\varepsilon x + O(T_\varepsilon) \\
\quad \downarrow \\
\quad \limsup_{x \to \infty} \frac{|M(x)|}{x} \leq && 2 \varepsilon \\
\quad \downarrow \\
\quad M(x) = o(x). \quad \bigcirc \bigcirc \quad \text{OK}
\end{align*}
\]

\text{NEXT, ASSUME (i) (ii) \quad Must prove \quad g(x) = o(1).}

To achieve this, we require 2 lemmas.
Lemma 1

\[ q(x) = \frac{f(x)}{\ln x} = O\left(\frac{1}{\ln x}\right) \quad x \geq 2 \]

(\text{So } q(x) = o(1) \iff f(x) = o(\ln x))

**Proof**

Recall that \( \sum_{d|N} \mu(d) = \begin{cases} 1 & N = 1 \\ 0 & N > 1 \end{cases} \) (lec 20 p. 29).

Then that

\[ 1 = \sum_{n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor \]

via \( \sum \mu(n) \) and the hyperbola method ala lec 20 p. 37. This idea can be generalized using

\[ 1 = \sum_{n \leq x} \frac{\mu(n)}{n} \left\{ \frac{1}{k} \right\} \leq \sum_{N \leq x} \frac{1}{N} \left\{ \sum_{n|N} \mu(n) \right\} \]

\[ = \sum_{n \leq x} \frac{\mu(n)}{n} \left[ \ln \left(\frac{x}{n}\right) + \gamma + O\left(\frac{n}{x}\right) \right] \]

lec 18 p. 40 bottom
\[ \sum_{n \leq x} \frac{\mu(n)}{n} (\ln x - \ln n) \]
\[ + \gamma \sum_{n \leq x} \frac{\mu(n)}{n} \]
\[ + O(1) \sum_{n \leq x} \frac{1}{n} \frac{n}{x} \]
\[
\Downarrow \]
\[ 1 = (\ln x) g(x) - \sum_{n \leq x} \frac{\mu(n) \ln n}{n} + \gamma g(x) + O(1) \]
\[ |g(x)| \leq 1 \]
\[
\Downarrow \]
\[ j = (\ln x) g(x) - f(x) + O(1) \quad \text{by (3)} \]
\[
\Downarrow \]
\[ (\ln x) g(x) - f(x) = O(1) \]
\[
\Downarrow \]
\[ g(x) - \frac{f(x)}{\ln x} = O\left(\frac{1}{\ln x}\right) \quad \Box \]

**Must now seek to prove** \( f(x) = o(\ln x) \).
Lemma 2

For \( x \geq 2 \), we have:

\[
\sum_{n \leq x} \frac{1}{n} g\left(\frac{x}{n}\right) = 1
\]

pf

On 6 middle, we saw that

\[
1 = \sum_{nk \leq x} \frac{\mu(n)}{nk}
\]

\{sum the other direction!\}

\[
= \sum_{k \leq x} \frac{1}{k} \left\{ \sum_{u \leq \frac{x}{k}} \frac{\mu(u)}{u} \right\}
\]

\[
= \sum_{k \leq x} \frac{1}{k} g\left(\frac{x}{k}\right) \quad \text{(by def of } g\text{)}
\]
Recall that

\[-\mu(N) \ln N = \sum_{k \leq N} \frac{\Lambda(k) \mu(k)}{k} \]

by Lec 20 p. 33. Accordingly

\[f(x) = \sum_{N \leq x} \frac{\mu(N) \ln N}{N} \quad \text{\textit{\`a la 3}}\]

\[-= \sum_{k \leq x} \frac{\Lambda(k) \mu(k)}{k} \]

\[\{ \text{think hyperbolas} \}\]

\[-= \sum_{k \leq x} \frac{\Lambda(k)}{k} \left\{ \sum_{l \leq x/k} \frac{\mu(l)}{l} \right\} \]

\[f(x) = -\sum_{k \leq x} \frac{\Lambda(k)}{k} g\left(\frac{x}{k}\right) \]

recall

\[\Lambda(1) = 0\]

\[\Lambda(p^m) = \ln p, \quad m \geq 1\]

Integration by parts with \(\psi(x)\) is cumbersome since \(g(x)\) is not continuous. Avoid it!!
To prove $f(x) = o(ln x)$ as $x \to \infty$, there is no loss of generality in taking $x = \text{integer}$. See 3.

We have

$$\Psi(v) = \sum_{n \leq v} \Lambda(n)$$

and $\Psi(v) = 0$, $v < 2$. Since we have assumed (i), (ii), it makes sense to write

$$\Psi(v) = v + vE(v), \quad v > 0$$

with

$$\lim_{v \to \infty} E(v) = 0$$

and

$$E(v) \equiv -1 \quad \text{for} \quad 0 < v < 2.$$  

For convenience, declare:

$$E(0) = 0.$$

We still have $\Psi(0) = O + O E(0)$.

Choose $M$ so that

$$|E(v)| \leq M, \quad \text{all} \quad v \geq 0.$$
In the box, notice that \( x = \text{integer} \)

\[
\sum_{k \leq x} \frac{A(k)}{k} \psi \left( \frac{x}{k} \right)
\]

\[
= \sum_{k \leq x} \psi(k) - \psi(k-1) \frac{k}{k} \psi \left( \frac{x}{k} \right)
\]

\[
\sum_{1 \leq k \leq x} \frac{1 + k \varepsilon(k) - (k-1) \varepsilon(k-1)}{k} \psi \left( \frac{x}{k} \right)
\]

\[
= \sum_{k=1}^{x} \frac{1}{k} \psi \left( \frac{x}{k} \right)
\]

\[
+ \sum_{k=1}^{x} \left[ \varepsilon(k) - \varepsilon(k-1) \right] \frac{k}{k} \psi \left( \frac{x}{k} \right)
\]

\[
+ \sum_{k=1}^{x} \frac{\varepsilon(k-1)}{k} \psi \left( \frac{x}{k} \right)
\]

\[
= 1 \quad \text{by lemma 2 on } \textcircled{7}
\]

\[
+ \sum_{k=1}^{x} \varepsilon(k) \psi \left( \frac{x}{k} \right) - \sum_{k=2}^{x} \varepsilon(k-1) \psi \left( \frac{x}{k} \right)
\]

\[
+ \sum_{k=2}^{x} \frac{\varepsilon(k-1)}{k} \psi \left( \frac{x}{k} \right)
\]

\[
\varepsilon(0) = 0
\]
\[ \begin{align*}
&= 1 + \sum_{k=1}^{x} \varepsilon(k) g\left(\frac{x}{k}\right) - \sum_{\ell=1}^{x-1} \varepsilon(\ell) g\left(\frac{x}{\ell+1}\right) \\
&\quad + \sum_{\ell=1}^{x-1} \frac{\varepsilon(\ell)}{\ell+1} g\left(\frac{x}{\ell+1}\right) \\
&= 1 + \sum_{k=1}^{x} \varepsilon(k) g\left(\frac{x}{k}\right) - \sum_{\ell=1}^{x} \varepsilon(\ell) g\left(\frac{x}{\ell+1}\right) \\
&\quad + \sum_{\ell=1}^{x} \frac{\varepsilon(\ell)}{\ell+1} g\left(\frac{x}{\ell+1}\right) \\
\text{IF WE DECLARE} \\
g(v) = 0 \text{ for } v < 1 \\
\text{THEN} \\
&= 1 + \sum_{k=1}^{x} \varepsilon(k) \left[ g\left(\frac{x}{k}\right) - g\left(\frac{x}{k+1}\right) \right] \\
&\quad + \sum_{\ell=1}^{x} \frac{\varepsilon(\ell)}{\ell+1} g\left(\frac{x}{\ell+1}\right) \\
(-f(x)) &= 1 + \sum_1 + \sum_2, \text{ say.}
\end{align*} \]

Fix any tiny \( \varepsilon > 0 \). Let \( |\varepsilon(k)| \leq \varepsilon \) for all \( k \geq K_\varepsilon + 1 \). Keep \( x \geq K_\varepsilon + 100 \).
Notice that:

\[ \left| \Sigma_2 \right| \leq \sum_{k=1}^{K_E} \frac{g(n)}{k+1} \cdot 1 \]

\[ + \sum_{K_E+1 \leq k \leq x} \frac{\varepsilon}{k+1} \cdot 1 \]

\[ = C_E + \varepsilon \ln x + O(1) \]

\[ \leq C'_E + \varepsilon \ln x \]

To estimate \( \Sigma_1 \), notice that

\[ g \left( \frac{x}{k} \right) - g \left( \frac{x}{k+1} \right) = \sum_{\frac{x}{k+1} < n \leq \frac{x}{k}} \frac{\mu(n)}{n} \]

by \( \text{(3)} \) and \( \text{(11)} \) lines 5 + 6. Accordingly,

\[ \left| g \left( \frac{x}{k} \right) - g \left( \frac{x}{k+1} \right) \right| \leq \begin{cases} 2 \\ \leq \sum_{\frac{x}{(k+1)x} \leq \frac{x}{k}} \frac{1}{n} \end{cases} \]
In the foregoing, \( \mathbb{Z} \cap \left( \frac{x}{k+1}, \frac{x}{k} \right] \) may well be empty. Empty sums are zero.

Continue to keep \( x \geq K_\varepsilon + 100 \). By def.

\[
\sum_1 = \sum_{k=1}^{x} \varepsilon(k) \left[ g\left(\frac{x}{k}\right) - g\left(\frac{x}{k+1}\right) \right].
\]

Hence:

\[
|\sum_1| \leq \sum_{1 \leq k \leq K_\varepsilon} \varepsilon M \cdot 2
\]

\[
+ \sum_{K_\varepsilon + 1 \leq k \leq x} \varepsilon \left( \sum_{\frac{x}{k+1} \leq \frac{x}{k}} \frac{1}{n} \right)
\]

The union over \( k \) with \( \left( \frac{x}{k+1}, \frac{x}{k} \right] \) is disjoint.

\[
\leq 2 \varepsilon M K_\varepsilon + \varepsilon \sum_{n \leq \frac{x}{K_\varepsilon + 1}} \frac{1}{n}
\]
\[ \leq 2M \kappa_\varepsilon + \varepsilon (\ln x + O(1)) \]

\[ \downarrow \]

\[ |\Sigma_1| \leq C''_{\varepsilon} + \varepsilon \ln x \]

Upon reviewing boxes 8, 10, and 12 (middle), and line 2 above, we see that

\[ |F(x)| \leq 1 + C''_{\varepsilon} + 2\varepsilon \ln x \]

for \( x \geq K_\varepsilon + 100 \). Since \( \varepsilon > 0 \) was arbitrary on 11, we get \( f(x) = o(\ln x) \), as required on 10 bottom. By lemma 1, on 5, we therefore have \( g(x) = o(1) \), as needed.

This completes the proof of THM on 3.
Finally, a very brief comment about $s(T)$.

We recall that:

$$s(T) = \frac{1}{\pi} \arg \int \frac{1}{T} \log \frac{1}{t} \, dt$$

'la lec 15 (26) (middle). Cf. also here lec 15 pp. 22 - 25 and lec 27 (3) (FACT) - (6) (top).

When $T \neq \infty$, we have

$$s(T) = -\frac{1}{\pi} \text{Im} \int_{\frac{1}{2}}^{\infty} \frac{1}{T} \log (\sigma + iT) \, d\sigma \ .$$

If $T = \infty$, one defines $s(T)$ by right continuity to preserve lec 15 (26) (lines 6+7).

One knows that:

$$s(T) = O(\ln T) \ .$$
Define \( \text{Log } S(s) \) in the usual up and across way beginning at point \( A \in \mathbb{R} \), \( A > 1 \).
Keep \( T \geq 2 \), \( T \) odd \( \gamma \), and \(-1 \leq \sigma \leq 2 \).
We then have:

\[
\text{Log } S(s) = \mathcal{O}(\ln T) + \sum_{\rho} \text{Log} (s - \rho) \quad \text{if} \quad 1 \gamma - \tau \leq 1
\]

Here \( s = \sigma + iT \) and \( \text{Log} = \) the standard principal value.

\[ \text{pf} \]

Review Lec 27 (bot) \( \oplus \) \( \ominus \) and Lec 15 (bot) \( \ominus \).

\[
\text{Log } S(s) = \mathcal{O} + \int_{\text{Re}} \frac{1}{T} (s + iT) \text{du}
\]

\[
= \int_{\text{Re}} \mathcal{O}(\gamma - u) \text{du} + \int_{\text{Re}} \frac{1}{\gamma} (s + iT) \text{du}
\]

\[
= \mathcal{O}(1) + \int_{\text{Re}} \left[ \mathcal{O}(\ln T) + \sum_{\rho} \frac{1}{u + iT - \rho} \right] \text{du}
\]

\[ \text{Lec } 15 \text{ (8)} \]
\[ = O(1) + O(\ln T) \]
\[ + \sum_{\nu} \frac{\log(s-\rho) - \log(2+i(T-\rho))}{\nu-\tau} \]
\[ \nu - \tau \leq 1 \]
\[ = O(\ln T) + \sum_{\nu} \log(s-\rho) \]
\[ \nu - \tau \leq 1 \]
\[ \{ \text{since } 0 \leq \Re(\rho) \leq 1 \} \]

For \( \nu \neq 0 \) and \( \nu \neq 0 \) it is customary to define:
\[ S_{\nu}(s) = -\frac{1}{\pi} \Re \int_{-\frac{1}{2}}^{\frac{1}{2}} (s - \frac{1}{2}) \frac{\pi}{\nu} (\nu + i\tau) d\nu \]

For \( \nu = 0 \), use right continuity (cf. page 18 box).

It is easy to see that \( S_{\nu}(s) \) is well-defined and nicely continuous insofar as \( T \) remains away from all \( \gamma \) on lec 15, pp. 22 (bot) - 23 (top).
Notice here that:

\[ S_1 = \frac{1}{\pi} \text{Re} \int_{\frac{1}{2}}^{\infty} (\sigma - \frac{1}{2}) \, d \log I(\sigma + iT) \]

\[ = -\frac{1}{\pi} \text{Re} \left[ 0 - 0 - \int_{\frac{1}{2}}^{\infty} \log I(\sigma + iT) \, d\sigma \right] \]

\[
\left\{ \begin{array}{l}
\text{by integ by parts and } \\
\log I(s) = O(2^{-\sigma}) \quad \sigma \geq 2
\end{array} \right\}
\]

\[ \downarrow \]

\[ S_1(T) = \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \ln |I(\sigma + iT)| \, d\sigma \]

This is for \( T \neq \text{all } \gamma \). By baby analysis and

\[ \int_{0}^{1} |\ln u| \, du < \infty \]

the box remains true for all \( T \geq 2 \) (i.e., there is good continuous behavior).
Notice further (for \( T \neq \text{all } \gamma \)) that:

\[
S_1(T) = \frac{1}{\pi} \text{Re} \int_{\frac{1}{2}}^{\infty} \log f(\sigma+iT) \, d\sigma - O(2^{-\sigma}) \quad \sigma \geq 2
\]

\[
\Rightarrow S_1'(T) = \frac{1}{\pi} \text{Re} \int_{\frac{1}{2}}^{\infty} \frac{\frac{d}{dx} f(\sigma+iT)}{f(\sigma+iT)} \, d\sigma \quad \{ \text{Leibnitz's rule} \}
\]

\[
= -\frac{1}{\pi} \text{Im} \int_{\frac{1}{2}}^{\infty} \frac{\frac{d}{dx} f(\sigma+iT)}{f(\sigma+iT)} \, d\sigma
\]

\[
= S(T) \quad \text{by (15) middle}
\]

\[
\{ \text{with good uniform convergence} \}
\]

\[
\text{locally wrt } T
\]

\[
S_1'(T) = S(T) \quad T \neq \text{all } \gamma
\]

\[
S_1(T) = c_1 + \int_{\frac{1}{2}}^{T} S(u) \, du \quad \text{all } T \geq 2
\]

\[
(\text{continuous wrt } T)
\]
THM (very fundamental)

In the zero-counting formula

\[ N(T) = \frac{T}{2\pi} \ln \left( \frac{T}{2\pi e} \right) + \frac{T}{8} + O\left(\frac{1}{T}\right) + S(T) \]

\[ \text{as } T \to \infty \]

à la Lec 15 (22) + (26), we have

\[ S(T) = O(\ln T) \quad \text{and} \quad \int_2^T S(u) \, du = O(\ln T). \]

\[ \underline{\underline{\text{Proof:}}} \]

Only the last assertion remains to be proved.

WLOG \( T \neq \infty \) all \( y \). Apply (18) box. Get:

\[ S_1(T) = \frac{1}{\pi} \int_2^T \frac{\ln |5(y+it)|}{\ln |3(y+it)|} \, dy + \frac{1}{\pi} \int_{\frac{T}{2}}^{2} \frac{\ln |3(y+it)|}{\ln |y+it|} \, dy \]

\[ \leq O(2^{-\sigma}) \]

\[ = O(1) + \frac{1}{\pi} \int_{\frac{T}{2}}^{2} \frac{\ln |5(y+it)|}{\ln |y+it|} \, dy \]
\[ = O(1) + \frac{1}{\pi} \int_2^1 \left[ O(\ln T) + \sum_{\gamma} \ln |u + i\tau - \gamma| \right] du \]

\[ \text{by p.} \text{[10]} \text{THM} \]

\[ \text{observe too that we have} \]
\[ |u - \beta| \leq 1 (u - \beta) + i(\tau - \gamma) | \leq 3 \]

\[ = O(1) + O(\ln T) + O(\ln T) \]

\[ \Rightarrow O(\ln T) \]

It is hardly surprising that the implied constants for \( \ln T \) in p. [10] THM can be made explicit.

The relation \( \int_2^1 S(u) \, du = O(\ln T) \) qualitatively states that the average value of \( S(u) \) is zero.
These last 2 points are important. In the early 1950s, Alan Turing used these facts to develop a numerical criterion (now known as Turing’s Law) by which the Riemann Hypothesis on interval \([T_1, T_2]\) can be checked simply by locating enough (that is to say, a requisite number of) sign-changes for the real-valued function

\[\Xi\left(\frac{1}{2} + it\right) \text{ or, better,} \quad \frac{G\left(\frac{1}{2} + it\right)}{G\left(\frac{1}{2}\right)} \cdot \frac{1}{\Gamma\left(\frac{1}{2}\right)}\]

in an interval slightly bigger than \([T_1, T_2]\). \(*\)

Recall Lec 23, p. 8 lines 1-3 and (c); Lec 15, p. 26 (line -3).

Of course, there is nothing to guarantee in advance that the requisite number will be found. One simply has to try!!!

The point here is 4-fold:

(A) if the requisite number is reached (by the machine), one is assured by Turing’s theorem that all zeros are accounted for, and that there are none having \(\text{Re}(s) = \frac{1}{2}\).

(B) there is no need to check RH for \(T < T_1\) first.

\[\ast\] with special attention paid to the pattern near \([T_1, T_2]\).
(c) there is no need to compute any \( s \)-values with \( \text{Re}(s) \neq \frac{1}{2} \).

(d) there is no need to "zero in" on the crossings in \([T_1, T_2]\) attached to the sign-changes detected by the machine.

To understand why Turing's Law is at least believable, simply pretend that one somehow knew that \( S(t) \) was exactly zero in some short intervals centered at \( T_1 \) and \( T_2 \). See p. 20 THEOREM [the formula for \( N(T) \)] and ponder the logical consequences which ensue.

For further details about Turing and \( S(t) \), see the paper of Hejhal and Odlyzko in the Turing Centenary volume "Alan Turing: His Work and Impact," published by Elsevier. The story is a VERY interesting one. \(^{\leftrightarrow}\) with links to Lec 29 p. 29 and S. Skewes.

Turing's Law is used in all modern computational work aimed at verifying the RH. When the approach is successful, the zeros of \( \zeta \) in the range \([T_1, T_2]\) are also known to be simple. \(^{\leftrightarrow}\) C.F. (A) on 22.
An alternate ending for Lec 30 ("skip Turing, stay with Euler").

After developing (3) - (14) in Lec 30, go back to Euler's Opera Omnia (e.g., vol. 8, p. 271) and say:

We now adopt the idea of Newman's general theorem on page 9 of Lec 28 — motivated by Newman's Amer Math Monthly article cited on p. 10.

Take \( \text{Re}(z) > 0 \) initially and define

\[
g_N(z) = \sum_{n=1}^{N} \frac{\mu(n)}{n^{1+z}}
\]

\[
g(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1+z}} = \frac{1}{\zeta(1+z)} \quad \text{if} \quad (1+iy) \neq 0
\]

Form analytic continuations. Keep \( N \) and \( R \) large; use exactly the same path as on p. 10 of Lec 28 (it depends on \( R \)). Get

\[
g(0) - g_N(0) = \frac{1}{2\pi i} \oint_C (g - g_N) N^z \left( 1 + \frac{zR}{R^2} \right) \frac{dz}{z}
\]

---

* This is the comment referred to on p. 24 of Lec 28.
\[
\begin{align*}
&= \frac{1}{2\pi i} \int_{C_+} (g - g_N) N^x \left( 1 + \frac{\xi^2}{R^2} \right) \frac{d\xi}{\xi} \\
&\quad - \frac{1}{2\pi i} \int_{C_-} g_N(\xi) N^x \left( 1 + \frac{\xi^2}{R^2} \right) \frac{d\xi}{\xi} \\
&\quad + \frac{1}{2\pi i} \int_{C_-} g(\xi) N^x \left( 1 + \frac{\xi^2}{R^2} \right) \frac{d\xi}{\xi}.
\end{align*}
\]

On \( C_+ \), observe that:

\[
|g(\xi) - g_N(\xi)| = \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1+\xi}} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\xi}} \\
\leq \int_N^\infty u^{-1-x} du = \frac{N^{-x}}{x}.
\]

\[
|N^x| = N^x,
\]

\[
\left| 1 + \frac{\xi^2}{R^2} \right| = \left| 1 + \frac{\xi^2}{R^2} \right| = \frac{2|x|}{R},
\]

\[
\left| \frac{1}{\xi} \right| = \frac{1}{R},
\]

\[
|d\xi| = R d\theta \quad (\xi = Re^{i\theta}).
\]
Hence,

\[
\left| \frac{1}{2\pi i} \int_{C_R} (g - gn) N^2 \left( 1 + \frac{z^2}{R^2} \right) \frac{dz}{z} \right|
\]

\[
\leq \frac{1}{2\pi} \int_{C_R} \frac{N^{-x}}{x} N^x \frac{2x}{R} \frac{Rd\theta}{R}
\]

\[
= \frac{1}{R}.
\]

Exactly as on p. 13 of Sec 28, we have

\[
\frac{1}{2\pi i} \int_{C} g_n(z) N^2 \left( 1 + \frac{z^2}{R^2} \right) \frac{dz}{z}
\]

\[
= \frac{1}{2\pi i} \int \left[ \sum_{|z| \geq R} \right] \frac{dz}{z}.
\]

(Left half of \(|z| = R\))

Along \(|z| = R, x < 0\), we have:

\[ |g_N(x)| = \left| \sum_{1}^{N} \frac{\delta_c(x)}{u^2 + \frac{1}{x}} \right| \]

\[ \sum_{1}^{N} \frac{1}{u^2 + \frac{1}{x}} = \sum_{1}^{N} \frac{1}{u^2 + \frac{1}{x}} \]

\[ = \left\{ \frac{N}{N}, |x| < 1 \right\} + \int_{1}^{N} \frac{1}{u^2 + \frac{1}{x}} \, du \]

\[ = \frac{N}{|x|} + \frac{N}{|x|} \]

\[ |N^x| = N^x = N^{-|x|} \]

\[ \left| 1 + \frac{\bar{z}^2}{R^2} \right| = \left| \frac{\bar{z}^2 - |z|^2}{\bar{z}^2} \right| = \frac{2|z|}{R} \]

\[ \left| \frac{1}{\bar{z}} \right| = \frac{1}{R} \]

\[ 1 d\bar{z} = R \, d\theta \]

So,

\[ \left| \frac{1}{2\pi i} \int_{C} g_N(z) N^x \left( 1 + \frac{\bar{z}^2}{R^2} \right) \frac{dz}{\bar{z}} \right| \]

\[ = \frac{1}{2\pi} \int_{C} N^{|x|} \left( \frac{1}{N} + \frac{1}{|x|} \right)^N \frac{2|z|}{R} \frac{R \, d\theta}{R} \]

(\text{semicircle})
\[ \begin{align*}
\int_{\text{semicircle}} \left( \frac{1}{N + \frac{1}{|x|}} \right)^{\frac{1}{2}} \frac{1}{R} \, d\theta &= \frac{1}{\pi} \int_{\text{semicircle}} \left( \frac{1}{N + \frac{1}{R}} \right) \, d\theta \\
\end{align*} \]

Finally, imitate Lec 28 pages 15+16. Remember that \( R \) and \( \delta \) are held fixed. Get:

\[ \lim_{N \to \infty} \frac{1}{2\pi i} \int_{C_-} g(z) N^z \left( 1 + \frac{z^2}{R^2} \right) \frac{dz}{z} = 0. \]

Conclude that:

\[ \lim_{N \to \infty} \sup_{z} |g(z) - g_N(z)| = \frac{1}{R} + O + \frac{1}{R} + O. \]

Since \( R \) is arbitrary, deduce that

\[ \lim_{N \to \infty} \sup_{z} |g(z) - g_N(z)| = 0. \]

IE: \[ \lim_{N \to \infty} g_N(0) = g(0). \]
But $g(0) = 0$ since $\zeta(1 + \tau)$ has a simple pole at $\tau = 0$. Hence,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$$

and Euler was right!!

Thanks to part I (i.e. 0-14) of Lec 30, this proves the PNT!

"VERY NICE INDEED!"

---

**NOTE:** clearly, the same argument utilized above adapts to show that

$$\sum_{n=1}^{\infty} \mu(n) n^{-1-i\tau} = \frac{1}{\zeta(1+i\tau)}$$

for $\tau \neq 0$. (Just declare $a_n = \mu(n)n^{-i\tau}$.)
Addendum B

[regarding Lecture 28]

Newman's proof is clearly very slick. One seeks to understand the GENERAL THM on page 9 better—especially its genesis.

In regard to the genesis issue, one is inevitably forced to keep things a bit speculative. With this in mind, it is helpful to back up and recall several very basic facts from Fourier analysis (on a fundamentally formal level).

Given appropriately decaying \( f \) on \([0, \infty)\).

Let

\[ \mathcal{L}(s) = \int_0^\infty e^{-sx} f(x) \, dx = \text{Laplace transform} \]

To get the inversion formula, note that

\[
\mathcal{L}(k+it) = \int_0^\infty \left[ e^{-kx} f(x) \right] e^{-itx} \, dx
\]

\[
= \int_{-\infty}^\infty e^{-kx} \left[ \frac{f(x)}{0} \right] e^{-itx} \, dx
\]

\[
e^{-kx} \left[ \frac{f(x)}{0} \right] = \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{L}(k+it) e^{itx} \, dt \quad (x \neq 0)
\]
\[
\begin{bmatrix}
  f(x) \\
  0
\end{bmatrix}
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{L}(k+it) e^{(k+it)x} \, dt
\]

\[
\begin{bmatrix}
  F(s) \\
  0
\end{bmatrix}
= \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \mathcal{L}(s) e^{sx} \, ds
\]

\[
F(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \mathcal{L}(s) e^{sx} \, ds, \quad x > 0.
\]

This is the familiar formula typically \( k > 0 \). For suitable \( f \), one can reason just as well with \( k = 0 \). In both cases, the \( s \)-integral is technically a principal value with \([-R, R]\) and \( R \to \infty \). See Lec 25 p. 5 middle.

Let \( h(x) \) be \( C^1 \) and appropriately decaying. Note that

\[
\int_0^\infty h'(x) e^{-sx} \, dx = \int_0^\infty e^{-sx} \, dh(x)
\]

\[
= 0 - h(0) + s \int_0^\infty h(x) e^{-sx} \, dx
\]

\[
\mathcal{L}_h(s) = \frac{h(0) + \mathcal{L}_h(s)}{s}.
\]
Assume now that our original \( F \) satisfies

\[
(A) \quad \int_{0}^{\infty} f(u) \, du = 0.
\]

Putting

\[
h(x) = \int_{0}^{x} f(u) \, du
\]

and observing that

\[
h(x) - \int_{0}^{\infty} f(u) \, du = \int_{0}^{x} + \int_{x}^{\infty} = 0,
\]

we deduce that

\[
h(0) = 0, \quad h'(x) = f(x) \quad \text{for } a.e. \quad [a, e_{0}]
\]

\[
\mathcal{L}_{h}(s) = \frac{1}{s} \mathcal{L}_{f}(s)
\]

\[
h(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \mathcal{L}_{h}(s) e^{sx} \, ds
\]

\[
h(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{1}{s} \mathcal{L}_{f}(s) e^{sx} \, ds
\]

\[
\int_{0}^{\infty} f(u) \, du = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{1}{s} \mathcal{L}_{f}(s) e^{sx} \, ds
\]

\[
\left( \text{the RHS being a } [R, \infty) \right)
\]

\[
\text{principal value}
\]
For suitable \( f \), we can reason just as well with \( k = 0 \), thus getting

\[
\int_0^x f(u) \, du = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{X(s)}{s} e^{sx} \, ds
\]

under \((\ast)\).

In alternate notation,

\[
\int_0^T f(u) \, du = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{X(s)}{s} e^{sx} \, ds
\]

\[
\int_0^T f(u) \, du = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{X(z)}{z} e^{zT} \, dz
\]

\[
\downarrow
\]

\[
\lim_{R \to \infty} \int_0^T f(u) \, du = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{-iR}^{iR} \frac{X(z)}{z} e^{zT} \, dz
\]

\[
\int_0^T f(u) \, du = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{X(z)}{z} e^{zT} \omega \left( \frac{u}{R} \right) \, dz
\]

wherein \( \omega(u) = \chi_{[-1,1]}(u) \), \( u \in \mathbb{R} \).
The foregoing format [in the box] is highly suggestive given that it is a time-honored trick in Fourier transform theory to replace the earlier [[even]] \( \omega \) with other "more interesting" choices.

The case of Fejér summability corresponds, for instance, to taking \( w(y) = \max\{0, 1-|y|\} \); see Lec 25 p. 10.

The essential point is that keeping \( w(0) = 1 \) and \( |\omega| = O(1) \) guarantees that

\[
\lim_{R \to \infty} \int_{-R}^{R} F(y) w\left( \frac{y}{R} \right) \, dy = \int_{-\infty}^{\infty} F(y) \, dy
\]

for every \( F \in L_1(\mathbb{R}) \). Indeed, let \( |\omega| \leq M \) and \( |\omega(u) - 1| < \epsilon \) for \( |u| < \delta \). Then:
\[ \left| \int_{-\infty}^{\infty} F(y) \, dy - \int_{-\infty}^{\infty} F(y) \, \omega \left( \frac{y}{R} \right) \, dy \right| \]

\[ \leq \int_{|y| \geq R} |F(y)|(1 + M) \, dy + \int_{|y| < R} |F(y)| E \, dy \]

\[ \leq (1 + M) \int \left| F(y) \right| \, dy + E \int_{|y| \leq R} \left| F(y) \right| \, dy. \]

Inspired by the Fejér case, it is more-or-less mandatory to observe that for any sensible \( \omega \), we have:

\[ \int_{-\infty}^{\infty} f_1(y) \, \tilde{f}_2(y) \, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F_1(u) \tilde{F}_2(u)}{u} \, du \]

\{ see Lec 25 pp. 58 \}

\[ \int_{-\infty}^{\infty} L(y) \, \omega \left( \frac{y}{R} \right) \, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{L}(u) R \tilde{\omega} \left( -Ru \right)}{u} \, du \]

\{ but \( \omega \) is even \}

\[ \int_{-\infty}^{\infty} L(y) \, \omega \left( \frac{y}{R} \right) \, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{L}(u) R \tilde{\omega} \left( Ru \right) \, du \]
\[
\int_{-\infty}^{\infty} L(y) \omega\left(\frac{y}{R}\right) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{L}(\frac{v}{R}) \tilde{\omega}(v) dv
\]

whereupon the expression \( \int_{R} L(y) \omega\left(\frac{y}{R}\right) dy \) with given \( L \in L_1(\mathbb{R}) \) is again seen to converge (as \( R \to \infty \)) to

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{L}(0) \tilde{\omega}(v) dv = \tilde{L}(0) \omega(0) = \int_{-\infty}^{\infty} L(y) dy
\]

(here)

We've switched to \( L \) (in lieu of \( F \)) to be generally suggestive of \( \mathcal{H} \) box. Issues with \( 2\pi \) are ignored.

For \( \max \{0, 1-|y|\} \), one knows that \( \tilde{\omega}(v) = \left( \frac{\sin \frac{v}{2}}{\frac{v}{2}} \right)^2 \)

by Lec 25 (\( l \)) line 4.

The Fourier transform of \( \max \{0, 1-y^2\} \) is slightly more complicated, \( \text{viz.} \)

\[
\frac{H}{V^2} \left( \frac{\sin v}{v} - \cos v \right) = \frac{H}{\sqrt{3}} \int_{0}^{v} (\xi \sin \xi) d\xi
\]
We stress that ALL of the foregoing is just rudimentary, completely classical Fourier transform theory!

In the Landau paper from 1932 attached to Sec 28, it is nearly self-evident that a Fejér type \( w(u) \) is lurking in the background. In (B7) (box), (B8) (lines 1+2), and Landau 526 lines 6-11.

Being aware of this, and guided by a concomitant desire to exploit the Cauchy integral theorem in the counterpart of (B4) (box), it stands to reason that an \( w \) which is analytic prior to "turning off" is best.

One can guess that this thought motivated Newman's choice of \( w(u) = \max \{ 0, 1-\frac{u^2}{R^2} \} \).

**Bottom line**: the \( e^{\frac{it}{\varepsilon}} (1 + \frac{\varepsilon^2}{R^2}) \frac{1}{\varepsilon} \) is thus completely natural in (B4) (box).
It is worthwhile at this juncture to quickly record the counterpart of all this for

\[ \mathcal{M}(s) = \int_1^\infty x^{-s} dA(x) \]

where \( A(s) = 0 \) and, for instance, \( A(x) = \int_1^x \phi(v) dv \).

One gets:

\[ \mathcal{M}(s) = -\int_1^\infty \frac{A(x)}{x^{s+1}} dx \quad (\text{for } s > 1) \]

\[ \frac{\mathcal{M}(s)}{s} = \int_0^\infty \frac{A(e^u)}{e^{us}} du \]

\[ \frac{\mathcal{M}(k+it)}{k+it} = \int_0^\infty e^{-ku} A(e^u) e^{-itu} du \]

\[ \frac{\mathcal{M}(k+it)}{k+it} = \int_{-\infty}^0 e^{-ku} \left[ \begin{array}{c} A(e^u) \\ 0 \end{array} \right] e^{-itu} du \]

\[ e^{-ku} \left[ \begin{array}{c} A(e^u) \\ 0 \end{array} \right] = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\mathcal{M}(k+it)}{k+it} e^{itu} dt \]

\{ [-R, R] principal value \}

\[ \left[ \begin{array}{c} A(e^u) \\ 0 \end{array} \right] = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\mathcal{M}(k+it)}{k+it} e^{(k+it)u} du \]

\[ \left[ \begin{array}{c} A(e^u) \\ 0 \end{array} \right] = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{\mathcal{M}(s)}{s} e^{su} ds \]
\[ A(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{\mathcal{M}(s)}{s} x^s \, ds, \quad x > 1 \]

(the RHS being a \([\text{-}R, R]\) principal value)

This is a very familiar formula (typically having \( k > 1 \)). Indeed: recall PERRON'S FORMULA in Lec 19, pp. 12(bot) - 13(top). (Also Lec 17, 10(top).

Again, under certain hypotheses, it will be possible to proceed with \( k = 1 \). The situation is clearly analogous to \( \text{BY}(\text{box}) \). One is thus led to the expression

\[ \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{\mathcal{M}(z)}{z} \left( \frac{y}{R} \right)^z \, dz, \quad y > 1 \]

with the possible need therein to have "engineered matters" so as to have a removable singularity at, say, \( z = 1 \).

Compare: Landau 526 line 11.
During Lec 28, I mentioned that Newman’s General Theory could be strengthened rather easily. This was already indicated on pp. 17–19 (top) of Lec 28.

To bring matters into a still better form, two elementary lemmas are required.

Let $D = \{ |z| < 1, y > 0 \}$, $K = \{ |z| \leq 1, y \geq 0 \}$. Let $\Gamma = \partial D$ (counterclockwise) and

$$\Gamma_\varepsilon = \Gamma - (-\varepsilon, \varepsilon),$$

[Here $0 < \varepsilon < 1$.]

Let $\log w = \ln |w| + i\text{Arg}(w)$, with $-\pi < \text{Arg}(w) \leq \pi$.

**Lemma 1**

Let $F(\varepsilon)$ be continuous on $K$ and analytic on $D$. We then have

$$\frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \frac{F(\varepsilon)}{z} \, dz = F(0).$$
\textbf{Proof}

Notice that

\[
\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \oint_{C_\varepsilon} \frac{F(0)}{z} \, dz = \lim_{\varepsilon \to 0} \frac{F(0)}{2\pi i} \left[ \log (-\varepsilon) - \log (\varepsilon) \right] = \lim_{\varepsilon \to 0} \frac{F(0)}{2\pi i} \pi i = \frac{F(0)}{2}.
\]

Our task is to check that

\[
0 = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \oint_{C_\varepsilon} \frac{F(z) - F(0)}{z} \, dz.
\]

As such, we might as well simply hypothesize \( F(0) = 0 \) at the outset. We do so by the extended Cauchy integral theorem,

\[
\int_{C_\varepsilon} \frac{F(z)}{z} \, dz + \int_{-\varepsilon}^{\varepsilon} \frac{F(z)}{z} \, dz = 0.
\]

But,

\[
\left| \int_{C_\varepsilon} \frac{F(z)}{z} \, dz \right| \leq \max_{C_\varepsilon} |F| \cdot \int_{-\varepsilon}^{\varepsilon} \left| \frac{dz}{z} \right| = \pi \max_{C_\varepsilon} |F|.
\]
Hence,

\[ \left| \int_{\Gamma_{\varepsilon}} \frac{F(z)}{z} \, dz \right| \leq \pi \max_{\gamma_{\varepsilon}} |F| . \]

Since \( F(0) = 0 \), we have \( \max_{\gamma_{\varepsilon}} |F| \to 0 \) as \( \varepsilon \to 0 \).

**Lemma 2**

Let \( F(z) \) be continuous on \( K \) and analytic on \( D \). Assume that \( F(0) = 0 \) and that

\[ \int_{-1}^{1} \frac{|F(x)|}{x} \, dx < \infty . \]

We then have

\[ 0 = \frac{1}{2\pi i} \int_{\Gamma} \frac{F(z)}{z} \, dz . \]

in a [natural] Lebesgue integral sense.

**Proof**

That \( \frac{1}{2\pi i} \int_{\Gamma} \frac{F(z)}{z} \, dz \) exists as a Lebesgue integral is obvious. But, then, by a standard specialization,
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{F(z)}{z} \, dz = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\Gamma_n} \frac{F(z)}{z} \, dz = 0,
\]

thanks to lemma 1.

**IMPROVED THEOREM.**

Newman's General Theorem on p. 10 of Lee 28 is actually valid anytime the function \( g(s) \) on \( \{ \operatorname{Re}(s) > 0 \} \) admits a continuous extension to \( \{ \operatorname{Re}(s) \geq 0 \} \) having the additional property that

\[
\int_{-1}^{1} \left| \frac{g(\xi t) - g(0)}{t} \right| \, d\xi < \infty.
\]

**Pf.**

We first claim that one can take \( g(0) = 0 \), wlog. Suppose, e.g., that \( v = 1 \) is a point of continuity of \( g \). Let \( A = g(0) \neq 0 \) and define

\[
F_1(v) = \begin{cases} 
  f(v) - A & 0 < v < 1 \\
  f(v) & v \geq 1
\end{cases}.
\]

The \( \text{fcn} \) \( F_1 \) is still bounded + piecewise continuous on \( [0, \infty) \). We get: \( \{ \text{for \, \operatorname{Re}(s) > 0 \text{ \, initially} \} \)
\[ g_1(s) = g(s) - A \int_0^1 e^{-sv} dv \]

\[ = g(s) - AE(s) \quad E(s) \equiv \frac{1 - e^{-s}}{s} \]

The function \( E \) is entire and equals \( 1 + O(s) \) near \( s = 0 \).

Clearly, \( g_1(0) = 0 \) and \( g_1 \) is continuous for all \( \Re(s) \geq 0 \).

For \( |t| \leq 1 \), notice that

\[ |g(it) - g(0) - g_1(it)| = |A| |E(it) - 1| \]

\[ = O(|it|) \]

Hence,

\[ \int_{-1}^{1} \left| \frac{g_1(it)}{t} \right| dt < \infty \iff \int_{-1}^{1} \left| \frac{g(it) - g(0)}{t} \right| dt < \infty. \]

Switching to \( \xi = g_1, \xi_1 \) and establishing \( 0 = \int_0^\infty f_1(v) dv \)
will thus lead to \( 0 = -A + \int_0^\infty f_1(v) dv \), which is exactly what we want.

From this point on, we assume \( g(0) = 0 \).

Fix any large \( R \). Let \( \Pi \) be the (counterclockwise) path \( \xi|\xi| = R \), \( x > 0 \) \( U \xi = iy, -R \leq y \leq R \). Introduce arcs as shown:

\[ \Pi = C + U C_0 \]

\[ \text{Diagram: } \begin{array}{c}
\text{C} \quad \text{C}_+ \\
\text{C}_- \quad \text{C}_0
\end{array} \]
Keep $T$ big. Write

$$g_T(z) = \int_0^T e^{-2TV} f(v) dv$$

as usual. By a trivial variant of Lemma 21, we have:

$$0 = \oint \frac{g(z) e^{Tz} \left( 1 + \frac{z^2}{R^2} \right)}{z} \, dz$$  \(\text{as a Lebesgue integral}\)

The $g_T(z)$ is entire by the Cauchy integral formula, we know:

$$g_T(0) = \frac{1}{2\pi i} \int_{C_+} g_T(z) e^{Tz} \left( 1 + \frac{z^2}{R^2} \right) \, \frac{dz}{z}$$

$$+ \frac{1}{2\pi i} \int_{C_-} g_T(z) e^{Tz} \left( 1 + \frac{z^2}{R^2} \right) \, \frac{dz}{z}$$

We propose to subtract \((\star\star)\) from this equation for $g_T(0)$. 

\[ g_T(0) = \frac{1}{2\pi i} \int_{C} \left[ g_T(z) - g(z) \right] e^{\frac{Tz}{R^2}} \frac{dz}{z} \]

\[ + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{g(z)}{z} e^{\frac{Tz}{R^2}} \frac{dz}{z} \]

\[ + \frac{1}{2\pi i} \int_{C} g_T(z) e^{\frac{Tz}{R^2}} \frac{dz}{z} \]

\[ \equiv J_1 + J_2 + J_3 \]

The estimations for \(|J_1|\) and \(|J_3|\) go exactly like before.

\[ |J_1| \ll \frac{B}{R} \quad (10) \text{hot} \sim (12) \text{ in Lec 28} \]

\[ |J_3| \ll \frac{B}{R} \quad (13) \text{(middle)} \sim (14) \text{ in Lec 28} \]

To estimate \(|J_2|\), we write
\[ |J_2| = \frac{1}{2\pi} \int_{-R}^{R} \frac{g(iy)}{y} e^{i\gamma T \left(1 - \frac{y^2}{R^2}\right)} dy \]

Note the Lebesgue integrability of \( g(iy)/y \), and then apply the standard Riemann-Lebesgue lemma. See Lec 7 p. 22.

\[ \lim_{T \to \infty} |J_2| = 0, \quad \text{each } R \]

If \( J_2 = o(1) \), akin to Lec 28 p. 19 (line 2).

It follows that:

\[ \limsup_{T \to \infty} \left| g_T(0) \right| = \limsup_{T \to \infty} \left| J_1 + J_2 + J_3 \right| \]

\[ \leq \frac{B}{R} + 0 + \frac{B}{R} = \frac{2B}{R} \]

akin to Lec 28 p. 19 (line -3). Since \( R \) is arbitrary, \( \lim_{T \to \infty} g_T(0) = 0 = g(0) \) and we are done.
After completing this proof, it pays to step back and note how the correctness of (B4) (box) for a large class of functions $f_j$ together with (B5) (top) + (B8) (lines 4–8), clearly engender a kind of “moral encouragement” that a limit theorem like Newman's General Thm [or B14] might well prove feasible on a relatively simple technical level.

The miraculous cancellations that appeared "along the way" are perhaps best viewed in this light.
Closing Comments

[Assorted remarks, revisions, augmentations for Lecs 1-30]

Part I (Lecs. 1-16)

1. (Lec 1, p. 1) “The primary reference for these lectures will be Ingham, Distr. of Prime Numbers, from 1932.” This sentence somehow failed to make its way into the record of Lec 1. ②

2. (Lec 1, p. 18) The fact that \( \Psi(x) = \Theta(x) + O(x^{\frac{1}{2}}) \) is important and probably should have been highlighted via some kind of box.


On page 8 (bottom half), it is important to note that the stated integration-by-parts formula holds equally well when \( f \) is merely continuous and \( \text{piecewise } C^1 \) on \( [a, b] \). Compare Lec 11, ① bot-

② middle.
4. (Lec 5, p. 9) In regard to formula #1 on this page, I should have paused to note that letting $z \to 1$ immediately gives

$$\sum_{n=1}^{N} \frac{1}{n} = 1 + \ln N - \int_{1}^{N} \frac{r(t)}{t^2} \, dt$$

$\Rightarrow$ [Euler constant] $\gamma = 1 - \int_{1}^{\infty} \frac{r(t)}{t^2} \, dt$

$\Rightarrow$ $\sum_{n=1}^{N} \frac{1}{n} - \gamma = \ln N + \int_{N}^{\infty} \frac{r(t)}{t^2} \, dt$

hence

$$\sum_{n=1}^{N} \frac{1}{n} = \ln N + \gamma + \int_{N}^{\infty} \frac{r(t)}{t^2} \, dt$$

$$\sum_{n=1}^{N} \frac{1}{n} = \ln N + \gamma + O\left(\frac{1}{N}\right)$$

In formula #2 on p. 9, application of the above equation for $\gamma$ quickly leads to

$$f(z) - \gamma = \frac{1}{z-1} - \left\{ -z \int_{1}^{\infty} \frac{r(t)}{t^{z+1}} \, dt - \int_{1}^{\infty} \frac{r(t)}{t^2} \, dt \right\}$$

The brace is simply $H(z) - H(1)$ with

$$H(z) = \int_{1}^{\infty} \frac{r(t)}{t^{z+1}} \, dt$$
The function \( H(z) \) is analytic on \( \{ x > \delta > 0 \} \) as already noted. Accordingly, on page 10 (top) after introducing \( F \), one can simultaneously assert that \[ f(z) = \frac{1}{z-1} + y + O(z^{-1}) \text{ near } z = 1. \]

Compare Lec 18 (40) – (41).

5. (Lec 6, p. 10 top) This subtraction trick is the "\( z = \bar{z} \)" counterpart of what was just obtained for \( \sum_{n=1}^{N} \frac{1}{n} \) in item \#4. Its importance can thus be said to have been recognized very early on.

6. (Lec 8, p. 14) Taking \( s = \frac{1}{1+t} \) and \( N \to N-1 \) in E-M version I leads to:

\[
\left\{ \begin{array}{c}
\sum_{n=1}^{N} \frac{1}{n} = \frac{1}{2} + \frac{1}{2N} + \ln N - \int_{s}^{N} \frac{\beta(t)}{t^2} \, dt \\
\beta(t) \equiv t - \|t\| - \frac{1}{2}
\end{array} \right. \]

This agrees with item \#4 above.
7. (Lee 9, p. 21) Regarding Euler and the special values \( J(-2k) = 0, J(2k), J(0), J(1-2k) \) for \( k \geq 1 \), it is very illuminating to actually have a look in Euler's collected works (e.g.,

Leonhardi Euleri Opera Omnia, Series I, vol. 14, pp. 73-86 (1734);
114 (§19) (1736);
424-434, 440-443 (1740);
477-479 (1750).

vol. 15, pp. 72-78 (1749).

One keeps in mind here the function \( \phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{-s}) \Gamma(s) \). The paper by R. Ayoub, "Euler and the Riemann Zeta Function," Amer. Math. Monthly 81 (1974), 1067-1086 is also very worthwhile, as is A. Weil's, "Prehistory of the Zeta-Function" in Number Theory, Trace Formulas, and Discrete Groups (ed. by K. Aufbret et al.), Acad. Press, 1989, pp. 1-96.

8. (Lee 11, p. 24) Concerning the functional equation
\[
\xi(s) = \xi(1-s)
\]
and the alternate version
\[
J(1-s) = 2(2\pi)^{-s} \cos \left( \frac{\pi s}{2} \right) \Gamma(s) J(s)
\]
noted in LeC 16, p. 7 (top), a look in Euler is
again very revealing (Cf., e.g.,)

vol. 14 p. 443 (1740) \\
vol. 15 pp. 79–79 (1749) \\
131–138 (1772)

Also of interest:

vol. 16 (part 2, preface) pp. XXVIII, XXXI, \\
LXXXII – LXXXV.

and the aforementioned works by R. Ayoub and A. Weil.

The extent to which Euler more-or-less “stumbled on” the
functional equation already around 1740–1749 is
striking indeed!!

Remember Riemann is ≈ 1859.

9. (Lec11, p. 19) It is worthwhile to show how a bare bones
form of Poisson summation follows nearly immediately from
Euler–Maclaurin version I (lec 8, p. 19).

Given any $\varphi \in C^1(\mathbb{R})$ such that $\varphi \in L_1(\mathbb{R})$, $\varphi' \in L_1(\mathbb{R})$.

We then have:

(a) $\varphi \to 0$ as $x \to \pm \infty$

(b) $\sum_{n \geq 0} |\varphi(x+n)|$ conv uniformly on $\mathbb{R}$-compacta

(c) $\sum_{n \geq 0} \varphi(x+n) = \lim_{N \to \infty} \frac{N}{x} \sum_{k=1}^{N} \varphi(k) e^{2\pi i k x}$ each $x \in \mathbb{R}$.

* The TRICK will be very similar to pp. 3 (bottom) – 4 (line 8) of Lec11.
Since \( q \in L_1(\mathbb{R}) \), clearly \( \lim_{x \to \infty} \inf |q(x)| = 0 \). Since \( q \in L_1(\mathbb{R}) \), \( q(y) \) is uniformly Cauchy as \( y \to \infty \). Hence \( \lim_{x \to \infty} q(x) = 0 \). The case \( x \to -\infty \) is similar. \( \Rightarrow (a) \) OK

Notice that \( \int_0^1 \left( \sum_{n=0}^{\infty} |q(x+n)| \right) \, dx < \infty \). Hence \( \sum_{n=0}^{\infty} q(x+n) \)

conv absolutely almost everywhere on \([0, 1]\). \( \Rightarrow \) \( g \) is continuous at point \( x_0 \). Consider any \( x_1 \in [0, 1] \) with, say, \( x_1 > x_0 \). (The case \( x_1 < x_0 \) will be similar.) For \( N \geq M \) large,

observe that:

\[
\sum_{M}^{N} |q(x_1+n)| = \sum_{M}^{N} |q(x_0+n) + \int_{x_0}^{x_1} q'(v+n) \, dv|
\]

\[
\leq \sum_{M}^{N} |q(x_0+n)| + \sum_{M}^{N} \int_{x_0+n}^{x_1+n} |q'(w)| \, dw
\]

\[
\left\{ \begin{array}{c}
\text{but the intervals } [n, n+1] \text{ are non-overlapping} \\
\text{and } x_0+n \in [n, n+1]
\end{array} \right\}
\]

\[
\leq \sum_{M}^{N} |q(x_0+n)| + \int_{M}^{\infty} |q'(w)| \, dw.
\]

Negative \( N \) and \( M \) are treated similarly; the "new" \( w \)-integral will be

\[
\int_{-\infty}^{-N+1} |q'(w)| \, dw.
\]
This proves (b) on \( K = [0,1] \). By virtue of (a), we then get (b) on a general \( K \).

To prove (c), a standard translation shows that \( x = 0 \) wlog. Let \( M \) be big. We have:

\[
\frac{1}{2} \varphi(-M) + \sum_{|n| < M} \varphi(n) + \frac{1}{2} \varphi(M) = \int_{-M}^{M} \varphi(x) \, dx + \int_{-M}^{M} \varphi'(x) \varphi(x) \, dx
\]

\[
\{ \beta(x) = x - \|x\| - \frac{1}{2} \geq 0 \}
\]

Let \( M \to \infty \) remembering that \( 1/\beta(x) \leq \frac{1}{2} \). Get:

\[
\sum_{-M}^{M} \varphi(n) = \varphi(0) + \int_{-M}^{M} \varphi'(x) \beta(x) \, dx
\]

Let \( S_N(x) \) be the usual partial sum for \( \beta(x) \); recall the bounded convergence properties of \( S_N \). Hence:

\[
\lim_{N \to \infty} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{0} \varphi'(x) \beta(x) \, dx \right| \, dx = 0
\]

\[
\int_{-\infty}^{\infty} \varphi'(x) \beta(x) \, dx = \lim_{N \to \infty} \sum_{k=1}^{N} \int_{-\infty}^{\infty} \varphi'(x) \frac{\sin(2\pi k x)}{\pi k} \, dx
\]

When \( k \geq 1 \), we immediately check via (a) and integration by parts that
\[
\int_{-\infty}^{\infty} \frac{\sin(2\pi kx)}{-\pi k} \, d\hat{\varphi}(x) = \frac{1}{\pi k} \int_{-\infty}^{\infty} \varphi(x) (2\pi k) \cos(2\pi kx) \, dx
\]
\[
= \int_{-\infty}^{\infty} \varphi(x) \left[ e^{2\pi ikx} + e^{-2\pi ikx} \right] \, dx
\]
\[
= \hat{\varphi}(k) + \hat{\varphi}(-k) \quad .
\]

Hence:
\[
\sum_{n=-\infty}^{\infty} \varphi(n) = \hat{\varphi}(0) + \lim_{N \to \infty} \left\{ \sum_{1 \leq |k| \leq N} \hat{\varphi}(k) \right\}
\]
\[
= \lim_{N \to \infty} \sum_{-N}^{N} \hat{\varphi}(k) \quad .
\]

10. (Lec 16, p.7) Once the 2nd box was obtained on p. 7, had it not been in a rush, it would have been useful to stop for a moment and obtain

\[
\frac{\Gamma(0)}{\Gamma(0)} = \ln(2\pi)
\]

by letting \(s \to 0\). See Lec 18 (41) (42) and item #4 above.

Notice incidentally that

\[
\text{Res}_{s=0} \left[ \frac{x^{s+1}}{s(s+1)} \left( \frac{\Gamma(s)}{\Gamma(s)} \right) \right] = \left( \frac{\Gamma(0)}{\Gamma(0)} \right) x
\]

\[\text{for } p=1 \text{ in Lec 16}\]
Part II (Lec. 17–30)

110. (Lec 17+18, pp. 28–35) A slight improvement can be made on pp. 28(top) – 30(bottom) with very little effort. Recall that $x \geq 1 + \delta$, $1 < c \leq 2$, $T \geq 3$ as on pp. 10 + 11(top). Consider those $\beta$ for which $T - 1 \leq \gamma < T$

On p. 28(bot), it is preferable to replace $C$ by a new path $\tilde{C}$ which changes with $\beta$.

\[ \rho = \beta + i \gamma \]

\[ |x - (\beta + i \tau)| = \varepsilon \]

\[ \tilde{C} = C_1 \cup C_2 \]

- semicircular part
- linear part

Keep $0 < \varepsilon = \frac{c-1}{y} \leq \frac{1}{4}$. Notice that

\[ \beta + \varepsilon \leq 1 + \varepsilon < 1 + \frac{c-1}{2} = \frac{\varepsilon + 1}{2} < 3 \]
On \( C_1 \), clearly \(|s - \rho| \geq |s - (\beta + i \tau)| = \varepsilon \) by elementary geometry. On \( C_2 \), clearly \(|s - \rho| \geq |\sigma - \beta|\).

Observe that:

\[
\left| \frac{1}{2\pi i} \int_{C_1} \frac{x^s}{s(s-\rho)} \, ds \right| \leq \frac{1}{2\pi} \int_{C_1} \frac{x^\sigma}{|s(s-\rho)|} \, |ds| \\
\leq \frac{1}{2\pi} \int_{C_1} \frac{x^\sigma}{(\tau-1) \varepsilon} \, |ds| \\
= \frac{1}{2\pi} \frac{x^{\sigma+\varepsilon}}{(\tau-1)} = O\left(\frac{1}{T}\right) x^c.
\]

At the same time,

\[
\left| \frac{1}{2\pi} \int_{C_2} \frac{x^s}{s(s-\rho)} \, ds \right| \leq \frac{1}{2\pi} \int_{C_2} \frac{x^{\sigma_0}}{(\tau-1)|\sigma - \beta|} \, d\sigma_0 \\
\leq \frac{1}{2\pi \varepsilon (\tau-1)} \int_{C_2} x^{\sigma_0} \, d\sigma_0 \\
= \frac{1}{2\pi \varepsilon (\tau-1)} \int_{-\infty}^{\infty} x^{\sigma_0} \, d\sigma_0 \quad (x > 1) \\
= \frac{1}{2\pi \varepsilon (\tau-1)} \frac{x^c}{\ln x} \\
= O\left(\frac{1}{T}\right) \frac{x^c}{\varepsilon \ln x}. 
\]
Accordingly

\[ \frac{1}{2\pi i} \int_{\mathcal{C}_p} \frac{x^s}{s(s-\rho)} \, ds = O\left(\frac{1}{T}\right) X^c \left[ 1 + \frac{1}{x \ln x} \right]. \]

The case \( T < y \leq T + 1 \) is addressed similarly via

\[ \mathcal{C}_p \rightarrow \begin{array}{c}
-1 + it \\
\mathcal{C}_p \end{array} \rightarrow c + it. \]

As such, p. 30 line 4 now becomes

\[ O\left(\frac{\ln T}{T}\right) X^c \left[ 1 + \frac{1}{x \ln x} \right] \]

(provided \( \varepsilon \) is kept independent of \( \rho \)); line 7 becomes

\[ O\left(\frac{\ln T}{T}\right) X^c \left[ 1 + 1 + \frac{1}{x \ln x} \right] \]

and, lastly, p. 30 (bottom) becomes

\[ O\left(\frac{\ln T}{T}\right) X^c \left[ 1 + \frac{1}{x \ln x} \right]. \]
On pp. 31-32, we can now make the replacement

$$O\left(\frac{\ln T}{T}\right) x^c e^{c-1} \rightarrow O\left(\frac{\ln T}{T}\right) x^c \left[1 + \frac{1}{\ln x}\right].$$

Note that the term $O\left(\frac{x^c \ln x}{T(c-1)}\right)$ is still present, minimization of this term leads to

$$c = 1 + \frac{b}{\ln x} \quad (b = \text{tiny constant}).$$

As far as $\varepsilon$ goes, it is natural to put

$$\varepsilon = \frac{c-1}{4} = \frac{b}{4 \ln x}.$$ (clearly)

The error term on p. 33 (bottom) thus becomes

$$O\left(\frac{x \ln T}{T}\right) + O\left(\frac{x \ln^2 x}{T}\right) + O(\ln x) \min\left\{1, \frac{x}{T(x)}\right\}.$$ The implied constants will depend solely on $\delta_0$.

On p. 35, in the statement of the explicit formula, we finally reach:
\[ \psi^*(x) = x - \sum_{1 \leq r \leq T} \frac{x^r}{r} - \frac{\psi'(0)}{\psi(0)} - \frac{1}{2} \ln(1 - x^2) \]
\[ + O\left(\frac{x \ln^2 x}{T}\right) + O\left(\frac{x \ln T}{T}\right) \]
\[ + O\left(\ln x \cdot \min\left\{ 1, \frac{x}{T \cdot \psi(x)} \right\} \right) \]

This is the improved version that was tacitly referred to in the footnote on p. 370.

13. (Lec 19+20, p. 35) Concerning Prop 3 \((\sum \frac{\mu(n)}{n} = 0)\) and the Euler product developments for \(\psi(s)\) and \(1/\psi(s)\)

\[ \text{L. Euler, Opera Omnia, Ser. I} \]

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\[ \text{vol. 14 pp. 243-244 (1737)} \]

\[ \text{vol. 14 pp. 241-242 for } \lambda(n) \text{ with methodology.} \]

\( \{ \text{vol. 8 = Introductio in analysin infinituarum} \} \)
13. (Lec 19+20, p. 28) Concerning item D: during the lecture, I misstated the relation to \( \sum \frac{\mu(n) \ln n}{n} = -1 \) (i.e., p. 27 box). I wanted the implication for this aspect to "go" only one way.

This point is now correct in the revised pdf for Lec 19+20.

Incidentally: note that if \( \sum \frac{\mu(n) \ln n}{n} \) converges, its value must be \(-1\). Thanks to p. 27 (bot) and Lec 21, p. 11 (Fact 2b). Similarly for \( \sum \frac{\mu(n)}{n} \) and 0.

14. (Lec 24, p. 18) The current fraction [after much technical effort] is \( \frac{13}{84} = 0.154764 \). Note that \( \frac{1}{7} = 0.142857 \).

In 2005, the fraction was \( \frac{32}{205} = 0.156094 \).

15. (Lec 24, pp. 16-18) As the [upper] bound for \( \mu(n) \) gradually improves, it is only natural to wonder what can be obtained via Perron's formula (Lec 19, p. 4) in a variety of problems utilizing just a crude absolute value technique over a rectangle - akin to what we did in Lec 19, p. 18 ff with \( M(x) \).
My original thought was to give another homework problem or two touching on this matter & alas, time (and endurance?) constraints intervened.

In the for what it's worth "department," I'll now scratch the surface on this topic by sketching what happens for

$$\sum_{n \leq x} d(n)$$

Here, of course, we have $f(s) = \zeta(s)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$

d'a lec 19, pp. 14-15. Nothing is lost by taking $x$ to have form $N + \frac{1}{2}$. For Lec 19, p. 4, we want:

\[
\begin{align*}
\begin{cases}
\gamma(n) = \nu, \\
\gamma(n) = \nu \sim n^\frac{1}{2}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
x = \text{big}
\end{align*}
\]

Get:

$$\sum_{n \leq x} d(n) \approx \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s)^2 \frac{x^w}{w} \, dw + O\left[ \frac{x \ln^2 x}{T} \right]$$

$$+ O\left[ \frac{x^{1+\epsilon} \ln x}{T} \right] + O\left[ \frac{x^{1+\epsilon}}{T} \right]$$

We'll write this with a minor abuse of language
in the equivalent form

\[
\sum_{n \leq x} d(n) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \zeta(w) \frac{x^w}{w} dw + \mathcal{O}\left[ \frac{x^{1+\epsilon}}{\epsilon} \right].
\]

The residue at \( w = 1 \) was computed earlier as

\[x \ln x + (2\gamma - 1)x\]

in loc. cit. p. 90. It seems reasonable to now select any \( \lambda \in (0, 2] \) and push \( \text{Re}(w) = C \) over to \( \text{Re}(w) = 1 - \lambda \).

If \( \lambda = 1 \), we make a minor indentation at \( w = 0 \).

Prior to continuing, we recall that the fcn \( \mu(\sigma) \) satisfies \( \mu(0) = \mu(1-\sigma) + \frac{1}{2} - \sigma \), which can be rewritten as

\[
\mu(0) + \frac{1}{2} \sigma = \mu(1-\sigma) + \frac{1}{2} (1-\sigma).
\]

Put:

\[
k(\sigma) = \begin{cases} 
\frac{1}{2} - \sigma, & \sigma \leq 0 \\
\frac{1-\sigma}{2}, & 0 \leq \sigma \leq 1 \\
0, & \sigma > 1
\end{cases}, \quad \nu(\sigma) = \begin{cases} 
\frac{1}{2} - \sigma, & \sigma \leq \frac{1}{2} \\
0, & \sigma > \frac{1}{2}
\end{cases}.
\]

The fcn \( k \) and \( \nu \) both satisfy (**) \( \therefore \) it is obvious that \( k(\sigma) \leq \nu(\sigma) \). One hopes that \( \mu(\sigma) = \nu(\sigma) \).
Let $A > 0$ satisfy
\[ |\mathcal{I}(1-\lambda + it)| = O(1)(1+|t|)^A. \]

To avoid trivial in $|\lambda|$ when $c \to 1-\lambda$, we insist that
\[ X^{-\lambda} \sum_{A \leq t \leq X} 1 \quad \text{i.e.} \quad T^{-A} \leq X^3. \]

In short order, by employing convexity à la pp. 812 in Sec 24, estimate (A) transforms into
\[
\sum_{n \leq x} d(n) = O\left( \sum_{n \leq x} n^{-\lambda} \right) + O\left( T^{-A} \frac{x^{1-\lambda}}{T} \right)
+ O\left( \frac{x^{1+\varepsilon}}{T} \right),
\]

at least when $x > x_0(A, \varepsilon, \lambda)$. $[x_0 = \exp \left( \frac{2A}{\varepsilon} \right)$ is fine. The term $O(x^{1/10})$ is present whenever $\lambda \in \left[ \frac{9}{10}, \frac{1}{2} \right]$ in order to accommodate either the residue at $w = 0$ or an indentation made to avoid any issues at $w = 0$. \]
To optimize, we imagine \( \epsilon \) as 0 and set

\[
x^{1-\lambda} \frac{2A}{T^2} = \frac{x}{T}
\]

Thus

\[
T = \frac{2}{x^{2A+1}}
\]

which then leads to a collective error term of

\[
O \left( \frac{1}{x^{10}} \right) + O \left[ \frac{2A + (1-A) + 2\epsilon}{x^{2A+1}} \right]
\]

We remark here that \( T \geq x^2 \) and that \( \frac{2}{2A+1} \leq \frac{2}{2A} \),

this last relation showing that \( T \) is admissible.

Because of the \( 2\epsilon \), one is now free to substitute any number \( \geq \epsilon(1-\lambda) \) for \( A \) and still have a true result (i.e., valid remainder term).

We'll go with \( A = k(1-\lambda) \). Accordingly,

\[
0 < \lambda < 1 \quad \Rightarrow \quad A = \frac{3}{2} \quad \Rightarrow \quad O \left[ x^{\frac{1}{4A+2\epsilon}} \right]
\]

\[
1 \leq \lambda \leq 2 \quad \Rightarrow \quad A = 2 - \frac{\lambda}{2} \quad \Rightarrow \quad O \left[ x^{\frac{1}{2A+2\epsilon}} \right]
\]

The optimal estimate for \( \sum_{n \leq x} d(n) \) with this [crude]

Perron-type technique is therefore:

\[
O \left( \frac{1}{x^{10}} \right) + O \left[ \frac{2A + (1-A) + 2\epsilon}{x^{2A+1}} \right]
\]
\[
\sum_{n \leq x} d(n) = x \ln x + (2\gamma - 1) x + O(x^{\frac{1}{2} + 2\varepsilon})
\]

attained basically for any \( \gamma \in [1, 2] \).

Under the Lindelöf Hypothesis, one can take \( A = \sqrt{1 - \lambda} \). This produces

\[
0 < \lambda < \frac{1}{2} \Rightarrow A = 0 \Rightarrow O\left[ x^{l - \lambda + 2\varepsilon} \right]
\]

\[
\frac{1}{2} \leq \lambda \leq 2 \Rightarrow A \approx \lambda - \frac{1}{2} \Rightarrow O\left[ x^{\frac{1}{2} + 2\varepsilon} \right]
\]

In other words: we still get (\(*\)) only beginning already at \( \lambda = \frac{1}{2} \).

One can SUMMARIZE by saying that a crude application of Perron's method essentially leads to just

\[
\sum_{n \leq x} d(n) = x \ln x + (2\gamma - 1) x + O(\sqrt{x})
\]

a result which was proved earlier, almost effortlessly, in Lec 21, pages 2 + 3.

Indeed, to obtain a remainder term of size \( O(x^{1/3}) \) which is the classical nontrivial estimate, it is basically
necessary to exploit the functional equation along \( \text{Re}(w) = 1 - \lambda \) with \( \lambda > 1 \) and then use techniques resembling those in Lec 22, p. 15 (Lemma IV) and Lec 23, pp. 11-13. See Titchmarsh, Theory of \( L(s) \) § 12.2 with \( k = 2 \).

Compare: Landau, Vorlesungen, Sätze 508+509 (where an important more intrinsic method is used).

Out of curiosity, it is natural to wonder what taking \( f(s) = L(s)^{1/2} \) produces with this crude Perron-type technique. One would hope that \( [[x]] \) could be estimated reasonably accurately!

One has
\[
\begin{cases}
  a_n = 1, \quad q = 1, \quad \hat{E}(v) = 1, \\
  c = 1 + \frac{i}{\ln x}
\end{cases}
\]

\( x = \text{big} \)

It is convenient to keep \( \lambda \in (0, M] \) with the tacit restriction that \( |\lambda - 1| > 10^{-3} \) (say). Here \( M = \text{some large integer} \).
The earlier procedure is now easily mimicked. One insists that $T^A \leq x^2$ (at the very least). The collective error term is easily seen to be

$$O(1) + O(T^A x^{1-2}) + O(T^E x^{1+e} T) + O(T^E x^{1+e} T)$$

One optimizes with $T = x^{\frac{2}{A+1}}$. We have $T \leq x^M$. This leads to

$$O(1) + O \left[ x^{\frac{A+1-2}{A+1}} + M E \right]$$

We have:

- $0 < \frac{1}{2}$, $k(1-2) = \frac{2}{A} \Rightarrow O \left[ x^{\frac{2}{2A+1}} + M E \right]$
- $1 < A \leq M$ $k(1-2) = A - \frac{1}{2} \Rightarrow O \left[ x^{A+1} + M E \right]$
- $0 < A \leq \frac{1}{2}$, $k(1-2) = 0 \Rightarrow O \left[ x^{1-2} + M E \right]$
- $\frac{1}{2} \leq A \leq M$, $k(1-2) = A - \frac{1}{2} \Rightarrow O \left[ x^{2A+1} + M E \right]$

In each instance, as $A$ grows, the exponent decreases. Since $E$ is arbitrary, we get

$$\|x\| = x + O \left( x^{\frac{1}{2M+1} + \frac{E}{2}} \right)$$

Letting $M$ grow, it emerges that

$$\|x\| = x + O(x^E)$$

17. (Lec 26, p. 11) A very nice complex variable proof of this THEOREM is given in Titchmarsh's *Theory of *\( \zeta(s) \)*, § 4.14.

18. (Lec 28, p. 9) In Newman's General Thm, it goes virtually without saying that \( g(it) \) can be addressed simply by setting \( s = a + it \).

19. (Lec 29, p. 29) In regard to the use of other kernel functions \( k(w) \), it is worth mentioning that an interesting variant of the 1936 Ingham method is pushed through in Montgomery and Vaughan's *Multiplicative Number Theory*, vol. I, pages 477–479, 482 (bottom) – 483 (top).


20. (Lec 30, p. 3) Prior to beginning the proof of THM, it would have been wise to step back and draw attention to an immediate Corollary, viz., first
Let \( q = 1 \). If
\[
\sum_{i=1}^{\infty} \frac{\mu(n)(\ln n)^2}{n}
\]
converges,
then
\[
\sum_{i=1}^{\infty} \frac{\mu(n)}{n}
\]
also converges and its value must be 0; as such, one will again get (i\(^*\)) + (ii\(^*\)) in an elementary fashion.

PF
That
\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n}
\]
converges is self-evident by Dirichlet's test; see Lec 7, p. 1 (bottom). By item #13 above (i.e., Fact 2(b) in Lec 21), the summation's value is immediately ascertained to be 0.

21o (Lec 30, p. 3, proof of THM) The proof clearly possesses a certain beauty. After finishing it, however, one is left wondering how the "\( f \)-condition" of item #20 fits into the scheme overall.

This issue was ultimately clarified by Kienast in *Math. Annalen* 95 (1925) 427–445. See also Landau's *Handbuch der Primzahlen*, CHELSEA EDITION, Appendix (by P. Bateman), p. 941 & 159.
A bit of "diagram-chasing" is necessary in order to unravel Kienast's paper. Let \( \sim \) mean "elementarily equivalent"; let
\[
[M + k] \quad \text{mean} \quad M(x) = o(1) \frac{x}{(\ln x)^k}
\]
\[
[\Psi + k] \quad \text{mean} \quad \Psi(x) \sim x = o(1) \frac{x}{(\ln x)^k}
\]
\[
g_k \quad \text{mean} \quad \sum_{i=1}^{\infty} \frac{a(n) (\ln n)^k}{n} \quad \text{converges.}
\]
(Here \( k \geq 0 \).) Note that \( A \sim B \) and \( B \sim C \Rightarrow A \sim C \).

**Lemma 1** (baby calculus - very useful)

Let \( \varphi \in C^1[a, b] \) be positive and monotonic.

Let \( F \) be real and piecewise \( C^1 \) on \( [a, b] \).

Assume that \( |F(x)| \leq \varrho(x) \), where \( \varrho \) is positive and monotonic + \( C^1 \) on \( [a, b] \). We then have
\[
\left| \int_a^b \varphi(x) dF(x) \right| \leq 2 \left[ \varphi(a) \varrho(a) + \varphi(b) \varrho(b) \right] + \left| \int_a^b \varphi(x) d\varrho(x) \right|.
\]

**Proof**
The function \( F \) has only a finite number of actual discontinuities. Because of the \( \text{Lipschitz condition} \) on the "chunks" of the rest of the graph, \( F(x) \) has bounded total variation on \( [a, b] \). Any such function
is expressible as the difference of two monotonic increasing \( q_f(x) \). (The R-S integral of \( qdf \) is thus finite.)

We now exploit integration-by-parts twice.

\[
\int_a^b q(x) dF(x) = |q(b) F(b) - q(a) F(a) - \int_a^b F(x) q'(x) dx| \\
\leq |q(b) \Phi(b) + q(a) \Phi(a) + \int_a^b \Phi(x) q'(x) dx| \\
\begin{split}
\text{this is correct when the sign of } q' \\
\text{is fixed?}
\end{split}
\]

\[
= |q(b) \Phi(b) + q(a) \Phi(a) + \int_a^b \Phi(x) q'(x) dx| \\
\leq |2q(b) \Phi(b) + 2q(a) \Phi(a) + \int_a^b q(x) d\Phi(x)|. 
\]
Lemma 2

Let \( k \geq 0 \). Put \( F(x) = \sum_{n \leq x} \mu(n) (\log n)^k \). We then have

\[
M(x) = \frac{o(x)}{(\log x)^{k}} \iff F(x) = o(x)
\]

elementarily \( \text{i.e., via a sequence of elementary techniques} \).

Proof

\( k = 0 \) is trivial, so take \( k \geq 1 \) w.l.o.g.

Assume first that \( F(x) = o(x) \). Keep \( T \) large.

Notice that

\[
M(x) = O(1) + \int_{2}^{x} (\log t)^{-k} dF(t)
\]

\[
= O(1) + O_\varepsilon(1) + \int_{T}^{x} (\log t)^{-k} dF(t)
\]

\[
\left\{ \begin{align*}
& \text{apply Lemma 1 with } \Phi(t) = \varepsilon x \\
& \text{let } 1 \Theta \leq 1 \text{ as usual}
\end{align*} \right\}
\]

\[
= O_\varepsilon(1) + 2\Theta \left[ \int_{(\log T)^{-1}}^{1} (\log t)^{-k} \varepsilon t + O_\varepsilon(1) \right]
\]

\[
= O_\varepsilon(1) + 2\Theta \frac{\varepsilon T}{(\log T)^{k}} + 2\Theta \varepsilon \int_{2}^{T} \frac{dx}{(\log x)^k}
\]

\[\uparrow \text{ familiar}\]
\[= O_{\varepsilon}(1) + 2\Theta \frac{\varepsilon T}{(\ln T)^k} + 2\Theta \varepsilon \frac{O(1)}{(\ln T)^k} \]  \( \tag{27} \)

\[= O_{\varepsilon}(1) + \varepsilon O(1) \frac{T}{(\ln T)^k} \]

Hence \( M(T) = o(1) \frac{T}{(\ln T)^k} \). \( \square \)

Next, suppose that \( M(x) = \frac{a(x)}{(\ln x)^k} \). Keep \( T \) large.

Notice that

\[ F(T) = O(1) + \int_{2}^{T} (\ln x)^k dM(x) \]

\[= O(1) + O_{\varepsilon}(1) + \int_{T_{\varepsilon}}^{T} (\ln x)^k dM(x) \]

\[\left\{ \text{apply lemma 1 with } \Phi(x) = \varepsilon \frac{x}{(\ln x)^k} \right\} \]

\[\text{let } |\theta| \leq 1 \text{ as usual} \]

\[= O_{\varepsilon}(1) + 2\Theta \int_{T_{\varepsilon}}^{T} (\ln x)^k \varepsilon \frac{T}{(\ln T)^k} + O_{\varepsilon}(1) \]

\[+ \int_{T_{\varepsilon}}^{T} (\ln x)^k d\left(\frac{\varepsilon x}{(\ln x)^k}\right) \]

\[= O_{\varepsilon}(1) + 2\Theta \varepsilon T \]

\[+ 2\Theta \varepsilon \int_{\exp(k)}^{T} (\ln x)^k \left[ \frac{(\ln x)^k - k(\ln x)^{k-1}}{(\ln x)^{2k}} \right] dx \]
\[ = O_\varepsilon(1) + 2\Theta ET \]
\[ + 2\Theta E \int_{\exp(k)}^{T} \left[ 1 - \frac{k}{\ln x} \right] dx \]
\[ = O_\varepsilon(1) + 4\Theta ET. \]

Hence \( F(T) = o(T) \).

\[ \text{Going back to } [M+k], [\psi+k], g_k \text{ one observes that in his paper - Kienast either recalls or proves:} \]

\[ \text{\textbf{Satz 1.}} \quad [M+0] \sim [\psi+0] \sim g_0 \]

\[ \text{\textbf{Satz 3.}} \quad [\psi+k] \sim g_k \quad \text{all } k \geq 0 \]

\[ \text{\textbf{Lemma 2 + Satz 10.}} \quad \text{Anytime } [\psi+k] \text{ is true, we have } [M+(k+1)] \sim g_{k+1} \]

The following assertion is now the key!

\[ \underline{\text{CLAIM}} : \text{ we have } [M+g] \sim g_k \sim [\psi+g] \text{ for each } g \geq 0. \]
Proof

Suppose not. Let $k$ be the smallest case where the proposed 3-way relation is FALSE. By Satz 1, $k \geq 1$.

Suppose $[M+k]$ holds. Clearly $[M+(k-1)]$ holds.

But 3-way relation $[M+(k-1)] \equiv g_{k-1} \equiv [\Psi+(k-1)]$ is TRUE. Hence we get $[\Psi+(k-1)]$ elementarily.

By Satz 10, the truth of $[M+k] \Rightarrow$ that of $g_k$ elementarily. So we can write $[M+k] \Rightarrow g_k$, the "$\Rightarrow" meaning elementary.

Suppose next that $g_k$ holds. By Satz 3, we get $[\Psi+k]$ in an elementary fashion. So $g_k \Rightarrow [\Psi+k]$.

Finally, suppose $[\Psi+k]$ holds. By Satz 3, $g_k$ holds elementarily. Trivially, of course, $[\Psi+(k-1)]$ holds.

By Satz 10, we then have $g_k \Rightarrow [M+k]$ elementarily. So $[\Psi+k] \Rightarrow [M+k]$.

All told, we have seen $[M+k] \Rightarrow g_k \Rightarrow [\Psi+k] \Rightarrow [M+k]$. This contradicts the definition of $k$.

The CLAIM is thus proved. \( \blacksquare \)
(Lec 30, p. 3, "already knew") Both here and in regard to Lec 25, p. 20 line 2, it is thought-provoking to have a look at Euler [1737] Opera Omnia, Ser. I, Vol. 14, pp. 242 (thm 19, second sentence) and 243-244. Cf. also item #12 above.

Euler's assertion that \( \sum p^{-1-\varepsilon} = \ln \left( \frac{e}{\varepsilon} + O(1) \right) \) is but 1 or 2 steps away from heuristically concluding that \( T(e^u) \sim \ln u \), wherein

\[
T(x) \equiv \sum_{p \leq x} \frac{1}{p}
\]

— and, then, via an analogous reasoning, engendering sparking the suspicion that [in all likelihood] \( \pi(x) \) must be roughly of size \( \int_2^x \frac{dt}{\ln t} \sim \frac{x}{\ln x} \). Alas, none of this is said there explicitly.

In modern parlance, the \( T(e^u) \) deduction (based strictly on \( \varepsilon > 0 \)) is a standard example of a Karamata-type Tauberian theorem; in this regard, cf. also Titchmarsh, Theory of \( \zeta(s) \), (7.12.1) - (7.12.5).* See Ingham, p. 10, for an Euler-style proof that \( T(e^u) > \ln u - \frac{1}{2} \); note too p. 118 (not, "infinitely fewer than the integers") and Hardy, Divergent Series, theorem 108.

For some additional perspective on both aspects of this matter, \( T \) and \( \pi \), see Edwards, Riemann's Zeta Func, pp. 1-2.
Course Announcement – Math 8280 – Spring 2016
(Topics in Number Theory)
An Introduction to Analytic Number Theory
Instructor: D. A. Hejhal

Beginning in the 19th century, it began to be realized (by Riemann, among others) that certain fundamental questions involving the ordinary integers, more specifically the primes, were amenable to study by bringing to bear on them methods of analysis, especially complex analysis.

In very loose terms, analytic number theory is that part of number theory whose results are obtained principally with the aid of constructs and techniques having at least one foot in some aspect of (either classical or modern) analysis. Today, for instance, in addition to complex, both harmonic and spectral analysis have begun to be used.

The purpose of this course, which should probably carry a number closer to 8001 (8009 would be apt!) is to offer students conversant in the standard advanced undergraduate courses in real, complex, and Fourier analysis, plus a bit of modern algebra, a kind of “gentle” introduction to analytic number theory, by coming in chiefly from the multiplicative side — that is to say, primes and constructs like the Riemann zeta function, \( \zeta(s) \).

Analytic number theory is a subject steeped in history. One of its main theorems is the celebrated Prime Number Theorem from 1896 (and its counterpart for primes in arithmetic progressions). Several approaches to the PNT are now known. One of the best ways of getting a feel for analytic number theory, and what makes it tick, is to simply make a careful study of the various approaches to the PNT.

The basic plan of the first 2/3 of the course is to do exactly this — taking the time, where need be, to develop some interesting cognate material involving, e.g., aspects of complex (and real!) analysis pertinent for \( \zeta(s) \) and the study of its zeros. Connections with the Riemann Hypothesis and (so-called) “explicit formula” for the prime counting function \( \pi(x) \) will arise here.

Following that, as time permits, a few topics further afield (but still relatively gentle) will touched on. One possibility: some “nitty-gritty” numerical calculation of a few Riemann zeros and related zeros of \( L(s, \chi) \), where \( \chi \) is a multiplicative character.

The course format will primarily be lectures, guided in part by the classic books of Ingham and Davenport. Some unpublished course notes by A. Selberg and a recent AMS volume by Iwaniec/Kowalski will also prove useful.

To facilitate fixing a class time, any students interested in taking this course (or simply desiring further information) should contact the instructor in the very near future. {Email: hejhal@math.umn.edu}

Students interested in the course, but lacking some of the prerequisites, may, after a discussion with the instructor, be allowed to join the class and receive Math 5990 credit for it. (A slightly modified syllabus would then apply; early contact with the instructor is again encouraged.)