MATH 8280/5990 – SPRING 2016
INTROD. TO ANALYTIC NUMBER THEORY
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LECTURE 23: Landau's proof of the Hardy theorem. (Consult the Addendum in Lecture 24 for an improvement.)

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LECTURE 29: J. E. Littlewood's 1914 $\Omega_\pm$ estimate for $\psi(x) - x$ proved using a variant of Ingham's 1936 Fourier transform technique.

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Lecture 1 (20 Jan 2016)

These notes will be a kind of loose/informal diary of what was discussed in the lecture. They are not a textbook, nor will they strive for "completeness".

Analytic Number Theory = that part of number theory wherein results are obtained by use of functions and appropriate analytic techniques.

Traditionally, the "analytic techniques" have come from complex analysis (i.e., the theory of analytic functions). But, this focus has gradually been widened.

Multiplicative A.N.T. (loosely speaking) deals with aspects/functions which are intimately tied to prime numbers and the unique factorization property of the positive integers.

The first 2 lectures will use elementary analysis combined with arithmetic to get some nontrivial information about primes.

I assume the unique factorization thm as being known: i.e.,

$$n = p_1^{e_1} \cdots p_r^{e_r},$$

<u>uniquely</u> with primes $p_1 < \cdots < p_r$ and $e_j \geq 1$.

<u>Theorem 1 (Euclid)</u>

The number of primes is infinite.

<u>Pf</u>

Assume not. Let the full list be $\overset{2}{p_1} < \cdots < p_m$. Form $N = p_1 \cdots p_m + 1$ = integer. Factor N. Therefore, some p_ℓ divides N. But,

$$\frac{N}{p_\ell} = p_1 \cdots \hat{p}_\ell \cdots p_m + \frac{1}{p_\ell}.$$

$\boxed{\wedge \text{ means omit}}$

This is not an integer (since $p_1 = 2$).

Contrad! ∎

Def:

$$\pi(x) = N[p : p \leq x] \quad \text{for } x > 0.$$

By thm 1, $\pi(x) \to \infty$ as $x \to \infty$.

The idea of the proof of thm 1 can be used to prove $\pi(x) \gtrsim \ln \ln x$. FOR $x \geq 2$ See Hardy-Wright, thm 10. This is not very interesting.

Notation:

$$f(t) = O[g(t)] \quad \text{as } t \to \infty$$

means $|f(t)| \leq M|g(t)|$ for $t \geq t_0$, some large t_0. But:

$$f(t) = O[g(t)] \quad \text{for } t \geq Q$$

means $|f(t)| \leq \mathfrak{M}|g(t)|$ for all $t \geq Q$.

In most situations, one can "get" the 2nd version merely by inflating M appropriately. This is why I wrote $\mathfrak{M}$.

Def:

$$\theta(x) = \sum_{p \leq x} \ln p \, , \quad \psi(x) = \sum_{p^m \leq x} \ln p \qquad (x > 0).$$

Empty sums are defined to be 0; hence
$\theta(x) = \psi(x) = 0$ for $x < 2$.

Notation used by Chebyshev!

Obviously

$$\psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \cdots ,$$

where eventually $\theta(x^{1/N}) = 0$.

Thm 2

$$\psi(x) = \sum_{p \leq x} \left[\!\left[\frac{\ln x}{\ln p} \right]\!\right] \ln p \qquad (x \geq 2).$$

For $x < 2$, both sides are 0.

Proof

Choose any prime $p \leq x$. We seek its total contribution to the def. of $\psi(x)$. It will contribute a $\ln p$ so long as $p^m \leq x$, i.e $m \leq \frac{\ln x}{\ln p}$, i.e. $m \leq \left[\!\left[\frac{\ln x}{\ln p} \right]\!\right]$. Thus, we collectively get $\left[\!\left[\frac{\ln x}{\ln p}\right]\!\right] \ln p$. Now, just add over all relevant p. ▨

Thm 3

(a) $\psi(x) = \theta(x) + O[x^{\frac{1}{2}}(\ln x)^2]$, $x \geqq 2$;

(b) we get $\psi(x) = \theta(x) + O(x^{1/2})$ if we can somehow prove that $\theta(y) = O(y)$ for all $y \geqq 1$.

Pf

Need to estimate $\sum_{n=2}^{\infty} \theta(x^{1/n})$. But $\theta(x^{\frac{1}{n}}) = 0$ once $x^{1/n} < 2$, IE $n > \frac{\ln x}{\ln 2}$. Get:

$$\sum_{n=2}^{\infty} \theta(x^{1/n}) = \sum_{n=2}^{L} \theta(x^{1/n}), \quad L = \left[\frac{\ln x}{\ln 2}\right] + 10$$

$$\leqq (L-1)\,\theta(x^{1/2}) \leqq L \sum_{p \leqq x^{1/2}} \ln p$$

$$\leqq L \ln x \sum_{p \leqq x^{1/2}} 1 \leqq L \ln x \sum_{n \leqq x^{1/2}} 1$$

$$= O[x^{\frac{1}{2}}(\ln x)^2] \quad \{\text{very crudely}\}.$$

This proves (a). For (b), we need to look at $\sum_{n=3}^{\infty} \theta(x^{1/n}) = \sum_{n=3}^{L} \theta(x^{1/n})$.

A trivial modification shows that this sum is $O[x^{\frac{1}{3}}(\ln x)^2]$. Hence,

$$\begin{aligned}\psi(x) &= \theta(x) + \theta(x^{1/2}) + O[x^{\frac{1}{3}}(\ln x)^2] \\ &= \theta(x) + O(x^{1/2})\end{aligned}$$

by the hypothesis about $\theta(y)$. ▨

Thm 4 (Legendre – early 1800s)

Given $m \geqq 2$. We then have

$$m! = \prod p^{E_p}, \quad E_p = \sum_{j=1}^{\infty} \left[\!\left[\frac{m}{p^j} \right]\!\right].$$

Proof

We use "baby set theory" (i.e., classes).

Write $m! = \prod_{k=1}^{m} k$. The only primes that "go into" k are clearly $\leqq m$.

With this being the case, temporarily fix any prime $p \leqq m$. Choose L so that $p^L \leqq m < p^{L+1}$.

Say that $k \in [1, m]$ is of "type j" when

$$k = p^j(\text{integer relatively prime to } p).$$

Here $j \geqq 0$. THINK UNIQUE FACTORIZATION THM.

Let $v_j = \text{card}\{k \in [1, m] : k \text{ is of type } j\}$.

Notice that

$$\begin{aligned} v_0 + v_1 + v_2 + \cdots + v_L &= m & \text{type } 0, \ldots, L \\ v_1 + v_2 + \cdots + v_L &= \left[\!\left[\frac{m}{p}\right]\!\right] & 1, \ldots, L \\ v_2 + \cdots + v_L &= \left[\!\left[\frac{m}{p^2}\right]\!\right] & 2, \ldots, L \\ &\vdots \\ v_L &= \left[\!\left[\frac{m}{p^L}\right]\!\right]. & L \end{aligned}$$

Delete the first row <u>and</u> $\overbrace{\ }^{\text{then}}$ <u>add</u>! Get:

$$v_1 + 2v_2 + 3v_3 + \cdots + Lv_L = \sum_{j=1}^{L} \left[\!\left[\frac{m}{p^j}\right]\!\right].$$

But, by construction, we clearly have

$$E_p = v_1 + 2v_2 + \cdots + Lv_L.$$

(Just think about this a second!)

Now, just slap together all the p^{E_p}. ∎

clearly any primes $> m$ get $E_p = 0$, exactly as they should

Chebyshev became interested in quotients of various factorials which turn out to be integers.

EG, the binomial coefficient $\binom{2n}{n} = \frac{(2n)!}{n!n!}$.

He sought to use standard Stirling-type estimates to get information about the number of primes in certain intervals.

Clearly, thm 4 p. 6 will be vital here.

We note:

BABY LEMMA

Let x be a positive rational (say, $\frac{m}{n}$ in lowest terms). The relation

$$x = p_1^{A_1} \cdots p_r^{A_r} = p_1^{B_1} \cdots p_r^{B_r} \qquad (p_1 < \cdots < p_r)$$

holds with $A_j \in \mathbb{Z}$ and $B_j \in \mathbb{Z}$ only if $A_j = B_j$. If $x \in \mathbb{Z}^+$ (i.e., $n = 1$), each A_j is necessarily $\geqq 0$.

Pf

Let G be giant. Notice that

$$(p_1\cdots p_r)^G p_1^{A_1}\cdots p_r^{A_r} = (p_1\cdots p_r)^G p_1^{B_1}\cdots p_r^{B_r} = \text{integer}.$$

By unique factorization thm, $G + A_j = G + B_j$. ∎

The Baby Lemma says that unique factorization extends to $\mathbb{Q}^+$ in an obvious way.

It *also* implies that the numbers $\ln p_j$ are linearly independent over $\mathbb{Q}$.

Thm 5 (elementary integral calculus)

$$m \geq 2 \Rightarrow$$

$$\ln(m!) = m \ln m - m + O(\ln m).$$

Pf

Suppose $y = f(x)$ is continuous, non-neg, and increasing on $1 \leq x \leq N+1$. By drawing a picture, clearly

$$f(1) + \cdots + f(N) \leq \int_1^{N+1} f(t)\,dt \leq f(2) + \cdots + f(N+1)$$

$$\Rightarrow \quad 0 \leq \int_1^{N+1} f(t)\,dt - \sum_{j=1}^{N} f(j) \leq f(N+1) - f(1).$$

Simply put $f(t) = \ln t$ to get:

$$0 \lesssim \int_1^{N+1} \ln t \, dt - \sum_{j=1}^{N} \ln j \lesssim \ln(N+1)$$

$$\left\{ \text{but } \int_1^x \ln t \, dt = x \ln x - x + 1 \,,\; x \gtrsim 1 \right\}$$

$$\Downarrow$$

$$0 \lesssim (N+1)\ln(N+1) - N - \ln(N!) \lesssim \ln(N+1)$$

$$\Downarrow$$

$$(N+1)\ln(N+1) - N - \ln(N!) = \omega \ln(N+1)$$

for some $\omega \in [0,1]$

$$\Downarrow$$

$$\begin{aligned} \ln(N!) &= (N+1)\ln(N+1) - N - \omega\ln(N+1) \\ &= N\ln(N+1) + (1-\omega)\ln(N+1) - N \\ &= N\left[\ln N + \ln\left(1+\tfrac{1}{N}\right)\right] + O(\ln N) - N \\ &= N\ln N + O(1) + O(\ln N) - N \\ &= N\ln N - N + O(\ln N) \,. \quad \blacksquare \end{aligned}$$

Let

$$L(x) = \sum_{k \leq x} \ln k \quad \text{for } x > 0.$$

For $x < 2$, $L(x) = 0$. For $x \geq 2$, $L(x) = \ln(\llbracket x \rrbracket !)$.

<u>Thm 6</u>

$$L(x) = x \ln x - x + O(\ln x) \quad \text{for } x \geq 2.$$

<u>Pf</u>

By inflating the constant, WLOG $x \geq 100$.
Let $M \leq x < M+1$. By Thm 5,

$$L(x) = L(M) = M \ln M - M + O(\ln M).$$

The derivative of $t \ln t - t$ is $\ln t$. Hence $x \ln x - x$ differs from $M \ln M - M$ by at most $\ln(M+1)$. As such,

$$\begin{aligned} L(x) &= x \ln x - x + O(\ln(M+1)) + O(\ln M) \\ &= x \ln x - x + O(\ln x). \quad \blacksquare \end{aligned}$$

Thm 7

Define (following von Mangoldt):

$$\Lambda(n) = \begin{cases} \ln p, & n = p^j \ (j \geq 1) \\ 0, & \text{otherwise} \end{cases}.$$

Here $n \geq 1$. We have:

(a) $\psi(x) = \sum_{n \leq x} \Lambda(n)$;

(b) $L(x) = \sum_{k \leq x} [\![\frac{x}{k}]\!] \Lambda(k)$; ← for $\log [\![x]\!]!$

(c) $L(x) = \psi(x) + \psi(\frac{x}{2}) + \psi(\frac{x}{3}) + \cdots$.

Proof

When $x < 2$, each of (a)(b)(c) just states $0 = 0$.
So, WLOG, $x \geq 2$.

Assertion (a) is now a tautology. OK

For (b), write $M \leq x < M+1$. By 11 (top), obviously

$$L(x) = L(M) = \log(M!) .$$

By Thm 4, Legendre-style,

$$L(M) = \sum E_p \ln p = \sum_p \left(\sum_{j=1}^{\infty} [\![\frac{M}{p^j}]\!] \right) \ln p$$

$$= \sum_{\text{all } k \leq M} [\![\frac{M}{k}]\!] \Lambda(k) .$$

The key issue now boils down to:

if $p^j \leqq M$ $(j \geqq 1)$, **why is** $[\![\frac{x}{p^j}]\!] = [\![\frac{M}{p^j}]\!]$?

To verify this, let

$$\ell = [\![\frac{M}{p^j}]\!] \quad (\geqq 1).$$

We have

$$\frac{M}{p^j} = \ell + \phi \quad \text{with} \quad 0 \leqq \phi < 1.$$

Write $M = p^j(\text{integer}) + r$, $0 \leqq r \leqq p^j - 1$ à la Euclid.
Clearly,

$$0 \leqq \phi \leqq \frac{p^j - 1}{p^j}.$$

Also write $x = M + \theta$, $0 \leqq \theta < 1$. We then have:

$$\ell \leqq \frac{x}{p^j} = \frac{M + \theta}{p^j} = \frac{M}{p^j} + \frac{\theta}{p^j} = \ell + \phi + \frac{\theta}{p^j}$$

$$< \ell + \frac{p^j - 1}{p^j} + \frac{1}{p^j} = \ell + 1,$$

which gives $[\![\frac{x}{p^j}]\!] = \ell = [\![\frac{M}{p^j}]\!]$, as desired.
Accordingly:

$$L(x) = \sum_{k \leqq x} [\![\frac{x}{k}]\!] \Lambda(k). \quad \text{OK}$$

To prove (c), we start with $\sum_{m=1}^{\infty} \psi\left(\frac{x}{m}\right)$. cf. (a)

Take any <u>given</u> integer p^A with $A \geq 1$. We ask: how many times does $\Lambda(p^A)$ assert "I am present" within $\sum_{m=1}^{\infty} \psi\left(\frac{x}{m}\right)$? This number will clearly be the largest ℓ so that $p^A \leq \frac{x}{\ell}$. In other words, $\ell \leq \frac{x}{p^A}$ or $\ell \leq \left[\!\left[\frac{x}{p^A}\right]\!\right]$. The collective contribution of p^A to $\sum_{m=1}^{\infty} \psi\left(\frac{x}{m}\right)$ will therefore be $\left[\!\left[\frac{x}{p^A}\right]\!\right] \Lambda(p^A)$.

In view of (b), it is now evident that $\sum_{m=1}^{\infty} \psi\left(\frac{x}{m}\right)$ must reduce to $L(x)$. OK ∎

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about 1850

Chebyshev played with $\binom{2n}{n} = \frac{(2n)!}{n!\,n!}$ and was therefore motivated to examine

$$L(x) - 2L\left(\frac{x}{2}\right).$$

<u>N.B.</u> Theorem 7(b) applies to $L(x) - 2L\left(\frac{x}{2}\right)$, but it is <u>easier</u> to use Thm 7(c).

Suppose for a moment that $x \geqq 4$. A quick calculation with Thm 6 gives

$$L(x) - 2L(\tfrac{x}{2}) = x(\ln 2) + O(\ln x).$$

By inflating the constant à la ③(bottom), the same relation holds for $\underline{x \geqq 2}$.

On the other hand, by Thm 7(c),

$$L(x) - 2L(\tfrac{x}{2}) = \psi(x) - \psi(\tfrac{x}{2}) + \psi(\tfrac{x}{3}) - \psi(\tfrac{x}{4}) \pm \cdots$$

(where, as usual, the terms are eventually 0).

We therefore have:

$$x(\ln 2) + O(\ln x) = \psi(x) - \psi(\tfrac{x}{2}) + \psi(\tfrac{x}{3}) - \psi(\tfrac{x}{4}) \pm \cdots$$

for all $x \geqq 2$.

THIS LAST RELATION CAN BE MANIPULATED!

Recall that $\psi(y)$ is non-neg and monotonically increasing. See Thm 7 (a).

Accordingly:

$$x(\ln 2) + O(\ln x) = \psi(x) - \left[\psi\left(\tfrac{x}{2}\right) - \psi\left(\tfrac{x}{3}\right)\right] - \cdots$$

$$\Downarrow$$

$$\psi(x) \geqq x(\ln 2) + O(\ln x). \qquad \blacksquare$$

At the same time,

$$x \ln 2 + O(\ln x) = \left[\psi(x) - \psi\left(\tfrac{x}{2}\right)\right] + \left[\psi\left(\tfrac{x}{3}\right) - \psi\left(\tfrac{x}{4}\right)\right] + \cdots$$

$$\Downarrow$$

$$\psi(x) - \psi\left(\tfrac{x}{2}\right) \leqq x \ln 2 + O(\ln x). \qquad \blacksquare$$

In both instances, one keeps $x \geqq 2$.

Take $x \geqq 1000$ (say) and make an iteration as follows:

$$\psi(x) - \psi\left(\frac{x}{2}\right) \lesssim x \ln 2 + B \ln x \qquad \text{step } 1$$

$$\psi\left(\frac{x}{2}\right) - \psi\left(\frac{x}{4}\right) \lesssim \frac{x}{2} \ln 2 + B \ln \frac{x}{2} \qquad \text{step } 2$$

$$\vdots$$

$$\psi\left(\frac{x}{2^r}\right) - \psi\left(\frac{x}{2^{r+1}}\right) \lesssim \frac{x}{2^r} \ln 2 + B \ln \frac{x}{2^r} \qquad \text{step } r+1$$

$$\left\{ \begin{array}{l} \text{take } \frac{x}{2^r} \in [4, 8] \text{ for safety} \\ \text{hence } r = \frac{\ln x - \Omega}{\ln 2}, \quad \ln 4 \leqq \Omega \leqq \ln 8 \end{array} \right\}$$

ADD

$$\Downarrow$$

$$\psi(x) + O(1) \lesssim 2x \ln 2 + B(r+1) \ln x$$

$$\Downarrow$$

$$\psi(x) \lesssim x(\ln 4) + \frac{B}{\ln 2} (\ln x)^2$$

$$\Downarrow$$

$$\psi(x) \leq x(\ln 4) + O[\ln^2 x] .$$

To include $2 \leq x < 1000$, one can inflate the constant.

Theorem A (Chebyshev $\approx$ 1850)

For $x \geq 2$, we have:

$$x(\ln 2) + O(\log x) \leq \psi(x) \leq x(\ln 4) + O(\log^2 x) ;$$

$$x(\ln 2) + O(x^{1/2}) \leq \theta(x) \leq x(\ln 4) + O(x^{1/2}) .$$

Proof

The case of $\psi(x)$ was just done. OK

Since $0 \leq \theta(y) \leq \psi(y)$, clearly $\theta(y) = O(y)$ for all $y \geq 1$. Recalling Thm 3(b), we have

$$\theta(x) = \psi(x) + O(x^{1/2}) ,$$

which then produces the inequality for $\theta(x)$. OK

Notice that

$$\pi(x)\ln x = \sum_{p \leqq x} \ln x \gtrapprox \sum_{p \leqq x} \ln p = \theta(x) .$$

In light of this, theorem A assures us that

$$\pi(x) \gtrapprox (\ln 2 - \varepsilon)\frac{x}{\ln x} \qquad .693147^{+}$$

for $x \gtrapprox x_0(\varepsilon)$. Here $\varepsilon > 0$ is arbitrary.

(p. 3 line 5 is trivial in comparison.)

The celebrated Prime Number Theorem, which we seek to prove soon, states that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty .$$

We'll say more about $\pi(x)$ in Lecture 2.

Lecture 2 (22 Jan 2016)

Our primary goal today is to improve Theorem A from Lecture 1. We also wish to address some preliminary stuff as well.

Theorem 1

$x \geq 2$. We have

$$\frac{\theta(x)}{\ln x} \leq \pi(x) \leq x^{1-\delta} + \frac{\theta(x)}{\ln x}\frac{1}{1-\delta}$$

for any $0 < \delta < 1$.

Pf

$\pi(x)\ln x = \sum_{p \leq x} \ln x \geq \sum_{p \leq x} \ln p = \theta(x)$ is obvious. On the other hand,

$$\theta(x) - \theta(x^{1-\delta}) = \sum_{x^{1-\delta} < p \leq x} \ln p \geq \ln(x^{1-\delta})[\pi(x) - \pi(x^{1-\delta})]$$

$$\Downarrow$$

$$\pi(x) - \pi(x^{1-\delta}) \leq \frac{\theta(x) - \theta(x^{1-\delta})}{(1-\delta)\ln x}$$

$$\pi(x) \leq \pi(x^{1-\delta}) + \frac{\theta(x)}{(1-\delta)\ln x} \quad \{\text{since } \theta(y) \geq 0\}$$

$$\pi(x) \leq x^{1-\delta} + \frac{\theta(x)}{(1-\delta)\ln x} \quad \{\text{trivially: } \pi(y) \leq y\}$$

OK $\blacksquare$

Corollary 1

As $x \to \infty$,

$$\pi(x) \sim \frac{\theta(x)}{\ln x} \sim \frac{\psi(x)}{\ln x}.$$

Pf

By Lec 1 thm A, we know $c_1 x < \theta(x) < c_2 x$. By Lec 1 thm 3, we then get

$$\frac{\psi(x)}{\theta(x)} \to 1 \quad \text{as } x \to \infty.$$

We need to show $\frac{\pi(x) \ln x}{\theta(x)} \to 1$ too. But here we can apply Thm 1 above to get:

$$\theta(x) \leqq \pi(x) \ln x \lessapprox x^{1-\delta} \ln x + \frac{\theta(x)}{1-\delta}$$

$$1 \leqq \frac{\pi(x) \ln x}{\theta(x)} \lessapprox \frac{x^{1-\delta} \ln x}{\theta(x)} + \frac{1}{1-\delta}.$$

Just take δ smaller and smaller! Clearly

$$\limsup_{x \to \infty} \frac{\pi(x) \ln x}{\theta(x)} \leqq \frac{1}{1-\delta}$$

(again using $c_1 x < \theta(x) < c_2 x$), so we are done. ▨

This successively reducing the δ seems a bit ugly. We can junk it.

Corollary 2

For $x \geqq 2$,

$$1 \lessapprox \frac{\pi(x) \ln x}{\theta(x)} \lessapprox 1 + \frac{O(1)}{\sqrt{\ln x}} .$$

Pf

By inflating the constant in $O(1)$, we can assume $x \geqq x_0$ (suff. large) so that

$$3 \frac{\ln \ln x}{\ln x} < \frac{1}{2}, \text{ say.}$$

This is legal.

We propose to simply take $\delta = 3 \frac{\ln \ln x}{\ln x}$ in Theorem 1 above. Note that

$$\frac{1}{1-u} < 1 + 2u \quad \text{for} \quad 0 < u < \frac{1}{2} .$$

Plug in page 2 line 11 [which is just a rewrite of Thm 1]. Get:

$$1 \lessapprox \frac{\pi(x) \ln x}{\theta(x)} \lessapprox \frac{1}{\theta(x)} x \cdot e^{-\delta \ln x} \cdot \ln x + 1 + 2\delta$$

$$= \frac{1}{\theta(x)} x \cdot \frac{\ln x}{(\ln x)^3} + 1 + 6 \frac{\ln \ln x}{\ln x}$$

$$= \frac{x}{\theta(x)} \frac{1}{(\ln x)^2} + 1 + \frac{6 \ln \ln x}{\ln x} .$$

Remember that $c_1 x < \theta(x) < c_2 x$ and $x \geqq x_0$. The last expression is clearly $\lessapprox 1 + \frac{O(1)}{\sqrt{\ln x}}$. ∎

Theorem A′ (Chebyshev ≈ 1850)

For $x \geq 2$,

$$x(\ln 2) + O(\log x) \lesssim \psi(x) \lesssim x(\ln 4) + O(\ln^2 x)$$

$$x(\ln 2) + O(x^{1/2}) \lesssim \theta(x) \lesssim x(\ln 4) + O(x^{1/2})$$

$$[\ln 2 + o(1)]\frac{x}{\ln x} \lesssim \pi(x) \lesssim [\ln 4 + o(1)]\frac{x}{\ln x}.$$

Here $o(1)$ means "bounded but tends to <u>zero</u> as $x \to \infty$".

Pf

For the first 2 lines, see Lec 1 Thm A. The third line, pertaining to $\pi(x)$, follows from corollary 1 or 2. ▨

$$\ln 2 = .693147^{+}$$
$$\ln 4 = 1.386294^{+}$$

We would like to reduce the spread between these numbers!

still using elementary techniques...

Chebyshev <u>also</u> played with the combination

$$\frac{(30m)!\, m!}{(15m)!\,(10m)!\,(6m)!}.$$

It is NOT obvious this is an integer!

Note that

$$30-15-10-6+1 = 30\left[1-\tfrac{1}{2}-\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{30}\right]=0.$$

Correspondingly, let

$$\underline{\psi_\sigma(x)} = L(x) - L\left(\tfrac{x}{2}\right) - L\left(\tfrac{x}{3}\right) - L\left(\tfrac{x}{5}\right) + L\left(\tfrac{x}{30}\right)$$

$$\left\{ \text{recall } L(x) = \ln [\![x]\!]! = \sum_{n\lesssim x} [\![\tfrac{x}{n}]\!]\Lambda(n) \right\} \leftarrow \text{Lec 1}$$

$$= \sum_{n\lesssim x}\left([\tfrac{x}{n}] - [\tfrac{x}{2n}] - [\tfrac{x}{3n}] - [\tfrac{x}{5n}] + [\tfrac{x}{30n}]\right)\Lambda(n)$$

$$\equiv \sum_{n\lesssim x} \sigma\left(\tfrac{x}{n}\right)\Lambda(n).$$

$\sigma(x)$ has interesting properties (as one soon discovers). First of all,

$$\sigma(x) = [\![x]\!] - [\![\tfrac{x}{2}]\!] - [\![\tfrac{x}{3}]\!] - [\![\tfrac{x}{5}]\!] + [\![\tfrac{x}{30}]\!].$$

Note that $[\![y]\!]$ and $\sigma(y)$ are right continuous on $\mathbb{R}$. Also,

$$
\begin{aligned}
\sigma(x+30) = \; & [\![x]\!]+30 \\
& -[\![\tfrac{x}{2}]\!]-15 \\
& -[\![\tfrac{x}{3}]\!]-10 \\
& -[\![\tfrac{x}{5}]\!]-6 \\
& +[\![\tfrac{x}{30}]\!]+1 \quad = \sigma(x).
\end{aligned}
$$

So, $\sigma(x)$ is periodic 30. To appreciate σ, one simply breaks down and computes it by hand or via a computer.

$$[0,30) = [0,1) \cup [1,2) \cup [2,3) \cup \cdots \cup [29,30).$$

GET:

[0,1)	0	10	0	20	0
[1,2)	1	11	1	21	0
[2,3)	1	12	0	22	0
[3,4)	1	13	1	23	1
[4,5)	1	14	1	24	0
[5,6)	1	15	0	25	0
[6,7)	0	16	0	26	0
[7,8)	1	17	1	27	0
[8,9)	1	18	0	28	0
[9,10)	1	19	1	29	1

We thus find that $\sigma = 0$ or 1 for all x.

(It was <u>not</u> obvious apriori that, e.g., $\sigma \geq 0$.)

> <u>N.B.</u> Notice that the original factorial quotient on 5 line 2 has logarithm $\psi_\sigma(30m)$. Since $\sigma \in \{0,1\}$, the formula on 5 line 9 makes it clear that the original quotient is a positive integer!

Clearly,

$$\psi_\sigma(x) = \sum_{n \leq x} \sigma\left(\frac{x}{n}\right) \Lambda(n) \leq \sum_{n \leq x} \Lambda(n) = \psi(x).$$

Also:

$\sigma\left(\frac{x}{n}\right) = 1$ for $1 \leq \frac{x}{n} < 6$ is VERY convenient

IE get $\sigma\left(\frac{x}{n}\right) = 1$ for <u>all</u> $\frac{x}{6} < n \leq x$.

Notice that $\sigma\left(\frac{x}{n}\right) = 1$ in some other portions of $n \leq x$ TOO. But, for now, we <u>don't</u> use this.

As a tautology,

$$\psi_\sigma(x) = \sum_{\frac{x}{6} < n \leq x} \sigma\left(\frac{x}{n}\right)\Lambda(n) + \underbrace{\sum_{n \leq \frac{x}{6}} \sigma\left(\frac{x}{n}\right)\Lambda(n)}_{\text{non-negative!}}$$

$$\Downarrow$$

$$\psi_\sigma(x) \geq \sum_{\frac{x}{6} < n \leq x} 1\,\Lambda(n) = \psi(x) - \psi\left(\frac{x}{6}\right)$$

$$\Downarrow$$

$$\psi(x) \leq \psi\left(\frac{x}{6}\right) + \psi_\sigma(x).$$

So,

$$\psi_\sigma(x) \leq \psi(x) \leq \psi_\sigma(x) + \psi\left(\frac{x}{6}\right).$$

Recall (Lec 1, thm 6)

$$L(y) = y\ln y - y + O(\ln y) \qquad \text{"Stirling"}$$

for all $y \gtrsim 2$.

We substitute into

$$\psi_0(x) = L(x) - L\left(\tfrac{x}{2}\right) - L\left(\tfrac{x}{3}\right) - L\left(\tfrac{x}{5}\right) + L\left(\tfrac{x}{30}\right),$$

keeping $x \gtrsim 60$ for safety. Get:

$$\begin{aligned}
& x\ln x - x + O(\ln x) \\
& - \tfrac{x}{2}\ln\left(\tfrac{x}{2}\right) + \tfrac{x}{2} + O(\ln x) \\
& - \tfrac{x}{3}\ln\left(\tfrac{x}{3}\right) + \tfrac{x}{3} + O(\ln x) \\
& - \tfrac{x}{5}\ln\left(\tfrac{x}{5}\right) + \tfrac{x}{5} + O(\ln x) \\
& + \tfrac{x}{30}\ln\left(\tfrac{x}{30}\right) - \tfrac{x}{30} + O(\ln x)
\end{aligned}$$

$$\begin{aligned}
= \ & \left(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{30}\right) x\ln x && \longleftarrow 0 \\
& + x\left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} - 1 - \tfrac{1}{30}\right) && \longleftarrow 0 \\
& + x\left(\tfrac{1}{2}\ln 2 + \tfrac{1}{3}\ln 3 + \tfrac{1}{5}\ln 5 - \tfrac{1}{30}\ln 30\right) \\
& + O(\ln x)
\end{aligned}$$

$$\frac{1}{2}\ln 2 + \frac{1}{3}\ln 3 + \frac{1}{5}\ln 5 - \frac{1}{30}\ln 30$$

$$\approx .9212920229^{+}$$

$$\Downarrow$$

$$\boxed{\psi_0(x) \approx (.9212920229^{+})\,x + O(\ln x)}\,.$$

By $\psi_0(x) \leq \psi(x)$ on 7 + 8, we get:

$$(.921^{+})\,x + O(\ln x) \lesssim \psi(x)\,.$$

By $\psi(x) - \psi(\frac{x}{6}) \lesssim \psi_0(x)$ on 8, we get:

$$\psi(x) - \psi\left(\frac{x}{6}\right) \lesssim (.921^{+})\,x + B\ln x$$

for some B whenever $x \gtrsim 60$. We can now iterate this to get:

$$\psi(x) - \psi\left(\frac{x}{6}\right) \lesssim (.921^{+})x + B\ln x \qquad \text{step } 1$$

$$\psi\left(\frac{x}{6}\right) - \psi\left(\frac{x}{36}\right) \lesssim (.921^{+})\frac{x}{6} + B\ln\frac{x}{6} \qquad \text{step } 2$$

$$\vdots$$

$$\psi\left(\frac{x}{6^r}\right) - \psi\left(\frac{x}{6^{r+1}}\right) \lesssim (.921^{+})\frac{x}{6^r} + B\ln\frac{x}{6^r} \qquad \text{step } r+1$$

$$\left\{ \begin{array}{ll} \text{take } \frac{x}{6^r} \in [216, 1296] & \text{for safety} \\ \text{hence } r = \dfrac{\ln x - \Omega}{\ln 6}, & \ln 216 \lesssim \Omega \lesssim \ln 1296 \end{array} \right\}$$

ADD

$$\Downarrow$$

$$\psi(x) + O(1) \lesssim \frac{6}{5}(.921^{+})x + B(r+1)\ln x$$

$$\Downarrow$$

$$\psi(x) \lesssim \frac{6}{5}(.921^{+})x + \frac{B}{\ln 6}(\ln x)^2 .$$

Note:

$$\frac{6}{5}(.9212920229^{+}) = 1.105550427^{+} .$$

Theorem B (harder Chebyshev $\approx$ 1850) (using ψ_σ)

For $x \geqq 2$, we have

$$(.921292^+)x + O(\ln x) \leqq \psi(x) \leqq (1.105550^+)x + O(\ln^2 x)$$

$$(.921292^+)x + O(\sqrt{x}) \leqq \theta(x) \leqq (1.105550^+)x + O(\sqrt{x})$$

$$[.921292^+ + o(1)]\frac{x}{\ln x} \leqq \pi(x) \leqq [1.105550^+ + o(1)]\frac{x}{\ln x}.$$

Here

$$1.105550^+ \approx \frac{6}{5}(.921292^+).$$

Proof

See 10 + 11 and inflate the (implied) constants (as necessary) for $\psi(x)$. The rest is like Theorem A'. ∎

Addeudum (for completeness)

On (5), $\psi_\sigma(x) = \sum_{n \leq x} \sigma\left(\frac{x}{n}\right)\Lambda(n)$. Let the value of $\sigma(y)$ on $[k, k+1)$ be σ_k. Here $k \geq 0$. Note that index $k \geq 1$ corresponds to $k \leq \frac{x}{n} < k+1$ hence $\frac{x}{k+1} < n \leq \frac{x}{k}$ in $\psi_\sigma(x)$.

Hence:

$$\psi_\sigma(x) = \sum_{k=1}^{\infty} \sigma_k \left\{ \sum_{\frac{x}{k+1} < n \leq \frac{x}{k}} \Lambda(n) \right\} = \sum_{k=1}^{\infty} \sigma_k \left[\psi\left(\frac{x}{k}\right) - \psi\left(\frac{x}{k+1}\right)\right]$$

$$= \sigma_1(\psi_1 - \psi_2) + \sigma_2(\psi_2 - \psi_3) + \sigma_3(\psi_3 - \psi_4) + \cdots$$

{ in shorthand }

$$= \psi_1\sigma_1 + \psi_2(\sigma_2 - \sigma_1) + \psi_3(\sigma_3 - \sigma_2) + \cdots$$

{ notice that $\sigma_n - \sigma_{n-1}$ is periodic $\underline{30}$ }

[apply (6) bottom]

$$= \psi(x) + \psi\left(\tfrac{x}{2}\right)(0) + \psi\left(\tfrac{x}{3}\right)(0) + \psi\left(\tfrac{x}{4}\right)(0) + \psi\left(\tfrac{x}{5}\right)(0)$$
$$- \psi\left(\tfrac{x}{6}\right) + \psi\left(\tfrac{x}{7}\right) + \psi\left(\tfrac{x}{8}\right)(0) + \psi\left(\tfrac{x}{9}\right)(0) + \psi\left(\tfrac{x}{10}\right)(-1)$$
$$+ \psi\left(\tfrac{x}{11}\right) + \psi\left(\tfrac{x}{12}\right)(-1) + \psi\left(\tfrac{x}{13}\right) + \psi\left(\tfrac{x}{14}\right)(0) + \psi\left(\tfrac{x}{15}\right)(-1)$$
$$+ \psi\left(\tfrac{x}{16}\right)(0) + \psi\left(\tfrac{x}{17}\right) + \psi\left(\tfrac{x}{18}\right)(-1) + \psi\left(\tfrac{x}{19}\right) + \psi\left(\tfrac{x}{20}\right)(-1)$$
$$+ \psi\left(\tfrac{x}{21}\right)(0) + \psi\left(\tfrac{x}{22}\right)(0) + \psi\left(\tfrac{x}{23}\right) + \psi\left(\tfrac{x}{24}\right)(-1) + \psi\left(\tfrac{x}{25}\right)(0)$$
$$+ \psi\left(\tfrac{x}{26}\right)(0) + \psi\left(\tfrac{x}{27}\right)(0) + \psi\left(\tfrac{x}{28}\right)(0) + \psi\left(\tfrac{x}{29}\right)(1) + \psi\left(\tfrac{x}{30}\right)(-1)$$
$$+ \psi\left(\tfrac{x}{31}\right)(1) + \cdots$$

$$\psi_\sigma(x) = (1) - (6) + (7) - (10) + (11) - (12) + (13) - (15)$$
$$+ (17) - (18) + (19) - (20) + (23) - (24) + (29) - (30)$$
$$+ (31) \pm \cdots$$

NOTE THE ALTERNATE SIGNS

Some Additional Remarks.

It's not immediately clear what other combinations of $L(x/j)$ can be used — and how much of an improvement can be gained.

The standard reference is:

Diamond and Erdős, On sharp elementary prime number estimates, L'Enseignement Math. 26 (1980) 313-321.

This reference is usually regarded as saying that if one already knows that the PNT is true, then there is in principle no obstruction to building better and better combinations that lead to "$1-\varepsilon$ and $1+\varepsilon$".

But, the assertion does not take into account the possibility of exploiting recursive relations and additional positive terms like 7 bottom line (and 8 top, for $n \lessapprox \frac{x}{6}$).

J. J. Sylvester found improvements based on use of recursive relations. See:

Sylvester, Amer. J. Math. 4 (1881) 230–247

.9569$^+$
1.0442$^+$

; Messenger of Math. 21 (1892) 1–19, and 87–120.

Also Mathews, Th. of Numbers, pp. 287–294 from 1892.

It seems fair to say the overall status of things is not as clear as one would like.

Incidentally, see:

a classic!

Landau, Handbuch der Lehre von der Verteilung der Primzahlen, vol. 1, pp. 94–95 (1909)

for a tiny improvement in $\frac{6}{5}(.921292^+)$ based just on ⑦ (bottom line) & ⑧ (top). He got:

$$\frac{171}{175}\,\frac{6}{5}(.921292^+)$$

$$= 1.080280^+ \; .$$

We'll drop this stuff temporarily: it seems obvious that some essentially new idea would be needed to reach $1-\varepsilon, 1+\varepsilon$ via "elementary reasoning".

of Thm B

<u>Corollary</u> (related to Bertrand's Postulate)

There exists a positive c so that, for large x,

$$\sum_{x < p \leqq 2x} \ln p \geqq cx .$$

<u>Pf</u>

$LHS = \theta(2x) - \theta(x)$. By Thm B,

$$\theta(2x) - \theta(x) \geqq 2(.921292^{+})x - (1.105550^{+})x + O(\sqrt{x})$$

$$= \frac{4}{5}(.921292^{+})x + O(\sqrt{x})$$

$$\geqq (.737)x + O(\sqrt{x}) . \quad \blacksquare$$

Hence,

$$\sum_{x < p \leqq 2x} 1 \geqq \frac{cx}{\log(2x)} .$$

the Bertrand issue

Theorem 2

For $x \geqq 2$,

$$\sum_{p \leqq x} \frac{\ln p}{p} = \ln x + O(1)$$

$$\sum_{n \leqq x} \frac{\Lambda(n)}{n} = \ln x + O(1) .$$

Proof

$$L(x) = \sum_{n \leqq x} [\![\tfrac{x}{n}]\!] \Lambda(n) \quad \text{for all } x \geqq 2. \qquad \begin{Bmatrix} \text{Lec 1} \\ \text{Thm 7} \end{Bmatrix}$$

Hence,

$$x \ln x + O(x) = \sum_{n \leqq x} [\![\tfrac{x}{n}]\!] \Lambda(n)$$

by Lec 1 Thm 6. Temporarily write

$$[\![\tfrac{x}{n}]\!] = \frac{x}{n} - \varphi_n \ , \quad 0 \leqq \varphi_n < 1 .$$

Get

$$x \ln x + O(x) = \sum_{n \leqq x} \frac{x}{n} \Lambda(n) - \sum_{n \leqq x} \varphi_n \Lambda(n)$$

This term is non-neg and $O(x)$ by thm A'

$$\Downarrow$$

$$x \ln x + O(x) = x \sum_{n \leq x} \frac{\Lambda(n)}{n} + O(x)$$

$$\Downarrow$$

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \ln x + O(1) .$$

Notice however that

$$\sum_{p^2 \leq x} \frac{\ln p}{p^2} + \sum_{p^3 \leq x} \frac{\ln p}{p^3} + \cdots$$

$$\leq \sum_{p^2 \leq x} \frac{\ln p}{p^2} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right)$$

$$\leq \sum_{\text{all } p} \frac{\ln p}{p^2} \frac{1}{1 - \frac{1}{p}} \leq 2 \sum_{\text{all } p} \frac{\ln p}{p^2} < +\infty .$$

At once,

$$\sum_{p \leq x} \frac{\ln p}{p} = \ln x + O(1) \quad \text{too.} \quad \blacksquare$$

(Thm 2 ~ Mertens ≈ 1874)

Theorem 3

We have

$$\liminf_{x\to\infty} \frac{\pi(x)\ln x}{x} = \liminf_{x\to\infty} \frac{\psi(x)}{x} \leq 1$$

$$\limsup_{x\to\infty} \frac{\pi(x)\ln x}{x} = \limsup_{x\to\infty} \frac{\psi(x)}{x} \geq 1 .$$

Hence, if $\psi(x) \sim cx$, we must have $c = 1$.

Pf

By ② corollary 1, need only treat $\psi(x)$.

Recall

$$L(x) = x\ln x + O(x) = \sum_{k\leq x} \psi\left(\frac{x}{k}\right)$$

for $x \geq 2$. See Lec 1, thms 6 + 7.

Assume $\liminf \frac{\psi(x)}{x} = q > 1$. Hence $\psi(y) \geq (1+\overset{\text{pos}}{h})y$ for all $y \geq x_0$. For large x, we have:

$$x\ln x + O(x) = \sum_{1\leq k\leq \frac{x}{x_0}} \psi\left(\frac{x}{k}\right) + \sum_{\frac{x}{x_0}<k\leq x} \psi\left(\frac{x}{k}\right)$$

$$\geq (1+h)\sum_{1\leq k\leq\frac{x}{x_0}} \frac{x}{k} + \sum_{\frac{x}{x_0}<k\leq x} O(1)\psi(x_0)$$

$$= (1+h)\,x\sum_{k=1}^{x/x_0}\frac{1}{k} + O[\psi(x_0)]\,x$$

$$= (1+h)x\left[\ln\frac{x}{x_0} + O(1)\right] + O[\psi(x_0)]x$$

$$= (1+h)x\ln x + O_{x_0}(1)x \ ,$$

$O_{x_0}(1)$ meaning the implied constant depends on x_0.
As $x \to \infty$, we get $1 \gtrsim 1+h$, an obvious contradiction. OK

The lim sup case is similar (again by contrad!).
Just take $\psi(y) \lesssim (1-h)y$ for $y \gtrsim x_0$. OK

∎

Date Highlights

Euler suggests PNT $\approx 1740 \sim 1760$

The boy → **Gauss** suggests PNT ≈ 1790
(counts in tables)

Legendre looks like $\dfrac{x}{\log x - c}$ ≈ 1800

Gauss letter to Encke $\int_2^x \dfrac{dt}{\ln t}$ 1849

Chebyshev rigorous $c_1 \dfrac{x}{\ln x} < \pi(x) < c_2 \dfrac{x}{\ln x}$ ≈ 1850

Riemann **1859** introd. of complex variable
(etc)

Sylvester elementary refinements 1880~1892
à la Chebyshev

de la Vallée-Poussin, Hadamard proof of PNT 1896

PARTIAL DIARY ENTRY FOR
Lectures 3 and 4
(27 Jan and 29 Jan)

+ SOME NEW STUFF

In lecture #3 and part of #4, we reviewed some key points in complex analysis (a subject regarded by many mathematicians as the <u>most beautiful</u> in mathematics, not only aesthetically but also vis à vis logical unity / coherence).

List of Some Definitions and Thms

$C^\infty(D)$ ← the usual

$A^\infty(D) \equiv$ the subset of $C^\infty(D)$ consisting of those complex-valued $f = u + iv$ for which we have a complex derivative

$$f'(z_0) \equiv \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

at each $z_0 \in D$

"A" for analytic!

$A(D) \equiv$ the set of complex-valued f for which we have a complex derivative $f'(z_0)$ at each $z_0 \in D$ (no other assumptions about $f = u + iv$)

Proved using $\Delta x = \frac{\Delta z + \overline{\Delta z}}{2}$, $\Delta y = \frac{\Delta z - \overline{\Delta z}}{2i}$ that

$$A^{\infty}(D) = \{u + iv \in C^{\infty}(D) : \underbrace{u_x = v_y, \; u_y = -v_x}_{\text{C-R eqs.}}\}.$$

Noted: $A^{\infty}(D)$ is a subring of $C^{\infty}(D)$; also that quotients f_1/f_2, composites $G[f(z)]$, and local inverses $\{$for $w = f(z)$, $f'(z_0) \neq 0\}$ have good properties.

Also noted: $f \in A^{\infty}(D) \Rightarrow f'(z) = u_x + iv_x = \frac{\partial f}{\partial x}$.
And that $f' \in A^{\infty}(D)$ as well.

C-R equations are <u>equivalent</u> to

$$\frac{\partial f}{\partial \bar{z}} \equiv \frac{1}{2}(f_x + i f_y) = 0 \quad \text{on } D.$$

Showed standard examples of $f \in A^{\infty}$(suitable D).

z^n, rational fcns of z,
$\exp(z) \equiv e^x(\cos y + i \sin y)$
$\log z = \ln|z| + i\arg(z)$ locally near some $z_1 \neq 0$

In Lec #4, did standard hand-waving about approximating "sensible" $f \in C^{\infty}$(suitable D) by polynomials in $(z, \bar{z})$. Hence being able to <u>explain</u> $A^{\infty}(D)$ to "man in the street".

Showed that

$$f \in A^{\alpha}(D) \Rightarrow f(z)dz = (u+iv)(dx+idy)$$
$$\equiv (udx - vdy) + i(vdx + udy)$$

is locally exact (i.e., closed). Accordingly, Green's thm can be brought to bear for suitable independence of path results for $\int_\gamma f(z)dz$.

no need to belabor

Showed that

$$F \in A^{\alpha}(D) \Rightarrow F'(z)dz = d\mathcal{U} + i d\mathcal{V} \quad (F \equiv \mathcal{U} + i\mathcal{V})$$

in the sense of standard differentials on RHS.

This produced

$$\int_\gamma F'(z)dz = F(B) - F(A)$$

as the "fundamental thm of complex integral calculus".

If $H(z)$ is continuous on γ, explained $\int_\gamma H(z)dz$ and why

$$\left|\int_\gamma H dz\right| \leq \int_\gamma |H(z)| ds .$$

Proved the standard <u>Cauchy - Goursat thm</u> for $f \in A(D)$, $D \simeq$ domain <u>straddling</u> closed rectangle R. Got:

$$\oint_{\partial R} f(z)\,dz = 0 . \qquad (\approx 1900)$$

Used bisection and nested interval/box thm.

Immediately went further to get the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial R} \frac{f(z)}{z - z_0}\,dz$$

for $z_0 \in \mathrm{int}(R)$. Here $f \in A(D)$.

Used Leibnitz's rule from adv calc to establish the fund thm that

$$A(\mathcal{D}) = A^{\infty}(\mathcal{D})$$

on any domain $\mathcal{D}$.

Turned quickly to a host of CLASSICAL theorems (in the "Cauchy theory").

(1) Cauchy Integral Thm
(2) Cauchy Integral Formula for $f(z_0)$

(3) Cauchy Integral Formula for $f^{(n)}(z_0)/n!$

(4) Max. Modulus Thm for $|f(z_0)|$, $z_0 \in D$ (gave the slick proof with CIF).

(5) Proved standard Cauchy – Taylor development for $f \in A(D)$, $D = \{|z-z_0| < R\}$: $f = \sum_0^\infty \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$.

(6) Given uniformly conv. $S(z) = \sum_{n=1}^\infty g_n(z)$ with g_n continuous; remarked about $S(z)$ being continuous, and

$$\int_\gamma p(z)\,S(z)\,dz = \sum_{n=1}^\infty \int_\gamma p(z)\,g_n(z)\,dz$$

for sensible $p(z)$.

(7) Weierstrass M – test for uniform conv.

(8) Used pathwise connectedness and a "marble" to prove that, when $f \in A(D)$, having $f \equiv 0$ on $\{|z-z_0| < \varepsilon\} \Rightarrow f \equiv 0$ on all D.

(9) Stated Laurent series for $f \in A(D)$, $D = \{R_1 < |z-z_0| < R_2\}$.

(10) Mentioned in one microsecond the idea of isolated singularities. Never said the words remov singularity, pole of order N, essential singularity.

(11) WANTED TO also stress the availability of Abel's lemma for an <u>arbitrary</u> power series $\sum_{n=0}^{\infty} a_n z^n$ which converges at $z_1 \neq 0$.

(12) Likewise, WANTED TO develop the Cauchy residue theorem (CRT)

$$\oint_{\Gamma} f(z)\,dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}(f; z_j)$$

after defining "Residue". Also wanted to do 2 quick examples. *

(13) Very quickly outlined the Weierstrass Convergence Theorem for $\sum_{n=1}^{\infty} f_n(z)$, $f_n \in A(D)$, and its proof (see 9 below).

(Time and Fortitude ran out on (10), (11), (12).)

---

* Also missed the argument principle

$$\frac{1}{2\pi i}\oint_{\Gamma} \frac{f'(z)}{f(z)}\,dz = N_0(f) = \frac{1}{2\pi}\Delta_{\Gamma} \arg f \;; \quad f \neq 0 \text{ on } \Gamma.$$

Other topics reviewed:

Used nested interval/box thm, bisection, and a Cantor diagonal to prove the Bolzano-Weierstrass thm in $\mathbb{R}^2$. Similarly $\mathbb{R}^k$. Noted that same proof shows that any <u>bdd + closed</u> $E \subseteq \mathbb{R}^k$ is sequentially compact.

Discussed the Riemann integrability criterion for

$$\int_a^b f(x)\,d\alpha(x)$$

f bdd, α monotonic increasing on $[a,b]$

using $U(P,f,\alpha)$, $L(P,f,\alpha)$ $\{P = \text{partition}\}$ and

$$\underline{\int_a^b} f(x)\,d\alpha(x)\,, \quad \overline{\int_a^b} f(x)\,d\alpha(x).$$

Using uniform continuity of $f \in C[a,b]$, proved that we actually have

$$\int_a^b f(x)\,d\alpha(x) = \lim_{\|P\| \to 0} \sum_{j=1}^N f(c_j)\,\Delta\alpha_j$$

whenever $f \in C[a,b]$. $\{\|P\| = \text{largest } \Delta x_j\}$

For $\int_a^b f(x)\,\underline{\underline{dx}}$ àla Riemann, remarked that

$$f \text{ monotonic} \Rightarrow f \in \mathcal{R}[a,b] \qquad \text{(easy)}$$

$$f, g \in \mathcal{R}[a,b] \Rightarrow fg \in \mathcal{R}[a,b].$$

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \qquad a<c<b$$

in a sensible way

Used summation by parts to derive the <u>integration by parts</u> formula:

$$\int_a^b f(x)\,d\alpha(x) = \Big[f(x)\alpha(x)\Big]_a^b - \int_a^b \alpha(x) f'(x)\,\underline{\underline{dx}}$$

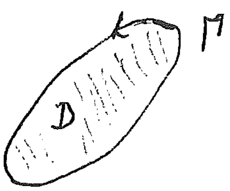
valid whenever $f \in C'[a,b]$ and $\alpha(x) \nearrow$ on $[a,b]$.

This Riemann integral DOES exist!

One gets an excellent review of the power of the structural properties of analytic functions by studying (developing) several results closely tied to the Weierstrass Convergence Thm.

It is always striking when one appears to be getting something for nothing.

<u>THEOREM</u> (the <u>standard</u> Weierstrass conv thm; similarly for multiply-connected D)



Let D be a simply-connected domain bounded by a piecewise smooth Jordan curve Γ.
Let $\{s_n(z)\}_{n=1}^{\infty}$ be a sequence of <u>analytic</u> functions on D which converges to a limit function $s(z)$ for each $z \in D$.
ASSUME THAT the convergence is <u>UNIFORM</u> on every <u>closed</u> subset K of D. Then:

(a) $s(z)$ must be analytic on D;

(b) we automatically have $s_n'(z) \to s'(z)$ <u>uniformly</u> on every closed $K \subsetneqq D$;

(c) similarly for $s_n^{(j)}(z) \to s^{(j)}(z)$, $j \gtrsim 2$.

$s_n(z)$ could, for instance, be $\sum_{k=1}^{n} f_k(z)$ with $f_k \in A(D)$

Weierstrass' Theorem is very well-known (and important). I sketched the proof of it in Lec #4. See 20¼ - 20¾ below for the essential details.

In the pages that follow, we give a review of some techniques which, taken together, permit one to formulate a less well-known [and much more impressive] variants of Weierstrass' theorem.

Lemma (stated in $\mathbb{R}^3$, but valid for $\mathbb{R}^k$, $k \geq 2$)

Let R be the box $[A_1, B_1] \times [A_2, B_2] \times [A_3, B_3]$. Let f be continuous on R. Then:

$$I(x,y) \equiv \int_{A_3}^{B_3} f(x,y,z)\,dz$$

is continuous on $\mathcal{R} = [A_1, B_1] \times [A_2, B_2]$, and

$$\iiint_R f(x,y,z)\,dV = \iint_{\mathcal{R}} I(x,y)\,dA$$

as in multi-variable calc.

Pf

As indicated, this is just a form of Fubini's thm from elem calc. The continuity of $I(x,y)$ is a familiar fact which is part of that theorem (or should be!) and is a simple consequence of the uniform continuity of f on R. ∎

Lemma (stated in $\mathbb{R}^3$ but valid in $\mathbb{R}^k$, $k \geq 2$)

Let R be the box $[A_1, B_1] \times [A_2, B_2] \times [A_3, B_3]$. Let $f(x,y;z)$ be continuous on R. In addition, let all partial derivatives of f wrt x & y also be continuous fcns of $(x,y;z)$ on R. $\{f_x, f_y, f_{xx}, f_{xy}, f_{yy}, \text{etc}\}$ Let

$$I(x,y) = \int_{A_3}^{B_3} f(x,y;z)\,dz .$$

Then $I(x,y)$ is C^∞ on $[A_1, B_1] \times [A_2, B_2]$ and we have

$$\frac{\partial I}{\partial x} = \int_{A_3}^{B_3} \frac{\partial f}{\partial x}(x,y;z)\,dz$$

$$\frac{\partial^2 I}{\partial x \partial y} = \int_{A_3}^{B_3} \frac{\partial^2 f}{\partial x \partial y}(x,y;z)\,dz$$

etc etc.

PF

This is Leibnitz's rule from advanced calc stated in iterated form — and relying on the foregoing lemma with a general $g(x,y;z)$. The proof is standard adv calc. ∎

Lemma

Let $F = \mathcal{U} + i\mathcal{V}$ (real) be a C^∞ function on, say, the open neighborhood $N = \{|z - z_0| < \delta\}$. The Cauchy-Riemann equations for $\mathcal{U}$ and $\mathcal{V}$ on N are equivalent to the relation

$$F_x + iF_y \equiv 0$$

on N. Hence $F_x + iF_y \equiv 0$ is the condition for F to belong to $A^\infty(N)$ (i.e., C^∞ + analytic).

Pf

Trivial calculation gives

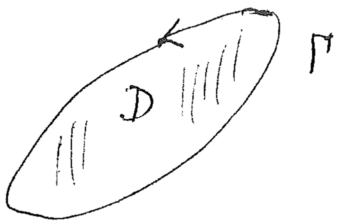
$$F_x + iF_y = \mathcal{U}_x + i\mathcal{V}_x + i(\mathcal{U}_y + i\mathcal{V}_y)$$
$$= (\mathcal{U}_x - \mathcal{V}_y) + i(\mathcal{U}_y + \mathcal{V}_x). \quad \blacksquare$$

Note:

$\dfrac{\partial F}{\partial \bar{z}} \equiv \dfrac{1}{2}(F_x + iF_y)$ is the standard definition.

Observe: $\dfrac{\partial(\bar{z})}{\partial \bar{z}} = 1$ and $\dfrac{\partial(z)}{\partial \bar{z}} = 0$.

LEMMA (about Cauchy-type integrals; similarly for multiply-connected D)



Let D be a simply-connected domain bounded by a piecewise smooth Jordan curve Γ. (See fig.) Let $\sigma(w)$ be a piecewise continuous complex-valued function on Γ.

Let

$$F(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{\sigma(w)}{w-z}\, dw \qquad \text{(DEF.)}$$

for $z \in D$. We then have:

(a) $F(z) \in C^\infty(D)$

(b) $F(z) \in A^\infty(D)$ (i.e., C^∞ + analytic)

(c)
$$F^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{\sigma(w)}{(w-z)^{n+1}}\, dw, \quad \begin{array}{l} n \geq 1 \\ z \in D \end{array}.$$

Proof

We can assume WLOG that Γ is smooth and that $\sigma(w)$ is continuous on Γ. Otherwise, in what follows, just split $F(z)$ into a SUM of several natural "chunk integrals".

Convert Γ to a parametric equation $w = w(t)$, $a \leqq t \leqq b$, $t \uparrow$. Get:

$$F(z) = \frac{1}{2\pi i}\int_a^b \frac{\sigma[w(t)]w'(t)}{w(t) - z}\,dt$$

Write $z = x+iy$. The integrand, as a fcn. of (x,y,t), satisfies the hypotheses of the Leibnitz Lemma on page 11 so long as $\alpha_1 \leqq x \leqq \beta_1$, $\alpha_2 \leqq y \leqq \beta_2$ is some small rectangle <u>inside</u> D.

Note that the numerator can be "pushed aside" since it has <u>no</u> dependence on z. Also, the partial derivatives of any

$$\frac{1}{W - x - iy} \qquad W \in \Gamma$$

wrt x and y are 100% trivial calc. Of course $\{(W-z)^{-1}$ lying in $\underline{A}^\infty\}$:

$$\frac{\partial}{\partial x}\left(\frac{1}{W-x-iy}\right) + i\frac{\partial}{\partial y}\left(\frac{1}{W-x-iy}\right)$$

$$= \frac{1}{(W-x-iy)^2} + i\,\frac{i}{(W-x-iy)^2} = 0!$$

(a) and (b) are now immediate on each $(\alpha_1,\beta_1)\times(\alpha_2,\beta_2)$, hence on all D by the Lemmas on 10 + 11 + 12.

To get (c), since we know $F \in A^\infty(D)$, just use

$$F^{(n)}(z) = \left(\frac{\partial}{\partial x}\right)^n F(z)$$

and the trivial fact that

$$\left(\frac{\partial}{\partial x}\right)^n (w-x-iy)^{-1} = n!\,(w-x-iy)^{-n-1}$$

[in addition to Leibnitz' Lemma on 11]. ∎

OK

Lemma (stated for $\mathbb{R}^2$, readily adapted to $\mathbb{R}^k$, $k \geq 1$)

Let $s_n(z) = s_n(x+iy)$ be a sequence of [complex-valued] functions on the closed rectangle $R: [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ OR, if you prefer, closed disk R. Assume that $s_n(z) \to$ some function $s(z)$ pointwise for $z \in R$. Assume further that, for some $M > 0$, we have *

$$|s_n(z_1) - s_n(z_2)| \leq M |z_1 - z_2|$$

for all $n \geq 1$ and $z_j \in R$. (Uniform Lipschitz condition!) THEN: the convergence of $s_n(z)$ to $s(z)$ is automatically uniform on R.

Pf

The procedure for this is standard. Choose any tiny $\varepsilon > 0$. Look at R and select a finite grid of points $\{P_1, \dots, P_L\} \subseteq R$ so that every point $z \in R$ is located within $\frac{\varepsilon}{5M}$ units of some P_ℓ. THIS IS CERTAINLY POSSIBLE!

Since L is finite, we can select N_ε so big

* The word "equicontinuity" may come to mind here.

$$|s_n(P_j^\circ) - s(P_j^\circ)| < \frac{\varepsilon}{10}$$

for all $n \geqq N_\varepsilon$ and all $j \in [1, k]$. Get:

$$|s_n(P_j^\circ) - s_m(P_j^\circ)| < \frac{\varepsilon}{5}$$

for all $n \geqq m \geqq N_\varepsilon$, $j \in [1, k]$.

Take *any* $z \in R$. Select P_ℓ so that

$$|z - P_\ell| \leqq \frac{\varepsilon}{5\mathfrak{M}}.$$

For $n \geqq m \geqq N_\varepsilon$, notice that

$$|s_n(z) - s_m(z)| \leqq |s_n(z) - s_n(P_\ell)| + |s_n(P_\ell) - s_m(P_\ell)| + |s_m(P_\ell) - s_m(z)|$$

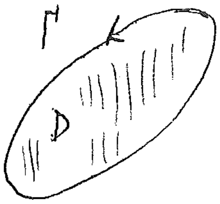
$$\leqq \mathfrak{M}\left(\frac{\varepsilon}{5\mathfrak{M}}\right) + \frac{\varepsilon}{5} + \mathfrak{M}\left(\frac{\varepsilon}{5\mathfrak{M}}\right)$$

$$= \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} < \varepsilon.$$

This is the standard **uniform** **Cauchy** condition for *uniform* conv of $\{s_i(z)\}_{i=1}^{\infty}$ over R. Indeed, by letting $n \to \infty$, we get

$$|s(z) - s_m(z)| \leqq \varepsilon \quad \text{anytime} \left\{ \begin{matrix} m \geqq N_\varepsilon \\ z \in R \end{matrix} \right\}.$$

Theorem (strengthened Weierstrass conv. thm; similarly for multiply-connected D)



Let D be a simply-connected domain bounded by a piecewise smooth Jordan curve Γ. Let $\{s_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions on D which converges pointwise to some [not necessarily analytic] function $s(z)$ for each $z \in D$.

Assume that, for each closed subset K of D, there exists a constant $M(K)$ so that

$$|s_n(z)| \leqq M(K)$$

whenever $z \in K$ and $n \geqq 1$. THEN:

(a) $s(z)$ must be analytic on D;

(b) $s_n(z)$ converges UNIFORMLY to $s(z)$ on every closed subset of D;

(c) we have $s_n'(z) \to s'(z)$ UNIFORMLY on every closed subset of D;

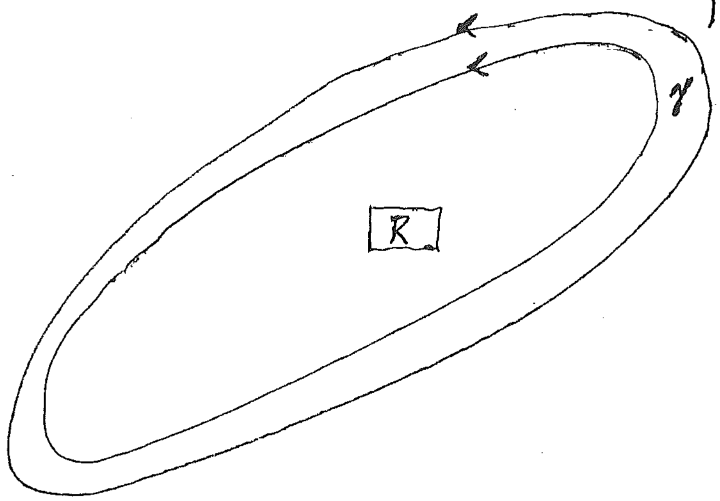
(d) similarly for $s_n^{(j)}(z) \to s^{(j)}(z)$, $j \geqq 2$.

Key Issue: where did all the UNIFORM conv. come from?

Pf

Not surprisingly, we rely on page 16 Lemma.

Every closed set $K \subseteq D$ is automatically bounded. Hence K is bdd + closed; hence, sequentially compact, etc. Any such K will lie a positive distance from Γ. Likewise from a piecewise smooth Jordan curve γ in D "paralleling Γ" extremely closely!



Note that γ is itself a closed subset of D.

Let $|s_n(z)| \leq M_0$ for $z \in \gamma$, $n \geq 1$.

Consider now any small closed rectangle R situated "inside" γ. Let $h = \operatorname{dist}(R, \gamma)$. For $\xi \in R$,

$$s_n'(\xi) = \frac{1}{2\pi i} \oint_\gamma \frac{s_n(z)}{(z-\xi)^2}\,dz \quad \text{by CIF for deriv.}$$

$$\Rightarrow \quad |s_n'(\xi)| \leq \frac{1}{2\pi} \int_\gamma \frac{M_0}{h^2}\,ds = \underbrace{\frac{1}{2\pi}\frac{M_0}{h^2}\ell(\gamma)}_{\text{call this } \tilde{M}}.$$

Connect any two points ξ_1 and ξ_2 of R by a line segment to get

$$s_n(\xi_2) - s_n(\xi_1) = \int_{\xi_1}^{\xi_2} s_n'(z)\,dz$$

linear

$$|s_n(\xi_2) - s_n(\xi_1)| \leq \int_{\xi_1}^{\xi_2} \tilde{\mathfrak{M}}\,ds = \tilde{\mathfrak{M}}\,|\xi_2 - \xi_1| .$$

Because of this, Lemma on page 16 applies.
Hence: $s_n(z) \to s(z)$ UNIFORMLY on R !!

Since R can be (reduced in size and) slid around, and γ can always be pushed closer to Γ, we get that $s_n(z) \to s(z)$ UNIFORMLY on each closed subset K of D. (It's best to fix K first, then select the Jordan curve γ.) This proves (b).

At this point, the standard Weierstrass convergence thm applies and, so, we get (a) + (c) + (d). OK ∎

A person wishing to keep matters completely **self-contained** would reason here alternatively as follows.

Let $\rightrightarrows$ signify uniform conv.

Refer to the picture on 19. Let R_0 be a closed rectangle just slightly bigger than R. 
Since we could have just as well used R_0 instead of R, we know that $s_n(z) \rightrightarrows s(z)$ **on** R_0.

By unif conv, $s(z)$ is continuous on R_0.

For $z_1 \in R$,

$$s_n(z_1) = \frac{1}{2\pi i} \oint_{\partial R_0} \frac{s_n(w)}{w - z_1} dw \qquad (n \geq 1).$$

By unif conv of $s_n(w)$ on ∂R_0 and use of test function $p(w) = \frac{1}{w - z_1}$, we get (via $n \to \infty$)

$$s(z_1) = \frac{1}{2\pi i} \oint_{\partial R_0} \frac{s(w)}{w - z_1} dw, \quad \text{each } z_1 \in R.$$

By the FACT on p. 13, $s(z)$ must belong to A^∞ on the interior of R.

Now just take the whole "picture frame" $\{R, R_0\}$ and slide it around slightly within the interior of γ. See picture on 19.

We get $s(z)$ = analytic on an open set containing R_0. We already know $s_n(z) \rightrightarrows s(z)$ on R_0.

Let $h = \mathrm{dist}(R, \partial R_0)$. For $z \in R$,

$$s_n'(z) - s'(z) = \frac{1}{2\pi i} \oint_{\partial R_0} \frac{s_n(w) - s(w)}{(w-z)^2}\, dw$$

{by CIF for deriv.}

$$|s_n'(z) - s'(z)| \leqq \frac{1}{2\pi} \oint_{\partial R_0} \frac{|s_n(w) - s(w)|}{h^2}\, |dw|$$

$$|s_n'(z) - s'(z)| \leqq \frac{1}{2\pi h^2}\, \ell(\partial R_0) \max_{\partial R_0} |s_n(w) - s(w)| .$$

But, choose N_ε so $|s_n(w) - s(w)| < \varepsilon$ for all $n \geqq N_\varepsilon$ and $w \in R_0$. Hence, $n \geqq N_\varepsilon \Rightarrow$

$$|s_n'(z) - s'(z)| < \frac{\ell(\partial R_0)}{2\pi h^2}\, \varepsilon \ , \quad z \in R .$$

This proves $s_n'(z) \rightrightarrows s'(z)$ on R. Similarly for $s_n^{(j)}(z)$.

Since R can be reduced in size and slid around, and γ can always be pushed closer to Γ, we immediately get (a), then (b)(c)(d). $\blacksquare$ $\blacksquare$

variant

The following version of the Weierstrass conv. theorem is <u>still</u> stronger and was proved by Vitali.

It shows how the <u>structure</u> of analytic functions can be exploited to induce a kind of "global rigidity".

<u>THEOREM</u> (<u>Vitali</u>). ← Similarly for multiply-connected D.

Let D and Γ be as on 18.

Let $\{s_n(z)\}_{n=1}^{\infty}$ be a sequence of <u>analytic</u> functions on D which satisfies the $M(K)$ hypothesis on 18 for each closed $K \subseteq D$.

ASSUME THAT $\lim_{n\to\infty} s_n(\xi_j)$ exists for each $j \geq 1$, where the ξ_j are <u>distinct</u> points of D tending to a point (say, ξ_∞) <u>of</u> <u>D</u>.

THEN:

(a) $\{s_n(z)\}_{n=1}^{\infty}$ <u>automatically</u> converges to a limit function $s(z)$ at each point of D;

(b) the convergence in (a) is UNIFORM on each closed $K \subseteq D$.

Pf

We first recall a simple fact about complex numbers.

Lemma

Given sequence $\{w_n\}_{n=1}^{\infty}$ in $\mathbb{C}$. This sequence converges to L as $n \to \infty$ if and only if every subsequence $\{w_{n_j} : n_1 < n_2 < n_3 < \cdots\}$ admits a further subsequence which converges to L.

Pf of lemma

The "only if" is obvious. For the "if", we use contradiction. Hence there must exist some bad $\varepsilon_0 > 0$ with no "N_{ε_0}". IE we can find arbitrarily big $\tilde{n}$ for which $|w_{\tilde{n}} - L| \geq \varepsilon_0$. Make a recursion construction to get $n_{j+1} > n_j \geq 1$ satisfying

$$|w_{n_j} - L| \geq \varepsilon_0 , \quad j \geq 1 .$$

By hypothesis, there exists a (increasing) subseq $\mathcal{S}$ of $\{n_j\}_{j=1}^{\infty}$ for which

$$\{w_m : m \in \mathcal{S}\} \to L .$$

But $\mathcal{S} \subseteq \{n_j\} \Rightarrow |w_k - L| \geq \varepsilon_0$ for each $k \in \mathcal{S}$.

Contradiction! ∎

We now turn to the proof of the THM.

The reasoning that follows is closely related to the Arzela–Ascoli theorem in real analysis (or point-set topology).

> See, e.g., Rudin, Principles of Math Analysis, 3rd ed, Theorem 7.25. Also 7.23.

Choose any $\delta > 0$. By taking a grid of points on D, we can clearly find a finite set $E_\delta \subseteq D$ such that every point of D lies within δ units of some point of E_δ. The set

$$\mathcal{C} = \bigcup_{k=1}^{\infty} E_{1/k}$$

is then countable and dense in D.

Let $Q_j = \lim_{n\to\infty} s_n(\xi_j)$ for each $j \geq 1$.

Also let $\mathcal{S}$ be any increasing subsequence of $\{n\}_{n=1}^{\infty}$.

By combining hypothesis $\mathfrak{M}(K)$, the Bolzano–Weierstrass thm, and the Cantor diagonal process, we can construct an increasing subsequence $\mathcal{S}_1$ of $\mathcal{S}$ such that

$$\lim_{n\to\infty} \{ s_n(P) : n \in \mathcal{S}_1 \} \text{ exists}$$

for EACH $P \in \mathcal{C}$.

At this juncture, we go back into the proof of p. 18 THM.

The key initial observation is this. Let R be any closed rectangle situated within γ. See picture on 19. Since $\mathcal{C}$ is dense in D, there exists a finite set of points $\{P_1, \ldots, P_L\} \subseteq R \cap \mathcal{C}$ satisfying the $\varepsilon/5M$-unit condition on 16 bottom. This assertion requires just a bit of care in handling points near ∂R; see also the very important page 20 (top 4 lines).

Keeping $n \in \mathcal{S}_1$, notice that lines 3–13 on 17 can now be re-used [since $P_j \in \mathcal{C}$ and $\lim_n \{s_n(P_j) : n \in \mathcal{S}_1\}$ exists!!!].

We conclude that $\{s_n : n \in \mathcal{S}_1\}$ is uniformly Cauchy on R.

Hence $\{s_n : n \in \mathcal{S}_1\} \rightrightarrows$ some $s(z)$ on each R.

By sliding R as in the proof of p. 18 THM (see especially 20), we get that $\{s_n : n \in \mathcal{S}_1\} \rightrightarrows s(z)$ on every closed $K \subseteq D$.

Note that the points $\{\xi_j\}_{j=1}^{\infty}$ will lie in some fixed closed $K \subsetneqq D$.

(9)

We can now apply either the traditional or p. (18) strengthened Weierstrass conv theorem to $\{s_n : n \in \underline{\underline{S_1}}\}$. The fcn $s(z)$ is thus analytic on D. Moreover, by substituting $z = \xi_j$, we find that

$$s(\xi_j) = Q_j \; , \quad j \geq 1 \, .$$

For Q_j, recall (23).

Let $\tilde{S}$ be any other increasing subseq of $\{n\}_{n=1}^{\infty}$. See (23). Form $\tilde{S}_1$ by analogy with S_1. The limit function for $\{s_n : n \in \tilde{S}_1\}$ will be $\tilde{s}(z)$. The function $\tilde{s}(z)$ is again analytic on D, and satisfies $\tilde{s}(\xi_j) = Q_j$.

The function $H(z) \equiv s(z) - \tilde{s}(z)$ is analytic on D and vanishes at each ξ_j. Hence also at ξ_∞. If $H(z) \not\equiv 0$ on D, any zero at ξ_∞ would need to be <u>isolated</u>. [[This follows from the local Taylor expansion.]] Since $\xi_j \to \xi_\infty$, we get an immediate violation. Hence: $H(z) \equiv 0$ and $s(z) \equiv \tilde{s}(z)$ on D.

CLAIM:

For each $r \in D$, $\{s_n(r) : \underline{\underline{n \geq 1}}\} \to s(r)$.

Pf of Claim

Just use the Lemma on 22. We need only show that any increasing subseq $\tilde{\mathcal{S}}$ of $\{n\}_{n=1}^{\infty}$ admits a subsequence $\tilde{\mathcal{S}}_0$ such that

$$\{s_n(r) : n \in \tilde{\mathcal{S}}_0\} \longrightarrow s(r).$$

But, $\tilde{\mathcal{S}}_1$ works in the role of $\tilde{\mathcal{S}}_0$ (since we just proved that $\tilde{s}(z) \equiv s(z)$). OK on the Claim. ▪

One is now exactly in the situation of p. 18 THM — and so we are done. ▪

PARTIAL DIARY ENTRY for

Lecture 5

(3 Feb 2016)

We first went over a number of elementary facts and properties. The goal today was to begin the Riemann zeta fcn in earnest.

Topic I

About Riemann-Stieltjes integrals.

Showed that even if $\alpha(x)$ is right continuous and $\nearrow$ on $[0,1]$, taking f to be piecewise continuous can lead to

$$\overline{\int_0^1} f(x)\,d\alpha(x) = 1\,, \quad \underline{\int_0^1} f(x)\,dx = 0\,.$$

Discouraging!! So, best to use R-S for continuous f when possible.

Showed:

$f \in C[1, N] \Rightarrow$

$$\int_1^N f(x)\, d[\![x]\!] = f(2) + \cdots + f(N)$$

note carefully

$g \in C[\beta, N] \quad (0 < \beta < 1) \Rightarrow$

$$\int_\beta^N g(x)\, d[\![x]\!] = g(1) + \cdots + g(N).$$

similarly for $g \in C[\beta, N+\beta]$

Hence, R-S has natural connection with sums!

Topic II

Abel's Lemma for power series.

Given $\sum_{n=0}^{\infty} a_n z^n$ which converges at $z_1 \neq 0$.

Then: $|a_n| \leq \dfrac{M}{|z_1|^n}$ for some M and all $n \geq 0$.

Hence, the orig power series conv uniformly and absolutely on each closed disk $\{|z| \leq |z_1| - \delta\}$.

And Weierstrass Conv Thm applies!! on $|z| < |z_1|$

Pf Trivial. ▥

Topic III

Another well-known result of Abel.

Thm (Abel)

Let $\sum_{n=0}^{\infty} a_n z^n$ converge at, say, $z=1$ (to S). — $S(z)$

Then:

$$\sum_{n=0}^{\infty} a_n x^n$$

conv. uniformly on $[0,1]$. Hence $\lim_{x\to 1^-} S(x) = S$.

(Similarly along $z = re^{i\theta}$.)

Pf

Uniform Cauchy estimate + Abel summation.

Must prove

$$|S_N(x) - S_M(x)| < \varepsilon \quad , \quad \text{all } N > M \geq \mathcal{N}_\varepsilon .$$

We know, of course,

$$|a_{M+1} + \cdots + a_N| < \varepsilon \quad \text{for } N > M \geq N_\varepsilon .$$

Claim that we can take $\mathcal{N}_\varepsilon = N_\varepsilon$. Put

$$T_k = a_{M+1} + \cdots + a_k \quad , \quad k \geq M+1 .$$

Get:

$$a_{M+1}x^{M+1} + \cdots + a_N x^N$$

$$= T_{M+1}x^{M+1} + (T_{M+2} - T_{M+1})x^{M+2} + \cdots +$$
$$(T_N - T_{N-1})x^N$$

$$= T_{M+1}(x^{M+1} - x^{M+2}) + \cdots$$
$$+ T_{N-1}(x^{N-1} - x^N) + T_N x^N .$$

Know $|T_k| < \varepsilon$, $k \geq M+1$. Get:

$$\text{ABS VALUE} < \varepsilon(x^{M+1} - x^{M+2}) + \cdots + \varepsilon(x^{N-1} - x^N)$$
$$+ \varepsilon x^N$$

$$\{0 \leq x \leq 1\} \quad x^j - x^{j+1} \geq 0$$

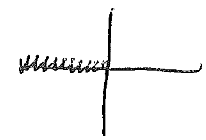
$$= \varepsilon x^{M+1} \leq \varepsilon .$$

Hence all is OK. ▥

This proof can clearly
be generalized to work in
<u>many</u> other settings!

Topic IV

Traditional to define principal value of $\arg(w)$ by declaring $-\pi < \mathrm{Arg}(w) < \pi$ and keeping w off the negative real axis $(-\infty, 0]$.



$$\mathrm{Log}(w) = \ln|w| + i\,\mathrm{Arg}(w)$$

Nice analytic fcn for $\mathbb{C} \sim (-\infty, 0]$.

$$\frac{d}{dw}\mathrm{Log}\, w = \frac{1}{w} \qquad \left\{ \begin{array}{l} \text{local inverses} \\ \text{are analytic, etc} \end{array} \right\}$$

So, $f(z) = \mathrm{Log}(1+z)$ is analytic for $|z| < 1$.

Cauchy - Taylor $\Rightarrow$

$$f(z) = \mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \pm \cdots, \quad |z| < 1.$$

get unif + abs conv for $|z| \leq 1 - \delta$

Thm 1 (basic def of $\zeta(z)$)

↑ RIEMANN ZETA FCN.

We write

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \qquad \{ n^{-z} \equiv \exp(-z \ln n) \}$$

for $Re(z) > 1$. The series conv unif and absolutely in every half-plane $\{Re(z) \geq 1+\delta\}$. Hence, $\zeta(z)$ is nicely analytic on $\{Re(z) > 1\}$.

Pf

Weierstrass M-test with $M_n = n^{-1-\delta}$. ▥

Also Weierstrass Conv Thm!

Thm 2

There exists a function $F(z)$ which is analytic on $\{Re(z) > 0\} - \{1\}$ such that $F(z) = \zeta(z)$ whenever $Re(z) > 1$. The fcn F is unique (numerically). We call it the analytic continuation of $\zeta(z)$. One can see that, near $z = 1$,

$$F(z) = \frac{1}{z-1} + [\text{something analytic}].$$

in, say, $|z-1| < \frac{1}{2}$

Pf

Suppose there were two: F_1 and F_2.
The fcn $F_1 - F_2$ is analytic on $\{Re(z) > 0\} - \{1\}$ but $\equiv 0$ for $Re(z) > 1$. By properties of analytic fcns, get $F_1 - F_2 \equiv 0$ everywhere.

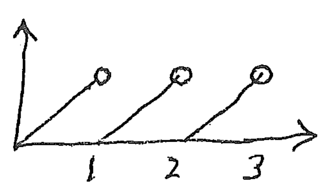
Hence F must be unique.

Must now find one F.

Take $R = N + \varepsilon$ for some tiny $\varepsilon > 0$. Keep $Re(z) > 1$.

$$1 + \int_1^R t^{-z} \, d[\![t]\!] = 1 + 2^{-z} + 3^{-z} + \cdots + N^{-z}.$$

Notice (see 2) that nothing is lost if we simply take $\varepsilon = 0$ rather than let $\varepsilon \to 0$. (Make sure you understand this; this type of trick is used a lot!)



Write $t = [\![t]\!] + r(t)$.

right continuous

$0 \leq r(t) < 1$

$\boxed{\text{diff of two increasing right continuous fcns}}$

Get:

$$\sum_{n=1}^{N} n^{-z} = 1 + \int_1^N t^{-z}\, d[\![t]\!]$$

$$= 1 + \int_1^N t^{-z}\, d(t - r(t))$$

$$= 1 + \int_1^N t^{-z}\, dt - \int_1^N t^{-z}\, dr(t)$$

R-S integral and t^{-z} nicely C^1 wrt t

$$\boxed{\frac{d}{dv} e^{cv} = c e^{cv} \text{ for } v \in \mathbb{R}} \Rightarrow \left\{ \begin{aligned} \frac{d}{dt} t^c &= \frac{1}{t} \frac{d}{d(\ln t)} t^c \\ &= \frac{1}{t} \frac{d}{d(\ln t)} e^{c(\ln t)} \\ &= \frac{1}{t} e^{c(\ln t)} \cdot c = \underline{c t^{c-1}} \\ &\text{for } c \in \mathbb{C} \text{ and } t > 0 \end{aligned} \right\}$$

$$= 1 + \frac{N^{1-z} - 1}{1-z} - \left[t^{-z} r(t) \right]_1^N + \int_1^N r(t)(-z) t^{-z-1}\, dt$$

parts

$$= 1 + \frac{1 - N^{1-z}}{z - 1} - 0 + 0 - z \int_1^N \frac{r(t)}{t^{z+1}}\, dt$$

$r(1) = r(N) = 0$

So, $Re(z) > 1$ gives (by taking $N \to \infty$)

$$\sum_{n=1}^{N} n^{-z} = 1 + \frac{1 - N^{1-z}}{z-1} - z \int_1^N \frac{r(t)}{t^{z+1}}\, dt$$

$$\zeta(z) = 1 + \frac{1}{z-1} - z \int_1^\infty \frac{r(t)}{t^{z+1}}\, dt$$

$$\left\{ \begin{aligned} |t^c| &= \left| e^{(\alpha + i\beta)\ln t} \right| = e^{\alpha \ln t} \\ &= t^\alpha = t^{Re(c)}, \quad t > 0 \end{aligned} \right\}.$$

In the formulae above, note that:

formula #1 holds for <u>any</u> $z \in \mathbb{C} \sim \{1\}$;

formula #2 holds for $Re(z) > 1$;

the <u>integral</u> in formula #2 is nicely absolutely + uniformly convergent so long as $x \geqq \delta > 0$ (!!!)

$\boxed{\text{HENCE analytic à la Weierstrass conv thm on } Re(z) > 0}$

Note: there is a Weierstrass M-test for improper integrals $\int_1^\infty$; one should review it. (Cf. any adv calc book.)

We can thus put

$$F(z) \equiv 1 + \frac{1}{z-1} - z\int_1^\infty \frac{r(t)}{t^{z+1}}\,dt$$

for all $\mathrm{Re}(z) > 0$ except $z = 1$. This choice works in Thm 2 on page 6. $\blacksquare$

$\boxed{z = x + iy}$

Thm 3

We have $|\zeta(z)| \leq \zeta(x)$ for all $\mathrm{Re}(z) > 1$.

We also have

$$|\zeta(z) - 1| < 2^{-x}\left(1 + \frac{2}{x-1}\right)$$

for all $\mathrm{Re}(z) > 1$. {Note that RHS is $< 3 \cdot 2^{-x}$ whenever $x > 2$.}

Pf

$$\left|\sum_{n=1}^{\infty} n^{-z}\right| \leq \sum_{n=1}^{\infty} |n^{-z}| = \sum_{n=1}^{\infty} n^{-x} \qquad (\text{see } 9)$$

$$= \zeta(x)$$

Also:

$$\left|\sum_{n=2}^{\infty} n^{-z}\right| \leq \sum_{n=2}^{\infty} n^{-x} < 2^{-x} + \int_2^{\infty} u^{-x}\,du \qquad (x>1)$$

$$\{\text{by baby areas}\}$$

$$= 2^{-x} + \left[\frac{u^{1-x}}{1-x}\right]_2^{\infty}$$

$$= 2^{-x} + \frac{2^{1-x}}{x-1}$$

$$= 2^{-x}\left\{1 + \frac{2}{x-1}\right\}.$$

∎

> By Thm 2,
>
> $$\zeta(x) = \frac{1}{x-1} + O(1)$$
>
> as $x \to 1^+$.

Thm 4 (very crude)

Keep $|z-1| \geq \frac{1}{3}$ say. Take any $0<\delta<1$. We then have

$$|\zeta(x+iy)| = O(1)\frac{1}{\delta}(1+|y|)$$

whenever $\delta \leq x \leq 1+\delta$. For $x \geq 1+\delta$, we have

$$|\zeta(x+iy)| = O(1)\frac{1}{\delta}.$$

{In $O(1)$, the implied constant is absolute.}

Pf

Use (10) line 5. Keep $\delta \leq x \leq 1+\delta$. Get:

$$\begin{aligned}
|\zeta(x+iy)| &\leq 1+3+|z|\int_1^\infty \frac{1}{t^{x+1}}\,dt \\
&\leq 4+(|x|+|y|)\int_1^\infty \frac{1}{t^{x+1}}\,dt \\
&\leq 4+(2+|y|)\frac{1}{x} \qquad (x>0) \\
&\leq 4(1+|y|)+2(1+|y|)\frac{1}{x} \\
&= (1+|y|)\left(4+\frac{2}{x}\right) \\
&\leq (1+|y|)\left(\frac{4}{\delta}+\frac{2}{\delta}\right) = \frac{6}{\delta}(1+|y|).
\end{aligned}$$

For $x \geq 1+\delta$, simply use $|\zeta(z)| \leq \zeta(x)$ (thm 3) and $\zeta(1+\delta) = \frac{1}{\delta}+O(1)$. ∎

We then paused to discuss infinite products, a topic which seems to have disappeared from UM's undergrad math curriculum!

We do not give a treatise; AND we will deal with products of COMPLEX numbers.
$\boxed{\text{using only REAL is much easier!!!}}$

Like with $\sum_{n=1}^{\infty} a_n$ vis à vis conv/div, matters should focus on the tail end of the series (or product), NOT on the first 10^{100} terms.

Unless $a_n \to 0$, $\sum_1^{\infty} a_n$ is div.

Unless $a_n \to 1$, $\prod_1^{\infty} a_n$ is (said to be) div!

We therefore focus on products with the first 10^{100} terms erased and presuppose that $a_n = 1 + b_n$, with $|b_n| \lesssim \lambda < 1$ for some λ.

Def

Given $a_n = 1 + b_n$ with $|b_n| \leqq \lambda < 1$.

We say

$$\prod_{n=1}^{\infty} a_n \quad \text{conv} \quad \text{to} \quad P$$

if

(a) $P \neq 0$

and

(b) $\frac{P_N}{P} \to 1$ as $N \to \infty$.

"Multiplicative style"

Here $P_N = a_1 \cdots a_N$.

If $a_n(z) = 1 + b_n(z)$, $|b_n(z)| \leqq \lambda < 1$, $z \in E$,

we say

$$\prod_{n=1}^{\infty} a_n(z) \quad \text{conv unif} \quad \text{to} \quad P(z)$$

if

$P(z) \neq 0$ and $\frac{P_N(z)}{P(z)} \to 1$ uniformly as $N \to \infty$.

Def

Given $a_n = 1 + b_n$ as above. We say

$$\prod_{n=1}^{\infty} a_n \quad \text{conv absolutely}$$

when

$$\prod_{n=1}^{\infty} (1 + |b_n|) \quad \text{converges.}$$

NOT $|a_n|$

N.B. see 24 below!

as above 14

Lemma

Suppose $\prod_{n=1}^{\infty} a_n(z)$ conv unif to $P(z)$ on E.

Then, there exist $c_j > 0$ so that

$$c_1 < |P(z)| < c_2 \quad \text{on } E.$$

Pf

Choose M so big that

$$\left| \frac{P_N(z)}{P(z)} - 1 \right| < 10^{-6} \qquad (z \in E)$$

for all $N \geq M$.

Get:

$$\left|\frac{P_M}{P} - 1\right| < 10^{-6} \qquad \left|\frac{P}{P_M} - 1\right| < 10^{-5}$$

$$\Downarrow$$

$$\frac{3}{4} < \left|\frac{P}{P_M}\right| < \frac{5}{4} \qquad \text{certainly}$$

$$\Downarrow$$

$$\frac{3}{4}|P_M| < |P| < \frac{5}{4}|P_M|$$

$$\Downarrow$$

$$\frac{3}{4}(1-\lambda)^M < |P| < \frac{5}{4}(1+\lambda)^M . \qquad \blacksquare$$

Corollary (important)

Notation as above. Suppose $\prod_{n=1}^{\infty} a_n(z)$ conv. unif to $P(z)$ on E. We then also have

$$P_N(z) \rightrightarrows P(z) \quad \text{on } E .$$

recall that this means UNIF CONV

Pf

Obvious because of Lemma on 15. ▨

Lemma

Recall $\mathrm{Log}(w)$ and $\mathrm{Arg}(w)$ on 5.

Suppose that $|w_1 - 1| < 1$ and $|w_2 - 1| < 1$.

Then:

$$w_1 w_2 \notin (-\infty, 0]$$

and

$$\mathrm{Arg}(w_1 w_2) = \mathrm{Arg}(w_1) + \mathrm{Arg}(w_2).$$

Pf

Write $w_j = R_j e^{i\theta_j}$. Clearly $-\frac{\pi}{2} < \theta_j < \frac{\pi}{2}$ and $R_j > 0$. Hence

$$w_1 w_2 = R_1 R_2 e^{i(\theta_1 + \theta_2)}$$

and

$$-\pi < \theta_1 + \theta_2 < \pi.$$

Done! ▨

$$\boxed{\begin{array}{l} \text{For } |w_j - 1| < \frac{1}{100}\text{, baby trig} \\ \Rightarrow \mathrm{Arg}(w_1 \cdots w_{100}) = \sum_{1}^{100} \mathrm{Arg}(w_j). \end{array}}$$

General Thm

Let E be some set which might possibly be just one point. Given $a_n(z) = 1 + b_n(z)$, $|b_n(z)| \leqq \lambda < 1$, $z \in E$, as above.

We then have:

$\prod_{n=1}^{\infty} a_n(z)$ conv unif to some $P(z)$ on E

if and only if

$\sum_{n=1}^{\infty} \mathrm{Log}(1 + b_n(z))$ conv unif to some $S(z)$ on E.

And, if so,

$$P(z) = \exp\{S(z)\}.$$

Pf

This thm is not (just) hand-waving trivia by "passing to logs". It is NOT true that

$$\mathrm{Log}(w_1 \cdots w_N) = \sum_{1}^{N} \mathrm{Log}(w_j)$$

in general, even if $w_j \approx 1$.

think, eg, $w_j = e^{2\pi i/N}$

Suppose first that $S_N(z) \rightrightarrows S(z)$, where

$$S_N = \sum_{n=1}^{N} \mathrm{Log}(1+b_n(z)) .$$

From unif conv (and λ), automatically (get) $|S(z)| < \mathfrak{M}$ for some $\mathfrak{M}$.

just imitate p. 15 Lemma

$$\Downarrow$$

We can now exponentiate freely.

$P_N = \exp(S_N)$
$P = \exp(S)$ makes sense on E

$$\frac{P_N}{P} \rightrightarrows 1 \quad \text{as} \quad N \to \infty , \quad z \in E$$

Hence: $\prod_1^\infty a_n(z)$ conv unif on E and $P = \exp(S)$. <THIS MUCH IS TRIVIAL.>

The problem is with the converse!

Suppose now that $\prod_{n=1}^{\infty} a_n(z)$ conv unif to $P(z)$ on E.

<u>Choose</u> M so large that

$$\left|\frac{P_N}{P} - 1\right| < 10^{-6}$$

for all $N \geq M$, $z \in E$. Do some baby algebra.

Get

$$\left|\frac{P_{N_2}}{P_{N_1}} - 1\right| < 10^{-5}, \quad \left|\frac{P}{P_{N_1}} - 1\right| < 10^{-5}$$

for all $N_2 \geq N_1 \geq M$. See 16 top.

Let:

$$\frac{P_N}{P_M}$$

$$\operatorname{Arg}\left[(1+b_{M+1})\cdots(1+b_N)\right] = \sum_{j=M+1}^{N} \operatorname{Arg}(1+b_j) + 2\pi i t_N$$

$t_N \in \mathbb{Z}$

for $N \geq M+1$.

<u>CLAIM</u>: $t_N = 0$ for <u>all</u> $N \geq M+1$.

<u>Pf of claim</u>

Take $N_2 = N_1 + 1$ and $N_1 \geq M$. Get: $|a_{N_2} - 1| < 10^{-5}$.

IE $|a_L - 1| < 10^{-5}$ for all $L \geq M+1$.

IE b_L

Clearly $t_{M+1} = 0$ by def.

Use induction. Suppose $0 = t_{M+1} = \cdots = t_N$.

Must prove $t_{N+1} = 0$.

Use Lemma on 17. Take:

$$w_1 = (1+b_{M+1}) \cdots (1+b_N) \longleftarrow \frac{P_N}{P_M} \qquad 10^{-5} \text{ etc}$$

$$w_2 = 1 + b_{N+1} \longleftarrow \frac{P_{N+1}}{P_N} \qquad 10^{-5} \text{ etc}$$

Get:

$$\mathrm{Arg}(w_1 w_2) = \mathrm{Arg}(w_1) + \mathrm{Arg}(w_2) \quad \text{by } |w_j - 1| < 10^{-5}$$

OR

$t_N = 0$

$$\mathrm{Arg}\left[(1+b_{M+1}) \cdots (1+b_{N+1})\right] = \sum_{j=M+1}^{N} \mathrm{Arg}(1+b_j) + 2\pi i(0)$$

$$+ \mathrm{Arg}(1+b_{N+1})$$

hence $t_{N+1} = 0$.

OK

We have just proved Claim for our given M.
But the same reasoning works with

$N_2 > N_1 \geqq M$ and N_1 in place of M.

IE

$$\operatorname{Arg}\left[(1+b_{N_1+1})\cdots(1+b_{N_2})\right] = \sum_{j=N_1+1}^{N_2} \operatorname{Arg}\left[1+b_j\right] + 0$$

hence

$$\boxed{\begin{array}{l} \operatorname{Log}\left[\dfrac{P_{N_2}(z)}{P_{N_1}(z)}\right] = \displaystyle\sum_{j=N_1+1}^{N_2} \operatorname{Log}\left[1+b_j(z)\right] \\ \text{so long as } N_2 > N_1 \geqq M \end{array}}$$

This is the key equation! Since $\dfrac{P_N(z)}{P(z)} \rightrightarrows 1$ on E and $c_1 < |P(z)| < c_2$ 15, we get a multiplicative Cauchy condition

$$\left|\frac{P_{N_2}(z)}{P_{N_1}(z)} - 1\right| < \varepsilon \text{ anytime } N_2 > N_1 \geqq \mathcal{N}_\varepsilon$$

(and, wlog, $\mathcal{N}_\varepsilon \geqq M$).

This shows that there is a uniform Cauchy condition for

$$\sum_{j=N_1+1}^{N_2} \mathrm{Log}(1+b_j(z)) \quad , \text{ i.e. } S_{N_2}(z) - S_{N_1}(z),$$

for $z \in E$. HENCE: $\sum_{j=1}^{\infty} \mathrm{Log}(1+b_j(z))$ conv uniformly on E to some $S(z)$.

By referring to 19, we again have $P = \exp(S)$.

Done! ∎

Important Remark.

If you know $P(z)$, note that you do NOT in general know $S(z)$ without further playing around with $\sum_{1}^{\infty} \mathrm{Log}(1+b_j(z))$. Indeed,

$n \in \mathbb{Z} \rightsquigarrow \exp[S(z) + 2\pi i n] = P(z)$ too.

I.E. which "branch" of $\log P(z)$ applies? YOU DO NOT KNOW THIS IN GENERAL, even if $E = \{\text{one point}\}$.

Thm (Yes, this **IS** a theorem!!)

Given $a_n(z) = 1 + b_n(z)$, $z \in E$, $|b_n(z)| \leqq \lambda < 1$ as usual.

If $\prod_{n=1}^{\infty} (1 + b_n(z))$ converges **absolutely** on E, then

$$\prod_{n=1}^{\infty} (1 + b_n(z)) \quad \text{converges on } E.$$

[Remember E could be one point.]

Pf

By hypothesis, we know $\prod_{n=1}^{\infty} (1 + |b_n(z)|)$ conv at each $z \in E$.

Apply 18. Get $\sum_{n=1}^{\infty} \ln(1 + |b_n|)$ conv on E.

But, baby calculus $\Longrightarrow$

$$\frac{1}{2} t \leqq \ln(1+t) \leqq t \quad \text{for } 0 \leqq t \leqq 1.$$

Hence:

$$\sum_{n=1}^{\infty} |b_n(z)| \quad \text{conv}, \quad \text{each } z \in E.$$

But, baby analytic functions $\Longrightarrow$

$$a_\lambda |w| \leqq |\operatorname{Log}(1+w)| \leqq b_\lambda |w| \quad \text{for } |w| \leqq \lambda < 1.$$

$0 < a_\lambda < b_\lambda < \infty$

EG use Taylor series or else

$$\operatorname{Log}(1+w) = \int_0^w \frac{d\xi}{1+\xi}.$$

line

Accordingly,

$$\sum_{n=1}^{\infty} \left| \operatorname{Log}(1+b_n(z)) \right| \text{ converges, each } z \in E.$$

This absolute conv $\Rightarrow$ ordinary conv. of $\sum_1^\infty \operatorname{Log}(1+b_n)$.

Now just apply 18 at each single point of E.

$\blacksquare$

Thm (Weierstrass M-test for products)

Given $a_n(z) = 1 + b_n(z)$, $|b_n(z)| \leq \lambda < 1$, $z \in E$.

Assume that

$$|b_n(z)| \leq M_n \quad \text{and} \quad \sum_1^\infty M_n < \infty \quad \text{on } E.$$

Then:

$$\prod_1^\infty (1 + b_n(z)) \quad \underline{\text{conv}} \quad \underline{\text{unif}} \quad \text{on } E.$$

(In fact, so does $\prod_1^\infty (1 + |b_n(z)|)$.)

Pf

Apply 18. Must show that $\sum_1^\infty \operatorname{Log}(1 + b_n(z))$ conv <u>unif</u> on E. Recall 24 last line! Get:

$$|\operatorname{Log}(1 + b_n(z))| \leq b_\lambda |b_n(z)| \leq b_\lambda M_n.$$

A <u>standard</u> Weierstrass M-test now applies to $\sum_1^\infty \operatorname{Log}(1 + b_n(z))$. Hence $\sum_1^\infty \operatorname{Log}(1 + b_n(z))$ conv uniformly as needed. OK

For the "in fact", just replace $b_n(z)$ by $|b_n(z)|$. The same M_n still work. ▮

Simple Exercise

Given $a_n(z) = 1 + b_n(z)$, $|b_n(z)| \leqq \lambda < 1$, $z \in E$.

(a) $\prod_1^\infty (1+|b_n|)$ conv pointwise on $E \Leftrightarrow \sum_1^\infty |b_n|$ does;

(b) $\prod_1^\infty (1+|b_n|)$ conv uniformly on $E \Leftrightarrow \sum_1^\infty |b_n|$ does;

(c) $\prod_1^\infty (1+|b_n|)$ conv unif on $E \Rightarrow \prod_1^\infty (1+b_n)$ does too.

See 18. Note $\mathrm{Log}(1+|w|) = \ln(1+|w|) \sim |w|$ as $w \to 0$.
Also recall uniform Cauchy condition for unif conv.

Mind-Twister Exercise $\left(\begin{array}{c}\text{otherwise known as} \\ \mathrm{Log}(1+w) \neq w\end{array}\right)$

Let E = one point. Keep $|b_n| \leqq \lambda < 1$. Put $a_n = 1 + b_n$.

(a) Find $\{b_n\}_{n=1}^\infty$ so that $\sum_1^\infty b_n$ conv, but $\prod_1^\infty a_n$ div!

(b) Find $\{b_n\}_{n=1}^\infty$ so that $\prod_1^\infty a_n$ conv, but $\sum_1^\infty b_n$ div!

Eye-opening Exercise

This exercise really goes with Lecture #6, but is placed here for convenience.

(A) Prove that $\prod_{n=1}^{\infty} \cos\left(\frac{z}{2^n}\right)$ is <u>unif</u> and <u>abs</u> conv on every closed disk $\{|z| \leq R\}$, hence its value $P(z)$ is some analytic fcn on $\mathbb{C}$.

(B) [MAIN PROBLEM!] Evaluate $P(z)$ in simple terms.

Recall that:

$$\cos(z) \equiv \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) \equiv \frac{e^{iz} - e^{-iz}}{2i}$$

when $z \in \mathbb{C}$. These trig fcns are analytic on $\mathbb{C}$. Standard identities are therefore true; eg $\sin^2 z + \cos^2 z = 1$.

Note too: $\sin(z) = 0 \Leftrightarrow z = n\pi$, etc.

PARTIAL DIARY ENTRY for

Lecture 6

(5 Feb 2016)

(I) More was discussed on infinite products, especially Weierstrass M-test. E.g., for

$$\prod_{n=1}^{\infty} \cos\left(\frac{z}{n}\right).$$

(II) $\prod_{n=1}^{\infty}(1+b_n(z))$ with $|b_n(z)| \leq \lambda < 1$ on E. We stressed that when unif conv holds, you get both

$$\frac{P_N(z)}{P(z)} \rightrightarrows 1 \qquad \text{AND} \qquad P_N(z) \rightrightarrows P(z)$$

since $c_1 < |P(z)| < c_2$ on E. Hence:

$b_n(z)$ continuous $\Rightarrow$ $P(z)$ continuous on E

$b_n(z)$ analytic $\Rightarrow$ $P(z)$ analytic (in the usual "on compacta" sense associated with Weierstrass convergence thm)

III Cauchy products. I remarked that

$$A = \sum_{n=0}^{\infty} a_n, \text{ abs conv} \qquad (a_n \in \mathbb{C})$$

$$B = \sum_{n=0}^{\infty} b_n, \text{ abs conv} \qquad (b_n \in \mathbb{C})$$

$$\Downarrow$$

$$AB = \sum_{n=0}^{\infty} \Big(\sum_{j+k=n} a_j b_k \Big) = \sum_{n=0}^{\infty} c_n, \text{ abs conv.}$$

(Same proof as in $\mathbb{R}$.)

IV $\mathrm{Re}(z) > 1$ say. Use III. Take $T \geq 2$.

$$\prod_{p \leq T} \frac{1}{1-p^{-z}} = \prod_{p \leq T} \left\{ 1 + p^{-z} + p^{-2z} + \cdots \right\}$$

$$= \sum_{\substack{n \geq 1 \\ n \text{ is factorizable} \\ \text{into primes which} \\ \text{are all } \leq T}} n^{-z}.$$

includes all $n \leq T$ obviously

Notice too:

$$\left| p^{-z} + p^{-2z} + \cdots \right| \leq p^{-x} + p^{-2x} + \cdots = \frac{p^{-x}}{1-p^{-x}} = \frac{1}{p^x - 1}$$

$$\Downarrow$$

$$|p^{-z} + p^{-2z} + \cdots| \leqq \frac{1}{2^x - 1} \quad \text{for all } p .$$

Hence, for $x \geqq 1 + \delta$, we have a good "λ" of

$$\frac{1}{2^{1+\delta} - 1} < 1 .$$

All of our earlier thms about infinite products apply — when we opt to let $T \to \infty$. All is well.

We clearly get: (Euler)

$$\boxed{\prod_p \frac{1}{1 - p^{-z}} = \zeta(z) = \sum_{n=1}^{\infty} n^{-z}}$$

with uniform + absolute convergence on each closed half-plane $\{x \geqq 1 + \delta\}$.

In particular, since LHS is nonzero (by def of conv infinite product), we get:

$$\zeta(z) \neq 0 \quad \text{for } \operatorname{Re}(z) > 1 .$$

IV I drew attention to Euler's identity

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \{1 + f(p) + f(p^2) + f(p^3) + \cdots\}$$

in Ingham p. 16 — under the assumption that

$\boxed{1}$ $$\sum_{1}^{\infty} |f(n)| < \infty$$

and f is multiplicative

$\boxed{2}$ $$\left\{ \begin{array}{l} f(1) = 1 \\ f(mn) = f(m) f(n) \quad \text{if } (m,n) = 1 \end{array} \right\}.$$

(Read proof there!)

V Defined a natural branch of log $\zeta(z)$ on $\mathrm{Re}(z) > 1$ by writing

used ③ BOX

$$\mathrm{Log}\, \zeta(z) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-z}.$$

This is NOT in general Log $\zeta(z)$!

For $z = x > 1$, however, one readily checks $\mathrm{Log}\, \zeta(z) = \ln \zeta(x)$. RECALL $\zeta(x) = \sum_{1}^{\infty} n^{-x} > 1$.

Clarification: (regarding $\mathcal{L}og\ \zeta(z)$)

Recall 2 last line and 3 lines 5~7. "All is well" because we are using Weierstrass M-test with

$$M_p = \frac{1}{p^{1+\delta}-1} \qquad (\text{for } x \gtrsim 1+\delta).$$

Our $\sum_p \operatorname{Log}(1+b_p(z))$ for the "infinite product equivalence thm" in Lec #5 is $-\sum_p \operatorname{Log}(1-p^{-z})$.

This infinite series converges to some $S(z)$.

The series is just

$$\sum_p \left\{ p^{-z} + \frac{1}{2} p^{-2z} + \frac{1}{3} p^{-3z} + \cdots \right\}$$

$$\equiv \sum_n \frac{\Lambda(n)}{\ln n} n^{-z}$$ ← nice analytic fcn

(with good abs conv). As in Lec 5, we always have:

$$P(z) = \exp\{S(z)\}.$$

So, here, 3 BOX,

$$\zeta(z) = P(z) = \exp\{S(z)\}.$$

Ie, there is no question $S(z) =$ some branch of $\log \zeta(z)$ on $\{\operatorname{Re}(z) > 1\}$.

Clearly, by inspection, $S(x) > 0$ for $x > 1$.

Hence, we do have: $S(x) \equiv \mathcal{L}og\ \zeta(x) = \operatorname{Log} \zeta(x)$.

$(x > 1)$

VI Clearly, in V,

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_n \frac{\Lambda(n)}{n^z}, \quad Re(z) > 1.$$

(by Weierstrass' conv thm for analytic fcns)

Thm (Hadamard)

For $x > 1$, $y \neq 0$

$$|\zeta(x)|^3 |\zeta(x+iy)|^4 |\zeta(x+2iy)| \geq 1.$$

Pf

Take ln. We Want:

$$3\ln|\zeta(x)| + 4\ln|\zeta(x+iy)| + \ln|\zeta(x+2iy)| \geq 0.$$

But,

$$\ln|\zeta(z)| = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-x} \cos(y \ln n)$$

$Re\{n^{-x} e^{-iy\ln n}\}$

by (4) $\{x > 1\}$.

Since, for any $\theta \in \mathbb{R}$,

$$3 + 4\cos\theta + \cos(2\theta)$$

$$= 3 + 4\cos\theta + 2\cos^2\theta - 1$$

$$= 2 + 4\cos\theta + 2\cos^2\theta$$

$$= 2(1+\cos\theta)^2 \geqq 0,$$

and $\frac{\Lambda(n)}{\ln n} \geqq 0$, a trivial substitution now gives what we claimed. ▥

<u>Corollary</u> (Hadamard ← famous result)

$\zeta(1+iy) \neq 0$ if $y \neq 0$.

<u>Pf</u>

Suppose we had $\zeta(1+iy) = 0$ <u>at</u> some $y \neq 0$.

$$\Downarrow$$

$$(x-1)^3 |\zeta(x)|^3 \frac{|\zeta(x+iy)|^4}{(x-1)^4} |\zeta(x+2iy)| \geq \frac{1}{x-1}$$

$$\{(x-1)|\zeta(x)|\}^3 \left| \frac{\zeta(x+iy)-\zeta(1+iy)}{x-1} \right|^4 |\zeta(x+2iy)| \geq \frac{1}{x-1}$$

let $x \to 1^+$

$$1^3 \cdot |\zeta'(1+iy)|^4 \cdot |\zeta(1+2iy)| \geq \infty \Rightarrow$$

a contradiction {since $\zeta(z)$ is nicely analytic for $Re(z) > 0$ except at $z=1$} !! ∎

Theorem (essentially like Ingham p. 27)

Let $0 < \delta < 1$. We then have:

(A) $|\zeta(x+iy)| \leqq A \ln|y|$ for $x \geqq 1$, $|y| \geqq 2$

(B) $|\zeta'(x+iy)| \leqq B \ln^2|y|$ for $x \geqq 1$, $|y| \geqq 2$

(C) $|\zeta(x+iy)| \leqq \dfrac{C}{\delta(1-\delta)} |y|^{1-\delta}$ for $x \geqq \delta$, $|y| \geqq 2$.

Here A, B, C are certain absolute constants.

Pf

$$\sum_{n=1}^{N} n^{-z} = 1 + \frac{1-N^{1-z}}{z-1} - z\int_1^N \frac{r(t)}{t^{z+1}}\,dt$$

$(z \neq 1)$ Lec 5, p. 8 + 9

$$\zeta(z) = 1 + \frac{1}{z-1} - z\int_1^\infty \frac{r(t)}{t^{z+1}}\,dt$$

$(\mathrm{Re}(z) > 1)$ Lec 5 p. 9

Then, we used this last formula to <u>define</u> $\zeta(z)$ for $\mathrm{Re}(z) > 0$. Lec 5 p. 9

By subtraction,

$$\zeta(z) - \sum_{n=1}^{N} n^{-z} = \frac{N^{1-z}}{z-1} - z\int_N^\infty \frac{r(t)}{t^{z+1}}\,dt\,.$$

This is a very useful TRICK!!

$$\zeta(z) = \sum_{n=1}^{N} n^{-z} + \frac{N^{1-z}}{z-1} - z\int_N^\infty \frac{r(t)}{t^{z+1}}\,dt\,,$$

$\mathrm{Re}(z) > \underline{\underline{0}}$. $(z \neq 1)$

We propose to <u>begin</u> with (C) [even though it looks to be the most complicated].

To prove (C), notice that it suffices to prove it for, say, $|y| \geq 100$.

In fact, for $2 \leq |y| \leq 100$, we can just use our old VERY CRUDE Theorem 4 from Lec 5, page 12.

In this connection, recall too that

$$|\zeta(z)| \leq \frac{1}{\delta} + O(1) \leq \frac{\text{const}}{\delta}$$

for all $\mathrm{Re}(z) \geq 1+\delta$. (Also on p. 12.)

Use p. 10 line 4 above. $|y| \geq 100, \; x \geq \delta$

$$|\zeta(x+iy)| \leq \sum_{n=1}^{N} n^{-x} + \frac{N^{1-x}}{|z-1|} + |z| \int_N^\infty \frac{1}{t^{x+1}} dt$$

$$|\zeta(x+iy)| \leq \sum_{n=1}^{N} n^{-\delta} + \frac{N^{1-\delta}}{|y|} + (x+|y|) \int_N^\infty \frac{dt}{t^{1+\delta}}$$

$$\sum_{n=1}^{N} n^{-\delta} < 1 + \int_1^N u^{-\delta} du \qquad \left\{ \begin{matrix} \text{by} \\ \text{areas} \end{matrix} \right\}$$

$$\sum_{n=1}^{N} n^{-\delta} < 1 + \left. \frac{u^{1-\delta}}{1-\delta} \right]_1^N$$

$$\sum_{n=1}^{N} n^{-\delta} < 1 + \frac{N^{1-\delta}}{1-\delta} < 2 \frac{N^{1-\delta}}{1-\delta}$$

$$|\zeta(x+iy)| \leq 2 \frac{N^{1-\delta}}{1-\delta} + \frac{N^{1-\delta}}{100} + (x+|y|) \frac{N^{-\delta}}{\delta}$$

For $x \geq 1+\delta$, we already know $|\zeta(x+iy)| \leq \frac{\text{const}}{\delta}$, hence (C) is certainly OK here [if C is taken sufficiently big].

For this reason, there is no harm in proceeding under the assumption

$$|y| \gtrsim 100, \quad \delta \lesssim x \lesssim 1+\delta .$$

Get:

$$|\zeta(x+iy)| \lesssim 2\frac{N^{1-\delta}}{1-\delta} + \frac{N^{1-\delta}}{100} + 2|y|\frac{N^{-\delta}}{\delta}$$

$$\lesssim 3\frac{N^{1-\delta}}{1-\delta} + 2\frac{|y|}{\delta}N^{-\delta}$$

$$\lesssim 3N^{-\delta}\left[\frac{N}{1-\delta} + \frac{|y|}{\delta}\right] .$$

This estimate can admittedly be improved. But, a sloppy one is sufficient.

Also, recall $\zeta(x-iy) = \overline{\zeta(x+iy)}$. Hence, wlog, $y \gtrsim 100$.

Let's try $N = G\frac{y}{\delta}$ where $1 \lesssim G \lesssim 10$, say, and we always adjust it to make $N \in \mathbb{Z}$.

(Note $\frac{y}{\delta} \gtrsim \frac{100}{\delta} \gtrsim 100$.)

Get:

$$|\zeta(x+iy)| \leqq 3\left(G\frac{y}{\delta}\right)^{-\delta}\left[N+\frac{y}{\delta}\right]\frac{1}{1-\delta} \quad \text{by (12)}$$

$$= 3\left(G\frac{y}{\delta}\right)^{-\delta}\left[G\frac{y}{\delta}+\frac{y}{\delta}\right]\frac{1}{1-\delta}$$

$$= \frac{3}{1-\delta}G^{-\delta}y^{-\delta}\delta^{\delta}(G+1)\frac{y}{\delta}$$

$$\boxed{\begin{array}{l}\delta^{\delta} = e^{\delta\ln\delta} \\ \text{is bdd away from } 0 \\ \text{and } \infty \text{ for } 0 \leqq \delta \leqq 1\end{array}}$$

$$\leqq \frac{c_1}{1-\delta}G^{-\delta}y^{1-\delta}\cdot\frac{2G}{\delta} \qquad \{1 \leqq G \leqq 10\}$$

$$\leqq \frac{c_2}{(1-\delta)\delta}y^{1-\delta}, \text{ AS REQUIRED.}$$

This proves (C).

It is important to note $\delta \in (0,1)$ is arbitrary. It could even be taken as a fcn of y.

(A) is now a trivial consequence of (C).

Indeed, since $\zeta(z)$ is a nice analytic fcn for $\mathrm{Re}(z) \leq 1$, $|y| \geq 2$, there is nothing to do for $\{1 \leq x \leq 2,\ 2 \leq |y| \leq 100\}$. For $\{x \geq 2,\ 2 \leq |y| \leq 100\}$ just use (10) last 3 lines; again nothing to do.

So, wlog, we can assume $|y| \geq 100$. Also $y \geq 100$.

Put $\delta \approx 1 - \frac{1}{\ln y}$ in (C). Note $\ln 100 = 4.605^{+}$.
Hence $.75 < \delta < 1$. By (C), get (see (9)):

$$|\zeta(x+iy)| \lesssim \frac{2C}{1-\delta} y^{1-\delta}, \quad x \geq 1 - \frac{1}{\ln y}$$

$$|\zeta(x+iy)| \lesssim 2C(\ln y)\, y^{\frac{1}{\ln y}}$$

$$|\zeta(x+iy)| \lesssim 2Ce(\ln y) \lesssim 6C(\ln y).$$

Now just specialize to $x \geq 1$. Done!

(B) is "almost" as trivial once we recall Cauchy's inequality for $|f'(z_0)|$. IE

$$f'(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2}\, dz$$

$$\Downarrow$$

$$|f'(z_0)| \lesssim \frac{1}{2\pi} \frac{M(R)}{R^2} (2\pi R)$$

where $M(R) \equiv \max_{|z-z_0|=R} |f|$

$$\Downarrow$$

$$\boxed{|f'(z_0)| \lesssim \frac{M(R)}{R}} .$$

Here are the details for (B).

First, since $f(z)$ is a nice analytic fcn for $Re(z) \gtrsim 1$, $|y| \lesssim 2$, there is nothing to do for $\{1 \lesssim x \lesssim 2, \ 2 \lesssim |y| \lesssim 150\}$. For $\{x \gtrsim 2, \ 2 \lesssim |y| \lesssim 150\}$, apply Cauchy's inequality with $R = 1/2$ and (10) last 3 lines. Again nothing to do. ⊛

So, wlog, take $|y| \gtrsim 100$. We can also assume $y \gtrsim 100$. We will use (C) with a δ similar to $1 - \frac{1}{\ln y}$.

⊛ I do want 150 here, i.e. a slight overshoot over 100.

Take $\delta = 1 - \frac{\lambda}{\ln y}$, where $0 < \lambda \le 1$. We'll choose λ in a few moments. Note that we have $.75 < \delta < 1$ by (14) line 8. Apply (C) on page (9).

Get:

$$|\zeta(x+iy)| \le \frac{2e}{1-\delta}\, y^{\frac{\lambda}{\ln y}}, \quad \text{all } x \ge 1 - \frac{\lambda}{\ln y},\ y \ge 100$$

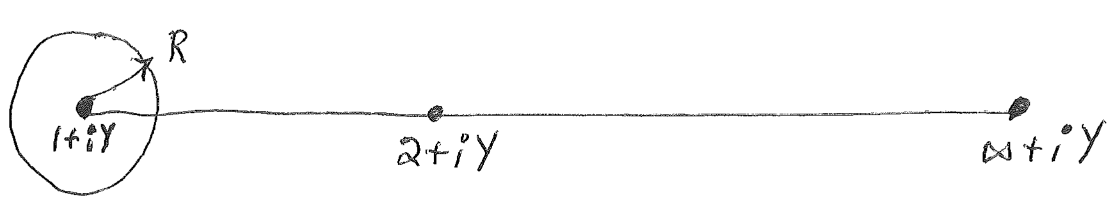
$$|\zeta(x+iy)| \le \frac{2e}{\lambda}(\ln y)\, e^{\lambda}$$

$$|\zeta(x+iy)| \le \frac{6e}{\lambda}(\ln y), \quad \text{for all } x \ge 1 - \frac{\lambda}{\ln y},\ y \ge 100.$$

We want to RIG THINGS so we can select $Y \ge 110$, then use $R = \frac{1}{10}\frac{\lambda}{\ln Y}$ (say) in Cauchy's inequality for a center z_0 along the segment $[1+iY, \infty+iY)$.

Note that $R \le \frac{1}{10}\frac{1}{\ln Y} \le \frac{1}{46} < \frac{1}{2}$.

$\boxed{\ln 100 = 4.605^{+}}$



As the circle slides along, its y-values clearly stay between $Y-1$ and $Y+1$.

Hence, obviously, $y \gtrsim 100$.

But we <u>must</u> make certain that no matter what happens, we have $x \gtrsim 1 - \frac{2}{\ln y}$ at <u>all</u> times on the circle.

<u>Baby Calculus Lemma</u>

Given $T \gtrsim 110$. Keep $y \in [T-1, T+1]$.

Then:

$$\frac{4}{5} \lessapprox \frac{\ln y}{\ln T} \lessapprox \frac{5}{4} .$$

<u>PF</u>

$$\frac{\ln(T-1)}{\ln T} \lessapprox \frac{\ln y}{\ln T} \lessapprox \frac{\ln(T+1)}{\ln T}$$

But, by theorem of the mean,

$$\ln(T+1) - \ln(T) \lessapprox \frac{1}{T}(1)$$
$$\ln(T) - \ln(T-1) \lessapprox \frac{1}{T-1}(1) .$$

So,

$$\ln(T+1) \lesssim \ln(T) + \frac{1}{T}$$
$$\ln(T-1) \gtrsim \ln(T) - \frac{1}{T-1}$$

$$\Downarrow$$

$$\ln(T+1) \lesssim \ln T + \frac{1}{110} < \ln T + \frac{1}{100}$$
$$\ln(T-1) \gtrsim \ln T - \frac{1}{109} > \ln T - \frac{1}{100}$$

$$\Downarrow$$

$$\frac{\ln T - \frac{1}{100}}{\ln T} \lesssim \frac{\ln y}{\ln T} \lesssim \frac{\ln T + \frac{1}{100}}{\ln T}$$

$\{$ but $\ln T \gtrsim \ln 100 = 4.605^{+} \}$

$$.99 \lesssim \frac{\ln y}{\ln T} \lesssim 1.01 \qquad \text{OK} \qquad \blacksquare$$

In the moving circle on 16 bottom, obviously

$$x \gtrsim 1 - \frac{1}{10}\frac{\lambda}{\ln Y}. \qquad \text{1-R}$$

But $\ln Y = \omega \ln y$ with $\frac{4}{5} \lesssim \omega \lesssim \frac{5}{4}$ by Calc Lemma.

So,

$$x \gtrsim 1 - \frac{1}{10}\frac{\lambda}{\omega \ln y} \qquad \text{on circle.}$$

Must make sure

$$\frac{\lambda}{\ln y} \gtrsim \frac{\lambda}{10\omega}\frac{1}{\ln y}$$

i.e.,

$$1 \gtrsim \frac{1}{10\omega} .$$

but $8 \leq 10\omega \lesssim 12.5$

Thus things are OK and actually λ is irrelevant.

So: just put $\lambda = 1$.

$$\boxed{\text{Get: } R = \frac{1}{10}\frac{1}{\ln Y} .}$$

By Cauchy's inequality 15, and 16 (middle), we find that:

$$|\zeta'(x+iY)| \lesssim \frac{6e\,[\ln(Y+1)]}{R} \lesssim \frac{12e\ln Y}{R}$$

$$\lesssim 120\,e\,(\ln Y)^2$$

for any $x \in [1, \infty)$. Recalling 15 lines 5–9, we have thus proved (B).

Two Remarks

① Take $\lambda \in [11, 12]$ say. Note $\ln 10^6 = 13.815^+$.
By mimicking ⑭ – ⑲, one easily sees that

$$|\zeta(x+iy)| = O(\ln y) \quad \text{for } x \gtrsim 1 - \frac{5}{\ln y},\ y \gtrsim 10^6$$

$$|\zeta'(x+iy)| = O(\ln^2 y) \quad \text{for same } (x, y).$$

One can take $R = \dfrac{\lambda}{2\ln y}$. Key necessity is

$$\frac{\lambda}{\ln y} \gtrsim \frac{5}{\ln y} + \frac{\lambda}{2\ln y}$$

or

$$\frac{\lambda}{\ln y} \gtrsim \frac{5}{\omega \ln y} + \frac{\lambda}{2\omega \ln y}$$

or

$$\lambda\left(1 - \frac{1}{2\omega}\right) \gtrsim \frac{5}{\omega}.$$

But, by ⑱, $\omega = 1 \pm [.01]$. OK

---

② Why do we use $\delta = 1 - \dfrac{\lambda}{\ln y}$?
Answer: go to ⑬ 5 lines from bottom.

We wonder: if δ is close to 1 and y is very large, what is the <u>smallest</u> that

$$\frac{y^{1-\delta}}{1-\delta}$$

can be? This is a trivial calc problem.

$\delta = 1-u \Rightarrow$ look at $\frac{y^u}{u} \Rightarrow$ look at

$$f(u) = u \ln y - \ln u \;, \quad 0 \leq u \leq 1$$

$$f' = \ln y - \frac{1}{u}$$

get $f' > 0 \Leftrightarrow u > \frac{1}{\ln y}$.

So key δ is $1 - \frac{1}{\ln y}$, which gives $e^1 \ln y$.

The insertion of λ allows us to "move around" a bit.

Lecture 7

(10 Feb)

Abel Summation Lemma

Let $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_N \geq 0$.

Let $c_k \in \mathbb{C}$ and $\left|\sum_1^n c_j\right| \leq M$ for $1 \leq n \leq N$.

Then:

$$|\varepsilon_1 c_1 + \cdots + \varepsilon_N c_N| \leq M\varepsilon_1 .$$

Pf

$$\text{Sum} = \varepsilon_1 s_1 + \varepsilon_2(s_2 - s_1) + \cdots + \varepsilon_N(s_N - s_{N-1})$$

$$= s_1(\varepsilon_1 - \varepsilon_2) + \cdots + s_{N-1}(\varepsilon_{N-1} - \varepsilon_N) + \varepsilon_N s_N$$

$$|\text{Sum}| \leq M(\varepsilon_1 - \varepsilon_2) + \cdots + M(\varepsilon_{N-1} - \varepsilon_N) + M\varepsilon_N$$

↑ equals $M\varepsilon_1$ ∎

---

2 immediate corollaries are:

Thm (Dirichlet Test for Unif Conv)

Let $1 \geq \varepsilon_1(\alpha) \geq \varepsilon_2(\alpha) \geq \cdots \geq 0$ and $\varepsilon_n(\alpha) \rightrightarrows 0$ for $\alpha \in E_1$. Let $\sum_1^\infty b_n(\beta)$ have unif bounded partial sums for $\beta \in E_2$. Then

$$\sum_{n=1}^\infty \varepsilon_n(\alpha) b_n(\beta)$$

conv unif on $E_1 \times E_2$.

Pf

Use uniform Cauchy criterion! ∎

Thm (Abel's Test for Unif Conv)

$\alpha \in E_1$, $\beta \in E_2$ again. Let $1 \geq \varepsilon_1(\alpha) \geq \varepsilon_2(\alpha) \geq \cdots \geq 0$, not nec going to 0. Let $\sum_1^\infty b_n(\beta)$ conv unif on E_2. Then

$$\sum_1^\infty \varepsilon_n(\alpha) b_n(\beta)$$

conv unif for $(\alpha,\beta) \in E_1 \times E_2$.

Pf

Use uniform Cauchy criterion! ∎

Use trivial geom series:

$$\left| \sum_{n=0}^{N} e^{2\pi i n \alpha} \right| \leq \frac{2}{2|\sin \pi \alpha|} = \frac{1}{|\sin \pi \alpha|}, \quad \alpha \notin \mathbb{Z}.$$

By Dirichlet's test, get

$$\sum_1^\infty \frac{1}{n} e^{2\pi i n \alpha}$$

conv unif for $\delta \leq \alpha \leq 1-\delta$. We connect this series with

$$-\mathrm{Log}(1-z).$$

$$-\mathrm{Log}(1-z) = z + \frac{z^2}{2} + \cdots, \quad |z|<1$$

$z = re^{i\alpha}$ as usual

$$-\mathrm{Log}(1-re^{i\alpha}) = re^{i\alpha} + \frac{1}{2}r^2e^{2i\alpha} + \frac{1}{3}r^3e^{3i\alpha} + \cdots$$

$0 \leqq r < 1$ Wish to let $r \to 1$.

2

Know $\sum_1^\infty \frac{1}{n} e^{2\pi i n \alpha}$ conv unif away from $\alpha \in \mathbb{Z}$, i.e. away from $z = 1$. So, by Abel's test, get

$$\sum_{n=1}^{\infty} \frac{r^n}{n} e^{2\pi i n \alpha}$$

conv unif $0 \leqq r \leqq 1$, $\delta \leqq \alpha \leqq 1-\delta$ (say). We conclude therefore that

$$-\mathrm{Log}(1-e^{2\pi i\alpha}) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n\alpha}}{n} \qquad \star$$

for $0 < \alpha < 1$.

One writes:

$$\begin{aligned} 1 - e^{2\pi i\alpha} &= e^{\pi i\alpha}\left(e^{-\pi i\alpha} - e^{\pi i\alpha}\right) \\ &= e^{\pi i\alpha}(-2i\sin\pi\alpha) \\ &= 2\sin\pi\alpha \cdot e^{\pi i\alpha - i\pi/2}, \quad 0<\alpha<1. \end{aligned}$$

Conclude at once: (see 3 ✱)

$$-\ln(2\sin\pi\alpha) = \sum_{n=1}^{\infty} \frac{\cos(2\pi n\alpha)}{n}$$

$$\alpha - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin(2\pi n\alpha)}{\pi n},$$

with unif conv for $\delta \leq \alpha \leq 1-\delta$. These are two very basic Fourier series, especially #2.

$$\alpha - [\![\alpha]\!] - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin(2\pi n\alpha)}{\pi n}, \quad \alpha \notin \mathbb{Z}$$

We (can) now start moving very majorly toward the proof of PNT.

Recall from Lec 6:

$$|\zeta(x+iy)| \leq A \ln|y|, \quad x \geq 1,\ |y| \geq 2$$

$$|\zeta'(x+iy)| \leq B \ln^2|y|, \quad x \geq 1,\ |y| \geq 2$$

$$|\zeta(x+iy)| \leq \frac{e}{\delta(1-\delta)} |y|^{1-\delta}, \quad x \geq \delta,\ |y| \geq 2,\ 0 < \delta < 1.$$

We also had:

$$\zeta(z) = \prod_p \frac{1}{1-p^{-z}} \qquad re(z)>1$$

$$Log\ \zeta(z) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-z}, \quad re(z)>1$$

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^z}, \quad re(z)>1$$

Hadamard

$$|\zeta(x)|^3 |\zeta(x+iy)|^4 |\zeta(x+2iy)| \geq 1, \quad x>1,\ y\neq 0.$$

I proved in Lec 6 that $\zeta(1+iy)\neq 0$.

Theorem (improvement over $\zeta(1+iy)\neq 0$)

$$\frac{1}{\zeta(x+iy)} = O(1)(\ln|y|)^7 \quad \text{for } x\geq 1,\ |y|\geq 3.$$

PF

Very close to Ingham!

$x\geq 2$: $\left|\frac{1}{\zeta(z)}\right| = \left|\prod_p (1-p^{-z})\right| \leq \prod_p (1+p^{-x}) \leq \prod_p \frac{1}{1-p^{-x}}$

$$\left|\frac{1}{\zeta(z)}\right| \leq \zeta(x) \leq \zeta(2).$$

So, wlog, $1\leq x\leq 2$. Also, wlog, $y\geq 3$.

Notice

$$[(x-1)\zeta(x)]^3 |\zeta(x+iy)|^4 |\zeta(x+2iy)| \geq (x-1)^3$$

use 4 bottom

$$\Rightarrow \quad |\zeta(x+iy)|^4 \geq \frac{(x-1)^3}{[(x-1)\zeta(x)]^3 A \ln(2y)}$$

trivial cont. fcn.

$$|\zeta(x+iy)|^4 \geq \frac{(x-1)^3}{A \ln y} \qquad (\text{not same } A)$$

$$|\zeta(x+iy)| \geq \frac{(x-1)^{3/4}}{A(\ln y)^{1/4}} \qquad \left\{ \begin{array}{l} 1 \leq x \leq 2 \\ y \geq 3 \end{array} \right\}.$$

But $|\zeta'(x+iy)| = O(\ln^2 y)$, all $x \geq 1$, $y \geq 3$.

Take any $c \in (1,2)$, $x \in [1,2]$. Get:

$$\zeta(x+iy) - \zeta(c+iy) = \int_c^x \zeta'(u+iy)\,du$$

$$|\zeta(x+iy) - \zeta(c+iy)| \leq A|x-c|\ln^2 y \qquad \text{4 bottom}$$

$$|\zeta(x+iy)| \geq |\zeta(c+iy)| - A|x-c|\ln^2 y.$$

But,

$$|\zeta(c+iy)| \geq \frac{(c-1)^{3/4}}{A(\ln y)^{1/4}} \quad \text{by above.}$$

Get:

$$|\zeta(x+iy)| \gtrsim \frac{(c-1)^{3/4}}{A(\ln y)^{1/4}} - A|c-x|\ln^2 y \text{ .}$$

If $x \gtrsim c > 1$, just use

$$|\zeta(x+iy)| \gtrsim \frac{(c-1)^{3/4}}{A(\ln y)^{1/4}} \text{ ,}$$

⑥ line 5

since it's better!! But, if $\underline{\underline{1 \lesssim x < c}}$, use

$$|\zeta(x+iy)| \gtrsim \frac{(c-1)^{3/4}}{A_1(\ln y)^{1/4}} - A_2|c-x|\ln^2 y$$

$$|\zeta(x+iy)| \gtrsim \frac{(c-1)^{3/4}}{A_1(\ln y)^{1/4}} - A_2(c-1)\ln^2 y$$

$$|\zeta(x+iy)| \gtrsim \frac{1}{A_1}\left[\frac{(c-1)^{3/4}}{(\ln y)^{1/4}} - A_3(c-1)\ln^2 y\right] \text{ .}$$

Now have a trivial-type calculus problem to make bracket as <u>large</u> as possible.

Standard trick:

$$\frac{(c-1)^{3/4}}{(\ln y)^{1/4}} = \mathcal{C}(c-1)\ln^2 y \text{ , } \quad \mathcal{C} = \text{adjustable}$$

$$\frac{1}{\mathcal{C}}\frac{1}{(\ln y)^{9/4}} = (c-1)^{1/4}$$

so we want

$$c - 1 = \frac{\sigma}{(\ln y)^9},$$

σ to be adjusted.

We declare:

$$c = 1 + \frac{\sigma}{(\ln y)^9} \qquad (y \gtrsim 3)$$

and keep σ small enough that $c \in (1, 2)$.

Get:

$$|\zeta(x+iy)| \gtrsim \frac{1}{A_1}(c-1)^{3/4}\left[\frac{1}{(\ln y)^{1/4}} - A_3 (c-1)^{1/4} \ln^2 y\right] \text{ by } (7)$$

$$= \frac{1}{A_1}\frac{\sigma^{3/4}}{(\ln y)^{27/4}}\left[\frac{1}{(\ln y)^{1/4}} - A_3 \frac{\sigma^{1/4}}{(\ln y)^{9/4}}(\ln y)^{8/4}\right]$$

$$= \frac{1}{A_1}\frac{\sigma^{3/4}}{(\ln y)^7}\left[1 - A_3 \sigma^{1/4}\right].$$

Want σ so small that $1 - A_3 \sigma^{1/4} \gtrsim \frac{1}{2}$. (In addition to keeping $1 < c < 2$.) Get:

$$|\zeta(x+iy)| \gtrsim \frac{\text{constant}}{(\ln y)^7}, \quad 1 \le x < 1 + \frac{\sigma}{(\ln y)^9}.$$

(c)

For $x > 1 + \frac{G}{(\ln y)^9}$, we use [[7 line 4]]

$$|\zeta(x+iy)| \geq \frac{(c-1)^{3/4}}{A(\ln y)^{1/4}} = \frac{\left(\frac{G}{\ln^9 y}\right)^{3/4}}{A(\ln y)^{1/4}}$$

$$= \frac{\text{constant}}{(\ln y)^7}.$$

So, in <u>all</u> cases,

$$|\zeta(x+iy)| \geq \frac{\text{const}}{(\ln y)^7} \qquad \begin{array}{l} x \in [1,2] \\ y \geq 3 \end{array}.$$

So:

$$\frac{1}{|\zeta(z)|} \leq (\text{const})(\ln y)^7. \qquad \blacksquare$$

One defines (following Riemann)

$$\psi_1(x) = \int_0^x \psi(v)\,dv \qquad \{\psi(v) = 0,\ v < 2\}$$

$$= \int_0^x \Big(\sum_{k \leq v} \Lambda(k)\Big)\,dv$$

$$= \sum_{k \leq x} \Lambda(k) \int_k^x dv = \sum_{k \leq x} \Lambda(k)(x-k).$$

We will also follow Riemann and begin writing

$$\boxed{s = \sigma + it}$$

instead of $z = x + iy$.

<u>Theorem</u> (Fund. Formula)

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]ds$$

for all $c > 1$, $x > 0$.

Note that:

$$\left|\frac{x^{s+1}}{s(s+1)}\right| = \frac{x^{c+1}}{|s||s+1|} \leq \frac{x^{c+1}}{|t|^2}.$$

There is no question that RHS converges!

Proof

We require a standard lemma from complex analysis.

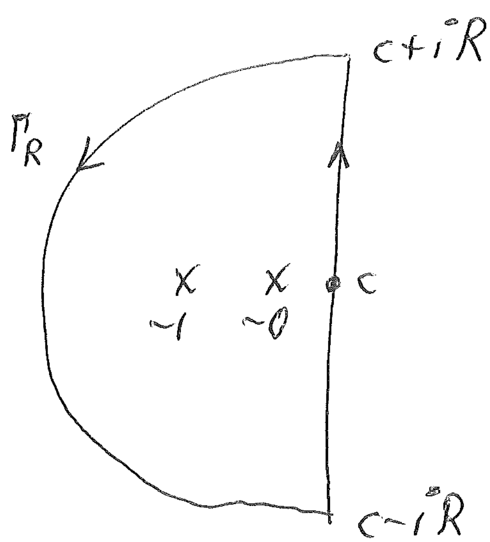
Lemma

Freeze y. Fix any $c > 0$. Then:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)}\,ds = \begin{cases} 0, & \text{if } y < 1 \\ 1 - y^{-1}, & \text{if } y \geq 1 \end{cases}.$$

Pf

$y \geq 1$ first.



Our Analytic fcn of s is $\dfrac{y^s}{s(s+1)}$.

(y fixed)

p. 31

Ingham uses Cauchy residue thm on this shape. We prefer Cauchy integral theorem + Cauchy integral formula!!

$$\text{CIT} \Rightarrow \oint = \oint_{|s|=\varepsilon} + \oint_{|s+1|=\varepsilon}$$

But,

$$\frac{1}{2\pi i}\oint_{|s|=\varepsilon} \frac{1}{s}\frac{y^s}{s+1}\,ds = \frac{y^0}{0+1} = 1 \quad \text{by CIF ;}$$

$$\frac{1}{2\pi i}\oint_{|s+1|=\varepsilon} \frac{1}{s+1}\frac{y^s}{s}\,ds = \frac{y^{-1}}{-1} = -y^{-1} \quad \text{by CIF .}$$

and

$$\left|\int_{\Gamma_R} \frac{y^s}{s(s+1)}\,ds\right| \leq \int_{\Gamma_R} \frac{y^c}{|s||s+1|}\,|ds|$$

correct since $y \geq 1$

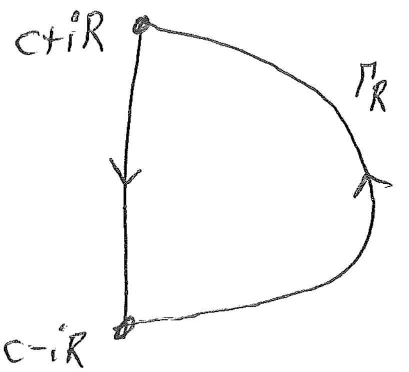
$$\lesssim (\text{const})\, y^c \frac{1}{R^2}\pi R$$

$$= O\left(\frac{y^c}{R}\right) \to 0 \quad \text{as } R \to \infty$$

We immediately get:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s(s+1)}\,ds + 0 = 1-y^{-1} .$$

For $y<1$, use shape:



$\left|\int_{\Gamma_R}\right| \to 0$ again since $y<1$

■ (on the lemma)

We are now ready for the Theorem. Start on RHS (10). <u>Freeze</u> $c>1$ and $x>0$.

Must evaluate:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left[+\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right]ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{x^{c+1+it}}{(c+it)(c+1+it)}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{c+it}}\,dt .$$

Want to integrate term-by-term.

Standard set-up applies.

$g(t)$ absolutely integrable on $\mathbb{R}$

$|f_n(t)| \leq M_n \quad \sum_1^\infty M_n < \infty$ for Weierstrass M-test

$$\Rightarrow \int_{-\infty}^{\infty} g(t) \sum_{n=1}^{\infty} f_n(t)\,dt = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(t) f_n(t)\,dt$$

(standard adv calculus.)

NOTE THAT:

$$\left\{ \begin{aligned} &\sum_1^\infty \int_{-\infty}^{\infty} |g(t)||f_n(t)|\,dt \\ &\quad \leq \sum_1^\infty \int_{-\infty}^{\infty} |g(t)| M_n\,dt < \infty \end{aligned} \right\}$$

We immediately get:

$$\text{13 bottom} = \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^1}{s(s+1)} (x/n)^s\,ds$$

$$= \sum_{n \leq x} \Lambda(n)\, x \left[1 - \frac{1}{x/n}\right] + 0 \quad \text{by Lemma}$$

$$= \sum_{n \leq x} \Lambda(n)[x-n] = \psi_1(x). \quad \blacksquare$$

Remark.

In Lec 8, I derived Thm 10 by elementary use of Fourier integrals. It is by no means essential to use complex variable. Riemann certainly knew this.

Once having Riemann's fund formula, one seeks to move $\{Re(s) = c\}$ over to the left.

THIS WILL USE COMPLEX VARIABLE!

I prefer to move the line to $\{Re(s) = 1\}$.

I need one more little lemma.

See Ingham p. 31.

CLAIM: $y > 0$ fixed, $c > 0$ fixed

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)(s+2)}\,ds = \begin{cases} 0, & \text{if } y < 1 \\ \frac{1}{2}(1-y^{-1})^2, & \text{if } y \geqq 1 \end{cases}$$

Pf

Very similar to what we did on 11 – 13.

Omit $y < 1$. For $y \geqq 1$, notice that

$$\frac{1}{2\pi i}\oint_{|s|=\varepsilon} \frac{1}{s}\,\frac{y^s}{(s+1)(s+2)}\,ds = \frac{y^0}{1\cdot 2} = \frac{1}{2}$$

$$\frac{1}{2\pi i}\oint_{|s+1|=\varepsilon} \frac{1}{s+1}\,\frac{y^s}{(s)(s+2)}\,ds = \frac{y^{-1}}{(-1)(1)} = -y^{-1}$$

$$\frac{1}{2\pi i}\oint_{|s+2|=\varepsilon} \frac{1}{s+2}\,\frac{y^s}{s(s+1)}\,ds = \frac{y^{-2}}{(-2)(-1)} = \frac{1}{2}y^{-2}$$

$$\text{Sum} = \frac{1}{2}(1-y^{-1})^2 \qquad \text{OK}$$

Keep $y \geqq 1$. Know: (any $\eta > 0$)

$$\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} \frac{y^s}{s(s+1)(s+2)}\,ds = \frac{1}{2}(1-y^{-1})^2$$

$$\xi = s+1$$

$$\frac{1}{2\pi i}\int_{\eta+1-i\infty}^{\eta+1+i\infty} \frac{y^{\xi-1}}{(\xi-1)\xi(\xi+1)}\,d\xi = \frac{1}{2}(1-y^{-1})^2$$

$$\underline{\underline{1+\eta > 1}}$$

The problem with moving $\{Re(s) = c\}$ in 10 directly to $Re(s) = 1$ is hitting the pole at $s = 1$. Must modify things slightly!

$\boxed{\text{near } s = 1}$

$$\zeta(s) = (s-1)^{-1}\left[1 + c_1(s-1) + c_2(s-1)^2 + \cdots\right]$$

$$\zeta(s) = (s-1)^{-1}\phi(s) \quad \text{say}$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \frac{\phi'(s)}{\phi(s)} \quad \text{near } s = 1$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + [\text{analytic}]$$

$$\Downarrow$$

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \quad \text{is } \underline{\text{analytic}} \text{ for } Re(s) \overset{?}{\geq} 1$$

Notice that: $(c > 1)$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{(s-1)s(s+1)}\,ds = \frac{1}{2}\left(1 - \frac{1}{x}\right)^2,$$

by 16 bottom for $x > 1$.

For $x > 1$, we thus have:

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]ds$$

$$\frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)}\left[\frac{1}{s-1}\right]ds$$

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right]ds.$$

Write

$$H(s) = -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \quad \text{for } Re(s) \geqq 1.$$

For $|t| \geqq 3$, know that:

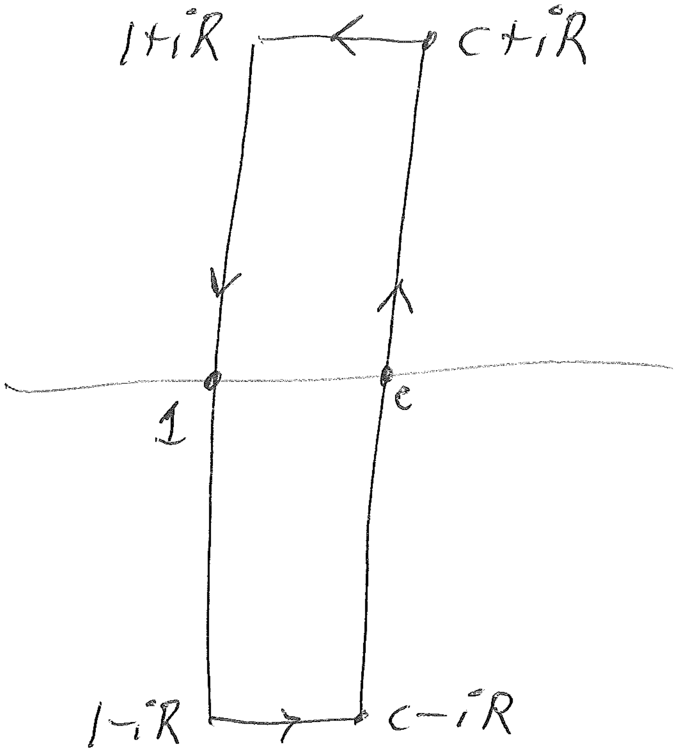
$$|H(s)| \leqq \text{constant} + O(1)\ln^2|t| \cdot \ln^7|t|$$

by 5 + 4 (bot). Ie,

$$|H(s)| \leqq O(1)(\ln|t|)^9$$

for $\sigma \geqq 1$, $|t| \geqq 3$.

Moving the contour in 18 line 4 is now trivial.



$x > 1$

$c > 1$

$R \approx$ giant

$$\left| \int_{\text{horizontal}} \frac{x^{s-1}}{s(s+1)} H(s)\, ds \right| \lessapprox \int_1^c O(1) \frac{x^{c-1}}{R^2} (\ln R)^9 \, d\sigma$$

$$\lessapprox O(1)(c-1)\, x^{c-1} \frac{(\ln R)^9}{R^2}$$

$$\to 0 \quad \text{as } R \to \infty .$$

(x, c frozen)

Get:

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{x^{s-1}}{s(s+1)} H(s)\,ds .$$

Notice here that $|H(1+it)| \leq A(\ln|t|)^9$.
The integrand on RHS has abs value

$$\leq O(1)\frac{1}{t^2}(\ln|t|)^9 . \qquad (|t| \geq 3)$$

THEOREM (almost the PNT)

$$\lim_{x\to\infty} \frac{\psi_1(x)}{x^2} = \frac{1}{2} .$$

Proof

Must look at

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{x^{it}}{(1+it)(2+it)} H(1+it)\,dt$$

$$\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{H(1+it)}{(1+it)(2+it)} e^{it(\ln x)}\,dt .$$

$(\ln x \to +\infty)$

But, the Riemann-Lebesgue lemma tells us that

$$\int_{-\infty}^{\infty} \mathcal{H}(t)\, e^{it\lambda}\, dt \longrightarrow 0$$

as $\lambda \to \pm\infty$ for any piecewise continuous absolutely integrable $\left(\int_{-\infty}^{\infty} |\mathcal{H}|\, dt < \infty\right)$ $\mathcal{H}$.

Since $\dfrac{H(1+it)}{(1+it)(2+it)}$ is $\underline{C^\infty}$ and $O(1)\dfrac{(\ln|t|)^9}{t^2}$ for $t \gtrsim 3$, we are done. $\blacksquare$

Proof of R-L lemma

Choose any $\varepsilon > 0$.

Choose G so big: $\int_{|t|>G} |\mathcal{H}(t)|\,dt < \frac{\varepsilon}{3}$.

Find a piecewise constant fcn $s(t)$ on $[-G,G]$ so that

$$\int_{-G}^{G} |\mathcal{H}(t) - s(t)|\,dt < \frac{\varepsilon}{3}.$$

For $s(t)$, notice that on each "step",

$$\int_{\alpha_j}^{\beta_j} c_j e^{it\lambda}\,dt = c_j \frac{e^{i\lambda\beta_j} - e^{i\lambda\alpha_j}}{i\lambda}$$

$$[\text{abs value}] \leq 2|c_j|\frac{1}{\lambda}.$$

Hence:

$$\int_{-G}^{G} s(t)e^{it\lambda}\,dt = O\left(\frac{1}{\lambda}\right).$$

Writing

$$\int_{-\infty}^{\infty} \mathcal{H}(t)e^{it\lambda}\,dt = \int_{|t|>G} \mathcal{H}(t)e^{i\lambda t}\,dt$$

$$+ \int_{-G}^{G} [\mathcal{H}(t) - s(t)]e^{i\lambda t}\,dt$$

(continued)

$$+ \int_{-G}^{G} s(t) e^{i\lambda t}\,dt,$$

we clearly get

$$\left| \int_{-\infty}^{\infty} \mathcal{H}(t) e^{it\lambda}\,dt \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

for all $|\lambda| > \lambda_\varepsilon$. Done! ∎

Note: if $\mathcal{H}$ is $\underline{C'}$, just do (G fixed)

$$\int_{-G}^{G} \mathcal{H}(t) e^{i\lambda t}\,dt = \int_{-G}^{G} \mathcal{H}(t)\,d\left(\frac{e^{i\lambda t}}{i\lambda}\right)$$

$$= O\left(\frac{1}{\lambda}\right) - \int_{-G}^{G} \frac{e^{i\lambda t}}{i\lambda} \mathcal{H}'(t)\,dt$$

$$= O\left(\frac{1}{\lambda}\right)$$

to make a short-circuit. IE, R-L lemma is <u>TRIVIAL</u> if $\mathcal{H} \in C^1$.

Lecture 8
(12 Feb)

We know $\psi_1(x) \sim \frac{x^2}{2}$, where $\psi_1(x) = \int_0^x \psi(v)\,dv$.

Theorem (equiv to PNT)

$$\psi(x) \sim x \quad \text{as} \quad x \to \infty .$$

Pf

Read Ingham p. 35 on your own. My method is closer to p. 64. Write $\psi_1(x) = \frac{x^2}{2} + R(x)$.

Keep $0 < h \leq \frac{x}{2}$ and x large. Obviously,

$$\frac{\psi_1(x+h) - \psi_1(x)}{h} = \frac{1}{h}\int_x^{x+h} \psi(v)\,dv \geq \psi(x)$$

$$\frac{\psi_1(x) - \psi_1(x-h)}{h} = \frac{1}{h}\int_{x-h}^{x} \psi(v)\,dv \leq \psi(x) .$$

This gives

$$\psi(x) \leq \frac{\frac{(x+h)^2}{2} - \frac{x^2}{2} + R(x+h) - R(x)}{h}$$

$$\psi(x) \leq x + \frac{h}{2} + \frac{R(x+h) - R(x)}{h} \quad ;$$

$$\psi(x) \gtrsim \frac{\frac{x^2}{2} - \frac{(x-h)^2}{2} + R(x) - R(x-h)}{h}$$

$$\psi(x) \gtrsim x - \frac{h}{2} + \frac{R(x) - R(x-h)}{h}.$$

Clearly:

$$\lesssim \psi(x) - x \lesssim \frac{h}{2} + \frac{|R(x+h)| + |R(x)|}{h}$$

$$-\frac{h}{2} - \frac{|R(x)| + |R(x-h)|}{h} \quad //.$$

Suppose $|R(y)| \lesssim E(y)$ with some <u>explicit</u> <u>monotonic increasing</u> fcn E. Get:

$$-\frac{h}{2} - \frac{2E(x)}{h} \lesssim \psi(x) - x \lesssim \frac{h}{2} + \frac{2E(2x)}{h}$$

$$\Rightarrow \boxed{|\psi(x) - x| \lesssim \frac{h}{2} + \frac{2E(2x)}{h}}.$$

But, given $\varepsilon > 0$, we know $|R(y)| \lesssim \varepsilon y^2$ for all $y \gtrsim A_\varepsilon$. Keep $x \gtrsim \underline{2000 A_\varepsilon}$ so that $x - h \gtrsim \frac{x}{2} \gtrsim 1000 A_\varepsilon$.

We are free to take $E(y) = \varepsilon y^2$ in the ranges which are relevant so long as we make doubly certain $0 < h \lesssim \frac{x}{2}$.

$$|\psi(x) - x| \lesssim \frac{h}{2} + \frac{2E(2x)}{h}$$

$$|\psi(x) - x| \lesssim \frac{h}{2} + \frac{8\varepsilon x^2}{h}$$

$$\left\{ \begin{array}{l} \text{wish to put } h = 4\sqrt{\varepsilon}\, x \\ \text{so just keep } \varepsilon < \frac{1}{100} \\ \text{and } x \text{ big} \end{array} \right\}$$

$$|\psi(x) - x| \lesssim 2\sqrt{\varepsilon}\, x + 2\sqrt{\varepsilon}\, x$$

$$|\psi(x) - x| \lesssim 4\sqrt{\varepsilon}\, x \quad , \quad \text{if } x \gtrsim x_\varepsilon \equiv 2000 A_\varepsilon$$
(say)

Since ε is arbitrary, we are done.

∎

From the earlier lectures (e.g. Lec 2, p. 2) we then get

$$\pi(x) \sim \frac{x}{\ln x} .$$

$\boxed{\text{PNT}}$

I remarked that, in Riemann's formula for $\psi_1(x)$, one would like to move $Re(s) = c$ over past $\sigma = \frac{1}{2}$.

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left[- \frac{\zeta'(s)}{\zeta(s)} \right] ds$$

__IF__ we expect the poles of $-\frac{\zeta'(s)}{\zeta(s)}$ to lie along $Re(s) = \frac{1}{2}$ (except for $s = 1$), it is reasonable [perhaps] for

$$\psi_1(x) \approx \frac{x^2}{2} + O(x^{3/2}) .$$

$E(x) = Cx^{3/2}$ on p. (3) line 4 would lead to

$$|\psi(x) - x| \leq (\text{constant})\, x^{3/4} .$$

Riemann was aware of this. By being less sloppy with $R(x)$ on (1) + (2), __perhaps__

$$|\psi(x) - x| \leq (\text{constant})\, x^{1/2}$$

could be obtained. THIS IS ALL JUST VERY ROUGH, THOUGH.

I recalled that:

$$\int_a^b [f(x)g'(x) + f'(x)g(x)]\,dx = [f(x)g(x)]_a^b$$

holds for $f \in C'[a,b]$ and $g \in C[a,b]$ but only piecewise C'.

This could also be viewed à la Riemann–Stieltjes integration by parts, by declaring

$$\alpha(x) = g(a) + \int_a^x g'(v)\,dv .$$

Of course: $\alpha(x) \equiv g(x)$.

Another thing I remarked was how Riemann's fundamental formula for $\Psi_1(x)$ was derivable by Fourier integrals.

$$\boxed{\text{"Fourier Integrals = Good."}}$$

Indeed,

R-S integral

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty x^{-s}\, d\psi(x) \quad , \quad Re(s) > 1$$

$$\left\{ \begin{matrix} \psi(u) = O(u) \\ \text{Chebyshev} \end{matrix} \right\}$$

$$= \left[x^{-s}\psi(x)\right]_1^\infty - \int_1^\infty \psi(x)\, d(x^{-s})$$

$$= 0 - 0 - \int_1^\infty \psi(x)(-s)x^{-s-1}\, dx$$

$$= s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx \quad \Longrightarrow$$

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx \quad \left\{ \begin{matrix} Re(s) > 1 \\ \text{Ingham p. 18} \end{matrix} \right\}.$$

But, $\psi_1(x) = \int_1^x \psi(v)\, dv$ for $x \geq 1$.

ψ_1 continuous piecewise C^1

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty x^{-s-1}\, d\left[\psi_1(x)\right]$$

$$= x^{-s-1}\psi_1(x)\Big]_1^\infty - \int_1^\infty \psi_1(x)\, d(x^{-s-1})$$

$$\left\{ \psi_1(x) = O(x^2) \quad \text{Chebyshev} \right\}$$

7

$$= 0 - 0 + (s+1)\int_1^{\infty} \frac{\psi_1(x)}{x^{s+2}}\,dx$$

$$\Downarrow$$

$$-\frac{1}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)} = \int_1^{\infty} \psi_1(x)\,x^{-s-2}\,dx\,, \quad Re(s) > 1$$

This is beginning to look like a Mellin transform. Ingham p. 32.

Recall Fourier inversion formula (heuristically).

$$\tilde{F}(p) = \int_{-\infty}^{\infty} F(v)\,e^{-ipv}\,dv \quad \leftarrow \text{Fourier transform}$$

$$\Rightarrow F(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(p)\,e^{ipv}\,dp\,.$$

This is very useful if

$$\mathfrak{M}(s) \equiv \int_0^{\infty} f(x)\,x^{-s}\,dx\,, \quad Re(s) > 1$$

WITH $f(x) \equiv 0$ near $x = 0$, $|f(x)| \lesssim O(1)$.

Why? Because:

$$\mathfrak{M}(c+it) = \int_0^{\infty} f(x)\,x^{-c-it}\,dx \qquad c > 1$$

$$= \int_{-\infty}^{\infty} f(e^v)\,e^{v(-c-it)}\,e^v\,dv$$

$$\mathcal{M}(c+it) = \int_{-\infty}^{\infty} \left[f(e^v) e^{-(c-1)v} \right] e^{-itv}\, dv$$ (8)

$\{ f(e^v) \equiv 0$ for v very negative $\}$

$$\Downarrow$$

$$f(e^v) e^{-(c-1)v} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}(c+it) e^{itv}\, dt$$

$$\Downarrow$$

$$f(e^v) e^v = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}(c+it) e^{cv} e^{itv}\, dt$$

$$f(e^v) e^v = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}(c+it) e^{(c+it)v}\, dt$$

$$f(e^v) e^v = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}(s) e^{sv}\, ds$$

$\{$ write $x = e^v$ $\}$

$$f(x) x = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}(s) x^s\, ds$$

$$\boxed{f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}(s) x^{s-1}\, ds}\ .$$

This is *essentially* what is called the Mellin inversion formula. $(s = 1 - \xi)$

Look at (7 top):

$$-\underbrace{\frac{1}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)}}_{\mathfrak{M}(s)} = \int_1^\infty \underbrace{\frac{\psi_1(x)}{x^2}}_{f(x)\ \ \{0 \text{ if } x<1\}} x^{-s}\,dx, \qquad Re(s)>1$$

so we get, by 8 box,

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[-\frac{1}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)}\right]x^{s-1}\,ds$$

which is <u>equivalent</u> to Riemann's

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[-\frac{1}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)}\right]x^{s+1}\,ds.$$

THUS: you really do <u>not</u> need complex variable (analytic function theory) to derive Riemann's fund formula.

Riemann knew this!

Ingham 37

FACT (very curious)

Suppose $\psi(x) \sim x$. Then we can see rather easily that $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}$. [Hence THIS fact is the essence of PNT!]

Pf

Suppose, e.g., that $\zeta(1+it_0) = 0$. Zero of multiplicity $m \geq 1$.

call this $\phi(s)$

$$f(s) \approx (s-s_0)^m \left[c_0 + c_1(s-s_0) + \cdots\right]$$

$$c_0 \neq 0$$

$$\frac{f'(s)}{f(s)} = \frac{m}{s-s_0} + \frac{\phi'(s)}{\phi(s)}$$

$$\frac{f'(s)}{f(s)} = \frac{m}{s-s_0} + O(1) \quad \text{near } s = s_0$$

recall Lec 7 p. 17

Get:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{m}{s-(1+it_0)} + O(1) \quad \text{near } 1+it_0.$$

But,

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx \qquad Re(s) > 1 \quad 6$$

$$\frac{1}{s-1} = \int_1^\infty \frac{x}{x^{s+1}}\,dx \qquad Re(s) > 1$$

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = \int_1^\infty \frac{\psi(x)-x}{x^{s+1}}\,dx .$$

Assume $\varepsilon > 0$ is small. Get

$$|\psi(x)-x| < \varepsilon x , \quad x \geq G_\varepsilon .$$

Hence:

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = \int_1^{G_\varepsilon} \frac{\psi(x)-x}{x^{s+1}}\,dx$$

$$+ \int_{G_\varepsilon}^\infty \frac{\psi(x)-x}{x^{s+1}}\,dx .$$

First integral on RHS is analytic for <u>all</u> $s \in \mathbb{C}$ (since G_ε = finite).

Let $s_0 = 1 + it_0$ and keep $Re(s) > 1$, $s \approx s_0$.

We have:

$$-\frac{1}{s}\left[\frac{m}{s-s_0} + O(1)\right] + O(1) = O(1) + \int_{G_\varepsilon}^{\infty} \frac{\psi(x)-x}{x^{s+1}}\,dx$$

$$-\frac{1}{s_0}\frac{m}{s-s_0} + O(1) = O(1) + \int_{G_\varepsilon}^{\infty} \frac{\psi(x)-x}{x^{s+1}}\,dx .$$

Take $s = \sigma + it_0$ and let $\sigma \to 1$. Get:

$$\frac{1}{|s_0|}\frac{m}{\sigma-1} + O(1) \lesssim \int_{G_\varepsilon}^{\infty} \frac{\varepsilon x}{x^{\sigma+1}}\,dx$$

$$\frac{1}{|s_0|}\frac{m}{\sigma-1} + O(1) \lesssim \varepsilon \int_{G_\varepsilon}^{\infty} x^{-\sigma}\,dx$$

$$\frac{1}{|s_0|}\frac{m}{\sigma-1} + O(1) \lesssim \varepsilon \left[\frac{x^{1-\sigma}}{1-\sigma}\right]_{G_\varepsilon}^{\infty}$$

$$\frac{1}{|s_0|}\frac{m}{\sigma-1} + O(1) \lesssim \varepsilon \frac{G_\varepsilon^{1-\sigma}}{\sigma-1} \qquad (\sigma > 1)$$

$$\Rightarrow \quad \frac{m}{|s_0|} \lesssim \varepsilon G_\varepsilon^{0} \quad \Rightarrow \quad \frac{|m|}{|s_0|} \leq \varepsilon .$$

Hence $\frac{1}{|s_0|} \leq \varepsilon$. But ε was arbitrary!

Contradiction. $\blacksquare$

I remarked in the lecture that I would now pause* for about 2 lectures to fill in some background stuff on Bernoulli numbers, Euler–Maclaurin summation, and special values of $\zeta(s)$.

It's very pretty to work with this very <u>explicit</u> stuff!!!

* possibly a mistake

Thm (Euler – Maclaurin Sum Formula, version 1)

$f \in C^1[0,N] \Rightarrow$

$$\tfrac{1}{2}f(0) + f(1) + \cdots + f(N-1) + \tfrac{1}{2}f(N)$$
$$= \int_0^N f(x)\,dx + \int_0^N f'(x)\left(x - [\![x]\!] - \tfrac{1}{2}\right)dx .$$

Pf

Let $\beta(x) = x - [\![x]\!] - \frac{1}{2}$ for a few moments.
Note that $\beta(x)$ is the difference of 2 right continuous increasing fcns. By def,

$$[\![x]\!] = x - \tfrac{1}{2} - \beta(x) .$$

$$f(1) + \cdots + f(N) = \int_0^N f(x)\,d[\![x]\!]$$ ← this is correct

$$= \int_0^N f(x)\,d\left(x - \tfrac{1}{2} - \beta(x)\right)$$

$$= \int_0^N f(x)\,dx - \int_0^N f(x)\,d\beta(x)$$

$$= \int_0^N f(x)\,dx - \left[f\beta\right]_0^N + \int_0^N \beta(x) f'(x)\,dx$$

{by R-S parts}

$$= \int_0^N f\,dx - f(N)\beta(N) + f(0)\beta(0) + \int_0^N \beta f'\,dx$$

$$= \int_0^N f\,dx + \tfrac{1}{2}f(N) - \tfrac{1}{2}f(0) + \int_0^N \beta f'\,dx$$

$$\Downarrow$$

$$\tfrac{1}{2}f(0) + f(1) + \cdots + f(N-1) + \tfrac{1}{2}f(N) = \int_0^N f\,dx + \int_0^N \beta f'\,dx.$$

$\blacksquare$

We intend to use $\beta(x)$ $*$

$$x - [\![x]\!] - \tfrac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{n\pi}, \quad x \notin \mathbb{Z}$$

repeatedly. We need a few facts.

$*$ $\boxed{\text{Recall that we got this equality via } -\mathrm{Log}(1-z).}$ Lec 7 p. 3

a nice way!

<u>Thm</u>

The partial sums $\displaystyle\sum_{n=1}^{N} \frac{\sin 2\pi n x}{\pi n}$ are **uniformly bounded** for all $x \in \mathbb{R}$.

<u>Pf</u>

Suffices to treat $\displaystyle\sum_{n=1}^{N} \frac{\sin(nt)}{n}$.

Use periodicity 2π and oddness wrt t.

Hence, wlog, $0 < t \lesssim \pi$.

Suffices to treat

$$\sigma_N(t) = \frac{t}{2} + \sum_{n=1}^{N} \frac{\sin nt}{n} .$$

But:

$$\sigma_N' = \frac{1}{2} + \sum_{1}^{N} \cos nt = \frac{1}{2}\left(\sum_{-N}^{N} e^{int} \right)$$

$$= \frac{1}{2} \frac{e^{-itN} - e^{it(N+1)}}{1 - e^{it}}$$

$$= \frac{1}{2} \frac{e^{-it(N+\frac{1}{2})} - e^{it(N+\frac{1}{2})}}{e^{-it/2} - e^{it/2}} = \frac{1}{2} \frac{\sin[(N+\frac{1}{2})t]}{\sin(t/2)} .$$

So,

$$\sigma_N(t) = \int_0^t \frac{\sin[(N+\frac{1}{2})v]}{2\sin(v/2)} \, dv .$$

Write:

$$\frac{1}{2\sin(v/2)} = \frac{1}{v} + h(v) , \qquad 0 < v \leq \pi .$$

Obviously $h(v)$ is C^∞. The fcn $h(v)$ is also
<u>analytic</u> near $v = 0$. Indeed,

$$\frac{1}{2\sin(\frac{v}{2})} - \frac{1}{v} = \frac{1}{2\left[\frac{v}{2} - \frac{1}{3!}\left(\frac{v}{2}\right)^3 + \cdots\right]} - \frac{1}{v}$$

$$= \frac{1}{v(1+b_2v^2+b_4v^4+\cdots)} - \frac{1}{v}$$

$$= \frac{1}{v}\left[1+A_2v^2+A_4v^4+\cdots\right] - \frac{1}{v}$$

$$= A_2v + A_4v^3 + \cdots \quad \text{near } v=0 \text{ in } \mathbb{C}.$$

So,

$$\sigma_N(t) = \int_0^t \left[\frac{1}{v} + h(v)\right]\sin(N+\tfrac{1}{2})v\,dv$$

$$= \int_0^t \frac{\sin[(N+\frac{1}{2})v]}{v}\,dv + \int_0^t h(v)\sin(N+\tfrac{1}{2})v\,dv.$$

But,

$$\left|\int_0^t h(v)\sin(N+\tfrac{1}{2})v\,dv\right| \leq \int_0^\pi |h(v)|\,dv < \infty$$

and

$$\int_0^t \frac{\sin(N+\frac{1}{2})v}{v}\,dv = \int_0^{(N+\frac{1}{2})t} \frac{\sin q}{q}\,dq.$$

By baby calculus, however,

$$\left|\int_0^R \frac{\sin q}{q}\,dq\right| \leq \text{constant}$$

for all $R \geq 0$. Just look at the graph of $\frac{\sin q}{q}$ and consider <u>signed</u> area.

Or use:

$$\int_1^R \frac{\sin q}{q}\,dq = \int_1^R \frac{d(-\cos q)}{q}$$

$$= -\left[\frac{\cos q}{q}\right]_1^R + \int_1^R \cos q \, d\left(\frac{1}{q}\right)$$

$$= O(1) - \int_1^R \frac{\cos q}{q^2}\,dq$$

$$= O(1) + O(1) \, .$$

One knows, in fact, that the improper integral $\int_0^\infty \frac{\sin q}{q}\,dq$ exists!

IN ANY EVENT, we clearly get (by 17)

$$|\sigma_N(t)| \leq \text{some constant}$$

for all $0 < t \leq \pi$. ∎

"Miracle #1" (by revisiting 16 – 18 with more real analysis)

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin q}{q}\,dq$$

Pf ← The standard proof in any Fourier series class.

On 16, we saw

$$\frac{1}{2} + \sum_{n=1}^{N} \cos nt = \frac{\sin[(N+\frac{1}{2})t]}{2\sin(t/2)}, \quad 0 < t \leq \pi.$$

For $t = 0$, use a limit. Integrate over $[0, \pi]$. Get:

$$\frac{\pi}{2} = \int_0^\pi \frac{\sin[(N+\frac{1}{2})v]}{2\sin(v/2)}\,dv.$$

Use 16 bottom – 17 with $h(v)$. Get:

$$\frac{\pi}{2} = \int_0^\pi \underline{\underline{\left(\frac{1}{v} + h(v)\right)}} \sin[(N+\tfrac{1}{2})v]\,dv$$

$$\frac{\pi}{2} = \int_0^\pi \frac{\sin[(N+\frac{1}{2})v]}{v}\,dv + \int_0^\pi h(v)\sin[(N+\tfrac{1}{2})v]\,dv$$

C^∞ and analytic near $v = 0$

$$\Downarrow$$

$$\frac{\pi}{2} \simeq \int_0^{\pi(N+\frac{1}{2})} \frac{\sin q}{q}\,dq$$

$$+ \int_0^{\pi} h(v) \sin\left[(N+\tfrac{1}{2})v\right] dv \,.$$

Recall R-L lemma for C^1

$$\int_0^{\pi} h(v) e^{i\lambda v} dv = \int_0^{\pi} h(v)\, d\left[\frac{e^{i\lambda v}}{i\lambda}\right]$$

$$= h(v)\frac{e^{i\lambda v}}{i\lambda}\Big]_0^{\pi}$$

$$- \int_0^{\pi} \frac{e^{i\lambda v}}{i\lambda} h'(v)\,dv$$

$$= O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{\lambda}\right)$$

as in Lec 7 p. 23.

Let $N \to \infty$ and use R-L lemma.

Get:

$$\frac{\pi}{2} = \int_0^{\infty} \frac{\sin q}{q}\,dq + 0\,.$$

OK!

Miracle #2 (by revisiting 16 – 18 with more real analysis)

I claim that p. 16 and 17 (middle) immediately imply

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin q}{q}\, dq$$

AND

$$\sum_{n=1}^\infty \frac{\sin(2\pi n q)}{\pi n} = \frac{1}{2} - q + [\![q]\!] \,, \quad q \notin \mathbb{Z}.$$

"No need for $-\mathrm{Log}(1-z)$"

Pf

Use p. 16 for $0 < t \leq 2\pi - \delta$.

Notice that $h(v)$ is C^∞ on $(0, 2\pi - \delta]$ and analytic near $v = 0$.

We still have

$$\sigma_N(t) = \int_0^t \frac{\sin[(N+\frac{1}{2})v]}{2\sin(v/2)}\, dv \quad \leftarrow 16$$

$$= \int_0^t \frac{\sin[(N+\frac{1}{2})v]}{v}\, dv$$

$$+ \int_0^t h(v) \sin[(N+\tfrac{1}{2})v]\, dv$$

àla 17 (middle).

Thus, for $0 < t \leq 2\pi - \delta$,

$$\frac{t}{2} + \sum_1^N \frac{\sin(nt)}{n} = \int_0^{(N+\frac{1}{2})t} \frac{\sin q}{q}\, dq$$

$$+ \int_0^t h(v) \sin\left[(N+\tfrac{1}{2})v\right] dv \, .$$

Freeze t temporarily and let $N \to \infty$.
Get:

$$\frac{t}{2} + \sum_1^\infty \frac{\sin(nt)}{n} = A + 0$$

cf. 20 middle
with minor change

where $A = \int_0^\infty \frac{\sin q}{q}\, dq$.

Thus:

$$\sum_1^\infty \frac{\sin(nt)}{n} = A - \frac{t}{2}, \quad \underline{\text{all}} \quad 0 < t < 2\pi .$$

<u>Plug</u> <u>in</u> $t = \pi$; this forces A to be $\frac{\pi}{2}$.

Let $t = 2\pi q$, $0 < q < 1$, to get

$$\sum_1^\infty \frac{\sin(2\pi n q)}{n} = \frac{\pi}{2} - \pi q, \quad \text{all} \quad 0 < q < 1 .$$

Hence

$$\sum_{1}^{\infty} \frac{\sin(2\pi n \varphi)}{\pi n} = \frac{1}{2} - \varphi \quad , \quad 0 < \varphi < 1 ,$$

and the rest is trivial by periodicity.

OK

NOTE:

On 22 line 4, if we keep t variable but inside $[\delta, 2\pi - \delta]$, this limit procedure is easily seen to be uniform wrt t as $N \to \infty$.

$t \geq \delta$ used in $\int_0^{(N+\frac{1}{2})t} \frac{\sin \xi}{\xi} d\xi$
$t \leq 2\pi - \delta$ used in $\int_0^t h(v) \sin\left[(N+\frac{1}{2})v\right] dv$ Review 20 (middle) with obvious changes.

Lecture 9 and 10

(17 and 19 Feb)

"Synopsis"

$$\sigma_N(t) = \frac{t}{2} + \sum_1^N \frac{\sin nt}{n}$$ look at this on $0 < t \lesssim 2\pi - \delta$

$$\sigma_N(t) = \int_0^t \frac{\sin[(N+\frac{1}{2})v]}{2\sin(v/2)}\,dv$$ Lec 8 p. 16 etc

but $\frac{1}{2\sin\frac{v}{2}} = \frac{1}{v} + h(v)$, $h(v)$ C^∞ on $[0, 2\pi - \delta]$ and analytic near $v = 0$

$$\Rightarrow \sigma_N(t) = \int_0^{(N+\frac{1}{2})t} \frac{\sin q}{q}\,dq + \int_0^t h(v)\sin[(N+\tfrac{1}{2})v]\,dv$$

Freeze t. Let $N \to \infty$. Use $\mathrm{Im}\left[\int_0^t h(v) e^{i\lambda v}\,dv\right] = O(\frac{1}{\lambda})$. Get:

$$\frac{t}{2} + \sum_1^\infty \frac{\sin nt}{n} = A + 0, \quad A \equiv \int_0^\infty \frac{\sin q}{q}\,dq$$

$$\Downarrow$$

$$\sum_1^\infty \frac{\sin nt}{n} = A - \frac{t}{2}, \quad \text{all } 0 < t < 2\pi.$$

Plug in $t = \pi$. Get $A = \frac{\pi}{2}$.

Write $t = 2\pi q$, $0 < q < 1$, get

$$\sum_1^\infty \frac{\sin(2\pi n q)}{n} = \frac{\pi}{2} - \pi q \Rightarrow \sum_1^\infty \frac{\sin(2\pi n q)}{-\pi n} = q - \frac{1}{2}.$$

By the way, regarding this, note that this method of proof immediately yields <u>uniform</u> convergence for $\delta \leq t \leq 2\pi - \delta$.

Lec 8, p. 23

$$\frac{t}{e^t - 1} \equiv \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n, \quad |t| < 2\pi, \quad t \in \mathbb{C}$$

def of <u>Bernoulli</u> numbers

k	B_k
0	1
1	$-1/2$
2	$1/6$
$\vdots$	$\vdots$

<u>Easy Lemma</u>

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t}{2}\operatorname{ctnh}\left(\frac{t}{2}\right) = \text{even}$$

$$1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} t^n$$

hence $B_3 = B_5 = B_7 = \cdots = 0$.

Put $t = 2iz$. Get:

$$iz \operatorname{ctnh}(iz) = 1 + \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \frac{B_k}{k!} (2iz)^k$$

$$\Downarrow$$

<u>Thus</u>

$$z \operatorname{ctn}(z) = 1 + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (-1)^n (2z)^{2n}, \quad |z| < \pi$$

and

$$\pi \operatorname{ctn}(\pi w) = \frac{1}{w}\left[\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} (-1)^n (2\pi w)^{2n}\right], \quad |w| < 1.$$

The function $\pi \operatorname{ctn}(\pi z)$ has familiar properties in complex analysis.

(A) periodic $z \to z+1$

(B) simple poles at $z = n$, $n \in \mathbb{Z}$

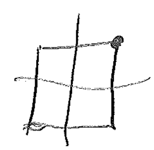
(C) residue always 1

(D) $\operatorname{ctn} \pi(x+iy) = -i + O(e^{-2\pi y})$ $\quad y \to +\infty$

(E) $\operatorname{ctn} \pi(x+iy) = i + O(e^{-2\pi |y|})$ $\quad y \to -\infty$

Standard Cauchy residue theorem set-up with

$$\frac{1}{2\pi i} \oint_{C_N} f(z)\, \pi \operatorname{ctn} \pi z \, dz$$



C_N = square with vertices $(\pm(N+\tfrac{1}{2}), \pm(N+\tfrac{1}{2}))$.

EG $f(z) = z^{-2m}$, $m \geq 1$. Use Thm on 2.

Take $N \to \infty$. Get:

$$\boxed{\zeta(2m) = 2^{2m-1} \pi^{2m} \frac{(-1)^{m+1} B_{2m}}{(2m)!}.} \quad \text{(Euler)}$$

In particular, note that $B_{2m} = (-1)^{m+1} |B_{2m}|$ since $\zeta(2m) > 0$.

Another interesting $f(z)$ is $f = \dfrac{1}{\xi - z}$.

Think $\mathcal{D} = \mathbb{C} \sim \mathbb{Z}$ and $\xi \in [\mathcal{D}$ compacta$]$.

By CRT,

$$-\pi \operatorname{ctn} \pi \xi + \sum_{|n| \leq N} \frac{1}{\xi - n} = \frac{1}{2\pi i} \int_{\mathcal{C}_N} \frac{\pi \operatorname{ctn} \pi z}{\xi - z} dz .$$

Write:

$$f(z) = -\frac{1}{z} + \left[\frac{1}{\xi - z} + \frac{1}{z} \right] = -\frac{1}{z} + \frac{\xi}{(\xi - z) z}$$

$$= -\frac{1}{z} + \underline{r(z)} , \quad r(z) = O(z^{-2}) .$$

Notice that:

$$\int_{\mathcal{C}_N} h(z) dz = \underline{0} \quad \text{for } \underline{\text{ANY}} \text{ even + continuous } h$$

$\{$ uses symmetry of $\mathcal{C}_N$ and $w = -z = z e^{i\pi}$ $\}$.

So, we still get:

$$\int_{\mathcal{C}_N} \frac{\pi \operatorname{ctn} \pi z}{\xi - z} dz = O\left(\frac{1}{N}\right) .$$

So,

$$\lim_{N \to \infty} \sum_{|n| \leq N} \frac{1}{\xi - n} = \pi \operatorname{ctn} \pi \xi , \quad \xi \in \mathcal{D}.$$

By reviewing the proof, we see the limit is <u>uniform</u> for $z \in [\mathcal{D}$ compacta$]$.

Tautology:

$$\frac{1}{z} + \sum_{\substack{-N \\ n \neq 0}}^{N} \left(\frac{1}{z-n} + \frac{1}{n} \right) = \sum_{-N}^{N} \frac{1}{z-n}.$$

THM $z \in \mathcal{D}$.

(i) $\pi \operatorname{ctn} \pi z = \lim_{N \to \infty} \sum_{-N}^{N} \frac{1}{z-n}$

(ii) $\pi \operatorname{ctn} \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right)$.

↑ unif conv on $\mathcal{D}$ compacta

Use thm on 2. Get:

$$\pi w \operatorname{ctn}(\pi w) = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} (2\pi)^{2n} (-1)^{n} w^{2n}, \quad |w| < 1.$$

But, now use the THM above:

$$\pi \operatorname{ctn}(\pi z) = \frac{1}{z} + \sum_{m \neq 0} \left(\frac{1}{z+m} - \frac{1}{m} \right) \qquad \text{Keep } 0 < |z| < 1$$

$$\frac{1}{m+z} = \frac{1}{m(1+\frac{z}{m})} = \frac{1}{m}\left[1 - \frac{z}{m} + \frac{z^2}{m^2} \pm \cdots\right]$$

$$\pi \operatorname{ctn} \pi z = \frac{1}{z} + \sum_{m \neq 0} \left[-\frac{z}{m^2} + \frac{z^2}{m^3} - \frac{z^3}{m^4} \pm \cdots \right]$$

$$\pi \operatorname{ctn} \pi z = \frac{1}{z} - 2z\,\zeta(2) - 2z^3\,\zeta(4) - \cdots$$

$$\pi z \operatorname{ctn} \pi z = 1 - 2z^2\,\zeta(2) - 2z^4\,\zeta(4) - \cdots, \quad |z|<1$$

$$\Downarrow$$

$$-2\,\zeta(2n) = \frac{B_{2n}}{(2n)!}(2\pi)^{2n}(-1)^{n}$$

$$\Downarrow$$

$$\zeta(2n) = \frac{B_{2n}}{(2n)!}(-1)^{n+1}\,2^{2n-1}\,\pi^{2n}$$

$= \underline{\underline{2^{nd}}}$ proof of Euler's formula.

NICE!

<u>Thm</u>

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

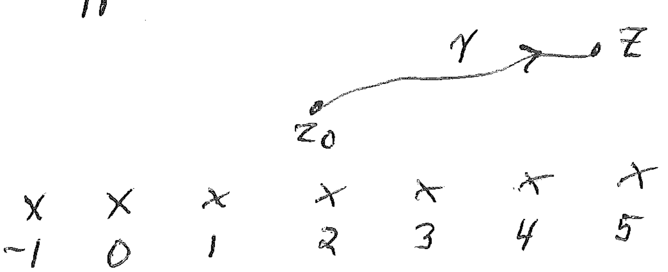
unif conv on $\mathcal{D}$ compacta

<u>Pf</u>

Differentiate THM on 5 by Weierstrass conv thm.

∎

Another nice trick. Take z_0 and z in upper half-plane. Connect via γ.



γ = compact subset

$$\frac{d}{dz}\log(\sin \pi z) = \pi \operatorname{ctn} \pi z$$

$$\Downarrow$$

(with some branches)

$$\log[\sin \pi z] - \log[\sin \pi z_0] = \int_\gamma \pi \operatorname{ctn} \pi w \, dw$$

$$\frac{\sin \pi z}{\sin \pi z_0} = \exp\left[\int_\gamma \pi \operatorname{ctn} \pi w \, dw\right]$$

substitute THM 5 (i)

$$= \exp\left[o(1) + \sum_{-N}^{N} \left(\log(z-n) - \log(z_0-n)\right)\right]$$

$o(1)$ = little "oh"

$$= [1+o(1)] \prod_{-N}^{N} \frac{z-n}{z_0-n}$$

$$= [1+o(1)] \frac{z}{z_0} \frac{\prod_1^N (k^2-z^2)}{\prod_1^N (k^2-z_0^2)}$$

$$= [1+o(1)] \frac{z}{z_0} \frac{\prod_1^N \left(1-\frac{z^2}{k^2}\right)}{\prod_1^N \left(1-\frac{z_0^2}{k^2}\right)}$$

$$\left\{ \begin{array}{l} z \in H,\ z_0 \in H \quad \text{think nonzero} \\ \text{infinite products etc etc} \end{array} \right\}$$

$$\frac{\sin \pi z}{\pi z} = \left(\frac{\sin \pi z_0}{\pi z_0} \right) \frac{1}{\prod_1^\infty \left(1 - \frac{z_0^2}{k^2}\right)} \left(\prod_{k=1}^\infty \left(1 - \frac{z^2}{k^2}\right) \right)$$

$$\frac{\sin \pi z}{\pi z} = C \prod_{k=1}^\infty \left(1 - \frac{z^2}{k^2}\right)$$

first for $Im(z) > 0$, then for ALL $z \in \mathbb{C}$ by analyticity of both sides

Let $z \to 0$. Get $C = 1$.

<u>THM</u>

$$\frac{\sin \pi z}{\pi z} = \prod_{k=1}^\infty \left(1 - \frac{z^2}{k^2}\right), \quad z \in \mathbb{C}.$$

Known to Euler

Baby Fact

Let $\beta(x) = x - [\![x]\!] - \frac{1}{2}$ for $x \in \mathbb{R}$.
Let F be Riemann integrable on $[a,b]$.
Then:

$$\int_a^b F(x)\beta(x)\,dx = \sum_{n=1}^{\infty} \int_a^b F(x) \frac{\sin 2\pi n x}{-\pi n}\,dx .$$

Proof

$$S_N(x) = -\sum_1^N \frac{\sin 2\pi n x}{\pi n} .$$

Know $S_N(x)$ is unif bdd all x, N. Also, $S_N(x) \rightrightarrows \beta(x)$ on compact subsets of $\mathbb{R}$ away from $\mathbb{Z}$.

Think $a < 1 < b$ (say) and $|F| \leq M$, say

$$\left| \int_a^b F(x)(\beta - S_N)\,dx \right| \leq \int_a^{1-\delta} |F||\beta - S_N|\,dx$$
$$+ \int_{1-\delta}^{1+\delta} |F||\beta - S_N|\,dx$$
$$+ \int_{1+\delta}^{b} |F||\beta - S_N|\,dx$$

$\Rightarrow$ done! ■

Take, e.g., $f \in C^{2k+1}[0,1]$. Know

$$\frac{1}{2}f(0) + \frac{1}{2}f(1) = \int_0^1 f\,dx + \int_0^1 f'\beta(x)\,dx$$

by E–M version I

$$= \int_0^1 f\,dx + \int_0^1 f'\left(-\sum_1^\infty \frac{\sin 2\pi n x}{\pi n}\right)dx$$

$$= \int_0^1 f\,dx + \sum_1^\infty \left(-\frac{1}{\pi n}\right)\int_0^1 f' \sin 2\pi n x\,dx$$

by Baby Fact.

Now just look at EACH integral

$$\int_0^1 \underline{f'} \sin(2\pi n x)\,dx$$

and repeatedly integrate by parts.

Step 1 $\quad \int_0^1 f'\,d\left(\frac{\cos 2\pi n x}{-2\pi n}\right)$

$$= f' \frac{\cos 2\pi n x}{-2\pi n}\Big]_0^1 + \int_0^1 \frac{\cos 2\pi n x}{2\pi n} f''\,dx$$

$$= \frac{f'(1) - f'(0)}{-2\pi n} + \frac{1}{2\pi n}\int_0^1 \cos 2\pi n x \cdot f''\,dx.$$

Step 2

$$\frac{f'(1)-f'(0)}{-2\pi n} + \frac{1}{2\pi n}\int_0^1 f''\,d\left[\frac{\sin 2\pi n x}{2\pi n}\right]$$

$$= \frac{f'(1)-f'(0)}{-2\pi n} + 0 - 0 - \frac{1}{(2\pi n)^2}\int_0^1 \sin 2\pi n x \cdot f'''\,dx$$

$$= \frac{f'(1)-f'(0)}{-2\pi n} - \frac{1}{(2\pi n)^2}\int_0^1 \underline{f'''(x)}\sin(2\pi n x)\,dx .$$

Clearly a recursion has begun!

$$\int_0^1 f'\sin(2\pi n x)\,dx = \sum_{k=1}^{R}\frac{(-1)^k}{(2\pi n)^{2k-1}}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right]$$

$$+ \frac{(-1)^R}{(2\pi n)^{2R}}\int_0^1 \underline{f^{(2R+1)}(x)}\sin(2\pi n x)\,dx .$$

So, on 10 line 5,

$$-\frac{2}{2\pi n}\int_0^1 f'(x)\sin(2\pi n x)\,dx$$

$$= \sum_{k=1}^{R}\frac{2(-1)^{k+1}}{(2\pi n)^{2k}}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right]$$

$$+ \frac{2(-1)^{R+1}}{(2\pi n)^{2R+1}}\int_0^1 \underline{f^{(2R+1)}(x)}\sin(2\pi n x)\,dx .$$

Thm (E-M version #2, prelim form)

$f \in C^{2R+1}[0,N]$. Then:

$$\sum_{n=0}^{N} f(n) = \frac{1}{2} f(0) + \frac{1}{2} f(N)$$

$$+ \int_0^N f\,dx$$

$$+ \sum_{k=1}^{R} \frac{2(-1)^{k+1}}{(2\pi)^{2k}} \zeta(2k) \left[f^{(2k-1)}(N) - f^{(2k-1)}(0) \right]$$

$$+ 2(-1)^{R+1} \int_0^N f^{(2R+1)}(x) \left[\sum_1^\infty \frac{\sin 2\pi n x}{(2\pi n)^{2R+1}} \right] dx$$

and (in the above) we actually have

$$\frac{2(-1)^{k+1}}{(2\pi)^{2k}} \zeta(2k) \equiv \frac{B_{2k}}{(2k)!}.$$

PF

Write

$$\frac{1}{2} f(0) + f(1) + \cdots + f(N-1) + \frac{1}{2} f(N) = \sum_{j=0}^{N-1} \frac{1}{2}\left[f(j) + f(j+1)\right],$$

then use 10 (top) + 11 (bottom). The final observation about B_{2k} follows from Euler's formula for $\zeta(2k)$. ▮

3

$\boxed{\text{Lec 10}}$ begins here.

Think about $\displaystyle\int_{C_N} f(z) \frac{e^{iaz}}{1-e^{2\pi i z}}\, dz$. Entertain yourself.

$\zeta(2k) \to 1$ as $k \to \infty$. So: $1 \sim \dfrac{(2\pi)^{2k}}{2} \dfrac{|B_{2k}|}{(2k)!}$;

IE $\quad |B_{2k}| \sim \dfrac{2(2k)!}{(2\pi)^{2k}}$.

$B_2 = \frac{1}{6}$ $\qquad B_{10} = \frac{5}{66}$ $\qquad$ there are long tables!

$B_4 = -\frac{1}{30}$ $\qquad B_{12} = -\frac{691}{2730}$

$B_6 = \frac{1}{42}$ $\qquad B_{14} = \frac{7}{6}$

$B_8 = -\frac{1}{30}$ $\qquad B_{16} = -\frac{3617}{510}$ $\qquad \left(B_1 = -\frac{1}{2}\right)$ p. 2

Bernoulli polynomials on $0 < x < 1$?

$B_n(x) =$ degree n, leading coefficient 1

$B_n(0) = B_n$

$$\frac{te^{tx}}{e^t - 1} \cong \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \qquad |t| < 2\pi$$

$B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$; ...

Let $\tilde{B}_n(x)$ be the period 1 periodic extension of $B_n(x)$ to $\mathbb{R} - \mathbb{Z}$.

Not hard to check:

$$\frac{\tilde{B}_{2R}(x)}{(2R)!} = 2(-1)^{R+1} \sum_{1}^{\infty} \frac{\cos(2\pi n x)}{(2\pi n)^{2R}}$$

$$\frac{\tilde{B}_{2R+1}(x)}{(2R+1)!} = 2(-1)^{R+1} \sum_{1}^{\infty} \frac{\sin(2\pi n x)}{(2\pi n)^{2R+1}} \quad \leftarrow \text{compare (12) line 6}$$

Note that $B_1(x)$ à la $R=0$ certainly fits!

THEOREM (E-M version 2)

$f \in C^{2R+1}[0,N]$, f complex OK. Then:

$$\sum_{j=0}^{N} f(j) = \tfrac{1}{2}f(0) + \tfrac{1}{2}f(N) + \int_0^N f(x)\,dx + \sum_{1}^{R} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(N) - f^{(2k-1)}(0)\right] + \text{Remainder}_R,$$

where

$$\left\{\begin{aligned} \text{Rem}_R &= \int_0^N f^{(2R+1)}(x)\,\frac{\tilde{B}_{2R+1}(x)}{(2R+1)!}\,dx \\ &= -\int_0^N f^{(2R)}(x)\,\frac{\tilde{B}_{2R}(x)}{(2R)!}\,dx, \quad \text{via easy integ by parts} \end{aligned}\right\}$$

Pf

Use (12) THM + the formulae at top of this page. ∎

Cor 1

$$|\mathrm{Rem}_R| \leq \frac{|B_{2R}|}{(2R)!} \int_0^N |f^{(2R)}(x)|\,dx .$$

Pf

Plug into 2nd form of Rem_R on (14).

$$|\mathrm{Rem}_R| \leq \int_0^N |f^{(2R)}| \cdot 2 \sum_1^\infty \frac{1}{(2\pi n)^{2R}}\,dx$$

$$= \frac{2}{(2\pi)^{2R}} \zeta(2R) \int_0^N |f^{(2R)}|\,dx$$

$$= \frac{|B_{2R}|}{(2R)!} \int_0^N |f^{(2R)}|\,dx \qquad \text{by (3) box.} \quad \blacksquare$$

Cor 2 (very useful in numerical work)

non-negative

$$\mathrm{Rem}_R = 2(-1)^{R+2} \int_0^N f^{(2R+2)}(x) \sum_{n=1}^\infty \frac{1-\cos 2\pi n x}{(2\pi n)^{2R+2}}\,dx$$

Pf

$$\mathrm{Rem}_R = \int_0^N f^{(2R+1)}(x) \left[2(-1)^{R+1} \sum_1^\infty \frac{\sin 2\pi n x}{(2\pi n)^{2R+1}} \right] dx$$

$$= \sum_n \frac{2(-1)^{R+1}}{(2\pi n)^{2R+1}} \int_0^N f^{(2R+1)}\, d\left(\frac{1-\cos 2\pi n x}{2\pi n}\right)$$

$$= \sum_n \frac{2(-1)^{R+1}}{(2\pi n)^{2R+1}} \left[0 - 0 - \int_0^N \frac{1-\cos 2\pi n x}{2\pi n} f^{(2R+2)}(x)\,dx \right]$$

$$= \sum_{n} \frac{2(-1)^{R+2}}{(2\pi n)^{2R+2}} \int_0^N (1-\cos 2\pi n x)\, f^{(2R+2)}(x)\,dx$$

$$= 2(-1)^{R+2} \int_0^N \sum_{n=1}^{\infty} \frac{1-\cos 2\pi n x}{(2\pi n)^{2R+2}}\, f^{(2R+2)}(x)\,dx\,. \qquad \blacksquare$$

<u>Cor 3</u> (very commonly used)

f complex. We always have:

$$|Rem_R| \leq 2 \frac{|B_{2R+2}|}{(2R+2)!} \int_0^N |f^{(2R+2)}|\,dx\,.$$

<u>Pf</u>

Use Cor 2.

$$|Rem_R| \leq 2 \int_0^N |f^{(2R+2)}| \cdot \sum_{n=1}^{\infty} \frac{2}{(2\pi n)^{2R+2}}\,dx$$

$$= 2 \cdot \frac{2}{(2\pi)^{2(R+1)}}\, \zeta(2R+2) \int_0^N |f^{(2R+2)}|\,dx$$

{apply ③ box}

$$= 2 \cdot \frac{|B_{2R+2}|}{(2R+2)!} \int_0^N |f^{(2R+2)}|\,dx \qquad \blacksquare$$

For the sake of clarity, notice too that

$$\mathrm{Rem}_R \equiv \frac{B_{2R+2}}{(2R+2)!}\left[f^{(2R+1)}(N) - f^{(2R+1)}(0)\right] + \mathrm{Rem}_{\underline{R+1}} .$$

Cor 3 can thus be obtained <u>equally</u> <u>well</u> by Cor 1; indeed,

$$|\mathrm{Rem}_R| \leqq \frac{|B_{2R+2}|}{(2R+2)!}\left|f^{(2R+1)}(N) - f^{(2R+1)}(0)\right| + \frac{|B_{2R+2}|}{(2R+2)!}\int_0^N |f^{(2R+2)}(x)|\,dx \leftarrow \text{Cor } 1$$

$$\leqq 2\,\frac{|B_{2R+2}|}{(2R+2)!}\int_0^N |f^{(2R+2)}(x)|\,dx .$$

THM (Corollary of E–M à la Euler)

$\zeta(z)$ is analytic on each half-plane $\{Re(z) > -\Delta\}$ except for a simple pole of residue 1 at $z = 1$. Hence,

$$\zeta(z) - \frac{1}{z-1} \quad \text{is analytic on } \mathbb{C}.$$

PF

E–M $\quad f(t) = (1+t)^{-z}$ $\quad$ keep $Re(z) > 1$ at first

$$f^{(j)}(t) = (-1)^j z(z+1)\cdots(z+j-1)(1+t)^{-z-j}$$

$$\Downarrow$$

$$\sum_{k=1}^{N+1} k^{-z} = \frac{1}{2} + \frac{1}{2}(1+N)^{-z} + \int_0^N (1+t)^{-z}\,dt$$

$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)!}\Big[(-1)\, z(z+1)\cdots(z+2k-2)(1+N)^{-z-(2k-1)}$$

$$+ z(z+1)\cdots(z+2k-2)\cdot 1\Big]$$

$$+ \int_0^N \frac{\tilde{B}_{2R+1}(t)}{(2R+1)!}(-1)\, z(z+1)\cdots(z+2R)\,(1+t)^{-z-2R-1}\,dt$$

Let $N \to \infty$.

$$\zeta(z) = \frac{1}{2} + \int_0^\infty (1+t)^{-z}\, dt$$
$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)!} z(z+1)\cdots(z+2k-2)\cdot 1$$
$$+ \int_0^\infty (-1) \frac{z(z+1)\cdots(z+2R)}{(1+t)^{z+2R+1}} \frac{\tilde{B}_{2R+1}(t)}{(2R+1)!}\, dt$$

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1}$$
$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)!} z(z+1)\cdots(z+2k-2)$$
$$+ (-1) z(z+1)\cdots(z+2R) \int_0^\infty \frac{1}{(1+t)^{z+2R+1}} \frac{\tilde{B}_{2R+1}(t)}{(2R+1)!}\, dt$$

Rem_R with $R \geq 1$

note $R=0$ is OK too; recall Lec 5 p. 9 line 3 and $r(t) = \frac{1}{2} + \beta(t)$

The integral containing $\tilde{B}_{2R+1}(t)$ is clearly nicely convergent, hence analytic, for compact subsets of $\{Re(z) > -2R\}$.

At once,

$$\zeta(z) - \frac{1}{z-1}$$

is analytic on each $\{Re(z) > -2R\}$ and we are done! ▩

<u>Examples</u> (Numerology!)

Recall B_k on p. 13.

$R = 3$ say $\Rightarrow$ get

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1} + \frac{B_2}{2!} z + \frac{B_4}{4!} z(z+1)(z+2)$$

$$+ \frac{B_6}{6!} z(z+1)\cdots(z+4)$$

$$+ (-1) z(z+1)\cdots(z+6) \int_0^\infty \frac{1}{(1+t)^{z+7}} \frac{\tilde{B}_7(t)}{7!} dt$$

$$Re(z) > -6$$

$$\zeta(0) = -\frac{1}{2}$$

$$\zeta(-1) = \frac{1}{2} - \frac{1}{2} + \frac{B_2}{2!}(-1) + 0 = -\frac{1}{12}$$

$$\zeta(-2) = \frac{1}{2} - \frac{1}{3} + \frac{B_2}{2!}(-2) + 0 = \frac{1}{6} + \frac{1}{12}(-2) = \underline{\underline{0}}$$

$$\zeta(-4) = \frac{1}{2} - \frac{1}{5} + \frac{B_2}{2!}(-4) + \frac{B_4}{4!}(-4)(-3)(-2)$$

$$+ 0$$

$$= \frac{1}{2} - \frac{1}{5} + \frac{1}{12}(-4) + \left(-\frac{1}{30}\right)\frac{(-1)4!}{4!} + 0$$

$$= \frac{3}{10} - \frac{1}{3} + \frac{1}{30} = \frac{9-10+1}{30} = \underline{\underline{0}}$$

One conjectures $\zeta(-2\ell) = 0$, $\ell \geq 1$.
Euler proved this (playing with the B_{2k}).

---

of course he used only $\zeta(x)$
and his "natural formulae".

$\Gamma(z)$, more accurately $\Gamma(\frac{z}{2})$, is part of the "modern Riemann zeta fcn". It's the Archimedean part.

$$\pi^{-z/2}\Gamma(\tfrac{z}{2})\zeta(z)$$

$$\Gamma(z) \equiv \int_0^\infty t^{z-1}e^{-t}\,dt, \quad Re(z)>0$$

GAMMA FCN.

$\Gamma(k) = (k-1)! \quad k \geq 1$

By Weierstrass M-test, the improper integral is unif conv on $\{Re(z)>0\}$-compacta.

$$|t^{z-1}| = t^{x-1}$$

Hence $\Gamma(z)$ analytic on $Re(z)>0$.

Easy integ by parts:

$$Re(z)>0 \Rightarrow \Gamma(z+1) = z\,\Gamma(z).$$

Hence

$$\Gamma(z+R) = z(z+1)\cdots(z+R-1)\,\Gamma(z), \quad R \geq 1.$$

$$\Rightarrow \boxed{\Gamma(z) = \frac{\Gamma(z+R)}{z(z+1)\cdots(z+R-1)}}$$

But, RHS is analytic for $\{Re(z)>-R\}$, except at $0,-1,-2,\ldots,-(R-1)$.

Hence, $\Gamma(z)$ is analytic on

$$\mathbb{C} - \{0, -1, -2, \dots\}.$$

Easy to check:

$z = -k$ is a <u>simple</u> pole

$$\text{Res} = \frac{(-1)^k}{k!}.$$

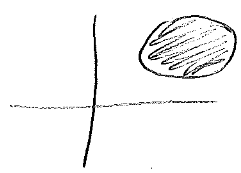
<u>Thm</u> (Euler)

$$\Gamma(z) = \lim_{N \to \infty} \frac{N!\, N^z}{z(z+1)\cdots(z+N)}$$

with <u>unif</u> conv on $\{\text{Re}(z) > 0\}$ compacta.

<u>Pf</u>

Keep $z \in K$ say.



$$e^{-t} = \lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n, \quad t > 0 \qquad \text{(baby calc)}$$

We hope to <u>approximate</u> $\Gamma(z)$ by

$$\int_0^n t^{z-1} \left(1 - \frac{t}{n}\right)^n dt.$$

$\exp[(z-1)\ln t]$

Baby calculus $\Rightarrow$

$$\text{previous integral} = n^z \int_0^1 v^{z-1}(1-v)^n dv$$

{ integrate by parts repeatedly }

NOT HARD

$$= n^z \frac{n!}{z(z+1)\cdots(z+n-1)} \left(\frac{1}{z+n}\right)$$

$$\int_0^1 v^{z+n-1}(1-v)^0 dv$$

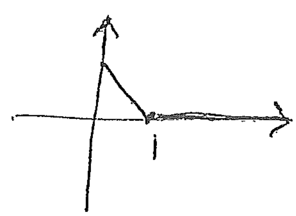
$$= \frac{n^z \cdot n!}{z(z+1)\cdots(z+n)}.$$

Must now prove

$$\lim_{N\to\infty} \int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt = \int_0^\infty t^{z-1}e^{-t}dt$$

<u>unif</u> on K.

Let $(1-u)_* = \begin{cases} 0, & u > 1 \\ 1-u, & 0 \le u \le 1 \end{cases}$.



We will look at

$$\int_0^\infty t^{z-1}\left(1-\frac{t}{N}\right)_*^N dt.$$

which equals $\int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt$

For one moment

Take $t > 0$ and keep $0 < t < n$.

$$\left(1-\frac{t}{n}\right)^n = e^{n\log\left(1-\frac{t}{n}\right)}$$
$$= e^{n\left[-\frac{t}{n}-\frac{1}{2}\frac{t^2}{n^2}-\cdots\right]}$$
$$= e^{-t}\, e^{-\frac{1}{2}\frac{t^2}{n}}\, e^{-\frac{1}{3}\frac{t^3}{n^2}}\cdots$$

$$\Downarrow$$

$$0 < \left(1-\frac{t}{n}\right)^n < \left(1-\frac{t}{n+1}\right)^{n+1} < \cdots < e^{-t}.$$

We thus see that

$$0 \leq \left(1-\frac{t}{N}\right)_*^N \nearrow \text{ to } e^{-t}$$

as $N \to \infty$. In addition, by the expansion above, we have uniform convergence as $N \to \infty$ on any $0 \leq t \leq \Delta$, Δ big.

The issue with $\int_0^\infty t^{z-1}\left(1-\frac{t}{N}\right)_*^N dt$ is now a simple manipulation with the dominated convergence thm for $z \in K$.

Just write

$$\int_0^\infty t^{z-1}\left(1-\frac{t}{N}\right)_*^N dt$$

$$= \int_0^\delta + \int_\delta^T t^{z-1}\left(1-\frac{t}{N}\right)_*^N dt + \int_T^\infty$$

$$\left\{ \frac{\text{then adjust } \delta \text{ and } T > 1 \text{ appropriately}}{\alpha = \inf \mathrm{Re}(K), \quad \beta = \sup \mathrm{Re}(K)} \right\}$$

$$\left|\int_0^\delta\right| \leq \int_0^\delta t^{\alpha-1} e^{-t}\, dt < \frac{\varepsilon}{100}$$

$$\left|\int_T^\infty\right| \leq \int_T^\infty t^{\beta-1} e^{-t}\, dt < \frac{\varepsilon}{100}$$

etc etc. ∎

Thm

$\Gamma(z) \neq 0$ for $\mathrm{Re}(z) > 0$.

Pf

$\Gamma(x) > 0 \Rightarrow \Gamma(z) \not\equiv 0$.

Notice that $\dfrac{N!\, N^z}{z(z+1)\cdots(z+N)} \neq 0$ on $\mathrm{Re}(z) > 0$.

By Hurwitz's thm in analytic fcns, the limit is either $\equiv 0$ or <u>never</u> zero. So, we are done thanks to THM 23. $\blacksquare$

Cor

$\Gamma(z) \neq 0$ for $z \in \mathbb{C}$.

Pf

$z = -k$ is no problem! $(k \geq 0)$

Suppose $\Gamma(z_0) = 0$, some $Re(z_0) \leq 0$. $z_0 \notin \mathbb{Z}$.

But, then, 22 box $\Rightarrow$

$$\Gamma(z_0 + m) = z_0(z_0+1)\cdots(z_0+m-1)\Gamma(z_0) = 0$$

for all m large. Contradicts Thm 26. $\blacksquare$

Hence, $\dfrac{1}{\Gamma(z)}$ is analytic <u>on $\mathbb{C}$</u> with zeros at $\{0, -1, -2, \ldots\}$, each of multiplicity 1.

Let K be any compact subset of

$$\mathbb{C} - \{0, -1, -2, \ldots\}.$$

Choose m so big that

$$m + \inf \operatorname{Re}(K) \geq 1.$$

The relation

$$\Gamma(z) = \frac{\Gamma(z+m)}{z(z+1)\cdots(z+m-1)}$$

holds first for $\operatorname{Re}(z) > 0$, then ALL z by analytic continuation.

One can apply Thm 23 to $\Gamma(z+m)$ for $z \in K$.

Thm

$$\Gamma(z) = \lim_{N \to \infty} \frac{N!\, N^z}{z(z+1)\cdots(z+N)}$$

for $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$ with unif conv on compacta.

PF

Exercise — using the procedure suggested. ▨

Lemma

Let $Q_n(z)$ be analytic on $|z-z_0| < 2h$.
Let $Q_n(z)$ conv unif on, say, $|z-z_0| = h$.
Then: $Q_n(z)$ conv unif on $|z-z_0| \leqq h$ too!

Pf

Apply max mod principle to $Q_m(z) - Q_n(z)$, $m > n$.
Get unif Cauchy condition for $|z-z_0| \leqq h$. ∎

Thm

$$\frac{1}{\Gamma(z)} = \lim_{N \to \infty} \frac{z(z+1)\cdots(z+N)}{N!\, N^z} \quad \text{on } \mathbb{C}$$

with unif conv on compacta.

Pf

Combine Thm 28 with Lemma. ∎

Thm (all well-known)

Let $D = \mathbb{C} \sim \{0, -1, -2, \dots\}$.

(a) $\Gamma(z) \neq 0$ on $\mathbb{C}$, $\frac{1}{\Gamma(z)}$ entire, simple zeros at $z = 0, -1, -2, \dots$

(b) $$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{k=1}^{\infty} \left(\frac{1}{z+k} - \frac{1}{k} \right)$$
with unif conv on D compacta

(c) $$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

(d) $$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + \tfrac{1}{2})$$

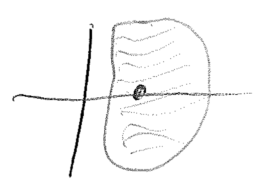
(e) $$\gamma = \lim_{N \to \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{N} - \ln N \right) = -\Gamma'(1)$$

Pf

Exercise!

Some pointers. In (b), by analytic continuation, it suffices to work on a compact subset of $\mathrm{Re}(z) > 0$ containing 1. Use thm 28.

$$f_n(z) \rightrightarrows \Gamma(z)$$



so

$$\frac{f_n'(z)}{f_n(z)} \rightrightarrows \frac{\Gamma'(z)}{\Gamma(z)}, \quad \frac{f_n'(1)}{f_n(1)} \to \frac{\Gamma'(1)}{\Gamma(1)}$$

Study

$$\frac{f_n'(z)}{f_n(z)} - \frac{f_n'(1)}{f_n(1)} \quad \text{as} \quad n \to \infty .$$

Get (b) and (e).

For (c), take $\mathrm{Im}(z) > 0$ w/log.* Use thm 28 and recall

$$\frac{\sin \pi z}{\pi z} = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) .$$

In (d), use thm 28 again with obvious factorings to get

$$\Gamma(2z) = A\, \Gamma(z)\, \Gamma(z + \tfrac{1}{2})\, 2^{2z} .$$

Evaluate A via the knowledge that $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$.

∎

* must prove

$$\Gamma(z)\Gamma(-z) = \frac{\pi}{-z \sin(\pi z)}$$

Thm (very well-known)

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-z/k}$$

with unif conv on $\mathbb{C}$ - compacta.

Pf

Work on $K = \{|z| \leq R\}$, R large.

$$\frac{1}{\Gamma(z)} = \lim_{N\to\infty} \frac{z \prod_{k=1}^{N}(z+k)}{N^{z} \prod_{k=1}^{N} k} \quad \text{29} \quad \text{uniformly}$$

$$= \lim_{N\to\infty} \frac{z \prod_{k=1}^{N}\left(1+\frac{z}{k}\right)}{e^{z \ln N}}$$

$$\left\{ \begin{array}{l} \sum_{1}^{N} \frac{1}{k} = \ln N + \gamma + \varepsilon_N \,, \quad \varepsilon_N \to 0 \\ \ln N = \sum_{k=1}^{N} \frac{1}{k} - \gamma - \varepsilon_N \end{array} \right\}$$

$$= \lim_{N\to\infty} \frac{z \prod_{k=1}^{N}\left(1+\frac{z}{k}\right)}{e^{z\left(\sum \frac{1}{k} - \gamma - \varepsilon_N\right)}}$$

$$= \lim_{N\to\infty} e^{\gamma z} e^{\varepsilon_N z} z \prod_{k=1}^{N} \left(1+\frac{z}{k}\right) e^{-z/k}$$

$$= z e^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-z/k}$$

Note:

for $k \gtrsim 100R$, say, get $\left|\frac{z}{k}\right| \lesssim \frac{1}{100}$

but $(1+w)e^{-w} = 1 + O(w^2)$ for $|w| \lesssim \frac{1}{100}$

$$\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}} = 1 + O\left(\frac{R^2}{k^2}\right)$$

$$\Downarrow$$

Weierstrass M-test for product is fine. ∎

We now return to 28 and wish to apply E-M version 2 to $\log \Gamma(z)$. We wish to do this for $z \in D = \mathbb{C} - \{0, -1, -2, \ldots\}$, keeping z on some compact subset of D initially.

Clearly, we need:

for some branch $\log \Gamma(z)$

$$\ln N! + z \ln N - \sum_{j=0}^{N} \log(z+j),$$

and for the logs, we should try to use Log.

For ease of calculation, focus on

$$T_N(z) \equiv \sum_{j=0}^{N} \mathrm{Log}(z+j) - \sum_{k=1}^{N} \ln k - z \ln N$$

Can use E−M

Can use E−M

$$\sum_{j=0}^{N-1} \mathrm{Log}(1+j)$$

The calculation is easy, but boring. We just give a few steps.

$$f(t) = \mathrm{Log}(t+z) \qquad z = \text{held fixed}$$

$$f'(t) = (t+z)^{-1}$$

$$f''(t) = (-1)(t+z)^{-2}$$

$$f^{(j)}(t) = (-1)^{j-1}(j-1)!\,(t+z)^{-j}$$

$$\Downarrow$$

$$\sum_{j=0}^{N} \mathrm{Log}(z+j) \approx \frac{1}{2}\mathrm{Log}\, z + \frac{1}{2}\mathrm{Log}(z+N)$$

see (14)

$$+ \int_0^N \mathrm{Log}(z+t)\,dt$$

$$+ \sum_{1}^{R} \frac{B_{2k}}{(2k)!}\Big[(2k-2)!\,(z+N)^{-(2k-1)} - (2k-2)!\,(z)^{-(2k-1)}\Big]$$

$$+ \int_0^N \frac{\tilde{B}_{2R+1}(t)}{(2R+1)!}\,(2R)!\,(t+z)^{-2R-1}\,dt$$

$$\left\{ \int_0^N \mathrm{Log}(z+t)\,dt = (z+N)\,\mathrm{Log}(z+N) - z\,\mathrm{Log}\, z - N \right\}$$

$$\sum_{j=0}^{N} \mathrm{Log}(z+j) \approx \frac{1}{2}\mathrm{Log}\, z + \left(z+N+\frac{1}{2}\right)\mathrm{Log}(z+N)$$

$$- z\,\mathrm{Log}\, z - N$$

$$+ \sum_{1}^{R} \frac{B_{2k}}{(2k)(2k-1)}\Big[(N+z)^{-2k+1} - z^{-2k+1}\Big]$$

$$+ \int_0^N \frac{\tilde{B}_{2R+1}(t)}{2R+1}\,(t+z)^{-2R-1}\,dt$$

---

to get $\sum_{k=1}^{N} \ln k$ just take $z=1$ and $N \hookrightarrow N-1$

$$\Downarrow$$

$$\sum_{k=1}^{N} \ln k = (N+\tfrac{1}{2}) \operatorname{Log} N - (N-1) + \sum_{1}^{R} \frac{B_{2k}}{(2k)(2k-1)} \left[N^{-2k+1} - 1 \right] + \int_0^{N-1} \frac{\tilde{B}_{2R+1}(t)}{2R+1} (t+1)^{-2R-1} dt$$

So, by flipping all signs in the above, get:

$$T_N(z) \simeq (z+N+\tfrac{1}{2}) \left\{ \operatorname{Log}(z+N) - \operatorname{Log}(N) \right\} - (z-\tfrac{1}{2}) \operatorname{Log} z - 1$$
$$+ \sum_{1}^{R} \frac{B_{2k}}{(2k)(2k-1)} \left[1 - z^{-2k+1} + (N+z)^{-2k+1} - N^{-2k+1} \right]$$
$$+ \frac{1}{2R+1} \int_0^{N} \tilde{B}_{2R+1}(t) (t+z)^{-2R-1} dt$$
$$- \frac{1}{2R+1} \int_0^{N-1} \tilde{B}_{2R+1}(t) (t+1)^{-2R-1} dt$$

$$\left\{ \text{use } \underset{z \in \mathbb{D}}{\operatorname{Log}(z+N) - \operatorname{Log}(N) \simeq \operatorname{Log}\left(\frac{z}{N}+1\right)} \right\}$$

$$T_N(z) \approx \left(z+N+\tfrac{1}{2}\right) \operatorname{Log}\left(1+\frac{z}{N}\right)$$

$$- \left(z-\tfrac{1}{2}\right)\operatorname{Log} z \; - 1$$

$$+ \sum_{1}^{R} \frac{B_{2k}}{(2k)(2k-1)} [\cdots]$$

$$+ \frac{1}{2R+1}\int_0^N \tilde{B}_{2R+1}(t)\,(t+z)^{-2R-1}\,dt$$

$$- \frac{1}{2R+1}\int_0^{N-1} \tilde{B}_{2R+1}(t)\,(t+1)^{-2R-1}\,dt \;.$$

Remember that $|\tilde{B}_{2R+1}(t)| \leqq$ some $\mathcal{B}_R$, all $t \in \mathbb{R}$.

Now let $N \to \infty$.

$$\left(z+\tfrac{1}{2}\right)\operatorname{Log}\left(1+\frac{z}{N}\right) \rightrightarrows 0$$

$$N \operatorname{Log}\left(1+\frac{z}{N}\right) \rightrightarrows z$$

Conclude:

$$T_N(z) \rightrightarrows z - \left(z-\tfrac{1}{2}\right)\operatorname{Log} z \; - 1$$

$$+ \sum_{1}^{R} \frac{B_{2k}}{(2k)(2k-1)}\left[1 - z^{-2k+1}\right]$$

$$+ \frac{1}{2R+1}\int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}}\,dt - \frac{1}{2R+1}\int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+1)^{2R+1}}\,dt \,.$$

Recall 33 bottom. Deduce:

$$-\log \Gamma(z) = z - \left(z-\tfrac{1}{2}\right) \operatorname{Log} z$$
$$- \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1}$$
$$+ \frac{1}{2R+1} \int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}}\, dt$$
$$+ C_R ,$$

some branch

where C_R = some appropriate <u>real</u> constant.

Thus, on D, $D = \mathbb{C} \sim \{0, -1, -2, \dots\}$,

$$\log \Gamma(z) = \left(z-\tfrac{1}{2}\right) \operatorname{Log} z - z$$
$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1}$$
$$- \frac{1}{2R+1} \int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}}\, dt$$
$$+ E_R ,$$

E_R = suitable <u>real</u> constant.

The preceding relation is an identity.

Note that RHS is real if $z = x > 0$.

<u>Hence</u> the branch of $\log \Gamma(z)$ under discussion is the one that reduces to $\ln \Gamma(x)$ for $z = x > 0$.

Also, note:

$$\left| \frac{1}{2R+1} \int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+x)^{2R+1}}\, dt \right|$$

$$\lesssim \frac{1}{2R+1} \int_0^\infty \frac{\mathcal{B}_R}{(t+x)^{2R+1}}\, dt$$

$$= \frac{1}{(2R+1)(2R)} \mathcal{B}_R \, x^{-2R}$$

$$= O\left(\frac{1}{x^{2R}}\right), \qquad \text{each } R.$$

So,

$$\ln \Gamma(x) = (x - \tfrac{1}{2}) \ln x - x + \sum_1^R \frac{B_{2k}}{(2k)(2k-1)} x^{-2k+1}$$

$$+ E_R + O(x^{-2R}) \qquad \text{as} \quad x \to +\infty.$$

At once, by comparing for different R's,

$$E_1 = E_2 = E_3 = \cdots = E_R = \cdots .$$

Call the common value E.

An easy substitution into

$$\Gamma(2x) = 2^{2x-1}\pi^{-1/2}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2}) \qquad (30)\,d$$

$$\Downarrow$$

$$\ln\Gamma(2x) = (2x-1)\ln 2 - \tfrac{1}{2}\ln\pi + \ln\Gamma(x) + \ln\Gamma(x+\tfrac{1}{2})$$

gives $(R=1)$

$$(2x-\tfrac{1}{2})\ln(2x) - 2x + O(\tfrac{1}{x}) + E$$

$$= (2x-1)\ln 2 - \tfrac{1}{2}\ln\pi + (x-\tfrac{1}{2})\ln x - x + O(\tfrac{1}{x}) + E + (x)\ln(x+\tfrac{1}{2}) - (x+\tfrac{1}{2}) + O(\tfrac{1}{x}) + E$$

$$\Downarrow$$

$$-\frac{1}{2}\ln 2 + O\left(\frac{1}{x}\right) + E$$
$$= -\ln 2 - \frac{1}{2}\ln\pi + 2E + O\left(\frac{1}{x}\right)$$

$$\frac{1}{2}\ln 2 + \frac{1}{2}\ln\pi + O\left(\frac{1}{x}\right) = E$$

$$\Rightarrow \quad E = \frac{1}{2}\ln(2\pi) . \quad \|$$

On 38 bottom, we therefore get:

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\operatorname{Log} z - z + \frac{1}{2}\ln(2\pi)$$
$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1}$$
$$- \frac{1}{2R+1}\int_0^{\infty} \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}}\,dt$$

AS AN IDENTITY ON D.

Theorem (Stirling – corollary of E–M)

Keep $z \in D$. $D = \mathbb{C} - \{0, -1, -2, \dots\}$.

We have

$$\log \Gamma(z) = \left(z - \tfrac{1}{2}\right) \operatorname{Log} z - z + \tfrac{1}{2} \ln(2\pi)$$
$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1}$$
$$- \frac{1}{2R+1} \int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}}\, dt,$$

where the branch of $\log \Gamma(z)$ reduces to $\ln \Gamma(x)$ when $z = x > 0$.

Pf

As above. ∎

FAMOUS

Corollary (Stirling's asymptotic formula)

Fix any $R \geqq 1$ and $\delta > 0$. Then:

$$\log \Gamma(z) = \left(z - \tfrac{1}{2}\right) \operatorname{Log} z - z + \tfrac{1}{2} \ln(2\pi)$$
$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1} + O_{R\delta}\left(\frac{1}{|z|^{2R+1}}\right)$$

as $z \to \infty$ in $|\operatorname{Arg}(z)| \leqq \pi - \delta$.

Pf

A simple absolute value estimate of

$$-\frac{1}{2R+1}\int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}}\,dt$$

compare (39) middle

does not work. One uses a minor trick instead. Namely:

$$\mathrm{Rem}_R = \frac{B_{2R+2}}{(2R+2)(2R+1)} z^{-2R-1}$$

$$-\frac{1}{2R+3}\int_0^\infty \frac{\tilde{B}_{2R+3}(t)}{(t+z)^{2R+3}}\,dt\,.$$

Notice that:

$$|\text{last term}| \leqq \frac{\mathcal{B}_{R+1}}{2R+3}\int_0^\infty \frac{dt}{|t+z|^{2R+3}}$$

$$\left\{\begin{array}{c} z = \rho e^{i\omega}, \quad |\omega| \leqq \pi - \delta, \\ \rho \text{ large} \\ \text{put } t = \rho v \end{array}\right\}$$

$$= \frac{\beta_{R+1}}{2R+3} \int_0^\infty \frac{\rho\, dv}{\rho^{2R+3} |v+e^{i\omega}|^{2R+3}}$$

$$= \frac{\beta_{R+1}}{2R+3} \rho^{-2R-2} \int_0^\infty \frac{dv}{|v+e^{i\omega}|^{2R+3}}$$

These integrals are $O_{R\delta}(1)$ for $|\omega| \leq \pi - \delta$

$$\Downarrow$$

$$|\mathrm{Rem}_R| \lesssim O_{R\delta}(z^{-2R-1}) + O_{R\delta}(1)|z|^{-2R-2}$$

$$= O_{R\delta}(1)|z|^{-2R-1}.$$

$\blacksquare$

Remark 1

Refer to 14 line 4 and 14 Thm (E-M vers 2). Though it may be tempting to allow $R=0$ on page 42, note that

$$\int_0^\infty \frac{\tilde{B}_1(t)}{t+z}\,dt$$

is NOT absolutely convergent. For this reason, allowing $R=0$ in Thm 42 is usually avoided. In the Corollary, it is of course OK, since

$$\frac{B_2}{2\cdot 1} z^{-1} + O_\delta\left(\frac{1}{|z|^3}\right) = O_\delta\left(\frac{1}{|z|}\right).$$

Remark 2 (classical Stirling) $R=0$ OK HERE.

By use of 15 Cor 2 (nontrivial), it is possible to show

$$\log N! = (N+\tfrac{1}{2})\ln N - N + \tfrac{1}{2}\ln(2\pi) + \sum_{1}^{R} \frac{B_{2k}}{(2k)(2k-1)} N^{-2k+1} + REM_R,$$

$$REM_R = \lambda \frac{B_{2R+2}}{(2R+2)(2R+1)} N^{-2R-1} \quad \underline{\text{with } 0<\lambda<1}.$$

Lecture 11
(Feb 24)

I began with a quick development of basic Fourier series — in a nonstandard way, i.e., via E-M summation.

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

On $[0,1]$ or <u>any</u> $[a, a+1]$:

$$\langle \varphi_m, \varphi_n\rangle = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

for

$$\varphi_n(x) = e^{2\pi i n x}, \quad n \in \mathbb{Z}.$$

So, we have the usual <u>idea</u> of trying to write f "most of the time" as $\sum_n c_n \varphi_n$, $c_n = \langle f, \varphi_n\rangle$.

<u>Lemma</u>

$f \in C[0,N]$. Assume f is only <u>piecewise</u> $\underline{C}^1$. not C^1

Then we still have

$$\frac{1}{2}f(0) + f(1) + \cdots + f(N-1) + \frac{1}{2}f(N) = \int_0^N f\,dx + \int_0^N f'\beta(x)\,dx,$$

$$\beta(x) \equiv x - [\![x]\!] - \frac{1}{2}.$$

Pf

Begin as before Lec 8 p. 14

$$f(1) + \cdots + f(N) = \int_0^N f(x)\, d[\![x]\!]$$

$$= \int_0^N f(x)\, d\left(x - \tfrac{1}{2} - \beta(x)\right) \qquad (R-S)$$

$$= \int_0^N f(x)\,dx - \int_0^N f(x)\, d\beta(x) .$$

Split $\int_0^N f\, d\beta$ into chunks corresponding to corners of f.

Then do the integration by parts and recombine.
Ambiguous f' at a finite # of corners does not affect

$$\int_0^N \beta f' dx .$$

$\Rightarrow$ All is fine. ▮

Take $N = 1$. Assume $f \in C[0,1]$, piecewise C^1.
Hence, by Lemmas

$$\tfrac{1}{2} f(0) + \tfrac{1}{2} f(1) = \int_0^1 f\,dx + \int_0^1 f'\left(- \sum_n \frac{\sin 2\pi n x}{\pi n}\right) dx$$

$$= \int_0^1 f\,dx + \sum_{n=1}^{\infty} \int_0^1 f' \frac{-\sin 2\pi n x}{\pi n}\, dx ,$$

the 2nd line by Lec 9, p. 9, Baby Fact.

Note that the error term after N is

$$\pm \int_0^1 f'(\beta - S_N)\,dx$$

i.e.

$$\text{ABS VALUE} \lesssim \mathfrak{M} \int_0^1 |\beta - S_N|\,dx\,,$$

$$\mathfrak{M} \equiv \sup_{[0,1]} |f'|\,.$$

The $|\beta - S_N|$ integral is an absolute expression, say ω_N, and $\omega_N \to 0$. So:

$$|\text{Error}| \lesssim \mathfrak{M}\omega_N\,.$$

Note TOO that $(n \geq 1)$

$$\int_0^1 f' \frac{\sin 2\pi n x}{-\pi n}\,dx = \frac{1}{2\pi i n} \int_0^1 f'(e^{-2\pi i n x} - e^{2\pi i n x})\,dx\,.$$

Write the last expr. as

$$\frac{1}{2\pi i n} \int_0^1 f' e^{-2\pi i n x}\,dx + \frac{1}{2\pi i(-n)} \int_0^1 f' e^{-2\pi i(-n)x}\,dx\,.$$

But,

$$\frac{1}{2\pi i n}\int_0^1 f' e^{-2\pi i n x}\,dx$$

$$= \frac{1}{2\pi i n}\int_0^1 e^{-2\pi i n x}\,df \qquad \text{(standard parts)}$$

$$= \frac{1}{2\pi i n}\left[e^{-2\pi i n x} f(x)\right]_0^1$$

$$- \frac{1}{2\pi i n}\int_0^1 f\, d(e^{-2\pi i n x})$$

$$\simeq \frac{f(1)-f(0)}{2\pi i n} + \int_0^1 f e^{-2\pi i n x}\,dx \,.$$

Similarly for $-n$. Now <u>add!</u> Get:

$$(\text{term } n) + (\text{term } -n) \cong c_n + c_{-n}$$

where

$$c_k = \int_0^1 f e^{-2\pi i k x}\,dx \,.$$

(2) bottom

So,

$$\frac{1}{2}f(0) + \frac{1}{2}f(1) = c_0 + \lim_{N\to\infty} \sum_{1\le|n|\le N} c_n$$

$$\Downarrow$$

$$\frac{1}{2}f(0) + \frac{1}{2}f(1) = \lim_{N\to\infty} \sum_{-N}^{N} c_n ,$$

any $f \in C[0,1]$, piecewise C'.

AHA! This is really a Fourier series $\underbrace{}_{\text{example}}$;

i.e.,

$$\lim_{N\to\infty} \sum_{-N}^{N} c_n e^{2\pi i n \cdot 0} \;!$$

The proof was just basic E-M, version I,

and

$$x - [\![x]\!] - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{\pi n} , \quad x \notin \mathbb{Z}.$$

NOTE THAT error term for $|n| > N$ is

$$\leq \mathfrak{M}\, \omega_N .$$

Initial Thm

Given $f \in C[0,1]$, piecewise C^1.

Let
$$c_k = \int_0^1 f e^{-2\pi i k x} dx = \langle f, \varphi_k \rangle.$$

Then:
$$\tfrac{1}{2} f(0) + \tfrac{1}{2} f(1) = \lim_{N \to \infty} \sum_{-N}^{N} c_k e^{2\pi i k 0},$$

$$|\text{Error}| \leq M \omega_N.$$

Pf

As above. ∎

Thm

Let $f \in C(\mathbb{R})$, periodic 1, piecewise C^1.

Then:

unif. conv.

$$\sum_{-N}^{N} c_k e^{2\pi i k x} \rightrightarrows f(x) \quad \text{on } \mathbb{R}.$$

Pf

periodic 1

$C[0,1]$, piecewise C^1

Fix any $x_0 \in \mathbb{R}$. Consider $g(x) = f(x + x_0)$ on $[0,1]$ in previous Thm. Note

$$c_k(g) = \int_0^1 g(x) e^{-2\pi i k x} dx = \int_0^1 f(x + x_0) e^{-2\pi i k x} dx$$

$$\{y = x + x_0\}$$

$$= \int_{x_0}^{x_0+1} f(y) e^{-2\pi i k y} e^{2\pi i k x_0} dy$$

$$= e^{2\pi i k x_0} \int_{x_0}^{x_0+1} f(y) e^{-2\pi i k y}\, dy$$

$$= e^{2\pi i k x_0} \int_0^1 f(y) e^{-2\pi i k y}\, dy$$

{ by periodicity of <u>integrand</u> }

$$= e^{2\pi i k x_0}\, c_k(f).$$

So:

$$f(x_0) = \lim_{N\to\infty} \sum_{-N}^{N} c_k(f) e^{2\pi i k x_0},$$

$$|\text{Error}| \leq \mathcal{M}\omega_N, \quad \mathcal{M} = \sup_{\mathbb{R}} |f'|.$$

Qed. ∎

The next thm is a commonly used <u>augmentation</u> of <u>thm</u> on 6 bottom.

<u>Theorem</u>

Let f belong to $C^2(\mathbb{R})$ and be periodic 1. We then have

$$|c_k| \leq \frac{1}{(2\pi k)^2} \int_0^1 |f''|\, dx, \quad k \neq 0.$$

This ensures that, on 6 bottom, $\sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$ conv <u>both</u> uniformly and absolutely to $f(x)$ on $\mathbb{R}$.

Pf compare 4

Simply integrate by parts twice:

$$c_k = \frac{1}{(2\pi i k)^2} \int_0^1 f'' e^{-2\pi i k x}\, dx, \quad k \neq 0.$$

The following is our **MAIN assertion** in this approach to FS based on E-M.

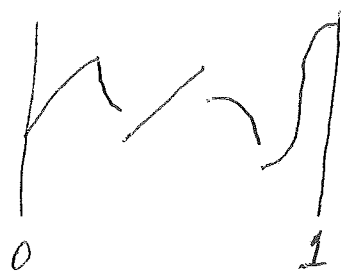
THEOREM (standard Fourier series thm in undergrad analysis)

Let f be given on $\mathbb{R}$ and be periodic 1. Assume f is piecewise C^1. (See picture.) and

Let $c_k = \int_0^1 f e^{-2\pi i k x}\, dx$

$$FS(f) \equiv \sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$$

as a formal sum.



We then have

$$\sum_{-N}^{N} c_k e^{2\pi i k x} \xrightarrow[\boxed{\text{unif conv}}]{} \!\!\!\rightarrow f(x) \quad \text{as } N \to \infty$$

away from the discontinuities of f. At the points of discontinuity, we have

$$\sum_{-N}^{N} c_k e^{2\pi i k x} \longrightarrow \frac{1}{2}\left[f(x+0) + f(x-0)\right].$$

Here $f(x+0)$, $f(x-0)$ are the one-sided limits.

(cont'd)

In addition, the partial sums $\sum_{-N}^{N} c_k e^{2\pi i k x}$ will be <u>uniformly</u> <u>bounded</u> on $\mathbb{R}$.

<u>Proof</u>

The thm is certainly correct if f has <u>no</u> discontinuities on $\mathbb{R}$. See (6) bottom.

We now do a trick. (using β)

<u>Baby Lemma</u>

Let $H(x) \equiv \lim_{N\to\infty} \sum_{-N}^{N} a_k e^{2\pi i k x}$, where the limit exists pointwise on all of $\mathbb{R}$. Assume that the partial sums $\sum_{-N}^{N}$ are <u>uniformly</u> <u>bounded</u> on $\mathbb{R}$. Finally, assume that the partial sums $\sum_{-N}^{N}$ <u>converge</u> <u>uniformly</u> away from $\{c_1, \ldots, c_m\}$ mod $\mathbb{Z}$ (m finite). Then:

(A) $H(x)$ is Riemann integrable on $[0,1]$;

(B) $a_k = \int_0^1 H(x) e^{-2\pi i k x}\,dx$, each $k \in \mathbb{Z}$.

$\boxed{\text{No! (B) is } \underline{\text{not}} \text{ a tautology!}}$

Pf of Lemma

The discontinuities of H are contained in $\{c_1, \ldots, c_m\}$ mod $\mathbb{Z}$ by the unif conv. $H(x)$ is also bounded by the unif boundedness of

$$S_N(x) = \sum_{-N}^{N} a_k e^{2\pi i k x}.$$

By baby calculus, H is Riemann integrable on any finite $[a,b]$. Hence $[0,1]$.

As we saw earlier, baby analysis $\Rightarrow$

$$\int_0^1 |H(x) - S_N(x)|\, dx \to 0 \quad \text{as } N \to \infty.$$

See Lec 9 p. 9.

By that same idea, we have:

$$\int_0^1 e^{-2\pi i m x} S_N(x)\, dx \longrightarrow \int_0^1 e^{-2\pi i m x} H(x)\, dx$$

for each $m \in \mathbb{Z}$. But LHS $= a_m + 0$!! for large N

Hence,

$$a_m = \int_0^1 H(x) e^{-2\pi i m x}\, dx. \quad \blacksquare$$

Before continuing, observe that: $(\ell \geq 1)$

$$\frac{e^{2\pi i \ell x}}{-2\pi i \ell} + \frac{e^{-2\pi i \ell x}}{-2\pi i(-\ell)} = -\frac{1}{2\pi i \ell}\left(e^{2\pi i \ell x} - e^{-2\pi i \ell x}\right)$$

p. 3 line 11

$$= \frac{\sin(2\pi \ell x)}{-\pi \ell}.$$

Also write

$$\tilde{\beta}(y) = \left\{ \begin{array}{l} 0, \ y \in \mathbb{Z} \\ \beta(y), \ y \notin \mathbb{Z} \end{array} \right\}.$$

We already know that

$$\tilde{\beta}(x) = \sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{-\pi m} = \sum_{n \neq 0} -\frac{1}{2\pi i n} e^{2\pi i n x},$$

all $x \in \mathbb{R}$. Unif conv away from $\mathbb{Z}$; partial sums unif bounded. Similarly

$$\tilde{\beta}(x-c) = \sum_{n \neq 0} -\frac{e^{-2\pi i n c}}{2\pi i n} e^{2\pi i n x},$$

all $x \in \mathbb{R}$. By Baby Lemma on 9, automatically,

$$\int_0^1 \tilde{\beta}(x) e^{-2\pi i n x} dx = \left\{ \begin{array}{l} 0, \ n = 0 \\ -\frac{1}{2\pi i n}, \ n \neq 0 \end{array} \right\}$$

$$\int_0^1 \tilde{\beta}(x-c)\, e^{-2\pi i n x}\, dx = \left\{ \begin{array}{ll} 0, & n = 0 \\ -\dfrac{e^{-2\pi i n c}}{2\pi i n}, & n \neq 0 \end{array} \right\}.$$

THUS:

$$FS[\tilde{\beta}(x)] \cong \sum_{n \neq 0} -\frac{1}{2\pi i n}\, e^{2\pi i n x}$$

$$FS[\tilde{\beta}(x-c)] \cong \sum_{n \neq 0} -\frac{e^{-2\pi i n c}}{2\pi i n}\, e^{2\pi i n x}.$$

Obviously, the "$\sim$" can be <u>removed</u> from β.

[These Fourier series can of course be checked directly, but we <u>prefer</u> the slick approach.]

We now return to the PROOF of p. 8 THM.
Let $f(x)$ have nontrivial <u>discontinuities</u> at points $\{c_1, \ldots, c_m\}$ mod $\mathbb{Z}$. Let the "right-left" jump be J_i. Saying $J_i = \underline{0}$ means $f(c_i+0) = f(c_i-0)$ but $f(c_i) \neq f(c_i+0)$.

Recall that

$$\beta(0+) - \beta(0-) = -\tfrac{1}{2} - \tfrac{1}{2} = -1.$$

Define: $\boxed{\text{THIS IS THE TRICK}}$

$$g(x) = f(x) + \sum_{i=1}^{m} J_i \beta(x - c_i) \quad , \quad x \in \mathbb{R}.$$

Fcn g is <u>very</u> interesting! It is obviously periodic 1. Also, it is obviously piecewise C^1. It <u>may</u> have discontinuities, but these lie in $\{c_1, \ldots, c_m\}$ mod $\mathbb{Z}$.

Note however that

$$\begin{aligned} g(c_i + 0) - g(c_i - 0) &= J_i - J_i + 0 \\ &= 0 \quad , \quad \text{each } 1 \le i \le m. \end{aligned}$$

The points c_i are thus "removable discontinuities" <u>if</u> g is redefined correctly at these points.

Apply p. 6 bottom THM to this modified g. We conclude that $FS(g)$ converges <u>uniformly</u> <u>over</u> <u>$\mathbb{R}$</u> to $\frac{1}{2}[g(x+0) + g(x-0)]$. The partial sums are automatically uniformly bounded on $\mathbb{R}$.

By linearity, however, as series,

$$FS(f) \equiv FS(g) - \sum_{i=1}^{m} J_i \, FS[\beta(x - c_i)] .$$

At once, the partial sums of $FS(f)$ are unif bounded on $\mathbb{R}$ (by the corresponding fact for β).

Also, $FS(f)$ conv uniformly away from the $\{c_i\}$ mod $\mathbb{Z}$ (by the corr fact for β).

At points $x \not\equiv c_1, \ldots, c_m$ mod $\mathbb{Z}$, clearly $FS(f)$ converges to

$$g(x) - \sum_{i=1}^{m} J_i \beta(x - c_i) = f(x).$$

(Big surprise!!)

At c_i, $FS(f)$ converges to

$$\frac{1}{2}\left[g(c_i+0) + g(c_i-0)\right] - 0 - \sum_{\ell \neq i} J_\ell \beta(c_i - c_\ell).$$

But:

$$g(c_i+0) = f(c_i+0) + J_i\left(-\tfrac{1}{2}\right) + \sum_{\ell \neq i} J_\ell \beta(c_i - c_\ell)$$

$$g(c_i-0) = f(c_i-0) + J_i\left(\tfrac{1}{2}\right) + \sum_{\ell \neq i} J_\ell \beta(c_i - c_\ell)$$

$$\frac{g(c_i+0) + g(c_i-0)}{2} = \frac{f(c_i+0) + f(c_i-0)}{2} + \sum_{\ell \neq i} J_\ell \beta(c_i - c_\ell)$$

$\Downarrow$

$FS(f)$ conv to $\dfrac{f(c_i+0)+f(c_i-0)}{2}$

at each c_i. (Again, big surprise!!)

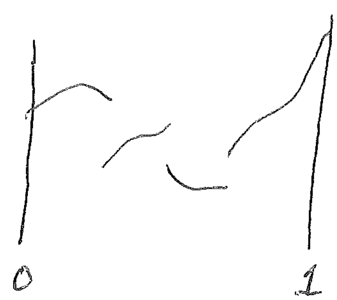
Thus, all is now proved. $\blacksquare$

---

famous formula

<u>THM</u> (Parseval's formula)

Let f be periodic 1, piecewise C^1 as in p. 8 THM.



We then have:

$$\int_0^1 |f(x)|^2\,dx = \sum_{k=-\infty}^{\infty} |c_k|^2 .$$

$\sum_{-N}^{N} c_k e^{2\pi i k x}$

<u>PF</u>

f is unif bdd on $\mathbb{R}$. We know $S_N(x)$ is unif bdd on $\mathbb{R}$ too. We also have $S_N(x) \rightrightarrows f(x)$ away from $\{c_1,\dots,c_m\}$ mod $\mathbb{Z}$. Apply the idea of Lec 9 p. 9 again! (See p. 10 above.)

We get:

$$\int_0^1 f(x)\overline{S_N(x)}\,dx \to \int_0^1 f(x)\overline{f(x)}\,dx \qquad (N\to\infty)$$

but

$$LHS = \int_0^1 f(x)\overline{\left(\sum_{-N}^{N} c_k e^{2\pi i k x}\right)}\,dx$$

$$= \sum_{-N}^{N} c_k \bar{c}_k = \sum_{-N}^{N} |c_k|^2 . \quad ▨$$

The Fourier theory so far has been a kind of $L_\infty \times L_1$ theory. In <u>traditional</u> real analysis courses, one investigates to see if an $L_2 \times L_2$ theory might be better (or more natural).

We will not bother to pursue the latter beyond 2 very quick remarks.

Use of completing the square on integrals like

$$\int_0^1 |f(x) - S_N(x)|^2\,dx \;, \qquad \int_0^1 \left|F(x) - \sum_{-N}^{N} A_k e^{2\pi i k x}\right|^2 dx$$

for a general <u>piecewise</u> <u>continuous</u>, periodic 1, $F(x)$ leads to

$$\sum_{-N}^{N} |c_k|^2 \leqq \int_0^1 |f(x)|^2 dx \qquad \text{(each } N\text{)}$$

$$\Downarrow$$

$$\boxed{\sum_{-\infty}^{\infty} |c_k|^2 \leqq \int_0^1 |f(x)|^2 dx}$$

(i.e. Bessel's inequality) .

Here $c_k = \int_0^1 f e^{-2\pi i k x} dx$.

(Actually, equality holds — but this is a harder theorem. One uses (15) Thm and "approximates" f by piecewise $\underline{\underline{C}}^1$ functions.) SEE ANY STANDARD BOOK ON F.S.

Our 2nd remark is a theorem.

<u>THM</u> (slight strengthening of p. (6) bottom)

Let $f \in C(\mathbb{R})$, periodic 1, and be piecewise C^1.
Let $c_k = \int_0^1 f e^{-2\pi i k x} dx$. The Fourier series

$$\sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$$

then converges <u>uniformly</u> to $f(x)$ on $\mathbb{R}$

<u>AND</u> we also have

$$\sum_{-\infty}^{\infty} |c_k| < \infty .$$

Ie, have nice ABS conv!

Pf

Take $k \neq 0$. By standard integ by parts,

$$c_k = \frac{1}{2\pi i k} \int_0^1 f'(x) e^{-2\pi i k x} dx . \qquad (4)$$

Again: NOTE THAT RHS is not affected by a few ambiguities in f'. Write the foregoing as

$$c_k = \frac{1}{2\pi i k} c_k(f')$$

and recall 17 box (Bessel's ineq). By Cauchy-Schwarz inequality,

$$\sum_{k=1}^{\infty} |c_k| = \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k} |c_k(f')|$$

$$\leq \frac{1}{2\pi} \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \sqrt{\sum_{k=1}^{\infty} |c_k(f')|^2} < +\infty .$$

Similarly for $k < 0$. ∎

Next topic: Poisson summation formula.

THM

Given $\varphi \in C^2(\mathbb{R})$ such that, say,

$$|\varphi(x)|,\ |\varphi'(x)|,\ |\varphi''(x)| \text{ all } = O\left[(1+|x|)^{-2}\right].$$

Let

$$\hat{\varphi}(p) = \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i p x}\, dx, \quad p \in \mathbb{R}.$$

Let

$$F(x) \equiv \sum_{n=-\infty}^{\infty} \varphi(x+n), \quad x \in \mathbb{R}.$$

We then have

$$F(x) = \sum_{-\infty}^{\infty} \hat{\varphi}(k) e^{2\pi i k x}$$

Poisson summation formula

for $x \in \mathbb{R}$, with absolute and uniform conv on both sides over every interval $[-\Delta, \Delta]$.

Proof

The series $\sum_{-\infty}^{\infty} \varphi^{(j)}(x+n)$, $0 \leq j \leq 2$, are clearly conv. both abs and uniformly on every $[-\Delta, \Delta]$. As such, we immediately get $F \in C^2(\mathbb{R})$. It is also apparent that $F(x+1) = F(x)$.

Apply Thm 7 bottom. Get:

$$F(x) = \sum_{-\infty}^{\infty} A_k e^{2\pi i k x} \quad \text{nicely},$$

$$A_k = \int_0^1 F(x) e^{-2\pi i k x}\,dx .$$

But:

$$A_k = \int_0^1 \Big(\sum_{-\infty}^{\infty} \varphi(x+n) \Big) e^{-2\pi i k x}\,dx$$

$$= \sum_{-\infty}^{\infty} \int_0^1 \varphi(x+n) e^{-2\pi i k x}\,dx \qquad \text{by unif conv}$$

$$= \sum_{-\infty}^{\infty} \int_n^{n+1} \varphi(y) e^{-2\pi i k y}\,dy = \int_{\mathbb{R}} \varphi(y) e^{-2\pi i k y}\,dy$$

$$= \hat{\varphi}(k) . \quad \blacksquare$$

<u>Example</u>

$$\varphi(x) = e^{-ax^2}, \quad a > 0$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \quad \Rightarrow$$

$$\int_{-\infty}^{\infty} e^{-ax^2} e^{-2\pi i p x}\,dx = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2 p^2}{a}}$$

(by elementary contour shift) .

Hence, by Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} e^{-a(x+n)^2} = \sqrt{\frac{\pi}{a}} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi^2 k^2}{a}} e^{2\pi i k x} .$$

Special Case:

$$\sum_{n=-\infty}^{\infty} e^{-\pi\beta n^2} = \sqrt{\frac{1}{\beta}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{\beta}} \qquad (\beta > 0).$$

The famous "θ" relation of Jacobi. $\theta(\beta) = \frac{1}{\sqrt{\beta}}\,\theta\left(\frac{1}{\beta}\right)$

We are now ready to derive (following Riemann) a slick formula for $\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

Easily check:

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} x^{\frac{s}{2}} e^{-x} \frac{dx}{x}, \qquad \mathrm{Re}(s) > 1 \text{ say}$$

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^{\infty} y^{\frac{s}{2}} e^{-\pi n^2 y} \frac{dy}{y}$$

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^{\infty} y^{\frac{s}{2}} \left[\sum_{n=1}^{\infty} e^{-\pi n^2 y} \right] \frac{dy}{y}$$

A nice positive fcn.
Clearly $O(1/\sqrt{y})$ for $y \to 0^+$
by Θ-relation.

$$\left\{ \begin{array}{l} \text{Note: the foregoing integral is} \\ \text{nicely convergent near } y = 0 \text{ because} \\ \displaystyle\int_0^1 y^{\frac{\sigma}{2}} \frac{1}{\sqrt{y}} \frac{dy}{y} < \infty \quad \text{for } \sigma > \underline{\underline{1}} \end{array} \right\}.$$

Write

$\boxed{\text{DO NOT CONFUSE WITH PNT } \psi \text{ and } \theta}$

$$\psi(y) = \sum_{n=1}^{\infty} e^{-\pi n^2 y} \quad \text{and} \quad \theta(y) = 2\psi(y) + 1 .$$

So:

$$\psi(y) + \frac{1}{2} = \frac{1}{\sqrt{y}}\left[\psi\left(\frac{1}{y}\right) + \frac{1}{2}\right], \quad y > 0$$

$$\psi(y) = -\frac{1}{2} + \frac{1}{2} y^{-1/2} + y^{-1/2}\,\psi\left(\frac{1}{y}\right).$$

Get:

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^1 y^{\frac{s}{2}}\,\psi(y)\,\frac{dy}{y} + \int_1^{\infty} y^{\frac{s}{2}}\,\psi(y)\,\frac{dy}{y}$$

put $y = \frac{1}{v}$ here

{now grind!}

$$= \int_1^\infty v^{-\frac{s}{2}} \left[-\frac{1}{2} + \frac{1}{2} v^{1/2} + v^{1/2} \psi(v) \right] \frac{dv}{v}$$

$$+ \int_1^\infty y^{\frac{s}{2}} \psi(y) \frac{dy}{y}$$

$$= -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty v^{\frac{1-s}{2}} \psi(v) \frac{dv}{v}$$

$$+ \int_1^\infty y^{\frac{s}{2}} \psi(y) \frac{dy}{y}$$

$$= -\left[\frac{1}{s} + \frac{1}{1-s} \right] + \int_1^\infty \left(y^{\frac{1-s}{2}} + y^{\frac{s}{2}} \right) \psi(y) \frac{dy}{y}$$

$$= -\frac{1}{s(1-s)} + \int_1^\infty \left(y^{\frac{1-s}{2}} + y^{\frac{s}{2}} \right) \psi(y) \frac{dy}{y} .$$

So, for $\mathrm{re}(s) > 1$,

$$\pi^{-\frac{s}{2}} \Gamma\left(\tfrac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(y^{\frac{1-s}{2}} + y^{\frac{s}{2}} \right) \psi(y) \frac{dy}{y} .$$

NOTE
invariance under
$s \leftrightarrow 1-s$

$O(e^{-\pi y})$
as $y \to +\infty$

The integral is analytic for <u>all</u> $s \in \mathbb{C}$.

The $\frac{1}{s(s-1)}$ is trivially analytic on $\mathbb{C} - \{0,1\}$.

<u>Theorem</u> (Functional Equation)

$\xi(s) \equiv \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ is analytic on $\mathbb{C} \smallsetminus \{0,1\}$ and satisfies

$$\xi(s) = \xi(1-s).$$

We also have for ξ:

$s=1$ simple pole, residue 1;
$s=0$ simple pole, residue -1.

<u>Pf</u>
The first part is just 23 bottom. OK

By 23 bottom, with $s = 1+h$, we get

$$\xi(1+h) = \frac{1}{(1+h)h} + O(1)$$

$$= \frac{1}{h} + O(1).$$

And, similarly, with $s = h$,

$$\xi(h) = \frac{1}{h(h-1)} + O(1) = -\frac{1}{h} + O(1).$$ ▥

Cor 1

$\xi_0(s) = s(s-1)\,\xi(s) = s(s-1)\,\pi^{-s/2}\Gamma(\frac{s}{2})\,\zeta(s)$ is an entire fcn which satisfies

$$\xi_0(s) = \xi_0(1-s), \quad \xi_0(1) = 1.$$

Pf

$$\xi_0(s) = 1 + s(s-1)\int_1^\infty \left(y^{\frac{1-s}{2}} + y^{\frac{s}{2}}\right)\psi(y)\,\frac{dy}{y} \quad \text{by } 23. \quad \blacksquare$$

Cor 2

$\zeta(-2k) = 0$ for $k \geq 1$ (simple zero).

Pf

$\xi(x) = \pi^{-x/2}\Gamma(\frac{x}{2})\,\zeta(x) > 0$ for $x > 1$. But $\xi(x) = \xi(1-x)$. Hence $\xi(x) > 0$ for $x < 0$. Let $x \to -2k$. Since $\Gamma(\frac{x}{2}) \to \Gamma(-k)$ simple pole, get $\zeta(x) \to 0$ à la simple zero. $\blacksquare$

Lemma

$$(1-2^{1-s})\zeta(s) = \sum_{k=1}^{\infty}\left((2k+1)^{-s}-(2k)^{-s}\right)$$

for $Re(s)>1$ and the RHS is actually analytic for $\{Re(s)>0\}$.

Pf

$Re(s)>1 \Rightarrow$

$$\zeta(s) = \sum_{k=1}^{\infty}(2k-1)^{-s} + \sum_{k=1}^{\infty}(2k)^{-s} \quad \text{trivially}$$

$$2^{1-s}\zeta(s) = 2\sum_{m=1}^{\infty}(2m)^{-s}$$

$$\text{difference} = \sum_{k=1}^{\infty}\left((2k-1)^{-s}-(2k)^{-s}\right) \quad \left\{\begin{matrix}\text{nice}\\ \text{abs}\\ \text{conv}\end{matrix}\right\}.$$

Keep $s\in K$ where K is a compact subset of $\{Re(s)>0\}$. Observe that:

$$(2k-1)^{-s}-(2k)^{-s} = (2k)^{-s}\left[1-\frac{1}{2k}\right]^{-s} - (2k)^{-s}$$

$$\left\{ (1+u)^{-s} = 1+(-s)u + O_K(1)u^2, \ |u|\leq\frac{3}{4} \right\}$$

$$(2k-1)^{-s}-(2k)^{-s} = (2k)^{-s}\left[\frac{s}{2k}+O(1)\frac{1}{k^2}\right]$$

$$= O(1)k^{-s-1} \quad \text{for } s\in K. \quad \blacksquare$$

<u>Corollary</u>

In the sense of analytic continuation,

$$\xi(x) \neq 0 \quad \text{for} \quad x \in \mathbb{R}$$

$$\xi_0(x) \neq 0 \quad \text{for} \quad x \in \mathbb{R}$$

$$J(x) < 0 \quad \text{for} \quad 0 < x < 1 .$$

<u>Proof</u>

That $J(x) < 0$ on $0 < x < 1$ is obvious from (26). Hence $\xi(x) \neq 0$ on $0 < x < 1$. The points $x = 0, 1$ are poles and take care of themselves. For $x > 1$ and $x < 0$, we have $\xi(x) > 0$ à la (25). Since $\xi_0(x) = x(x-1)\,\xi(x)$, the assertions for ξ_0 are immediate. ∎

We wish to bound the size of

$$F(z) \equiv z(z-1)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

(roughly) using Stirling, $F(z) = F(1-z)$, and basic properties of $\zeta(z)$.

Because of $F(z) = F(1-z)$, we can clearly restrict to $\mathrm{Re}(z) \gtrsim \frac{1}{2}$.

We had

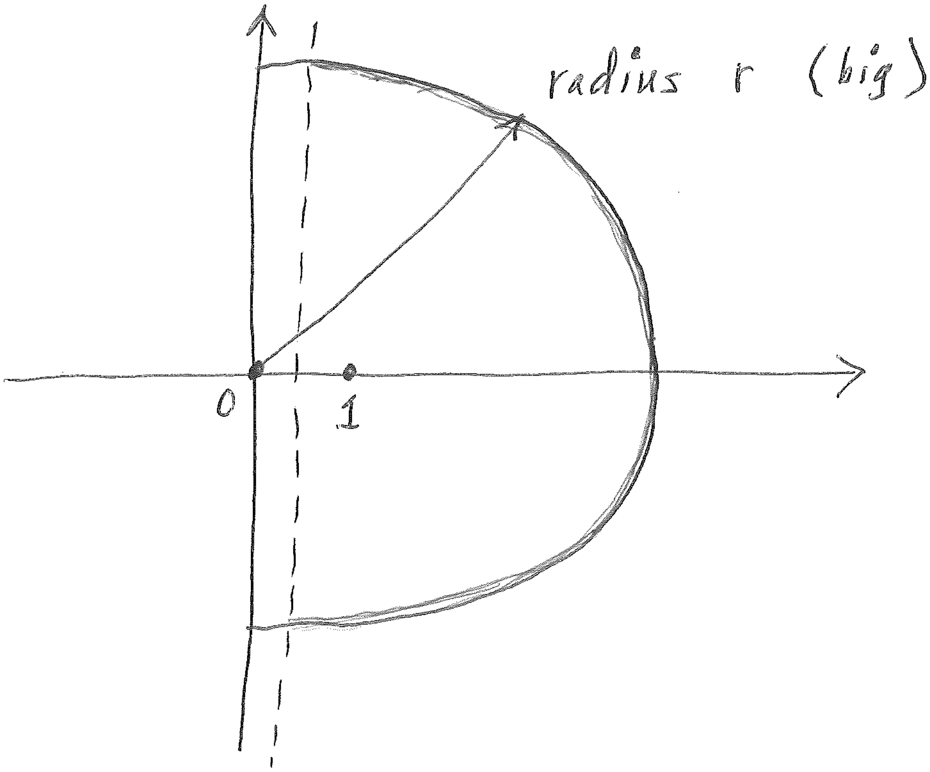
$$|\zeta(x+iy)| \lesssim \frac{C}{\delta(1-\delta)}|y|^{1-\delta} \quad \left\{\begin{matrix} x \gtrsim \delta \\ |y| \gtrsim 2 \end{matrix}\right\}$$

any $0 < \delta < 1$. Lec 6 page 9. EG $\delta = \frac{1}{2}$.

Also, we had

$$|\zeta(z) - 1| < \frac{3}{4} \quad \text{for} \quad \mathrm{Re}(z) > 2$$

by Lec 5 page 10.



$|z| = r$

$$F(z) = z(z-1)\,\pi^{-\frac{z}{2}}\,\Gamma\!\left(\tfrac{z}{2}\right)\zeta(z)$$

$$|F(z)| = |z|\,|z-1|\,\pi^{-\frac{x}{2}}\,\left|\Gamma\!\left(\tfrac{z}{2}\right)\right|\,|\zeta(z)|$$

$$|F(z)| = |z|^2\left[1+O\!\left(\tfrac{1}{r}\right)\right]\pi^{-\frac{x}{2}}\,\left|\Gamma\!\left(\tfrac{z}{2}\right)\right|\,|\zeta(z)|$$

Know:

$$|\zeta(z)-1| < \frac{3}{4} \quad \text{for } x > 2$$

$$|\zeta(x+iy)| \lesssim O(|y|^{1/2}), \quad x \gtrsim \frac{1}{2},\ |y| \gtrsim 2.$$

Also:

$$\ln\left|\Gamma\!\left(\tfrac{z}{2}\right)\right| = \mathrm{Re}\left\{\mathcal{L}og\,\Gamma\!\left(\tfrac{z}{2}\right)\right\}$$

↑ Stirling Lec 10 p. 42

and

$$\mathrm{Log}\,\Gamma\left(\frac{z}{2}\right) = \left(\frac{z}{2}-\frac{1}{2}\right)\mathrm{Log}\left(\frac{z}{2}\right) - \frac{z}{2} + \frac{1}{2}\ln(2\pi)$$
$$+ O\left(1/|z|\right)$$

$\{$ for, say, $|z| = r$, r large, $|\mathrm{Arg}\, z| \leqq \frac{3}{4}\pi \}$.

As in Ingham 56–57, we get

$$\ln\left|\Gamma\left(\frac{z}{2}\right)\right| \leqq \frac{r}{2}\ln r + A_1 r$$

$\{$ for, say, $|z| = r$, $|\mathrm{Arg}\, z| \leqq \frac{3}{4}\pi \}$

$$\Downarrow$$

$$\ln|F(re^{i\theta})| \leqq \frac{r}{2}\ln r + A_2 r$$

for $|z| = r$, $\mathrm{Re}(z) \geqq \frac{1}{2}$

$\boxed{\text{then, } \underline{\text{using}}\ F(z) = F(1-z),\ \text{similarly for } \mathrm{Re}(z) \leqq 1/2}$.

Also, looking at $\theta = 0$,

$$F(R) = R^2 \left[1 + O\left(\frac{1}{R}\right) \right] \pi^{-\frac{R}{2}} \Gamma\left(\frac{R}{2}\right) \zeta(R)$$

$$\geq (\text{constant})\, R^2\, \pi^{-\frac{R}{2}} \Gamma\left(\frac{R}{2}\right)$$

$$\left\{ \begin{array}{l} \text{but} \\ \ln \Gamma\left(\frac{R}{2}\right) \sim \left(\frac{R}{2} - \frac{1}{2}\right) \ln\left(\frac{R}{2}\right) - \frac{R}{2} \end{array} \right\}$$

$$\Downarrow$$

$$\ln F(r) \gtrsim \frac{r}{2} \ln r - A_3 r \qquad (r \text{ large}).$$

THM

Let $F(z) = z(z-1)\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)$. Let

$$M(r) = \max_{|z|=r} |F(z)| \qquad (r \text{ large}).$$

Then:

$$\ln M(r) \sim \frac{r}{2} \ln r.$$

Proof

As above. ▥

For any entire fcn $g(z)$, $g \not\equiv 0$, we write

$$M(r) = \max_{|z|=r} |g(z)| .$$

Then put:

$$\rho = \inf\{\omega : \underline{\ln} M(r) \leq r^{\omega}, \text{ all large } r\}$$

$$\tau = \inf\{\beta : \underline{\ln} M(r) \leq \beta r^{\rho}, \text{ all large } r\}.$$

Herein $\omega \geq 0$ and $\beta \geq 0$. Empty braces mean $\inf = +\infty$. We call:

ρ = ORDER of $g(z)$

τ = TYPE of $g(z)$.

$$\boxed{\text{For our } F(z), \text{ clearly } \rho = 1 \text{ and } \tau = +\infty.}$$

Lecture 12

(26 Feb)

2 Notes!

Lec 10 p. 42 Stirling (Corollary).

We also have:

Thm (Stirling)

$$\frac{\Gamma'(z)}{\Gamma(z)} = \mathrm{Log}\, z - \frac{1}{2z} + \sum_{k=1}^{R} \left(-\frac{B_{2k}}{2k}\right) z^{-2k} + O_{R\delta}\left(\frac{1}{|z|^{2R+1}}\right)$$

as $z \to \infty$ in $|\mathrm{Arg}\, z| \leq \pi - \delta$.

Pf

Call the $\tilde{B}_{2R+1}(t)$ integral term in 42 Thm $r(z)$.
Note $r(z)$ is <u>nicely analytic</u> <u>and</u> $r(z) = O(z^{-2R-1})$
by the Cor on 42. But:

$$r'(z) = \frac{1}{2\pi i} \oint_{|w-z|=1} \frac{r(w)}{(w-z)^2}\, dw.$$

Just use $|\mathrm{Arg}\, z| \leq \pi - 2\delta$ in place of $|\mathrm{Arg}\, z| \leq \pi - \delta$.
Done. ∎

About $\Gamma(z) \neq 0$, Lec 10 p. 26. One can avoid Hurwitz's thm.

Thm

Let $f_n(z) \rightrightarrows f(z)$ on $|z| < R$ compacta. We assume f_n, f are analytic. Let $f_n(z) \neq 0$ for all z and $f(z) \not\equiv 0$. Then: $f(z) \neq 0$.

Pf

Zeros of f are isolated. Hence finite # on each $|z| \leq R - \varepsilon$. Find $R_n \uparrow R$ so $f(R_n e^{i\theta}) \neq 0$.

Fix any N. Find $m, M > 0$ so $m \leq |f(R_N e^{i\theta})| \leq M$. By unif conv,

$$\frac{m}{2} \leq |f_n(R_N e^{i\theta})| \leq 2M \quad , \quad n \geq \mathcal{N}.$$

Apply max mod principle to f_n AND $1/f_n$. Get

$$|f_n(z)| \leq 2M \qquad n \geq \mathcal{N}$$

$$\left| \frac{1}{f_n(z)} \right| \leq \frac{2}{m}$$

on $|z| \leq R_N$. So,

$$\frac{m}{2} \leq |f_n| \leq 2M.$$

Let $n \to \infty$ to get $\frac{m}{2} \leq |f| \leq 2M$ on $|z| \leq R_N$. Now let $N \to \infty$. Done. ■

END OF NOTES

We then turned to the issue of the entire fcn

$$\xi_0(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

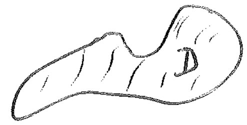
$$\xi_0(s) = \xi_0(1-s)$$

Lec 11 p. 25 31

order 1, type ∞

and trying to get a product expansion over the zeros. Ie, trying to get a "Hadamard factorization" of ξ_0 to justify Riemann's unproved assertion. [from 1859]

Standard lemmas.



Lemma 1

D = simply-connected domain.
Let $f = u + iv$ be analytic on D.
Then: u is harmonic on D (i.e. C^2 and $u_{xx} + u_{yy} = 0$).

Conversely, given real-valued U harmonic on D. We can cook up V, harmonic on D, so $F = U + iV$ is analytic on D.

Cor

Every harmonic U on D is actually C^∞.

Lemma 2 (mean-value property)

Let u be harmonic on D (as above).
Let $|z - z_0| \leqq R$ be contained in D.
Then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\varphi})\,d\varphi .$$

Lemma 3

D as above. Let g be analytic on D and $g(z) \neq 0$. We can always find an analytic function $\phi(z)$ on D such that $\exp(\phi) = g$.

[ϕ is unique up to $+2\pi i k$]

Theorem (Jensen's formula) ← Lemma 4

D as above. Let $\{|z| \leq R\} \subseteq D$. Let f be analytic on D, $f \neq 0$ on $|z| = R$, $f(0) \neq 0$.

Then:

$$\ln|f(0)| + \sum_{j=1}^{m} \ln \frac{R}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \ln|f(Re^{i\varphi})| \, d\varphi .$$

Here $\{a_1, \ldots, a_m\}$ are the zeros of f in $0 < |z| < R$ listed with multiplicity.

Pf

Wlog $D = \{|z| < R + \varepsilon\}$, ε tiny.

Wlog $f \neq 0$ on $\{R \leq |z| < R + \varepsilon\}$. Form analytic fcn:

$$F(z) = f(z) \prod_{j=1}^{m} \frac{R^2 - \bar{a}_j z}{R(z - a_j)} .$$

Get $|F| = |f|$ on $|z| = R$, $F(z) \neq 0$ on $|z| < R + \varepsilon$.

Apply Lemma 3 to get $\operatorname{Log} F(z)$. By lemma 2+1,

$$\ln|F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \ln|F(Re^{i\varphi})| \, d\varphi .$$

Done. ∎

If $f(0) = 0$, people usually just pass to $\dfrac{f(z)}{z^N}$.

Thm (Lemma 5)

Let f be entire of order $\leqq \rho$. $(f \not\equiv 0)$
Then, counting with multiplicity,

$$n(r) \equiv N[\underline{\text{zeros}} \text{ of } f \text{ in } |z| \leqq r] = O(r^{\rho+\varepsilon})$$

for all r large. Here $\varepsilon > 0$.

Pf

$f(0) = 0 \Rightarrow$ pass to $g = \dfrac{f(z)}{z^N}$. g is still entire and has order $\leqq \rho$.

WLOG $f(0) = 1$. Know $\ln M(R; f) \leqq R^{\rho+\varepsilon}$, large R.
Perturb R slightly to make $f(Re^{i\theta}) \neq 0$.

Apply Lemma 4 (Jensen):

$$0 + \sum_{j=1}^{m} \ln \frac{R}{|a_j|} = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\varphi})|\,d\varphi \leqq R^{\rho+\varepsilon}.$$

Hence:

$$n\left(\frac{R}{2}\right) \ln 2 \leqq R^{\rho+\varepsilon}$$

$$\Rightarrow n(r) = O(r^{\rho+\varepsilon}) \quad \text{all large } r. \quad \blacksquare$$

6

KEY THM (Lemma 6, Hadamard/Borel/Caratheodory)

D as above. f analytic on D.

Suppose $\{|z-z_0| \leqq R\} \subseteq D$. Let $f = \sum_0^\infty c_n (z-z_0)^n$

on the closed disk.

Assume further that

$$\operatorname{Re} f(z) \leqq \mathcal{M} \longleftarrow \text{KEY}$$

on the closed disk. Then:

(A) $|c_n| \leqq \frac{2}{R^n}(\mathcal{M} - \operatorname{Re} c_0)$, $n \geqq 1$

(B) $|f(z) - f(z_0)| \leqq \frac{2r}{R-r}\{\mathcal{M} - \operatorname{Re} c_0\}$, $|z-z_0| \leqq r < R$

(C) $\left|\frac{f^{(k)}(z)}{k!}\right| \leqq \frac{2R}{(R-r)^{k+1}}\{\mathcal{M} - \operatorname{Re} c_0\}$, $k \geqq 1$, .

PF

See Ingham 50-51. (There is a) famous trick in this proof (starts 50 bottom). ■

Lemma 7

Suppose we have an entire fcn f such that $f(0) \neq 0$. Let its zeros (listed with multiplicity) be $\{a_j\}$. WLOG $|a_j| \leqq |a_{j+1}|$. Assume that we know $n(r) = O(r^\beta)$ for large r. Then:

5

$$\sum_n \frac{1}{|a_n|^\gamma} < \infty \quad \text{for each } \gamma > \beta.$$

Pf

Take δ tiny. Look at $\int_\delta^T r^{-\gamma} d\mu(r)$ and integrate by parts. ∎

Corollary

Let f be entire and $f(0) \neq 0$. Then:

$$\sum \frac{1}{|a_n|^{\rho+\varepsilon}} < \infty \text{ , each } \varepsilon > 0 .$$

Pf

Lemma 5 + 7. ∎

Thus, we always have (for $f(z)$ entire)

$$\sum \frac{1}{|a_n|^{[\![\rho]\!]+1}} < \infty .$$

We let

$$\boxed{p = [\![\rho]\!]}$$

(Do not confuse p with a prime!)

when we play with a given f.

When $p =$ non-neg integer, following Weierstrass it is customary to define:

$$E(u;p) = \left\{ \begin{array}{ll} 1-u, & p = 0 \\ & \\ (1-u)\exp\left[u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}\right], & p \geqq 1 \end{array} \right\}.$$

Note that $E(z;p)$ is entire.

Take $|u| \leqq h < 1$. With <u>some</u> branch of log,

$$\begin{aligned} \log E(u;p) &= \operatorname{Log}(1-u) + u + \cdots + \frac{u^p}{p} \\ &= -\sum_{n=1}^{\infty} \frac{u^n}{n} + u + \cdots + \frac{u^p}{p} \\ &= -\sum_{n=p+1}^{\infty} \frac{u^n}{n}. \end{aligned}$$

Clearly,

$$|\log E(u;p)| \leqq \frac{|u|^{p+1}}{1-|u|} \qquad (p = 0 \text{ ok too}).$$

Hence:

$$\ln|E(u;p)| \leqq \frac{|u|^{p+1}}{1-h}, \qquad |u| \leqq h < 1.$$

Given $p \geq 0$. Also given $a_n \in \mathbb{C} - \{0\}$, $a_n \to \infty$, and

$$\sum_n \frac{1}{|a_n|^{p+1}} < \infty .$$

We call

$$\prod_{n=1}^{\infty} E\left(\frac{z}{a_n} ; p\right)$$

a CANONICAL PRODUCT of genus p.

<u>THM</u>

In the above, the canonical product of genus p converges <u>uniformly</u> & <u>absolutely</u> on $\mathbb{C}$-compacta. Hence it is an entire function with zeros <u>exactly at</u> $\{a_n\}$.

<u>Pf</u>

We use our standard reduction to the " $\sum_{k=1}^{\infty} Log(1+b_k(z))$ theorem ".

Take $K = \{|z| \leq R\}$. Restrict attention to $|a_n| > 1000R$. Hence, in product, each term has

$$\left|\frac{z}{a_n}\right| < \frac{1}{1000} \quad \text{for } z \in K .$$

Get:

$$\left|\log E\left(\frac{z}{a_n};p\right)\right| \leqq \frac{\left|\frac{z}{a_n}\right|^{p+1}}{1-\frac{1}{1000}}$$

$$\leqq (1.01)\left|\frac{z}{a_n}\right|^{p+1}$$

$$\leqq (1.01)\left(\frac{1}{1000}\right)^{p+1}$$

$$\leqq .002 \quad .$$

Therefore, the "log" is actually Log.

And:

$$\left|\text{Log}\, E\left(\frac{z}{a_n};p\right)\right| \leqq .002$$

$$\left|E\left(\frac{z}{a_n};p\right)-1\right| \leqq .01$$

i.e. "$|b_n(z)| \leqq .01$" (on K).

Must look at

$$\sum_n \left|\text{Log}\,(1+b_n(z))\right|$$

on K. This sum will be

$$\leqq \sum_n (1.01)\left(\frac{R}{|a_n|}\right)^{p+1} \quad \{\text{by the above}\}$$

for all $z \in K$. $\Rightarrow$ all is OK. ■

When we study canonical products, it is helpful to conceptualize them as

$$\prod E\left(\frac{z}{a_n}; p\right) \equiv \prod_{|a_n| \leq 1000R} E\left(\frac{z}{a_n}; p\right)$$

$$\cdot \prod_{|a_n| > 1000R} E\left(\frac{z}{a_n}; p\right)$$

THIS PART IS NONZERO

over $\{|z| \leq R\}$.

<u>THEOREM</u> (preliminary factorization)

Let f be entire. Let the order be $\rho < \infty$. Put $p = [\![\rho]\!]$. Let the <u>zeros</u> of f in $\mathbb{C} \sim \{0\}$ be $\{a_n\}$. [This set could be empty!]

We then have:

$$f(z) = z^N \exp[\phi(z)] \prod_{a_n \neq 0} E\left(\frac{z}{a_n}; p\right),$$

where ϕ is <u>some</u> entire fcn and where the product (if infinite) is abs + uniformly conv on $\mathbb{C}$ compacta.

Pf

Pass first to $g(z) \equiv \frac{f(z)}{z^N}$, as usual.
The fcn g is entire, order ρ, $g(0) \neq 0$.

Now review 9 and form

$$h(z) \equiv \frac{g(z)}{\prod_{a_n \neq 0} E(\frac{z}{a_n}; p)}.$$

See 11 top. Get $h(z) \neq 0$ for all $z \in \mathbb{C}$.
By Lemma 3 applied to $h(z)$, we can write
$h = \exp(\phi(z))$. Done. ■

Hadamard realized, in studying Riemann's work, that he needed some way of controlling $\phi(z)$ using <u>only</u> information about <u>Re</u> $\phi(z)$.

This is what led to p. 6 Key Thm!

Hadamard's Factorization Theorem ~1893

Given the situation of p. 11 THM.
We then have that $\phi(z)$ must be a
polynomial of degree $\leqq p$.

(Recall that $p = [\![\rho]\!]$.)

In the case of $\Xi_0(s)$, we had $\rho = 1$
and type $\tau = \infty$. So, here,

$$\Xi_0(s) = e^{As+B} \prod_n E\left(\frac{s}{a_n}; 1\right).$$

Lec 11 p. 25 31 32

The (current-day) proof of the HFT either follows
an approach of Landau or else one based
on the so-called Poisson-Jensen formula
[a very common identity used in Nevanlinna
theory]. The proof is function theory,
not number theory.

The Landau approach is remarkable for its simplicity. See:

Landau, Vorlesungen über Zahlentheorie, Satz 423 (from 1927)

$\boxed{\text{OR}}$

Landau, Math. Zeitschrift 26 (1927) 170–175.

INGHAM, pages 54(bottom) – 55(bottom) is compressed, but follows LANDAU.

I presented the details of this in Lecture #12 and the first half of Lecture #13. I OMIT them here!

Lecture 13

(2 Mar 2016)

I began by finishing the proof of the Hadamard factorization thm.

I then turned to some simple function-theoretic facts and some corollaries of HFT.

Simple Facts

(I) f order ρ, g order $\rho' < \rho$ $\Rightarrow$ $f+g$ order ρ

(II) f order ρ_1, g order ρ_2 $\Rightarrow$ fg order $\leq \max\{\rho_1, \rho_2\}$

(III) $p(z) \not\equiv 0$ polynomial, f order ρ

$\Rightarrow$ $p(z)f(z)$ order ρ too

(IV) f order ρ. Zeros at $\{a_1, \ldots, a_M\}$. Then

$$g(z) = \frac{f(z)}{(z-a_1)\cdots(z-a_M)} \quad \text{order } \rho.$$

(V) Let f have order ρ. Let $z_0 \in \mathbb{C}$. Then:

$h(z) = f(z+z_0)$ has order ρ.

{Must fiddle and use max mod principle.}

VI Let f be entire, even, and order ρ.
Then

$$g(z) = f(z^{1/2}) \quad \text{has order } \rho/2 .$$

Cor 1 to HFT

Let f be entire, order ρ, $\rho \notin \mathbb{Z}$. Let $a \in \mathbb{C}$. Then $f(z) = a$ has infinitely many roots.

Pf

$\rho \notin \mathbb{Z} \Rightarrow \rho > 0$. Let $g = f(z) - a$. g has order ρ. Suppose $g(z)$ has only finitely many zeros. $\rho(g) = \rho$, $p = [\![\rho]\!]$. Apply HFT.

$$g(z) = z^N e^{\phi(z)} \prod_{n=1}^{\mathcal{N}} E(\tfrac{z}{a_n}; p) \quad , \quad \mathcal{N} < \infty$$

But $p < \rho < p+1$. RHS has order AT MOST p since $\deg \phi \leq p$. {Recall def of E, Lec 12 p. 8.}

Contradiction! ∎

Cor 2 to HFT (a form of the baby Picard thm)

Let f be entire, order $\rho > 0$, $\rho \in \mathbb{Z}$. Then $f(z)$ assumes every $a \in \mathbb{C}$ with AT MOST ONE exception.

Pf

Suppose 2 exceptions: $f \neq \alpha$, $f \neq \beta$. Write

$$g(z) = \frac{f(z) - \alpha}{\beta - \alpha} .$$

Order ρ again. And $g \neq 0, 1$.

Apply HFT. Get $g = \exp(\phi)$, ϕ = polynomial, degree $\leq p$. HERE $p = \rho$.

But, order of $g(z)$ is $\rho = p$, so $\deg \phi = p$. Since $p \geq 1$, we can solve $\phi(z) = 2k\pi i$ for any integer k. Thus, we get many points z_k where $g(z_k) = 1$. Contrad! ∎

WE NOW GO TO $\Xi_0(s)$!!

THM

Recall $\xi_0(s) = s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ à la Lec 11 p. 25 + 31. We have $\xi_0(s) = \xi_0(1-s)$, $\xi_0(1) = 1$.
Also order $\rho = 1$ and type $\tau = \infty$.

$\xi_0(s)$ has infinitely many zeros; these lie exclusively in $\{0 < \mathrm{Re}(s) < 1\}$.

Pf

Let $f(z) = \xi_0(\frac{1}{2}+z)$ so f has order 1 AND $f(z)$ IS EVEN. Form $g(z) = f(z^{1/2})$, which has order $\underline{\underline{1/2}}$. Page 2 top.

By Cor 1, g has infinitely many zeros!
Hence so does f, and hence ξ_0.

By Lec 11 p. 27, $\xi_0 \neq 0$ along the real axis.

For $\mathrm{Re}(s) > 1$, we know $s(s-1) \neq 0$, $\pi^{-s/2} \neq 0$, $\Gamma(\frac{s}{2}) \neq 0$, $\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \neq 0$.
The same is true for $s = 1+it$, t real $\neq 0$.
(Recall $\zeta(1+it) \neq 0$ was proved Lec 6 p. 7.)

Hence, $\xi_0(s) \neq 0$ for all $Re(s) \geqq 1$.

By $\xi_0(s) = \xi_0(1-s)$, get same for $Re(s) \leqq 0$.

So, all zeros lie in $\{0 < Re(s) < 1\}$. ▨

$\xi_0(1) = \xi_0(0) = 1$, $\xi_0(s) = \xi_0(1-s)$.

HFT now implies

$$\boxed{\xi_0(s) = e^{A+Bs} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho}}$$

$$0 < Re(\rho) < 1.$$

It might be better to use a letter other than ρ (to avoid confusion with order ρ). For these zeros, But "everyone" uses ρ Riemann, Landau, etc etc.

Order $= 1$, $[\![\text{order}]\!] + 1 = 2$ for ξ_0.

So,
$$\sum_\rho \frac{1}{|\rho|^2} < \infty.$$

We had $\xi_0(x) \neq 0$ for $x \in \mathbb{R}$, so

$$\boxed{Im(\rho) \neq 0} .$$

Recall that:

$$\xi_0(s) = 1 + s(s-1)\int_1^\infty \left[y^{\frac{1-s}{2}} + y^{\frac{s}{2}}\right]\Psi(y)\,\frac{dy}{y}$$

$$= \sum_1^\infty e^{-\pi n^2 y}$$

àla Lec 11 p. 25. Clearly

$$\xi_0(\bar{s}) = \overline{\xi_0(s)}\ , \quad \text{all } s \in \mathbb{C} .$$

Hence any zeros occur in conjugate pairs. The CANONICAL PRODUCT on 5 is thus real-valued for $s \in \mathbb{R}$. So is $\xi_0(s)$. As such, we conclude $\underline{\underline{B}}$ must be real.

Letting $s \to 0$ on 5 middle, we get $e^A = 1$, so WLOG $A = 0$.

With <u>some</u> branches,

$$\log \xi_0(s) = A + Bs + \sum_\rho \log\left\{\left(1-\frac{s}{\rho}\right)e^{s/\rho}\right\}.$$

Keep s on some $\{|s| \leq R\}$ at first!

Following Riemann, get:

- $$\frac{\xi_0'(s)}{\xi_0(s)} = B + \sum_\rho \left\{\frac{1}{s-\rho} + \frac{1}{\rho}\right\}$$

$\left\{\begin{array}{l}\text{with nice convergence on} \\ \mathbb{C}\text{-compacta away from the } \rho\text{'s} \\ \text{(Weierstrass conv theorem)}\end{array}\right\}.$

But, 4 line 2,

$$\log \xi_0(s) = \log s + \log(s-1) - \frac{s}{2}\ln\pi + \log\Gamma\left(\frac{s}{2}\right) + \log \zeta(s)$$

$$\Downarrow$$

$$\frac{\xi_0'(s)}{\xi_0(s)} = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\ln\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{\zeta'(s)}{\zeta(s)}$$

$$\Gamma(z+1) \equiv z\,\Gamma(z)$$

$$\frac{\Gamma'}{\Gamma}(z+1) \equiv \frac{1}{z} + \frac{\Gamma'}{\Gamma}(z)$$

$$\Downarrow$$

$$\bullet \quad \frac{\xi_0'(s)}{\xi_0(s)} = \frac{1}{s-1} - \frac{1}{2}\ln\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \frac{\zeta'(s)}{\zeta(s)} \, .$$

On (7) bottom, if desired, one could also substitute

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right)$$

from Lec 10 p. (3D). We'll skip this for now.

Combine (7) line 5 with line 3 above. Get:

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \left(B + \frac{1}{2}\ln\pi\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)$$

$$+ \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

Thm (Riemann)

Away from the set $\{1\} \cup \{\rho\} \cup \{-2k\}_{k=1}^{\infty}$
we have

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \left(B + \frac{1}{2}\ln\pi\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)$$

$$+ \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Pf

As above. ■

$$3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2$$
$$\geq 0$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{2}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{2}^{\infty} \frac{\Lambda(n)}{n^\sigma} n^{-it}$$

$$\operatorname{Re}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] = \sum_{2}^{\infty} \frac{\Lambda(n)}{n^\sigma} \cos(t\ln n)$$

$$\sigma = \operatorname{Re}(s) > 1 .$$

Fact

$\sigma > 1$, $t \in \mathbb{R}$, $t \neq 0$

$$\mathrm{Re}\left[3\frac{\zeta'}{\zeta}(\sigma) + 4\frac{\zeta'}{\zeta}(\sigma+it) + \frac{\zeta'}{\zeta}(\sigma+2it)\right] \leqq 0.$$

Pf

As above. ∎

Abbreviate Thm on p. 9 as

$$\frac{\zeta'(s)}{\zeta(s)} = -g(s) + f(s)$$

$$\uparrow$$

$$\sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

For a few moments. Keep $\sigma > 1$.

We immediately get:

$$\mathrm{Re}[3f(\sigma) + 4f(\sigma+it) + f(\sigma+2it)]$$

$$\leqq \mathrm{Re}[3g(\sigma) + 4g(\sigma+it) + g(\sigma+2it)],$$

$t \neq 0$ real, $\sigma > 1$.

THM (classical zero-free region)

There exists an absolute constant $a > 0$ so that

$$\zeta(s) \neq 0 \quad \text{on} \quad \left\{ \sigma > 1 - \frac{a}{\log(|t|+2)} \right\}.$$

Proof

Essentially following INGHAM 59–60.

WLOG $t > 0$.

We can always re-adjust "a" to take care of bounded t. So, WLOG, we need only look at the case of $t > G$, $G =$ giant.

We play with 10 bottom for $1 < \sigma < 3$, $t > G$. Remember that

$$g(s) = \frac{1}{s-1} - b + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)$$

for some real constant b. Hence $g(s)$ is rather explicit.

Recall Stirling for $\frac{\Gamma'}{\Gamma}$; Lec 12 p. 1.

We clearly get

$$g(s) = O(\ln t) \qquad \underline{1 < \mathrm{Re}(s) < 3}$$

anytime $\mathrm{Im}(s) \geqq 100$, say.

Of course, for $1 < \sigma < 3$,

$$g(\sigma) = \frac{1}{\sigma - 1} + O(1).$$

On 10 bottom, we get:

$$\mathrm{Re}\left[3f(\sigma) + 4f(\sigma+it) + f(\sigma+2it)\right]$$

$$\leqq \frac{3}{\sigma-1} + A \ln t \qquad \left\{ \begin{matrix} 1 < \sigma < 3 \\ t > G \end{matrix} \right\}.$$

Here:

$$f(s) \equiv \sum_{\rho} \left\{ \frac{1}{s-\rho} + \frac{1}{\rho} \right\}. \quad \leftarrow 9 \text{ line } 5$$

For $\sigma > 1$ and $\rho = \beta + i\gamma$, note that

$$\mathrm{Re}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = \frac{\sigma - \beta}{(\sigma-\beta)^2 + (t-\gamma)^2} + \frac{\beta}{\beta^2 + \gamma^2} \geqq 0 \;!!!$$

Consider now <u>any</u> zero $\rho_0 = \beta_0 + i\gamma_0$ of ξ_0 with $\gamma_0 > G$.

last line

Apply 12 lines <u>8 + 9</u> keeping 12 bottom in mind.

KEY

Get:

$$4\,\frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+(t-\gamma_0)^2} \lessapprox \frac{3}{\sigma-1} + A\log t\,,$$

all $1<\sigma<3$, $t>G$.

Notice that

$$LHS \lessapprox \frac{4}{\sigma-\beta_0}$$

<u>trivially</u> for ANY $\sigma>1$. For $\sigma \gtrapprox 3$, get

$$LHS \lessapprox \frac{4}{3-\beta_0} \lessapprox 2\,.$$

By revising A, we can thus say

$$4\,\frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+(t-\gamma_0)^2} \lessapprox \frac{3}{\sigma-1} + \tilde{A}\ln t$$

for <u>ANY</u> $\sigma>1$ and $t>G$. $\left(\begin{array}{c}\text{And any } \rho_0 \\ \text{with } \gamma_0 > G\end{array}\right)$

Put $t = \gamma_0$ to see that

$$\frac{4}{\sigma - \beta_0} \leqq \frac{3}{\sigma - 1} + \tilde{A} \ln \gamma_0$$

for all $\sigma > 1$.

Let $\sigma \to 1$ to see that $\beta_0 = 1$ is impossible (a fact we <u>already</u> know).

One <u>expects</u> that $\sigma - \beta_0$ and $\sigma - 1$ of comparable size will be most <u>illuminating</u>. Take:

$$\sigma = 1 + \lambda(1 - \beta_0), \quad \lambda > 0$$
$$\sigma - \beta_0 = \sigma - 1 + 1 - \beta_0 = (\lambda + 1)(1 - \beta_0)$$

$$\Downarrow$$

get

$$\frac{4}{(1+\lambda)(1-\beta_0)} - \frac{3}{\lambda(1-\beta_0)} \leqq \tilde{A} \ln \gamma_0$$

<u>any</u> $\lambda > 0$.

$$\boxed{\text{For large } \lambda, \text{ obviously } \frac{4}{1+\lambda} - \frac{3}{\lambda} > 0 \text{ since } 4 > 3.}$$

$\lambda = 4$ gives

$$\frac{.80 - .75}{1-\beta_0} \lessapprox \tilde{A} \ln \gamma_0$$

$$\Downarrow$$

$$1-\beta_0 \gtrapprox \frac{.05}{\tilde{A}(\ln \gamma_0)} .$$

Hence,

$$\beta_0 \lessapprox 1 - \frac{.05}{\tilde{A}(\ln \gamma_0)} .$$

This is sufficient to prove Thm on p. 11. ∎

Lecture 14 Synopsis

(4 Mar 2016)

The aim in this lecture was to develop some standard estimates for $\psi(x) - x$ and $\pi(x) - li(x)$ based on given zerofree regions for $\zeta(s)$. Ingham 60–67.

I began by recalling the Hadamard/Borel/Caratheodory lemma from Ingham 50.

I then turned to the development of Ingham's general estimate on $\frac{\zeta'(s)}{\zeta(s)}$ for a given zerofree region $\sigma > 1 - \eta(|t|)$. See Ingham 60–62.

$0 < \eta(t) \leqq \frac{1}{2}$

$\eta(t)$ decreasing on $[0, \infty)$, C^1

$\eta'(t) \to 0$ as $t \to \infty$

$\eta(t) \geqq \frac{1}{G \ln t}$ for t large (G = big constant)

$$\Downarrow$$

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln^2 |t|) \quad \text{on} \quad \sigma > 1 - \alpha\eta(|t|)$$

for $|t|$ large and any $0 < \alpha < 1$.

One can then use $(c>1)$ Lec 7 p. (10)

$$\psi_1(x) \approx \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]ds$$

$(x \geqq 1)$ and begin to do contour shifts over to the left, beyond $s=1$, using the Cauchy Residue Theorem. Here one wants to move the path of integration over to

$$\sigma = 1 - q\,\eta(|t|)$$

for a fixed $0<q<1$. {due to (1) bottom}

Ingham 62(bottom) – 63 gets

$$\bullet \quad \psi_1(x) \approx \frac{x^2}{2} + O\left[x^2 e^{-q\,\omega(x)}\right], \longleftarrow \boxed{\text{THM 21}}$$

$$\omega(x) \equiv \min_{t \geqq e} \left[\eta(t)\ln x + \ln t\right].$$

↑ I prefer "e" over Ingham's "1"

The introduction of $\omega(x)$ is slightly "slick". Classical estimates simply did "each $\eta(t)$"

separately, using <u>whatever</u> technique was natural.

Concerning $\omega(x)$, I noted:

<u>Lemma</u>

Keep $x \geqq 1$. Then:

(a) $\omega(x)$ strictly $\uparrow$ $\boxed{\omega(1)=1 \text{ clearly}}$

(b) $\ln x - \omega(x)$ strictly $\uparrow$

(c) $1 < \omega(x) < 1 + \ln x$, $x > 1$.

<u>Pf</u>

Let $1 \leqq x_2 < x_1$. Let $\omega(x_2)$ "occur" for t_2, $\omega(x_1)$ "occur" for t_1. Get:

$$\omega(x_1) \leqq \eta(t_2)\ln x_1 + \ln t_2 \quad \text{a priori}$$

$$= \eta(t_2)\ln x_2 + \ln t_2$$
$$+ \eta(t_2)[\ln x_1 - \ln x_2]$$

$$\leqq \omega(x_2) + \frac{1}{2}(\ln x_1 - \ln x_2) \quad \text{see p. 1}$$

$$< \omega(x_2) + \ln x_1 - \ln x_2$$

$$\Rightarrow \quad \omega(x_1) - \ln x_1 < \omega(x_2) - \ln x_2$$

$$\Rightarrow \quad \ln x_1 - \omega(x_1) > \ln x_2 - \omega(x_2)$$, i.e (b).

Similarly :

$$\omega(x_2) \leqq \eta(t_1)\ln x_2 + \ln t_1 \qquad \text{a priori}$$

$$< \eta(t_1)\ln x_1 + \ln t_1 = \omega(x_1)$$

$\Rightarrow$ $\omega(x)$ strictly $\uparrow$, i.e. (a) .

And,

$\omega(1) = 1$ by def

but, $\omega(x) \uparrow$ strictly, so $\omega(x) > 1 \quad (x>1)$

and, $\ln x - \omega(x) \uparrow$ strictly, so $\ln x - \omega(x) > -1 \quad (x>1)$

IE, for $x > 1$,

$1 < \omega(x) < \ln x + 1$. This is (c). ■

<u>Theorem</u>

Given $\eta(|t|)$ as above. For large x, we have

$$\psi(x) = x + O[xe^{-\frac{\alpha}{2}\omega(x)}]$$

$$\pi(x) = li(x) + O[xe^{-\frac{\alpha}{2}\omega(x)}] .$$

Here $0 < \alpha < 1$.

Pf

Essentially like Ingham 64-65.

Keep $x \geqq 1000$ say. Take $0 < h < \frac{x}{2}$. Know

$$\frac{1}{h}\int_{x-h}^{x} \psi(u)\,du \leqq \psi(x) \leqq \frac{1}{h}\int_{x}^{x+h} \psi(u)\,du$$

$$\frac{\psi_1(x) - \psi_1(x-h)}{h} \leqq \psi(x) \leqq \frac{\psi_1(x+h) - \psi_1(x)}{h}$$

$$\left\{ \begin{array}{l} \text{here } x-h > \frac{x}{2} \geqq 500 \\ x+h < \frac{3}{2}x \end{array} \right\}$$

let's look at upper side first

$$\psi(x) \leqq \frac{\frac{1}{2}\left[(x+h)^2 - x^2\right] + O\left[(x+h)^2 e^{-q\omega(x+h)}\right] + O\left[x^2 e^{-q\omega(x)}\right]}{h}$$

$$\{\omega(u) \text{ strictly } \uparrow\}$$

$$\psi(x) \leqq x + \frac{h}{2} + \frac{O\left[x^2 e^{-q\omega(x)}\right] + O\left[x^2 e^{-q\omega(x)}\right]}{h}$$

$$\Rightarrow \psi(x) \leqq x + \frac{h}{2} + \frac{1}{h} O\left[x^2 e^{-q\omega(x)}\right]$$

next, do lower side ; get

$$\psi(x) \gtrsim x - \frac{h}{2} - \frac{1}{h}\left[O(x^2 e^{-\alpha\omega(x)}) + O(x^2 e^{-\alpha\omega(\frac{x}{2})})\right]$$

$x - h > \frac{x}{2}$ and
$\omega(u) \uparrow$ strictly

$$\left\{ \begin{array}{l} \ln u - \omega(u) \uparrow \text{ strictly } \Rightarrow \\ \ln x - \omega(x) > \ln\frac{x}{2} - \omega(\frac{x}{2}) \\ \omega(\frac{x}{2}) > \omega(x) - \ln 2 \\ e^{-\alpha\omega(\frac{x}{2})} < 2^{\alpha} e^{-\alpha\omega(x)} < 2 e^{-\alpha\omega(x)} \end{array} \right\}$$

$$\Rightarrow \psi(x) \gtrsim x - \frac{h}{2} - \frac{1}{h} O[x^2 e^{-\alpha\omega(x)}] .$$

So,

$$\psi(x) = x + O(h) + O\left[\frac{1}{h} x^2 e^{-\alpha\omega(x)}\right] .$$

Put

$$h = \frac{x}{3} e^{-\frac{\alpha}{2}\omega(x)} \quad , \quad \text{~~say~~} .$$

cf. 3 Lemma (c)

Get:

$$\psi(x) = x + O\left[x e^{-\frac{a}{2}\omega(x)}\right].$$

Recall Lec 1 p. 4 middle. Then define:

$$\Pi(x) = \sum_{2 \leqq n \leqq x} \frac{\Lambda(n)}{\ln n} \qquad (x \geqq 2)$$

$\boxed{\text{Ingham p. 18}}$

$$= \sum_{p^m \leqq x} \frac{1}{m}$$

$$= \pi(x) + \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \tfrac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots .$$

Note that

Lec 1 p. 5

$$x^{\frac{1}{M}} < 2 \qquad \text{for} \quad M = \left[\!\left[\frac{\ln x}{\ln 2} \right]\!\right] + 10 .$$

Get:

$$\Pi(x) = \int_c^x \frac{1}{\ln t}\, d\psi(t) \qquad 1 < c < 2$$

$$= \frac{\psi(t)}{\ln t}\Big]_c^x - \int_c^x \psi(t)\, d\left(\frac{1}{\ln t}\right)$$

$$= \frac{\psi(x)}{\ln x} - 0 - \int_c^x \frac{\psi(t)(-1)}{(\ln t)^2} \frac{1}{t}\, dt$$

$$= \frac{\psi(x)}{\ln x} + \int_c^x \frac{\psi(t)}{t(\ln t)^2}\, dt .$$

Let $c \to 2$ to get

$$\pi(x) = \frac{\psi(x)}{\ln x} + \int_2^x \frac{\psi(t)}{t(\ln t)^2}\, dt .$$ Ingham 64 middle

Of course, we also have

$$li(x) = \int_2^x \frac{1}{\ln u}\, du$$ by our def (compare Ingham p. 3)

$$= \frac{x}{\ln x} - \frac{2}{\ln 2} - \int_2^x u\, d\left(\frac{1}{\ln u}\right)$$

$$= \frac{x}{\ln x} - \frac{2}{\ln 2} + \int_2^x \frac{u}{u(\ln u)^2}\, du .$$

So,

$$\boxed{\pi(x) - li(x) = \frac{\psi(x) - x}{\ln x} + \frac{2}{\ln 2} + \int_2^x \frac{\psi(t) - t}{t(\ln t)^2}\, dt}$$

which is a very useful identity, clearly.

We get: $\boxed{\beta \equiv \frac{\alpha}{2}}$ $\boxed{0<\beta<\frac{1}{2}}$

$$|\pi(x) - li(x)| \lessapprox O(1) + \frac{O[xe^{-\beta\omega(x)}]}{\ln x} \quad \text{by 7 top}$$

$$+ \int_2^x \frac{O[te^{-\beta\omega(t)}]}{t(\ln t)^2}\,dt$$

{the implied constant needs inflation for small t}

$$\lessapprox O[xe^{-\beta\omega(x)}] + O(1)$$

$$+ O(1)\int_2^x e^{-\beta\omega(t)}\,dt$$

$$\left\{ \begin{array}{l} 1 < \omega(t) < 1 + \ln t \quad \text{p. 3} \\ \hline xe^{-\beta\omega(x)} \gtrapprox xe^{-\beta(1+\ln x)} \\ \qquad = e^{-\beta}x^{1-\beta} \end{array} \right\}$$

$$\lessapprox O[xe^{-\beta\omega(x)}] + O(1)\int_2^x e^{-\beta\omega(t)}\,dt$$

$\{$ but $\ln u - w(u) \nearrow$ strictly, p. 3 $\}$

$$\lessapprox O[xe^{-\beta w(x)}] + \int_2^x O(1)\, e^{\beta(\ln t - w(t))} \frac{dt}{t^\beta}$$

$$\lessapprox O[xe^{-\beta w(x)}] + \int_2^x O(1)\, e^{\beta(\ln x - w(x))} \frac{dt}{t^\beta}$$

$$\lessapprox O[xe^{-\beta w(x)}] + O(1)\, x^\beta e^{-\beta w(x)} \left[\frac{t^{1-\beta}}{1-\beta}\right]_2^x$$

$$\lessapprox O[xe^{-\beta w(x)}] + O(1)\, x^\beta e^{-\beta w(x)} \frac{x^{1-\beta}}{1-\beta}$$

$\boxed{0 < \beta < \frac{1}{2}}$

$$\lessapprox O[xe^{-\beta w(x)}] .$$

So,

$$\Pi(x) - li(x) \approx O[xe^{-\frac{\alpha}{2} w(x)}] .$$

But

$$\Pi(x) - \pi(x) = \sum_{m=2}^{\infty} \frac{1}{m} \pi(x^{1/m}) \qquad \text{see } 7$$

$$= O\left[\frac{x^{1/2}}{\ln x}\right] + O[Mx^{1/3}]$$

$M = [\![\frac{\ln x}{\ln 2}]\!] + 10$

$$= O\left[\frac{x^{1/2}}{\ln x}\right] .$$

Hence, for large x,

$$\pi(x) - li(x) = O\left[x e^{-\frac{a}{2}\omega(x)}\right]$$

$\{$noting 9 2 lines from bottom$\}$.

■

Example I

$$\eta(t) = \frac{1}{G \ln t}, \quad G \text{ big}, \quad t \gtrsim e$$

in accordance with Lec 13, p. 11 Thm.

$$\omega(x) = \min_{t \gtrsim e} \left\{ \frac{1}{G \ln t} \ln x + \ln t \right\} \qquad (2)$$

Trivial calc problem with $u \gtrsim 1$ and

$$\frac{1}{G} \frac{\ln x}{u} + u .$$

Get

$$\min \approx 2\sqrt{\frac{\ln x}{G}} \qquad (x \text{ large}).$$

hence $\frac{\ln x}{G} \gtrsim 1$

So, by Lec 13 p. 11 and p. 4 Thm above,

$$\boxed{\begin{aligned} \psi(x) &= x + O\left[x e^{-c\sqrt{\ln x}}\right] \\ \pi(x) &= \mathrm{li}(x) + O\left[x e^{-c\sqrt{\ln x}}\right] \end{aligned}}$$

for suitably small $c > 0$.

The estimates in the box are the famous classical estimates of de la Vallée Poussin ~ 1899.

Example II

Lec 13 p. 4

Assume the Riemann Hypothesis, i.e. $\mathrm{Re}(\rho) = 1/2$ for all zeros of $\xi_0(s)$.

Here $\eta(t) = 1/2$.

$$w(x) = \min_{t \geq e} \left\{ \tfrac{1}{2} \ln x + \ln t \right\} = \tfrac{1}{2} \ln x + 1.$$

$\uparrow$ 2

In this situation, we get

$$\psi(x) = x + O\left[x e^{-\frac{\alpha}{2}\frac{1}{2}\ln x}\right]$$

$$= x + O\left[x^{1-\frac{\alpha}{4}}\right]$$

$$= x + O\left[x^{\frac{3}{4}+\varepsilon}\right]$$

$\alpha = 1 - 4\varepsilon$

$\varepsilon = \frac{1}{4}(1-\alpha)$

$$\boxed{\pi(x) = li(x) + O\left[x^{\frac{3}{4}+\varepsilon}\right]} \, .$$

WE EXPECT AN EXPONENT MORE LIKE $\frac{1}{2}+\varepsilon$, NOT $\frac{3}{4}+\varepsilon$. (Under RH.)

To fix this, we must use a more refined technique. The idea on page (2) top is too crude! Not enough structure!!

Riemann recognized this fact. IE, a need for a more explicit formula for $\psi_1(x)$.

Lecture 15 Synopsis

(9 Mar 2016)

Before starting, I noted a simple lemma having relevance to $\xi_0(s)$.

Lemma

Let $f(z)$ be entire, with order $\rho \in [1,2)$. Let $\{a_n\}$ be the nonzero zeros of f (listed with multiplicity). We must then have

$$\sum \frac{1}{|a_n|} = +\infty$$

if either

(a) $1 < \rho < 2$

(b) $\rho = 1$ but type $\tau = +\infty$.

Pf

Apply Hadamard factorization. $p = [\![\rho]\!] = 1$.

$$f(z) = z^{g} e^{Q(z)} \prod_n E\left(\frac{z}{a_n}; 1\right)$$

$$\deg Q \leq 1$$

$\begin{pmatrix} \text{Lec } 12 \\ \text{p. } 13 \end{pmatrix}$

Know $\sum \frac{1}{|a_n|^2} < \infty$ by Lec 12 p. 7.

Assume that $\sum \frac{1}{|a_n|} < \infty$. Take R large.

Observe that, for $|z| = R$,

recall Lec 5 16 Corollary

$$|f(Re^{i\theta})| = R^q |e^{Az+B}| \cdot \prod_n |1 - \tfrac{z}{a_n}| |e^{\frac{z}{a_n}}|$$

$$\lesssim R^q e^{O(R)} \prod_n \left(1 + \frac{|z|}{|a_n|}\right) e^{\frac{|z|}{|a_n|}}$$

$$\lesssim R^q e^{O(R)} \prod_n e^{\frac{|z|}{|a_n|}} e^{\frac{|z|}{|a_n|}}$$

$$\{\text{since } 1+u \lesssim e^u,\ u \gtrsim 0\}$$

$$\lesssim R^q e^{O(R)} \prod_n e^{2R \frac{1}{|a_n|}}$$

$$= R^q e^{O(R)} e^{2R\left(\sum_n \frac{1}{|a_n|}\right)}$$

$$\lesssim e^{O(R)} \quad \Longrightarrow$$

$$M(R;f) \lesssim e^{O(R)}$$

$$\ln M(R;f) \lesssim O(R).$$

This is a contradiction to both (a) and (b). We conclude that

$$\sum \frac{1}{|a_n|} = +\infty \quad \text{whenever (a) or (b) holds.}$$

$\blacksquare$

Lec 11 pp. 25, 31
Lec 13 p. 4

Since $\xi_0(z)$ had $\rho = 1$, $\tau = \infty$, we conclude at once that

$$\sum \frac{1}{|a_n|} = +\infty,$$

whereupon $\xi_0(z)$ must have infinitely many zeros (each lying in $0 < x < 1$).

IE

There is no need to use the $\rho = \frac{1}{2}$ trick following from $\xi_0(z + \frac{1}{2}) = \text{EVEN}$.

Lec 13 p. 4

One might also recall that, for $0 < \rho \leq 1$, an entire f has infinitely many zeros by Lec 13, p. 2. Here, of course, $\sum \frac{1}{|a_n|} < \infty$.

I then remarked that I. M. Vinogradov showed (with method of trigonometric sums) that

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma \geq 1 - c(\ln t)^{-2/3}$$

t large. [As noted in Ingham, 2^{nd} edition, p. xii, there is some question about this; only $(\ln t)^{-2/3}(\ln \ln t)^{-1/3}$ is properly justified.]

Taking $\eta(t) = \dfrac{1}{G(\ln t)^{2/3}}$ $\quad (t \geq e)$, Lec 14

leads to

$$\omega(x) = \min_{t \geq e} \left\{ \frac{\ln x / G}{(\ln t)^{2/3}} + \ln t \right\},$$

i.e. trivial calc for $u \geq 1$ on the fcn.

$$g(u) = \frac{\ln x / G}{u^{2/3}} + u \qquad \text{(compare Lec 14, (11))}$$

$\Rightarrow$ min for $u = (\text{const})(\ln x)^{3/5}$

if x = large

$\Rightarrow \omega(x) = (\text{const})(\ln x)^{3/5}$

$$\Downarrow$$

$$\boxed{\begin{aligned} \psi(x) - x &= O\left[x e^{-c(\ln x)^{3/5}}\right] \\ \pi(x) - \mathrm{li}(x) &= O\left[x e^{-c(\ln x)^{3/5}}\right] \end{aligned}}$$

with suitable $c > 0$. Compare Lec 14, 12.

The box needs a slight adjustment if only

$$\sigma \gtrapprox 1 - c(\ln t)^{-2/3}(\ln\ln t)^{-1/3} \quad \text{zerofree}$$

holds rigorously.

The "3/5" is about where things <u>still</u> lie <u>in 2016</u> unconditionally. On this point, note that:

$$x^{1/2} = x \cdot x^{-\frac{1}{2}} = x e^{-\frac{1}{2}(\ln x)^{1}}.$$

There is obviously "some" distance yet to go!!!

$$\underset{\sim 1900}{\frac{1}{2}} \longrightarrow \underset{\sim 1958}{\frac{3}{5}} \longrightarrow \underset{\text{when??}}{1}$$

The rest of the lecture was devoted to some important preparations for getting an explicit formula for $\psi_1(x)$.

Know

$$\xi_0(s) = e^{Bs} \prod_\rho \left(1-\frac{s}{\rho}\right) e^{s/\rho} \qquad B \in \mathbb{R}$$

$$\xi_0(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$\frac{\Gamma'(z)}{\Gamma(z)} = \operatorname{Log} z - \frac{1}{2z} - \sum_{k=1}^{R} \frac{B_{2k}}{2k} z^{-2k} + O_{R\delta}\left(|z|^{-2R-1}\right)$$

$$|\operatorname{Arg} z| \leqq \pi - \delta$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s} - \frac{1}{s-1} + \left(B + \frac{1}{2}\ln\pi\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

$$\operatorname{Im}(\rho) \neq 0, \quad 0 < \operatorname{Re}(\rho) < 1, \quad \rho = \beta + i\gamma$$

by virtue of

Lec 12 1 stirling Γ'/Γ

Lec 13 4 ~ 9

and $\frac{\Gamma'}{\Gamma}(1+z) = \frac{1}{z} + \frac{\Gamma'}{\Gamma}(z)$.

We also have :

$$\frac{\xi_0'(s)}{\xi_0(s)} = B + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

$$\frac{\xi'(s)}{\xi(s)} = -\left[\frac{1}{s} + \frac{1}{s-1} \right] + B + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

$$\left\{ \xi(s) \equiv \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \right\}$$

$$\sum \frac{1}{|\rho|^2} < \infty \quad , \quad \sum \frac{1}{\gamma^2} < \infty$$

by Lec 13 7, 5 (bottom), and $\mathrm{Im}(\rho) = \gamma \neq 0$.

THM

For large t,

$$N[\rho : |\gamma - t| \leqq 1] = O(\ln t).$$

Pf

We follow von Mangoldt's method.

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_2^\infty \frac{\Lambda(n)}{n^s} \qquad \text{Re}(s) > 1$$

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leqq \sum_2^\infty \frac{\Lambda(n)}{n^\sigma} \qquad \text{Re}(s) > 1$$

Keep $s = \sigma + it$, $t \geqq$ big G.

Keep $1 < \sigma \leqq 10$, say.

Apply Riemann's formula for $\frac{\zeta'}{\zeta}(s)$ on 6 bottom.

TAKE REAL PART ONLY!!!

Get

$$O(1) = O(1) + \overset{\text{Stirling}}{O(\ln t)} + \sum_\rho \left\{ \text{Re}\left(\frac{1}{s-\rho}\right) + \text{Re}\left(\frac{1}{\rho}\right) \right\}$$

$$O(\ln t) = \sum_{\substack{\text{all} \\ \rho}} \frac{\sigma - \beta}{(\sigma-\beta)^2 + (t-\gamma)^2} + \sum_{\substack{\text{all} \\ \rho}} \frac{\beta}{\beta^2 + \gamma^2}$$

but $\sigma - \beta > 0$ since $0 < \beta < 1$

$$\Downarrow$$

put $\sigma = 2$ to get

$$O(\ln t) = \sum_{\text{all } \rho} \frac{2-\beta}{(2-\beta)^2 + (t-\gamma)^2}$$

$$\Downarrow$$

$$\boxed{\sum_{\text{all } \rho} \frac{1}{1+(t-\gamma)^2} = O(\ln t).}$$

Restrict box to those terms with $|\gamma - t| \leqq 1$.

Thus,

$$N[|\gamma - t| \leqq 1] = O(\ln t),$$

as promised. {Valid for $\underline{\underline{t \geqq 2}}$ by inflation of constant.}

<u>Corollary</u>

$$N[\rho : 0 < \gamma \leqq T] = O(T \ln T), \quad T \geqq 2.$$

Pf

WLOG $T =$ giant.

By theorem, know that:

$$N[\tfrac{u}{2} < \gamma \leq u] = O(u \ln u)$$

for all $u \geq 4$ say. Now just apply the standard

$$\left\{ \frac{T}{2^{k+1}} < \gamma \leq \frac{T}{2^k} \right\}$$

summation. ∎

We now go back to $\frac{\zeta'(s)}{\zeta(s)}$ but don't take the real part!

The formula on 6 bottom is valid at $3+it$ and also for any $s = \sigma + it$ with $-1 \leq \sigma \leq 2$, $t \geq 6$, $t \neq$ any γ. Get:

$$\frac{\zeta'(s)}{\zeta(s)} = O\left(\frac{1}{t}\right) + O(\ln t) + \sum_{\rho} \left(\frac{1}{\sigma + it - \rho} + \frac{1}{\rho} \right)$$

$$\frac{\zeta'}{\zeta}(3+it) = O\left(\frac{1}{t}\right) + O(\ln t) + \sum_{\rho} \left(\frac{1}{3+it-\rho} + \frac{1}{\rho} \right)$$

{ Subtract }

$\Downarrow$

$$\frac{\zeta'(s)}{\zeta(s)} + O(1) = O\left(\frac{1}{t}\right) + O(\ln t) + \sum_{\rho}\left(\frac{1}{\sigma+it-\rho} - \frac{1}{3+it-\rho}\right)$$

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln t) + \sum_{|\gamma-t|>1}\left[\frac{3-\sigma}{(\sigma+it-\rho)(3+it-\rho)}\right] + \sum_{|\gamma-t|\leqq 1}\left[\frac{1}{\sigma+it-\rho} - \frac{1}{3+it-\rho}\right]$$

$$\left\{ \begin{array}{l} \underline{\text{but}}, \text{ for } |\gamma-t| \leqq 1, \\ \left|\frac{1}{3+it-\rho}\right| = \frac{1}{|(3-\beta)+i(t-\gamma)|} \leqq \frac{1}{3-\beta} \\ \leqq \frac{1}{2} \end{array} \right\}$$

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln t) + \sum_{|\gamma-t|>1}\left[\frac{3-\sigma}{(\sigma+it-\rho)(3+it-\rho)}\right] + \sum_{|\gamma-t|\leqq 1}\frac{1}{\sigma+it-\rho}$$

but, for $|\gamma - t| > 1$, $-1 \leqq \sigma \leqq 2$,

$$\left| \frac{3-\sigma}{(\sigma-\beta+i(t-\gamma))((3-\beta)+i(t-\gamma))} \right| \leqq \frac{4}{|t-\gamma|^2}$$

while, by 9 box,

$$\sum_{\text{all } \rho} \frac{1}{1+(t-\gamma)^2} = O(\ln t)$$

$$\Downarrow$$

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln t) + \sum_{\substack{\rho \\ |\gamma - t| \leqq 1}} \frac{1}{s-\rho} \, .$$

One remarks here that the "1" can be replaced (if convenient) by any positive constant. Just review the earlier steps!

Theorem (Very Important and Basic)

Let $-1 \leqq \sigma \leqq 2$ and t be large, with $t \neq$ all γ. We then have

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln t) + \sum_{\substack{\rho \\ |\gamma - t| \leqq 1}} \frac{1}{s-\rho}$$

for $s = \sigma + it$. The "1" can be replaced by any positive constant (as convenient).

Pf

As above. ∎

On p. 7, recall that $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ had $\xi(s) = \xi(1-s)$ and simple poles at $s = 1$ and $s = 0$.

Lec 11, p. ~~24~~ 23

Recall too:

$$\xi(s) = (s-q)^M \phi(s), \quad \phi(s) = \text{power series in } (s-q) \text{ with } \phi(q) \neq 0$$

$$\Rightarrow \frac{\xi'(s)}{\xi(s)} = \frac{M}{s-q} + [\text{analytic near } s = q].$$

And, remember that

$$\xi(x) \neq 0, \quad x \in \mathbb{R} \qquad \text{Lec 11 (27)}.$$

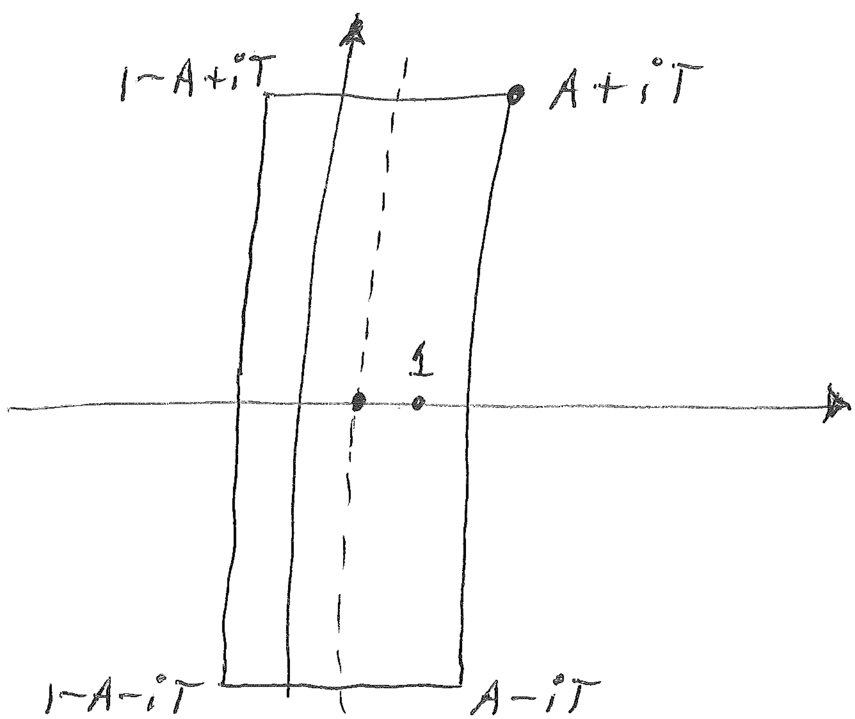
We propose to try to count the zeros of $\xi(s)$ — following Riemann.

Let $T \geq 2$, say, $T \neq$ all γ.

Let $A > 1$.

Draw the rectangle

$$R(A,T) = [1-A, A] \times [-T, T]$$



Put

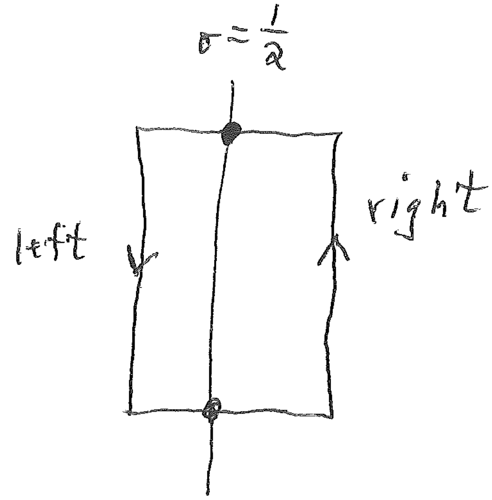
$$N(T) = N[\rho : 0 < \gamma \leq T] .$$

By 13 bottom and CRT, we get

$$\frac{1}{2\pi i} \oint_{\partial R(A,T)} \frac{\xi'(s)}{\xi(s)}\, ds = 2N(T) - 2 .$$

$\boxed{\xi(\bar{s}) = \overline{\xi(s)} \text{ by Lec 11, } 23}$

from $s = 0, 1$ simple poles of ξ



Note:

$$\frac{1}{2\pi i} \int_{\text{left}} \frac{\xi'(s)}{\xi(s)}\, ds \overset{s = 1-u}{=} \frac{1}{2\pi i} \int_{\substack{u\text{-image} \\ \text{of left}}} \frac{\xi'}{\xi}(1-u)\,(-du)$$

$$= -\frac{1}{2\pi i} \int_{\substack{u\text{-image} \\ \text{of left}}} \frac{\xi'}{\xi}(1-u)\, du$$

easily check u-image of "left" is exactly "right" (in the correct direction)

$$= -\frac{1}{2\pi i}\int_{\text{right}} \frac{\xi'}{\xi}(1-u)\,du \,.$$

But,

$$\xi(z) = \xi(1-z)$$

$$\Rightarrow \log \xi(z) = \log \xi(1-z)$$ suitable branches

$$\Rightarrow \frac{\xi'}{\xi}(z) = -\frac{\xi'}{\xi}(1-z) \,.$$

So:

$$\frac{1}{2\pi i}\int_{\text{left}} \frac{\xi'}{\xi}(s)\,ds = \frac{1}{2\pi i}\int_{\text{right}} \frac{\xi'(u)}{\xi(u)}\,du \,.$$

Hence: 15 line 4

$$2N(T) - 2 = \frac{2}{2\pi i}\int_{\text{right}} \frac{\xi'(s)}{\xi(s)}\,ds$$

$$N(T) = 1 + \frac{1}{2\pi i} \int_{\text{right}} \frac{\xi'(s)}{\xi(s)} ds .$$

Notice that $A > 1$ has <u>not</u> been specified yet in $R(A, T)$.

Put $\xi(s) = G(s)\zeta(s)$, $G(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$.

$$\Rightarrow \frac{\xi'(s)}{\xi(s)} = \frac{G'(s)}{G(s)} + \frac{\zeta'(s)}{\zeta(s)} .$$

The function $G(s)$ is analytic on $Re(s) > 0$ and <u>not</u> <u>zero</u>.

By CIT,

$$\frac{1}{2\pi i} \int_{\text{right}} \frac{G'}{G}(s) ds = \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{G'}{G}(s) ds .$$

So:

$$N(T) = 1 + \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{G'}{G}(s) ds + \frac{1}{2\pi i} \int_{\text{right}} \frac{\zeta'(s)}{\zeta(s)} ds .$$

To handle the $\frac{G'}{G}$ integral, recall our use of the <u>uniquely determined</u> branch $\mathcal{L}og\,\Gamma(z)$ near p. 42 of Lec 10. We had $\mathcal{L}og\,\Gamma(x) = \mathrm{Log}\,\Gamma(x) = \ln \Gamma(x)$ for $x > 0$.

We can therefore unambiguously declare

$$\log G(s) = -\frac{s}{2}\ln\pi + \mathcal{L}og\,\Gamma\left(\frac{s}{2}\right)$$

for <u>$\mathrm{Re}(s) > 0$</u>. At once:

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{G'}{G}(s)\,ds$$

$$= \frac{1}{2\pi i}\Big[\log G(s)\Big]_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}$$

$$\left\{\begin{array}{l} \text{but} \qquad\qquad\qquad \boxed{\sigma > 0} \\ G(\bar{s}) = \overline{G(s)} \\ \log G(\bar{s}) = \overline{\log G(s)} \\ \log G(u) = \ln|G(u)| + i\,\mathrm{Arg}\,G(u) \end{array}\right\}$$

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\frac{G'}{G}(s)\,ds$$

$$= \frac{1}{2\pi i}2i\,\mathrm{Arg}\,G(\tfrac{1}{2}+iT)$$

$$= \frac{1}{\pi}\mathrm{Arg}\,G(\tfrac{1}{2}+iT)$$ see (18) middle

$$= \frac{1}{\pi}\left[-\frac{T}{2}\ln\pi + \mathrm{Arg}\,\Gamma(\tfrac{1}{4}+i\tfrac{T}{2})\right].$$

So far, then, we have:

$$\boxed{N(T) = 1 + \frac{1}{\pi}\mathrm{Arg}\,G(\tfrac{1}{2}+iT) + \frac{1}{2\pi i}\int_{\text{right}}\frac{\zeta'(s)}{\zeta(s)}\,ds.}$$

This box clearly holds for any $T > 0$, $T \neq$ all γ. There was nothing used about $T \geqq 2$ yet.

We now PAUSE to apply Stirling to

$$\frac{1}{\pi}\left[-\frac{T}{2}\ln\pi + \operatorname{Arg}\Gamma\left(\frac{1}{4}+i\frac{T}{2}\right)\right].$$

Here we keep $T \gtrsim 2$ and imagine $T \gtrsim 6$ if necessary (along the way).

$$\operatorname{Arg}\Gamma\left(\frac{1}{4}+i\frac{T}{2}\right) = \operatorname{Im}\operatorname{Log}\Gamma\left(\frac{1}{4}+i\frac{T}{2}\right)$$

$$= \operatorname{Im}\Big[\left(\frac{1}{4}+i\frac{T}{2}-\frac{1}{2}\right)\operatorname{Log}\left(\frac{1}{4}+i\frac{T}{2}\right)$$
$$-\left(\frac{1}{4}+i\frac{T}{2}\right)$$
$$+\ln\sqrt{2\pi} + O\left(\frac{1}{T}\right)\Big]$$

{Lec 10 p. 42}

$$= \operatorname{Im}\Big[\left(-\frac{1}{4}+i\frac{T}{2}\right)\left\{\operatorname{Log}\left(i\frac{T}{2}\right)\left(1+\frac{1}{2iT}\right)\right\}$$
$$-\frac{1}{4}-i\frac{T}{2}+\ln\sqrt{2\pi}+O\left(\frac{1}{T}\right)\Big]$$

$$= \operatorname{Im}\Big[\left(-\frac{1}{4}+i\frac{T}{2}\right)\left\{\ln\left(\frac{T}{2}\right)+i\frac{\pi}{2}\right.$$
$$\left.+\operatorname{Log}\left(1+\frac{1}{2iT}\right)\right\}$$
$$-\frac{1}{4}-i\frac{T}{2}+\frac{1}{2}\ln(2\pi)+O(T^{-1})\Big]$$

$$\left\{ \mathrm{Log}\left(1+\frac{1}{2iT}\right) = \frac{1}{2iT} + O(T^{-2}) \right\}$$

$$= \frac{T}{2}\ln\frac{T}{2} - \frac{\pi}{8} + O(T^{-1})$$

$$- \frac{T}{2} + O(T^{-1})$$

$$= -\frac{\pi}{8} + \frac{T}{2}\ln\left(\frac{T}{2e}\right) + O(T^{-1}).$$

This is valid for $T \gtrsim 6$, then by constant inflation for $T \gtrsim 2$. Compare: Ingham 69 line 6.

Get:

$$\frac{1}{\pi}\operatorname{Arg} G\left(\tfrac{1}{2}+iT\right)$$

$$= \frac{1}{\pi}\left[-\frac{T}{2}\ln\pi + \operatorname{Arg}\Gamma\left(\tfrac{1}{4}+i\tfrac{T}{2}\right)\right] \qquad (19)$$

$$= -\frac{T}{2\pi}\ln\pi + \left(-\frac{1}{8}\right) + \frac{T}{2\pi}\ln\frac{T}{2e} + O(T^{-1})$$

$$= \frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) - \frac{1}{8} + O(T^{-1}),$$

$$T \gtrsim 2.$$

On 19, for $T \geqq 2$ ($T \neq$ all γ), we therefore have

$$\boxed{N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + O(T^{-1}) + \frac{1}{2\pi i}\int_{\text{right}} \frac{\zeta'(s)}{\zeta(s)}\,ds}.$$

To address $\frac{1}{2\pi i}\int_{\text{right}} \frac{\zeta'(s)}{\zeta(s)}\,ds$, we proceed in 2 ways.

First, recall that:

$$|\zeta(z) - 1| < 3 \cdot 2^{-x}, \quad x \geqq 2;$$

{ Lec 5 p. 10 }

$$\operatorname{Log} \zeta(z) = \sum_{2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-z}, \quad x > 1$$

{ Lec 6, p. 4 + 3 }

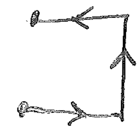
$$\operatorname{Log} \zeta(z) = O(2^{-x}), \quad x \text{ large};$$

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{2}^{\infty} \frac{\Lambda(n)}{n^z} \quad , \quad x > 1$$

$\{$ Lec 6, p. 6 $\}$

$$\frac{\zeta'(z)}{\zeta(z)} = O(2^{-x}) \quad , \quad x \text{ large.}$$

We can now <u>freeze</u> T and let $A \to \infty$ in $R(A, T)$ to get



$$\frac{1}{2\pi i} \int_{\text{right}} \frac{\zeta'}{\zeta}(s)\, ds = \frac{1}{2\pi i} \int_{\frac{1}{2}}^{\infty} \frac{\zeta'}{\zeta}(u - iT)\, du$$

$$+ \frac{1}{2\pi i} \int_{\infty}^{\frac{1}{2}} \frac{\zeta'}{\zeta}(u + iT)\, du$$

$$+ \; 0 \longleftarrow \{ 2^{-A} \to 0 \}$$

$$\left\{ \text{but } \frac{\zeta'}{\zeta}(u - iT) = \overline{\frac{\zeta'}{\zeta}(u + iT)} \right\}$$

$$= \frac{1}{2\pi i} \int_{\infty}^{\frac{1}{2}} \left[\frac{\zeta'}{\zeta}(u + iT) - \overline{\frac{\zeta'}{\zeta}(u + iT)} \right] du$$

$$= \frac{1}{2\pi i} \int_{\infty}^{\frac{1}{2}} 2i \,\mathrm{Im}\, \frac{\zeta'}{\zeta}(u + iT)\, du$$

$$= -\frac{1}{\pi}\int_{1/2}^{\infty} \operatorname{Im} \frac{\zeta'}{\zeta}(u+iT)\,du .$$

Thus,

$$\boxed{\frac{1}{2\pi i}\int_{\text{right}} \frac{\zeta'}{\zeta}(s)\,ds = -\frac{1}{\pi}\operatorname{Im}\int_{\frac{1}{2}}^{\infty} \underbrace{\frac{\zeta'}{\zeta}(u+iT)}_{O(2^{-u})}\,du .}$$

(with A fixed)

The 2^{nd} way is more basic. One starts with $\mathcal{L}og\ \zeta(s)$ on $Re(s) > 1$ (see 22) and forms an analytic continuation along the line segments $[\frac{1}{2}+iT, A+iT]$ and $[\frac{1}{2}-iT, A-iT]$ in an obvious way (starting at $A \pm iT$).

THIS IS LEGAL SINCE $T \neq$ all γ. No zeros of $\zeta(s)$ will be hit.

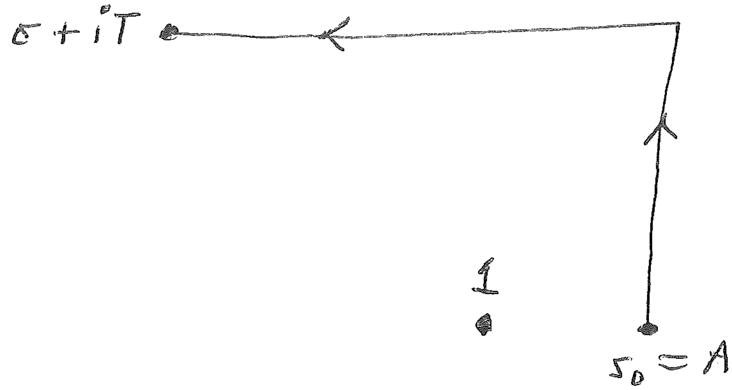
This branch of $\log \zeta(s)$ clearly satisfies

$$\log \zeta(\bar{s}) = \overline{\log \zeta(s)} .$$

SEE Lec 6, 12 line 10

[Analytic $f(z)$ with $f(x) \in \mathbb{R} \Rightarrow f(\bar{z}) \equiv \overline{f(z)}$.]

One typically says $\mathcal{L}og\,\zeta(s)$ has been found by continuing "up" from $s_0 = A$, then "across" to s from $A \pm iT$.



$(A > 1)$

With this convention,

$$\frac{1}{2\pi i}\int_{\text{right}} \frac{\zeta'}{\zeta}(s)\,ds = \frac{1}{2\pi i}\int_{\text{right}} d\left[\mathcal{L}og\,\zeta(s)\right]$$

$$= \frac{1}{2\pi i}\left[\mathcal{L}og\,\zeta(\tfrac{1}{2}+iT) - \mathcal{L}og\,\zeta(\tfrac{1}{2}-iT)\right]$$

$$= \frac{1}{2\pi i}\,\underline{2i}\,\mathrm{Im}\left[\mathcal{L}og\,\zeta(\tfrac{1}{2}+iT)\right] \Rightarrow$$

$$\boxed{\frac{1}{2\pi i}\int_{\text{right}} \frac{\zeta'}{\zeta}(s)\,ds = \frac{1}{\pi}\arg\zeta(\tfrac{1}{2}+iT)}$$

in an obvious "up and across" sense.

<u>THM</u> (Very important and basic) ← Essentially Riemann

Let $T > 0$, $T \neq$ any γ.

Let

$$\xi(s) = G(s)\zeta(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

We then have:

$$N(T) = N[\rho : 0 < \gamma \leq T]$$

$$= 1 + \frac{1}{\pi}\arg G\left(\tfrac{1}{2}+iT\right) + S(T)$$

wherein

$$S(T) = \frac{1}{\pi}\arg \zeta\left(\tfrac{1}{2}+iT\right) \quad \text{"up and across"}$$

$$= -\frac{1}{\pi}\operatorname{Im}\int_{1/2}^{\infty}\frac{\zeta'}{\zeta}(\sigma+iT)\,d\sigma$$

and $\arg G(\frac{1}{2}+iT)$ is defined àla Stirling.

For $T \gtrsim 2$, we have:

$$\frac{1}{\pi}\arg G\left(\tfrac{1}{2}+iT\right) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) - \frac{1}{8} + O\left(\frac{1}{T}\right).$$

C^{∞} fcn of T

Pf

As above. ▮

THM (Very Important and Basic)

Introduce $S(T)$ for $T > 0$, $T \neq$ all γ as on p. 26. We then have

$$S(T) = O(\ln T), \quad T \geqq 2.$$

Pf

Apply Thm on 13. Remember that

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \mathrm{Re}(s) > 1$$

$$= O(2^{-\sigma}), \quad \sigma \geqq 2.$$

See 23 (top). Get:

$$S(T) = -\frac{1}{\pi} \mathrm{Im} \int_{1/2}^{2} \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma$$

$$- \frac{1}{\pi} \mathrm{Im} \int_{2}^{\infty} \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma$$

$$= -\frac{1}{\pi}\,\mathrm{Im}\int_{1/2}^{2}\Big[O(\ln T) + \sum_{|\gamma - T| \leqq 1}\frac{1}{s-\rho}\Big]\,d\sigma$$

$$+ O(1)\int_{2}^{\infty} 2^{-\sigma}\,d\sigma$$

$\{$ here $s = \sigma + iT, \ \rho = \beta + i\gamma \}$

$$= O(\ln T)$$

$$-\frac{1}{\pi}\,\mathrm{Im}\sum_{|\gamma - T| \leqq 1}\left(\int_{1/2}^{2}\frac{1}{s-\rho}\,d\sigma\right)$$

$$+ O(1)$$

$\Bigg\{$ **but** $\displaystyle\int_{\frac{1}{2}+iT}^{2+iT}\frac{1}{s-\rho}\,ds, \quad T \neq \text{all } \gamma$

$\displaystyle = \underline{L}og(2+iT-\rho) - \underline{L}og(\tfrac{1}{2}+iT-\rho)$

$\Rightarrow$ imaginary part has absolute $\leqq \pi$ $\Bigg\}$

$$= O(\ln T) + O(1)\sum_{|\gamma - T| \leqq 1} 1 \ + O(1)$$

$= O(\ln T)$ by 8 THM. $\blacksquare$

THM ← stated by Riemann

$$N(T) = N[\rho : 0 < \gamma \leq T] \quad \leftarrow \text{definition}$$

$$= \frac{T}{2\pi} \ln\left(\frac{T}{2\pi e}\right) + O(\ln T)$$

for all $T \geq 2$. {Ingham p. 68 thm 25}

Proof

For $T \neq$ all γ, just combine 26 + 27.

If $T =$ some γ, just use the right continuity of $N(t)$ as a counting function. ∎

For later use, notice that:

$t = 2\pi u$

$u = t/2\pi$

$$\frac{d}{dt}\left(\frac{t}{2\pi} \ln\left(\frac{t}{2\pi e}\right)\right) = \frac{1}{2\pi} \frac{d}{du}\left(u \ln\left(\frac{u}{e}\right)\right)$$

$$= \frac{1}{2\pi} \ln u$$

$$= \frac{1}{2\pi} \ln\left(\frac{t}{2\pi}\right).$$

Lecture 16

(11 March)

$\boxed{c>1}$

We seek to use

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] ds \qquad (x \geq 1)$$

to get an explicit formula for $\psi_1(x)$. $\boxed{\text{Lec 7 p. 10}}$

We shall use an appropriate rectangle

$$R: [-\Delta, c] \times [-T, T]$$

and let $T \to \infty$, $\Delta \to \infty$.

Lemma $\boxed{\rho = \beta + i\gamma}$

(a) $\displaystyle\sum_{0<\gamma\leq T} \frac{1}{\gamma} = O(\ln^2 T)$ for $T \geq 2$.

(b) $\displaystyle\sum_{\gamma>T} \frac{1}{\gamma^2} = O\left(\frac{\ln T}{T}\right)$

Proof

We know $N(t) = \frac{t}{2\pi}\ln\left(\frac{t}{2\pi e}\right) + O(\ln t)$, $t \geq 2$, by Lec 15 p. 29. Recall that $N(t)$ is <u>right</u> continuous.

In both (a) and (b), WLOG $T \geq 1000$.

2

Write $N(t) = \frac{t}{2\pi}\ln\left(\frac{t}{2\pi e}\right) + R(t)$, $R(t)$ right continuous. For (a), get:

$$\sum_{0<\gamma\leq T}\frac{1}{\gamma} = O(1) + \int_2^T \frac{1}{t}\,dN(t)$$

$$= O(1) + \int_2^T \frac{1}{t}\left\{\frac{1}{2\pi}\ln\frac{t}{2\pi}\right\}dt \quad \leftarrow \text{Lec 15 p. 29}$$

$$+ \int_2^T \frac{1}{t}\,dR(t) \quad \leftarrow R(t) = O(\ln t)$$

$$= O(1) + O(\ln T)\int_2^T \frac{1}{t}\,dt$$

$$+ \frac{R(T)}{T} - \frac{R(2)}{2} - \int_2^T R(t)\frac{(-1)}{t^2}\,dt$$

$$= O(\ln^2 T) + \frac{O(\ln T)}{T} + O(1)$$

$$+ O(1)\int_2^T \frac{\ln t}{t^2}\,dt$$

$$= O(\ln^2 T) + O(1)\ln T \cdot \int_2^\infty \frac{1}{t^2}\,dt$$

$$= O(\ln^2 T). \quad \text{OK}$$

For (b),

$$\sum_{\gamma>T}\frac{1}{\gamma^2} = \int_T^\infty \frac{1}{t^2}\,dN(t) \quad \{\text{this is correct even if } T = \text{some } \gamma\}$$

(3)

$$= \int_T^\infty \frac{1}{t^2}\left\{\frac{1}{2\pi}\ln\frac{t}{2\pi}\right\}dt$$

$$+ \int_T^\infty \frac{1}{t^2}\, dR(t)$$

$$= O(1)\int_{T/2\pi}^\infty \frac{\ln u}{u^2}\,du$$

$$+ \left.\frac{R(t)}{t^2}\right]_T^\infty - \int_T^\infty R(t)(-2)t^{-3}\,dt$$

$$= O(1)\int_{\mathcal{U}}^\infty \ln u\; d\left(\frac{1}{u}\right) \qquad \mathcal{U} \equiv \frac{T}{2\pi}$$

$$+ O\left(\frac{\ln T}{T^2}\right) + O(1)\int_T^\infty \frac{\ln t}{t^3}\,dt$$

$$= O(1)\left[\left[\frac{\ln u}{u}\right]_{\mathcal{U}}^\infty - \int_{\mathcal{U}}^\infty \frac{1}{u}\frac{1}{u}\,du\right]$$

$$+ O\left(\frac{\ln T}{T^2}\right) + O(1)\int_T^\infty \frac{\ln t}{t^3}\,dt$$ ← we'll use parts again

$$= O(1)\frac{\ln T}{T} + O(1)\int_T^\infty \ln t\; d(t^{-2})$$

$$= O(1)\frac{\ln T}{T} + O(1)\left[\left[\frac{\ln t}{t^2}\right]_T^\infty - \int_T^\infty t^{-3}\,dt\right]$$

$$= O(1)\frac{\ln T}{T} + O(1)\frac{\ln T}{T^2} = O(1)\frac{\ln T}{T}. \quad \blacksquare$$

<u>Lemma</u>

For $m \geqq 2$, we can always find some

$$T_m \in (m, m+1)$$

so that

$$\left| \frac{\zeta'(\sigma + iT_m)}{\zeta(\sigma + iT_m)} \right| \leqq A_1 \ln^2 T_m \quad \text{for } -1 \leqq \sigma \leqq 2.$$

Here $A_1 =$ a suitable absolute constant.

<u>Pf</u>

WLOG $m \geqq 1000$. $\quad (\ln 1000 = 6.90^+)$

By Lec 15 Thm p. 8, see also p. 29, we know:

$$N[m-2 \leqq \gamma \leqq m+2] = O(\ln m).$$

Write this as

$$N[m-2 \leqq \gamma \leqq m+2] \leqq B \ln m.$$

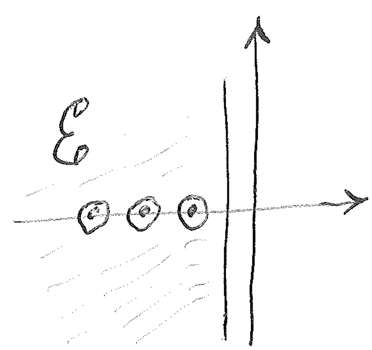
WLOG $B \geqq 1$. Divide $(m, m+1]$ into $2[\![B \ln m]\!]$ <u>equal</u> left-open subintervals. Some interval must therefore contain NO γ. Let $T_m =$ midpoint of <u>this</u> subinterval. By construction,

$$|\gamma - T_m| \geq \frac{1}{4\cdot 2[\![B\ln m]\!]} \geq \frac{1}{8B\ln m}$$

for all γ. Apply Lec 15, p. 13 (the partial fraction thm). With $t = T_m$, we clearly get

$$\frac{\zeta'}{\zeta}(\sigma + iT_m) = O(\ln T_m) + O(\ln m)\cdot O(\ln m)$$
$$= O(\ln^2 T_m)$$

for $-1 \leq \sigma \leq 2$. ▥



Lemma

Consider the domain

$$\mathcal{E} = \{\mathrm{Re}(s) < -1\} - \bigcup_{k=1}^{\infty} \{|s+2k| \leq \tfrac{1}{2}\}.$$

We have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq A_2 \ln(|s| + 10)$$

for $s \in \mathcal{E}$. Here A_2 = suitable absolute constant.

Pf

Recall the functional equation of $\xi(s)$, $\zeta(s)$.
Get:

$$\zeta(s) = \frac{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)}{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}\zeta(1-s)$$

$$= \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(1-s)$$

$$\left\{ \text{but } \Gamma(s) = 2^{s-1}\pi^{-1/2}\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(\tfrac{s}{2}+\tfrac{1}{2}\right) \quad \begin{matrix}\text{Lec 9}\\ \text{p. 30 (d)}\end{matrix} \right\}$$

$$= \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)}{\pi^{1/2}2^{1-s}\Gamma(s)}\zeta(1-s)$$

$$= \pi^{s-1}2^{s-1}\frac{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)}{\Gamma(s)}\zeta(1-s)$$

$$\left\{ \text{but } \Gamma\left(\tfrac{1}{2}-\tfrac{s}{2}\right)\Gamma\left(\tfrac{1}{2}+\tfrac{s}{2}\right) = \frac{\pi}{\sin\pi\left(\frac{1}{2}-\frac{s}{2}\right)} \quad \begin{matrix}\text{Lec 9}\\ \text{p. 30 (c)}\end{matrix} \right\}$$

$$= \pi^{s-1}2^{s-1}\frac{1}{\Gamma(s)}\frac{\pi}{\cos\frac{\pi s}{2}}\zeta(1-s)$$

$$\Rightarrow \zeta(1-s) = \pi^{-s}2^{1-s}\Gamma(s)\cos\frac{\pi s}{2}\cdot\zeta(s)$$

$\Downarrow$

Ingham p. 41

$$\boxed{\zeta(1-s) = 2\cdot(2\pi)^{-s}\cos\frac{\pi s}{2}\cdot\Gamma(s)\,\zeta(s)}\,.$$

Here s = any generic value in $\mathbb{C}$. Take logarithmic derivatives to get

$$-\frac{\zeta'}{\zeta}(1-s) = -\ln 2\pi - \frac{\pi}{2}\tan\frac{\pi s}{2} + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)}$$

flip $s \hookrightarrow 1-s$

$\Downarrow$

$$-\frac{\zeta'(s)}{\zeta(s)} = -\ln 2\pi - \frac{\pi}{2}\operatorname{ctn}\frac{\pi s}{2} + \frac{\Gamma'(1-s)}{\Gamma(1-s)} + \frac{\zeta'}{\zeta}(1-s)$$

$\Downarrow$

$$\boxed{\frac{\zeta'(s)}{\zeta(s)} = \ln 2\pi + \frac{\pi}{2}\operatorname{ctn}\frac{\pi}{2}s - \frac{\Gamma'}{\Gamma}(1-s) - \frac{\zeta'}{\zeta}(1-s)}\,.$$

Ingham 73

Recall that $\pi \operatorname{ctn} \pi z$ is periodic $z \to z+1$,

$$\pi \operatorname{ctn} \pi z = \lim_{N \to \infty} \sum_{-N}^{N} \frac{1}{z-n},$$

and

$$|\operatorname{ctn} \pi z + i| = O(e^{-2\pi y}) \quad \text{for } y \geq 1.$$

similarly $y \leq -1$

See Lec 9, pp. 3 (D) (A), 5 THM.

For $s \in \mathcal{E}$, p. 7 2nd box gives:

$$\frac{\zeta'(s)}{\zeta(s)} = O(1) + O(1) + O(1)\left|\frac{\Gamma'}{\Gamma}(1-s)\right|$$

$$\boxed{\left|\frac{\zeta'(z)}{\zeta(z)}\right| \leq \sum_{2}^{\infty} \frac{\Lambda(n)}{n^x}, \quad x \geq 2}$$

$$= O(1) + O(1)\left|\operatorname{Log}(1-s) + O(1)\right|$$

$\boxed{\text{Stirling, Lec 12, p. 1}}$

$$= O(1) + O(1)\,|\ln|1-s||$$

note $|1-s| > 2$ on $\mathcal{E}$

$$\leqq O(1) + O(1)\ln(|s|+10)$$

$|s| > 1$ on $\mathcal{E}$

$$\leqq O(1)\ln(|s|+10),$$

as was to be proved. ■

For our rectangle R on ① we take

$$c = 2$$
$$\Delta = 2m+1, \quad m \text{ big}$$
$$T = T_m.$$

We know that

$$\psi_1(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] ds.$$

Here $x \geqq 1$. Notice that

$$\left|\int_{2+iT_m}^{2+i\infty} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] ds\right| \leqq \int_{T_m}^{\infty} \frac{x^3}{t^2} O(1)\,dt$$

$$= O(1)\frac{x^3}{T_m}.$$

Similarly for $\int_{2-i\infty}^{2-iT_m}$. Thus,

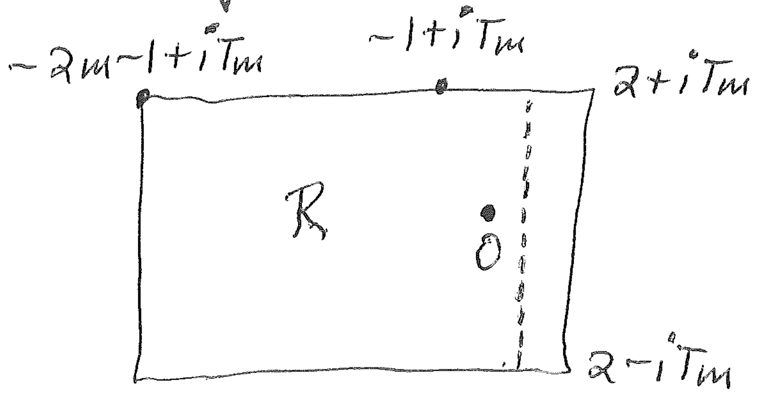
$$\psi_1(x) + O\left(\frac{x^3}{T_m}\right) = \frac{1}{2\pi i}\int_{2-iT_m}^{2+iT_m}.$$

By Cauchy Residue Thm, we have:

$$\frac{1}{2\pi i}\oint_{\partial R} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] ds$$

$$= \text{Res}(\text{at } 1) + \text{Res}(\text{at } 0) + \text{Res}(\text{at } -1)$$

$$+ \sum_{k=1}^{m} \text{Res}(\text{at } -2k) + \sum_{|\gamma|<T_m} \text{Res}(\text{at } \rho)$$



$$\Downarrow$$

$$\frac{1}{2\pi i}\oint_{\partial R} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] ds$$

Lec 9, p. 20

$$= \frac{x^2}{2} + x^1\left[-\frac{\zeta'(0)}{\zeta(0)}\right] + x^0\left[\frac{\zeta'(-1)}{\zeta(-1)}\right]$$

$$+ \sum_{k=1}^{m} (-1)\frac{x^{1-2k}}{(2k)(2k-1)}$$

$$+ \sum_{|\gamma|<T_m} (-1)\frac{x^{\rho+1}}{\rho(\rho+1)}$$

$\rho = \beta + i\gamma$
as usual
$0 < \beta < 1$

Note that:

$$LHS = \psi_1(x) + O\left(\frac{x^3}{T_m}\right)$$

$$+ \frac{1}{2\pi i}\int_{\substack{\text{horiz} \\ t=T_m}} + \frac{1}{2\pi i}\int_{\substack{\text{vertical} \\ \sigma=-2m-1}}$$

$$+ \frac{1}{2\pi i}\int_{\substack{\text{horiz} \\ t=-T_m}}.$$

See (10) bottom.

Apply (4) + (5) to [horiz, $t = T_m$]. Get:

$$\int_{\substack{\text{horiz}\\ t=T_m}} = O(1)\int_{-2m-1}^{-1} \frac{x^{1+\sigma}}{T_m^2}\ln m\, d\sigma$$

$$+\, O(1)\int_{-1}^{2}\frac{x^{1+\sigma}}{T_m^2}\underline{\ln^2 m}\, d\sigma$$

$\{\ |x^{s+1}| = x^{\sigma+1}$ and $x \geqq 1\ \}$

uses $x \geqq 1$

$$= O(1)\,\frac{1}{m^2}(\ln m)\,O(m)$$

$$+\, O(1)\,\frac{x^3}{m^2}\ln^2 m$$

$$= O(1)\,\frac{\ln m}{m} + O(1)\,x^3\,\frac{\ln^2 m}{m^2}.$$

Similarly for [horiz, $t = -T_m$].

Apply 5 to [vertical, $\sigma = -2m-1$]. Get:

$$\int_{\substack{\text{vert} \\ \sigma=-2m-1}} = O(1)\int_{-T_m}^{T_m} \frac{x^{1+(-2m-1)}}{m^2}\ln m \, dt$$

$\boxed{x \geq 1}$

$$\leq O(1)\int_{-T_m}^{T_m} \frac{1}{m^2}\ln m \, dt$$

$$= O(1)\frac{\ln m}{m}.$$

We conclude that on 11 bottom:

$$\frac{1}{2\pi i}\oint_{\partial R} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] ds$$

$$= \psi_1(x) + O\left(\frac{x^3}{T_m}\right)$$

$$+ O(1)\,x^3\left(\frac{\ln m}{m}\right)^2 + O(1)\frac{\ln m}{m}.$$

Combining 11 top with 13 bottom, we get

$$\psi_1(x) + O(1)\frac{\ln m}{m} + O\left(\frac{x^3}{m}\right)$$

$$= \frac{x^2}{2} + Ax + B$$

$$+ \sum_{k=1}^{m} (-1) \frac{x^{1-2k}}{(2k)(2k-1)}$$

$$+ \sum_{|\gamma| < T_m} (-1) \frac{x^{\rho+1}}{\rho(\rho+1)}$$

$$\left\{ \begin{aligned} A &= -\frac{\zeta'(0)}{\zeta(0)} \\ B &= \frac{\zeta'(-1)}{\zeta(-1)} \end{aligned} \right\}$$

$$\left\{ \text{but } \sum \frac{1}{|\rho|^2} < \infty \right\}$$

$$\Downarrow \longleftarrow \boxed{\text{LET } m \to \infty}$$

$$\psi_1(x) = \frac{x^2}{2} + Ax + B - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{(2k)(2k-1)}$$

$$- \sum_{\text{all } \rho} \frac{x^{\rho+1}}{\rho(\rho+1)}$$

for <u>EACH</u> $x \geq 1$, both series ABS conv.

compare Lec 7 p. 10 thm

Remark.

One definitely wants to keep $x \gtrsim 1$.
Indeed, for $0 < x < 1$, IE $\frac{1}{x} > 1$, we notice that

$$\sum_{k=1}^{\infty} \frac{x^{-2k}}{(2k)(2k-1)} = +\infty .$$

THM (Riemann's explicit formula for $\psi_1(x)$)

For each $x \gtrsim 1$, we have

$$\psi_1(x) = \frac{x^2}{2} + Ax + B - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{(2k)(2k-1)} - \sum_{\text{all } \rho} \frac{x^{\rho+1}}{\rho(\rho+1)}$$

wherein $A = -\frac{\zeta'(0)}{\zeta(0)}$, $B = \frac{\zeta'(-1)}{\zeta(-1)}$.

Ingham 73

Pf

As above. See 14 bottom. ▨

Important Procedural Remark.

To keep things completely clear logically, notice that our proof of p. 15 THM technically only relied on $0 \leqq \beta \leqq 1$. IE, we did not need to know $\zeta(1+iy) \neq 0$.

To verify this, observe that certain "expungements" can be safely made:

Lec 6 pp. 6–8 Hadamard
(note that pp. 9–21 do not use 6–8)

Lec 7 pp. 5–9 (middle), 15–23 using $1/\zeta(z)$
(NOTE Lec 7 pp. 9 (bot) – 14 on $\psi_1(x)$ is OK)

Lec 8 pp. 1–4, 10–13 (top) on $\psi(x)$ + PNT
(NOTE Lec 9+10 is E–M, no $1/\zeta(z)$.)
(NOTE Lec 11 got functional eq for $\zeta(z)$, never needed $1/\zeta(z)$.)

In Lec 13, p. 4 THM, state only that $0 \leqq Re(\rho) \leqq 1$. Expunge $0 < Re(\rho) < 1$. Also on p. 5 in connection with HFT.

Expunge Lec 13 pp. 9 (bot) – 15, plus all of Lec 14 (related to zerofree regions).

With these expungements, a quick review shows that Lec 15 goes thru perfectly well — knowing only that $0 \leqq \beta \leqq 1$ and $\mathrm{Im}(\rho) \neq 0$.

Pages 1 – 15 above are then recovered without difficulty.

This being said, we can now get a "new" proof of the PNT as follows:

- Develop the explicit formula for $\psi_1(x)$ as on p. 15. (Note that this requires the CRT.)
- Do the Hadamard trick to get $\zeta(1+iy) \neq 0$. See Lec 6, pp. 6 – 8.
- Use the functional equation of $\xi_0(z)$ to get $0 < \beta < 1$. See Lec 11, pp. 24–25, also 27.
- Choose R so big that $\sum_{|\gamma| > R} \frac{1}{|\rho|^2} < \varepsilon$.
- Exploit the explicit formula to get

$$\limsup_{x \to \infty} \left| \frac{\psi_1(x)}{x^2} - \frac{1}{2} \right| \leqq 0 + 0 + 0 + \limsup_{x \to \infty} \left| \sum_\rho \frac{x^{\rho-1}}{\rho(\rho+1)} \right|$$

(continued)

R held fixed

$$\leq 0 + \limsup_{x \to \infty} \sum_{|\gamma| \leq R} \frac{x^{\beta - 1}}{|\rho||\rho+1|}$$

$$+ \limsup_{x \to \infty} \sum_{|\gamma| > R} \frac{x^{\beta-1}}{|\rho||\rho+1|}$$

$\{$ <u>but</u> $|\rho+1| \geq |\rho|$ since $\beta \geq -\frac{1}{2}$ $\}$

$\beta < 1$

$$\leq 0 + 0 + \limsup_{x \to \infty} \sum_{|\gamma| > R} \frac{1}{|\rho|^2}$$

$$< \varepsilon .$$

- Hence $\psi_1(x) \sim \frac{x^2}{2}$ and we can repeat Lec 8 pp. 1–3. ■

This proof corresponds to Ingham 82 (middle).

Loosely Put:

Explicit Formula for $\psi_1(x)$ plus $\zeta(1+iy) \neq 0$, $y \in \mathbb{R}$, immediately implies the PNT.

It is now customary to define

$$\boxed{\Theta = \sup\{\mathrm{Re}(\rho)\}.}$$

The Riemann Hypothesis is equivalent to stating that $\Theta = \frac{1}{2}$. Obviously $\frac{1}{2} \leqq \Theta \leqq 1$.

THM

$$\psi_1(x) = \frac{x^2}{2} + O(x^{\Theta+1}) \quad \text{for large } x.$$

PF

Obvious from p. 15 Thm since $\sum_\rho \frac{1}{|\rho|^2} < \infty$.

////

THM (Very Basic and Interesting)

$$\psi(x) = x + O(x^\Theta \ln^2 x)$$

$$\pi(x) = li(x) + O(x^\Theta \ln x).$$

Here $li(x) \equiv \int_2^x \frac{dt}{\ln t}$.

Corollary

Assume RH. Then:

$$\psi(x) = x + O(x^{\frac{1}{2}} \ln^2 x)$$

$$\pi(x) = li(x) + O(x^{\frac{1}{2}} \ln x).$$

These have never been improved.

Proof of Theorem

Know

$$\psi_1(x) = \frac{x^2}{2} + Ax + B + E(x) - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)},$$

$$E(x) \equiv -\sum_{k=1}^{\infty} \frac{x^{1-2k}}{(2k)(2k-1)}$$

by p. 15.

Note that $E(x) = b_1 x^{-1} + b_3 x^{-3} + b_5 x^{-5} + \cdots$ is a nice power series in x^{-1}.

Also know: Lec 14, p. 5

$$\frac{\psi_1(x) - \psi_1(x-h)}{h} \lessapprox \psi(x) \lessapprox \frac{\psi_1(x+h) - \psi_1(x)}{h} \qquad (x \text{ large})$$

for all $1 \lessapprox h \leq \frac{x}{2}$ (say).

Look at upper part of the inequality.

$$\frac{\frac{(x+h)^2}{2} - \frac{x^2}{2}}{h} = x + \frac{h}{2}$$

$$\frac{A(x+h) + B - Ax - B}{h} = A$$

$$\frac{E(x+h) - E(x)}{h} = E'(x+\tilde{h}) \;, \quad 0 < \tilde{h} < h$$

$$= O(x^{-2}) \quad \text{by Taylor series}$$

$$\left| \frac{(x+h)^{p+1} - x^{p+1}}{hp(p+1)} \right| \lessapprox \frac{(x+h)^{\Theta+1} + x^{\Theta+1}}{h\,\gamma^2}$$

{very crude}

$$\lessapprox \frac{(\text{constant})\, x^{\Theta+1}}{h\,\gamma^2}$$

---

less crudely,

$$\left| \frac{(x+h)^{p+1} - x^{p+1}}{h\,p(p+1)} \right| = \frac{1}{h} \left| \int_x^{x+h} \frac{u^p}{p}\, du \right|$$

$$\left\{ \begin{array}{c} \text{no ambiguity: } u^s \equiv \exp\{s \ln u\} \\ u > 1 \end{array} \right\}$$

$$\leqq \frac{1}{h} \frac{1}{|p|} \int_x^{x+h} u^{\Theta}\, du$$

$$\left\{ \tfrac{1}{2} \lessapprox \Theta \lessapprox 1 \right\}$$

$$\lessapprox \frac{1}{h} \frac{1}{|\gamma|} (x+h)^{\Theta} h$$

$$\leqq \frac{(\text{constant})\, x^{\Theta}}{|\gamma|}$$

Hence,

$$\left| \frac{(x+h)^{p+1} - x^{p+1}}{h\,p(p+1)} \right| \lessapprox (\text{const}) \min\left[\frac{x^{\Theta+1}}{h\gamma^2} , \frac{x^{\Theta}}{|\gamma|} \right]$$

$$\leq (\text{const}) \frac{x^{\Theta}}{|\gamma|} \min\left(\frac{x}{h|\gamma|}, 1\right)$$

$$= (\text{const}) \frac{x^{\Theta}}{|\gamma|} \begin{cases} 1 & \text{if } |\gamma| < \frac{x}{h} \\ \frac{x}{h|\gamma|} & \text{if } |\gamma| > \frac{x}{h} \end{cases}$$

We thus get:

$$\psi(x) \leq x + \frac{h}{2} + A + O(x^{-2})$$

$$+ O(1) \sum_{|\gamma| < \frac{x}{h}} \frac{x^{\Theta}}{|\gamma|}$$

$$+ O(1) \sum_{|\gamma| > \frac{x}{h}} \frac{x^{\Theta+1}}{h\gamma^2} .$$

The lower part of 20 bot will give similar; simply replace $x + \frac{h}{2}$ by $x - \frac{h}{2}$.

Get:

$$\psi(x) = x + O(h) + O(1) + O(x^{-2})$$
$$+ O(1) \sum_{|\gamma| < \frac{x}{h}} \frac{x^{\Theta}}{|\gamma|}$$
$$+ O(1) \sum_{|\gamma| > x/h} \frac{x^{\Theta+1}}{h\gamma^2}.$$

Here $1 \leq h \leq \frac{x}{2}$ and p. 1 LEMMA applies.

$$\psi(x) = x + O(h) + O(1) x^{\Theta} \ln^2\left(\frac{x}{h}\right)$$
$$+ O(1) x^{\Theta+1} \frac{1}{h} \frac{\ln(x/h)}{x/h}$$

$$= x + O(h) + O(1) x^{\Theta} \ln^2\left(\frac{x}{h}\right)$$
$$+ O(1) x^{\Theta} \ln\left(\frac{x}{h}\right) \leftarrow$$ LIKE A MIRACLE

$$= x + O(h) + O(1) x^{\Theta} \ln^2\left(\frac{x}{h}\right)$$

We get $\psi(x) = x + O(x^{\Theta} \ln^2 x)$ with $h = 1$!

Do (related) calculus problem for safety:

$$\text{let } h \equiv \frac{x}{t}, \quad 2 \lesssim t \lesssim x$$

$$h + x^{\theta} \ln^2\left(\frac{x}{h}\right) \equiv \frac{x}{t} + x^{\theta} \ln^2(t)$$

$$\underline{\theta = 1} \Rightarrow \quad x\left[\frac{1}{t} + \ln^2 t\right] \qquad \Rightarrow (\text{const})\, x$$

minimum at $t = 2$

$$\underline{\theta < 1} \;(x \text{ large}) \quad x^{\theta}\left[\frac{x^{1-\theta}}{t} + \ln^2 t\right]$$

deriv of bracket:

$$-\frac{x^{1-\theta}}{t^2} + \frac{2\ln t}{t} < 0$$

iff

$$\frac{2\ln t}{t} < \frac{x^{1-\theta}}{t^2}$$

iff

$$2t\ln t < x^{1-\theta}$$

$$\Rightarrow t_{critical} \sim \frac{\frac{1}{2}x^{1-\theta}}{(1-\theta)\ln x}$$

$$\Rightarrow \text{bracket min is} \approx (1-\theta)^2 \ln^2 x$$

$$\Rightarrow \text{OVERALL } (\text{const})\, x^{\theta} \ln^2 x.$$

We now continue via

$$\Pi(x) - li(x) = \frac{\psi(x) - x}{\ln x} + O(1)$$
$$+ \int_2^x \frac{\psi(t) - t}{t(\ln t)^2}\, dt$$

$$\Pi(x) = \pi(x) + \sum_{n=2}^{\infty} \frac{1}{n} \pi(x^{1/n})$$

à la Lec 14 pp. 8 + 7 + 10 (bottom)

$$\Downarrow$$

$$\pi(x) - li(x) = O\left(\frac{x^{1/2}}{\ln x}\right) + \frac{\psi(x) - x}{\ln x}$$
$$+ \int_2^x \frac{\psi(t) - t}{t(\ln t)^2}\, dt$$

$$|\pi(x) - li(x)| \lessapprox O\left(\frac{x^{1/2}}{\ln x}\right) + O(1)\, \frac{x^{\Theta} \ln^2 x}{\ln x}$$
$$+ O(1) \int_2^x \frac{t^{\Theta} \ln^2 t}{t(\ln t)^2}\, dt$$

$$\leq O(1)\, x^{\theta} \ln x + O(1) \int_2^x t^{\theta-1}\, dt$$

$$= O(1)\, x^{\theta} \ln x + O(1) \frac{1}{\theta} x^{\theta}$$

$$= O(1)\, x^{\theta} \ln x \,. \qquad \blacksquare$$

2 HW problems

[1] Prove rigorously that, for large x, the number of primes in $(1, x]$ <u>exceeds</u> that in $(x, 2x]$.

compare Lec 2 p. 20

[2] Regarding Legendre and Ingham p. 2 (bottom).
Prove that there is exactly one constant C such that

$$\left| \pi(x) - \frac{x}{\ln x - C} \right| = O\left(\frac{x}{\ln^3 x}\right)$$

and that value is 1.

Lectures 17 and 18

(23 and 25 March)

Before proceeding to the *explicit formula* for $\psi(x)$, we treat an important connection between Θ (Lec 16, p. 19) and the PNT.

Let $\Theta' = \inf\{w > 0 : \psi(x) - x = O(x^{w}),\ x \geq 2\}$.

Thm $\boxed{\text{Ingham 84}}$

$$\Theta' = \Theta.$$

Pf

By Lec 16 p. 19 2nd Thm, $\Theta' \leq \Theta$.

Suppose now that $\psi(x) = x + O(x^{w})$, $x \geq 2$. $\boxed{w > 0}$

For $\mathrm{Re}(s) > 1$, we immediately check

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty u^{-s}\,d\psi(u) = s\int_1^\infty \psi(u)\,u^{-s-1}\,du$$

$$\frac{s}{s-1} = s\int_1^\infty u \cdot u^{-s-1}\,du$$

so

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} - 1 = s\int_1^\infty \frac{\psi(u) - u}{u^{s+1}}\,du.$$

see Lec 8 p. 11

The RHS is <u>analytic</u> wrt s for $Re(s) > \omega$.

Lec 7 p. 17

Note that $\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}$ has a removable singularity at $s = 1$. We thus find that

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \text{analytic for } Re(s) > \omega .$$

Clearly, for any ρ with $\Xi_0(\rho) = 0$,* we must then get $Re(\rho) \leqq \omega$. Hence: $\Theta \leqq \omega$. Hence $\Theta \leqq \Theta'$. ▮

In the near future, we will improve the theorem on p. 1.

One interprets p. 1 THM as saying (loosely) that Θ <u>controls</u> the size of the remainder term in $\psi(x) - x$ or $\pi(x) - li(x)$. See here Lec 14 p. 8 BOX and 10 bottom. Also Lec 16 pp. 26 – 27. All of this will (soon) be improved/sharpened.

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* Recall Lec 13, p. 4 THM. <u>Also</u>: Lec 8, p. 10.

The discussion that I gave (in Lec 17+18) of the explicit formula for $\psi(x)$ can be seen as something having 2 basic stages:

(A) an initial "fleshing it out" in the style of Landau; (Landau, Vorlesungen über Z...)

(B) tightening that up — and strengthening it.

I follow the same procedure in these notes, but make some slight changes to streamline (AND strengthen) things.

We had

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$

à la Lec 7. Here $c>1$, $x>0$. By a purely formal differentiation wrt x, one expects that

$$\psi(x) \text{ is (basically) associated with } \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\left(-\frac{\zeta'}{\zeta}\right) ds.$$

Danger exists here since, for each x,

$$\int_{c-i\infty}^{c+i\infty} \frac{x^c}{|s|}(1)\,|ds| = +\infty .$$

Ie absolute convergence fails! $\boxed{:(}$

By Lec 7, we expect to study

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds$$

for $y > 0$.

<u>Fundamental Lemma</u> (Perron + others)

Keep $0 < y < \infty$, $0 < c \leqq 2$, $T \geqq 3$.

Let

$$\eta(y) = \left\{ \begin{array}{ll} 1, & y > 1 \\ 1/2, & y = 1 \\ 0, & 0 < y < 1 \end{array} \right\}.$$

We then have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds = \eta(y) + \text{Remainder},$$

where

$$|\text{Remainder}| \lesssim \begin{Bmatrix} \dfrac{y^c}{\pi T\,|\ln y|} \,, & y \neq 1 \\[2ex] \dfrac{c}{\pi T} \,, & y = 1 \end{Bmatrix}.$$

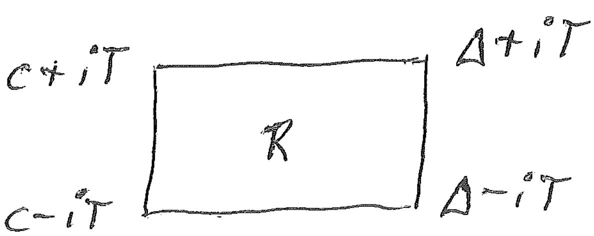
One also has, if one is willing to be quite crude,

$$|\text{Remainder}| \lesssim O_A(1)$$

for $0 < y < A$, with an "implied" constant which depends solely on A.

Proof

Take $0<y<1$ first. Look at $\dfrac{1}{2\pi i}\displaystyle\int_{\partial R} \frac{y^s}{s}\,ds$ on



$(s = \sigma + it)$

with T frozen and let $\Delta \to \infty$. Note that $y^\Delta \to 0$ along $[\Delta - iT, \Delta + iT]$. By the Cauchy Integral Thm, we get

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} = \frac{1}{2\pi i}\int_{c-iT}^{\infty - iT} + \frac{1}{2\pi i}\int_{\infty + iT}^{c+iT}.$$

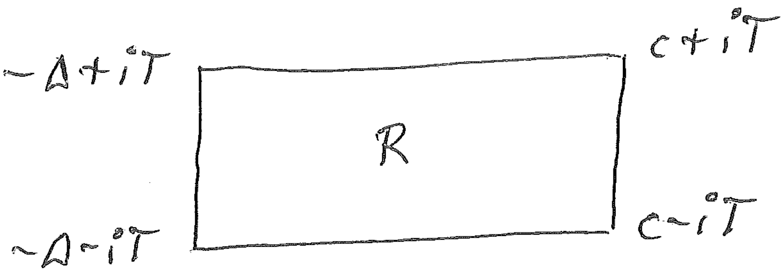
At once,

$$|LHS| \lessapprox \frac{2}{2\pi}\int_c^\infty \frac{y^\sigma}{|s|}\,d\sigma \qquad (s = \sigma \pm iT)$$

$$\lessapprox \frac{1}{\pi T}\int_c^\infty e^{-\sigma|\ln y|}\,d\sigma$$

$$= \frac{1}{\pi T}\frac{e^{-c|\ln y|}}{|\ln y|} = \frac{1}{\pi T}\frac{y^c}{|\ln y|}.$$

Take $1 < y < \infty$ next. Here we use $\frac{y^s}{s}$ and



$(s = \sigma + it)$.

Notice that $\operatorname{Res}\left\{\frac{y^s}{s};\ s=0\right\} = 1$ (simple pole).
Freeze T and let $\Delta \to \infty$ again. By the
Cauchy Residue Theorem (or Cauchy Integral Formula), get:

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} = 1 - \frac{1}{2\pi i}\int_{-\infty-iT}^{c-iT} - \frac{1}{2\pi i}\int_{c+iT}^{-\infty+iT}$$

$$= 1 - \frac{1}{2\pi i}\int_{-\infty-iT}^{c-iT} + \frac{1}{2\pi i}\int_{-\infty+iT}^{c+iT}.$$

At once,

$$|\text{Remainder}| \lesssim \frac{2}{2\pi}\int_{-\infty}^{c} \frac{y^\sigma}{|s|}\,d\sigma \qquad (s = \sigma \pm iT)$$

$$\lesssim \frac{1}{\pi T}\int_{-\infty}^{c} e^{\sigma(\ln y)}\,d\sigma \qquad (y>1)$$

$$= \frac{1}{\pi T}\frac{e^{c(\ln y)}}{\ln y} = \frac{y^c}{\pi T(\ln y)} .$$

For $y=1$, we notice that $(0<c\leq 2)$

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{1^s}{s}\,ds = \frac{1}{2\pi i}\left[\operatorname{Log} s\right]_{c-iT}^{c+iT}$$

$$= \frac{1}{2\pi i}\left[\operatorname{Log}(c+iT) - \overline{\operatorname{Log}(c+iT)}\right]$$

$$= \frac{1}{\pi}\operatorname{Arg}(c+iT)$$

$$= \frac{1}{\pi}\arctan\frac{T}{c}$$

$$= \frac{1}{\pi}\left[\int_0^\infty \frac{dq}{1+q^2} - \int_{T/c}^\infty \frac{dq}{1+q^2}\right]$$

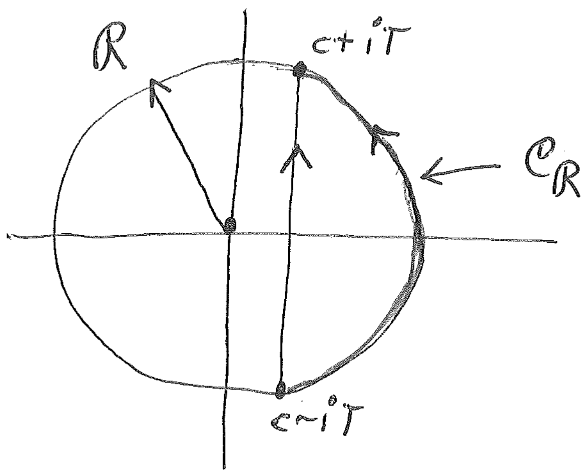
$$= \frac{1}{2} - \frac{1}{\pi}\int_{T/c}^\infty \frac{dq}{1+q^2}$$

$$\Downarrow$$

$$|\text{Remainder}| < \frac{1}{\pi}\frac{1}{(T/c)} = \frac{c}{\pi T} .$$

To conclude, we assume that $0 < y < A$ for some $A \geqq 2$ (wlog).

For $0 < y < 1$, look at



radius $R = \sqrt{c^2 + T^2}$

and get

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} = \frac{1}{2\pi i}\int_{\mathcal{C}_R} \frac{y^s}{s}\,ds \qquad \text{by CIT}$$

$\Rightarrow$

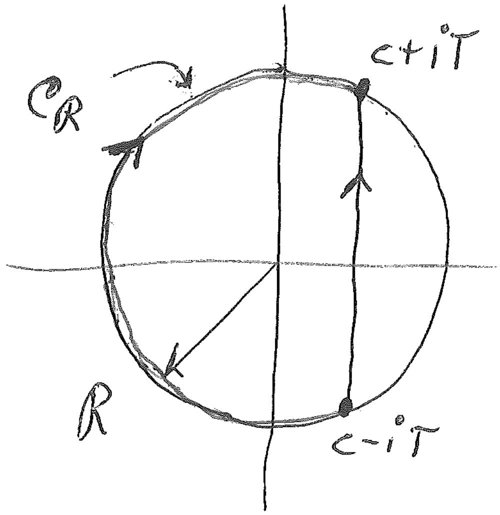
$$|\text{Remainder}| \lessapprox \frac{1}{2\pi}\int_{\mathcal{C}_R} \frac{y^\sigma}{|s|}|ds|$$

$$\lessapprox \frac{1}{2\pi}\, y^c\, \frac{1}{R}(\pi R)$$

$$\lessapprox \frac{1}{2}$$

since $0 < c \lessapprox 2$ (and $0 < y < 1$).

Next, consider $1 \leq y < A$. Use



$R = \sqrt{c^2+T^2}$

and get

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} = 1 + \frac{1}{2\pi i}\int_{C_R} \frac{y^s}{s}\,ds$$

$\eta(1) = \frac{1}{2}!$

$\Rightarrow$

$$|\text{Remainder}| \leq \frac{1}{2} + \frac{1}{2\pi}\int_{C_R} \frac{y^\sigma}{|s|}\,|ds|$$

$$\leq \frac{1}{2} + \frac{1}{2\pi} y^c \frac{1}{R}(2\pi R)$$

$$\leq \frac{1}{2} + y^c$$

{but $0 < c \leq 2$ and $1 \leq y < A$}

$$\leq \frac{1}{2} + A^2 .$$

$\blacksquare$

Corollary of Lemma 4

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds = \eta(y) = \begin{cases} 1, & y>1 \\ 1/2, & y=1 \\ 0, & y<1 \end{cases}$$

for $0<y<\infty$, $0<c\leq 2$.

Guided by p. 3 bottom, we now turn our attention to

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]ds$$

under the hypothesis that

$$x \geq 1+\delta_0 \quad \text{AND} \quad 1<c\leq 2 .$$

Here δ_0 is some small positive constant (e.g. $1/2$) which we fix once and for all.

> **N.B.** Landau, Vorlesungen über Z... likes $c=2$ and arbitrary $x>1$.

Since we plan to use p. 4 Fund Lemma, we also insist that

$$T \geqq 3.$$

Conceptually, in phase (A) on p. 3, it is simplest to declare $c = 2$ [for instance] like Landau does, and then stay with that.

To facilitate making improvements, we prefer however [in contrast to what was done in the lectures] to keep c free for the time being.

Let $\mathcal{F}$ be the set $\{p^m : p = \text{prime}, m = \text{pos. integer}\}$.
Obviously $\mathcal{F} \subseteq \mathbb{Z} \cap [2, \infty)$.

Let $\|u\|' = \left\{ \begin{matrix} |u|, & \text{if } u \neq 0 \\ \infty, & \text{if } u = 0 \end{matrix} \right\}$. Here $u \in \mathbb{R}$.

For $x \geqq 1 + \delta_0$, let $\xi(x) = \min\limits_{\lambda \in \mathcal{F}} |x - \lambda|$. Also write

$$\langle x \rangle = \min\left\{ \frac{1}{100}, \|\xi(x)\|' \right\}.$$

Notice that $\langle x \rangle = \frac{1}{100}$ unless $\|\xi(x)\|' < \frac{1}{100}$, which would mean that $x \notin \mathcal{F}$ but x lies LESS THAN $\frac{1}{100}$ units from $\mathcal{F}$.

In particular, we see that:

$$\langle x \rangle = \frac{1}{100} \quad \text{anytime } x \in \mathbb{Z}.$$

In all cases, obviously:

$$0 < \langle x \rangle \leqq \frac{1}{100}.$$

Lemma 2

Keep $x \geqq 1+\delta_0$, $1 < c \leqq 2$, $T \geqq 3$.

Let

$$\psi^*(x) = \frac{\psi(x+0) + \psi(x-0)}{2}.$$

We then have:

(i) $$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds = \psi^*(x) + O\left[\frac{x^c \ln x}{T(c-1)}\right] + O\left[\frac{x \ln x}{T\langle x\rangle}\right];$$

(ii) $$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds = \psi^*(x) + O\left[\frac{x^c \ln x}{T(c-1)}\right] + O(\ln x).$$

The implied constants will depend on [at most] δ_0.

Proof

We use p. 4 LEMMA. Get:

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right) ds$$

$$= \sum_{1}^{\infty}\Lambda(n)\,\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{(x/n)^s}{s}\,ds$$

$$= \sum_{1}^{\infty}\Lambda(n)\left\{\eta\left(\frac{x}{n}\right) + \begin{bmatrix} \dfrac{\Omega}{\pi T}\dfrac{(x/n)^c}{|\ln x - \ln n|}, & n \neq x \\ \dfrac{2\Omega}{\pi T}, & n = x \\ O_A(1), & \dfrac{x}{A} < n < \infty \end{bmatrix}\right\}.$$

Here $|\Omega| \leq 1$. Clearly

$$\sum_{1}^{\infty}\Lambda(n)\,\eta\left(\frac{x}{n}\right) = \psi^{*}(x).$$

We first look at (i). This will use chunks #1 and #2 in the big bracket.

Make the obvious

$$1 \leq n \leq [\![x]\!] - 1$$

$$n = [\![x]\!] \longleftarrow n_1$$

$$n = [\![x]\!] + 1 \longleftarrow n_2$$

$$n \geq [\![x]\!] + 2$$

splitting.

I

$$\sum_{n=[\![x]\!]+2}^{\infty} \Lambda(n) \frac{\Omega}{\pi T} \frac{(x/n)^c}{|\ln x - \ln n|}$$

$$= \sum_{n=[\![x]\!]+2}^{2[\![x]\!]+4} + \sum_{n=2[\![x]\!]+5}^{\infty}$$

Must bound the expressions in absolute value!

$n \geq 2[\![x]\!] + 5 \implies n \geq 2([\![x]\!]+1) > 2x$.
So, $\ln(n/x) > \ln 2$. Get:

$$\leq \frac{x^c}{T} \sum_{n=2[\![x]\!]+5}^{\infty} \frac{\Lambda(n)}{n^c} \frac{1}{\ln 2}$$

$$= O\left(\frac{x^c}{T}\right)\left[\frac{1}{c-1} + O(1)\right] = O\left(\frac{x^c}{T}\right)\frac{1}{c-1} \quad ||$$

since $\quad -\dfrac{\zeta'(\sigma)}{\zeta(\sigma)} = \sum_{1}^{\infty} \dfrac{\Lambda(n)}{n^\sigma} = \dfrac{1}{\sigma-1} + O(1)$ for $1 < \sigma \lesssim 2$. ② top.

Next, $[\![x]\!]+2 \lesssim n \lesssim 2[\![x]\!]+4$. Get:

$$\lesssim \frac{1}{T} \sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \Lambda(n)\left(\frac{x}{n}\right)^c \frac{1}{\ln n - \ln x}$$

$$\lesssim \frac{1}{T} O(\ln(10x)) \sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{1}{\ln n - \ln x}$$

$$\left\{\begin{array}{l} \text{apply mean value thm to } \ln n - \ln x; \\ \text{get final sum} \\ \qquad \lesssim O(1) \displaystyle\sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{1}{\frac{1}{x}(n-x)} \\ \qquad \lesssim O(1)\, x \cdot O(1) \ln(10x) \; ; \\ \text{note here that this estimate is} \\ \text{trivial if, say, } x \lesssim 1000 \end{array}\right\}$$

$$\lesssim \frac{x}{T} O(1)(\ln 10x)^2 \lesssim \frac{x}{T} O(1) \ln^2 x.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow x \gtrsim 1+\delta_0$

But:

$$\frac{x^c}{c-1} \gtrsim (\text{const})\, x \ln x \qquad \text{for } x \gtrsim 1+\delta_0.$$

$\boxed{\text{TRICK}}$

This follows by elem calculus ; the "const" will depend on δ_0.

$$\left\{ \begin{array}{l} \text{For } x > e \text{, the fcn } \dfrac{x^v}{v} \text{ on } 0 < v \leqq 1 \\ \text{has its MIN at } v = \dfrac{1}{\ln x} . \end{array} \right\}$$

just like Lec 6 p. 21 lines 2-10

Accordingly : for $x \geqq 1 + \delta_0$

$$\frac{x^c \ln x}{T(c-1)} \geqq (\text{const}) \frac{x \ln^2 x}{T} .$$

Hence,

$$\sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{1}{T} \Lambda(n) \left(\frac{x}{n}\right)^c \frac{1}{\ln n - \ln x}$$

$$= O(1) \frac{x^c \ln x}{T(c-1)} . \quad \|$$

The "implied" constant will depend on δ_0.

Pause For A Moment!

In Lec 17 with $c = 2$, what I remarked following Landau was that :

(proceeding somewhat crudely...)

$$\sum_{[\![x]\!]+2}^{\infty} \frac{1}{T}\Lambda(n)\left(\frac{x}{n}\right)^2 \frac{1}{\ln n - \ln x}$$

$$= O\left(\frac{x^2}{T}\right) + \sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{1}{T}\Lambda(n)\frac{x^2}{n^2}\frac{1}{\ln n - \ln x}$$

$$= O\left(\frac{x^2}{T}\right) + \frac{x^2}{T}\sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{\Lambda(n)}{n^2}\frac{1}{(\text{const})\frac{1}{x}(n-x)}$$

$$= O\left(\frac{x^2}{T}\right) + \frac{x^2}{T}\sum_{[\![x]\!]+2}^{2[\![x]\!]+4} \frac{\ln(10x)}{x^2}\frac{(\text{const})x}{n-x}$$

$$\leqq O\left(\frac{x^2}{T}\right) + (\text{const})\frac{x^2}{T}\ln(10x)\cdot O(1)$$

$$= O(1)\frac{x^2 \ln x}{T} \qquad \text{for } x \geqq 1+\delta_0$$

END of PAUSE. $\left\{ \text{See Landau, } \underline{\text{Vorlesungen}}, \text{ proof of Satz 451 near (564).} \right\}$

Step (II).
$$\sum_{n=1}^{[\![x]\!]-1} \Lambda(n)\frac{\Omega}{\pi T}\frac{(x/n)^c}{(\ln x - \ln n)}$$

$$= \sum_{n<\frac{1}{2}[\![x]\!]} + \sum_{\frac{1}{2}[\![x]\!] \leqq n \leqq [\![x]\!]-1}$$

Expect things to be similar to step (I).

At once,

$$\sum_{n<\frac{1}{2}[\![x]\!]} \Lambda(n) \frac{1}{T} \frac{(x/n)^c}{(\ln \frac{x}{n})}$$

$$\leqq \sum_{n<\frac{1}{2}[\![x]\!]} \Lambda(n) \frac{1}{T} \frac{x^c}{n^c} \frac{1}{\ln 2}$$

$$= O\left(\frac{x^c}{T}\right) \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^c}$$

$$= O\left(\frac{x^c}{T}\right) \left[\frac{1}{c-1} + O(1) \right] \leqq O\left(\frac{x^c}{T(c-1)}\right).$$

Like 15 bottom.

Also,

$$\sum_{\frac{1}{2}[\![x]\!] \leqq n \leqq [\![x]\!]-1} \Lambda(n) \frac{1}{T} \frac{(x/n)^c}{\ln x - \ln n}$$

$$\leqq \frac{O(\ln(10x))}{T} (\text{const}) \sum_{\frac{1}{2}[\![x]\!] \leqq n \leqq [\![x]\!]-1} \frac{x}{x-n}$$

$$\leqq O(1) \frac{x \ln(10x)}{T} \cdot \ln(10x)$$

$$\leqq \frac{x}{T} O(1) \ln^2 x \qquad \text{for } x \geqq 1+\delta_0 .$$

Line 6 on 17 is still valid, so we get:

$$\sum_{\frac{1}{2}[\![x]\!] \leq n \leq [\![x]\!]-1} \frac{1}{T}\Lambda(n)\left(\frac{x}{n}\right)^{c} \frac{1}{\ln x - \ln n}$$

$$= O(1)\frac{x^{c}\ln x}{T(c-1)} .$$

Recall $n_1 = [\![x]\!]$, $n_2 = [\![x]\!]+1$ on (15) top.

By (14), we must still check that:

LHS

$$\sum_{n\in\{n_1,n_2\}} \Lambda(n)\begin{bmatrix} \frac{1}{T}\frac{(x/n)^{c}}{|\ln x - \ln n|} &, n \neq x \\ \frac{1}{T} &, n = x \end{bmatrix}$$

$$= O\left[\frac{x^{c}\ln x}{T(c-1)}\right] + O\left[\frac{x\ln x}{T\langle x\rangle}\right].$$

Step III A

Suppose first that $x =$ integer (≥ 2).

Here $\langle x\rangle = \frac{1}{100}$ by (12); $n_1 = x$, $n_2 = x+1$;

and

$$\text{LHS} = \Lambda(n_1)\frac{1}{T} + \Lambda(n_2)\frac{1}{T}\frac{(x/n_2)^{c}}{|\ln x - \ln n_2|}$$

(21)

$$= \frac{O(\ln x)}{T} + \frac{O(\ln(x+1))}{T} \frac{1}{\ln(x+1) - \ln x}$$

$$= \frac{O(\ln x)}{T} + \frac{O[x \ln(x+1)]}{T} \quad \left\{ \begin{array}{l} \text{by mean} \\ \text{value thm} \end{array} \right\}$$

$$= \frac{O(x \ln x)}{T},$$

which is subsumed by **both** $\frac{x^c \ln x}{T}$ and $\frac{x \ln x}{T\langle x \rangle}$. OK III A

Step III B

Suppose next that $x \neq$ integer $(x \geq 1 + \delta_0)$.

For **each** j, notice that (in LHS on (20)):

$$\Lambda(n_j) \frac{1}{T} \left(\frac{x}{n_j}\right)^2 \frac{1}{|\ln x - \ln n_j|}$$

$$= \Lambda(n_j) \frac{O(1)}{T} \frac{x}{|x - n_j|} \quad \left\{ \begin{array}{l} \text{by mean} \\ \text{value thm} \end{array} \right\}.$$

If $|x - n_j| \geq \frac{1}{100}$, the foregoing bound is

ie, TERM

$$O(\ln x)\,\frac{x}{T}\,(100) \;=\; O\left[\frac{x\ln x}{T}\right].$$

This is obviously subsumed by $O\left[\frac{x\ln x}{T\langle x\rangle}\right]$.

$$\boxed{\langle x\rangle \lessapprox \frac{1}{100} \quad \text{by } 12}$$

On the other hand, suppose $|x - n_j| < \frac{1}{100}$.
The TERM on 21 bottom is either

$$0 \qquad \text{or else} \qquad O(\ln x)\,\frac{O(1)}{T}\,\frac{x}{\langle x\rangle}$$

by recalling the def of $\langle x\rangle$ again. Here, then, we again get something subsumed by $O\left[\frac{x\ln x}{T\langle x\rangle}\right]$.

Both cases in the "either" can occur so the $O\left[\frac{x\ln x}{T\langle x\rangle}\right]$ is essentially sharp.

Bottom Line:

$$\left[\begin{array}{c}\text{for } x \neq \text{integer} \\ \text{LHS on } 20\end{array}\right] = O(1)\,\frac{x\ln x}{T\langle x\rangle}.$$

OK

III B

By the 2 OK's, we get: in III

$$\sum_{n \in \{n_1, n_2\}} \Lambda(n) \begin{bmatrix} \frac{1}{T} \frac{(x/n)^c}{|\ln x - \ln n|} & , \; n \neq x \\ \frac{1}{T} & , \; n = x \end{bmatrix}$$

$$= O(1) \frac{x \ln x}{T\langle x \rangle} \, . \quad ////$$

All told, chunks #1 and #2 in the big bracket on 14 lead to an error term [à la 15, 17, 19, 20, and line 3 above] of

$$O(1) \frac{x \ln x}{T\langle x \rangle} + O(1) \frac{x^c \ln x}{T(c-1)} \quad \text{for } x \gtrsim 1 + \delta_0 \, ,$$

where the "implied constants" depend on δ_0.

Assertion (i) on p. 13 is thus proved.

13

One expects that (ii) will be VERY similar — with use of chunk #3 in the big bracket on 14 at an appropriate point.

Note that steps I and II are OK <u>as</u> <u>is</u>!

We need only Fiddle with step III. See 20 and 14 bracket.

$$n_1 = [\![x]\!] \,,\quad n_2 = [\![x]\!]+1 \qquad \left\{ \begin{array}{l} n_j \text{ is relevant} \\ \underline{\text{only}}\ \underline{\text{if}}\ \Lambda(n_j) \neq 0 \end{array} \right\}$$

Want $\frac{x}{A} < n_j < \infty$

$$\Downarrow$$

$A = 1$ OK for $j = 2$

for $j=1$, n_1 is <u>relevant</u> only if $[\![x]\!] \geqq 2$
whereupon $A = 2$ is adequate

$$\Downarrow$$

NEW step III

$$= \sum_{n \in \{n_1, n_2\}} \Lambda(n) \left[O_{\underline{2}}(1) \text{ wlog} \right]$$

$$\leqq O(1) \ln(x+1) \leqq O(1) \ln x \,, \qquad \|$$

for $x \geqq 1 + \delta_0$.

Page 13 assertion (ii) follows at once. ∎

<u>In</u> <u>Lec</u> <u>17</u>, following Landau $(c=2)$,
we got $O\left(\frac{x^2 \ln x}{T}\right) + O(\ln x)$
for assertion (ii).

To continue, consider $\{x, c, T\}$ as on 10 + 11 top.
Assume that

$$T \neq \text{all } \gamma. \qquad \boxed{\rho = \beta + i\gamma}$$

Form $\overset{\text{(rectangle)}}{R}$ as on p. 6 with $A = 2m+1$. Treat $\{x, c, T\}$ as frozen for a few moments. By letting $m \to \infty$, we immediately get

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds + \frac{1}{2\pi i}\int_{c+iT}^{-\infty+iT}$$

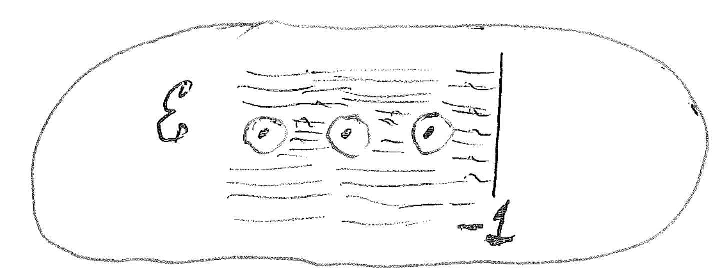
$$+ \frac{1}{2\pi i}\int_{-\infty-iT}^{c-iT} = \sum \text{Res}$$

$$= x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{k=1}^{\infty} \frac{x^{-2k}}{(-2k)} - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho}$$

thanks to

$$\frac{\zeta'(s)}{\zeta(s)} = O(1)\ln(|s| + 10) \quad \text{for } s \in \mathcal{E},$$

a fact proved in Lec 16, pp. 5 ~ 9.



Unfreeze
x, c, T!

The horizontal integrals have $s = \sigma \pm iT$ and are absolutely convergent (since $x > 1$).

To estimate them, it suffices to look at

$$\left| \frac{1}{2\pi i} \int_{c+iT}^{-\infty+iT} \frac{x^s}{s} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds \right| .$$

For $-1 \leq \sigma \leq 2$, we know that

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln T) + \sum_{|\gamma - T| \leq 1} \frac{1}{s-\rho}$$

when $s = \sigma + iT$. See Lec 15 p. 13. One inflates the implied constant in $O(\ln T)$ to handle moderate T.

The portion of the horiz integral arising from $(-\infty + iT, -1 + iT]$ is clearly

$$O(1) \int_{-\infty}^{-1} \frac{x^\sigma}{|s|} O(\ln|s|)\, d\sigma \qquad \left\{ \begin{array}{l} \text{by } \frac{\zeta'}{\zeta} \text{ estimate} \\ \text{over } \mathcal{E} \end{array} \right\}$$

$$\leq O(1) \frac{\ln T}{T} \int_{-\infty}^{-1} x^\sigma \, d\sigma$$

$$= O(1) \frac{\ln T}{T} \frac{x^{-1}}{\ln x} \leq O(1) \frac{\ln T}{T} x^{-1}, \qquad \{x \geq 1 + \delta_0\}.$$

To treat the stretch $[-1+iT, c+iT]$, we must look at

$$\left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s} \Big[O(\ln T) + \sum_{|\gamma - T| \leq 1} \frac{1}{s-\rho} \Big] ds \right|$$

wherein $s = \sigma + iT$. We stress that the bracket is a <u>continuous</u> function of σ.

The portion with $O(\ln T)$ clearly gives:

$$O(\ln T) \int_{-1}^{c} \frac{x^\sigma}{T} d\sigma$$

$$= O\left(\frac{\ln T}{T}\right) \int_{-1}^{c} x^\sigma d\sigma$$

$$= O\left(\frac{\ln T}{T}\right) \int_{-\infty}^{c} x^\sigma d\sigma$$

$$= O\left(\frac{\ln T}{T}\right) \frac{x^c}{\ln x}$$

$$= O\left(\frac{\ln T}{T}\right) x^c \qquad \{x \geq 1 + \delta_0\}.$$

Compare 26 bottom.

To handle the rest, we look at

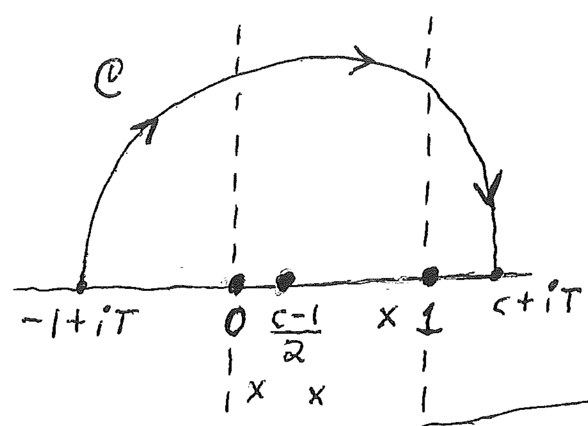
$$\sum_{\substack{\rho \\ |\gamma - T| \leq 1}} \left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s(s-\rho)}\, ds \right|$$

using the Cauchy integral theorem on each integral separately.

> We will not seek the best possible estimate (especially for c very close to 1), just something that leads to an explicit formula for $\psi(x)$ of a quality comparable to the best that is currently known for practical use.

Remember that $1 < c \leq 2$. Also that $\rho = \beta + i\gamma$ has $0 \leq \beta \leq 1$. We propose to deform $[-1+iT, c+iT]$.

$T - 1 \leq \gamma < T \implies$



radius $\frac{c+1}{2}$
center $\frac{c-1}{2}$

> A rectangle of height $\frac{1}{2}(c+1)$ can also be used.

x = typical $\beta + i\gamma$'s

$T < \gamma \lesssim T+1 \Rightarrow$ make similar $\mathcal{C}$, but go <u>down</u>.

For $s \in \mathcal{C}$, notice first that:

$$|s-\rho| \gtrapprox |s-(\beta+iT)| \,.$$

Elem geometry shows that the sliding circle

$$\{\, |s-(\eta+iT)| = \tfrac{1}{2}(c-1) \,\}$$

never intersects $\mathcal{C}$ for $0 \leq \eta \leq 1$. [At $\eta = 0$, $\frac{c-1}{2} < 1$.]

Accordingly, $s \in \mathcal{C} \Rightarrow |s-\rho| \gtrapprox \frac{1}{2}(c-1)$ for $1 < c \leq 2$.

For <u>EACH</u> ρ, get:

$$\left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s(s-\rho)}\, ds \right|$$

$$= \left| \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{x^s}{s(s-\rho)}\, ds \right| \qquad \text{(by CIT)}$$

$$\lessapprox \frac{1}{2\pi} \int_{\mathcal{C}} \frac{x^\sigma}{|s||s-\rho|} |ds|$$

$$\lessapprox \frac{1}{2\pi} \int_{\mathcal{C}} \frac{x^c}{(T-\frac{3}{2})\frac{1}{2}(c-1)} |ds|$$

$$\lessapprox (\text{const}) \frac{x^c}{T} \frac{1}{c-1} (\pi c + \pi)$$

$$\lessapprox (\text{const}) \frac{x^c}{T} \frac{1}{c-1} \quad //$$

Since $N[\rho : |\gamma - T| \leqq 1] = O(\ln T)$ by Lec 15 p. 8, we now get:

$$\left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s} \left[\sum_{|\gamma - T| \leqq 1} \frac{1}{s-\rho} \right] ds \right| \leftarrow \text{on } 27$$

$$\leqq O(\ln T) \frac{x^c}{T} \frac{1}{c-1} .$$

By 27 bottom, we get:

$$\left| \frac{1}{2\pi i} \int_{-1+iT}^{c+iT} \frac{x^s}{s} \left[O(\ln T) + \sum_{|\gamma - T| \leqq 1} \frac{1}{s-\rho} \right] ds \right|$$

$$= O\left(\frac{\ln T}{T}\right) \frac{x^c}{c-1} \; // .$$

By 26, we thus see that:

$$\frac{1}{2\pi i} \int_{-\infty \pm iT}^{c \pm iT} \frac{x^s}{s} \left(- \frac{\zeta'(s)}{\zeta(s)} \right) ds = O\left(\frac{\ln T}{T}\right) \frac{x^c}{c-1} \; //$$

for $1 < c \leqq 2$, $x \geqq 1 + \delta_0$, $T \geqq 3$, $T \neq$ all γ.

Referring to 25 middle and 13, we find that:

$$\psi^*(x) + O\left[\frac{x^c \ln x}{T(c-1)}\right] + \left\{\begin{matrix} O(\ln x) \\ \\ O\left[\frac{x \ln x}{T\langle x\rangle}\right] \end{matrix}\right\}$$

$$+ \; O\left(\frac{\ln T}{T}\right)\frac{x^c}{c-1} \quad \longleftarrow \text{by } 30$$

$$= x - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{x^{-2k}}{k} - \sum_{|\gamma| \leqq T}\frac{x^{\rho}}{\rho}$$

for $x \geqq 1+\delta_0$, $T \geqq 3$, $T \neq$ all γ, $1 < c \leqq 2$.

Notice that, for <u>any</u> small $h \in (0,1)$,

$$\left|\sum_{|\gamma - t| \leqq h}\frac{x^{\rho}}{\rho}\right| \leqq O(\ln t)\frac{x}{t}$$

$$\leqq O\left(\frac{\ln t}{t}\right)x$$

anytime $t \geqq \frac{5}{2}$. Similarly for $|\gamma + t| \leqq h$.
Here $x \geqq 1+\delta_0$, as usual.

Holding $\{x, c\}$ fixed for a moment, we can now apply <u>right continuity</u> in T to address cases on 31 top where $T \geq 3$, but $T =$ some γ.

GET: (for <u>any</u> $T \geq 3$)

$$\psi^*(x) = x - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \ln(1 - x^{-2}) - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho}$$

$$+ O\left(\frac{\ln T}{T}\right) \frac{x^c}{c-1}$$

$$+ O\left(\frac{x^c \ln x}{T(c-1)}\right) + \left\{ \begin{matrix} O(\ln x) \\ \\ O\left[\frac{x \ln x}{T\langle x \rangle}\right] \end{matrix} \right\}$$

for $x \geq 1 + \delta_0$, $1 < c \leq 2$.

$\boxed{\text{NB}}$

Compare: Ingham, p. 77, wherein $c = 2$,

AND Landau, <u>Vorlesungen</u>, Satz 452 (wherein $c = 2$). Note how our formula above is <u>better</u> <u>than</u> both, via taking $c = 3/2$.

To optimize 32 middle, one basically wants to <u>minimize</u>

$$\frac{x^c}{c-1},$$

i.e., put $c \approx 1 + \frac{b}{\ln x}$ where b = a tiny constant. Here $x \geqq 1 + \delta_0$ and we need to have $1 < c \leqq 2$. See 17 top. Since c is free in 32 middle, this choice of c is <u>completely legal</u>.

For $x \geqq 1 + \delta_0$, $T \geqq 3$, we thus get:

$$\psi^*(x) = x - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln(1 - x^{-2}) - \sum_{|\gamma| \leqq T} \frac{x^\rho}{\rho}$$

$$+ O\left(\frac{x \ln T \cdot \ln x}{T}\right) + O\left(\frac{x \ln^2 x}{T}\right)$$

$$+ O(\ln x) \min\left\{1, \frac{x}{T\langle x \rangle}\right\}.$$

The implied constants will depend [solely] on δ_0.

The "trivial terms"

$$-\frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln(1-x^{-2}) \qquad \longleftarrow \text{ on } 33$$

define an obvious power series in x^{-2} and are often simply replaced by $O(1)$. Insofar as that is done, once any given remainder term on 33 bot drops below Ωx^{-6}, say, (with $|\Omega| \leqq 1$) that term takes on a patently secondary role vis à vis x. ⟵ $\boxed{\text{especially if } x \to \infty}$

That being said, we now observe that:

(A) for $x \geqq 1+\delta_0$ and $T \gtrsim 3x^{10}$ (say),

$$\frac{x \ln T \cdot \ln x}{T} + \frac{x \ln^2 x}{T} \leqq \frac{x(\ln T + \ln x)^2}{T}$$

$$= O(x^{-8}) \quad ;$$

(B) for $x \geqq 1+\delta_0$ and $3 \leqq T \lessapprox 3x^{10}$,

$$\underline{c(\delta_0)}\,\frac{x}{T}(\ln T + \ln x)^2 \lessapprox \frac{x}{T}(\ln T \cdot \ln x + \ln^2 x) \leqq \frac{x}{T}(\ln T + \ln x)^2,$$

this final chunk scales like $\langle 1, 144\rangle \frac{x \ln^2 x}{T}$ if $x \geqq 3$

where $c(\delta_0)$ is some appropriately tiny positive constant.

<u>Theorem</u> (Standard statement of Explicit Formula for $\psi(x)$)

p^m set

Let $x \geq 1+\delta_0$, $T \geq 3$. Define $\mathcal{F}$ and $\langle x \rangle$ as on 12. We then have:

$$\psi^*(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln(1-x^{-2})$$

$$+ O\left[\frac{x}{T}(\ln T + \ln x)^2\right]$$

$$+ O(\ln x)\min\left\{1, \frac{x}{T\langle x \rangle}\right\},$$

wherein

$$\psi^*(x) = \frac{\psi(x+0) + \psi(x-0)}{2}.$$

The implied constants will depend on [at most] δ_0. In addition, one has:

$$\frac{\zeta'(0)}{\zeta(0)} = \ln(2\pi).$$

Proof

See 33 (bottom) and, then, the obvious relation

$$\frac{x \ln T \cdot \ln x}{T} + \frac{x \ln^2 x}{T} \leq \frac{x(\ln T + \ln x)^2}{T}$$

used on 34. This proves the formula for $\psi^*(x)$. OK

We'll verify $\frac{\Gamma'(0)}{\Gamma(0)} = \ln(2\pi)$ in a theorem stated several pages below. ← $\boxed{\text{See 41.}}$

■

N. B.

The formula on 35 middle can be found many places; e.g., in Davenport, Mult. Number Theory, 2nd ed., p. 109 (9)(10) OR Prachar, Primzahlverteilung, Satz 4.5 on pp. 231–2.

We define:

$$\sum_{\rho} \frac{x^{\rho}}{\rho} \equiv \lim_{T\to\infty} \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho} .$$

Recall (31) bottom concerning slight "sloppiness" in T.

Riemann

Corollary 1.

For each $x \geq 1+\delta_0$, we have

$$\psi^{*}(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln(1-x^{-2}) .$$

In this regard, we also have (in an obvious sense)

$$\sum_{|\gamma|>T} \frac{x^{\rho}}{\rho} = O\left[\frac{x}{T}(\ln T+\ln x)^2\right] + O(\ln x)\min\left\{1, \frac{x}{T\langle x\rangle}\right\} .$$

If desired, the 1^{st} term on the RHS can be replaced by

$$O\left[\frac{x}{T}(\ln T\cdot\ln x + \ln^2 x)\right] . \quad *$$

Pf

Straightforward. See (33) bottom for the last assertion.

▨

---

* As already hinted in the two boxes on (28), the term $\frac{x\ln T\cdot\ln x}{T}$ can in fact be improved slightly. This will not affect the estimates for $\psi(x)-x$ though. See (34). Also p. (39).

Corollary 2.

Let $[x_1, x_2]$ be any closed interval in $[1+\delta_0, \infty)$.

(a) If $[x_1, x_2] \cap \mathcal{F} = \phi$, then $\sum \frac{x^\rho}{\rho}$ converges as a symmetric limit <u>uniformly</u> on $[x_1, x_2]$ as a symmetric limit in T.

(b) In <u>every</u> instance, the partial sums $\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho}$ are <u>uniformly bounded</u> on $[x_1, x_2]$ for all $T \geqq 3$.

Pf

For (a), use corollary 1.

For (b), rearrange (35) middle and use the "1" in the minimum. ▮

Thm (recall Lec 16, p. (19))

The explicit formula for $\psi(x)$ immediately gives

$$\psi(x) = x + O(x^\Theta \ln^2 x). \qquad (x \geqq 2)$$

Pf

By Lec 16, p. (1), assertion (a), we have:

$$\sum_{|\gamma| \leq T} \frac{1}{|\gamma|} = O(\ln^2 T), \quad T \geqq 3.$$

We stress that the foregoing bound is essentially sharp due to

$$\int_3^T \frac{1}{t}\, d\left(\frac{t}{2\pi}\ln\frac{t}{2\pi e}\right) = \int_3^T \frac{1}{t}\left[\frac{1}{2\pi}\ln\frac{t}{2\pi}\right]dt$$

Lec 15 p. 29

$$\{t = 2\pi u\}$$

$$= \frac{1}{2\pi}\int_{3/2\pi}^{T/2\pi} \frac{\ln u}{u}\, du$$

$$\sim \frac{1}{4\pi}\left(\ln\frac{T}{2\pi}\right)^2 \sim \frac{1}{4\pi}(\ln T)^2.$$

Apply 35 with, say, $x \gtrsim 100$ and $T = x^2$.

Get:

$$|\psi^*(x) - x| \lesssim \sum_{|\gamma| \leq x^2} \frac{x^\Theta}{|\gamma|} + O(1) + O\left[\frac{x\ln^2 x}{x^2}\right]$$

$$+ O(\ln x)$$

$$\lesssim O(1)\, x^\Theta \ln^2 x .$$

But $\psi(x) = \psi^*(x) + O(\ln x)$. Hence,

$$|\psi(x) - x| \lesssim O(1)\, x^\Theta \ln^2 x$$

as promised. ▮

We now PAUSE for some elementary facts (better late than never) related to Lec 5 and Lec 16, p. 7.

THM

Let $\gamma \approx$ the Euler constant. Near $z=1$, we then have:

$$\zeta(z) = \frac{1}{z-1} + \gamma + O(|z-1|) .$$

Proof

Recall Lec 5, pp. 8 - 10 with $r(t) \equiv t - [\![t]\!]$.

The function $G(z) \equiv \zeta(z) - \frac{1}{z-1}$ is analytic on $\{x>0\}$.

$$G(z) = 1 - z\int_1^\infty \frac{r(t)}{t^{z+1}}\,dt$$

Apply Lec 5 p. 8 but take $z=1$. Get:

$$\sum_{n=1}^{N} \frac{1}{n} = 1 + \ln N - \int_1^N \frac{r(t)}{t^2}\,dt$$

$$\Downarrow$$

$$\gamma = 1 - \int_1^\infty \frac{r(t)}{t^2}\,dt .$$

Accordingly,

$$G(1) = 1 - \int_1^\infty \frac{r(t)}{t^2}\, dt = \gamma .$$

By Taylor series,

$$G(z) = \sum_{k=0}^\infty \frac{G^{(k)}(1)}{k!} (z-1)^k$$

$$= \gamma + O(z-1) \quad \text{near } z=1,$$

and we are done. $\blacksquare$

THEOREM

$$\zeta'(0) = -\frac{1}{2} \ln(2\pi), \quad \frac{\zeta'(0)}{\zeta(0)} = \ln(2\pi).$$

PF

Know $\zeta(0) = -\frac{1}{2}$ by Lec 9, pp. 18 + 20.

Now have:

$$\zeta(s) = \frac{1}{s-1}\left[1 + \gamma(s-1) + O(s-1)^2\right]$$

by 40. Accordingly:

$$\log \zeta(s) = -\log(s-1) + \gamma(s-1) + O(s-1)^2$$

$$\Downarrow$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + O(s-1) .$$

This <u>sharpens</u> Lec 7, p. 17. We can now apply {the functional equation}

$$\frac{\zeta'(z)}{\zeta(z)} = \ln 2\pi + \frac{\pi}{2} \operatorname{ctn} \frac{\pi z}{2} - \frac{\Gamma'}{\Gamma}(1-z) - \frac{\zeta'}{\zeta}(1-z)$$

from Lec 16 p. 7. Recall

$$\Gamma(1) = 1 , \quad \Gamma'(1) = -\gamma$$

by Lec 10, p. 30 assertion (e) [[and p. 22]]. Take $z \to 0$ to get:

$$\frac{\zeta'(0)}{\zeta(0)} = \ln 2\pi - \frac{\Gamma'(1)}{\Gamma(1)}$$

$$+ \lim_{z \to 0} \left[\frac{1}{z} + O(z) + \left\{ \frac{1}{-z} - \gamma + O(z) \right\} \right]$$

$$= \ln 2\pi + \gamma - \gamma + 0 = \ln 2\pi .$$

Multiply by $\zeta(0)$ to get $\zeta'(0) = -\frac{1}{2} \ln(2\pi)$. ∎

$\boxed{\text{END-OF-PAUSE.}}$

We closed Lec 18 with a statement of, and very brief sketch-of-the-proof for, the so-called PERRON SUMMATION FORMULA associated with a general Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and $\sum_{n \leq x} a_n \quad (x \geq 10)$.

The formula is based on p. 4 Lemma [above] and its verification parallels pp. 14 ~ 24.

Due to the (already excessive!) length of these notes for Lec 17 + 18, we postpone this "Perron matter" until the notes for Lec 19.

~

Lectures 19 and 20

(30 March and 1 April)

SYNOPSIS

The development in Lectures 17+18, pp. 4 ~ 24 is highly suggestive and capable of a major generalization.

We touched on this already in Lec 18 at the very end. See Lec 18 p. 43.

This brought us to the Perron summation formula with error term.

We'll sketch this formula emphasizing the similarity to Lec 17+18, pp. 4 ~ 24.

We begin with a slight revision of Lec 17, p. 4 Lemma. Compare Ingham p. 75.

FACT

In Lec 17, p. (4) Lemma, things can be broadened to have

$$0 < y < \infty\,, \quad 0 < c \lessapprox 3\,, \quad T \gtrapprox 3$$

getting

$$|\text{Remainder}| \lessapprox \begin{cases} O(1)\,\dfrac{y^c}{T|\ln y|}\,, & y \neq 1 \\ O(1)\,\dfrac{1}{T}\,, & y = 1 \\ O(1)\,y^c\,, & \text{any } y \end{cases} .$$

The "implied" constant in the $O(1)$ is absolute. One also has [the somewhat cruder]

$$|\text{Rem.}| \lessapprox O(1)\,\frac{y^c}{1 + T|\ln y|} .$$

Proof

Simply review pp. (5) ~ (9) and modify several lines. For the final [cruder] assertion, divide into $T|\ln y| > 1$ and $T|\ln y| \lessapprox 1$. ▥

Recall Lec 17 p. 12. We generalize this!

Let $\{a_n\}_{n=1}^{\infty}$ be given. Assume that $a_n \neq 0$ infinitely often as $n \to \infty$.

Let $\mathcal{F}$ be any subset of $\mathbb{Z}^+$ which <u>includes</u> the set $\{n : a_n \neq 0\}$.

Define $\|u\|' = \begin{cases} |u|, & u \neq 0 \\ \infty, & u = 0 \end{cases}$ for $u \in \mathbb{R}$.

For $x \geqq \frac{3}{2}$, let $\xi(x) = \min\limits_{\lambda \in \mathcal{F}} |x - \lambda|$. Also write

$$\langle x \rangle = \min\left\{ \frac{1}{100}, \|\xi(x)\|' \right\}.$$

Notice that $\langle x \rangle = \frac{1}{100}$ unless $\|\xi(x)\|' < \frac{1}{100}$, which would mean $x \notin \mathcal{F}$, but lies <u>LESS</u> THAN $\frac{1}{100}$ units from $\mathcal{F}$.

We clearly have:

$$\langle x \rangle = \frac{1}{100} \quad \text{anytime } x \in \mathbb{Z} \quad (x \geqq \tfrac{3}{2})$$

$$0 < \langle x \rangle \leqq \frac{1}{100} \quad \text{always}.$$

THEOREM (Perron summation formula with error term)

Given a Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which is absolutely convergent on $\{Re(s) > 1\}$. Assume that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^w} = O\left(\frac{1}{(w-1)^\alpha}\right) \quad \text{for } 1 < w \leq 2.$$

Here $\alpha \geq 0$. Assume further that $|a_n| \leq \Phi(n)$, where $\Phi(v)$ is some continuous, nonnegative, monotonic increasing function on $\{1 \leq v < \infty\}$.

Consider $\{c, x, T\}$ such that

$$1 < c \leq 2, \quad x \geq 10, \quad T \geq 3.$$

Taking $\sigma = Re(s)$ and $\sigma + c > 1$, we then have the following relations insofar as $-9 \leq \sigma \leq 10$ (say):

(1) in a style reminiscent of the explicit formula,

$$\sum_{n<x} a_n n^{-s} + \left\{ \begin{matrix} 0, & x \notin \mathbb{Z} \\ \frac{1}{2} a_x x^{-s}, & x \in \mathbb{Z} \end{matrix} \right\}$$

$$= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w}\, dw + O\left[\frac{x^c}{T(\sigma+c-1)^\alpha}\right]$$

$$+ O\left[\frac{\Phi(2x)\, x^{1-\sigma} \ln x}{T}\right] + \text{(see next page)}$$

$$+\ O(1)\,\frac{\Phi(2x)}{x^{\sigma}}\,\min\left\{1,\ \frac{x}{T\langle x\rangle}\right\}$$

[see 3 for $\langle x\rangle$] ;

(2) in a style closer to that of an <u>apriori</u> bound,

$$\sum_{n<x} a_n n^{-s} + \left\{\begin{matrix} 0, & x\notin \mathbb{Z} \\ \frac{1}{2}a_x x^{-s}, & x\in \mathbb{Z}\end{matrix}\right\}$$

$$= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\,\frac{x^w}{w}\,dw$$

$$+\ O(1)\sum_{n=1}^{\infty} |a_n|\,n^{-\sigma}\,\frac{(x/n)^c}{1+T\,|\ln \frac{x}{n}|}\ .$$

In (1), the "implied" constants are absolute apart from a <u>mild dependence</u> on α and the implied constant associated with $O[(\omega-1)^{-\alpha}]$. In (2), the implied constant is absolute.

<u>Proof</u>

It will be convenient to let $N =$ the integer nearest to x (with $x = k+\frac{1}{2} \Rightarrow N = k$).

We look first at:

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\,dw$$

$$= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \sum_{n=1}^{\infty} \frac{a_n}{n^{s+w}}\frac{x^w}{w}\,dw$$

$$= \sum_{n=1}^{\infty}\frac{a_n}{n^s}\left(\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{(x/n)^w}{w}\,dw\right)$$

in the setting of p. 2 FACT. Think

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}\left\{\eta\left(\frac{x}{n}\right) + \text{REM.}\right\},$$

where REM. takes various formats.

Assertion (2) follows immediately. $\boxed{\text{OK}}$

For (1), we need to adapt Lec 17 pp. 14 – 24.
We merely sketch this.

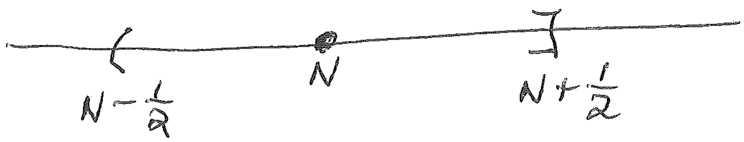
First of all, corresponding to Lec 17 p. 15 top, we split things according to

$1 \leq n \leq N-1$

$n = N$

$N+1 \leq n < \infty$.

Note that, by hypothesis, $N - \frac{1}{2} < x \leq N + \frac{1}{2}$.
(When $x = k + \frac{1}{2}$, $N = k$ and all is OK.)



For $1 \leq n < \frac{x}{2}$, one easily checks

$$\sum_{1 \leq n < \frac{x}{2}} \frac{a_n}{n^s} \{\text{REM.}\} = O\left[\frac{x^c}{T(\sigma+c-1)^q}\right] . \quad //$$

<u>N. B.</u> $-9 \leq \sigma \leq 10$, $1 < c \leq 2 \Rightarrow -8 < \sigma + c \leq 12$ apriori.
All is OK if $1 < \sigma + c \leq 2$. For $2 < \sigma + c \leq 12$, one easily adjusts the implied constant in $O[(\omega-1)^{-q}]$ to accommodate $2 < \omega \leq 12$. This clearly involves q. Inflation by 11^q is sufficient.

For <u>$2x < n < \infty$</u>, we also get

$$O\left[\frac{x^c}{T(\sigma+c-1)^q}\right] . \quad //$$

For $\frac{x}{2} \lesssim n \leqq N-1$, we write $n = N-r$ and IMITATE Lec 17 p. 19 bottom. We get

$$\sum_{\frac{x}{2} \lesssim n \leqq N-1} \frac{a_n}{n^s} \{\text{REM.}\}$$

$$= O(1) \frac{\Phi(N)}{x^\sigma} \frac{x \ln x}{T} \qquad \left\{ \begin{array}{l} \text{recall } x \gtrsim 10, \\ -9 \leqq \sigma \leqq 10 \end{array} \right\}$$

$$= O(1) \frac{\Phi(2x)}{x^\sigma} \frac{x \ln x}{T}$$

via trivial insertion of absolute values.

Not surprisingly, noting Lec 17 p. 16 middle, we find that

$$\sum_{N+1 \leqq n \lesssim 2x} \frac{a_n}{n^s} \{\text{REM.}\}$$

$$= O(1) \frac{\Phi(2x)}{x^\sigma} \frac{x \ln x}{T}$$

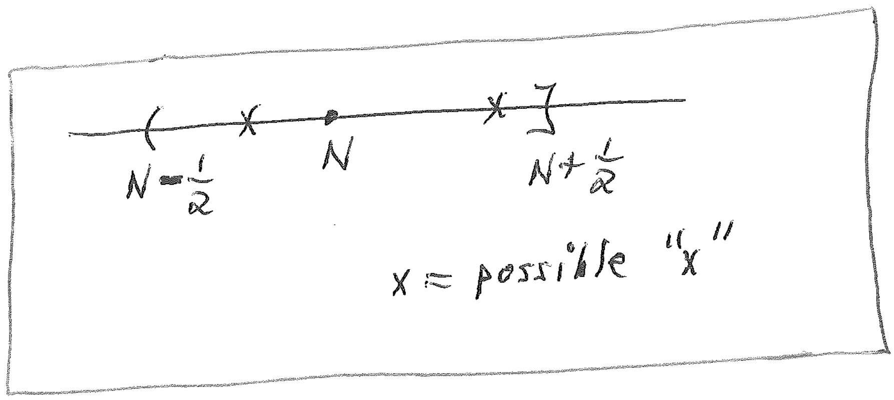
as well.

It remains to discuss

$$\frac{|a_N|}{N^\sigma}\,|\mathrm{REM.}| \qquad \{n = N\}.$$

$$\boxed{\begin{array}{c} x \geqq 10 \\ \Downarrow \\ N \geqq 10 \end{array}}$$

If $a_N = 0$, we get 0 which is subsumed by anything. Suppose therefore that $a_N \neq 0$. Hence $N \in \mathcal{F}$.



If $|x - N| \geqq \frac{1}{100}$, we have:

$$\frac{|a_N|}{N^\sigma}\,|\mathrm{REM.}| = O(1)\,\frac{\Phi(N)}{N^\sigma}\,\frac{1}{T}\,\frac{1}{|\ln x - \ln N|}$$

$$\leqq O(1)\,\frac{\Phi(N)}{x^\sigma}\,\frac{1}{T}\,\frac{1}{\frac{1}{x}|x - N|}$$

$$= O(1)\,\frac{\Phi(N)}{x^\sigma}\,\frac{x}{T}$$

$$= O(1)\,\frac{\Phi(2x)}{x^\sigma}\,\frac{x}{T}$$

$$= O(1)\,\Phi(2x)\,\frac{x^{1-\sigma}}{T}\,.$$

This is subsumed by $O(1)\,\Phi(2x)\,\dfrac{x^{1-\sigma}\ln x}{T}$ without further ado.

OK

$\boxed{\text{cf. page 8 above; also 4 bottom.}}$

To finish up, we therefore take $|x-N| < \frac{1}{100}$ wlog. We still have $a_N \neq 0$ and $N \in \mathcal{F}$.

Must consider 2 cases.

<u>Case I</u> $N = x$.

Here $\langle x \rangle = \frac{1}{100}$ (cf. 3) and

$y = \frac{x}{N}$

$$\frac{|a_N|}{N^\sigma}\,|\mathrm{REM}_0| = O(1)\,\frac{|a_N|}{N^\sigma}\min\left\{1, \frac{1}{T}\right\} \quad \text{by } 2$$

$$= O(1)\,\frac{\Phi(2x)}{x^\sigma}\min\left\{1, \frac{1}{T}\right\}.$$

Of course, $T \geq 3$. In any event, this last expression is safely subsumed by

$$O(1)\,\frac{\Phi(2x)}{x^\sigma}\min\left\{1, \frac{x}{T\langle x\rangle}\right\}. \;//$$

OK

Though not really necessary, we remark that:

$$\frac{|a_N|}{N^\sigma}|REM_0| = O(1)\frac{\Phi(2x)}{x^\sigma}\frac{1}{T}$$

$$= O(1)\,\Phi(2x)\,\frac{x^{-\sigma}(x\ln x)}{T};$$

i.e. matters are also subsumed by the term

$$O(1)\,\Phi(2x)\,\frac{x^{1-\sigma}\ln x}{T}$$

in case I

à la 8 above (cf. also 4).

Case II $N \neq x$ but $|x-N| < \frac{1}{100}$.

By 2,

$$\frac{|a_N|}{N^\sigma}|REM_0| = O(1)\frac{|a_N|}{N^\sigma}\min\left\{\left(\frac{x}{N}\right)^c, \frac{(x/N)^c}{T|\ln x - \ln N|}\right\}$$

$$\boxed{1 < c \leqq 2,\ -9 \leqq \sigma \leqq 10,\ x \gtrapprox 10,\ T \gtrapprox 3}$$

$$= O(1)\frac{|a_N|}{N^\sigma}\left(\frac{x}{N}\right)^c \min\left\{1, \frac{1}{\frac{T}{x}|x-N|}\right\}$$

$$= O(1)\frac{|a_N|}{x^\sigma}\min\left\{1, \frac{x}{T\langle x\rangle}\right\}$$

$\langle x\rangle = |x-N|$ here; see 3

$$= O(1)\,\frac{\Phi(2x)}{x^\sigma}\min\left\{1, \frac{x}{T\langle x\rangle}\right\}.$$

OK

By combining the three OK's [on 10 + here], we deduce that:

$$\frac{|a_N|}{N^\sigma}\,|REM_0| = O(1)\,\Phi(2x)\,\frac{x^{1-\sigma}\ln x}{T} + O(1)\,\frac{\Phi(2x)}{x^\sigma}\min\left\{1, \frac{x}{T\langle x\rangle}\right\}$$

exactly as needed.

This completes the proof of (1) on 4.

Page 4 THM is clearly little more than a pedestrian revamp of Lec 17+18 pp. 4 – 24. Note that:

$$\sum_{n<x} a_n n^{-s} + \left\{\begin{matrix} 0, & x \notin \mathbb{Z} \\ \frac{1}{2}a_x x^{-s}, & x \in \mathbb{Z}\end{matrix}\right\} = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\,dw$$

pointwise w.r.t. x. This is the original version of the Perron summation formula.

On the matter of assertion (2), we leave it as an easy exercise to obtain

*

$$|\text{remainder term}| = O\left[\frac{x^c}{T(\sigma+c-1)^\alpha}\right]$$
$$+ O(1)\,\Phi(2x)\,x^{-\sigma}\,\frac{x\ln[\min(T, x^{1/2})]}{T}$$
$$+ O(1)\,\Phi(2x)\,x^{-\sigma}\min\left\{1, \frac{x}{T\,\xi_0(x)}\right\}$$

where

$$\xi_0(x) \equiv \min\left\{\frac{1}{100}, \xi(x)\right\}$$

and $\xi(x) = \min_{\lambda\in\mathcal{F}} |x-\lambda|$ as on 3. Of course,

$$\min(T^{1/2}, x^{1/2}) \lesssim \min(T, x^{1/2}) \lesssim \min(T, x)$$
$$\Downarrow$$
$$\tfrac{1}{2}\ln[\min(T,x)] \lesssim \ln[\min(T, x^{1/2})] \lesssim \ln[\min(T,x)].$$

This allows line 6 to be cleaned up slightly.

---

* As suggested, treat $T < x^{1/2}$ and $T \gtrsim x^{1/2}$ separately. Note that $\xi_0(x)$ can be 0.

For $T > x^{1/2}$, assertion (1) is better than lines 5–7 on 13. Cf. $\langle x \rangle$ versus $\xi_0(x)$. For $T < x^{1/2}$, lines 5–7 will sometimes give the better result. EG take $x \notin \mathbb{Z}$ and $T = \exp[(\ln x)^\delta]$, δ tiny.

NEW TOPIC.

Let

$$\mu(n) = \begin{cases} 0, & \text{if } n \text{ is NOT squarefree} \\ (-1)^\nu, & \text{if } n = p_1 \cdots p_\nu \text{ (distinct primes)} \end{cases}.$$

Of course, $\nu = 0$ for $n = 1$. It is completely standard by Euler's identity (see Ingham 16 + Lec 6 p. 4) to verify that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$. It is also standard to verify that

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$

wherein $d(n) \equiv N[k \geq 1 : k | n]$. Thus $d(6) = 4$ via $\{1, 2, 3, 6\}$. Elementary number theory gives

$$d(n) = (1+E_1)\cdots(1+E_r) \quad \text{for} \quad n = p_1^{E_1}\cdots p_r^{E_r} .$$

Here $p_1 < p_2 < \cdots < p_r$ are primes. An easy analysis shows

$$d(n) = O(n^{\varepsilon})$$

for every $\varepsilon > 0$. Indeed: $\{\text{wlog } n \geq 4\}$

$$E_1 + \cdots + E_r \leq (1.5) \ln n$$

$$1 + m \leq (p^{\varepsilon})^m \quad \text{for all} \quad p \geq 2^{1/\varepsilon} \quad (m \geq 0)$$

$$\prod_{j=1}^{r} (1+E_j) = \prod_{p_j < 2^{1/\varepsilon}} (1+E_j) \cdot \prod_{p_j \geq 2^{1/\varepsilon}} (1+E_j)$$

$$\leq \prod_{p_j < 2^{1/\varepsilon}} (1+E_j) \cdot \prod_{p_j \geq 2^{1/\varepsilon}} p_j^{\varepsilon E_j}$$

$$\leq \prod_{p_j < 2^{1/\varepsilon}} (3(\ln n)) \cdot n^{\varepsilon}$$

$$\leq O(n^{2\varepsilon}) . \quad \{\text{EACH } \varepsilon\}$$

For real $y \geq 1$, we define

$$M(y) = \sum_{n \leq y} \mu(n)$$

$$T(y) = \sum_{n \leq y} d(n) .$$

Theorem

The flip-flopping function $\mu(n)$ has average 0 in the sense that

$$\lim_{x \to \infty} \frac{M(x)}{x} = 0 .$$

IN FACT:

$$M(x) = O\left[x e^{-c(\ln x)^{1/10}}\right]$$

with some suitably small $c > 0$.

Pf

We use (Perron) p. 4 version (1) with $x = m + \frac{1}{2}$, m large. We take

$$a_n = \mu(n),\ \Phi(v) = 1,\ q = 1,\ s = 0,\ \sigma = 0 .$$

We note that

$$\frac{1}{\zeta(s)} = O[\ln^7 t] \quad \text{for } \sigma \geq 1 - \lambda(\ln t)^{-9}$$

whenever λ is appropriately small and $t \geq 3$. In checking this, wlog t = giant. We remember that:

$$\frac{1}{\zeta(\sigma+it)} = O(\ln^7 t), \quad \sigma \geqq 1,\ t \geqq 3 \qquad \text{Lec 7 p. 5}$$

$$|\zeta(\sigma+it)| \geqq A(\ln t)^{-7} \quad \text{here}$$

$$|\zeta(\sigma+it)| \leqq \frac{C}{\delta(1-\delta)} |t|^{1-\delta} \qquad \sigma \geqq \delta,\ |t| \geqq 2 \qquad \text{Lec 6 p. 9}$$

$$|\zeta(\sigma+it)| \leqq A_2 \ln|t|, \quad \sigma \geqq 1 - \frac{5}{\ln|t|}, \quad |t| \geqq t_0 \qquad \text{Lec 6 (14) (20)}$$

$$|\zeta'(\sigma+it)| \leqq A_3 \ln^2|t|, \quad \sigma \geqq 1 - \frac{5}{\ln|t|}, \quad |t| \geqq t_0 \qquad \text{Lec 6 (20)}$$

and

$$s_1 = 1 + it, \quad s_2 = \sigma_2 + it, \quad \tfrac{1}{2} < \sigma_2 < 1 \Rightarrow$$

$$|\zeta(s_2)| \geqq |\zeta(s_1)| - |\zeta(s_2) - \zeta(s_1)|$$

$$\geqq |\zeta(s_1)| - \int_{s_2}^{s_1} |\zeta'(w)|\, d\sigma \qquad \{w = \sigma + it\}.$$

Get: $(t \geqq t_0)$

$$|\zeta(\sigma_2+it)| \geqq A(\ln t)^{-7} - \int_{\sigma_2}^{1} A_3 \ln^2 t\, d\sigma$$

$$\geqq A(\ln t)^{-7} - A_3 \ln^2 t\,(1-\sigma_2)$$

KEEP $\sigma_2 > 1 - \frac{\lambda}{(\ln t)^9}$ $\qquad$ $\underline{\lambda \text{ tiny}}$

$$\Downarrow$$

$$\geqq (A - \lambda A_3)(\ln t)^{-7} \geqq \tfrac{1}{2} A (\ln t)^{-7}.$$

In other words,

$$\frac{1}{\zeta(\sigma+it)} = O(\ln^7 t)\,, \quad \sigma \gtrsim 1 - \frac{\lambda}{(\ln t)^9}\,, \quad t \gtrsim 3$$

for <u>some</u> tiny λ.

By p. (4) (1), with $\mathcal{F} = \mathbb{Z}^+$ and $c \in (1,2]$,

$$M(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{1}{\zeta(w)}\frac{x^w}{w}\,dw$$
$$+ O\left[\frac{x^c}{T(c-1)}\right] + O\left[\frac{x\ln x}{T}\right]$$
$$+ O(1)\min\left\{1, \frac{x}{T(\frac{1}{2})}\right\}.$$

To optimize $O\left[\frac{x^c}{T(c-1)}\right]$ in regard to c, we put

$$c = 1 + \frac{1}{\ln x}.$$

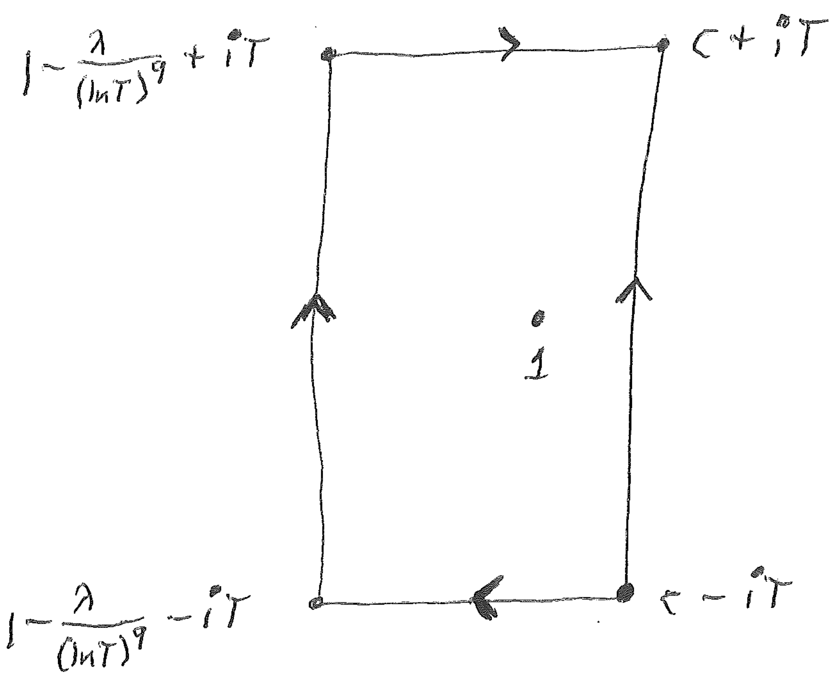
Since $x = m + \frac{1}{2}$ is big, this is legal. Get:

$$M(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} + O\left[\frac{x\ln x}{T}\right] + O\left[\frac{x\ln x}{T}\right]$$
$$+ O(1)\frac{x}{T}$$

$$\Downarrow$$

$$M(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \quad + \quad O\left[\frac{x}{T}(\ln x)\right].$$

We now deform $[c-iT, c+iT]$ remembering pp. 17, 18 top. Of course, $1/\mathfrak{Z}(w)$ has a <u>simple zero</u> at $w=1$ (in the sense of a removable singularity).



use a rectangle!

By either reducing λ or inflating T, we (and do) can hypothesize that $1/\mathfrak{Z}(w)$ is nicely <u>analytic</u> on this closed rectangle. By 18 top, the issue occurs only for $|\mathrm{Im}\, w| \leqq 3$.

By the Cauchy Integral Thm, looking at $\frac{1}{\zeta(w)} \frac{x^w}{w}$, we get:

$$|M(x)| \lessapprox O(1)(\ln T)^7 x^{1-\frac{\lambda}{(\ln T)^9}} \int_3^T \frac{1}{v}\,dv$$

$$+ O(1)(\text{const})\, x^{1-\frac{\lambda}{(\ln T)^9}} \int_0^3 (\text{const})\,dv$$

$$+ O(1)\left|(c-1) + \frac{\lambda}{(\ln T)^9}\right| (\ln T)^7 \frac{x^c}{T} \qquad (c>1)$$

$$+ O\left[\frac{x \ln x}{T}\right] \leftarrow \text{(19) top}$$

$$= O(1)(\ln T)^8 x^{1-\frac{\lambda}{(\ln T)^9}}$$

$$+ O(1)\frac{1}{\ln x}\frac{x(\ln T)^7}{T} \longleftarrow \boxed{c = 1 + \frac{1}{\ln x}}$$

$$+ O(1)\frac{\lambda}{(\ln T)^2}\frac{x}{T} + O\left[\frac{x \ln x}{T}\right]$$

$$= O(1)\,x \cdot (\ln T)^8 x^{-\frac{\lambda}{(\ln T)^9}}$$

$$+ O(1)\,x \cdot \frac{(\ln T)^7}{T \ln x}$$

$$+ O(1)\,x \cdot \frac{\lambda}{T(\ln T)^2}$$

$$+ O(1)\,x \cdot \frac{\ln x}{T}$$

$$= O(x)\left[(\ln T)^8 x^{-\frac{\lambda}{(\ln T)^9}} + \frac{(\ln T)^7}{T \ln x} + \frac{\lambda}{T(\ln T)^2} + \frac{\ln x}{T} \right].$$

Continue to keep x large. Also keep $T \geqq e^{10}$.
Since λ is tiny, obviously

$$\frac{\ln x}{T} \geqq \frac{\lambda}{T} \geqq \frac{\lambda}{T(\ln T)^2}.$$

We can therefore erase the term $\frac{\lambda}{T(\ln T)^2}$.

ASSUME NOW THAT $\underline{T \leqq x}$.

At once:

$$|M(x)| \lessapprox O(x)(\ln x)^8 x^{-\frac{\lambda}{(\ln T)^9}}$$
$$+ O(x)(\ln x)^6 \frac{1}{T}$$
$$+ O(x)(\ln x) \frac{1}{T}$$

$$\leq O(x)(\ln x)^8\left[x^{-\frac{\lambda}{(\ln T)^9}} + \frac{1}{T}\right]$$

$$= O(x)(\ln x)^8\left[e^{-\frac{\lambda \ln x}{(\ln T)^9}} + e^{-\ln T}\right].$$

But,

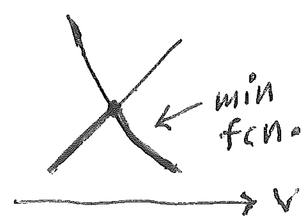
$$e^{-\min\{A,B\}} \leq e^{-A} + e^{-B} \leq 2e^{-\min\{A,B\}}$$

for $A>0$, $B>0$.

So,

$$|M(x)| = O(x)(\ln x)^8 e^{-\min\left\{\frac{\lambda \ln x}{(\ln T)^9},\ \ln T\right\}}.$$

Optimize by graphing "$\frac{\lambda \ln x}{v}$ versus v", and thus setting



$$\frac{\lambda \ln x}{(\ln T)^9} = \ln T$$

I.E.

$$\ln T = (\lambda \ln x)^{\frac{1}{10}}$$

I.E.

$$T = \exp\left[(\lambda \ln x)^{\frac{1}{10}}\right].$$

$\boxed{\text{clearly } T \leq x}$

Hence:

$$|M(x)| = O(x)(\ln x)^8 e^{-(\lambda \ln x)^{\frac{1}{10}}}$$

$$= O(x)\, e^{\underline{8 \ln \ln x}}\, e^{-\lambda^{\frac{1}{10}}(\ln x)^{\frac{1}{10}}}$$

$$= O(x)\, e^{-\frac{1}{2}(\lambda \ln x)^{1/10}} \quad \text{for large } x.$$

∎

Corollary

For $x \geq 3$ and <u>any</u> big Δ, we have

$$M(x) = O\left(\frac{x}{\ln^{\Delta} x}\right).$$

Pf

$\Delta \ln \ln x < \frac{1}{2}(\lambda \ln x)^{\frac{1}{10}}$ once x is big enough.

∎

We now give some easy corollaries of p. (23) Corollary.

Proposition 1

The series $\sum_{1}^{\infty} \frac{\mu(n)}{n^s}$ converges at each point of $\{Re(s)=1\}$. The convergence is uniform on compact subsets.

Pf

Consider a general point $s_0 = 1+it$ $(t \in \mathbb{R})$. Know $M(x) = 1$ for $1 \leq x < 2$. Keep U large and in $\mathbb{Z}^+$.

$$\sum_{n=1}^{U} \frac{\mu(n)}{n^{s_0}} = 1 + \int_1^U x^{-s_0}\, dM(x)$$

$$= 1 + \left[x^{-s_0} M(x)\right]_1^U - \int_1^U M(x)\, d(x^{-s_0})$$

$$= U^{-s_0} M(U) + s_0 \int_1^U \frac{M(x)}{x^{s_0+1}}\, dx$$

$$= O(1)\, U^{-1} \frac{U}{\ln^4 U} + s_0 \int_1^U \frac{M(x)}{x^{2+it}}\, dx$$

$$= O(1) \frac{1}{\ln^4 U} + s_0 \int_1^U \frac{M(x)}{x^{2+it}}\, dx$$

But $\int_3^\infty \frac{M(x)}{x^{2+it}}\, dx$ is nicely majorized by $\int_3^\infty \frac{O(1)}{x(\ln x)^4}\, dx$.

A moment's thought now gives the 2 statements in the proposition. ▥

Notice that we get
$$\sum_{1}^{\infty} \frac{\mu(n)}{n^{s_0}} = s_0 \int_1^{\infty} \frac{M(x)}{x^{s_0+1}}\,dx .$$

Prop 2

In connection with $\sum_{1}^{\infty} \frac{\mu(n)}{n^s}$, we have uniform conv on compact subsets on $\{ \mathrm{Re}(s) \geq 1 \}$.

Pf

Easy modification of Prop 1. ▥

Again,
$$\sum_{1}^{\infty} \frac{\mu(n)}{n^s} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}}\,dx = \frac{1}{\zeta(s)},$$
this time for $\mathrm{Re}(s) \geq 1$.

Prop 3

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 .$$

(Euler, 1748 not rigorously)

Pf

Let $s \to 1$ in the 2nd box and use unif conv. ▨

Prop 4

Let $\ell \geq 1$. The series $\sum_1^\infty \frac{\mu(n)(\ln n)^\ell}{n^s}$ conv unif on compact subsets of $\{\mathrm{Re}(s) \geq 1\}$.

Pf

Imitate Prop 1 + 2 with $\Delta \geq \ell + 4$ in 23 Corollary.

EG

$$\sum_1^u \frac{\mu(n)(\ln n)^\ell}{n^{s_0}} = \int_1^u \frac{(\ln x)^\ell}{x^{s_0}}\, dM(x)$$

$$= \left[M(x) \frac{(\ln x)^\ell}{x^{s_0}} \right]_1^u$$

$$- \int_1^u M(x)\, d\left[\frac{(\ln x)^\ell}{x^{s_0}} \right]$$

$$= \text{etc.}$$ ▨

Using the Weierstrass conv thm (for <u>analytic</u> fcns), notice that

$$-\frac{\zeta'(s)}{\zeta(s)^2} = -\sum_{n=1}^\infty \frac{\mu(n)\ln n}{n^s}, \quad \mathrm{Re}(s) > 1.$$

(Clean up by erasing the minus signs.)

By virtue of the unif conv in Prop 4, we immediately get

$$\frac{\zeta'(1+it)}{\zeta(1+it)^2} = \sum_{n=1}^{\infty} \frac{\mu(n)\ln n}{n^{1+it}}, \quad t \neq 0.$$

Letting $t \to 0$ gives

$$\boxed{\sum_{n=1}^{\infty} \frac{\mu(n)\ln n}{n} = -1}.$$

Indeed:

$$\zeta(s) = \frac{1}{s-1}\left[1 + \gamma(s-1) + O(s-1)^2\right] \qquad \text{Lec 18 p. 40}$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + O(s-1) \qquad \text{Lec 18 p. 42}$$

$$\frac{\zeta'(1+w)}{\zeta(1+w)} = -\frac{1}{w} + \gamma + O(w) \qquad w \to 0$$

$$\frac{1}{\zeta(1+w)} = w\left[1 - \gamma w + O(w^2)\right] \qquad w \to 0$$

$$\frac{\zeta'(1+w)}{\zeta(1+w)^2} = (-1)\left[1 - 2\gamma w + O(w^2)\right] \qquad w \to 0.$$

OK

These facts using 23 Corollary are nice and are of interest because of several <u>facts</u> (very old ones) which we will not prove fully at this time.

$\boxed{A}$ The Riemann Hypothesis is <u>equivalent</u> to the statement that $M(x) = O(x^{\frac{1}{2}+\varepsilon})$.

$\boxed{B}$ The RH is <u>equivalent</u> to the statement that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ <u>converges</u> for all $\{\mathrm{Re}(s) > \frac{1}{2}\}$.

$\boxed{C}$ By elementary techniques (no use of complex variable), one can show

$$M(x) = o(x) \text{ is } \underline{\text{equiv}} \text{ to } \psi(x) \sim x.$$

$\boxed{D}$ By elementary techniques, one can show that the following are equivalent

(i) $\psi(x) \sim x$;

(ii) $M(x) = o(x)$;

(iii) $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$. $\boxed{\text{so Euler "knew" the PNT}}$

And, <u>supplementing</u> this, that 27 box $\Rightarrow$ (iii)(ii)(i).

In Lec 20, I proposed to look at (just) $\boxed{C}$.
There is a certain amount of "fun" in doing so. Plus being instructive!

It is convenient to first review some preliminaries involving elementary number theory and $\mu(n)$. We do so quickly.

Given $f(n)$ for $n \in \mathbb{Z}^+$. Recall that f is <u>multiplicative</u> when

$$f(n_1 n_2) = f(n_1) f(n_2) \quad \text{if } (n_1, n_2) = 1.$$

To avoid trivialities, we assume $f(1) = 1$.

Note that $\mu(n)$ and $d(n)$ are multiplicative. Cf. 14 + 15 top.

Review proposition

Prop R1

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}.$$

Pf

$1 = \frac{1}{\zeta(s)} \zeta(s)$. Think of s real, $s > 1$. So,

$$1 = \sum_1^\infty \frac{\mu(k)}{k^s} \cdot \sum_1^\infty \frac{1}{\ell^s} .$$

By absolute conv of both series, get *

$$\left\{ \begin{array}{l} 1, \ n=1 \\ 0, \ n>1 \end{array} \right\} = \sum_{k\ell = n} \mu(k) \cdot 1 \quad \left\{ \begin{array}{l} k \geq 1 \\ \ell \geq 1 \end{array} \right\}$$

$$= \sum_{k|n} \mu(k) . \quad \blacksquare$$

(ABS CONV ensures that an infinite series can be summed/rearranged in <u>any</u> order.)

<u>Prop R2</u> (Möbius inversion)

Let f be <u>any</u> fcn defined on $\mathbb{Z}^+$. Let

$$g(n) = \sum_{d|n} f(d) .$$

Then:

$$f(n) = \sum_{d|n} \mu(d) \, g\left(\frac{n}{d}\right) .$$

<u>Pf</u>

For an elementary proof, see R1 and any basic

---

* Think $\sum_1^\infty c_n n^{-s} = \left(\sum_1^\infty a_k k^{-s}\right)\left(\sum_1^\infty b_\ell \ell^{-s}\right) \Leftrightarrow c_n = \sum_{k\ell=n} a_k b_\ell .$

book on number theory. Eg Hardy and Wright.
The "slick informal" proof goes like so:

$$\sum_{1}^{\infty} \frac{g(n)}{n^s} = \zeta(s) \sum_{1}^{\infty} \frac{f(l)}{l^s} \qquad s > G \quad (G = \text{giant})$$

$$\Rightarrow \quad \frac{1}{\zeta(s)} \sum_{1}^{\infty} \frac{g(m)}{m^s} = \sum_{1}^{\infty} \frac{f(n)}{n^s}$$

$$\Rightarrow \quad f(n) = \sum_{lm=n} \mu(l) g(m) = \sum_{l|n} \mu(l) g\left(\frac{n}{l}\right).$$

$\boxed{n \text{ frozen}}$

∎

Prop R3 (converse of Möbius inversion)

f, g on $\mathbb{Z}^+$. Assume $f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$.
Then:
$$g(n) = \sum_{d|n} f(d).$$

Pf
Look in, eg, Hardy and Wright (using Prop R1).
Slick/informal proof:

$$\sum_{1}^{\infty} \frac{f(n)}{n^s} = \frac{1}{\zeta(s)} \sum_{k=1}^{\infty} \frac{g(k)}{k^s} \qquad s > G \text{ giant}$$

$$\Rightarrow \quad \sum_{1}^{\infty} \frac{g(n)}{n^s} = \zeta(s) \sum_{1}^{\infty} \frac{f(k)}{k^s}$$

$$\Rightarrow \quad g(n) = \sum_{lk=n} 1 \cdot f(k) = \sum_{k|n} f(k). \quad ∎$$

Prop R4

Let f be *multiplicative* (on $\mathbb{Z}^+$). Then, so is

$$g(n) = \sum_{d|n} f(d).$$

Pf

$g(1) = f(1) = 1$ OK. Given $n_j \geqq 2$ for $j = 1, 2$.

Suppose $n_1 n_2$ has $(n_1, n_2) = 1$. Use elem numb th.

$$g(n_1 n_2) = \sum_{d|n_1 n_2} f(d)$$

THINK prime factorizations of n_1 and n_2

$\{$ such d are *uniquely* $d_1 d_2$ with $d_1 | n_1$, $d_2 | n_2 \}$

$$= \sum_{d_j | n_j} f(d_1 d_2) = \sum_{d_j | n_j} f(d_1) f(d_2)$$

$$= \left(\sum_{d_1 | n_1} f(d_1) \right) \left(\sum_{d_2 | n_2} f(d_2) \right)$$

$$= g(n_1) g(n_2).$$

The case $n_1 = 1$, $n_2 \geqq 2$ / $n_2 = 1$, $n_1 \geqq 2$ is trivial. ∎

~

OK, then. To prove $\boxed{C}$, there are 2 halves:

$$\psi(x) \sim x \Rightarrow M(x) = o(x) \quad \text{and}$$
$$M(x) = o(x) \Rightarrow \psi(x) \sim x.$$

We begin with $\psi(x) \sim x \Rightarrow M(x) = o(x)$.

FACT

$$\sum_{n \leq x} \mu(n) \ln \frac{x}{n} = M(x) \ln x + \sum_{k\ell \leq x} \Lambda(k)\mu(\ell).$$

Here $x \in \mathbb{R}$, $x \geq 2$.

Pf

$$\left(\frac{1}{\zeta}\right)' = -\frac{1}{\zeta^2}\zeta' = \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{1}{\zeta(s)} \Rightarrow$$

$$-\sum_{n=1}^{\infty} \frac{\mu(n)\ln n}{n^s} = \left[\sum_{k=1}^{\infty} \frac{\Lambda(k)}{k^s}\right]\left[\sum_{\ell=1}^{\infty} \frac{\mu(\ell)}{\ell^s}\right]$$

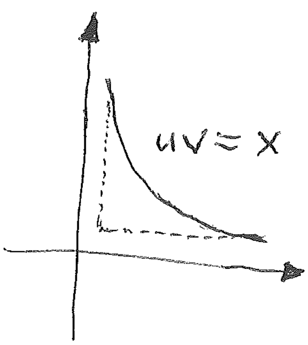
$$\boxed{-\mu(n)\ln n = \sum_{k\ell = n} \Lambda(k)\mu(\ell)}.$$

$$\mu(n)\ln\frac{x}{n} = \mu(n)\ln x + \sum_{kl=n}\Lambda(k)\mu(l)$$

{now sum!}

$$\sum_{n\le x}\mu(n)\ln\frac{x}{n} = M(x)\ln x + \sum_{n\le x}\sum_{kl=n}\Lambda(k)\mu(l)$$

$k\ge 1$
$l\ge 1$
$(k,l)\in\mathbb{R}^2$
is a lattice point



$$= M(x)\ln x + \sum_{\substack{\text{all } (k,l) \\ \text{lattice points} \\ kl\le x}}\Lambda(k)\mu(l)$$

$\left\{\begin{array}{l}\text{view in first quadrant} \\ \text{of } uv\text{-plane, and} \\ \text{under hyperbola } v=\dfrac{x}{u}\end{array}\right\}.$ **KEY**

FACT

$x \in \mathbb{R}$, $x \geq 2$. Then:

$$\sum_{n \leq x} \ln \frac{x}{n} = x + O(\ln x).$$

Pf

By Lec 1, $\ln [\![x]\!]! = x \ln x - x + O(\ln x)$. See Lec 1 p. 11, Thm 6. But, in the above,

$$\begin{aligned} \text{LHS} &= [\![x]\!] \ln x - \ln [\![x]\!]! \\ &= [\![x]\!] \ln x - x \ln x + x + O(\ln x) \\ &= ([\![x]\!] - x) \ln x + x + O(\ln x) \\ &= O(\ln x) + x + O(\ln x) \\ &= x + O(\ln x). \end{aligned}$$

$\blacksquare$

FACT

$$M(x)\ln x = O(x) - \sum_{\ell \leq x} \mu(\ell)\,\psi\!\left(\tfrac{x}{\ell}\right). \qquad x \in \mathbb{R},\ x \geq 2$$

PF

Use 33 Fact. Note there that

$$\Big|\sum_{n\leq x}\mu(n)\ln\tfrac{x}{n}\Big| \leq \sum_{n \leq x}\ln\tfrac{x}{n} = O(x)$$

by 35. Get:

$$M(x)\ln x = O(x) - \sum_{k\ell \leq x}\Lambda(k)\,\mu(\ell)$$

view the hyperbola region in First quadrant 34

given any $\ell \leq x$, note that k then must range in $[1, x/\ell]$

$$M(x)\ln x = O(x) - \sum_{\ell \leq x}\mu(\ell)\Big(\sum_{k\leq x/\ell}\Lambda(k)\Big)$$

$$= O(x) - \sum_{\ell \leq x}\mu(\ell)\,\psi\!\left(\tfrac{x}{\ell}\right). \quad ▨$$

FACT

$$\left|\sum_{m\leqq x}\frac{\mu(m)}{m}\right| < 1 \quad \text{for all} \quad \begin{matrix} x\in\mathbb{R} \\ x\geqq 2\end{matrix}.$$

Pf

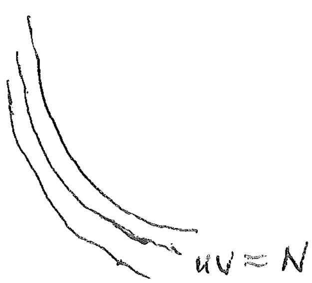
WLOG $x\in\mathbb{Z}$. $x=1$ gives Sum $=1$.

$x=2$ OK; SUM $=\frac{1}{2}$. So, wlog, $x\geqq 3$.

Note:

$$\sum_{m\leqq x}\mu(m)\left[\!\left[\frac{x}{m}\right]\!\right] = \sum_{mk\leqq x}\mu(m)\cdot 1$$

as in the p. 34 hyperbola region



$\left\{\begin{array}{l}\text{write } N=mk \\ \text{for } \underline{\text{each}}\ N,\ \text{note how } m \\ \text{ranges over the divisors of } N\end{array}\right\}$ and $k\equiv\frac{N}{m}$!

$$\equiv \sum_{N\leqq x}\left\{\sum_{m\mid N}\mu(m)\right\}$$

$$= \sum_{N\leqq x}\left\{\begin{matrix}1, & N=1 \\ 0, & N>1\end{matrix}\right\} \quad \leftarrow 29 \text{ RI}$$

$$= 1.$$ ////

Accordingly,

$$\sum_{m=1}^{x} \mu(m)\left(\frac{x}{m} - r\left(\frac{x}{m}\right)\right) = 1$$

$$\{\, r(t) = t - [\![t]\!] \,\}$$

$$x \sum_{m=1}^{x} \frac{\mu(m)}{m} = 1 + \sum_{1}^{x} \mu(m) r\left(\frac{x}{m}\right)$$

$$\left\{ \begin{array}{l} m=1 \Rightarrow \mu(1)r(x) = 0 \\ m=x \Rightarrow \mu(x)r(1) = 0 \\ \text{in general, } 0 \leq r\left(\frac{x}{m}\right) < 1 \end{array} \right\}$$

$$x \left| \sum_{1}^{x} \frac{\mu(m)}{m} \right| \lesseqqgtr 1 + (x-2) = x-1 < x$$

$$\Downarrow$$

$$\left| \sum_{m=1}^{x} \frac{\mu(m)}{m} \right| < 1 \quad \text{all } x \geq 2 \,. \qquad ▩$$

Recall 36 : get

$$M(x)\ln x = O(x) - \sum_{\ell \leq x} \mu(\ell)\,\psi\left(\frac{x}{\ell}\right) \qquad \begin{array}{l} x \in \mathbb{R} \\ x \geq 2 \end{array}$$

$\{$ use 37 $\}$

$$= O(x) + \sum_{\ell \leq x} \mu(\ell)\left(\frac{x}{\ell} - \psi\left(\frac{x}{\ell}\right)\right) .$$

$\boxed{\text{Very Slick}}$

FACT

$\psi(x) \sim x$ as $x \to \infty$ $\Longrightarrow$

$$M(x) = o(x).$$

Pf

Use 38 bottom.

Choose any tiny $\varepsilon > 0$. Let $|\psi(y) - y| < \varepsilon y$ for all $y \geqq A \geqq 5$, say. (A depends on ε.)

Assume $x \geqq 1000A$. Get:

$$|M(x)| \ln x \leqq O(x) + \sum_{l \leqq \frac{x}{A}} |\mu(l)| \left| \frac{x}{l} - \psi\left(\frac{x}{l}\right) \right|$$

$$+ \sum_{\frac{x}{A} < l \leqq x} |\mu(l)| \left| \frac{x}{l} - \psi\left(\frac{x}{l}\right) \right|$$

$\Bigg\{$ since $\psi(x) \sim x$ as $x \to \infty$, one knows trivially that

$$|\psi(y)| \leqq \mathfrak{M} y$$

for ALL $y \geqq 1$ with some $\mathfrak{M}$

also recall Chebyshev estimate for ψ from Lec 1 p. 18 $\Bigg\}$

$$\lessapprox O(x) + \sum_{l \lessapprox \frac{x}{A}} \varepsilon \frac{x}{l}$$

$$+ (1+M) \sum_{\frac{x}{A} < l \lessapprox x} \frac{x}{l}$$

$$\lessapprox O(x) + (1) \varepsilon x \left[\ln \frac{x}{A} + O(1) \right]$$

$$+ O(1)\, x \left[\ln \frac{x}{x/A} + O(1) \right]$$

KEY

$$\Downarrow$$

$$|M(x)| \ln x \lessapprox O(x) + O(x) + \varepsilon x \ln x$$

$$+ O(x)$$

$$\Downarrow$$

$$|M(x)| \lessapprox O\left(\frac{x}{\ln x}\right) + \varepsilon x .$$

yes!

Hence $M(x) = o(x)$ as $x \to \infty$. ▨

We'll treat $M(x) = o(x) \Rightarrow \psi(x) \sim x$ recall p. 33

in the next set of notes. [redacted]

(40 pages is LONG ENOUGH!)

for a synopsis

Lecture 21 Synopsis
(6 April)

We must now prove that

$$M(x) = o(x) \implies \psi(x) \sim x$$

by elementary methods. See Lec 20 p. 33.

We need to make a detour into the Dirichlet divisor problem.

$$T(x) = \sum_{n \leqq x} d(n) = \sum_{\substack{(k,\ell) \\ \text{lattice point} \\ k\ell \leqq x}} 1$$

Trivially,

$$r(t) = t - [\![t]\!]$$
$$0 \leqq r(t) < 1$$

$$\begin{aligned} T(x) &= \sum_{m \leqq x} \left[\!\left[\frac{x}{m}\right]\!\right] \\ &= \sum_{m \leqq x} \frac{x}{m} - \sum_{m \leqq x} r\left(\frac{x}{m}\right) \\ &= x \ln x + O(x) + O(x) \\ &= x \ln x + O(x). \end{aligned}$$

One checks that

$$\operatorname{card}\{(m_1, m_2) : m_1 m_2 \leqq x\}$$

$$= \operatorname{card}\{m_1 \leqq \llbracket x^{1/2} \rrbracket,\ m_2 \leqq \llbracket x^{1/2} \rrbracket\}$$

$$+ \operatorname{card}\{m_1 \leqq \llbracket x^{1/2} \rrbracket,\ m_2 > \llbracket x^{1/2} \rrbracket,\ m_1 m_2 \leqq x\}$$

$$+ \operatorname{card}\{m_1 > \llbracket x^{1/2} \rrbracket \text{ and } m_1 m_2 \leqq x\}$$

but $m_2 \leqq \frac{x}{m_1}$ and $m_1 > \llbracket x^{1/2} \rrbracket$

$\Rightarrow m_1 > x^{1/2} \Rightarrow m_2 \leqq \frac{x}{m_1} < x^{1/2}$

$\Rightarrow m_2 \leqq \llbracket x^{1/2} \rrbracket$ automatically

$$= A + B + \overset{m_1 \leftrightarrow m_2}{B} \quad \text{(in obvious notation)}$$

$$= 2(A+B) - A$$

$$= 2 \sum_{m_1 \leqq \llbracket x^{1/2} \rrbracket} \left\llbracket \frac{x}{m_1} \right\rrbracket - \llbracket \sqrt{x} \rrbracket^2$$

$$= 2 \sum_{k \leqq \llbracket \sqrt{x} \rrbracket} \frac{x}{k} - 2 \sum_{k \leqq \llbracket \sqrt{x} \rrbracket} r\left(\frac{x}{k}\right) - x + O(\sqrt{x})$$

Lec 18 p. 40 bottom re: γ

$$= 2x\left[\ln \llbracket \sqrt{x} \rrbracket + \gamma + O\left(\frac{1}{\sqrt{x}}\right)\right] + O(\sqrt{x}) - x$$

$$= 2x \ln \llbracket \sqrt{x} \rrbracket + 2\gamma x + O(\sqrt{x}) - x$$

$$\left\{ \begin{aligned} \ln(\sqrt{x}-\theta) &= \ln \sqrt{x}\left(1-\frac{\theta}{\sqrt{x}}\right) \qquad 0 \leq \theta < 1 \\ &= \frac{1}{2}\ln x + O\left(\frac{1}{\sqrt{x}}\right) \end{aligned} \right\}$$

$$= x \ln x + \underline{O(\sqrt{x})} + (2\gamma - 1)x .$$

Thm (Dirichlet)

For $x \geq 1$,

$$T(x) = \sum_{n \leq x} d(n) = x \ln x + (2\gamma - 1)x + O(\sqrt{x}) .$$

Dirichlet divisor problem
= best exponent α
$\alpha = \frac{1}{4} + \varepsilon$??

Pf

As above. ▨

Fact

Let $\Delta(x) = \sum_{n \leq x} (\ln n - d(n) + 2\gamma)$, $x \geq 1$.

Then:

$$\Delta(x) = O(\sqrt{x}) .$$

Pf

WLOG x = integer. Just use THM and $\ln x!$.

▨

Note that:

$$\zeta = \zeta^2 \cdot \frac{1}{\zeta}$$

$$\Rightarrow \boxed{1 = \sum_{k\ell = n} d(k)\mu(\ell)}$$

$$-\frac{\zeta'}{\zeta} = (-\zeta') \cdot \frac{1}{\zeta}$$

$$\Rightarrow \boxed{\Lambda(n) = \sum_{k\ell = n} (\ln k)\mu(\ell)}$$

---

and, as before,

$$\left(\frac{1}{\zeta}\right)' = -\frac{\zeta'}{\zeta^2} = \left(-\frac{\zeta'}{\zeta}\right) \cdot \frac{1}{\zeta}$$

$$\Rightarrow \boxed{-\mu(n)\ln n = \sum_{k\ell = n} \Lambda(k)\mu(\ell)}$$

Lec 20 33 .

Fact

$$\psi(x) - x + 2\gamma = \sum_{\ell k \leq x} \mu(\ell)\{\ln k - d(k) + 2\gamma\},$$

$x \in \mathbb{Z}^+$.

Proof

$$\underset{\substack{\uparrow \\ \psi(x)}}{\sum_{n \lessapprox x} \Lambda(n)} \approx \sum_{n \lessapprox x} \left(\sum_{kl \approx n} (\ln k)\, \mu(l) \right) \quad \text{by } 4$$

$$\approx \sum_{\substack{(k,l) \\ kl \lessapprox x}} \mu(l) \ln k$$

$$\approx \sum_{lk \lessapprox x} \mu(l) \ln k$$

$$\underset{\substack{\uparrow \\ -x}}{-\sum_{n \lessapprox x} 1} \approx -\sum_{n \lessapprox x} \left(\sum_{kl \approx n} d(k)\, \mu(l) \right) \quad \text{by } 4$$

$$\approx -\sum_{lk \lessapprox x} \mu(l)\, d(k)$$

$$2\gamma \sum_{lk \lessapprox x} \mu(l) \approx 2\gamma \sum_{\substack{N \lessapprox x \\ \text{with} \\ N \approx kl}} \mu(l) \quad \longleftarrow \quad \text{Lec 20 } 37 \text{ middle (hyperbola)}$$

$$\approx 2\gamma \sum_{N \lessapprox x} \sum_{l \mid N} \mu(l)$$

$$\approx 2\gamma \sum_{N \lessapprox x} \begin{Bmatrix} 1, & N \approx 1 \\ 0, & N > 1 \end{Bmatrix}$$

$$\approx 2\gamma .$$

Add together. ALL IS FINE! ∎

FACT

$$M(x) = o(x) \implies \psi(x) \sim x.$$

Compare Lec 20 p. 39

Proof

Keep $x \in \mathbb{Z}^+$. Choose any large integer G, $100 \leqq G \leqq x$. Recall 4 bottom.

$$\psi(x) - x + 2\gamma = \sum_{\substack{\ell k \leqq x \\ k \leqq G}} \mu(\ell) \{\ln k - d(k) + 2\gamma\}$$

$$+ \sum_{\substack{\ell k \leqq x \\ k > G}} \mu(\ell) \{\ln k - d(k) + 2\gamma\}.$$

For the $k \leqq G$ piece, note:

$$\sum_{\substack{\ell k \leqq x \\ 1 \leqq k \leqq G}} = \sum_{k=1}^{G} \sum_{1 \leqq \ell \leqq \frac{x}{k}} \mu(\ell) \{\ln k - d(k) + 2\gamma\}$$

$$= \sum_{k=1}^{G} (\ln k - d(k) + 2\gamma) M\left(\frac{x}{k}\right).$$

G is held fixed. As $x \to \infty$, by hypothesis, the RHS $= o(x)$. //

Next, for $k > G$, notice that: (on 6)

$$lk \leqq x \quad \Rightarrow \quad 1 \leqq l < \frac{x}{G} \quad \text{a priori}$$

$$\Downarrow$$

for each $l \in [1, \frac{x}{G})$, we look at

$$G < k \leqq \frac{x}{l}$$

$$\Downarrow$$

$$\sum_{\substack{lk \leqq x \\ k > G}} = \sum_{1 \leqq l < \frac{x}{G}} \sum_{k \in (G, \frac{x}{l}]} \mu(l)\left(\ln k - d(k) + 2\gamma\right)$$

$$= \sum_{1 \leqq l < \frac{x}{G}} \mu(l)\left[\Delta\left(\tfrac{x}{l}\right) - \Delta(G)\right]$$

see 3 bottom

$$= \sum_{1 \leqq l < \frac{x}{G}} \mu(l)\left[O(1)\sqrt{\tfrac{x}{l}} + O(1)\sqrt{G}\right]$$

$$\{|\mu(l)| \leqq 1\}$$

$$= O(1) \sum_{l < \frac{x}{G}} \sqrt{\tfrac{x}{l}} + O(1)\,\frac{x}{G}\sqrt{G}$$

$$= O(1)\, x^{1/2} \Big\{ 2\sqrt{\tfrac{x}{G}} + O(1) \Big\} + O(1) \frac{x}{\sqrt{G}}$$

$$\frac{x}{G} \geqq 1$$

$$= O(1) \frac{x}{\sqrt{G}} + O(x^{1/2}) \quad //.$$

Hence : 6 middle

$$|\psi(x) - x + 2\gamma| \leqq o(x) + O(1) \frac{x}{\sqrt{G}} \quad \text{as } x \to \infty.$$

By moving G upward in successive jumps, we get :

$$\psi(x) - x = o(x) \,, \quad \text{i.e.,}$$

$$\psi(x) \sim x \,. \quad \blacksquare$$

OK

Remark:

Recall Perron's formula $x \geq 10$, $x \notin \mathbb{Z}$

$$\sum_{n<x} a_n n^{-s} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s+w) \frac{x^w}{w}\, dw$$

$$\sigma + c > 1$$

Lec 19 p. 4 etc.

$s = 0 \Rightarrow$

$$\sum_{n<x} d(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(w)^2 \frac{x^w}{w}\, dw.$$

An easy manipulation* gives:

$$\mathrm{Res}\left[\zeta(w)^2 \frac{x^w}{w};\ w=1\right] = x \ln x + (2\gamma - 1)x.$$

BIG SURPRISE!!! (See THM on 3.)

$\boxed{\text{* Lec 18 p. 40 note the } \gamma}$

In the remainder of Lec 21, I pretty much followed Ingham 86 ~ 89 (middle), 90 ~ 91 (top).

There's <u>no</u> point repeating that discussion here except for a couple of special items that I plan to refer to later.

<u>Definitions:</u>

Dirichlet Series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$

Generalized Dirichlet Series $\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$

with $0 < \lambda_1 < \lambda_2 < \cdots \to \infty$

[usually $\lambda_1 = 1$]

one often puts $\lambda_n = e^{\mu_n}$ $\boxed{\sum_1^{\infty} a_n e^{-s\mu_n}}$

"Dirichlet integral" $\int_1^{\infty} \frac{a(x)}{x^s} dx$

Here $a(x)$ = a <u>piecewise</u> <u>continuous</u> fcn on $[1, \infty)$ with a discrete set of (possible) jump points which run off to $+\infty$. Also

$$\int_1^{\infty} \equiv \lim_{R \to \infty} \int_1^R . \qquad (R = \text{real})$$

Fact 1 (for Dirichlet series)

Let $\sum_1^\infty a_n n^{-s}$ conv at s_0. Then:

(a) the series conv **absolutely** on $\{Re(s) > Re(s_0) + 1\}$

(b) there is **uniform** conv on any half-plane $\{Re(s) \geq Re(s_0) + 1 + \delta\}$.

$\boxed{\text{uses } \zeta(\sigma - \sigma_0)}$

Fact 2 (for generalized D. S.)

Given $\sum_1^\infty a_n \lambda_n^{-s}$ with, say, $\lambda_1 = 1$. Assume that we have convergence at s_0. Then:

(a) we have **pointwise** conv on $\{Re(s) > Re(s_0)\}$;

(b) we have **uniform** conv on sectors of the form $\{|Arg(s - s_0)| \leq \frac{\pi}{2} - \delta\}$.

Proof (sketch)

WLOG $s_0 = 0$.

(a) Let $T = \sum_{n=1}^\infty a_n$. Let $F(x) = \sum_{\lambda_n \leq x} a_n$. $F(x)$ is a right continuous step function. $\lim_{x \to \infty} F(x) = T$.

Fix any s with $Re(s) > 0$. Integrate by parts in the standard way.

$$\lim_{N\to\infty} \left\{ a_1 + \int_1^N x^{-s}\, dF(x) \right\} \qquad a_1 = 1$$

$$= \lim_{N\to\infty} \left\{ a_1 + [x^{-s}F(x)]_1^N + s\int_1^N F(x)x^{-s-1}dx \right\}$$

$$= \lim_{N\to\infty} \left\{ N^{-s}F(N) + s\int_1^N F(x)x^{-s-1}dx \right\}$$

$$= s\int_1^\infty \frac{F(x)}{x^{s+1}}\,dx \quad \longleftarrow \text{integ is absolutely conv}$$

(b) For uniform conv, by modifying a_1, WLOG $T = \underline{\underline{0}}$. Note that $N^{-s}F(N)$ [above] tends uniformly to 0 on $\{Re(s) \geq 0\}$. The issue is how fast

$$s\int_1^N \frac{F(x)}{x^{s+1}}\,dx \longrightarrow s\int_1^\infty \frac{F(x)}{x^{s+1}}\,dx$$

on $\{ |Arg(s)| \leq \frac{\pi}{2} - \delta,\ s \neq 0 \}$. Write $\alpha = \frac{\pi}{2} - \delta$.

Assume that $|F(x)| < \varepsilon$ for $x \geq N_\varepsilon$ ($N_\varepsilon \in \mathbb{Z}^+$). We know $\lim_{x\to\infty} F(x) = 0$ since $T = 0$.

Get:

$$\left|\text{relevant remainder}\right| = |s| \left| \int_N^{\infty} \frac{F(x)}{x^{s+1}}\, dx \right|$$

$\{\text{keep } N \geqq N_\varepsilon\}$

$$\leqq |s| \int_N^{\infty} \frac{\varepsilon}{x^{\rho\cos\theta+1}}\, dx$$

$\{\text{we write } s = \rho e^{i\theta},\ r > 0,\ |\theta| \leqq \varphi\}$

$$= \rho\,\varepsilon \int_N^{\infty} x^{-\rho\cos\theta-1}\, dx$$

$$= \rho\,\varepsilon\, \frac{N^{-\rho\cos\theta}}{\rho\cos\theta}$$

$$\leqq \frac{\varepsilon}{\cos\theta}\, N^{-\rho\cos\theta}$$

$\{\cos t \text{ decreases on } [0, \varphi]\}$

$$\leqq \frac{\varepsilon}{\cos\varphi} \cdot 1 \quad .$$

By recalibrating ε for our given $\varphi = \frac{\pi}{2} - \delta$, we are done. ∎

Fact 3 (for generalized D.S.)

In Fact 2, the associated summation fcn $f(s)$ is analytic on $\{Re(s) > Re(s_0)\}$, with unif conv on compacta. Hence,

$$f^{(k)}(s) = \sum_{n=1}^{\infty} \frac{a_n(-\ln\lambda_n)^k}{\lambda_n^s} \qquad \boxed{k \geq 1}$$

with unif conv on compacta too.

Fact 4 (for generalized D.S.)

Every $\sum_1^\infty a_n \lambda_n^{-s}$ with, say, $\lambda_1 = 1$ has an abscissa of convergence $\sigma_c \in [-\infty, +\infty]$ so that

$$\sum_1^\infty a_n \lambda_n^{-s} \text{ conv on } \{Re(s) > \sigma_c\}$$

$$\sum_1^\infty a_n \lambda_n^{-s} \text{ diverges on } \{Re(s) < \sigma_c\}.$$

[No assertion about $Re(s) = \sigma_c$.]

FACTS 2 – 4 have easy analogs for Dirichlet integrals

10 bottom

$$f(s) = \int_1^\infty \frac{a(x)}{x^s}\,dx .$$

$s_0 = 0$. Define $F(x) = \int_1^x a(y)\,dy$. Note that $F(x)$ is continuous and piecewise C^1 on $[1, \infty)$.

Also:

$$\int_1^R x^{-s} \underbrace{a(x)\,dx} = \int_1^R x^{-s}\,dF(x) \qquad (R>1)$$

$$\Downarrow$$

$$f(s) = s\int_1^\infty \frac{F(x)}{x^{s+1}}\,dx \qquad \text{etc etc}$$

insofar as $\mathrm{Re}(s) > 0$.

IMPORTANT NOTE:

Lec 11 p. 26

The standard example

$$f(s) = (1-2^{1-s})\,\zeta(s) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s}$$

has $\sigma_c = 0$ [via $s = \varepsilon$] yet no singularity anywhere along $\mathrm{Re}(s) = 0$.

Not like a Taylor series.

USEFUL REMARK (not stated in Lec)

$f(s)$ = generalized D.S. or Dirichlet integral

Assume $\sigma_c \neq +\infty$. Take any $A > \sigma_c$.

Then:

$$|f(\sigma + it)| = O(1 + |s|)$$

on $\{\mathrm{Re}(s) \geqq A\}$.

PF

Do a trivial translation to position things so that the point $s_0 = 0$ is a point of convergence. Look at

$$s \int_1^\infty \frac{F(x)}{x^{s+1}}\, dx$$

on 12 + 15. Keep $\mathrm{Re}(s) \geqq A > 0$. Obviously

$$|\text{previous expression}| \lessapprox |s| \int_1^\infty \frac{|F(x)|}{x^{\sigma+1}}\, dx$$

$$\lessapprox |s| \int_1^\infty \frac{C}{x^{\sigma+1}}\, dx$$

$$= |s| \frac{C}{\sigma} \lessapprox |s| \frac{C}{A}. \quad \text{OK}$$

Landau's Thm

$f(s)$ = generalized D.S. or Dirichlet integral. Assume $\sigma_c \in \mathbb{R}$. Assume also that the a_n or $a(x)$ are real and eventually of fixed sign. Then, as an analytic function (cf. 14), $f(s)$ must have a true singularity at the point $s = \sigma_c$.

Proof

Famous. As in Ingham 88–89. ∎

THM

Introduce $\Theta = \sup \mathrm{Re}(\rho)$ for ζ as usual. Then:

$$\psi(x) - x = \Omega_{\pm}(x^{\Theta - \varepsilon})$$

$$\Pi(x) - \mathrm{li}(x) = \Omega_{\pm}(x^{\Theta - \varepsilon})$$

for each tiny $\varepsilon > 0$ (as $x \to +\infty$). Here

$$\Pi(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\ln n} = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \cdots .$$

The "implied constant" in each of the foregoing can be taken arbitrarily large.

Also

$$li(x) \equiv \int_2^x \frac{dt}{\ln t}.$$

This uses $\zeta(x) \neq 0$ on $0 < x < 1$.

Lec 11 p. 27

Proof

Ingham 90–91 top. ■

plus baby calculus

STANDARD DEFINITION.

Recall:

Ingham 86

$h(x)$ real;

$g(x) > 0$.

$h(x) = \Omega_+[g(x)] \iff$

$h(x) > c\,g(x)$ infinitely often as $x \to +\infty$

for some constant $c > 0$. Similarly for $h(x) = \Omega_-[g(x)]$ and $\Omega[g(x)]$.

Lecture 22 Synopsis

(8 April)

We will basically <u>not</u> repeat Ingham 91(middle) – 92 (top) here. The reasoning in the book is quite clear. We get:

$\gamma_1 \cong 14.134725$

$$\limsup_{x \to \infty} \frac{\psi(x) - x}{x^{1/2}} \geq \frac{m_1}{|\frac{1}{2} + i\gamma_1|}$$

$$\liminf_{x \to \infty} \frac{\psi(x) - x}{x^{1/2}} \leq -\frac{m_1}{|\frac{1}{2} + i\gamma_1|}$$

if RH is true [$\frac{1}{2} + i\gamma_1$ = first zero on critical line; m_1 = its multiplicity]. If RH is false, $\Theta > \frac{1}{2}$ and the aforementioned lim sup / lim inf are $+\infty$ and $-\infty$; see Lec 21 p. 17.

Littlewood proved in 1914 that (actually) a much better result can be obtained if RH holds. See Ingham p. 100. If time permits, we will discuss his result later.

We turn now to a proof of Hardy's theorem that $\zeta(s)$ has <u>infinitely</u> <u>many</u> zeros <u>along</u> $\mathrm{Re}(s) = \frac{1}{2}$. (1914)

We follow the approach of Landau $\approx$ 1927 in his <u>Vorlesungen</u>.

We (must) begin with some calculus lemmas.

<u>Fact 1</u>

Let $f \in C^1[a,b]$ and $\alpha(x)$ be monotonic on $[a,b]$ (either increasing or decreasing).
Then:

$$\int_a^b f(x)\,d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x) f'(x)\,dx,$$

the "dx" integral existing as a nice Riemann integral.

<u>Pf</u>

See Lec 3 p. 8 bottom; also (Lec 3) 7 middle – 8 middle.

∎

Fact 2 (1^{st} mean-value thm)

Let $g \in C[a,b]$ (and real). Let $\alpha(x) \nearrow$ on $[a,b]$. Then

$$\int_a^b g(x)\,d\alpha(x) = g(\xi)\int_a^b d\alpha(x)$$

for some $\xi \in [a,b]$.

Pf

If $\alpha(b) = \alpha(a)$, matter is trivial.

If $\alpha(b) > \alpha(a)$, take $m = \min g$, $M = \max g$ on $[a,b]$. Note

$$LHS \in [m(\alpha(b)-\alpha(a)),\ M(\alpha(b)-\alpha(a))].$$

So, we can write $LHS = C\,[\alpha(b)-\alpha(a)]$ for a unique $C \in [m, M]$. Apply intermediate value thm to g. Get $g(\xi) = C$. ■

Fact 3A (rudimentary 2^{nd} mean value thm)

Let f be monotonic on $[a,b]$. Let α be real and in $C'[a,b]$. There then exists $\xi \in [a,b]$ so that

$$\int_a^b f(x)\,d\alpha(x) = f(a)\int_a^\xi d\alpha(x) + f(b)\int_\xi^b d\alpha(x).$$

$\boxed{\text{here } d\alpha(x) = \alpha'(x)\,dx}$

Pf

The ideas in Lec 3 7 - 8 assure us that

$$(R-S) \int_a^b H(x)\, d\alpha(x) = (R) \int_a^b H(x)\, \alpha'(x)\, dx$$

holds whenever H is <u>either</u> continuous or monotonic. To be strictly correct, one writes

$$\alpha = \alpha_1 - \alpha_2$$

with $\alpha_j \in C^1$ and $\alpha_j \nearrow$.

To prove the relation stated in Fact 3A, we flip f to $-f$ <u>if need be</u> and declare $f \nearrow$ wlog. By Fact 1, (i.e. integ by parts)

$$\int_a^b f(x)\, d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\, df(x).$$

f increasing

By Fact 2,

$$\int_a^b \alpha(x)\, df(x) = \alpha(\xi) \int_a^b df = \alpha(\xi)\,[f(b) - f(a)].$$

Hence:

$$\int_a^b f\,d\varphi = f(b)\varphi(b) - f(a)\varphi(a)$$
$$\qquad - \varphi(\xi)\,[f(b) - f(a)]$$

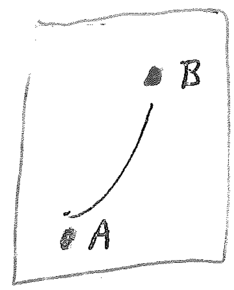
$$= f(a)\,[\varphi(\xi) - \varphi(a)] + f(b)\,[\varphi(b) - \varphi(\xi)]$$

$$= f(a)\int_a^{\xi} d\varphi + f(b)\int_{\xi}^{b} d\varphi \,. \qquad \blacksquare$$

<u>Fact 3B</u> (2^{nd} mean-value thm)

Let f be monotonic $\nearrow$ on $[a,b]$. Let $\varphi \in C'[a,b]$ and <u>real</u>. Let $A \leq f(a+)$, $B \geq f(b-)$.
We can then find $\xi \in [a,b]$ so that

$$\int_a^b f\,d\varphi = A\int_a^{\xi} d\varphi + B\int_{\xi}^{b} d\varphi \,.$$



SIMILARLY for $f \downarrow$ on $[a,b]$.

<u>Pf</u>

Let $f_0(x) = \left\{ \begin{array}{l} A,\ x = a \\ f(x),\ a < x < b \\ B,\ x = b \end{array} \right\}$. Note that $f_0 \nearrow$.

Apply Fact 3A to get

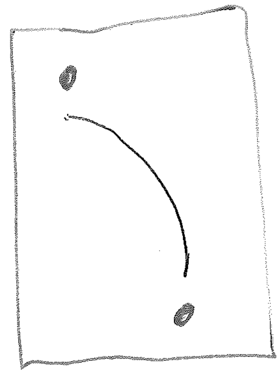
$$\int_a^b f_0\,d\varphi = A\int_a^{\xi} d\varphi + B\int_{\xi}^{b} d\varphi \,.$$

But,

$$\int_a^b f_0\,d\varphi = \int_a^b f_0(x)\varphi'(x)\,dx = \int_a^b f(x)\varphi'(x)\,dx$$
$$= \int_a^b f\,d\varphi$$

by (4) top. $\blacksquare$

NOTE: for $f \downarrow$, one takes
$$A \geq f(a+), \quad B \leq f(b-).$$



Lemmas I – IV are in the spirit of van der Corput ≈ 1921

Lemma I

Let $F \in C'[a,b]$ and real. Assume that $F'(x)$ is monotonic on $[a,b]$. Assume further that
$$F'(x) \geq m > 0 \quad \text{OR} \quad F'(x) \leq -m < 0$$
for all $x \in [a,b]$. Then:
$$\left| \int_a^b e^{iF(x)}\,dx \right| \leq \frac{4}{m}.$$

Pf

WLOG $F'(x)$ is monotonic $\nearrow$ (simply flip F to $-F$ if need be).

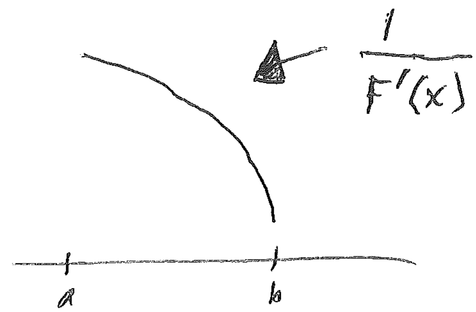
Suppose first that $F'(x) \geqq m > 0$.

$$\int_a^b \cos F \, dx = \int_a^b \frac{1}{F'} (\cos F \cdot F') \, dx$$

$$= \int_a^b \frac{1}{F'} \, d(\sin F)$$

$\left\{ \frac{1}{F'} \text{ is monotonic } \downarrow \text{ and positive} \right\}$

Recall Fact 3B.



$\left\{ \text{Put } \underline{A} = \frac{1}{F'(a)} \text{ and } \underline{B} = 0. \right\}$

$$\int_a^b \frac{1}{F'} d(\sin F) = \frac{1}{F'(a)} \int_a^{\xi} d(\sin F) + 0 \int_{\xi}^b d(\sin F)$$

$$\Rightarrow \quad \left| \int_a^b \cos F \, dx \right| \leqq \frac{2}{|F'(a)|} \leqq \frac{2}{m}$$

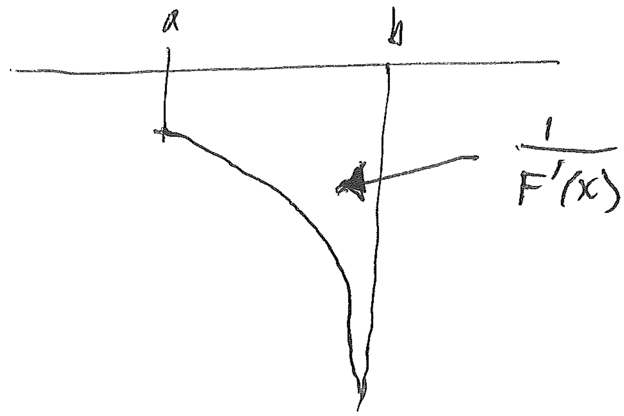
Similarly:

$$\left|\int_a^b \sin F\,dx\right| \lesssim \frac{2}{m}.$$

Now suppose $F'(x) \lesssim -m < 0$ on $[a,b]$.

$$\int_a^b \cos F\,dx = \int_a^b \frac{1}{F'}\,d(\sin F) \quad \text{as before}$$

$$\left\{ \begin{array}{l} F' \text{ is negative, monotonic } \nearrow \\ \frac{1}{F'} \text{ is negative, monotonic } \downarrow \end{array} \right\}$$



Fact 3B $\left\{ \begin{array}{l} A = 0 \\ B = \frac{1}{F'(b)} \end{array} \right\}$

$$\int_a^b \frac{1}{F'}\,d(\sin F) = 0\int_a^{\xi} d(\sin F) + \frac{1}{F'(b)}\int_{\xi}^b d(\sin F)$$

$$\Rightarrow \left|\int_a^b \cos F\,dx\right| \lesssim \frac{2}{|F'(b)|} \lesssim \frac{2}{m}.$$

Similarly:

$$\left|\int_a^b \sin F \, dx\right| \leq \frac{2}{m} . \quad ∎$$

Lemma II

F, G real. $G(x) > 0$. F, G in $C^1[a,b]$.

Assume $\frac{F'(x)}{G(x)}$ is monotonic on $[a,b]$.

Assume further that

$$\frac{F'(x)}{G(x)} \geq m > 0 \quad \text{OR} \quad \frac{F'(x)}{G(x)} \leq -m < 0$$

for all $x \in [a,b]$. Then:

$$\left|\int_a^b G(x) e^{iF(x)} dx\right| \leq \frac{4}{m} .$$

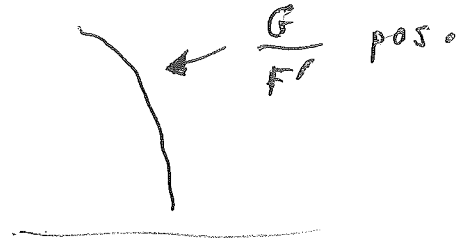
Pf

Flip F to $-F$ if need be; (so) take $\frac{F'}{G}$ monotonic $\uparrow$ on $[a,b]$ wlog.

Suppose first that $\frac{F'(x)}{G(x)} \geq m > 0$.

$$\int_a^b G(x) \cos F(x) \, dx = \int_a^b \frac{G(x)}{F'(x)} \, d(\sin F)$$

Mimic 7.



$$\Downarrow$$

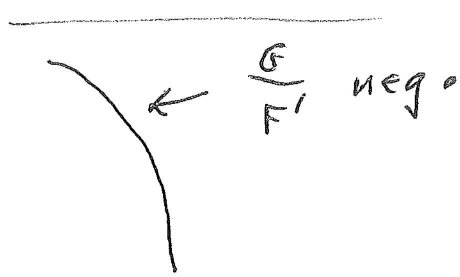
$$\left|\int_a^b G\cos F\,dx\right| \lesssim \frac{2}{m}.$$

Then:

$$\left|\int_a^b G\sin F\,dx\right| \lesssim \frac{2}{m}.$$

Suppose next that $\frac{F'(x)}{G(x)} \lesssim -m < 0$.

Mimic 8.



$$\Downarrow$$

$$\left|\int_a^b G\cos F\,dx\right| \lesssim \frac{2}{m}.$$

Then

$$\left|\int_a^b G\sin F\,dx\right| \lesssim \frac{2}{m}. \qquad \blacksquare$$

Lemma III

F real. $F \in C^2[a,b]$. $F''(x) \geq r > 0$ OR $F''(x) \leq -r < 0$ for all $x \in [a,b]$. Then:

$$\left| \int_a^b e^{iF(x)} dx \right| \leq \frac{8}{\sqrt{r}} .$$

Pf

Note that $\left| \int_a^b e^{iF} dx \right| \leq (b-a)$ trivially.

Hence, wlog,

$$\frac{8}{\sqrt{r}} < b-a .$$

Case 1

$F'(x) > 0$ on $a < x < b$.

Know $F''(x) \neq 0$ on $[a,b]$ and $|F''(x)| \geq r > 0$.
Hence $F'(x)$ is monotonic on $a \leq x \leq b$.
Suppose, e.g., $F'(x)$ is decreasing.
Clearly

$$F'(x) - F'(b) \geq r\delta \quad \text{for } a \leq x \leq b - \delta .$$

We assume here $\delta < b-a$! Hence, $F'(x) \geq r\delta$.

Write

$$I = \int_a^b e^{iF}dx = \int_a^{b-\delta} e^{iF}dx + \int_{b-\delta}^b e^{iF}dx$$

to get

$$|I| \lesssim \left| \int_a^{b-\delta} e^{iF}dx \right| + \delta$$

{ apply Lemma I }

$$|I| \lesssim \frac{4}{r\delta} + \delta .$$

We propose to take $\delta = \frac{2}{\sqrt{r}}$. We have

$$b-a > \frac{8}{\sqrt{r}} > \frac{2}{\sqrt{r}} = \delta$$

as needed. So, for 11 bottom

$$|I| \lesssim \frac{4}{\sqrt{r}} . \quad \text{OK}$$

Similarly for $F'(x)$ increasing.

<u>Case 2</u>

$$F'(x) < 0 \quad \text{on} \quad a < x < b.$$

Here, just flip F to $-F$ and use Case 1. OK

For all <u>other</u> cases, we must then have $F'(c) = 0$ for some $c \in (a, b)$.

We know $F''(x) \neq 0$ on $[a, b]$ and $|F''(x)| \geq r > 0$.
Hence $F'(x)$ is <u>strictly</u> <u>monotonic</u> on $[a, b]$.
The point c is therefore unique.

<u>Case 3</u>

$F'(x) > 0$ on $[a, c)$, $F'(x) < 0$ on $(c, b]$.

We maintain that

$$\left| \int_a^c e^{iF} dx \right| \leq \frac{4}{\sqrt{r}} \quad \text{and} \quad \left| \int_c^b e^{iF} dx \right| \leq \frac{4}{\sqrt{r}}.$$

Take, say, $[c, b]$. If $b - c \leq \frac{4}{\sqrt{r}}$, we are fine. Suppose therefore

$$b - c > \frac{4}{\sqrt{r}}.$$

Know $F'(c) - F'(x) \geq r(x - c)$ for $c < x \leq b$.

Hence

$$F'(c) - F'(x) \geq r\delta \quad \text{for } c + \delta \leq x \leq b.$$

We <u>assume</u> here $\delta < b - c$. Get: $F'(x) \leq -r\delta$.

See 11 bottom.

Get:

$$\left|\int_c^b e^{iF}dx\right| \leq \left|\int_c^{c+\delta} e^{iF}dx\right| + \left|\int_{c+\delta}^b e^{iF}dx\right|$$

$$\leq \delta + \frac{4}{r\delta} \quad \text{by Lemma I.}$$

(the bound)

Propose to put $\delta = \frac{2}{\sqrt{r}}$ to get $\frac{4}{\sqrt{r}}$.

We need:

$$\frac{2}{\sqrt{r}} < b-c, \quad \underline{\underline{\text{but}}} \quad b-c > \frac{4}{\sqrt{r}}$$

(by hypothesis). Hence:

$$\left|\int_c^b e^{iF}dx\right| \leq \frac{4}{\sqrt{r}}.$$

it's essentially case 1

The case $[a,c]$ is similar, of course.

GET:

$$\left|\int_a^b e^{iF}dx\right| \leq \frac{4}{\sqrt{r}} + \frac{4}{\sqrt{r}} = \frac{8}{\sqrt{r}}.$$

<u>Case 4</u>

$F'(x) < 0$ on $[a,c)$, $F'(x) > 0$ on $(c,b]$.

Here, just flip F to $-F$ and use Case 3. ∎

Lemma IV

F real, $G > 0$. $F \in C^2$, $G \in C^1$ on $[a,b]$.
Assume $F''(x) \geqq r > 0$ or $F''(x) \leqq -r < 0$, all x.
Assume also that $\frac{F'}{G}$ is monotonic on $[a,b]$
and $|G(x)| \leqq M$. Then:

$$\left| \int_a^b G(x) e^{iF(x)} dx \right| \leqq \frac{8M}{\sqrt{r}} .$$

Pf

Imitate proof of Lemma III. WLOG, $\frac{8}{\sqrt{r}} < b-a$.
Use Lemma II. Etc.

EG case 1 $\Longrightarrow$ $M\delta + \dfrac{4}{\left(\frac{r\delta}{M}\right)} = M\left[\delta + \dfrac{4}{r\delta}\right]$
$\Rightarrow$ etc.

In case 3, p. 13 middle, refer to:

$$\frac{4M}{\sqrt{r}} \quad \text{and} \quad \frac{4M}{\sqrt{r}} .$$

$\blacksquare$

Lecture 23 Synopsis

(13 April)

We use a series of Facts (in the writing style of Landau) to establish Hardy's theorem that

$$N_{critical}(T) \to \infty \quad \text{as} \quad T \to \infty.$$

Here $N_{critical}(T) = N[\rho : Re(\rho) = \frac{1}{2}, 0 < Im(\rho) \leq T]$.

Fact 1

$T \geq 2$, $\alpha \in \mathbb{R}$. Then

$$\left| \int_T^{2T} t^{\frac{1}{8}} e^{\frac{i}{2}(t\ln t + \alpha t)} dt \right| = O(T^{5/8})$$

with an implied constant which is absolute.
(No dependence on α.)

Pf

Lec 22 p. 15 Lemma IV.

$$\left.\begin{array}{l} G(t) = t^{1/8} \\ M = (2T)^{1/8} \end{array}\right| \begin{array}{l} 2F(t) = t\ln t + \alpha t \\ 2F'(t) = 1 + \ln t + \alpha \\ 2F''(t) = \frac{1}{t} \end{array} \Rightarrow r = \frac{1}{4T} \text{ for } [T, 2T]$$

$$\frac{F'(t)}{G(t)} = \frac{1}{2}\,\frac{1+q+\ln t}{t^{1/8}}$$

real

$$\frac{d}{dt}\left(\frac{A+\ln t}{t^{1/8}}\right) = \frac{t^{1/8}(t^{-1}) - (A+\ln t)\frac{1}{8}t^{-7/8}}{t^{1/4}}$$

$$= \frac{t^{-7/8}}{t^{1/4}}\left[1 - \frac{A+\ln t}{8}\right]$$

critical pt $\Leftrightarrow$ $8 = A + \ln t$ (etc)

so $\frac{F'(t)}{G(t)}$ has AT MOST ONE crit pt on $[T, 2T]$

$$\frac{M}{\sqrt{r}} = (\text{constant})\,T^{5/8}$$

Apply Lemma IV from Lec 22 either once or twice. ∎

> NOTE:
> Analogous Fact holds for $[T, T+H]$, any $H \in [1, T]$.

Recall

$$\xi(s) = G(s)\,\zeta(s) = \pi^{-s/2}\,\Gamma\left(\tfrac{s}{2}\right)\zeta(s)$$

$$\xi(s) = \xi(1-s)$$

à la Lec 11 eg 24 + 27. DEFINE:

following Landau

$$\boxed{f(s) = e^{-\frac{\pi i}{4}\left(s-\frac{1}{2}\right)}\,\xi(s)}$$

for $\operatorname{Im}(s) \geq 1$.

Fact 2

(a) $f(s)$ is analytic on $\{\operatorname{Im}(s) \geq 1\}$

(b) $|f(\sigma+it)| = e^{\frac{\pi}{4}t}\,|\xi(\sigma+it)|$

(c) $|f(1-\sigma+it)| = |f(\sigma+it)|$

(d) $f(\tfrac{1}{2}+it) =$ real for $t \in [1,\infty)$

(e) $\xi(\tfrac{1}{2}+it) =$ real for $t \in \mathbb{R}$.

Pf

Easy. ▨

Fact 3

Given any $-\infty < \sigma_1 < \sigma_2 < \infty$. We then have

$$|\Gamma(\sigma+it)| = \sqrt{2\pi}\,|t|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|}\left(1+O\left(\frac{1}{|t|}\right)\right)$$

uniformly on $\{\sigma_1 \leq \sigma \leq \sigma_2,\ |t| \geq \frac{1}{10}\}$.

Pf

Standard corollary of Stirling's formula for $\mathcal{L}og\,\Gamma(\sigma+it)$. Lec 10 around 42. ∎

Recall that:

$$|\zeta(\sigma+it)| \lesssim \frac{e}{\delta(1-\delta)}\,t^{1-\delta} \qquad \sigma \geq \delta,\ t \geq 3$$

$$|\zeta(\sigma+it)| = O(\ln t) \qquad \sigma \geq 1-\frac{c}{\ln t},\ t \geq 3$$

$$|\zeta'(\sigma+it)| = O(\ln^2 t) \qquad \sigma \geq 1-\frac{c}{\ln t},\ t \geq 3$$

$$\mathcal{L}og\,\zeta(s) = O_\varepsilon(1) \quad \text{for } \sigma \geq 1+\varepsilon.$$

Here $0<\delta<1$, $c=$ small, $0<\varepsilon<\frac{1}{2}$. See Lec 6 9 20 4.

Fact 4

On $\{-\frac{1}{4} \leqq \sigma \leqq \frac{5}{4}\ ,\ t \geqq 1\}$, we have

$$|f(s)| = O(t^{1/2}) \,.$$ $\boxed{\text{CRUDE BOUND}}$

Pf

By Fact 2(c) (3), wlog $\frac{1}{2} \leqq \sigma \leqq \frac{5}{4}$. Apply p. (4) bottom with $\delta = \frac{1}{2}$. Get:

$$\begin{aligned}
|f(\sigma+it)| &= c\, e^{\frac{\pi}{4}t} \left| \pi^{-\frac{s}{2}} \Gamma\left(\tfrac{s}{2}\right) \zeta(s) \right| \\
&\leqq c\, e^{\frac{\pi}{4}t} \left| \Gamma\left(\tfrac{\sigma}{2} + i\tfrac{t}{2}\right) \right| |\zeta(\sigma+it)| \\
&\leqq c\, e^{\frac{\pi}{4}t} \left(\tfrac{t}{2}\right)^{\frac{\sigma}{2}-\frac{1}{2}} e^{-\frac{1}{4}\pi t} |\zeta(\sigma+it)| \qquad \text{Fact 3} \\
&\leqq c \left(\tfrac{t}{2}\right)^{\frac{\sigma}{2}-\frac{1}{2}} t^{Q}
\end{aligned}$$

where "c" can change from line to line and

$$Q \equiv \begin{cases} 1/2, & \sigma \leqq 1 \\ \frac{1}{100}, & \sigma > 1 \end{cases} \,.$$

The extreme exponents are $1/2$ and $\frac{1}{8} + \frac{1}{100}$, so we are done. ∎

Fact 5

For $\sigma = \frac{5}{4}$ and $-\frac{1}{4}$, we have

$$|f(\sigma + it)| = O(t^{1/8})$$

in Fact 4.

Pf

on 5

Just review the proof and recall $|\zeta(s)| \leq \zeta(\sigma)$ whenever $\sigma > 1$. Get

$$|f(\tfrac{5}{4} + it)| = O(t^{1/8}).$$

Treat $\sigma = -1/4$ via Fact 2(c). ■

Fact 6

On $\{-\frac{1}{4} \leq \sigma \leq \frac{5}{4},\ t \geq 1\}$, we actually have

$$|f(\sigma + it)| = O(t^{1/8})$$

for <u>any</u> σ.

Pf

This is an <u>immediate</u> consequence of Facts 4 + 5 when the <u>Phragmén – Lindelöf</u> <u>principle</u> for (general) analytic functions is applied. To avoid interruptions, we prove P-L in Lec 24.

■

Fact 7

For $t \geqq \frac{1}{10}$ and some $\beta \in \mathbb{C}$ with $|\beta| = \sqrt{2\pi}$ we have:

$$\Gamma\left(\frac{5}{8} + it\right) \approx \beta e^{-\frac{\pi}{2}t} t^{\frac{1}{8}} e^{it \ln\left(\frac{t}{e}\right)} \left[1 + O\left(\frac{1}{t}\right)\right].$$

Pf

Kindergarten calculation with Stirling's formula; see Lec 10 around 42. ■

In what follows, we plan to compare

$$\int_T^{2T} \left|f\left(\tfrac{1}{2}+it\right)\right| dt \quad \text{with} \quad \left|\int_T^{2T} f\left(\tfrac{1}{2}+it\right) dt\right|.$$

{ Also similarly for $[T, T+H]$, $1 \leqq H \leqq T$. }

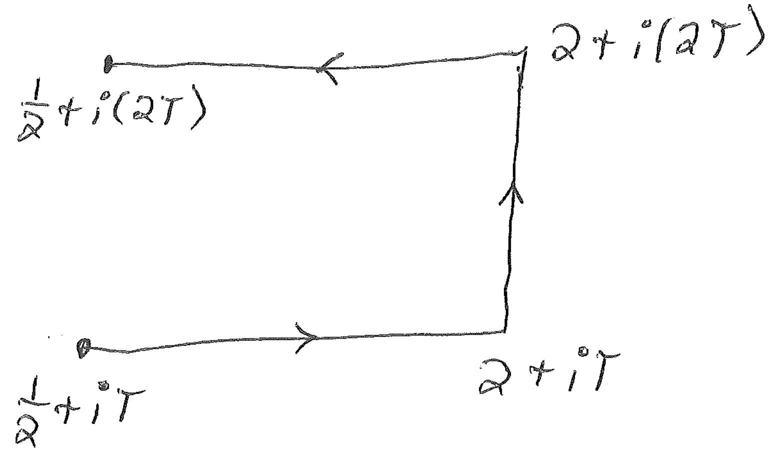
in Lec 24

Fact 8

$$\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i(2T)} \zeta(s)\,ds = iT + O(T^{1/2}).$$

Pf

Cauchy integral theorem $\Rightarrow$ use new path



Recall (4) bottom $\delta = 1/2$. Get

$$\left| \int_{\text{horizontal}} \zeta(s)\,ds \right| = O(T^{1/2}). \quad ///$$

Along $\sigma = 2$, we get the revised integral

$$\int_{2+iT}^{2+i(2T)} \Big[1 + \sum_{n=2}^{\infty} n^{-2-it} \Big] \underbrace{i\,dt}_{\boxed{ds}}$$

$$= i(2T-T)$$

$$+ \sum_{n=2}^{\infty} n^{-2} i \int_T^{2T} e^{-it\ln n}\, dt$$

$$= iT + \sum_{n=2}^{\infty} n^{-2} i \left[\frac{e^{-it\ln n}}{-i\ln n} \right]_T^{2T}$$

$$= iT + O(1) \sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$$

$$= iT + O(1).$$

Adding things, we are done. ∎

$\left\{ \begin{array}{l} \text{Note that } [T, T+H] \text{ gives } \underline{iH} + O(T^{1/2}) \\ \quad \text{insofar as } 1 \leqq H \leqq T. \end{array} \right\}$

<u>Fact 9</u>

For large T, one has

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|\, dt > \frac{1}{2} T.$$

<u>Pf</u>

Trivial corollary of Fact 8. ∎

Thm Hardy 1914

$N_{critical}(T) \to \infty$ as $T \to \infty$.

In fact,

$$N_{crit}(T) \gtrsim c \ln T \quad \text{for } T \text{ large.}$$

Pf

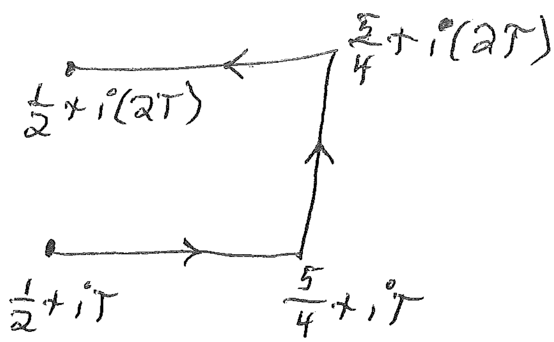
We study $f(s)$ on $[\frac{1}{2}, \frac{5}{4}] \times [T, 2T]$. Know

$$f(s) = e^{-\frac{\pi i}{4}(\sigma - \frac{1}{2})} e^{\frac{\pi t}{4}} \xi(s). \qquad (3)$$

Apply CIT to

$$i \int_T^{2T} f(\tfrac{1}{2} + it)\, dt = \int_{\frac{1}{2} + i(T)}^{\frac{1}{2} + i(2T)} f(s)\, ds$$

$=$



By (6) Fact 6,

$$\int_{\text{horizontal}} f(s)\,ds = O(T^{1/8}).$$

For the $\left(\frac{5}{4}\right)$ contribution, note that

vertical

$$\int_{\frac{5}{4}+i(T)}^{\frac{5}{4}+i(2T)} f\left(\tfrac{5}{4}+it\right) \underbrace{i\,dt}_{ds}$$

has:

$$f\left(\tfrac{5}{4}+it\right) = e^{-\frac{3\pi i}{16}} e^{\frac{\pi t}{4}} \pi^{-\frac{1}{2}\left(\frac{5}{4}+it\right)}$$

$$\cdot \Gamma\left(\tfrac{5}{8}+i\tfrac{t}{2}\right) \zeta\left(\tfrac{5}{4}+it\right)$$

$$= C e^{\frac{\pi t}{4}} e^{-i\frac{t}{2}\ln\pi} \Gamma\left(\tfrac{5}{8}+i\tfrac{t}{2}\right) \zeta\left(\tfrac{5}{4}+it\right)$$

complex and nonzero

changes from line to line

see (7) Fact 7

$$= C e^{\frac{\pi t}{4}} e^{-i\frac{t}{2}\ln\pi} \beta_1 e^{-\frac{\pi}{4}t} t^{1/8} e^{i\frac{t}{2}\ln\left(\frac{t}{2e}\right)}$$

$$\cdot \left[1+O\left(\tfrac{1}{t}\right)\right] \cdot \zeta\left(\tfrac{5}{4}+it\right)$$

$$= \mathcal{C}\, t^{\frac{1}{8}} e^{i\frac{t}{2}\ln\left(\frac{t}{2e\pi}\right)} \cdot \left[1+O\left(\frac{1}{t}\right)\right]$$

$$\cdot\, \zeta\left(\frac{5}{4}+it\right)$$

$$= \mathcal{C}\, t^{\frac{1}{8}} e^{i\frac{t}{2}\ln\left(\frac{t}{2\pi e}\right)} \cdot \zeta\left(\frac{5}{4}+it\right)$$

$$+\ O(t^{1/8}) \cdot O\left(\frac{1}{t}\right) \cdot \exp\left[O(1)\right]$$

④ bottom

$$f \ = \mathcal{C}\, t^{\frac{1}{8}} e^{i\frac{t}{2}\ln\left(\frac{t}{2\pi e}\right)} \left\{ \sum_{n=1}^{\infty} n^{-\frac{5}{4}-it} \right\}$$

$$+\ O(t^{-7/8}) \ .$$

Of course,

$$\int_{\frac{5}{4}+iT}^{\frac{5}{4}+i(2T)} O(t^{-7/8})\, dt \ = \ O(T^{1/8}) \ .$$

We now need to focus on

$$\mathcal{C} \int_{T}^{2T} \sum_{n=1}^{\infty} n^{-\frac{5}{4}} e^{-it\ln n}\, t^{\frac{1}{8}} e^{i\frac{t}{2}\ln\left(\frac{t}{2\pi e}\right)}\, dt$$

$$= c \sum_{n=1}^{\infty} n^{-\frac{5}{4}} \int_T^{2T} t^{\frac{1}{8}} e^{i\frac{t}{2}\ln\left(\frac{t}{2\pi e n^2}\right)} dt$$

$$= c \sum_{n=1}^{\infty} n^{-\frac{5}{4}} O(T^{5/8}) \quad \text{by (1) Fact 1 !!}$$

$$= O(T^{5/8}) .$$

It follows, by (11) + (12) + the above, that

$$\int_{\frac{5}{4}+iT}^{\frac{5}{4}+i(2T)} f(s)\, ds = O(T^{5/8}) .$$

By (10) bottom + (11) top, we finally get:

$$i\int_T^{2T} f(\tfrac{1}{2}+it)\, dt = O(T^{1/8}) + O(T^{5/8})$$
$$= O(T^{5/8}) .$$

Remark.

Landau uses Fact 6 for (11) line 2, i.e. $\int_{\text{horiz}} f(s)ds$.
Exploitation of the weaker Fact 4 produces $O(T^{1/2})$.
This is sufficient since

$$O(T^{5/8}) + O(T^{1/8}) + O(T^{1/2}) = O(T^{5/8}) .$$

Phragmén-Lindelöf can thus be avoided.

$$\boxed{\int_T^{2T} f(\tfrac{1}{2}+it)\,dt = O(T^{5/8})}$$

$\left\{ \begin{array}{l} \text{with main contribution due} \\ \text{to 13 lines 1~3 and} \\ \text{Fact 1} \end{array} \right\}$.

On the other hand, by Fact 2, on 3,

$$
\begin{aligned}
|f(\tfrac{1}{2}+it)| &= e^{\frac{\pi t}{4}} |\xi(\tfrac{1}{2}+it)| \\
&= e^{\frac{\pi t}{4}} \left| \pi^{-\frac{1}{2}(\frac{1}{2}+it)} \Gamma(\tfrac{1}{4}+i\tfrac{t}{2}) \zeta(\tfrac{1}{2}+it) \right| \\
&\geqq c\, e^{\frac{\pi t}{4}} \sqrt{2\pi}\, t^{\frac{1}{4}-\frac{1}{2}} e^{-\frac{\pi}{2}\frac{t}{2}} [1+O(\tfrac{1}{t})] |\zeta(\tfrac{1}{2}+it)| \\
&\geqq c\, t^{-1/4} |\zeta(\tfrac{1}{2}+it)|
\end{aligned}
$$

for t large. Hence:

$$
\begin{aligned}
\int_T^{2T} |f(\tfrac{1}{2}+it)|\,dt &\geqq c\,T^{-1/4} \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|\,dt \\
&\geqq c\,T^{-1/4}(T/2) \qquad \text{Fact 9 } 9 \\
&\geqq c\,T^{3/4} \quad ////.
\end{aligned}
$$

Accordingly, for each large T,

$$\left| \int_T^{2T} f(\tfrac{1}{2}+it)\,dt \right| < \frac{1}{2}\int_T^{2T} |f(\tfrac{1}{2}+it)|\,dt .$$

As such, there must be some point in $(T, 2T]$ where the real-valued continuous function $f(\frac{1}{2}+it)$ undergoes a change of sign.

> Remember that $f(s)$ is nicely analytic à la local Taylor series!

In other words: $(T, 2T]$ contains at least one odd order zero of $f(\frac{1}{2}+it)$.

By 3 top, hence likewise for $\zeta(\frac{1}{2}+it)$.

By studying the cases $T = 2^k$, we clearly get

$$N_{crit}(T) \to \infty$$

and, indeed,

$$N_{crit}(T) \gtrsim c \ln T \quad (\text{all large } T).$$

∎

A moment's thought about p. 15 shows that we have actually proved:

$$\#\{\underline{\text{distinct}}\ \rho : \mathrm{Re}(\rho) = \tfrac{1}{2},\ 0 < \mathrm{Im}(\rho) \leq T\}$$

$$\geq c \ln T.$$

Some further refinements were left for discussion in Lec 24.

[End of Lec 23]

Lecture 24 Synopsis

(15 Apr)

Function theory — centered on max mod principle, Phragmén–Lindelöf principle, Lindelöf mu-function, Littlewood's formula for $\int_{\alpha}^{\beta} N(\sigma; T_1, T_2)\, d\sigma$.

__Thm__ (Max Mod Principle)

Let D = bdd domain in $\mathbb{C}$.

Let F be analytic on D. Let

$$\limsup_{z \to \xi} |F(z)| \leq M, \quad \underline{\text{all}}\ \xi \in \partial D.$$

Then:

$$|F(z)| \leq M, \quad \text{all } z \in D.$$

If equality ever holds, then $F(z) \equiv M e^{i\theta}$ for some $\theta \in \mathbb{R}$.

__Pf__

As in function theory, with standard use of

$$F(z_0) = \frac{1}{2\pi} \int_0^{2\pi} F(z_0 + re^{i\varphi})\, d\varphi$$

for $0 < r < \operatorname{dist}(z_0, \partial D)$. ∎

Thm (Phragmén – Lindelöf)

Let D = bdd simply-connected domain. Let F be analytic on D. Let $|F| \leq G$ for some big constant G. Let

$$\overline{\lim_{z \to \xi}} |F(z)| \leq M, \quad \text{all } \xi \in \partial D \sim \{a_1, \dots, a_m\}.$$

Then:

$$|F(z)| \leq M, \quad \text{all } z \in D.$$

Pf

EG $m = 1$. $z - a_1 \neq 0$ on D. Construct single-valued branch $\log(z - a_1)$. Also $(z - a_1)^\varepsilon$. Let $F_\varepsilon = F \cdot \left(\frac{z - a_1}{R}\right)^\varepsilon$. Here $R = 2\,\mathrm{diam}\,(D)$.

Note $\overline{\lim_{z \to a_1}} |F_\varepsilon| = 0$. And $\overline{\lim_{z \to \xi}} |F_\varepsilon| \leq M \cdot 1$. Hence $|F_\varepsilon(z)| \leq M$. Fix any $z \in D$. Get

$$|F(z)| \leq M \left| \frac{R}{z - a_1} \right|^\varepsilon.$$

Let $\varepsilon \to 0$. ∎

(Simply-connected D taken for maximal simplicity in the proof.)

Counterexample if no G exists.

$$F(z) = \exp\left(\frac{i}{z}\right), \quad D = \{|z|<1, y>0\}$$

$$\mathcal{M} = e^{1}, \quad a_1 = \{0\}$$

$$e^{\frac{1}{y}} \to \infty \quad \text{as } y \to 0^+$$

(a, b finite)

Fact

Given $E = \{a<x<b, y>0\}$. Let F be analytic on E. Let $|F(z)| \leq G$.

Let

$$\overline{\lim_{z\to\xi}} |F(z)| \leq \mathcal{M}, \quad \text{all } \xi \in \partial E \cap \mathbb{C}.$$

Then $|F(z)| \leq \mathcal{M}$ on E.

Pf

Apply p. 2 after passing to a change of variable $\mathfrak{z} = \dfrac{1}{z+c}$, with c big enough to have $c+a>0$. The new domain $E_{\mathfrak{z}}$ is bounded. (c = real !!!) ∎

Fact

Let $E = \{-\frac{\pi}{2} < x < \frac{\pi}{2},\ y > 0\}$. The fcn $w = \sin z$ maps E in a 1-1 way onto $\{\mathrm{Im}(w) > 0\}$. ∂E corresponds to $\mathbb{R}$ in a nice fashion.

Proof

Look at the formula

$$\sin(x+iy) = \sin x \cdot \cosh y + i \cos x \cdot \sinh y.$$

Use standard fcn theory. $\blacksquare$

Note that:

$$F(z) = e^{-i\sin(z)} \qquad (z \in E)$$

has $|F(z)| > 1$, although $\lim_{z \to \xi} |F(z)| = 1$, each $\xi \in \partial E \cap \mathbb{C}$. Also, for fixed x in $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have:

$$|F(x+iy)| = e^{\cos x \sinh y}$$

<u>and</u> $\cos x \cdot \sinh y \sim \cos x \cdot \frac{1}{2} e^{y}$.

$$(y \to \infty)$$

<u>Thm</u> (compare Ingham p. 95) (classical thm P-L)

Let $E = \{q_1 < x < q_2,\ y > 0\}$. Let F be analytic on E; let

$$\overline{\lim_{z \to \xi}}\ |F(z)| \leq \mathcal{M}, \quad \text{all } \xi \in \partial E \cap \mathbb{C};$$

$$|F(x+iy)| \leq \mathcal{C} \exp[e^{cy}], \quad \text{some } \mathcal{C},$$

$$\text{some } 0 < c < \frac{\pi}{q_2 - q_1}.$$

Then:

$$|F(z)| \leq \mathcal{M} \text{ on } E.$$

<u>Pf</u>

wlog $q_1 = -\frac{\pi}{2}$, $q_2 = \frac{\pi}{2}$. Take $c < b < 1$.

Study

$$F_\varepsilon(z) \equiv F(z)\, e^{i\varepsilon \sin(bz)} \quad \text{on } E.$$

By formula for $\sin(x+iy)$ on 4, get

$$|F_\varepsilon(z)| = |F(z)|\, e^{-\varepsilon \cos(bx)\sinh(by)}$$

$$\overline{\lim_{z \to \xi}}\ |F_\varepsilon(z)| \leq \mathcal{M} \cdot 1$$

but

$$e^{cy} - \varepsilon\left(\cos b\tfrac{\pi}{2}\right)\sinh(by) \to -\infty$$

as $y \to +\infty$

$\Downarrow$

$|F_\varepsilon| \rightrightarrows 0$ as $y \to \infty$

$\Downarrow$

$|F_\varepsilon| \leq$ some G on E

and $\overline{\lim_{z \to \xi}} |F_\varepsilon| \leq M$, all $\xi \in \partial E \cap \mathbb{C}$.

Apply (3). Get $|F_\varepsilon(z)| \leq M$ on E,

so

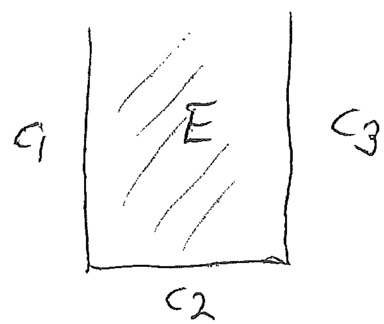
$$|F(z)| \leq M e^{\varepsilon \cos(bx)\sinh(by)}, \text{ each } z.$$

Let $\varepsilon \to 0^+$. Get $|F(z)| \leq M$. ∎

7

Corollary

$E = \{a_1 < x < a_2,\ y > 0\}$. F analytic on E. Let $|F(x+iy)| = O(e^{\sigma y})$, some giant σ. Let $\overline{\lim_{z \to \xi}} |F|$ be bdd à la



Then:

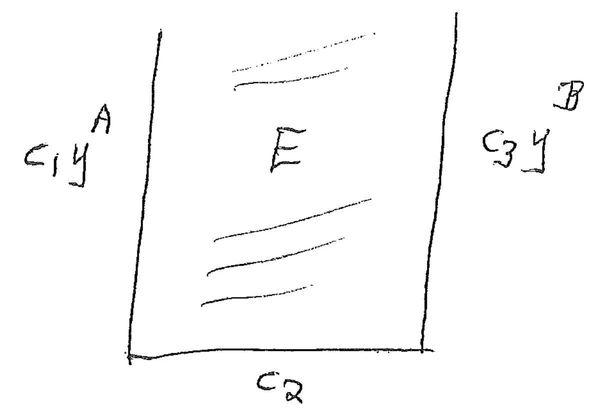
$$|F(z)| \leq \max\{c_1, c_2, c_3\} \quad \text{on } E.$$

Thm (convexity thm)

Given $E = \{a < x < b,\ y > y_0\}$ with a $y_0 > 0$. Let F be analytic on E and have

$$|F(x+iy)| = O(e^{Gy}), \quad G = \text{giant}.$$

Let $\overline{\lim\limits_{z \to \xi}}\, |F(z)|$ be bounded à la sizes



$$\left\{ \begin{array}{c} A \in \mathbb{R} \\ B \in \mathbb{R} \\ \text{negative OK} \\ c_j > 0 \end{array} \right\}.$$

We can then find a constant M depending in an *explicit* way on

$$\{E,\ A,\ B,\ \max\{c_1, c_2, c_3\}\}$$

such that

$$|F(x+iy)| \lesssim M\, y^{A\left(\frac{b-x}{b-a}\right) + B\left(\frac{x-a}{b-a}\right)}.$$

Pf (sketch)

WLOG $c_1 = c_2 = c_3 = 1$ and $a = 0$, $b = 1$.

Introduce (on E)

$$\mathrm{Log}(-iz) = \mathrm{Log}\, z - i\frac{\pi}{2}.$$

Look at

$$g(z) = \exp\left[(A(1-z) + Bz) \mathrm{Log}(-iz) \right].$$

Write, for $0 < x < 1$, $y > y_0$,

$$\mathrm{Log}(y - ix) = \ln y + \mathrm{Log}\left(1 - \frac{ix}{y}\right)$$
$$= \ln y - \frac{ix}{y} + O\left(\frac{1}{y^2}\right).$$

Get $[0 < x < 1,\ y > y_0]$:

$$|g(x+iy)| = y^{A(1-x) + Bx} \exp\left[O(1) \right].$$

depends on A, B, E

Form

$$H(z) \equiv \frac{F(z)}{g(z)} \quad \text{on } E.$$

$$\overline{\lim_{z \to \xi}} \; |H(z)| \lesssim \text{some } \beta \,, \qquad \xi \in \partial E \cap \mathbb{C}$$

while

$$|H(x+iy)| \lesssim \frac{O(1)\, e^{Gy}}{y^{A(1-x)+Bx} \exp[O(1)]}$$

$$\lesssim O(1)\, e^{2Gy} \quad \text{on} \quad E$$

$$\Downarrow$$

$$|H(z)| \leq \beta \,, \text{ all } z \in E$$

$$|F(z)| \leq \beta |g(z)|$$

$$|F(z)| \lesssim y^{A(1-x)+Bx} \beta \exp[O(1)] . \quad \blacksquare$$

Lindelöf mu-fcn. ← 1908

Let $F(z)$ be analytic on positive

$$E_0 = \{ \alpha < x < \beta \,,\; y > y_0 \} .$$

Assume:

$$|F(x+iy)| \lesssim O(e^{Gy}) \quad \text{on} \quad E_0 .$$

(Some giant G.)

We define, for $\alpha < x < \beta$,

$$\mu(x) = \inf\{\omega : |F(x+iy)| = O(y^{\omega})\}.$$

Here we allow $\mu(x) = \pm\infty$ in an obvious sense.

Tautologically, for each x,

$$\mu(x) = \overline{\lim_{y\to\infty}} \frac{\ln|F(x+iy)|}{\ln y}.$$

NOTE:

$$\alpha = -1, \quad \beta = 1, \quad y_0 = 1$$

$$F(z) = e^{-iz^2}$$

$$|F(z)| = e^{2xy}$$

$$|F(x+iy)| \lesssim e^{2y} \quad \text{on } E_0$$

$$\mu(x) = \left\{ \begin{array}{ll} -\infty, & -1 < x < 0 \\ 0, & x = 0 \\ +\infty, & 0 < x < 1 \end{array} \right\}.$$

Fact

Suppose that $\mu(x) < +\infty$ for all $x \in (\alpha, \beta)$. If $\mu(x_0) = -\infty$ for some $x_0 \in (\alpha, \beta)$, we must then have $\mu(x) \equiv -\infty$ on (α, β).

Pf

Simply apply p. 8 THM with appropriate a, b, A, B and let one of A or B tend incrementally to $-\infty$. ∎

Thm (convexity of μ)

Given F on E_0 as above. Assume that $-\infty < \mu(x) < +\infty$ for each $x \in (\alpha, \beta)$. The fcn $\mu(x)$ is then convex on (α, β); i.e.,

$$\mu[(1-t)x_1 + t x_2] \leq (1-t)\mu(x_1) + t\mu(x_2)$$

for $t \in [0,1]$ and $x_1 < x_2$ in (α, β).

Pf

Easy consequence of p. 8 THM. ∎

It is a standard thm of basic analysis that every (finite) convex fcn $\phi(x)$ on (a, β) is automatically continuous.

Let $F(z) = \zeta(z)$.

The Euler-Maclaurin development (in the style of Euler) given in Lec 9 p. 19 immediately shows that $\underline{\mu(x) < +\infty}$ for every $x \in \mathbb{R}$.

Cf. Lec 5 pp. 10 (line 5) + 12 (thm) for $x > 0$.

Recall $\mathcal{L}og\, \zeta(z)$ in Lec 6, pp. 3 + 4, in connection with

$$\zeta(z) = \prod_p \frac{1}{1-p^{-z}}, \quad Re(z) > 1.$$

Clearly:

$$\mathcal{L}og\, \zeta(z) = O_\varepsilon(1) \quad \text{for } x \geqq 1+\varepsilon.$$

Hence: $-A_\varepsilon \leqq \ln|\zeta(x+iy)| \leqq A_\varepsilon$ here.

Fact

$F(z)$ on 10

Given $\zeta(z)$. We have $-\infty < \mu(x) < +\infty$ for all $x \in \mathbb{R}$. In fact: $\mu(x) = 0$ for all $x > 1$.

Pf

Obvious by p. 13 and the Fact on 12. ∎

Now exploit $\xi(s) = \pi^{-s/2}\,\Gamma\!\left(\frac{s}{2}\right)\zeta(s)$ and $\xi(s) = \xi(1-s)$ à la Lec 11 p. 24.

Recall:

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}\,|t|^{\sigma - \frac{1}{2}}\,e^{-\frac{\pi}{2}|t|}\left(1 + O\!\left(\frac{1}{|t|}\right)\right)$$

for any $\sigma_1 \leq \sigma \leq \sigma_2$ and $|t| \geq 1$. See Lec 23 p. 4 Fact 3; also Lec 10 p. 42 for Stirling.

Get:

$$\pi^{-\frac{\sigma}{2}} \left| \Gamma\left(\frac{\sigma}{2}+i\frac{t}{2}\right)\right| \left|\zeta(\sigma+it)\right|$$

$$= \pi^{-\frac{1-\sigma}{2}} \left| \Gamma\left(\frac{1-\sigma}{2}-i\frac{t}{2}\right)\right| \left|\zeta(1-\sigma-it)\right|$$

$$\Downarrow$$

{ by (14) line -4 }

$$\pi^{-\frac{\sigma}{2}} \sqrt{2\pi} \left(\frac{t}{2}\right)^{\frac{\sigma}{2}-\frac{1}{2}} e^{-\frac{\pi}{4}t} \left|\zeta(\sigma+it)\right|$$

$$\sim \pi^{-\frac{1-\sigma}{2}} \sqrt{2\pi} \left(\frac{t}{2}\right)^{\frac{1-\sigma}{2}-\frac{1}{2}} e^{-\frac{\pi}{4}t} \left|\zeta(1-\sigma+it)\right|$$

{compare Lec 23 p. (5) }

$$\left|\zeta(\sigma+it)\right| \sim c(\sigma)\, t^{\frac{1}{2}-\sigma} \left|\zeta(1-\sigma+it)\right|$$

as $t \to +\infty$.

THM

For $F(s) = \zeta(s)$, we have

$$\mu(\sigma) = \mu(1-\sigma) + \frac{1}{2} - \sigma .$$

Pf

As above. ∎

By 13 (top), 14 (top), 15 THM, we get:

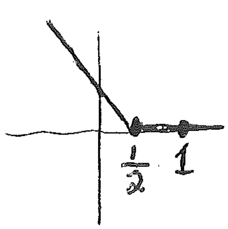
$$\mu(\sigma) = \begin{cases} 0, & \sigma \geq 1 \\ \frac{1}{2} - \sigma, & \sigma \leq 0 \end{cases}.$$

Application of p. 12 THM then gives:

$$\mu(\sigma) \leq \frac{1}{2} - \frac{\sigma}{2} \quad \text{for } 0 < \sigma < 1.$$

The <u>exact</u> value of $\mu(\sigma)$ at any given $\sigma \in (0,1)$ remains a mystery.

Lindelöf has <u>conjectured</u> (1908) that



$$\mu(\sigma) = \begin{cases} \frac{1}{2} - \sigma, & 0 < \sigma < \frac{1}{2} \\ 0, & \frac{1}{2} \leq \sigma < 1 \end{cases}.$$

It is known that the Riemann Hypothesis (Lec 16, p. 19, $\Theta = \frac{1}{2}$) <u>implies</u> Lindelöf's

$\boxed{\text{Lec 14, p. 12}}$

conjecture. See, e.g., Titchmarsh's book on $\zeta(s)$.

Fact

Lindelöf's conjecture is equivalent to proving that $\mu(\frac{1}{2}) = 0$.

Pf

Clearly Lindelöf $\Rightarrow$ $\mu(\frac{1}{2}) = 0$.

Now suppose $\mu(\frac{1}{2}) = 0$. By convexity 12 and $\mu(\sigma) = 0$ when $\sigma > 1$, we get $\mu \leq 0$ on $[\frac{1}{2}, 1]$.

If we had $\mu(x_0) < 0$ for some $x_0 \in (\frac{1}{2}, 1]$, application of 12 again would give

$$\mu(x_0) < 0,\ \mu(2) = 0 \Rightarrow \mu(\tfrac{3}{2}) < 0.$$

Contrad!! Hence $\mu = 0$ on $[\frac{1}{2}, 1]$.

By p. 15 THM, get $\mu = \frac{1}{2} - \sigma$ on $[0, \frac{1}{2}]$.

Hence all is OK. ▨

The **best** that is currently known is that $\mu(\frac{1}{2})$ is at most a specific fraction somewhat less than $\frac{1}{6}$.

It has sometimes been claimed that $\mu(\frac{1}{2}) \lesssim \frac{1}{8}$, but this has never panned out [i.e., proven to be correct]. The conventional wisdom is that achieving even this would be a "major advance".

Now we turn to Littlewood's formula.

Let $(\alpha, \beta) \times (T_1, T_2)$ be a given rectangle. We'll call it R. Let $f(s)$ be analytic on $R \cup \partial R$. Let $f(\beta + it) \neq 0$. Also let

$$f(\sigma + iT_1) \neq 0, \quad f(\sigma + iT_2) \neq 0.$$

We are completely happy if f vanishes at some points of $\{\sigma = \alpha\}$. $\{t \neq T_1, T_2\}$

Begin by defining a single-valued branch of $\log f(s)$ on a narrow open set containing $\{\sigma = \beta,\ T_1 \leq t \leq T_2\}$. For t-values *not* matching the *ordinate* of a zero of $f(s)$ on $R \cup \partial R$, define $\phi(s) \equiv \mathcal{L}og\, f(s)$ by *horizontal* analytic continuation starting with $\log f(s)$. Compare Lec 15 p. 25.

Once that is done, (we) then use continuity FROM *ABOVE* wrt t to take care of the ordinates of f-zeros. {Note that this makes good sense even for $\sigma = \alpha$.}

<u>THM</u> (Littlewood)

Given R, f as above. Let *

$N(u; T_1, T_2)$ = # of zeros of $f(s)$ on $R \cup \partial R$ having abscissa $\geq u$ (and counted WITH multiplicity).

We then have:

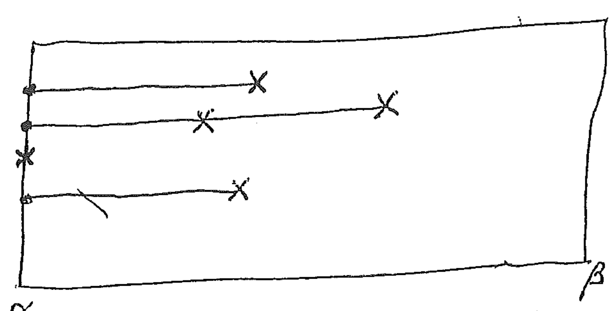
$$-\frac{1}{2\pi i}\oint_{\partial R} \phi(s)\,ds = \sum_{j=1}^{N} [\mathrm{Re}(\rho_j) - \alpha]$$

$$= \int_\alpha^\beta N(\sigma; T_1, T_2)\,d\sigma$$

using an obvious ρ_j notation for the zeros of f.

<u>Pf</u>

Make the connected open set <u>R'</u> by drawing



in an obvious way. The x's corr to ρ_j.

---

* Note that $N(u; T_1, T_2)$ is <u>right</u> continuous.

Write

$$f(s) = f_0(s)(s-\rho_1)\cdots(s-\rho_N)$$

$$\left\{ \begin{array}{l} f_0(s) \text{ analytic and nonzero} \\ \quad \text{on } R \cup \partial R \end{array} \right\}.$$

The branch $\mathcal{L}og\, f_0(s)$ is uniquely defined on $R \cup \partial R$ once it is "started" on $\sigma = \beta$.

Let us agree that the standard principal value $\mathrm{Log}\, z$ has $-\pi < \mathrm{Arg}(z) \leq \pi$. Then:

$$\mathrm{Log}(-q) = \lim_{\varepsilon \to 0^+} \mathrm{Log}(-q + i\varepsilon)$$

for every $q > 0$.

There is no loss of generality in presupposing that

$$\overset{\phi(s)}{\mathcal{L}og\, f(s)} = \mathcal{L}og\, f_0(s) + \sum_{j=1}^{N} \mathrm{Log}(s - \rho_j)$$

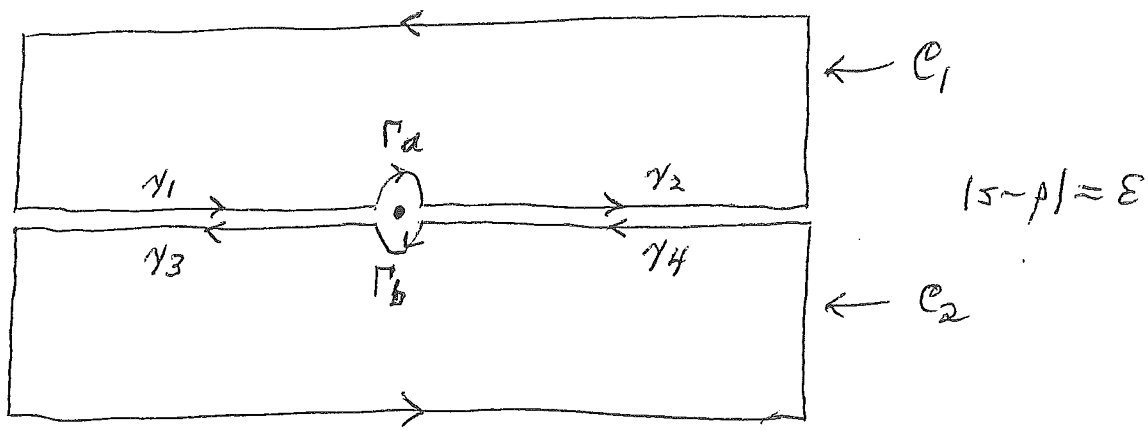
first along $\sigma = \beta$, then throughout $\underline{\underline{R'}}$.

Naturally, along $\partial R'$, one must be more careful [utilizing, e.g., the continuity from above idea].

also in $\mathrm{Log}\, z$

Take just one zero ρ_j and drop the j.

For simplicity, take $\alpha < Re(\rho) < \beta$. The case $Re(\rho) = \alpha$ is an easy adaptation.



$$\int_{\mathcal{C}_1} + \int_{\gamma_1} + \int_{\Gamma_a} + \int_{\gamma_2} Log(s-\rho)\, ds = 0$$

{by CIT}

$$\int_{\mathcal{C}_2} + \int_{\gamma_3} + \int_{\Gamma_b} + \int_{\gamma_4} Log(s-\rho)\, ds = 0$$

$$\left| \int_{\Gamma_a} Log(s-\rho)\, ds \right| \leq \int_{\Gamma_a} \left[\ln \frac{1}{\varepsilon} + 2\pi \right] |ds|$$

$$= O\left(\varepsilon \ln \frac{1}{\varepsilon}\right) \to 0$$

$$\left| \int_{\Gamma_b} Log(s-\rho)\, ds \right| = O\left(\varepsilon \ln \frac{1}{\varepsilon}\right) \to 0$$

similarly

Obviously:

$$\int_{\gamma_2} + \int_{\gamma_4} = 0 \qquad (\operatorname{Arg}(s-\rho) = 0).$$

But,

$$\int_{\gamma_1} \operatorname{Log}(s-\rho)\,ds = \int_{q}^{\operatorname{Re}(\rho)-\varepsilon} [\ln|s-\rho| + i\pi]\,d\sigma$$

$$\int_{\gamma_3} \operatorname{Log}(s-\rho)\,ds = -\int_{q}^{\operatorname{Re}(\rho)-\varepsilon} [\ln|s-\rho| - i\pi]\,d\sigma$$

$$\Rightarrow \int_{\gamma_1} + \int_{\gamma_3} = 2\pi i\,[\operatorname{Re}(\rho) - q] + O(\varepsilon).$$

Hence, collectively, we get:

$$\int_{C_1} + \int_{C_2} + 2\pi i\,[\operatorname{Re}(\rho) - q] = o(1)$$

$$\Downarrow$$

$$\oint_{\partial R} \operatorname{Log}(s-\rho)\,ds = -2\pi i\,[\operatorname{Re}(\rho) - q].$$

This will hold for <u>each</u> ρ_j.

Of course, by CIT,

$$\oint_{\partial R} \mathcal{L}og\, f_0(s)\, ds = 0 .$$

Adding produces:

(21) line −6

$$\oint_{\partial R} \mathcal{L}og\, f(s)\, ds$$

$$= -2\pi i \sum_{j=1}^{N} [\operatorname{Re}(\rho_j) - \sigma]$$

OR

$$-\frac{1}{2\pi i} \oint_{\partial R} \phi(s)\, ds = \sum_{j=1}^{N} [\operatorname{Re}(\rho_j) - \sigma] .$$

OK

If one writes

$$N(\sigma; T_1, T_2) = \sum_{\substack{\text{each} \\ \rho_j}} N_{\rho_j}(\sigma; T_1, T_2)$$

<u>in an obvious way</u>, we clearly get

$$\int_\alpha^\beta N(\sigma; T_1, T_2)\, d\sigma = \sum_j \left[\mathrm{Re}(\rho_j) - \alpha\right].$$

Here, of course, one can suppress any ρ_j having $\mathrm{Re}(\rho_j) = \alpha$. ∎

<u>Corollary</u> (Littlewood)

$$2\pi \int_\alpha^\beta N(\sigma; T_1, T_2)\, d\sigma$$

$$= \int_{T_1}^{T_2} \ln|f(\alpha + it)|\, dt - \int_{T_1}^{T_2} \ln|f(\beta + it)|\, dt$$

$$- \int_\alpha^\beta \arg f(\sigma + iT_1)\, d\sigma + \int_\alpha^\beta \arg f(\sigma + iT_2)\, d\sigma,$$

wherein $\arg f$ comes from $\mathcal{L}og\, f(s)$ à la (19).

<u>Pf</u>

Use (20) and take the appropriate real part. ∎

$$\boxed{\mathrm{Re}\left[i \oint_{\partial R} \phi(s)\, ds\right]}$$

Addendum

(a remark about Lec 23)

I commented that the technique of Lec 23 actually gives $\gtrsim (\text{const}) T^{\omega}$ zeros on the critical line for some small ω. I claim that $\omega \approx 1/8$ works.

More precisely, I claim that:

$$\#\{\text{online } \zeta\text{-zeros with } U < \gamma \leq 2U\} \gtrsim (\text{small constant})\, U^{1/8}$$

once U is large enough.

Let H be any number in $[T^{\frac{51}{100}}, T]$. Keep T large. Note that Lec 23, Fact 1, holds equally well for

$$\int_T^{T+H}.$$

Lec 23 Facts 2 – 7 require no change. On ⑦ (bottom) of Lec 23, look at

$$\int_T^{T+H} |f(\tfrac{1}{2}+it)|\,dt \quad \text{vs.} \quad \left|\int_T^{T+H} f(\tfrac{1}{2}+it)\,dt\right|.$$

Analog of Fact 8 is

$$\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i(T+H)} J(s)\,ds \approx iH + O(T^{1/2}).$$

note role of $n=1$

(also)
See Lec 23 p. 9 middle. The analog of Fact 9 is:

$$\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|\,dt > \tfrac{1}{2}H$$

<u>once</u> T is large enough.

On Lec 23 pp. 10 ~ 11, use $[\frac{1}{2}, \frac{5}{4}] \times [T, T+H]$.
On 12, get

$$\int_{\frac{5}{4}+iT}^{\frac{5}{4}+i(T+H)} O(t^{-7/8})\,dt = O(HT^{-7/8}) = O(T^{1/8})$$

since $H \leq T$. On 13 line 3, get $O(T^{5/8})$ again. Hence, on 14 top,

$$\int_T^{T+H} f(\tfrac{1}{2}+it)\,dt = O(T^{5/8}).$$

On 14 (bottom), we get

$$\int_T^{T+H} |f(\tfrac{1}{2}+it)|\,dt \gtrsim_< T^{-1/4}\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|\,dt \gtrsim_< T^{-1/4}(H/2) \geq c_2 H T^{-1/4}.$$

Observe, however, that

$$T^{5/8} \leqq c_2 H T^{-1/4}$$

anytime

$$H \geqq \frac{1}{c_2} T^{7/8} .$$

This suggests keeping

(*) $$H \geqq G T^{7/8}$$

for some giant constant G. Doing so clearly produces

$$\left| \int_T^{T+H} f(\tfrac{1}{2}+it)\,dt \right| < \frac{1}{2} \int_T^{T+H} |f(\tfrac{1}{2}+it)|\,dt$$

once T is large enough.

Hence, under (*), we find at least ONE true change of sign for $f(\frac{1}{2}+it)$ in $[T, T+H]$. See Lec 23 15 (lines 3–5).

All this being said, let U (now) be large and take:

$$H = G(2U)^{7/8} .$$

Let

$$U_n = U + nH, \qquad 0 \leq n \leq [\![\tfrac{U}{H}]\!] .$$

Look at the disjoint intervals

$$(U_{n-1}, U_n] \qquad (n \geq 1) .$$

We clearly get <u>at least</u> $[\![\frac{U}{H}]\!]$ true changes of sign of $f(\frac{1}{2}+it)$ [[hence, <u>distinct zeros</u>]] on $(U, 2U]$. This number clearly exceeds

$[\![\frac{U}{H}]\!]$ $\qquad$ $(\text{small constant})\, U^{1/8}$.

OK

T^{w} A review of this proof shows that a <u>similar</u> estimate holds for a wider class of Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad (a_1 \neq 0)$$

having functional equation similar to that of $\xi(s)$. The <u>total</u> number of zeros will still be $\sim (\text{constant})\, T \ln T$. And the existence of an Euler product will <u>NOT</u> be required.

Going back to $\zeta(s)$, I also noted that with a <u>much</u> harder proof, A. Selberg proved ☆

$$N_{crit}(T) > (\text{tiny constant})\, T \ln T .$$

(1942)

73–74

In the early 1970s, N. Levinson used a different [but related] approach to get

$$> \frac{1}{3}\left(\frac{T}{2\pi} \ln \frac{T}{2\pi e}\right) .$$

Conrey pushed this to

$$> 40\% \left(\frac{T}{2\pi} \ln \frac{T}{2\pi e}\right) .$$

☆ Hardy and Littlewood reached $> cT$ in 1921.

Lecture 25 Synopsis

(Wed, 20 Apr)

The lecture covered a variety of topics.

First, regarding Lindelöf's μ-function for $\zeta(s)$. Cf. Lec 24 p. 11 ff.

note Lec 24, p. 13 lines 4–8

Thm

Consider $f(s) = \zeta(s)$ for $\mathrm{Im}(s) \geq 1$, say.

(a) $\mu(\sigma) + (\sigma - \frac{1}{2}) = \mu(1-\sigma)$

(b) $\mu(\sigma) = 0, \quad \sigma > 1$

(c) $\mu(\sigma) = \frac{1}{2} - \sigma, \quad \sigma < 0$

(d) $\mu(\sigma)$ is convex on every $[a,b]$

(e) $\mu(\sigma)$ is continuous on $\mathbb{R}$

(f) $\mu(\sigma) \geq 0$

(g) $\mu(\sigma)$ is monotonic decreasing

(h) $\mu(\frac{1}{2}) \leq \frac{1}{4}$

(i) Lindelöf's conjecture is true $\Leftrightarrow \mu(\frac{1}{2}) = 0$.

Pf

(a) Lec 24 15.

(b) Lec 24 13 bot + 14.

(c) Combine (a) + (b). See Lec 24 16.

(d) Lec 24 12.

(e) Lec 24 (13) top.

(f) Know $\mu(\sigma) = 0$, $\sigma > 1$. Hence $\mu(\sigma) = 0$, $\sigma \geqq 1$.

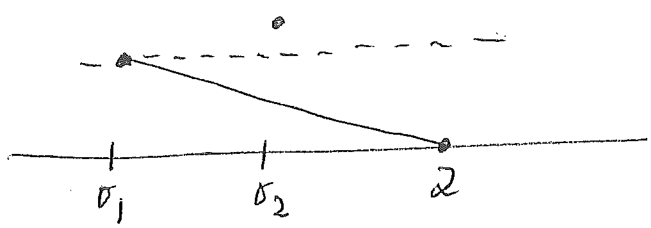
<u>Suppose</u> $\mu(\sigma_1) < 0$ with some $\sigma_1 < 1$. Take $\sigma_2 = 2$ and apply convexity over $[\sigma_1, \sigma_2]$.

Get $\mu(\frac{3}{2}) < 0$. Contrad!

(g) Know $\mu(\sigma) \geqq 0$. And $\mu(\sigma) = 0$, $\sigma \geqq 1$.

<u>Suppose</u> $\sigma_1 < \sigma_2$ has $0 \leqq \mu(\sigma_1) < \mu(\sigma_2)$.

So, $\sigma_2 < 1$. Look at convexity over $[\sigma_1, 2]$.



This violates convexity (at σ_2).

(h) Lec 24 p. (16) line 4, put $\sigma = 1/2$.

(i) Lec 24 p. (17). ▨

— — — —

Recall <u>Lindelöf's Conjecture</u>

$$\mu(\sigma) = \begin{cases} 0, & \frac{1}{2} \leqq \sigma < \infty \\ \frac{1}{2} - \sigma, & -\infty < \sigma \leqq \frac{1}{2} \end{cases}.$$

It is known that RH $\Rightarrow$ Lindelöf Conjecture. $\boxed{\text{Lec 24 (16)}}$

2nd topic.

I briefly discussed the following thm.

Thm

Let $f(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ be a given generalized Dirichlet series with $1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$.

Suppose the series converges at $s_0 \in \mathbb{C}$.

Then:

(a) the series conv uniformly on every Stolz angle

$$\left\{ |\operatorname{Arg}(s - s_0)| \leqq \frac{\pi}{2} - \delta \right\} ;$$

(b) the series conv uniformly on every "super" Stolz angle

$$\left\{ |t - t_0| \leqq e^{M(\sigma - \sigma_0)} - 1 \right\}$$

$(M > 0)$.

The proof (omitted here) is an interesting exercise. Of course, (a) is known already by Lec 21, p. 11 Fact 2. Concerning (b), I simply remark: just study Lec 21, p. 13, line 7 when (wlog) $s_0 = 0$. For $\sigma > A$ big [but frozen], notice that:

$$\sigma \lessapprox e^{M\sigma} \qquad (M \geqq 1 \text{ wlog})$$

$$|t| \lessapprox e^{M\sigma} - 1 \lessapprox e^{M\sigma}$$

$$|s| \lessapprox 2e^{M\sigma} \quad \text{a priori}$$

$$\Downarrow$$

get a $\frac{\varepsilon \cdot 2}{A} e^{-\sigma(\ln N - M)}$ term!

Needless to say, by a minor expungement and insertion (of a new "λ"), we can actually allow ANY λ_1 in the above Thm; we do not need $\lambda_1 = 1$ ONLY $\lambda_1 > 0$.

Because of 3 Thm, Stolz angles or "super" Stolz angles are natural vehicles on which to discuss, e.g., identity theorems of the sort

$f_1(\xi_k) = f_2(\xi_k)$, all $k \geqq 1$ $\Rightarrow$ $a_{n1} \equiv a_{n2}$.

3rd topic.

We did a quick review of basic Fourier transforms and related analysis.

$$\hat{f}(p) \equiv \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i p x}\, dx \qquad p \in \mathbb{R}$$

$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$, f piecewise C^1 basically

$$\frac{f(x+0)+f(x-0)}{2} = \int_{-\infty}^{\infty} \hat{f}(p)\, e^{2\pi i p x}\, dp$$

$$RHS \equiv \lim_{R\to\infty} \int_{-R}^{R} \hat{f}(p)\, e^{2\pi i p x}\, dp$$

$$\tilde{f}(u) \equiv \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx$$

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(u)\, e^{iux}\, du$$

For "nice" functions f, g (real or complex) on $\mathbb{R}$, we define the <u>convolution</u>

$$H(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\, dt .$$

$H(x)$ is a reasonable function, due to

$$|H(x)| \leq \int_{-\infty}^{\infty} |f(t)| |g(x-t)|\, dt$$

$$\int_{-\infty}^{\infty} |H(x)|\, dx \leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(t)| |g(x-t)|\, dt \right) dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t)| |g(x-t)|\, dx\, dt$$

{Fubini}

$$= \int_{-\infty}^{\infty} |f(t)| \left(\int_{-\infty}^{\infty} |g(\xi)|\, d\xi \right) dt$$

$$= \left(\int_{-\infty}^{\infty} |f(t)|\, dt \right) \left(\int_{-\infty}^{\infty} |g(\xi)|\, d\xi \right)$$

$$< \infty .$$

Often, f and g are initially kept in the Schwartz class $\mathcal{S}$.

One easily checks that $H(x)$ is continuous and bounded if <u>either</u> f or g is known to be bounded. This is true in all "sensible" cases.

review 6 middle

What is also checked quite easily by Fubini and the key fact that

$$e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)} \qquad \begin{matrix} \theta \in \mathbb{R} \\ \phi \in \mathbb{R} \end{matrix}$$

is the <u>relation</u> from Fourier transform theory

$$\hat{H}(p) = \hat{f}(p)\hat{g}(p)$$

$$\left(\text{also } \tilde{H}(u) = \tilde{f}(u)\tilde{g}(u) \right).$$

This is WHY the convolution $H = f * g$ is so useful!

— — —

Another useful property goes as follows. Assume $f, g, \hat{f}, \hat{g}$ are all "nice". Then, observe that:

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$$

$$= \int_{-\infty}^{\infty} f(x) \overline{\left[\int_{-\infty}^{\infty} \hat{g}(p) e^{2\pi i p x} dp \right]}\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \left[\int_{-\infty}^{\infty} \overline{\hat{g}(p)}\, e^{-2\pi i p x} dp \right] dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \overline{\hat{g}(p)}\, e^{-2\pi i p x}\, \underline{dx}\, dp$$

(by Fubini)

$$= \int_{-\infty}^{\infty} \overline{\hat{g}(p)} \left[\int_{-\infty}^{\infty} f(x) e^{-2\pi i p x} dx \right] dp$$

$$= \int_{-\infty}^{\infty} \hat{f}(p) \overline{\hat{g}(p)}\, dp \quad .$$

Thus,

$$\int_{-\infty}^{\infty} |f_1(x) - f_2(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}_1(p) - \hat{f}_2(p)|^2 dp$$

for nice f_j. In particular:

$$\int_{-\infty}^{\infty} |f_1(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}_1(p)|^2 dp \, .$$

This is the <u>Plancherel formula</u>.

Let $\chi_E(x) = \begin{Bmatrix} 1, & x \in E \\ 0, & x \notin E \end{Bmatrix}$. One easily checks:

$$\widetilde{\chi_{[-c,c]}(x)} = 2\,\frac{\sin cu}{u}$$

$$\widetilde{\max(0, b-|x|)} = 2\,\frac{1-\cos bu}{u^2} \approx \frac{4\sin^2(\frac{b}{2}u)}{u^2}$$

$$\{u = 2\pi p\}$$

$$\widehat{\chi_{[-c,c]}(x)} = \frac{\sin 2\pi p c}{\pi p}$$

$$\widehat{\max(0, b-|x|)} = \frac{1-\cos 2\pi p b}{2\pi^2 p^2} = \frac{\sin^2(\pi p b)}{\pi^2 p^2}.$$

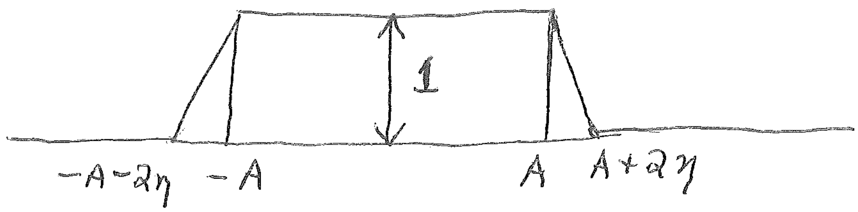
It will be convenient to consider the convolution

$$T(x) \equiv \frac{1}{2\eta}\,\chi_{[-\eta,\eta]}(x) * \chi_{[-A-\eta,\,A+\eta]}(x).$$

THM

$A > 0$, $\eta > 0$. $T(x)$ as above. Then:

(1) $T(x)$ is the trapezoid



(2)
$$\widetilde{T(x)} = \frac{\cos(Au) - \cos((A+2\eta)u)}{\eta u^2}$$

(3)
$$\widetilde{T(x)} = 2\,\frac{\sin(\eta u)\sin((A+\eta)u)}{\eta u^2}.$$

Pf

By 7 + 9,

$$\widetilde{T(x)} = \frac{1}{2\eta}\, 2\,\frac{\sin \eta u}{u}\, 2\,\frac{\sin(A+\eta)u}{u}$$

$$= 2\,\frac{\sin(\eta u)\,\sin((A+\eta)u)}{\eta u^2}, \qquad (11)$$

so (3) is OK. Of course,

$$\cos(\theta-\phi) - \cos(\theta+\phi) = 2\sin\theta\sin\phi$$

$$\Downarrow$$

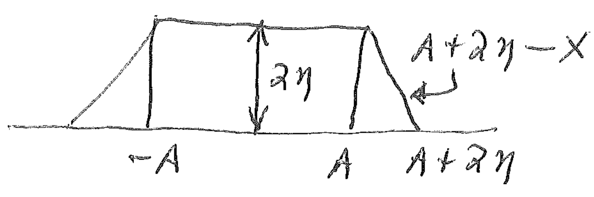
$$\cos(Au) - \cos((A+2\eta)u) = 2\sin((A+\eta)u)\sin(\eta u)$$

$$= 2\sin(\eta u)\sin((A+\eta)u),$$

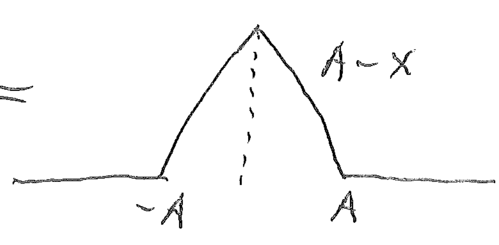
so (2) is OK too.

For (1), define $g(x)$ to be the trapezoid (shown) on (10). Look at:

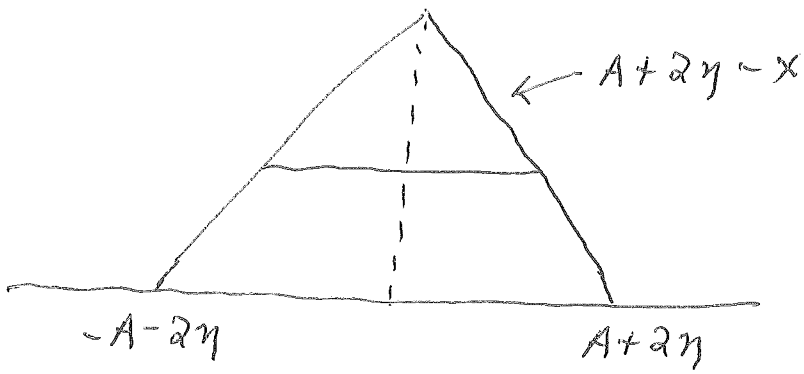
$2\eta\, g(x) =$



$\max(0, A-|x|) =$



$$2\eta\, q(x) + \max(0, A-|x|) =$$



$$= \max(0, A+2\eta-|x|)$$

$$\Downarrow$$

$$\boxed{2\eta\, q(x) = \max(0, A+2\eta-|x|) - \max(0, A-|x|)}$$

$$2\eta\, \widetilde{q(x)} = 2\left[\frac{1-\cos((A+2\eta)u)}{u^2}\right] - 2\left[\frac{1-\cos(Au)}{u^2}\right]$$

$$= 2\left[\frac{\cos(Au)-\cos((A+2\eta)u)}{u^2}\right]$$

$$\Rightarrow \widetilde{q(x)} = \frac{\cos(Au)-\cos((A+2\eta)u)}{\eta u^2}.$$

By (2) on p. 10,

$$\widetilde{q(x)} = \widetilde{T(x)} \quad . \qquad (\text{all } u \in \mathbb{R})$$

Apply 5 last line to this situation. Get

$$q(x) = T(x), \quad \text{each } x \in \mathbb{R},$$

since q and T are continuous on $\mathbb{R}$. Cf. also the $\tilde{f}_j$ counterpart of 8 line -4. In any case, (1) is now true. ∎

4^{th} Topic.

THM (an important L_2 estimate for Dirichlet polynomials).

We have:

$$\int_{\alpha}^{\alpha+H} \left| \sum_{j=1}^{N} b_j e^{-i\lambda_j t} \right|^2 dt = H \sum_{j=1}^{N} |b_j|^2 + \frac{O(1)}{\delta} \sum_{j=1}^{N} |b_j|^2$$

anytime

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N$$

$$|\lambda_k - \lambda_j| \geq \delta \ , \quad \text{all } k \neq j$$

$$b_j \in \mathbb{C} \ , \quad \alpha \in \mathbb{R} \ , \quad H > 0 \ .$$

The "implied" constant in $O(1)$ is absolute; it can be taken to be $\frac{4\pi}{\sqrt{3}}$.

Pf

Some easy wlog's (giving same implied constant in $O(1)$).

First one: $\alpha = -\frac{H}{2}$. Second one: $H = 2$.

Must prove:

$$\int_{-1}^{1} \Big| \sum_{j=1}^{N} b_j e^{-i\lambda_j t} \Big|^2 dt = 2 \sum_{j=1}^{N} |b_j|^2 + \frac{O(1)}{\delta} \sum_{j=1}^{N} |b_j|^2 .$$

Take $T(t)$ on (10) with $A=1$, $\eta=\eta$. Let

$$\mathcal{F} = \int_{-1}^{1} \Big| \sum b_j e^{-i\lambda_j t} \Big|^2 dt .$$

Clearly

$$\mathcal{F} \lesseqqgtr \int_{\mathbb{R}} \Big| \sum b_j e^{-i\lambda_j t} \Big|^2 T(t)\, dt$$

$$\mathcal{F} \lesseqqgtr \sum_{j,k} b_j \bar{b}_k \int_{\mathbb{R}} T(t) e^{-i(\lambda_j - \lambda_k)t} dt$$

$\{$ put $d_{jk} = \lambda_j - \lambda_k$ $\}$

$$\mathcal{F} \leqq \sum_{j,k} b_j \bar{b}_k \frac{2}{\eta} \frac{\sin(\eta d_{jk})}{d_{jk}} \frac{\sin[(1+\eta) d_{jk}]}{d_{jk}}$$

by 10 (3)

$$\mathcal{F} \leqq \sum_{j=1}^{N} |b_j|^2 \left\{ \frac{2}{\eta} \eta (1+\eta) \right\}$$

$$+ \frac{2}{\eta} \sum_{j \neq k} b_j \bar{b}_k \frac{\sin(\eta d_{jk})}{d_{jk}} \frac{\sin[(1+\eta) d_{jk}]}{d_{jk}}$$

$$\leqq (2+2\eta) \sum_{1}^{N} |b_j|^2$$

$$+ \frac{2}{\eta} \sum_{j \neq k} Re(b_j \bar{b}_k) \boxed{*} \boxed{*}$$

$$\left\{ \text{but } 2 Re(b_j \bar{b}_k) \leqq |b_j|^2 + |b_k|^2 \right\}$$

$$\leqq (2+2\eta) \sum_{1}^{N} |b_j|^2 + \frac{1}{\eta} \sum_{j \neq k} \frac{|b_j|^2 + |b_k|^2}{d_{jk}^2}$$

$$= (2+2\eta) \sum_{1}^{N} |b_j|^2 + \frac{2}{\eta} \sum_{j \neq k} \frac{|b_j|^2}{d_{jk}^2}$$

$$\leq (2+2\eta) \sum_{1}^{N} |b_j|^2$$

$$+ \frac{2}{\eta} \sum_{1}^{N} |b_j|^2 \left(2 \sum_{m=1}^{\infty} \frac{1}{m^2 \delta^2} \right)$$

$$\left\{ \begin{array}{l} \text{somewhat crudely} \\ \text{via } |\lambda_j - \lambda_k| \geq \delta > 0 \\ \text{for } j \neq k \end{array} \right\}$$

$$= (2+2\eta) \sum_{1}^{N} |b_j|^2 + \frac{4}{\eta \delta^2} \sum_{1}^{N} |b_j|^2 \frac{\pi^2}{6}$$

$$\left\{ \text{let } D = \frac{\pi}{\sqrt{3}} \right\}$$

$$= (2+2\eta) \sum_{1}^{N} |b_j|^2 + \frac{2D^2}{\eta \delta^2} \sum_{1}^{N} |b_j|^2$$

$$= \left(2 + 2\eta + \frac{2D^2}{\eta\delta^2}\right) \sum_{1}^{N} |b_j|^2 .$$

To minimize RHS, take

$$\eta = \frac{D}{\delta} .$$

Get:

$$\mathcal{F} \leqq \left(2 + \frac{4D}{\delta}\right) \sum |b_j|^2$$

or

$$\int_{-1}^{1} \left| \sum_{j=1}^{N} b_j e^{-i\eta_j t} \right|^2 dt \leqq 2 \sum_{1}^{N} |b_j|^2$$

$$+ \frac{4\pi/\sqrt{3}}{\delta} \sum_{1}^{N} |b_j|^2 .$$

This, of course, is the <u>upper</u> bound posited on page 14.

[The remainder of the proof was $\overbrace{\ }^{\text{(actually)}}$ done in Lec 26, but we include it here!]

The <u>lower</u> bound for $\mathcal{F}$ is similar but slightly harder. Must prove:

$$\mathcal{F} \geqq \left(2 - \frac{4D}{\delta}\right) \sum_{1}^{N} |b_j|^2 .$$

If $\delta \leq 2D = \frac{2\pi}{\sqrt{3}}$, matters are trivial.

So, wlog, $\delta > 2D$. Hence, $\frac{1}{2} > \frac{D}{\delta}$.

We consider $T(t)$ on 10 with $A = 1 - 2\eta$, $0 < \eta < \frac{1}{2}$, and observe that $A + 2\eta = 1$. Here

$$\mathcal{J} \geq \int_{-\infty}^{\infty} T(t) \left| \sum_{1}^{N} b_j e^{-i\lambda_j t} \right|^2 dt$$

$$= \sum b_j \bar{b}_k \int_{-\infty}^{\infty} T(t) e^{-i(\lambda_j - \lambda_k)t} \, dt$$

$$= \sum b_j \bar{b}_k \, 2 \frac{\sin(\eta d_{jk})}{\eta d_{jk}} \frac{\sin((1-\eta) d_{jk})}{d_{jk}}$$

$$= \sum_{j=1}^{N} |b_j|^2 (2 - 2\eta)$$

$$+ \sum_{j \neq k} Re(b_j \bar{b}_k) \, 2 \frac{\sin \eta d_{jk}}{\eta d_{jk}} \frac{\sin((1-\eta) d_{jk})}{d_{jk}}$$

{ as on 16 }

$$\geq (2 - 2\eta) \sum_{1}^{N} |b_j|^2 - \sum_{j \neq k} \left(|b_j|^2 + |b_k|^2 \right) \frac{1}{\eta d_{jk}} \frac{1}{d_{jk}}$$

$$= (2 - 2\eta) \sum_{1}^{N} |b_j|^2 - \frac{2}{\eta} \sum_{1}^{N} |b_j|^2 \left(\sum_{k \neq j} \frac{1}{d_{jk}^2} \right)$$

{ as on 16 bottom }

$$\geq (2-2\eta)\sum_1^N |b_j|^2 - \frac{2}{\eta}\sum_1^N |b_j|^2 \frac{1}{\delta^2} 2\left(\frac{\pi^2}{6}\right)$$

$\{$ cf. 17 middle $\}$ $\{ D = \frac{\pi}{\sqrt{3}} \}$

$$= \sum_1^N |b_j|^2 \left(2 - 2\eta - \frac{2D^2}{\eta\delta^2}\right).$$

Take $\eta = \frac{D}{\delta}$. Since $\frac{1}{2} > \frac{D}{\delta}$, η is admissible.

Get

$$\mathcal{J} \geq 2\sum_1^N |b_j|^2 - \frac{4D}{\delta}\sum_1^N |b_j|^2,$$

with $4D \approx \frac{4\pi}{\sqrt{3}}$, parallel to 18 lines 4–5. This is the lower bound promised. ▨

Let $\mathcal{C}$ be the constant in the $O(1)$ on 14. The theorem on 14 is very closely related to the <u>generalized Hilbert inequality</u>

$$(*) \qquad \left|\sum_{j\neq k} \frac{z_j \bar{z}_k}{\lambda_j - \lambda_k}\right| \leq \frac{\mathcal{C}/2}{\delta} \sum_{j=1}^N |z_j|^2 \qquad (z_j \in \mathbb{C}).$$

One readily checks that $(*) \Rightarrow$ thm on 14. Selberg has noted [in a very slick proof] that the thm on 14 $\Rightarrow$ $(*)$.

By choosing majorant/minorant functions more sophisticated than trapezoids, one finds that the best C is 2π.

Note:

$$\frac{4\pi}{\sqrt{3}} = 2\pi(1.1547^{+}),$$

which is not too bad!

Lecture 26 Synopsis

(Fri, 22 Apr)

Recall E-M version 2 from Lec 9, pp. 12 + 14.
Taking $R=0$ led to

$$\tilde{B}_1(x) = -2\sum_1^\infty \frac{\sin 2\pi n x}{2\pi n} = x - [\![x]\!] - \frac{1}{2}$$

for $x \notin \mathbb{Z}$. See Lec 9, pp. 13 bottom, 14 top.

In Lec 9, p. 19, we saw that

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1} - z\int_1^\infty \frac{\tilde{B}_1(u)}{u^{z+1}}\,du \quad , \quad x>1 .$$

In fact, the derivation on 18 bottom, with
$N \hookrightarrow N-1$ and $u = 1+t$, produced:

$$\sum_{k=1}^N k^{-z} = \frac{1}{2} + \frac{1}{z-1} + \frac{1}{2}N^{-z} + \frac{N^{1-z}}{1-z} - z\int_1^N \frac{\tilde{B}_1(u)}{u^{z+1}}\,du .$$

It is natural to <u>subtract</u> these 2 formulae.

Get:

$$\zeta(z) \sim \sum_1^N k^{-z} = -\frac{1}{2}N^{-z} - \frac{N^{1-z}}{1-z} - z\int_N^\infty \frac{\tilde{B}_1(u)}{u^{z+1}}\,du,$$

where the final term is nicely analytic on $\{\mathrm{Re}(z)>0\}$ thanks to $|\tilde{B}_1(u)| \leq 1/2$.

It follows that

$$(\star)\quad \zeta(z) = \sum_1^N n^{-z} - \frac{N^{1-z}}{1-z} - \frac{1}{2}N^{-z} - z\int_N^\infty \frac{\tilde{B}_1(u)}{u^{z+1}}\,du$$

on $\{\mathrm{Re}(z)>0\} \sim \{1\}$. Compare Lec 6, 10 line 4.

In numerical work (evaluating $\zeta(z)$), one often uses the counterpart of $(\star)$ associated with $\tilde{B}_{2R+1}(u)$ and the remainder term

$$(-1)\,z(z+1)\cdots(z+2R)\int_N^\infty \frac{\tilde{B}_{2R+1}(u)}{(2R+1)!}\,\frac{1}{u^{z+2R+1}}\,du.$$

Cf. Lec 9 pp. 18 + 19. One takes R and N appropriately large.

These comments hint that controlling the size of $\Xi(s)$ in the critical strip $\{0 < \mathrm{Re}(s) < 1\}$ comes down to doing the same for CERTAIN sums

$$\sum_{n=1}^{N} n^{-\sigma} n^{-it} \equiv \sum_{n=1}^{N} n^{-\sigma} e^{-it \ln n} .$$

The size of N will depend at least loosely on the magnitude of $|t|$. (See, e.g., p. 17.)

The issue of numerical calculation of $\Xi(s)$ deserves a separate lecture!! It will, however, play no role in the remaining lectures in this course.

In the present lecture, the goal is to simply obtain an important formula of Hardy and Littlewood growing out of 2 (*).

We need a preliminary.

THM

Let $f(x)$ be real and C^1 on $[a,b]$. Let $f'(x)$ be monotonic here. Assume, say, that $0 \leqq f'(x) \leqq \delta < 1$. Then:

$$\sum_{a<n\leqq b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)}\,dx + \frac{O(1)}{1-\delta},$$

with an absolute "implied" constant.

Proof

By inflating $O(1)$, wlog, a and b are integers and $b-a \geqq 100$. Indeed, we can also assume that $a=0$.

also Lec 8 14

By E-M version I (Lec 9 14 $R=0$), know:

$$\sum_{0\leqq n\leqq b} e^{2\pi i f(n)} = O(1)+O(1)+\int_0^b e^{2\pi i f(x)}\,dx$$
$$+\int_0^b \left(x-[\![x]\!]-\tfrac{1}{2}\right) 2\pi i f' e^{2\pi i f}\,dx.$$

So,

$$\sum_{0<n\leq b} e^{2\pi i f(n)} = O(1) + \int_0^b e^{2\pi i f(x)}\,dx$$

$$+ 2\pi i \int_0^b \left(\sum_1^\infty \frac{\sin 2\pi n x}{-\pi n} \right) f' e^{2\pi i f}\,dx$$

purists may prefer an "m" here!

{see Lec 9 p. 9}

$$= O(1) + \int_0^b e^{2\pi i f(x)}\,dx$$

$$- 2i \sum_{n=1}^\infty \frac{1}{n} \int_0^b \sin(2\pi n x) f' e^{2\pi i f}\,dx$$

$$= O(1) + \int_0^b e^{2\pi i f(x)}\,dx$$

$$+ \sum_{n=1}^\infty \frac{1}{n} \int_0^b \left(e^{-2\pi i n x} - e^{2\pi i n x}\right) f' e^{2\pi i f}\,dx \,.$$

This last sum

$$= \sum_1^\infty \frac{1}{n} \left[\int_0^b e^{2\pi i (f(x) - nx)} f'\,dx - \int_0^b e^{2\pi i (f(x)+nx)} \cdot f'\,dx \right]$$

$$= \sum_{1}^{\infty} \frac{1}{n} \left(\frac{1}{2\pi i} \int_0^b \frac{f'(x)}{f'(x)-n} \, d e^{2\pi i [f(x) - nx]} \right.$$

$$\left. - \frac{1}{2\pi i} \int_0^b \frac{f'(x)}{f'(x)+n} \, d e^{2\pi i [f(x) + nx]} \right).$$

We note herein that $n \neq 0$, f' is monotonic, and $0 \leqq f'(x) \leqq \delta < 1$.

Recall 2^{nd} mean value thm (Lec 22 5):

$$\int_A^B g(x) \, d\varphi(x) = g(A) \int_A^{\xi} d\varphi + g(B) \int_{\xi}^{B} d\varphi$$

monotonic

φ real and $C'[A, B]$.

Complex φ are treated via $\varphi = \varphi_1 + i\varphi_2$.

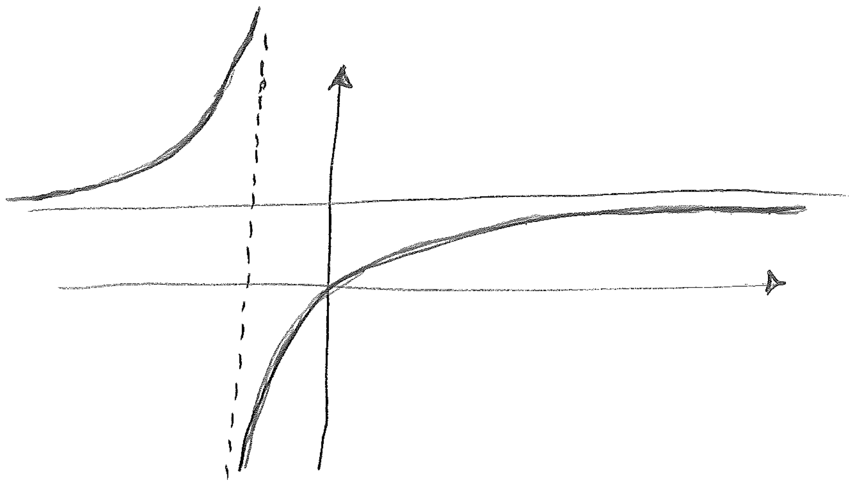
Also, notice that:

$$\left(\frac{u}{u+v}\right)' = \frac{(u+v)\cdot 1 - u\cdot 1}{(u+v)^2} = \frac{v}{(u+v)^2}$$

for $v \in \mathbb{Z} \sim \{0\}$.

This derivative has <u>fixed</u> <u>sign</u> for $u \neq -v$.

EG $\frac{u}{u+1}$



So, each $\frac{f'(x)}{f'(x) \pm n}$ is <u>monotonic</u> on $[0,b]$.

Look at:

$$\int_0^b \frac{f'(x)}{f'(x)+n}\, d\,e^{2\pi i(f(x)+nx)} \qquad (n \geq 1)$$

treated as $q_1 + i q_2$

and apply <u>2nd mean value thm</u> àla 6 bottom.

Get:

$$\boxed{0 \leq f' \leq \delta}$$

$$\int_0^b \frac{f'(x)}{f'(x)+n}\, d e^{2\pi i (f(x)+nx)}$$

$$= O\left(\frac{1}{n}\right).$$

Hence, on 6 top, we see that line 2

contributes

$$\sum_1^\infty \frac{1}{n} O\left(\frac{1}{n}\right) = O(1). //$$

We now look at

$$\int_0^b \frac{f'(x)}{f'(x)-n}\, d e^{2\pi i (f(x)-nx)} \qquad (n \geq 1)$$

treated as $\varphi_1 + i\varphi_2$

analogously. Since $0 \leq f'(x) \leq \delta < 1$,

$$n = 1 \implies O\left(\frac{1}{1-\delta}\right)$$

$$n \geq 2 \implies O\left(\frac{1}{n-1}\right)$$

$$\Downarrow$$

we see that on 6 top, in <u>line 1</u>, the collective contribution is:

$$O\left(\frac{1}{1-\delta}\right) + \sum_{n=2}^{\infty} \frac{1}{n} O\left(\frac{1}{n-1}\right)$$

$$= O\left(\frac{1}{1-\delta}\right). \quad //$$

Note how $n=1$ plays a special role in this portion of things.

Reviewing 5, we conclude that:

$$\sum_{0<n\leq b} e^{2\pi i f(n)} = O(1) + \int_0^b e^{2\pi i f(x)}\,dx$$

$$+ O(1) + O\left(\frac{1}{1-\delta}\right).$$

This proves p. 4 THM. $\blacksquare$

Remark

One can obviously do $-\delta \leqq f'(x) \leqq 0$ in much the same way.

Still keeping f' monotonic, by splitting the original sum into 2 chunks, if need be, one can handle $-\delta \leqq f'(x) \leqq \delta$ as well.

Additional Remark

More general forms of p. 4 THM certainly suggest themselves. Cf., eg, Lec 22 5 + 9 bottom half. *

Theorems of this sort (with summations) arose in the work of van der Corput from 1921/22.

---

* Also, Titchmarsh, Theory of $\zeta(s)$, near §4.10.

THEOREM (Hardy–Littlewood, Math. Zeit. 10 (1921).)

Given any $\sigma_0 \in (0, \frac{1}{10}]$, say. Given any $C > 1$.
Keep

$$\sigma_0 \leqq \sigma \lessapprox 2, \quad |t| \gtrapprox 100.$$

Then:

$$\zeta(s) = \sum_{n \leqq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})$$

whenever $x > C\frac{|t|}{2\pi}$.

Pf
Note the formal similarity to 2 (*).
Clearly, via a conjugation, WLOG $t \gtrapprox 100$.
We immediately see by 2 (*) that:

$$\zeta(s) = \sum_1^N n^{-s} - \frac{N^{1-s}}{1-s} + O(N^{-\sigma})$$
$$+ O(|s|) \int_N^\infty u^{-\sigma-1}\, du$$

$$= \sum_1^N n^{-s} - \frac{N^{1-s}}{1-s} + O(|s| N^{-\sigma})$$

{since $\sigma_0 \lessapprox \sigma \lessapprox 2$}.

Think of N as being giant and much greater than x. Let

$$A(v) = \sum_{x<n\leq v} n^{-it} = \sum_{x<n\leq v} e^{-2\pi i\left(\frac{t \ln n}{2\pi}\right)} .$$

Apply p. 4 + 10 line 2.

$$f(q) = -\frac{t \ln q}{2\pi} \qquad (q \geq x)$$

$$f'(q) = -\frac{t}{2\pi q} \qquad \text{monotonic}$$

$$0 \leq -f'(q) \leq \frac{t/2\pi}{x} < \frac{1}{c} < 1$$

So,

$$A(v) = \int_x^v e^{-2\pi i\left(\frac{t \ln q}{2\pi}\right)} dq + O(1)$$

$$= \int_x^v e^{-it \ln q}\, dq + O(1)$$

$$= \int_x^v q^{-it}\, dq + O(1)$$

$$= \frac{v^{1-it} - x^{1-it}}{1-it} + O(1) .$$

But, now,

$$\sum_{x < n \leq N} n^{-\sigma - it} = \int_x^N v^{-\sigma}\, dA(v)$$

$$= v^{-\sigma} A(v)\Big]_x^N - \int_x^N A(v)(-\sigma) v^{-\sigma-1}\, dv$$

$$= \frac{A(N)}{N^{\sigma}} + \sigma \int_x^N \frac{A(v)}{v^{\sigma+1}}\, dv$$

$$= N^{-\sigma}\left[\frac{N^{1-it} - x^{1-it}}{1-it} + O(1)\right]$$

$$+ \sigma \int_x^N v^{-\sigma-1}\left[\frac{v^{1-it} - x^{1-it}}{1-it} + O(1)\right] dv$$

$$= \frac{N^{1-\sigma-it}}{1-it} - \frac{N^{-\sigma} x^{1-it}}{1-it} + O(N^{-\sigma})$$

$$+ \sigma \int_x^N \frac{v^{-\sigma-it}}{1-it}\, dv$$

$$- \sigma \int_x^N \frac{v^{-\sigma-1} x^{1-it}}{1-it}\, dv$$

$$+ O(1)(x^{-\sigma} - N^{-\sigma}) .$$

Remember that $N > x$.

Get:

$$\sum_{x < n \leq N} n^{-s} = \frac{N^{1-\sigma-it}}{1-it} - \frac{N^{-\sigma} x^{1-it}}{1-it} + O(x^{-\sigma})$$

$$+ \frac{\sigma}{1-it} \int_x^N v^{-s}\, dv$$

$$- \frac{\sigma}{1-it} x^{1-it} \int_x^N v^{-\sigma-1}\, dv$$

$$= \frac{N^{1-\sigma-it}}{1-it} - \frac{N^{-\sigma} x^{1-it}}{1-it} + O(x^{-\sigma})$$

$$+ \frac{\sigma}{1-it} \left[\frac{N^{1-s} - x^{1-s}}{1-s} \right]$$

$$- \frac{\sigma}{1-it} x^{1-it} \left[\frac{N^{-\sigma} - x^{-\sigma}}{-\sigma} \right]$$

$$= \frac{N^{1-\sigma-it}}{1-it} - \frac{N^{-\sigma} x^{1-it}}{1-it} + O(x^{-\sigma})$$

$$+ \frac{\sigma}{1-it} \frac{1}{1-s} N^{1-s} - \frac{\sigma}{1-it} \frac{1}{1-s} x^{1-s}$$

$$+ \frac{x^{1-it}}{1-it} N^{-\sigma} - \frac{x^{1-it}}{1-it} x^{-\sigma}$$

{note the cancellation!}

$$= \frac{N^{1-s}}{1-it}\left[1+\frac{\sigma}{1-s}\right] + O(x^{-\sigma})$$

$$- \frac{\sigma}{1-it}\frac{1}{1-s}x^{1-s} - \frac{x^{1-it}}{1-it}x^{-\sigma}$$

$$= \frac{N^{1-s}}{1-it}\left[1+\frac{\sigma}{1-s}\right] + O(x^{-\sigma})$$

$$- \frac{x^{1-s}}{1-it}\left[\frac{\sigma}{1-s}+1\right]$$

$$\left\{ 1+\frac{\sigma}{1-s} = \frac{1-s+\sigma}{1-s} = \frac{1-it}{1-s} \right\}$$

$$= \frac{N^{1-s}}{1-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \ .$$

So, with our $N > x$, we get:

$$\boxed{\begin{aligned} \sum_{x<n\leq N} n^{-s} &= \frac{N^{1-s}}{1-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \\ &= \int_x^N q^{-s}\,dq + O(x^{-\sigma}) \end{aligned}} \ .$$

very natural term

Recall 11 bottom. Thus,

$$\zeta(s) = \sum_{n \leqq x} n^{-s} + \sum_{x < n \leqq N} n^{-s}$$

$$- \frac{N^{1-s}}{1-s} + O(|s| N^{-\sigma})$$

by 15

$$= \sum_{n \leqq x} n^{-s} + \underline{\underline{\frac{N^{1-s}}{1-s}}} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})$$

$$- \underline{\underline{\frac{N^{1-s}}{1-s}}} + O(|s| N^{-\sigma})$$

$$= \sum_{n \leqq x} n^{-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})$$

$$+ O(|s| N^{-\sigma}).$$

Now let $N \to \infty$ (to eliminate it).

Get:

$$\zeta(s) = \sum_{n \leqq x} n^{-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}),$$

exactly as promised. ▥

Take $x > C\frac{t}{2\pi}$ with, say, $C \approx \pi$ and t big.
Apply p. 11 THM. Hence:

$$\zeta(\sigma+it) = \sum_{n \lesssim t} n^{-\sigma-it} - \frac{t^{1-s}}{1-s} + O(t^{-\sigma})$$

$$= \sum_{n \lesssim t} n^{-\sigma-it} - \frac{t^{1-\sigma}t^{-it}}{1-\sigma-it} + O(t^{-\sigma})$$

$$\Downarrow$$

$$|\zeta(1+it)| \lesssim \ln t + O(1) + O(t^{-1}) \text{ crudely;}$$

$$|\zeta(\tfrac{1}{2}+it)| \lesssim 2\sqrt{t} + O(1) + O(t^{-1/2}) \text{ crudely.}$$

Of course, by Lec 25 1 (h) [or Lec 24 16 line 4],
we already know:

$$|\zeta(\tfrac{1}{2}+it)| = O(t^{\frac{1}{4}+\varepsilon}), \text{ each } \varepsilon.$$

This <u>hints</u> that p. 11 THM may be improved.
It <u>can</u> be — but the argument is much harder.

$\boxed{\text{We only need p. 11 THM.}}$

Lecture 27 Synopsis

(Wed, 27 Apr)

We seek to develop the famous thm of Bohr-Landau about zeros of $\zeta(s)$ to the right of $\mathrm{Re}(s) = \frac{1}{2}$.

Recall, from Lec 24, (19) (20), that we had

$$N(u; T_1, T_2) = \# \text{ of zeros of } f(s) \text{ on } R \cup \partial R$$

having abscissa $> u$ (and counted WITH multiplicity)

wherein $R = (\alpha, \beta) \times (T_1, T_2)$; $f(s)$ is analytic on $R \cup \partial R$, $f(\beta + it) \neq 0$, $f(\sigma + iT_1) \neq 0$, $f(\sigma + iT_2) \neq 0$. One defines $\phi(s) = \mathcal{L}og\, f(s)$ via horizontal analytic continuation starting at $\sigma = \beta$ insofar as $t \neq$ ordinate of a zero of f; otherwise by an obvious right continuity from above.

In this framework, we got Littlewood's formula

$$-\frac{1}{2\pi i} \oint_{\partial R} \phi(s)\, ds = \int_{\alpha}^{\beta} N(u; T_1, T_2)\, du$$

and, then, the simplified version in Lec 24, (25).

The Bohr – Landau theorem will arise by specializing $f(s)$ to be $\zeta(s)$ in the foregoing — and playing with appropriate α and β.

Not-too-surprisingly, matters will need to be looked at along the way with the aid of some of our previously obtained estimates.

1914

THEOREM (basic form of Bohr – Landau thm)

Consider $\zeta(s)$. For $T \geqq 2$, say, [not the ordinate of a zero of ζ], we have:

$$N(\tfrac{1}{2}+\varepsilon;\, h, T) = O_\varepsilon(T)$$

for each $\varepsilon > 0$. Here h is a tiny number such that $\zeta \neq 0$ on $\{ Re(s) \geqq 0,\ 0 \leqq Im(s) \leqq h \}$; see Lec 11, p. 27 and Lec 13, pp. 6 (top) + 9 (top).

We stress here that, in toto, we have

$$N(0;\, h, T) \sim \frac{T}{2\pi} \ln\left(\frac{T}{2\pi e}\right)$$

by Lec 15, pp. 29 + 22 (box). As such, by the functional equation of ζ, only 0% of the complex zeros of ζ can lie outside $|Re(s) - \tfrac{1}{2}| \leqq \varepsilon$.

The relevant "PREVIOUSLY OBTAINED" estimates referenced on p. 2 are those found on

page 8 of Lec 15 (a priori upper bound)
page 13 of Lec 15 (the partial fraction thing)
page 14 of Lec 25 (L_2 estimate)
page 11 of Lec 26 (Hardy – Littlewood estimate).

In connection with the first two — from Lec 15 — we note the following:

FACT

Keep $T \gtrsim 2$, say, and not the ordinate of a zero of Ξ. Use the standard UP AND ACROSS definition of $\mathcal{L}og\ \Xi(s)$ beginning at some point $A \in \mathbb{R}$, $A >> 1$. Then:

$$\arg \Xi(\sigma + iT) = O(\ln T),$$

for all $-1 \lesssim \sigma \lesssim 2$.

Pf

by the Dirichlet series!

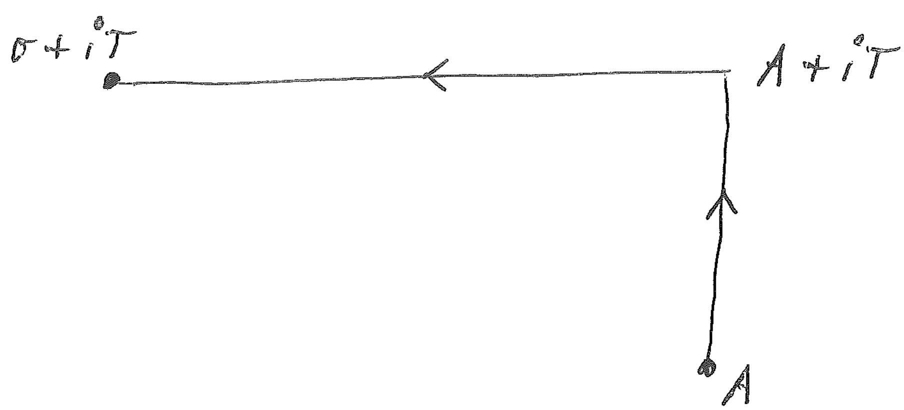
We know $|\mathcal{L}og\, \zeta(s)| = O_\eta(1)$, $\sigma \geq 1+\eta$.

hence also for $\operatorname{Arg} \zeta(s)$

Use

$$N[\,|\gamma - t| \leq 1\,] = O(\ln t)$$

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln t) + \sum_{\substack{\rho \\ |\gamma - t| \leq 1}} \frac{1}{s-\rho} \qquad \{-1 \leq \sigma \leq 2\}$$



$$\operatorname{Arg} \zeta(\sigma + iT) = \operatorname{Im} \int_A^\sigma \frac{\zeta'}{\zeta}(u+iT)\, du$$

$$+ \operatorname{Im} \mathcal{L}og\, \zeta(A+iT)$$

this is $O(2^{-A})$ by the Dirichlet series for $\mathcal{L}og\, \zeta$

$\{$we can let $A \to \infty\}$

$$\operatorname{Arg} \zeta(\sigma + iT) = -\operatorname{Im} \int_\sigma^\infty \frac{\zeta'}{\zeta}(u+iT)\,du$$

$$= -\operatorname{Im} \int_\sigma^2 \; - \operatorname{Im} \int_2^\infty \frac{\zeta'}{\zeta}(u+iT)\,du$$

$O(2^{-4})$ by the Dirichlet series for $-\frac{\zeta'}{\zeta}$

$$= -\operatorname{Im} \int_\sigma^2 \left\{ O(\ln T) + \sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \frac{1}{u+iT-\rho} \right\} du$$

$$+ \, O(1)$$

$$= O(\ln T)$$

$$- \sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \left\{ \operatorname{Arg}(2+iT-\rho) - \operatorname{Arg}(\sigma+iT-\rho) \right\}$$

$$+ \, O(1)$$

Arg = ordinary principal value

$$= O(\ln T) + O(\ln T) + O(1)$$

$$= O(\ln T)$$

{ much like Lec 15, p. 28 } . ∎

By Littlewood's identity, Lec 24, p. 25, know:

$$\tfrac{1}{2} \leqq \alpha < \beta \leqq 2$$

$$\Downarrow$$

$$2\pi \int_\alpha^\beta N(\sigma; T_1, T_2)\, d\sigma$$

$$= \int_{T_1}^{T_2} \ln|\zeta(\alpha+it)|\, dt$$

$$- \int_{T_1}^{T_2} \ln|\zeta(\beta+it)|\, dt$$

$$+ O[(\beta-\alpha) \ln T_2] \longleftarrow \text{by FACT on (3)}$$

at least for T_j which are not the ordinates of ζ- zeros. We'll keep $T_2 > T_1 \geqq 2$. Cf. also h on page 2.

Must focus on

$$\int_{T_1}^{T_2} \ln|\zeta(\beta+it)|\,dt$$

see 15 middle

(since β will be taken $\geq \frac{3}{2}$ later).

We propose to look first at

$$\int_{T/2}^{T} \underline{|\zeta(\sigma+it)|}^2\,dt$$

TRICK

with $T \geq 1000$ and $\frac{1}{2} \leq \sigma \leq 2$.

Use H-L estimate from Lec 26 11, $\sigma_0 = \frac{1}{10}$.
Take $C = \pi$. We get:

$$\zeta(\sigma+it) = \sum_{\underline{n \leq T}} n^{-\sigma-it} - \frac{T^{1-\sigma-it}}{1-\sigma-it} + O(T^{-\sigma})$$

for $t \in [\frac{1}{2}T, T]$.

So,

$$\zeta(\sigma+it) = \sum_{n \lessapprox T} n^{-\sigma} n^{-it} + \frac{O(T^{1-\sigma})}{T} + O(T^{-\sigma})$$

$$= \sum_{n \lessapprox T} n^{-\sigma} e^{-it\ln n} + O(T^{-\sigma})$$

for $\frac{1}{2}T \leqq t \leqq T$. We'll write this as

$$\zeta(\sigma+it) = \underline{\underline{\Sigma + R}}.$$

Hence,

$$|\zeta(\sigma+it)|^2 = (\Sigma + R)(\bar{\Sigma} + \bar{R})$$

$$= \Sigma\bar{\Sigma} + 2\,\mathrm{Re}(R\bar{\Sigma}) + |R|^2$$

$$= \Sigma\bar{\Sigma} + O(1)T^{-\sigma}|\Sigma| + O(1)T^{-2\sigma}$$

for $t \in [\frac{1}{2}T, T]$ and $\frac{1}{2} \leqq \sigma \leqq 2$.

Put:

$$\Sigma = \sum_{n \leq T} n^{-\sigma} e^{-i\lambda_n t} \quad \text{with}$$

$$\lambda_n \equiv \ln n ,$$

and then use Lec 25 p. 14 (the L_2 estimate). Here, of course, $n > m \Rightarrow$

$$\lambda_n - \lambda_m = \ln n - \ln m$$

$$= \frac{1}{\tilde{n}}(n-m) , \quad \tilde{n} \in (m, n)$$

$$\geq \frac{1}{T} .$$

We can apply Lec 25 14 with $\delta = \frac{1}{T}$. So,

$$\int_{T/2}^{T} |\Sigma|^2 dt = \frac{T}{2} \sum_{n \leq T} n^{-2\sigma}$$

$$+ \frac{O(1)}{1/T} \sum_{n \leq T} n^{-2\sigma}$$

{ this may be improvable, but we prefer to stay with a crude bound! }

$$\Downarrow$$

$$\int_{T/2}^{T} |\Sigma|^2\,dt = O(T) \sum_{n \leqq T} n^{-2\sigma} \quad .$$

Suppose now that $\sigma > \frac{1}{2}$. In that case, we go further and get

$$\int_{T/2}^{T} |\Sigma|^2\,dt = O(T)\,\zeta(2\sigma)$$

$$= O(T)\,\frac{1}{2\sigma - 1} \, . \qquad \boxed{\sigma \in (\tfrac{1}{2}, 2]}$$

At the same time, (see ⑧ bottom)

$$\int_{T/2}^{T} T^{-\sigma} |\Sigma|\,dt \leqq \left\{ \int_{T/2}^{T} T^{-2\sigma}\,dt \right\}^{\frac{1}{2}} \cdot \left\{ \int_{T/2}^{T} |\Sigma|^2\,dt \right\}^{\frac{1}{2}}$$

{by Cauchy - Schwarz}

$$\leqq \sqrt{T^{1-2\sigma}}\,\sqrt{\frac{O(T)}{2\sigma - 1}}$$

$$= O(1)\,\frac{T^{1-\sigma}}{\sqrt{2\sigma - 1}} = O(1)\,\frac{T^{1-\sigma}}{2\sigma - 1} \, .$$

Referring to 8 bottom again, we get

$$\int_{T/2}^{T} |\zeta(\sigma+it)|^2 dt$$

$$= \frac{O(T)}{2\sigma-1} + \frac{O(T^{1-\sigma})}{2\sigma-1} + O(T^{1-2\sigma})$$

for $\frac{1}{2} < \sigma \leqq 2$. Accordingly (via $2\sigma \geqq 1$):

$$\boxed{\int_{T/2}^{T} |\zeta(\sigma+it)|^2 dt \leq \frac{O(1)}{\sigma-\frac{1}{2}} T}$$

for $\frac{1}{2} < \sigma \leqq 2$. The case $\sigma = \frac{1}{2}$ is also OK, but [obviously] not very informative.

Suppose NEXT that we only know $\sigma \geqq \frac{1}{2}$.

Observe that 10 line 1 is still usable.

Being totally crude, we can say:

$$\sum_{n \leqq T} n^{-2\sigma} \leq \sum_{n \leqq T} \frac{1}{n} \leqq C + \ln T.$$

Hence,

$$\int_{T/2}^{T} |\Sigma|^2 \, dt = O(T \ln T)$$

$\{$ can then mimic 10 bot + 11 top $\}$

$$\Downarrow$$

$$\int_{T/2}^{T} |\zeta(\sigma+it)|^2 \, dt = O(T \ln T) + O(1)\, T^{1-\sigma} \sqrt{\ln T} + O(T^{1-2\sigma})$$

$$\boxed{\int_{T/2}^{T} |\zeta(\sigma+it)|^2 \, dt \lesssim O(1)\, T \ln T}$$

for $\frac{1}{2} \lesssim \sigma \lesssim 2$.

We have obtained the above box, and 11 box, assuming $T \gtrsim 1000$ (see 7).

THEOREM (standard a priori estimate)

For $T \geq 3$ and $\sigma \in [\frac{1}{2}, 2]$, we have:

$$\int_2^T |J(\sigma+it)|^2 dt = O(T) \min\{\ln T, \frac{1}{\sigma-\frac{1}{2}}\}.$$

PF

Matters are obvious for $3 \leq T \leq 10^6$. In fact, here,

$$\min\{\ln T, \frac{1}{\sigma-\frac{1}{2}}\} \geq \min\{\ln 3, \frac{1}{3/2}\}$$
$$= \frac{2}{3}$$

and it suffices to adjust the implied constant in $O(T)$.

For $T > 10^6$, choose ℓ so that

$$\frac{T}{2^{\ell+1}} \leq 1000 < \frac{T}{2^\ell}.$$

Apply 11 box and 12 box to $\frac{T}{2^k}$ for $k \in [0, \ell]$.
Add! Get:

$$\int_{1000}^{T} |\zeta(\sigma+it)|^2 dt = O(T) \min\left\{\ln T, \frac{1}{\sigma-\frac{1}{2}}\right\}.$$

To replace 1000 by 2, repeat the observation used for $3 \leqq T \leqq 10^6$. ▨

With more work, one can prove that:

$$\int_2^T |\zeta(\sigma+it)|^2 dt \sim T\,\zeta(2\sigma), \quad \sigma > \frac{1}{2}$$

and

$$\int_2^T |\zeta(\tfrac{1}{2}+it)|^2 dt \sim T \ln T.$$

See Titchmarsh, Theory of $\zeta(s)$, first few sections in chapter 7. We won't need these more precise results.

— — —

On p. 6 bottom, fix $T_1 \in [2, 2.5]$ to be well away from the ordinate of any ζ-zero.

> Since numerical work shows that
>
> $$\rho_1 = \frac{1}{2} + i[14.134725^+],$$ Lec 13 (5) Lec 22 (1)
>
> one can declare $T_1 = 2$. We prefer not to use this, however.

Keep $T = T_2 \geq 3$ and then take

$$\frac{1}{2} < \alpha \leq 1 \quad \text{and} \quad \beta = 2 \ (\text{say}).$$

One gets:

$$2\pi \int_\alpha^2 N(\sigma; T_1, T)\, d\sigma = \int_{T_1}^{T} \ln|\zeta(\alpha + it)|\, dt - \int_{T_1}^{T} \ln|\zeta(2+it)|\, dt + O(\ln T).$$

BABY LEMMA

(a) for $x \in \mathbb{R}$, $e^x \geqq 1+x$;

(b) for $y \geqq 0$, $\ln y \leqq \frac{1}{2}(y^2-1)$.

clearly very crude

PF

We give 2 proofs of (a). The first notes that $g(x) = e^x$ is concave upward since $g'' > 0$. Hence $g(x)$ sits on or above the tangent line at each point x_0. Take $x_0 = 0$. Get: $g(x) \geqq 1+x$ by inspection.

The 2nd proof is more boring. For $x > 0$, apply mean value thm to get

$$e^x - 1 = e^{\tilde{x}}(x-0) = e^{\tilde{x}} \cdot x \quad , \quad 0 < \tilde{x} < x$$

$$e^x - 1 \geqq e^0 \cdot x = x . \quad \text{OK}$$

For $x < 0$, apply mean value theorem to get

$$1 - e^x = e^{\tilde{x}}(0-x) = (-x)e^{\tilde{x}} \quad , \quad x < \tilde{x} < 0$$

$$1 - e^x \leqq (-x)e^0$$

$$e^x - 1 \geqq x . \quad \text{OK}$$

In (b), wlog $y > 0$. Put $y = e^u$ with $u \in \mathbb{R}$.

Must check that

$$u \lesssim \frac{1}{2}(e^{2u}-1)$$

$$2u \lesssim e^{2u}-1$$

$$e^{2u} \gtrsim 1+2u \, .$$

But this is obvious by (a). ∎

By BABY LEMMA, then,

$$\int_{T_1}^{T} \ln|\zeta(\sigma+it)|\,dt \lesssim \frac{1}{2}\int_{T_1}^{T}\left(|\zeta(\sigma+it)|^2-1\right)dt$$

$$\leq \frac{1}{2}\int_{T_1}^{T} |\zeta(\sigma+it)|^2\,dt$$

$$\lesssim \frac{1}{2}\int_{2}^{T} |\zeta(\sigma+it)|^2\,dt$$

$$\leq \frac{O(T)}{\sigma-\frac{1}{2}} \quad \text{by (13)} \, .$$

In addition,

$$\mathcal{L}og\ \zeta(2+it) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-2-it} ;$$

$$\int_{T_1}^{T} \ln|\zeta(2+it)|\,dt = \operatorname{Re}\int_{T_1}^{T} \mathcal{L}og\ \zeta(2+it)\,dt$$

$$= \operatorname{Re}\int_{T_1}^{T} \sum_{2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-2-it}\,dt$$

$$= O(1) \sum_{2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-2} \left[\frac{e^{-it\ln n}}{-i\ln n}\right]_{T_1}^{T}$$

$$= O(1) \sum_{2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-2} \frac{1}{\ln n}$$

$$= O(1) . \quad ///$$

Page (15) bottom then gives:

$$2\pi \int_{\alpha}^{2} N(\sigma; T_1, T)\,d\sigma \lesssim \frac{O(T)}{\alpha - \frac{1}{2}} + O(1) + O(\ln T)$$

$$\lesssim \frac{O(T)}{\alpha - \frac{1}{2}} \quad /// .$$

here $\alpha \in (\frac{1}{2}, 1]$

Write $\alpha = \frac{1}{2} + 2\omega$, $0 < \omega \leq \frac{1}{4}$. Notice that

$$N(\tfrac{1}{2} + 2\omega; T_1, T) \leq \frac{1}{\omega} \int_{\frac{1}{2}+\omega}^{\frac{1}{2}+2\omega} N(\sigma; T_1, T)\, d\sigma$$

since $N(\sigma; T_1, T)$ is monotonic decreasing in σ.

Thus:

$$N(\tfrac{1}{2} + 2\omega; T_1, T) \leq \frac{1}{\omega} \int_{\frac{1}{2}+\omega}^{2} N(\sigma; T_1, T)\, d\sigma$$

$$\leq \frac{1}{\omega} \frac{O(T)}{\omega} = \frac{4}{(2\omega)^2} O(T).$$

In other words,

$$\boxed{N(\alpha; T_1, T) \leq \frac{O(T)}{(\alpha - \frac{1}{2})^2}, \quad \frac{1}{2} < \alpha \leq 1}.$$

This <u>proves</u> the THM on p. 2. ∎

Because our use of 16(b) was so crude, one <u>suspects</u> that the foregoing box can be improved in the α-aspect.

cf. also 18 bottom

This is indeed correct. We claim, in fact, that

$$\int_{T_1}^{T} \ln|\zeta(\alpha+it)|\,dt = O(T)\ln\left(\frac{1}{\alpha-\frac{1}{2}}\right).$$

On (19), this leads to

$$\boxed{N(\alpha;T_1,T) \leq O(T)\,\frac{1}{\alpha-\frac{1}{2}}\ln\left(\frac{1}{\alpha-\frac{1}{2}}\right)}.$$

There are 2 approaches to proving line 2.

Method I

By (18) (bottom), wlog $\alpha \in (\frac{1}{2}, \frac{3}{5}]$.

Under this hypothesis, we have $r \equiv \dfrac{1}{\alpha-\frac{1}{2}} \geq 10$.

Notice that $\ln y \leq \frac{1}{2\lambda}(y^{2\lambda}-1)$ for $0<\lambda\leq 1$ [and $y \geq 0$]. See (16). We'll keep $\lambda < 1$. Get:

$$\int_{T_1}^{T} \ln|\zeta|\,dt \leq \frac{1}{2\lambda}\int_{T_1}^{T}\left(|\zeta|^{2\lambda}-1\right)dt$$

$$\leq \frac{1}{2\lambda}\int_{T_1}^{T}|\zeta|^{2\lambda}\,dt$$

$$\leq \frac{1}{2\lambda}\left(\int_{T_1}^{T}|\zeta|^{2\lambda\frac{1}{\lambda}}dt\right)^{\lambda}\left(\int_{T_1}^{T}1\,dt\right)^{1-\lambda}$$

{ by **Hölder's** inequality }

$$\boxed{\begin{aligned} p &= \frac{1}{\lambda} \\ q &= \frac{1}{1-\lambda}\end{aligned}}$$

$$\leq \frac{1}{2\lambda}\left(\frac{O(T)}{\sigma-\frac{1}{2}}\right)^{\lambda} T^{1-\lambda}$$

by (13)

$$\leq \frac{C}{\lambda}\left(\frac{1}{\sigma-\frac{1}{2}}\right)^{\lambda} T$$

$$= \frac{C}{\lambda} r^{\lambda}\cdot T \quad .$$

Put $\lambda = \dfrac{1}{\ln r}$. This is <u>admissible</u> since $r \geq 10$.

Get:

$$\int_{T_1}^{T} \ln|\zeta(\sigma+it)|\,dt \leq C(\ln r)\,e\,T$$

$$\leq O(T)\ln\left(\frac{1}{\sigma-\frac{1}{2}}\right) \quad .$$

OK

Method II

We use Jensen's inequality common in probability and measure theory.

To recall it, let $\Phi(v)$ be non-negative and convex on $\mathbb{R}$. (Hence Φ is automatically continuous.) Let Y be an extended real-valued random variable on a probability space (X, μ). Assume that $\mathbb{E}(Y)$ exists (i.e. that $Y \in L_1(\mu)$). Then:

$$\Phi(\mathbb{E}(Y)) \leq \mathbb{E}(\Phi(Y)).$$

One simply approximates $\mathbb{E}(Y)$ by an obvious Riemann sum and uses

$$\left\{ \begin{array}{l} \Phi\left(\sum_{j=1}^{N} t_j v_j\right) \leq \sum_{j=1}^{N} t_j \Phi(v_j) \\ \text{for } t_j \in [0,1] \text{ with } t_1 + \cdots + t_N = 1 \end{array} \right\}.$$

Put $\Phi(v) = \exp(v)$ to get

$$\exp\left(\frac{1}{H}\int_0^H f(t)\,dt\right) \leq \frac{1}{H}\int_0^H \exp[f(t)]\,dt$$

anytime $f \in L_1[0, H]$.

A trivial specialization gives:

$$\exp\left(\frac{1}{T-T_1}\int_{T_1}^{T} 2\ln|\zeta(\sigma+it)|\,dt\right)$$

$$\leqq \frac{1}{T-T_1}\int_{T_1}^{T} |\zeta(\sigma+it)|^2\,dt$$

$$\leqq \frac{1}{T-T_1}\,\frac{O(T)}{\sigma-\frac{1}{2}} \quad \text{by } (13)$$

$$\leqq \frac{O(1)}{\sigma-\frac{1}{2}}.$$

Accordingly:

$$\frac{1}{T-T_1}\int_{T_1}^{T} 2\ln|\zeta(\sigma+it)|\,dt$$

$$\leqq \ln\left(\frac{B}{\sigma-\frac{1}{2}}\right) \qquad \left\{\begin{matrix}\text{some}\\ B \geqq 1\end{matrix}\right\}$$

$$\Downarrow$$

$$\int_{T_1}^{T} \ln|\zeta(\sigma+it)|\,dt \leqq O(T)\ln\left(\frac{1}{\sigma-\frac{1}{2}}\right)$$

for $\sigma \in (\frac{1}{2}, 1]$. OK

Incidentally, observe how 13 THM gets applied in both methods I + II. Switching to $\ln T$ in place of $\frac{1}{\sigma-\frac{1}{2}}$ produces the following:

method I

get $\frac{c}{\lambda}(\ln T)^{\lambda} T$ on 21 middle

$\Rightarrow$ take $\lambda = \frac{1}{\ln \ln T}$ (T giant)

$\Rightarrow \int_{T_1}^{T} \ln|\zeta(\sigma+it)|\,dt = O(T \ln \ln T)$

uniformly for $\sigma \in [\frac{1}{2}, 1]$

method II

get $\leqq O(1) \ln T$ on 23 middle

$\Rightarrow \frac{1}{T-T_1}\int_{T_1}^{T} 2\ln|\zeta|\,dt \leqq \ln(B \ln T)$

$\Rightarrow \int_{T_1}^{T} \ln|\zeta(\sigma+it)|\,dt = O(T \ln \ln T)$

uniformly for $\sigma \in [\frac{1}{2}, 1]$.

HENCE:

$$\boxed{\int_{T_1}^{T} \ln|\zeta(\sigma+it)|\,dt = O(T) \ln\left[\min\left(\ln T, \frac{1}{\sigma-\frac{1}{2}}\right)\right]}.$$

Closing Remark.

The estimate

$$N\left(\tfrac{1}{2}+\varepsilon\,;\,T_1,T\right) = O(T)\,\frac{1}{\varepsilon}\ln\left(\frac{1}{\varepsilon}\right)$$

mentioned on p. 20 (line 4) was obtained by Littlewood in 1924. Proc. Cambr. Philo. Soc. 22 (1924)

It is possible to expunge the term $\ln\frac{1}{\varepsilon}$.

This was shown by A. Selberg around 1942 with the aid of some fundamentally new ideas. By use of a so-called mollifier method, Selberg was able to demonstrate that

$$\int_{\frac{1}{2}}^{2} N(\sigma\,;\,T_1,T)\,d\sigma = O(T).$$

Compare 15 (bottom) + 18 (middle).

See also Titchmarsh, Theory of $\zeta(s)$, around §9.24.

Lecture 28

(Fri, Apr 29)

A short presentation of D. T. Newman's proof of PNT. **It's fun. And it's slick.**

{ Bak and Newman, *Complex Analysis*, chap 19, 3rd ed. Also: Monthly 87 (1980) 693–696. }

Known Facts

1. $\zeta(s) = \sum_{1}^{\infty} n^{-s}$ analytic $re(s) > 1$
2. $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ nice convergence $re(s) > 1$ Lec 6 (3)
3. $\zeta(s) - \frac{1}{s-1}$ analytic $re(s) > 0$ Lec 5 (6) + (10)
4. $\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$ near $s = 1$ Lec 17 (40)
5. $\operatorname{Log} \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s}$ $re(s) > 1$ Lec 6 (4)
6. $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}$ $re(s) > 1$ Lec 6 (6)
7. $\zeta(s) \neq 0$ on $Re(s) \geq 1$ Lec 6 (6) (7)

Newman likes

$$\phi(s) \equiv \sum_p \frac{\ln p}{p^s}, \quad re(s) > 1.$$

Obviously,

$$-\frac{\zeta'(s)}{\zeta(s)} = \phi(s) + \underbrace{\sum \frac{\ln p}{p^{2s}} + \sum \frac{\ln p}{p^{3s}} + \cdots}$$

and the underlined fcn is analytic on $Re(s) > \frac{1}{2}$.

Indeed,

$$(\ln p) \sum_{n=2}^{\infty} p^{-n\sigma} = (\ln p) \frac{p^{-2\sigma}}{1-p^{-\sigma}}$$

$$\leqq (\ln p) \frac{p^{-2\sigma}}{1-2^{-\sigma}} \qquad \sigma \geqq 1+\varepsilon$$

etc etc.

Let $E(s)$ mean a fcn analytic on $\mathrm{Re}(s) > \frac{1}{2}$.
Not necessarily the same one each time...

$$\boxed{-\frac{\zeta'(s)}{\zeta(s)} = \phi(s) + E(s).}$$

FACT 1

Write

$$\zeta(s) = (s-1)^{-1}\left[1+\gamma(s-1) + O(s-1)^2\right] \quad \text{near } s=1.$$

Take logarithmic derivative. Get

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \left(\gamma + O(s-1)\right).$$

$$\boxed{\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + O(s-1) \quad \text{near } s=1.}$$

FACT 2

see Lec 17 p. 42

Recall

$$\psi(x) \equiv \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \ln p \qquad (x > 1)$$

$$= \theta(x) + \theta(x^{\frac{1}{2}}) + \theta(x^{\frac{1}{3}}) + \cdots$$

with

$$\theta(x) = \sum_{p \leq x} \ln p \quad .$$

See Lec 1 4 12 .

Additional Known Fact for $x \geq 2$

8. $c_1 x \leq \psi(x) \leq c_2 x$, $c_3 x \leq \theta(x) \leq c_4 x$ $(c_j > 0)$

$$\psi(x) = \theta(x) + O(x^{1/2})$$

"Chebyshev"

See Lec 1 4 5 16 – 18 .

$$\boxed{\begin{gathered} \psi(x) = \theta(x) + R(x) \\ R(x) = O(x^{1/2}) \\ x \geq 1 \ . \end{gathered}}$$

FACT 3

Recall

$\mathrm{re}(s) > 1 \quad \Rightarrow \qquad\qquad (\psi(x) = 0,\ x < 2)$

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty x^{-s}\, d\psi(x)$$

$$= \left[x^{-s}\psi(x) \right]_1^\infty - \int_1^\infty \psi(x)\, d(x^{-s})$$

$$= 0 + s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx$$

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx$$

and $\displaystyle \frac{1}{s-1} = \int_1^\infty \frac{x}{x^{s+1}}\, dx$

$$\boxed{-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = \int_1^\infty \frac{\psi(x) - x}{x^{s+1}}\, dx \qquad \mathrm{re}(s) > 1 .}$$

$\leftarrow$ FACT 4

37 line 7

See Ingham 18(17), 91(8) and Lec 8 (11).

But,

$$\psi(x) - x = \theta(x) - x + R(x) \qquad \text{see } 3 \text{, Fact 3}$$

and

$$\int_1^\infty \frac{R(x)}{x^{s+1}}\,dx = \text{analytic on } Re(s) > \tfrac{1}{2}$$
$$\text{(since } R(x) = O(\sqrt{x})\text{)}.$$

So,

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = \int_1^\infty \frac{\theta(x) - x}{x^{s+1}}\,dx + E(s)$$

à la 2

<u>FACT 5</u>

$$\int_1^\infty \frac{\theta(x) - x}{x^{s+1}}\,dx = -\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} + E(s)$$

on $Re(s) > 1$ and we get:

(a) LHS has a meromorphic continuation to $Re(s) > 1/2$

(b) LHS has <u>no</u> pole at $s = 1$

(c) LHS has <u>no</u> poles on $\{Re(s) \geqq 1\}$.

(a) is obvious by 1.

(b) is easy by 2 (bottom) ; (c) then follows by 1. ∎

Next:

$$\int_1^\infty \frac{\theta(x)-x}{x^s}\frac{dx}{x} = \int_0^\infty \frac{\theta(e^v)-e^v}{e^{sv}}\,dv$$

$$\left\{ \begin{matrix} x = e^v \\ v = \ln x \end{matrix} \right\}$$

shift $s \to s+1$

$$\Rightarrow \text{get } \int_0^\infty \frac{\theta(e^v)-e^v}{e^{(s+1)v}}\,dv$$

$$= \int_0^\infty e^{-sv}\left[\frac{\theta(e^v)}{e^v}-1\right]dv.$$

<u>FACT 6</u>

$$\int_0^\infty e^{-sv}\left[\frac{\theta(e^v)}{e^v}-1\right]dv$$

(a) is analytic on $\mathrm{Re}(s) \geq 0$

(b) is meromorphic on $\mathrm{Re}(s) > -\frac{1}{2}$

(c) <u>secretly</u> has poles at $\rho - 1$, where

$$\xi_0(\rho) = 0, \quad \xi_0(s) = s(s-1)\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s)$$

à la Lec 13 4 5.

On the other hand, following Newman,

$$\phi(s) = \int_1^\infty u^{-s}\, d\theta(u) \,, \quad re(s)>1 \qquad \begin{pmatrix}\theta(u)=0\\ u<2\end{pmatrix}$$

$$= u^{-s}\theta(u)\Big]_1^\infty - \int_1^\infty \theta(u)\, d(u^{-s})$$

$$= s\int_1^\infty \frac{\theta(u)}{u^{s+1}}\, du$$

$$= s\int_0^\infty e^{-sv}\theta(e^v)\, dv$$

but

$$\frac{s}{s-1} = s\int_0^\infty e^{-sv} e^v\, dv$$

$$\Downarrow$$

$$\phi(s) - \frac{s}{s-1} = s\int_0^\infty e^{-vs}\left[\theta(e^v) - e^v\right] dv$$

$$\frac{\phi(s)}{s} - \frac{1}{s-1} = \int_0^\infty e^{-sv}\left[\theta(e^v) - e^v\right] dv \qquad re(s)>1$$

$$\boxed{\frac{\phi(s+1)}{s+1} - \frac{1}{s} = \int_0^\infty e^{-sv}\left[\frac{\theta(e^v)}{e^v} - 1\right] dv \qquad re(s)>0}$$

FACT 7

same fcn
as in Fact 6

DEFINITION

$$g(s) \equiv \frac{\phi(s+1)}{s+1} - \frac{1}{s} \quad ;$$

$$f(v) \equiv \frac{\theta(e^v)}{e^v} - 1 \qquad (v \geq 0) .$$

By 7, Fact 7, know

$$g(s) = \int_0^\infty e^{-sv} f(v)\, dv , \qquad re(s) > 0$$

in the style of a Laplace transform.

6 Fact 6 allows us to better understand g.

THEOREM

g and f as above. Then:

(i) $g(s) = \int_0^\infty e^{-sv} f(v)\, dv$, $re(s) > 0$

(ii) $f(v)$ is bounded and piecewise C^∞

(iii) $g(s)$ is meromorphic on $Re(s) > -\frac{1}{2}$

BUT HAS NO POLES on $Re(s) \geq 0$.

Proof

(i) as above. (ii) by Chebyshev on 3.

(iii) see Fact 6. ∎

NEWMAN'S GENERAL THM

Let $f(v)$ be ANY bounded, piecewise continuous fcn on $[0,\infty)$. Let

$$g(s) = \int_0^\infty e^{-sv} f(v)\,dv, \quad \mathrm{re}(s) > 0.$$

ASSUME THAT g extends to a single-valued analytic function on a connected open set slightly bigger than $\mathrm{Re}(s) \geqq 0$. (Call it g again.) Then:

$$\int_0^\infty f(v)\,dv \text{ exists and equals } g(0).$$

Pf

$\int_0^\infty f(v)\,dv$ means $\lim\limits_{T\to\infty} \int_0^T f(v)\,dv$!

To convey the function theory flavor, switch to $z = x+iy$ instead of s.

Let

$$g_T(z) = \int_0^T e^{-zv} f(v)\,dv, \quad T > 0.$$

The fcn $g_T(z)$ is entire for each T. [Simply view as a standard limit of Riemann sums. $\underbrace{}_{(N\to\infty)}$
Recall Lec 3 (18).]

Must prove:

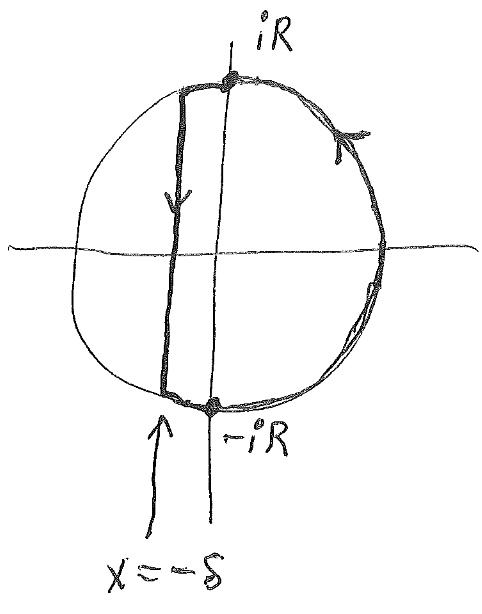
$$\lim_{T\to\infty} g_T(0) = g(0).$$

Take R giant and <u>freeze</u> it!

Select a tiny $\delta > 0$ (depending on R) so that $g(z)$ is nicely analytic on

$$\{|s| \leqq R\} \cap \{x \geqq -\delta\}.$$

⟦ That means on a slightly bigger open set! ⟧



C = heavy path
C_+ = portion with $x > 0$
C_- = portion with $x < 0$

Apply Cauchy integral formula to

$$\left[g(z) - g_T(z) \right] \underline{e^{zT} \left(1 + \frac{z^2}{R^2}\right)}.$$

KEY

Newman's Trick

Get:

$$g(0) - g_T(0) = \frac{1}{2\pi i} \oint_C [g - g_T] e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}.$$

We estimate RHS in several steps.

First over C_+. Let $B = \sup\limits_{v \geq 0} |f(v)|$.

On C_+,

$$|g(z) - g_T(z)| = \left| \int_T^\infty e^{-zv} f(v)\, dv \right|$$

$$\leq B \int_T^\infty e^{-xv}\, dv$$

$$= B \frac{e^{-xT}}{x}$$

$$|e^{zT}| = e^{xT}$$

$$\left|1+\frac{z^2}{R^2}\right| = \left|1+\frac{z^2}{z\bar{z}}\right| = \left|1+\frac{z}{\bar{z}}\right|$$

$$= \frac{|\bar{z}+z|}{R}$$

$$= \frac{2|x|}{R} = \frac{2x}{R}$$

$$|g-g_T|\,|e^{zT}|\,\left|1+\frac{z^2}{R^2}\right| \leq \frac{2B}{R}$$

$$\Downarrow$$

$$\left|\frac{1}{2\pi i}\int_{C_+}(g-g_T)\,e^{Tz}\left(1+\frac{z^2}{R^2}\right)dz\right|$$

$$\leq \frac{1}{2\pi}\int_{C_+}\frac{2B}{R}\frac{|dz|}{R}$$

$$= \frac{1}{2\pi}\frac{2B}{R}\frac{\pi R}{R} = \frac{B}{R} \,.$$

For C_-, we write:

$$I_1 = \frac{1}{2\pi i} \int_{C_-} g(z)\, e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

$$I_2 = \frac{1}{2\pi i} \int_{C_-} g_T(z)\, e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \,.$$

g_T is entire (10).

For I_2, by deformation of contour, note that

$$I_2 = \frac{1}{2\pi i} \int_{iR}^{-iR} [\dots] \frac{dz}{z}$$

$\begin{pmatrix}\text{left half} \\ |z| = R\end{pmatrix} \leftarrow$ see (10)

here, on $|z| = R$, have

$$|g_T(z)| \leqq \int_0^T e^{-vx} |f(v)|\, dv \qquad \text{(9) bot}$$

$$\{x < 0\}$$

$$\leqq B \int_0^T e^{v|x|}\, dv$$

$$= B\, \frac{e^{T|x|} - 1}{|x|} \leqq B\, \frac{e^{T|x|}}{|x|}$$

$$|e^{zT}| = e^{xT} = e^{-|x|T}$$

$$\left|1+\frac{z^2}{R^2}\right| = \left|1+\frac{z^2}{z\bar{z}}\right| = \left|1+\frac{z}{\bar{z}}\right|$$

$$= \frac{2|x|}{R}$$

take *product* to get

$$|[\cdots]| \leq B\frac{e^{T|x|}}{|x|}e^{-|x|T}\frac{2|x|}{R} = \frac{2B}{R}$$

$$\Downarrow$$

$$\left|\frac{1}{2\pi i}\int_{\substack{\text{left half}\\|z|=R}}[\cdots]\frac{dz}{z}\right| \leq \frac{1}{2\pi}\frac{2B}{R}\pi = \frac{B}{R}.$$

I_2

Must now do I_1. We'll do Newman's method first and, then, note an *alternate* reasoning.

interesting

R is frozen, as is δ. (10)

Look at the integrand

$$e^{zT} g(z)\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\,dz$$

on curve C_-. Each chunk

$$\left\{ g(z),\ 1+\frac{z^2}{R^2},\ \frac{1}{z} \right\}$$

is bounded by something. So is e^{zT};

$$|e^{zT}| = e^{xT} \leqq e^0 = 1.$$

Switch now to a parametric representation of C_-, say $z = z(\lambda)$, $0 \leqq \lambda \leqq 1$, $\lambda \uparrow$.

Get new integral

$$\int_0^1 B(\lambda)\, e^{z(\lambda)T}\, z'(\lambda)\, d\lambda$$

continuous + bounded

can now apply an elementary bounded convergence thm for Riemann integrals, since

$$\left| e^{z(\lambda)T} \right| = e^{x(\lambda)T} , \quad \lambda \in [0,1]$$

$$\leq 1$$

AND

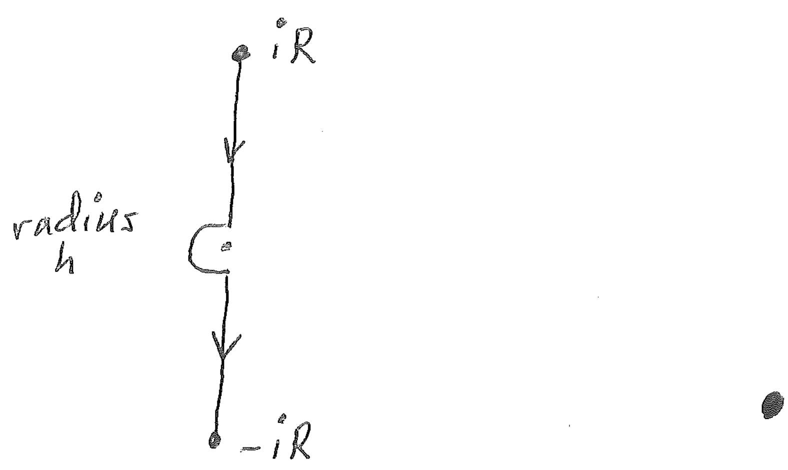
$$\lim_{T \to \infty} e^{x(\lambda)T} = 0 \quad \text{pointwise on } 0 < \lambda < 1 .$$

In fact, this last limit is <u>uniform</u> on each $[\varepsilon, 1-\varepsilon]$. Get:

$$\lim_{T \to \infty} \frac{1}{2\pi i} \int_{C_-} g(z)\left(1 + \frac{z^2}{R^2}\right) e^{zT} \frac{dz}{z} = 0 .$$

A highly suggestive <u>alternate</u> approach to I_1 goes as follows.

Take $h > 0$ microscopic. Make a new path $C_-(h)$ à la



By the extended (limit) form of the CIF, we have

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_{C_+} (g - g_T)\, e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

$$+ \frac{1}{2\pi i} \int_{C_-(h)} (g - g_T)\, e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

see 9

anytime g is ONLY known to be continuous on $\{x \geqq 0\}$ **and** **analytic** **near** $z = 0$.

The I_2 part of the $C_-(h)$ integral again gives $\Theta \frac{B}{R}$, $|\Theta| \leqq 1$. See 13 14.

For the I_1 portion, use $C_-(h)$ as given:

$$\frac{1}{2\pi i}\int_{iR}^{ih} g(z)\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} \quad \longleftarrow I_{11}$$

$$+\ \frac{1}{2\pi i}\int_{\substack{|z|=h \\ \text{left}}} g(z)\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} \quad \longleftarrow I_{12}$$

$$+\ \frac{1}{2\pi i}\int_{-ih}^{-iR} g(z)\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} \quad \longleftarrow I_{13}$$

Note:

$$I_{11} = (\text{const})\int_h^R g(iy)\, e^{iyT}\left(1-\frac{y^2}{R^2}\right)\frac{1}{y}\,dy$$

$$= o(1) \quad \text{by } \underline{\text{Riemann}}\text{-}\underline{\text{Lebesgue}} \text{ lemma}$$

$\{h, R \text{ fixed}\}$

$I_{13} = o(1)$ similarly

$I_{12} = o(1)$ by a mimic of 15 (bot) + 16

$\{h > 0 \text{ fixed}\}$

So,

$$I_1 = o(1) \quad . \quad //$$

End of Alternate Approach!

Remember R = giant, but fixed.

Get:

$$\limsup_{T \to \infty} \left| \frac{1}{2\pi i} \int_C (g - g_T) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right|$$

$$= \limsup_{T \to \infty} |g(0) - g_T(0)|$$

$$\leq \frac{B}{R} + \frac{B}{R} + 0 \quad \text{by } (12), (14), (16) \text{ or line 2 above}$$

$$= \frac{2B}{R} .$$

Since R is arbitrary, deduce that

$$\limsup_{T \to \infty} |g(0) - g_T(0)| = 0 . \quad \blacksquare$$

Corollary

$$\int_1^\infty \frac{\theta(x) - x}{x^2}\, dx \quad \text{is convergent.}$$

Pf

Recall 7 (bottom) + 8. Then apply Newman's general thm. Get

$$\int_0^\infty \left[\frac{\theta(e^v)}{e^v} - 1 \right] dv \quad \text{converges}$$

$$\{x = e^v,\ v = \ln x\}$$

$$\int_1^\infty \frac{\theta(x) - x}{x} \frac{dx}{x} \quad \text{converges.} \quad \blacksquare$$

FACT

Suppose $H(x)$ is piecewise continuous on $[1,\infty)$. Suppose $H(x) \nearrow$. Suppose

$$\int_1^\infty \frac{H(x)-x}{x^2}\,dx \quad \text{converges}$$

as an improper integral. Then

$$H(x) \sim x \quad \text{as} \quad x \to \infty .$$

Pf

Suppose $H(x) \geqq \lambda x$ frequently as $x \to \infty$ for some $\lambda > 1$. Notice that, AT SUCH x,

$$H(u) \geqq \lambda x \quad \text{on} \quad [x, \lambda x] \qquad (\text{by } H \nearrow)$$

$$H(u) - u \geqq \lambda x - u \quad \text{here}$$

$$\int_x^{\lambda x} \frac{H(u)-u}{u^2}\,du \geqq \int_x^{\lambda x} \frac{\lambda x - u}{u^2}\,du$$

put $u = xw$

$$= \int_1^\lambda \frac{\lambda x - xw}{x^2 w^2}(x\,dw)$$

$$= \int_1^\lambda \frac{\lambda - w}{w^2}\,dw > 0 .$$

This violates

$$\left| \int_{y_1}^{y_2} \frac{H(u)-u}{u^2}\, du \right| < \varepsilon$$

for all $y_2 \gtrsim y_1 \gtrsim y_\varepsilon$.

Now let $H(x) \lesssim \eta x$ frequently as $x \to \infty$ for some $\eta < 1$. Look AT SUCH x.

$H(u) \lesssim \eta x$ on $[\eta x, x]$ (by $H \uparrow$)

$H(u) - u \lesssim \eta x - u$ here

$$\int_{\eta x}^{x} \frac{H(u)-u}{u^2}\, du \lesssim \int_{\eta x}^{x} \frac{\eta x - u}{u^2}\, du$$

put $u = xw$

$$= \int_{\eta}^{1} \frac{\eta x - xw}{x^2 w^2} (x\, dw)$$

$$= \int_{\eta}^{1} \frac{\eta - w}{w^2}\, dw$$

$$= -\int_{\eta}^{1} \frac{w-\eta}{w^2}\, dw < 0 .$$

This violates the Y_1, Y_2 condition above.

So,

$$H(x) \sim x . \quad \blacksquare$$

Corollary (PNT)

$$\theta(x) \sim x .$$

Pf

Combine 20 + 21. $\blacksquare$

REMARKS.

$\boxed{1}$ Clearly, a very nice proof! ☺

$\boxed{2}$ It is reasonable to conjecture Newman (actually) began with 4 box, the FACT on 21, and Landau, Gött. Nachr. 1932 [attached below].

$\boxed{3}$ Various extensions of the THM on 9 have been made based on the idea of 17, 18, 19 top.

$\boxed{4}$ We'll <u>return</u> to page 9 THM a bit later, in a comment about lecture 30.

~

Göttinger Nachr. $\boxed{1932}$

pp. 525–527

$\boxed{\text{This theorem is essentially the Wiener–Ikehara Tauberian theorem (when you let } \lambda \to \infty).}$

Über Dirichletsche Reihen.

Von

Edmund Landau.

Vorgelegt in der Sitzung am 25. November 1932.

$\boxed{\text{No complex variables;}}$ only harmonic analysis.

Durch Weiterführung der N. Wienerschen Methode bewiesen Herr Heilbronn und ich[1]) den

Satz: *Es gibt zwei für $\lambda > 0$ definierte positive Funktionen $P_1(\lambda)$ und $P_2(\lambda)$ mit*

$$\lim_{\lambda=\infty} P_1(\lambda) = \lim_{\lambda=\infty} P_2(\lambda) = 1$$

und folgender Eigenschaft.

Die Dirichletsche Reihe

note

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad \underline{\underline{a_n \geqq 0}},$$

konvergiere für $\sigma > 1$.

(Trivialerweise ist also,

$$e^{-y} \sum_{\lambda_n \leqq y} a_n = H(y) \quad \text{für} \quad y \gtreqless 0$$

gesetzt[2]),

$$f(s) = s \int_{-\infty}^{\infty} H(y) e^{-y(s-1)} dy \quad \text{für} \quad \sigma > 1.)$$

<u>Für $|t| \leqq 2\lambda$</u>, $\sigma = 1 + \varepsilon$, $\varepsilon > 0$, sei bei $\varepsilon \to 0$ gleichmäßig in t

$$h_\varepsilon(t) = f(s) - \frac{1}{s-1} \to h(t).$$

Dann ist

$$P_1(\lambda) \geqq \overline{\lim_{y=\infty}}\, H(y) \geqq \underline{\lim_{y=\infty}}\, H(y) \geqq P_2(\lambda).$$

---

1) *Bemerkungen zur vorstehenden Arbeit von Herrn Bochner* (Mathematische Zeitschrift, im Druck). Wegen aller historischen Bemerkungen verweise ich auf diese Arbeit. MZ 37 (1933) 10–16

2) Für $y_2 \geqq y_1$ ist also $H(y_2) \geqq H(y_1)\, e^{y_1 - y_2}$.

Landau, Collected Works vol. 9

Ich teile hier einen noch mehrfach vereinfachten Beweis unseres Satzes mit, bei dem ich o. B. d. A. $\lambda_1 = 0$ annehmen darf.

Hilfssatz:

$$\lim_{y=\infty} \int_{-\infty}^{\lambda y} H\left(y-\frac{v}{\lambda}\right)\frac{\sin^2 v}{v^2}\,dv = \int_{-\infty}^{\infty} \frac{\sin^2 v}{v^2}\,dv = \pi. \quad ^{3)}$$

Beweis: Für $\varepsilon > 0$ ist

$$\frac{1}{2}\int_{-2\lambda}^{2\lambda} e^{yti}\left(1-\frac{|t|}{2\lambda}\right)\frac{h_\varepsilon(t)-1}{1+\varepsilon+ti}\,dt$$

$$= \frac{1}{2}\int_0^{\infty}(H(u)-1)\,e^{-\varepsilon u}\,du \int_{-2\lambda}^{2\lambda}\left(1-\frac{|t|}{2\lambda}\right)e^{t(y-u)i}\,dt$$

$$= \int_0^{\infty} H(u)\,e^{-\varepsilon u}\,\frac{\sin^2\lambda(y-u)}{\lambda(y-u)^2}\,du - \int_0^{\infty} e^{-\varepsilon u}\,\frac{\sin^2\lambda(y-u)}{\lambda(y-u)^2}\,du.$$

$\varepsilon \to 0$ ist links und im Subtrahendus rechts, also im Minuendus rechts unter dem Integralzeichen ausführbar.

$$\frac{1}{2}\int_{-2\lambda}^{2\lambda} e^{yti}\left(1-\frac{|t|}{2\lambda}\right)\frac{h(t)-1}{1+ti}\,dt$$

$$= \int_0^{\infty} H(u)\,\frac{\sin^2\lambda(y-u)}{\lambda(y-u)^2}\,du - \int_0^{\infty}\frac{\sin^2\lambda(y-u)}{\lambda(y-u)^2}\,du$$

$$= \int_{-\infty}^{\lambda y} H\left(y-\frac{v}{\lambda}\right)\frac{\sin^2 v}{v^2}\,dv - \int_{-\infty}^{\lambda y}\frac{\sin^2 v}{v^2}\,dv.$$

Bei $y \to \infty$ strebt das Integral links als FOURIERkonstante gegen 0, das letzte Integral gegen π.

Beweis des Satzes: 1) Mit $a = \sqrt{\lambda}$ ist

$$\pi \geqq \overline{\lim_{y=\infty}} \int_{-a}^{a} H\left(y-\frac{v}{\lambda}\right)\frac{\sin^2 v}{v^2}\,dv \geqq \overline{\lim_{y=\infty}} \int_{-a}^{a} H\left(y-\frac{a}{\lambda}\right)e^{-\frac{2a}{\lambda}}\frac{\sin^2 v}{v^2}\,dv,$$

$$\overline{\lim_{y=\infty}}\, H(y) \leqq \pi e^{\frac{2a}{\lambda}} : \int_{-a}^{a}\frac{\sin^2 v}{v^2}\,dv = P_1(\lambda) \to 1.$$

3) Gebraucht wird nur $\pi > 0$, nicht π = LUDOLPHsche Zahl.

2) Nach 1) ist $H(y)$ beschränkt und zuletzt $< 2\,P_1(\lambda)$. Mit $b = \frac{4}{\pi}P_1(\lambda) + \sqrt{\lambda}$ ist also

$$\pi = \lim_{y=\infty} \int_{-\infty}^{\sqrt{y}} H\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2}\, dv$$

$$\leqq 2\,P_1(\lambda) \int_{-\infty}^{-b} \frac{dv}{v^2} + \lim_{y=\infty} \int_{-b}^{b} H\left(y + \frac{b}{\lambda}\right) e^{\frac{2b}{\lambda}} \frac{\sin^2 v}{v^2}\, dv + 2\,P_1(\lambda) \int_{b}^{\infty} \frac{dv}{v^2},$$

$$\underline{\lim}_{y=\infty} H(y) \geqq e^{-\frac{2b}{\lambda}} \left(1 - \frac{4\,P_1(\lambda)}{\pi b}\right) = P_2(\lambda) \to 1.$$

Lecture 29

(Wed, 4 May)

We wish to prove

$$\psi(x) - x = \Omega_{\pm}(x^{\frac{1}{2}} \log\log\log x)$$

using the Ingham method from 1936, not the one in the book, pp. 92 - 100 (which follows Littlewood).

Ingham 1936 = Acta Arithmetica 1 (1936) 201 - 211

We use a variant in technique stressed by A. Selberg.

— — — — — —

(17)

Since $\psi(x) - x = \Omega_{\pm}(x^{\Theta-\delta})$ (Ing 90) à la Lec 21, we take $\Theta = \frac{1}{2}$ wlog. Ie, we assume RH.

Know:

Lec 13 p. (5)

① $\sum \frac{1}{|\rho|^2} < \infty$ for $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$

② $\psi_1(x) = \frac{x^2}{2} - \frac{\zeta'(0)}{\zeta(0)}x + B - \sum_{k=1}^{\infty}\frac{x^{1-2k}}{2k(2k-1)} - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)}$

$B = \frac{\zeta'(-1)}{\zeta(-1)}$ = unimportant, $x \geqq 1$

Lec 16 p. (15) Ing 73

(3) For $x \geq 1+\delta_0$,

$$\psi^*(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} - \sum_\rho \frac{x^\rho}{\rho}$$

with some reasonable conditional convergence on the ρ-sum over compact subsets of $[1+\delta_0, \infty)$.

Also:

$$\frac{\zeta'(0)}{\zeta(0)} = \ln(2\pi)\ ;$$

Lec 18 (35)
Ing 77

$$\psi^*(x) \equiv \frac{\psi(x+0)+\psi(x-0)}{2}\ .$$

Let:

$$E_1(x) = -\frac{\zeta'(0)}{\zeta(0)}x + B - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{2k(2k-1)}$$

$$E(x) = -\frac{\zeta'(0)}{\zeta(0)} + \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k}\ .$$

These fcns are clearly C^∞ and we have

$$\boxed{E = E_1'}\ .$$

Obviously

$$\psi_1(x) - \frac{x^2}{2} - E_1(x) = -\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)}\ ;$$

$$\psi^*(x) - x - E(x) = -\sum_\rho \frac{x^\rho}{\rho}\ .$$

DEF (modified remainder terms)

$$P(x) = \psi^*(x) - x - E(x)$$

$$P_1(x) = \psi_1(x) - \frac{x^2}{2} - E_1(x)$$

Notice that $x + E(x)$ and $\frac{x^2}{2} + E_1(x)$ are C^∞, and that P and P_1 are piecewise C^1. In addition, P_1 is continuous.

FACT 1

For $x \geq 2$,

$$P_1(x) = b_1 + \int_2^x P(t)\,dt .$$

b_1 = some real constant.

Pf

$$\int_2^x P(t)\,dt = \int_2^x [\psi(t) - t - E(t)]\,dt \leftarrow \boxed{\text{by def of } \psi^* \text{ and baby integrals}}$$

$$= \int_2^x [\psi(t) - t - E_1'(t)]\,dt$$

$$= \text{constant} + \psi_1(x) - \frac{x^2}{2} - E_1(x)$$

$$\left\{ \psi_1(x) = \int_1^x \psi(v)\,dv ,\ \psi(v) = 0 \text{ for } v < 2 \right\} .$$

∎

By (2) bottom, we also have:

$$P_1(x) = -\sum_{\rho} \frac{x^{\frac{3}{2}+i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)}$$

$$\rho \equiv \frac{1}{2}+i\gamma, \text{ as usual}$$.

The series is unif conv on $[2, \infty)$ compacta.

FACT 2 (generalization of Dirichlet's "pigeon hole" principle) ← see Ing 94

Let $a_1, \dots, a_N$ be real numbers. Let T_0 and $\delta_1, \dots, \delta_N$ be positive. Then, there exist integers x_j and a number t_0 so that

$$|t_0 a_j - x_j| < \delta_j \quad \text{all } j \in [1, N]$$

$$T_0 \leqq t_0 \leqq T_0 \prod_{k=1}^{N} \left(1 + [\![\tfrac{1}{\delta_k}]\!]\right) .$$ ★

The number t_0 can be taken to be a multiple of T_0.

$$\left\{ \text{Note:} \quad 1 + [\![\frac{1}{\delta_k}]\!] \leqq 1 + \frac{1}{\delta_k} . \right\}$$

★ $[\![x]\!]$ can be replaced if desired by $[\![x]\!]' = \lim_{\varepsilon \to 0} [\![x - \varepsilon]\!]$. Useful for Ing 94.

PF

Classical pigeon hole principle:

$m+1$ things dumped into m boxes

$\Rightarrow$ some box contains ≥ 2 things.

Look at $[0,1)^N$. Partition this into

$$\underbrace{1+[\![\tfrac{1}{\delta_1}]\!]}\ \times \cdots \times\ \underbrace{1+[\![\tfrac{1}{\delta_N}]\!]} \leftarrow \text{total } m$$

subboxes (each semi-open; also disjoint). Note $\delta_j > 1 \Rightarrow 1+[\![\frac{1}{\delta_j}]\!] = 1 \Rightarrow$ no action in coordinate $\#j$.

Look at the $m+1$ points

$$(ta_1, \ldots, ta_N) \bmod 1 \quad \leftarrow \text{in } [0,1)^N$$

For $t = qT_0$, $0 \leq q \leq m$. Apply classical pigeon hole principle. Get obvious $0 \leq q' < q'' \leq m$, $t' < t''$,

$$\left| t''a_j - t'a_j - \text{integer} \right| < \frac{1}{1+[\![\frac{1}{\delta_j}]\!]},$$

each j. But $1+[\![u]\!] > u$ * when $u > 0$. Let $t_D = t'' - t' = (q''-q')T_0$. This works. ▥

* For $[\![u]\!]'$, have $1+[\![u]\!]' \geq u$.

To continue, we now define

$$\boxed{F(v) = \alpha + \int_1^v \frac{P(e^u)}{\sqrt{e^u}}\,du}\ , \quad v \geqq 1\ ,$$

where α = a suitable constant (yet to be assigned).

Note

$$F(v) = \alpha + \int_e^{e^v} \frac{P(x)}{\sqrt{x}}\,\frac{dx}{x} \qquad \left\{ \begin{matrix} x = e^u \\ u = \ln x \end{matrix} \right\}$$

$$= \alpha + \int_e^{e^v} x^{-\frac{3}{2}} P(x)\,dx$$

$$= \alpha + \int_e^{e^v} x^{-3/2}\,dP_1(x) \qquad \text{(R-S style)}$$

$$= \alpha + x^{-3/2} P_1(x) \Big]_e^{e^v} - \int_e^{e^v} P_1(x)\left(-\tfrac{3}{2}\right) x^{-\frac{5}{2}}\,dx$$

$$= \alpha + b_2 + e^{-\frac{3}{2}v} P_1(e^v) + \frac{3}{2}\int_e^{e^v} P_1(x)\,x^{-5/2}\,dx$$

we propose to plug in 4 top.

So, let's just look at:

(7)

$$x_0^{-3/2} P_1(x_0) + \frac{3}{2}\int_e^{x_0} P_1(x)\, x^{-5/2}\, dx$$

$$= \sum_{\gamma} \frac{(-1)}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)} \Big[x_0^{-3/2} x_0^{\frac{3}{2}+i\gamma} + \frac{3}{2}\int_e^{x_0} x^{\frac{3}{2}+i\gamma}\, x^{-\frac{5}{2}}\, dx \Big].$$

But,

$$\frac{3}{2}\int_e^{x_0} x^{\frac{3}{2}+i\gamma}\, x^{-\frac{5}{2}}\, dx = -\int_e^{x_0} x^{\frac{3}{2}+i\gamma}\, d(x^{-3/2})$$

$$= -\Big[x^{\frac{3}{2}+i\gamma} x^{-\frac{3}{2}} \Big]_e^{x_0} \quad \begin{pmatrix}\text{by}\\ \text{parts}\end{pmatrix}$$

$$+ \int_e^{x_0} x^{-\frac{3}{2}}\, d(x^{\frac{3}{2}+i\gamma})$$

$$= -x_0^{-3/2} x_0^{\frac{3}{2}+i\gamma}$$

$$+ e^{-3/2} e^{\frac{3}{2}+i\gamma}$$

$$+ (\tfrac{3}{2}+i\gamma)\int_e^{x_0} x^{-\frac{3}{2}} x^{\frac{1}{2}+i\gamma}\, dx$$

$$= -x^{-\frac{3}{2}}\, x^{\frac{3}{2}+i\gamma} + e^{i\gamma}$$

$$+ \left(\tfrac{3}{2}+i\gamma\right)\left[\frac{x^{i\gamma}}{i\gamma}\right]_e^{x}$$

$$\Downarrow$$

$$\frac{(-1)}{\left(\frac{1}{2}+i\gamma\right)\left(\frac{3}{2}+i\gamma\right)}$$ [big bracket on 7 top]

$$= \frac{(-1)}{\left(\frac{1}{2}+i\gamma\right)\left(\frac{3}{2}+i\gamma\right)}\left[e^{i\gamma} + \left(\tfrac{3}{2}+i\gamma\right)\left(\frac{x^{i\gamma}}{i\gamma} - \frac{e^{i\gamma}}{i\gamma}\right)\right]$$

$$= x^{i\gamma}\,\frac{(-1)}{(i\gamma)\left(\frac{1}{2}+i\gamma\right)} + \frac{e^{i\gamma}}{(i\gamma)\left(\frac{1}{2}+i\gamma\right)}$$

$$- \frac{e^{i\gamma}}{\left(\frac{3}{2}+i\gamma\right)\left(\frac{1}{2}+i\gamma\right)}$$

$$\Downarrow$$

see 6 middle

$$F(v) = \alpha + \underline{b_3} - \sum_{\gamma} \frac{1}{(i\gamma)\left(\frac{1}{2}+i\gamma\right)}\, x^{i\gamma} \qquad x \equiv e^{v}$$

$$\Downarrow$$

it is now natural to

<u>declare</u> $\alpha = -b_3$

$$\Downarrow$$

$$\boxed{\begin{aligned} F(v) &= -b_3 + \int_1^v \frac{P(e^u)}{\sqrt{e^u}}\,du \\ F(v) &= -\sum_{\gamma} \frac{e^{i\gamma v}}{(i\gamma)(\frac{1}{2}+i\gamma)} \end{aligned}}$$

The sum over γ is unif conv for $v \in \mathbb{R}$.

[Remember $\gamma \in \mathbb{R}$.]

Recollection of baby Fourier analysis.

$$\tilde{f}(u) = \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx$$

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(u)\, e^{iux}\, du$$

$$\int_{-\infty}^{\infty} \max(0, b-|x|)\, e^{-iux}\, dx = \left(\frac{\sin(\frac{u}{2}b)}{u/2}\right)^2$$

$$\max(0, b-|x|) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{u}{2}b}{\frac{u}{2}}\right)^2 e^{iux}\,du$$

$$\Downarrow$$

$$\max\left(0, 1-\frac{|x|}{2\pi}\right) = \int_{-\infty}^{\infty}\left(\frac{\sin\pi u}{\pi u}\right)^2 e^{iux}\,du$$

$$\max\left(0, 1-\frac{|u|}{2\pi}\right) = \int_{-\infty}^{\infty}\left(\frac{\sin\pi x}{\pi x}\right)^2 e^{iux}\,dx .$$

We let $k(x) = \left(\frac{\sin\pi x}{\pi x}\right)^2$. Thus,

$$\tilde{k}(u) = \max\left(0, 1-\frac{|u|}{2\pi}\right) .$$

Crucial Facts:

$k(x) \geqq 0$, support of $\tilde{k} = [-2\pi, 2\pi]$

$k \in C^\infty(\mathbb{R})$, $\tilde{k} \geqq 0$,

$k, k' = O(x^{-2})$ for $|x| \geqq 1$.

$\{ k(x) =$ Fejer kernel $\}$

WANT TO NOW CONSIDER:

$$\int_{t-A}^{t+A} \frac{P(e^v)}{\sqrt{e^v}}\, k[N(v-t)]\, dv$$

with $\begin{cases} A = \underline{\text{fixed}} \text{ positive integer;} \\ N = \text{positive integer (kept very large);} \\ t = \text{real and very large.} \end{cases}$

Recall

$$F(v) = b_4 + \int_1^v \frac{P(e^u)}{\sqrt{e^u}}\, du \qquad \text{(9) box}$$

Need:

$$\int_{t-A}^{t+A} k[N(v-t)]\, dF(v) \qquad (R\text{-}5)$$

$$= k[N(v-t)]\, F(v)\Big]_{t-A}^{t+A}$$

$$- \int_{t-A}^{t+A} F(v)\, k'[N(v-t)]\, N\, dv$$

$$\left\{ \underline{\text{but } k(\pm NA) = 0} \right\}$$

$$= -N\int_{t-A}^{t+A} k'[N(v-t)]\,F(v)\,dv. \qquad (12)$$

We can now substitute

$$F(v) = -\sum_{\gamma} \frac{e^{i\gamma v}}{(i\gamma)(\frac{1}{2}+i\gamma)} \qquad \text{à la } (9).$$

Do this manipulation <u>term-by-term</u>. Clearly there is no harm in taking $\gamma > 0$ and then getting the case $\gamma < 0$ by taking a conjugate.

Accordingly, with $\gamma > 0$, get:

$$\frac{N}{(i\gamma)(\frac{1}{2}+i\gamma)} \int_{t-A}^{t+A} k'[N(v-t)]\,e^{i\gamma v}\,dv$$

$$\left\{ w = N(v-t),\quad v = t + \frac{w}{N},\quad dv = \frac{dw}{N} \right\}$$

$$= \frac{N}{(i\gamma)(\frac{1}{2}+i\gamma)} \int_{-NA}^{NA} k'(w)\,e^{i\gamma(t+\frac{w}{N})}\,\frac{dw}{N}$$

$$= \frac{e^{i\gamma t}}{(i\gamma)(\frac{1}{2}+i\gamma)} \int_{-NA}^{NA} k'(w)\,e^{i\frac{\gamma}{N}w}\,dw$$

$O(w^{-2})$ for $|w|$ large

$$= \frac{e^{i\gamma t}}{(i\gamma)(\frac{1}{2}+i\gamma)} \left[\int_{-\infty}^{\infty} k'(w) e^{i\frac{\gamma}{N}w}\, dw + O\left(\frac{1}{NA}\right) \right].$$

Pause for a Second!!

$$\int_{-\infty}^{\infty} k'(w) e^{iqw}\, dw = \int_{-\infty}^{\infty} e^{iqw}\, dk(w)$$

$$= -\int_{-\infty}^{\infty} k(w)\, d_w(e^{iqw})$$

$$= -iq \int_{-\infty}^{\infty} k(w) e^{iqw}\, dw$$

$$= -iq \max\left(0, 1-\frac{|q|}{2\pi}\right) \quad (10)$$

We therefore have:

$$(12)\ \text{line } 1 = \sum_{\gamma>0} \frac{e^{i\gamma t}}{(i\gamma)(\frac{1}{2}+i\gamma)} \left[-i\frac{\gamma}{N} \max\left(0, 1-\frac{|\frac{\gamma}{N}|}{2\pi}\right) + O\left(\frac{1}{NA}\right) \right]$$

$$+ \text{ CONJUGATE}$$

$$\simeq \sum_{0<\gamma<2\pi N} \frac{e^{i\gamma t}}{(i\gamma)(\frac{1}{2}+i\gamma)} \frac{(-i\gamma)}{N} \left(1-\frac{\gamma}{2\pi N}\right)$$

$$+ \sum_{\text{all } \gamma} \frac{1}{\gamma^2} O\left(\frac{1}{NA}\right)$$

$$+ \text{ CONJUGATE}$$

$$= -\frac{1}{N} \sum_{0<\gamma<2\pi N} \frac{e^{i\gamma t}}{\frac{1}{2}+i\gamma} \left(1-\frac{\gamma}{2\pi N}\right)$$

$$+ \text{ CONJUGATE}$$

$$+ O\left(\frac{1}{NA}\right) .$$

Here, note that

$$2\,\mathrm{Re}\left\{\frac{e^{i\phi}}{\frac{1}{2}+i\gamma}\right\} = 2\left\{\frac{\frac{1}{2}\cos\phi + \gamma\sin\phi}{\frac{1}{4}+\gamma^2}\right\}$$

$$= \frac{\cos\phi + (2\gamma)\sin\phi}{\frac{1}{4}+\gamma^2} .$$

So, we get

$$\underline{\text{(12) line 1}} = -\frac{1}{N}\sum_{0<\gamma<2\pi N}\frac{\cos(\gamma t)+2\gamma\sin(\gamma t)}{\frac{1}{4}+\gamma^2}\left(1-\frac{\gamma}{2\pi N}\right)$$

$$+\ O\left(\frac{1}{NA}\right)$$

$$\Downarrow$$

$$\int_{t-A}^{t+A}\frac{P(e^v)}{\sqrt{e^v}}\,k\,[N(v-t)]\,dv$$

$$= O\left(\frac{1}{NA}\right)$$

$$-\frac{1}{N}\sum_{0<\gamma<2\pi N}\frac{\cos(\gamma t)+2\gamma\sin(\gamma t)}{\frac{1}{4}+\gamma^2}\left(1-\frac{\gamma}{2\pi N}\right).$$

THE FINITENESS
OF THIS RANGE
IS CRUCIAL. (10) lines 5+7

This is Ingham's Trick.

Remember that "A" is fixed.

Also note that

$$\frac{1}{N} \sum_{0<\gamma<2\pi N} \frac{\cos(\gamma t)}{\frac{1}{4}+\gamma^2}\left(1-\frac{\gamma}{2\pi N}\right) = O\left(\frac{1}{N}\right).$$

And:

$$\frac{2\gamma}{\frac{1}{4}+\gamma^2} - \frac{2\gamma}{\gamma^2} = 2\gamma \cdot O(\gamma^{-4})$$
$$= O(\gamma^{-3})$$

$$\Downarrow$$

$$\frac{1}{N} \sum_{0<\gamma<2\pi N} \left[\frac{2\gamma}{\frac{1}{4}+\gamma^2} - \frac{2\gamma}{\gamma^2}\right] \sin(\gamma t)\left(1-\frac{\gamma}{2\pi N}\right)$$
$$= O\left(\frac{1}{N}\right).$$

$$\Downarrow$$

{ by recalling (11) }

$$\int_{t-A}^{t+A} \frac{P(e^v)}{\sqrt{e^v}}\, k[N(v-t)]\, dv$$

$$= O\left(\frac{1}{N}\right) - \frac{2}{N} \sum_{0<\gamma<2\pi N} \frac{\sin \gamma t}{\gamma}\left(1-\frac{\gamma}{2\pi N}\right)$$

$$= \Theta \frac{\mathcal{B}}{N} - \frac{2}{N} \sum_{0<\gamma<2\pi N} \frac{\sin \gamma t}{\gamma}\left(1-\frac{\gamma}{2\pi N}\right)$$

with $\mathcal{B} \geqq 1$, $|\Theta| \leqq 1$.

NOTE THAT the last sum can be taken over $0<\gamma \leqq 2\pi N$, if desired.

Ingham 1936	Here:
T	N
η	A
ω	t

$\rightsquigarrow$ $\boxed{\begin{array}{l} w = N(v-t) \\ v = t + \frac{w}{N} \end{array}}$

See Ingham p. 207 eq (11).

Ingham's statement is thus effectively:

$$\int_{-T\eta}^{T\eta} G\left(\omega + \frac{w}{T}\right) \mathcal{K}(w) \frac{dw}{T}$$

$$= O\left(\frac{1}{T}\right) - \frac{2}{T} \sum_{0<\gamma<2\pi T} \frac{\sin \gamma\omega}{\gamma}\left(1-\frac{\gamma}{2\pi T}\right).$$

As noted already on 1 (top), we plan to use a variant of Ingham's approach.

To continue, we now follow a direct approach with certain CHOICES. We'll hide the rough calculations which motivated these!!

Also, we will not seek an optimal constant in

$$\psi(x) - x = \Omega_{\pm}(x^{\frac{1}{2}} \log\log\log x)$$

on 1. That constant is conjectured to be arbitrarily large. [Ing, 2nd edition, p. xiv.]

Let G = a sufficiently large constant. We hold G frozen.

$0 < \gamma_n \leq 2\pi N$

Apply 4 FACT 2 to get:

$$\left| t_0 \frac{\gamma_n}{2\pi} - (\text{integer}) \right| < \frac{\gamma_n}{2\pi G N}$$

$$T_0 \leq t_0 \leq T_0 \prod_{0<\gamma_n \leq 2\pi N} \left(1 + \frac{2\pi G N}{\gamma_n}\right).$$

We'll select To in a few moments; it will be very big.

standard $|\Theta| \leqq 1$ meaning

Get:

$$t_0 \gamma_n = 2\pi(\text{integer}) + \Theta \frac{\gamma_n}{GN}$$

$$0 < \frac{\gamma_n}{GN} \leqq \frac{2\pi N}{GN} = \frac{2\pi}{G} < 10^{-6} \text{ (say)}.$$

$G > 2\pi(10^6)$

Put

$$\boxed{h = \frac{1000}{GN}}.$$

Let

$$\boxed{t_1 = t_0 + h}$$

← àla Selberg

(similarly, at very end, consider $t_1 = t_0 - h$).

Clearly:

$$0 < h < 10^{-6} \quad (\text{since } N = \text{giant}).$$

$\boxed{0 < \gamma_n \leq 2\pi N}$

$$\gamma_n t_1 \equiv \gamma_n t_0 + \gamma_n h$$

$$= 2\pi(\text{integer}) + \underbrace{- \frac{\gamma_n}{GN} + \frac{1000\,\gamma_n}{GN}}$$

but

$$\frac{999\,\gamma_n}{GN} \leq \underbrace{- \frac{\gamma_n}{GN} + \frac{1000\,\gamma_n}{GN}} \leq \frac{1001\,\gamma_n}{GN} \leq \frac{2\pi(1001)}{G} < .002$$

$$\boxed{\underline{\text{and}} \quad \frac{999\,\gamma_n}{GN} = \frac{999}{1000}\gamma_n(h)}$$

$$\Downarrow$$

$$\left[\begin{array}{c}\text{mod } 2\pi \text{ part} \\ \text{of } \gamma_n t_1\end{array}\right] \in \left[\frac{999}{1000}\gamma_n h\,,\ .002\right)$$

$\boxed{t_0 = \text{NOT useful}}$

$$\Downarrow$$

$$\sin(\gamma_n t_1) \geq \sin\left[\frac{999}{1000}\gamma_n h\right] > 0$$

KEY!!

$\Downarrow$ ← baby calc

$$\sin(\gamma_n t_1) \gtrapprox \frac{998}{1000}\gamma_n h > 0$$

$$\frac{\sin(\gamma_n t_1)}{\gamma_n} \gtrapprox \frac{998}{1000} h > 0$$

$$\sum_{0<\gamma_n \lessapprox 2\pi N} \frac{\sin(\gamma_n t_1)}{\gamma_n}\left(1-\frac{\gamma_n}{2\pi N}\right)$$ ← as on 17 top

$$\gtrapprox \sum_{0<\gamma_n \lessapprox 2\pi N} (.998) h \left(1-\frac{\gamma_n}{2\pi N}\right)$$

$$\gtrapprox \sum_{0<\gamma_n \lessapprox \pi N} (.49) h$$

$$\approx (.49)\frac{1000}{GN}\mathcal{N}(\pi N)$$

where $\mathcal{N}(v) = \#\{0<\gamma \lessapprox v, \underline{\text{with}} \text{ multiplicity}\}$

$$\approx \frac{v}{2\pi}\ln\frac{v}{2\pi} + O(v), \quad v \gtrapprox 2$$

{recall Lec 15 p. 29}

$N = \text{giant}$

$$\mathfrak{N}(\pi N) = \frac{N}{2}\ln\frac{N}{2} + O(N)$$

$$\mathfrak{N}(\pi N) = \frac{N}{2}\ln N + O(N)$$

also $\mathfrak{N}(2\pi N) = N\ln N + O(N)$

$$\Downarrow$$

$$\sum_{0<\gamma_n \leqq 2\pi N} \frac{\sin(\gamma_n t_1)}{\gamma_n}\left(1 - \frac{\gamma_n}{2\pi N}\right)$$

$$\gtrapprox (.49)\frac{1000}{GN}\left[\frac{N}{2}\ln N + O(N)\right]$$

$$\gtrapprox (240)\frac{1}{G}\ln N$$

{ rather crudely ! }

$$\Downarrow$$

$$\boxed{\boxed{\text{17 top at } t_1} \lessapprox \frac{\beta}{N} - (480)\frac{\ln N}{GN} \lessapprox -450\frac{\ln N}{GN}}$$

for N suff large, G frozen, $\beta < \frac{\ln N}{G}$.

So, with our $t_0 \hookrightarrow t_1$ construction, it emerges that

$$\boxed{\int_{t_1-A}^{t_1+A} \frac{P(e^v)}{\sqrt{e^v}} k[N(v-t_1)]\,dv \leqq -\mathcal{C}\frac{\ln N}{N}}$$

holds for all N suff. large with $\mathcal{C} \approx \frac{450}{G}$, i.e. some (fixed) appropriately small $\mathcal{C}$.

For convenience, we now just declare

$$T_0 = \prod_{0<\gamma_n \leqq 2\pi N} \left(1+\frac{2\pi G N}{\gamma_n}\right)$$

on p. 18 bottom. Recall too that $k(w) \geqq 0$.

We seek to transform the $\sim \mathcal{C}\frac{\ln N}{N}$ estimate into information about negative values of P.

$$T_0 \lesssim t_0 \lesssim T_0^2 \quad \text{on } (18) \text{ bot}$$

$$T_0 \lesssim \underline{t_1} \lesssim 2T_0^2$$

$$\frac{2\pi GN}{\gamma_n} \leq 1 + \frac{2\pi GN}{\gamma_n} \leq \frac{4\pi GN}{\gamma_n} \quad \text{trivially}$$

$$\prod_{0<\gamma_n\lesssim 2\pi N} \frac{2\pi GN}{\gamma_n} \lesssim T_0 \lesssim \prod_{0<\gamma_n\lesssim 2\pi N} \frac{4\pi GN}{\gamma_n}$$

$$(G)^{\mathfrak{N}(2\pi N)} \prod_{0<\gamma_n\lesssim 2\pi N} \frac{2\pi N}{\gamma_n} \lesssim T_0 \lesssim (2G)^{\mathfrak{N}(2\pi N)} \prod_{0<\gamma_n\lesssim 2\pi N} \quad (*)$$

---

positive

$$\log(T_0) = \left[N\log N + O(N)\right]\left[\ln G + \Theta \ln 2\right]$$

$$+ \sum_{0<\gamma_n\lesssim 2\pi N} \ln\left(\frac{2\pi N}{\gamma_n}\right)$$

---

best written as $\quad (\eta = \text{tiny})$

$$\int_\eta^{2\pi N} \log\left(\frac{2\pi N}{x}\right) d\mathfrak{N}(x)$$

$$\boxed{\xi(x) \neq 0 \text{ on } \mathbb{R}}$$

$$\sum_{0<\gamma_n \leq 2\pi N} \ln\left(\frac{2\pi N}{\gamma_n}\right)$$

$$= 0 - 0 - \int_\eta^{2\pi N} \mathcal{N}(x)\left(-\frac{1}{x}\right) dx \qquad \text{by parts}$$

$$= \int_\eta^{2\pi N} \frac{\mathcal{N}(x)}{x}\, dx$$

$$\int_\eta^{2\pi N} \frac{\mathcal{N}(x)}{x}\, dx \sim \int_{2\pi}^{2\pi N} \frac{1}{2\pi} \ln\frac{x}{2\pi}\, dx$$

$$= \int_1^N \ln y \, dy$$

$$= N\ln N - N + 1$$

$$= N\ln N + O(N)$$

$$\boxed{\sum_{0<\gamma_n \leq 2\pi N} \ln\left(\frac{2\pi N}{\gamma_n}\right) = N\ln N + O(N)}$$

$$\Rightarrow \ln T_0 = N\log N \cdot (\ln G + \lambda)$$

with some $\lambda \in [1,2]$

G frozen + big

$$\Rightarrow \boxed{\ln\ln T_0 \sim \ln N} \quad \text{as } N \to \infty .$$

Needless to say, as $x \to \infty$,

$$\ln\ln x^{\alpha} \sim \ln\ln x \sim \ln\ln x^{\beta}$$

for any $0 < \alpha < 1 < \beta$. This has relevance for 24 line 1+2. EG $\ln\ln t_0 \sim \ln\ln t_1 \sim \ln\ln T_0$.

Looking at t_1 and 23 BOX, let

$$\| \; \mathfrak{M} = \inf_{[t_1-A,\, t_1+A]} \frac{P(e^v)}{\sqrt{e^v}} \; . \; \|$$

Get

$$\mathfrak{M} \int_{t_1-A}^{t_1+A} \underbrace{k[N(v-t_1)]}_{\text{non-neg}} dv \lessapprox -\mathcal{C}\,\frac{\ln N}{N}$$

$$\left\{ w = N(v-t_1), \quad v = t_1 + \frac{w}{N} \right\}$$

$$\mathfrak{M} \int_{-NA}^{NA} k(w)\,\frac{dw}{N} \lessapprox -\mathcal{C}\,\frac{\ln N}{N}$$

$$\left\{ \text{here } A \geqq 1,\ N \text{ giant},\ \int_{-\infty}^{\infty} k(w)\,dw = 1 \quad \text{cf. 10 line 3} \right\}$$

$$\Downarrow$$

$$\mathcal{M} \lessapprox -\mathcal{C} \ln N$$

$$\Downarrow$$

$$\mathcal{M} \lessapprox -\frac{99}{100}\mathcal{C} \ln \ln T_0$$ see 25 bottom

but

$0 < h < 10^{-6}$ 19

$A =$ pos. integer 11 (fixed)

$t_0 - 2A \lessapprox t_1 - A \lessapprox t_1 + A \lessapprox t_0 + 2A$ crudely

$\ln\ln\ln [e^{t_1 - A}, e^{t_1 + A}] \subseteq \ln\ln\ln [e^{t_0 - 2A}, e^{t_0 + 2A}]$

$\ln\ln\ln [e^{v}\text{-range}]$

numerically asymptotic to

$\ln\ln\ln (e^{t_0})$

$= \ln\ln t_0$

by 26 top, get:

numerically asymptotic to

$$\boxed{\ln\ln T_0}$$

$\left\{ \begin{array}{c} \text{but} \\ \text{see} \\ \text{line 2} \\ \text{above} \end{array} \right\} \Rightarrow$ Yes!

It follows that:

$$\frac{P(x)}{\sqrt{x}} = \Omega_-[\ln\ln\ln x]$$

holds with a constant of, say, $-\frac{98}{100}C$.

One similarly establishes

$$\frac{P(x)}{\sqrt{x}} = \Omega_+[\ln\ln\ln x]$$

using $t_0 - h$ as t_1. See 19 second box!

On 2 + 3,

$$E(x) = O(1) \quad \text{for } x \geq 2 \quad \text{(obviously)}$$
$$P(x) = \psi^*(x) - x - E(x) \quad \text{by def}$$
$$P(x) = \psi(x) - x + O(\ln x).$$

Thus:

$$\psi(x) - x = \Omega_\pm(x^{\frac{1}{2}}\ln\ln\ln x). \quad \boxed{\text{OK}}$$

We won't bother to explain the corresponding results for

$\pi(x)$ Lec 14 p. 7 8
Lec 21 p. 17

$$\Pi(x) - li(x)$$

and, then, in regard to

$$\pi(x) - li(x) .$$

See Ingham (book) p. 103 theorem 35. One also recalls Ingham, p. 90 theorem 32! The issue of making things effective [e.g., by tampering with the kernel function $k(w)$], which was highlighted on p. 105, is quite interesting and was the subject of some famous work by S. Skewes (late 1930s – early 1950s).

> "The Skewes Number"
> (eg, see google.com)

Ingham, 2nd edition, page xiii (also viii) references this (work). The Selberg trick is also touched on there...

> See also the Hejhal–Odlyzko paper in the TURING centenary volume!!

Lecture 30 Synopsis

(Fri, 6 May 2016)

In this lecture — the last lecture — I thought that it would be fun to return to Euler and the PNT.

At the very end, to keep things more current, I (relented and) interjected a quick note about $S(T)$ [Lec 15, p. 26] which plays a role in ongoing computer tests of the Riemann Hypothesis.

EULER FIRST!

Recall

$$\mu(n) = \begin{cases} (-1)^{\nu}, & n = p_1 \cdots p_\nu, \text{ distinct } p_i \\ \\ 0, & n \text{ not squarefree} \end{cases}$$

[$\mu(1) = 1$]. See Lec 17 p. 14. We have:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \mathrm{Re}(s) > 1$$

$$M(x) = \sum_{n \leqq x} \mu(n), \quad x \geqq 1, \ x \in \mathbb{R}.$$

Taking $s > 1$ and letting $s \to 1^+$, one would suspect (with Euler, 1748) that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 .$$

Of course, the "rub" here is that the <u>convergence</u> of the LHS is far from obvious!

> NOTE. <u>If</u> the LHS converges, its value must be 0. This follows immediately from Lec 21, p. 11 FACT 2(b) [with $\delta = \frac{\pi}{2}$]; see also Lec 25 p. 3 on "super Stolz".

In Lec 20 p. 28 $\boxed{D}$, I pointed out that it was known very early — using only ELEMENTARY techniques — that the following statements are equivalent:

$\boxed{\text{recall Lec 2 p. 2}}$

(i) $\psi(x) \sim x$ (i.e., the PNT);

(ii) $M(x) = o(x)$;

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$.

See Lec 20 pp. 33 – 40 for (i) $\Rightarrow$ (ii); then Lec 21 pp. 1 – 8 for (ii) $\Rightarrow$ (i).

We now "finish" the job. ← And bring in Euler.

THM.

(i)(ii) $\Longleftrightarrow$ (iii).

I.E., Euler "already knew" the PNT.

Proof

Following Landau's, Handbuch der Primzahlen, 568 + 569, we set:

$$g(x) = \sum_{n \leqq x} \frac{\mu(n)}{n} \qquad (x \geqq 1)$$

$$f(x) = \sum_{n \leqq x} \frac{\mu(n) \ln n}{n}.$$

Recall that $|g(x)| < 1$ for $x \geqq 2$; Lec 20 p. 37.

ASSUME (iii): I.E., $g(x) = o(1)$.

Need to show $M(x) = o(x)$. Keep $x \geqq 2$. Get:

$$\sum_{n \leqq x} \mu(n) = 1 + \int_1^x v\,dg(v)$$

$$= 1 + [vg(v)]_1^x - \int_1^x g(v)\,dv$$

$$= 1 + xg(x) - 1 - \int_1^x g(v)\,dv$$

$$= x\,g(x) - \int_1^x g(v)\,dv \,.$$

But $|g(t)| \leqq \varepsilon$ for $t \geqq T_\varepsilon$. For $x \geqq T_\varepsilon$, we now get:

$$|M(x)| \leqq \varepsilon x + \int_1^{T_\varepsilon} |g(v)|\,dv + \int_{T_\varepsilon}^x \varepsilon\,dv$$

$$\leqq \varepsilon x + O(T_\varepsilon) + \varepsilon x$$

$$= 2\varepsilon x + O(T_\varepsilon)$$

$$\Downarrow$$

$$\limsup_{x\to\infty} \frac{|M(x)|}{x} \leqq 2\varepsilon$$

$$\Downarrow$$

$$M(x) = o(x) \,. \qquad \text{OK}$$

NEXT, ASSUME (i)(ii). Must prove $g(x) = o(1)$. To achieve this, we require 2 lemmas.

Lemma 1

$$g(x) - \frac{f(x)}{\ln x} = O\left(\frac{1}{\ln x}\right).$$

$\boxed{x \geq 2}$

(So: $g(x) = o(1) \Leftrightarrow f(x) = o(\ln x)$.)

Pf

Recall that $\displaystyle\sum_{d|N} \mu(d) = \left\{\begin{matrix} 1, & N=1 \\ 0, & N>1 \end{matrix}\right\}$ (Lec 20 p. 29).

Then that

$$1 = \sum_{n \leq x} \mu(n) \left[\!\left[\frac{x}{n} \right]\!\right]$$

via $\displaystyle\sum_{nk \leq x} \mu(n)$ and the hyperbola method à la Lec 20 p. 37. This idea can be generalized using

$k \equiv \frac{N}{n}$

$$1 = \sum_{nk \leq x} \frac{\mu(n)}{nk} \longleftarrow \boxed{\sum_{N \leq x} \frac{1}{N} \left\{ \sum_{n|N} \mu(n) \right\}}$$

$$\Downarrow$$

$$1 = \sum_{n \leq x} \frac{\mu(n)}{n} \left\{ \sum_{k \leq \frac{x}{n}} \frac{1}{k} \right\}$$

$$= \sum_{n \leq x} \frac{\mu(n)}{n} \left[\ln\left(\frac{x}{n}\right) + \gamma + O\left(\frac{n}{x}\right) \right]$$

Lec 18 p. 40 bottom

$$\approx \sum_{n \leqq x} \frac{\mu(n)}{n} (\ln x - \ln n)$$

$$+ \gamma \sum_{n \leqq x} \frac{\mu(n)}{n}$$

$$+ O(1) \sum_{n \leqq x} \frac{1}{n} \frac{n}{x}$$

$$\Downarrow$$

$$1 = (\ln x) g(x) - \sum_{n \leqq x} \frac{\mu(n) \ln n}{n} + \gamma g(x) + O(1)$$

$|g(x)| \leqq 1$

$$\Downarrow$$

$$1 = (\ln x) g(x) - f(x) + O(1) \quad \text{by } 3$$

$$\Downarrow$$

$$(\ln x) g(x) - f(x) = O(1)$$

$$\Downarrow$$

$$g(x) - \frac{f(x)}{\ln x} = O\left(\frac{1}{\ln x}\right). \quad \blacksquare$$

Must now seek to prove $f(x) = o(\ln x)$.

Lemma 2

For $x \geqq 2$, we have:

$$\sum_{n \leqq x} \frac{1}{n} g\left(\frac{x}{n}\right) = 1 .$$

Pf

On (5) middle, we saw that

$$1 = \sum_{nk \leqq x} \frac{\mu(n)}{nk}$$

think hyperbolas

{sum the other direction!}

$$= \sum_{k \leqq x} \frac{1}{k} \left\{ \sum_{n \leqq \frac{x}{k}} \frac{\mu(n)}{n} \right\}$$

$$= \sum_{k \leqq x} \frac{1}{k} g\left(\frac{x}{k}\right) \quad \text{(by def of } g\text{)}.$$

▥

Recall that

$$-\mu(N)\ln N = \sum_{k\ell=N} \Lambda(k)\mu(\ell)$$

by Lec 20 p. 33. Accordingly

$$f(x) = \sum_{N \leqq x} \frac{\mu(N)\ln N}{N} \qquad \text{à la } 3$$

$$= -\sum_{k\ell \leqq x} \frac{\Lambda(k)\mu(\ell)}{k\ell}$$

{ think hyperbolas }

$$= -\sum_{k \leqq x} \frac{\Lambda(k)}{k} \left\{ \sum_{\ell \leqq \frac{x}{k}} \frac{\mu(\ell)}{\ell} \right\}$$

$$\boxed{f(x) = -\sum_{k \leqq x} \frac{\Lambda(k)}{k}\, g\left(\frac{x}{k}\right)}$$

recall
$\Lambda(1) = 0$
$\Lambda(p^m) = \ln p\,, \quad m \geqq 1$

Integration by parts with $\psi(v)$ is cumbersome since $g(*)$ is <u>not</u> continuous. <u>Avoid it!!</u>

To prove $f(x) = o(\ln x)$ as $x \to \infty$, there is no loss of generality in taking $x =$ integer. See 3.

We have

$$\psi(v) = \sum_{n \leqq v} \Lambda(n)$$

and $\psi(v) = 0$, $v < 2$. Since we have assumed (i)(ii), it makes sense to write

$$\psi(v) = v + v\,\mathcal{E}(v) \ , \quad v > 0$$

with

$$\lim_{v \to \infty} \mathcal{E}(v) = 0$$

and

$$\mathcal{E}(v) \equiv -1 \quad \text{for} \quad 0 < v < 2 \ .$$

For convenience, declare:

$$\mathcal{E}(0) = 0 \ .$$

We still have $\psi(0) = 0 + 0\,\mathcal{E}(0)$.

CHOOSE $\mathcal{M}$ so that

$$|\mathcal{E}(v)| \leqq \mathcal{M} \ , \quad \text{all } v \geqq 0 \ .$$

In 8 box, notice that (x = integer)

$$\sum_{k \leq x} \frac{\Lambda(k)}{k} g\left(\frac{x}{k}\right)$$

$$= \sum_{k \leq x} \frac{\psi(k) - \psi(k-1)}{k} g\left(\frac{x}{k}\right)$$

$$\{ \psi(k) = k + k\,\varepsilon(k) \}$$

$$= \sum_{1 \leq k \leq x} \frac{1 + k\,\varepsilon(k) - (k-1)\,\varepsilon(k-1)}{k} g\left(\frac{x}{k}\right)$$

$$= \sum_{k=1}^{x} \frac{1}{k} g\left(\frac{x}{k}\right)$$

$$+ \sum_{k=1}^{x} [\varepsilon(k) - \varepsilon(k-1)]\, g\left(\frac{x}{k}\right)$$

$$+ \sum_{k=1}^{x} \frac{\varepsilon(k-1)}{k} g\left(\frac{x}{k}\right)$$

$= 1$ ← by Lemma 2 on 7

$$+ \sum_{k=1}^{x} \varepsilon(k)\, g\left(\frac{x}{k}\right) - \sum_{k=2}^{x} \varepsilon(k-1)\, g\left(\frac{x}{k}\right)$$

$\varepsilon(0) = 0$

$$+ \sum_{k=2}^{x} \frac{\varepsilon(k-1)}{k} g\left(\frac{x}{k}\right)$$

$$= 1 + \sum_{k=1}^{x} \varepsilon(k)\, g\left(\frac{x}{k}\right) - \sum_{\ell=1}^{x-1} \varepsilon(\ell)\, g\left(\frac{x}{\ell+1}\right)$$

$$+ \sum_{\ell=1}^{x-1} \frac{\varepsilon(\ell)}{\ell+1}\, g\left(\frac{x}{\ell+1}\right)$$

$$= 1 + \sum_{k=1}^{x} \varepsilon(k)\, g\left(\frac{x}{k}\right) - \sum_{\ell=1}^{x} \varepsilon(\ell)\, g\left(\frac{x}{\ell+1}\right)$$

$$+ \sum_{\ell=1}^{x} \frac{\varepsilon(\ell)}{\ell+1}\, g\left(\frac{x}{\ell+1}\right)$$

IF WE DECLARE
$g(v) \equiv 0$ for $v < 1$

$$= 1 + \sum_{k=1}^{x} \varepsilon(k) \left[g\left(\frac{x}{k}\right) - g\left(\frac{x}{k+1}\right) \right]$$

$$+ \sum_{\ell=1}^{x} \frac{\varepsilon(\ell)}{\ell+1}\, g\left(\frac{x}{\ell+1}\right)$$

$$(-f(x)) = 1 + \Sigma_1 + \Sigma_2 \text{ , say.}$$

Fix any tiny $\varepsilon > 0$. Let $|\varepsilon(k)| \lesssim \varepsilon$ for all $k \gtrsim K_\varepsilon + 1$. Keep $x \gtrsim K_\varepsilon + 100$.

Notice that:

$$|\Sigma_2| \leq \sum_{l=1}^{K_\varepsilon} \frac{M}{l+1} \cdot 1 \qquad \boxed{|g(v)| \leq 1}$$

$$+ \sum_{K_\varepsilon+1 \leq l \leq x} \frac{\varepsilon}{l+1} \cdot 1$$

$$\leq C_\varepsilon + \varepsilon \ln x + O(1)$$

$$\leq C_\varepsilon' + \varepsilon \ln x .$$

To estimate Σ_1, notice that

$$g\left(\frac{x}{k}\right) - g\left(\frac{x}{k+1}\right) = \sum_{\frac{x}{k+1} < n \leq \frac{x}{k}} \frac{\mu(n)}{n}$$

by p. 9 and 11 lines 5+6. Accordingly,

$$\left|g\left(\frac{x}{k}\right) - g\left(\frac{x}{k+1}\right)\right| \leq \left\{ \begin{matrix} 2 \\ \\ \sum_{\left(\frac{x}{k+1}, \frac{x}{k}\right]} \frac{1}{n} \end{matrix} \right\} .$$

In the foregoing, $\mathbb{Z} \cap (\frac{x}{k+1}, \frac{x}{k}]$ may well be empty. Empty sums are zero.

Continue to keep $x \geqq K_\varepsilon + 100$. By def,

$$\Sigma_1 = \sum_{k=1}^{x} \varepsilon(k) \left[g\left(\frac{x}{k}\right) - g\left(\frac{x}{k+1}\right) \right]. \qquad (11)$$

Hence:

$$|\Sigma_1| \leqq \sum_{1 \leqq k \leqq K_\varepsilon} \mathfrak{M} \cdot 2$$

$$+ \sum_{K_\varepsilon + 1 \leqq k \leqq x} \varepsilon \left(\sum_{(\frac{x}{k+1}, \frac{x}{k}]} \frac{1}{n} \right)$$

THE UNION OVER k WITH $(\frac{x}{k+1}, \frac{x}{k}]$ IS DISJOINT

$$\leqq 2 \mathfrak{M} K_\varepsilon + \varepsilon \sum_{n \leqq \frac{x}{K_\varepsilon + 1}} \frac{1}{n}$$

$$\leqq 2\mathfrak{M} K_\varepsilon + \varepsilon\left(\ln x + O(1)\right)$$

$$\Downarrow$$

$$|\Sigma_1| \lessapprox C''_\varepsilon + \varepsilon \ln x \,.$$

Upon reviewing 8 box, 10, (11,) 12 (middle), and line 2 above, we see that

$$|f(x)| \lessapprox 1 + C'''_\varepsilon + 2\varepsilon \ln x$$

for $x \gtrapprox K_\varepsilon + 100$. Since $\varepsilon > 0$ was arbitrary on 11, we get $f(x) = o(\ln x)$, as required on 6 bottom. By Lemma 1, on 5, we therefore have $g(x) = o(1)$, as needed.

This completes the proof of THM on 3. ∎

I couldn't resist

Finally, a very brief comment about $S(T)$!

vis à vis verifications of RH

We recall that:

$$S(T) = \frac{1}{\pi} \arg \zeta\left(\tfrac{1}{2} + iT\right)$$

àla Lec 15 26 (middle). Cf. also here Lec 15 pp. 22 – 25 and Lec 27 3 (FACT) – 6 (top).

When $T \neq$ all γ, we have

$$S(T) = -\frac{1}{\pi} \operatorname{Im} \int_{\frac{1}{2}}^{\infty} \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma .$$

If $T =$ some γ, one defines $S(T)$ by right continuity to preserve Lec 15 26 (lines 6+7).

One knows that:

$$\boxed{S(T) = O(\ln T)} .$$

THM

Define $\mathcal{L}og\,\zeta(s)$ in the usual up and across way beginning at point $A \in \mathbb{R}$, $A \gg 1$. Keep $T \geqq 2$, $T \neq$ all γ, and $-1 \leqq \sigma \leqq 2$. We then have:

$$\mathcal{L}og\,\zeta(s) = O(\ln T) + \sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \operatorname{Log}(s-\rho).$$

Here $s = \sigma + iT$ and Log = the standard principal value.

Pf

Review Lec 27 (3)(bot) – (5) and Lec 15 (22)(bot).

$$\mathcal{L}og\,\zeta(s) = 0 + \int_{\infty}^{\sigma} \frac{\zeta'}{\zeta}(u+iT)\,du$$

$$= \int_{\infty}^{2} O(2^{-u})\,du + \int_{2}^{\sigma} \frac{\zeta'}{\zeta}(u+iT)\,du$$

$$= O(1) + \int_{2}^{\sigma} \Big[O(\ln T) + \sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \frac{1}{u+iT-\rho} \Big] du$$

Lec 15 (13) (8)

$$= O(1) + O(\ln T)$$

$$+ \sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \left[\underline{\underline{\mathrm{Log}}}(s-\rho) - \underline{\underline{\mathrm{Log}}}(2+iT-\rho) \right]$$

$$= O(\ln T) + \sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \mathrm{Log}(s-\rho)$$

$$\{ \text{since } 0 \leqq \mathrm{Re}(\rho) \leqq 1 \} .$$

For $T \neq$ all γ, it is customary to define:

$$\underline{\underline{S_1}}(T) = -\frac{1}{\pi} \mathrm{Re} \int_{\frac{1}{2}}^{\infty} (\sigma - \tfrac{1}{2}) \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma .$$

For $T =$ some γ, use right continuity (cf. page 18 box).

It is easy to see that $S_1(T)$ is well-defined and nicely continuous insofar as T remains away from all γ. Lec 15, pp. 22 (bot) – 23 (top).

Notice here that:

$$S_1 = -\frac{1}{\pi} \operatorname{Re} \int_{\frac{1}{2}}^{\infty} \left(\sigma - \tfrac{1}{2}\right) d \operatorname{Log} \zeta(\sigma + iT)$$

$$= -\frac{1}{\pi} \operatorname{Re} \left[0 - 0 - \int_{\frac{1}{2}}^{\infty} \operatorname{Log} \zeta(\sigma + iT)\, d\sigma \right]$$

$$\left\{ \begin{array}{l} \text{by integ by parts and} \\ \qquad \operatorname{Log} \zeta(s) = O(2^{-\sigma}), \quad \sigma \geqq 2 \end{array} \right\}$$

$$\Downarrow$$

$$\boxed{S_1(T) = \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \ln |\zeta(\sigma + iT)|\, d\sigma} \; .$$

This is for $T \neq$ all γ. By baby analysis and

$$\int_0^1 |\ln u|\, du < \infty ,$$

the box remains true for all $T \geqq 2$ (i.e., there is good continuous behavior).

Notice further (for $T \neq$ all γ) that:

$$S_1(T) = \frac{1}{\pi} \operatorname{Re} \int_{\frac{1}{2}}^{\infty} \mathcal{L}og\, \zeta(\sigma + iT)\, d\sigma \qquad (18) \text{ line 3}$$

$$O(2^{-\sigma}) \quad \sigma \geqq 2$$

$$\Rightarrow \quad \underline{S_1'(T)} = \frac{1}{\pi} \operatorname{Re} \int_{\frac{1}{2}}^{\infty} \frac{\zeta'}{\zeta}(\sigma + iT)\, i\, d\sigma \qquad \left\{ \begin{matrix} \text{Leibnitz's} \\ \text{rule} \end{matrix} \right\}$$

$$= -\frac{1}{\pi} \operatorname{Im} \int_{\frac{1}{2}}^{\infty} \frac{\zeta'}{\zeta}(\sigma + iT)\, d\sigma$$

$$= S(T) \quad \text{by (15) middle}$$

$$\left\{ \begin{matrix} \text{with good uniform convergence} \\ \text{locally wrt } T \end{matrix} \right\}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} S_1'(T) &= S(T), \quad T \neq \text{all } \gamma \\ S_1(T) &= c_1 + \int_2^T S(u)\, du, \quad \text{all } T \geqq 2 \end{aligned}}$$

(continuous wrt T)

THM (very fundamental)

In the zero-counting formula

$$N(T) = \underbrace{\frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + O\left(\frac{1}{T}\right)}_{C^\infty} + S(T)$$

àla Lec 15 22 + 26, we have

$$S(T) = O(\ln T), \quad \int_2^T S(u)\,du = O(\ln T).$$

Pf.

Only the last assertion remains to be proved. WLOG $T \neq$ all γ. Apply 18 box. Get:

$$S_1(T) = \frac{1}{\pi}\int_2^\infty \ln|\zeta(\sigma+iT)|\,d\sigma + \frac{1}{\pi}\int_{\frac{1}{2}}^2 \ln|\zeta|\,d\sigma$$

$O(2^{-\sigma})$

$$= O(1) + \frac{1}{\pi}\int_{\frac{1}{2}}^2 \ln|\zeta(u+iT)|\,du$$

$$= O(1) + \frac{1}{\pi}\int_{\frac{1}{2}}^{2} \Big[O(\ln T) + \sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \ln|u + iT - \rho| \Big] du$$

by p. 16 THM

observe too that we have

$$|u-\beta| \leqq |(u-\beta)+i(T-\gamma)| \leqq 3$$

$$= O(1) + O(\ln T) + O(\ln T)$$

$$= O(\ln T). \quad \blacksquare$$

It is hardly surprising that the implied constants for $\ln T$ in p. 20 THM can be made explicit.

The relation $\int_2^T S(u)\,du = O(\ln T)$ qualitatively states that the <u>average</u> value of $S(u)$ is $\underline{\underline{0}}$.

These last 2 points are important. In the early 1950s, Alan Turing used these facts to develop a numerical criterion (now known as Turing's Law) by which the Riemann Hypothesis on interval $[T_1, T_2]$ can be checked simply by locating enough (that is to say, a requisite number of) sign-changes for the REAL-VALUED function

$$\xi(\tfrac{1}{2}+it) \quad \text{or, better,} \quad \frac{G(\frac{1}{2}+it)}{|G(\frac{1}{2}+it)|}\,\zeta(\tfrac{1}{2}+it)$$

in an interval slightly bigger than $[T_1, T_2]$. *

recall Lec 23, p. 3 lines 1-3 and (e); Lec 15, p. 26 (line -3)

OF COURSE, there is nothing to guarantee in advance that the requisite number will be found. One simply has to try!!!

The point here is 4-fold:

(A) if the requisite number is reached (by the machine), one is assured by Turing's theorem that all zeros are accounted for, and that there are none having $Re(s) \neq \frac{1}{2}$;

(B) there is no need to check RH for $T < T_1$ first;

---

* with special attention paid to the pattern near T_1, T_2

(C) there is no need to compute any $\mathfrak{Z}$-values with $Re(s) \neq \frac{1}{2}$;

(D) there is no need to "zero in" on the crossings in $[T_1, T_2]$ attached to the sign-changes detected by the machine.

[To understand why Turing's Law is at least believable, simply pretend that one somehow knew that $S(t)$ was exactly zero in some short intervals centered at T_1 and T_2. See p. 20 THEOREM [the formula for $N(T)$] and ponder the logical consequences which ensue!]

For further details about Turing and $\mathfrak{Z}(s)$, see the paper of Hejhal and Odlyzko in the Turing Centenary volume "Alan Turing: His Work and Impact" published by Elsevier. The story is a VERY interesting one. ← with links to Lec 29 p. 29 and S. Skewes

Turing's Law is used in all modern computational work aimed at verifying the RH. When the approach is successful, the zeros of $\mathfrak{Z}$ in the range $[T_1, T_2]$ are also known to be simple. Cf. (A) on 22.

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Addendum A

[concerning Lecture 30]

An alternate ending for Lec 30 ("skip Turing, stay with Euler"). *

After developing (3) ~ (14) in Lec 30, go back to Euler's Opera Omnia (e.g. vol. 8, p. 291) and say:

We now adapt the idea of Newman's general thm on page (9) of Lec 28 — motivated by Newman's Amer Math Monthly article cited on p. (1).

Take $\mathrm{Re}(z) > 0$ initially and define

$$g_N(z) = \sum_{n=1}^{N} \frac{\mu(n)}{n^{1+z}}$$

$$g(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1+z}} = \frac{1}{\zeta(1+z)} \, .$$

$\zeta(1+iy) \neq 0$

Form analytic continuations. Keep N and R large. Use exactly the same path as on p. (10) of Lec 28 (it depends on R). Get:

$$g(0) - g_N(0) = \frac{1}{2\pi i} \oint_C (g - g_N) N^z \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

---

* This is the comment referred to on p. (24) of Lec 28.

$$= \frac{1}{2\pi i}\int_{C_+} (g-g_N)\, N^z\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

$$-\frac{1}{2\pi i}\int_{C_-} g_N(z)\, N^z\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

$$+\frac{1}{2\pi i}\int_{C_-} g(z)\, N^z\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\ .$$

On C_+, observe that:

$$|g(z)-g_N(z)| = \left|\sum_{N+1}^{\infty}\frac{\mu(n)}{n^{1+z}}\right| \leq \sum_{N+1}^{\infty}\frac{1}{n^{1+x}}$$

$$\leq \int_N^{\infty} u^{-1-x}\,du = \frac{N^{-x}}{x}\ ;$$

$$|N^z| = N^x\ ;$$

$$\left|1+\frac{z^2}{R^2}\right| = \left|1+\frac{z}{\bar{z}}\right| = \frac{2|x|}{R}\ ;$$

$$\left|\frac{1}{z}\right| = \frac{1}{R}\ ;$$

$$|dz| = R\,d\theta \quad (z = Re^{i\theta})\ .$$

Hence,

$$\left| \frac{1}{2\pi i} \int_{C_+} (g - g_N) N^z \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right|$$

$$\lessapprox \frac{1}{2\pi} \int_{C_+} \frac{N^{-x}}{x} N^x \frac{2x}{R} \frac{R\,d\theta}{R}$$

$$\approx \frac{1}{R} \,. \quad //$$

Exactly as on p. 13 of Lec 28, we have

$$\frac{1}{2\pi i} \int_{C_-} g_N(z) N^z \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

$$= \frac{1}{2\pi i} \int_{\left(\substack{\text{left half} \\ \text{of } |z| = R}\right)} [\cdots] \frac{dz}{z} \,.$$

Along $\{|z| = R,\ x < 0\}$, we have:

A4

$$|g_N(z)| = \left| \sum_1^N \frac{\mu(n)}{n^{1+z}} \right|$$

$$\leq \sum_1^N \frac{1}{n^{1+x}} = \sum_1^N n^{|x|-1}$$

$$\leq \left\{ \begin{matrix} 1, |x|<1 \\ \frac{N^{|x|}}{N}, |x| \geq 1 \end{matrix} \right\} + \int_1^N u^{|x|-1}\, du$$

$$\leq \frac{N^{|x|}}{N} + \frac{N^{|x|}}{|x|} \quad ;$$

$$|N^z| = N^x = N^{-|x|} \quad ;$$

$$\left| 1 + \frac{z^2}{R^2} \right| = \left| 1 + \frac{z^2}{z\bar{z}} \right| = \frac{2|x|}{R} \quad ;$$

$$\left| \frac{1}{z} \right| = \frac{1}{R} \quad ;$$

$$|dz| = R\, d\theta \quad .$$

So,

$$\left| \frac{1}{2\pi i} \int_{C_-} g_N(z) N^z \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right|$$

$$\leq \frac{1}{2\pi} \int_{\text{(semicircle)}} N^{|x|} \left(\frac{1}{N} + \frac{1}{|x|} \right) N^{-|x|} \frac{2|x|}{R} \frac{R\, d\theta}{R}$$

$$= \frac{1}{\pi} \int_{(\text{semicircle})} \left(\frac{1}{N} + \frac{1}{|x|} \right) \frac{|x|}{R} \, d\theta$$

$$\leqq \frac{1}{\pi} \int_{(\text{semicircle})} \left(\frac{1}{N} + \frac{1}{R} \right) d\theta$$

$$= \frac{1}{N} + \frac{1}{R} \text{.} \quad //$$

Finally, imitate Lec 28 pages 15 + 16. Remember that R and δ are held fixed. Get:

$$\lim_{N \to \infty} \frac{1}{2\pi i} \int_{C_-} g(z) N^z \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} = 0 \text{.} \quad //$$

Conclude that:

$$\limsup_{N \to \infty} |g(0) - g_N(0)| \leqq \frac{1}{R} + 0 + \underbrace{\frac{1}{R}}_{} + 0 \text{.}$$

Since R is arbitrary, deduce that

$$\limsup_{N \to \infty} |g(0) - g_N(0)| = 0 \text{.}$$

IE,

$$\lim_{N \to \infty} g_N(0) = g(0) \text{.}$$

But $g(0)=0$ since $\zeta(1+z)$ has a simple pole at $z=0$. Hence,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$$

and Euler was right!!

Thanks to part I (i.e. 1 – 14) of Lec 30, this proves the PNT!

"VERY NICE INDEED!"

NOTE: clearly, the same argument utilized above adapts to show that

$$\sum_{1}^{\infty} \mu(n)\, n^{-1-i\tau} = \frac{1}{\zeta(1+i\tau)}$$

for $\tau \neq 0$. (Just declare $a_n = \mu(n)\, n^{-i\tau}$.)

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Addendum B

[regarding Lecture 28]

Newman's proof (in Lec 28) is clearly very slick. One seeks to understand the GENERAL THM on page (9) better — especially its genesis.

In regard to the genesis issue, one is inevitably forced to keep things a bit speculative.

With this in mind, it is helpful to back up and recall several very basic facts from Fourier analysis (on a fundamentally formal level).

Given appropriately decaying f on $[0,\infty)$.

Let

$$\mathcal{L}(s) = \int_0^\infty e^{-sx} f(x)\,dx = \text{Laplace transform.}$$

To get the inversion formula, note that

$$\mathcal{L}(k+it) = \int_0^\infty [e^{-kx} f(x)]\, e^{-itx}\,dx$$

$$= \int_{-\infty}^\infty e^{-kx} \begin{bmatrix} f(x) \\ 0 \end{bmatrix} e^{-itx}\,dx$$

$$e^{-kx} \begin{bmatrix} f(x) \\ 0 \end{bmatrix} = \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{L}(k+it)\, e^{itx}\,dt \qquad (x \neq 0)$$

$$\begin{bmatrix} f(x) \\ 0 \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{L}(k+it)\,e^{(k+it)x}\,dt$$

$$\begin{bmatrix} f(x) \\ 0 \end{bmatrix} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathcal{L}(s)\,e^{sx}\,ds$$

$$\boxed{f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathcal{L}(s)\,e^{sx}\,ds\,, \quad x > 0}\,.$$

This is the familiar formula; typically $k > 0$. For suitable f, one can reason just as well with $k = 0$. In both cases, the s-integral is technically a principal value with $[-R, R]$ and $R \to \infty$. See Lec 25, p. 5 middle.

(or continuous + piecewise C^1)

Let $h(x)$ be C^1 + appropriately decaying. Note that

$$\int_0^\infty h'(x)\,e^{-sx}\,dx = \int_0^\infty e^{-sx}\,dh(x)$$

$$= 0 - h(0) + s\int_0^\infty h(x)\,e^{-sx}\,dx$$

$$\Downarrow$$

$$\boxed{\mathcal{L}_h(s) = \frac{h(0) + \mathcal{L}_{h'}(s)}{s}}\,.$$

Assume now that our original f satisfies

$$(*) \qquad \int_0^\infty f(u)\,du = 0 \, .$$

Putting

$$h(x) = \int_0^x f(u)\,du$$

and observing that

$$h(x) - \int_\infty^x f(u)\,du = \int_0^X + \int_X^\infty = 0 \, ,$$

we deduce that

$$h(0) = 0 \, , \quad h'(x) = f(x) \quad [a.e.]$$

$$\mathcal{L}_h(s) = \frac{1}{s}\mathcal{L}_f(s)$$

$$h(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathcal{L}_h(s)\,e^{sx}\,ds$$

$$h(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{1}{s}\mathcal{L}_f(s)\,e^{sx}\,ds$$

$$\int_0^x f(u)\,du = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{1}{s}\mathcal{L}_f(s)\,e^{sx}\,ds$$

(the RHS being a $[-R, R]$ principal value).

For suitable f, we can reason just as well with $k=0$, thus getting

$$\int_0^x f(u)\,du = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\mathcal{L}(s)}{s} e^{sx}\,ds$$

under $(\ast)$.

In alternate notation,

$$\int_0^T f(u)\,du = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\mathcal{L}(s)}{s} e^{sT}\,ds$$

$$\int_0^T f(u)\,du = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\mathcal{L}(z)}{z} e^{zT}\,dz$$

$$\Downarrow$$

$$\int_0^T f(u)\,du = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{-iR}^{iR} \frac{\mathcal{L}(z)}{z} e^{zT}\,dz$$

$$\boxed{\int_0^T f(u)\,du = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\mathcal{L}(z)}{z} e^{Tz}\,\omega\left(\frac{y}{R}\right) dz \quad \text{wherein } \omega(u) = \chi_{[-1,1]}(u),\ u\in\mathbb{R}}$$

.

The foregoing format [in the box] is highly suggestive given that it is a time-honored trick in Fourier transform theory to replace the earlier ⟦even⟧ ω with other "more interesting" choices.

$$\boxed{\text{The case of Fejér summability corresponds, for instance, to taking } \omega(y) = \max\{0, 1-|y|\}\text{; see Lec 25 p. 9.}}$$

The essential point in these things is that keeping

$$\omega(0) = 1 \qquad \text{and} \qquad |\omega| = O(1)$$

guarantees that

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} F(y)\,\omega\left(\frac{y}{R}\right) dy = \int_{-\infty}^{\infty} F(y)\,dy$$

for every $F \in L_1(\mathbb{R})$. Indeed, let $|\omega| \leq M$ and $|\omega(u) - 1| < \varepsilon$ for $|u| < \delta$. Then:

$$\left| \int_{-\infty}^{\infty} F(y)\,dy - \int_{-\infty}^{\infty} F(y)\,\omega\left(\frac{y}{R}\right) dy \right|$$

$$\leqq \int_{|y| \geqq R\delta} |F(y)|\,(1+\mathcal{M})\,dy$$

$$+ \int_{|y| < R\delta} |F(y)|\,\varepsilon\,dy$$

$$\leqq (1+\mathcal{M}) \int_{|y| \geqq R\delta} |F(y)|\,dy + \varepsilon \int_{-\infty}^{\infty} |F(y)|\,dy .$$

Inspired by the Fejér case, it is more-or-less <u>mandatory</u> to observe that [for any sensible ω] we have:

$$\int_{-\infty}^{\infty} f_1(y)\,\overline{f_2(y)}\,dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}_1(u)\,\overline{\tilde{f}_2(u)}\,du$$

{see Lec 25 pp. 5 8}

$$\int_{-\infty}^{\infty} L(y)\,\omega\left(\frac{y}{R}\right) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{L}(u)\,R\,\tilde{\omega}(-Ru)\,du$$

{but ω is even}

$$\int_{-\infty}^{\infty} L(y)\,\omega\left(\frac{y}{R}\right) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{L}(u)\,R\,\tilde{\omega}(Ru)\,du$$

(B7)

$$\boxed{\int_{-\infty}^{\infty} L(y)\,\omega\!\left(\frac{y}{R}\right)dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{L}\!\left(\frac{v}{R}\right)\tilde{\omega}(v)\,dv}\,,$$

whereupon the expression $\int_{\mathbb{R}} L(y)\,\omega\left(\frac{y}{R}\right)dy$ {with given $L \in L_1(\mathbb{R})$ } is <u>again</u> seen to converge (as $R \longrightarrow \infty$) to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{L}(0)\,\tilde{\omega}(v)\,dv = \tilde{L}(0)\,\omega(0) = \int_{-\infty}^{\infty} L(y)\,dy\,.$$

We've switched (here) to L (in lieu of F) to be generally suggestive of (B4) box. Issues with 2π are ignored.

think $L \leftrightarrow \frac{\lambda(z)}{z}e^{Tz}$

For $\max\{0, 1-|y|\}$, one knows that $\tilde{\omega}(v) = \left(\frac{\sin\frac{v}{2}}{\frac{v}{2}}\right)^2$ by Lec 25 (9) line 4.

The Fourier transform of $\max\{0, 1-y^2\}$ is slightly more complicated; viz.,

$$\frac{4}{v^2}\left(\frac{\sin v}{v} - \cos v\right) \equiv \frac{4}{v^3}\int_0^v (\xi \sin \xi)\,d\xi\,.$$

We stress that ALL of the foregoing is just rudimentary / completely classical Fourier transform theory! B8

In the Landau paper from 1932 attached to Lec 28, it is nearly self-evident that a Fejér type $w(u)$ is lurking in the background; cf. B7 (box), B5 (lines 1+2), and Landau 526 lines 6+11.

Being aware of this, and guided by a concomitant desire to exploit the Cauchy integral theorem in the counterpart of B4 (box), it stands to reason that an ω which is analytic prior to "turning off" is best.

One can guess that this thought motivated Newman's choice of $\omega(u) = \max\{0, 1-u^2\}$.

BOTTOM LINE: the fcn $e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}$ is thus completely natural in B4 (box).

It is worthwhile at this juncture to quickly record the counterpart of all this for

$$\mathfrak{M}(s) = \int_1^\infty x^{-s}\, dA(x),$$

where $A(1) = 0$ and, for instance, $A(x) = \int_1^x \phi(v)\,dv$.

One gets:

$$\mathfrak{M}(s) = s\int_1^\infty \frac{A(x)}{x^{s+1}}\,dx \qquad (\mathrm{re}(s) > 1)$$

$$\frac{\mathfrak{M}(s)}{s} = \int_0^\infty \frac{A(e^u)}{e^{us}}\,du$$

$$\frac{\mathfrak{M}(k+it)}{k+it} = \int_0^\infty e^{-ku} A(e^u)\, e^{-itu}\,du$$

$$\frac{\mathfrak{M}(k+it)}{k+it} = \int_{-\infty}^\infty e^{-ku}\begin{bmatrix} A(e^u) \\ 0 \end{bmatrix} e^{-itu}\,du$$

$$e^{-ku}\begin{bmatrix} A(e^u) \\ 0 \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\mathfrak{M}(k+it)}{k+it}\, e^{itu}\,dt$$

{ [-R, R] principal value }

$$\begin{bmatrix} A(e^u) \\ 0 \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\mathfrak{M}(k+it)}{k+it}\, e^{(k+it)u}\,dt$$

$$\begin{bmatrix} A(e^u) \\ 0 \end{bmatrix} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\mathfrak{M}(s)}{s}\, e^{su}\,ds$$

$$\boxed{\begin{array}{c} A(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\mathfrak{M}(s)}{s}\, x^s\, ds\ ,\quad x>1 \\ \text{(the RHS being a } [-R,R] \text{ principal value)} \end{array}}$$

This is a very familiar formula (typically having $k>1$). Indeed: recall PERRON'S FORMULA in Lec 19, pp. 12 (bot) ~ 13 (top). (Also Lec 17, 10 top.)

Again, under certain hypotheses, it will be possible to proceed with $k=1$. The situation is clearly analogous to B4 (box). One is thus led to the expression

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{\mathfrak{M}(z)}{z}\, y^z\, \omega\!\left(\frac{y}{R}\right) dz\ ,\quad y>1\ ,$$

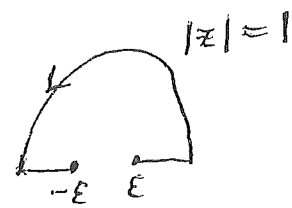
with the possible need therein to have "engineered matters" so as to have a removable singularity at, say, $z=1$. Compare: Landau 526 line 11.

<u>During</u> Lec 28, I mentioned that Newman's General Thm could be strengthened rather easily. This was already indicated on pp. 17 – 19 (top) of Lec 28.

To bring matters into a <u>still</u> <u>better</u> form, two elementary lemmas are required.

Let $D = \{|z| < 1,\ y > 0\}$, $K = \{|z| \leq 1,\ y \geq 0\}$.
Let $\Gamma = \partial D$ (counterclockwise) and

$$\Gamma_\varepsilon = \Gamma - (-\varepsilon, \varepsilon).$$



[Here $0 < \varepsilon < 1$.]

Let $\operatorname{Log} w = \ln|w| + i \operatorname{Arg}(w)$, with $-\pi < \operatorname{Arg}(w) \leq \pi$.

LEMMA 1

Let $F(z)$ be continuous on K and analytic on D. We then have

$$\frac{1}{2} F(0) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{F(z)}{z}\, dz.$$

Pf

Notice that

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{F(0)}{z}\, dz$$

$$= \lim_{\varepsilon \to 0} \frac{F(0)}{2\pi i} \left[\operatorname{Log}(-\varepsilon) - \operatorname{Log}(\varepsilon) \right]$$

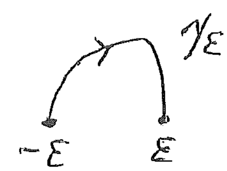
$$= \lim_{\varepsilon \to 0} \frac{F(0)}{2\pi i} [\pi i] = \frac{F(0)}{2} .$$

Our task is to check that

$$0 = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{F(z) - F(0)}{z}\, dz .$$

As such, we might as well simply hypothesize $F(0) = 0$ at the outset. We do so. By the extended Cauchy integral theorem,

$$\int_{\Gamma_\varepsilon} \frac{F(z)}{z}\, dz + \int_{\gamma_\varepsilon} \frac{F(z)}{z}\, dz = 0 .$$



But,

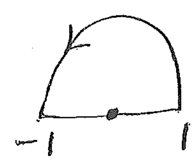
$$\left| \int_{\gamma_\varepsilon} \frac{F(z)}{z}\, dz \right| \leqq \max_{\gamma_\varepsilon} |F| \cdot \int_{\gamma_\varepsilon} \frac{|dz|}{\varepsilon} = \pi \max_{\gamma_\varepsilon} |F| .$$

Hence,

$$\left| \int_{\Gamma'_\varepsilon} \frac{F(z)}{z}\, dz \right| \lesssim \pi \max_{\gamma_\varepsilon} |F| \,.$$

Since $F(0) = 0$, we have $\max_{\gamma_\varepsilon} |F| \to 0$ as $\varepsilon \to 0$.

∎



LEMMA 2

Let $F(z)$ be continuous on K and analytic on D. Assume that $F(0) = 0$ and that

$$\int_{-1}^{1} \left| \frac{F(x)}{x} \right| dx < \infty \,.$$

We then have

$$0 = \frac{1}{2\pi i} \int_{\Gamma'} \frac{F(z)}{z}\, dz$$

in a [natural] Lebesgue integral sense.

Pf

That $\frac{1}{2\pi i} \int_{\Gamma'} \frac{F(z)}{z}\, dz$ exists as a Lebesgue integral is obvious. But, then, by a standard specialization,

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{F(z)}{z}\,dz = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{n}}} \frac{F(z)}{z}\,dz = 0 ,$$

thanks to Lemma 1. ▮

IMPROVED THEOREM.

Newman's General Theorem on p. 9 of Lec 28 is actually valid anytime the function $g(s)$ on $\{Re(s) > 0\}$ admits a continuous extension to $\{Re(s) \geq 0\}$ having the additional property that

$$\int_{-1}^{1} \left| \frac{g(it) - g(0)}{t} \right| dt < \infty .$$

Pf

We first claim that one can take $g(0) = 0$, wlog. Suppose, e.g., that $v = 1$ is a point of continuity of f. Let $A = g(0) \neq 0$ and define

$$f_1(v) = \begin{cases} f(v) - A , & 0 \leq v < 1 \\ f(v) , & v \geq 1 \end{cases} .$$

The fcn f_1 is still bounded + piecewise continuous on $[0, \infty)$. We get: {for re(s) > 0 initially}

$$g_1(s) = g(s) - A\int_0^1 e^{-sv}\,dv$$
$$= g(s) - A\,E(s)\,, \quad E(s) \equiv \frac{1-e^{-s}}{s}\,.$$

The fcn E is entire and equals $1 + O(s)$ near $s = 0$. Clearly, $g_1(0) = 0$ and g_1 is continuous for all $\{\mathrm{Re}(s) \geqq 0\}$. For $|t| \leqq 1$, notice that

$$|g(it) - g(0) - g_1(it)| = |A|\,|E(it) - 1|$$
$$= O(|t|)\,.$$

Hence,

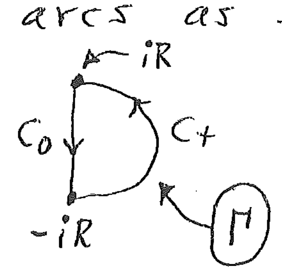
$$\int_{-1}^1 \left|\frac{g_1(it)}{t}\right| dt < \infty \iff \int_{-1}^1 \left|\frac{g(it) - g(0)}{t}\right| dt < \infty\,.$$

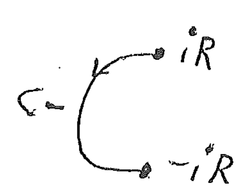
Switching to $\{g_1, f_1\}$ and establishing $0 = \int_0^\infty f_1(v)\,dv$ will thus lead to $0 = -A + \int_0^\infty f(v)\,dv$, which is exactly what we want.

From this point on, we assume $g(0) = 0$.

Fix any large R. Let Γ be the (counterclockwise) path $\{|z| = R,\ x > 0\} \cup \{z = iy,\ -R \leqq y \leqq R\}$. Introduce arcs as shown:

$$\boxed{\Gamma = C_+ \cup C_0}$$





Keep T big. Write

$$g_T(z) = \int_0^T e^{-zv} f(v)\,dv$$

as usual. By a trivial variant of Lemma 2, we have:

$$(**) \qquad 0 = \oint_\Gamma \frac{g(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)}{z}\,dz \, .$$

(as a Lebesgue integral)

The fcn $g_T(z)$ is entire; by the Cauchy integral formula, we know:

$$g_T(0) = \frac{1}{2\pi i}\int_{C_+} g_T(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

$$+ \frac{1}{2\pi i}\int_{C_-} g_T(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} \, .$$

We propose to subtract (**) from this equation for $g_T(0)$.

Get:

$$g_T(0) = \frac{1}{2\pi i}\int_{C_+} [g_T(z) - g(z)]\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

$$+ \frac{1}{2\pi i}\int_{-iR}^{iR} \frac{g(z)}{z}\, e^{Tz}\left(1+\frac{z^2}{R^2}\right) dz$$

this arc is $-C_0$

$$+ \frac{1}{2\pi i}\int_{C_-} g_T(z)\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

$$\equiv J_1 + J_2 + J_3\ , \quad \text{say.}$$

The estimations for $|J_1|$ and $|J_3|$ go exactly like before.

$|J_1| \leqq \frac{B}{R}$ 11 bot ~ 12 in Lec 28

$|J_3| \leqq \frac{B}{R}$ 13 (middle) ~ 14 in Lec 28

To estimate $|J_2|$, we write

$$|J_2| = \frac{1}{2\pi} \left| \int_{-R}^{R} \frac{g(iy)}{y} e^{iyT} \left(1 - \frac{y^2}{R^2}\right) dy \right| ,$$

note the Lebesgue integrability of $g(iy)/y$, and then apply the standard Riemann-Lebesgue lemma. Get:

see Lec 7 p. 22

$$\lim_{T \to \infty} |J_2| = 0 , \quad \text{each } R.$$

IE $J_2 = o(1)$, akin to Lec 28 p. 19 (line 2).

It follows that:

$$\limsup_{T\to\infty} |g_T(0)|$$
$$= \limsup_{T\to\infty} |J_1 + J_2 + J_3|$$
$$\leq \frac{B}{R} + 0 + \frac{B}{R} = \frac{2B}{R}$$

akin to Lec 28 p. 19 (line −3). Since R is arbitrary, $\lim_{T\to\infty} g_T(0) = 0 = g(0)$ and we are done. ▨

After completing this (last) proof, it pays to step back and note how the correctness of B4 (box) for a large class of functions f, together with B5 (top) + B8 (lines 4~8), clearly engender a kind of "moral encouragement" that a limit theorem like Newman's General Thm [or B14] might well prove feasible on a relatively simple technical level.

The miraculous cancellations that appeared "along the way" are perhaps best viewed in this light.

6/5/2016

Closing Comments

[Assorted remarks, revisions, augmentations for Lecs 1–30]

Part I (Lecs. 1–16)

1. (Lec 1, p. 1) "The primary reference for these lectures will be Ingham, Distr. of Prime Numbers, from 1932." This sentence somehow failed to make its way into the record of Lec 1. ☹

2. (Lec 1, p. 18) The fact that $\psi(x) = \theta(x) + O(x^{\frac{1}{2}})$ is important and probably should have been highlighted via some kind of box.

3. (Lec 3, pp. 7+8) Apostol's book, *Mathematical Analysis*, 1st edition, 1957, can be cited as a good reference for Riemann–Stieltjes integrals.

On page 8 (bottom half), it is important to note that the stated integration-by-parts formula holds equally well when f is merely continuous + piecewise C^1 on $[a, b]$. Compare Lec 11, (1) bot– (2) middle.

4. (Lec 5, p. 9) In regard to formula #1 on this page, I should have paused to note that letting $z \to 1$ immediately gives

$$\sum_1^N n^{-1} = 1 + \ln N - \int_1^N \frac{r(t)}{t^2}\,dt$$

$$\Rightarrow \text{[Euler constant]} \quad \gamma = 1 - \int_1^\infty \frac{r(t)}{t^2}\,dt$$

$$\Rightarrow \sum_1^N n^{-1} - \gamma = \ln N + \int_N^\infty \frac{r(t)}{t^2}\,dt \,,$$

hence

$$\boxed{\begin{aligned} \sum_{n=1}^N \frac{1}{n} &= \ln N + \gamma + \int_N^\infty \frac{r(t)}{t^2}\,dt \,; \\ \sum_{n=1}^N \frac{1}{n} &= \ln N + \gamma + O\left(\frac{1}{N}\right) \end{aligned}} \,.$$

In formula #2 on p. 9, application of the above equation for γ quickly leads to

$$\zeta(z) - \gamma = \frac{1}{z-1} - \left\{ z\int_1^\infty \frac{r(t)}{t^{z+1}}\,dt - \int_1^\infty \frac{r(t)}{t^2}\,dt \right\}.$$

The brace is simply $H(z) - H(1)$ with

$$H(z) \equiv z\int_1^\infty \frac{r(t)}{t^{z+1}}\,dt \,.$$

The function $H(z)$ is analytic on $\{x > \delta > 0\}$ as already noted. Accordingly, on page 10 (top), after introducing F, one can simultaneously assert that

$$\boxed{\zeta(z) = \frac{1}{z-1} + \gamma + O(z-1) \quad \text{near } z = 1} \; .$$

Compare Lec 18 40 – 41.

5. (Lec 6, p. 10 top) This subtraction trick is the "$z = z$" counterpart of what was just obtained for $\sum_1^N n^{-1}$ in item #4. Its importance can thus be said to have been recognized very early on.

6. (Lec 8, p. 14) Taking $f = \frac{1}{1+t}$ and $N \hookrightarrow N-1$ in E-M version I leads to:

$$\left\{ \begin{aligned} &\sum_{n=1}^{N} \frac{1}{n} = \frac{1}{2} + \frac{1}{2N} + \ln N - \int_1^N \frac{\beta(t)}{t^2}\, dt \\ &\beta(t) \equiv t - [\![t]\!] - \frac{1}{2} \end{aligned} \right\} .$$

This agrees with item #4 above.

7. (Lec 9, p. 21) Regarding EULER and the special values $\zeta(-2k) = 0$, $\zeta(2k)$, $\zeta(0)$, $\zeta(1-2k)$ for $k \geq 1$, it is very illuminating to actually have a look in Euler's collected works. (cf., e.g.,

Leonhardi Euleri, Opera Omnia, Series I,

vol 14 pp. 73 - 86 (1734);
114 (§19) (1736);
424 - 434, 440 - 443 (1740);
477 - 479 (1750);

vol 15 pp. 72 - 78 (1749).

One keeps in mind here the (modified) function $\phi(s) \equiv \sum_{1}^{\infty} (-1)^{n-1} n^{-s} = (1-2^{1-s})\zeta(s)$. The paper by R. Ayoub, "Euler and the Riemann Zeta Function", Amer. Math. Monthly 81 (1974) 1067 - 1086 is also very worthwhile, as is A. Weil's, "Prehistory of the Zeta-Function", in Number Theory, Trace Formulas, and Discrete Groups (ed. by K. Aubert, et al.), Acad. Press, 1989, pp. 1 - 9.

8. (Lec 11, p. 24) Concerning the functional equation $\xi(s) = \xi(1-s)$ and the alternate version

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$$

noted in Lec 16, p. 7 (top), a look in Euler is again very revealing. (cf., e.g.,

vol. 14 p. 443 (1740);

vol. 15 pp. 79–90 (1749);

131–138 (1772).

Also of interest:

vol. 16 (part 2, preface) pp. XXVIII, XXXII,
LXXXII–LXXXV,

and the aforementioned works by R. Ayoub and A. Weil.
The extent to which Euler more-or-less "stumbled on" the functional equation already around 1740–1749 is striking indeed!! Remember Riemann is ≈ 1859.

9. (Lec 11, p. 19) It is worthwhile to show how a bare bones form of Poisson summation follows nearly immediately from Euler–Maclaurin version I (Lec 8, p. 14). *

Given any $\varphi \in C^1(\mathbb{R})$ such that $\varphi \in L_1(\mathbb{R})$, $\varphi' \in L_1(\mathbb{R})$.
We then have:

(a) $\varphi \to 0$ as $x \to \pm\infty$;

(b) $\sum_{-\infty}^{\infty} |\varphi(x+n)|$ conv uniformly on $\mathbb{R}$-compacta;

(c) $\sum_{-\infty}^{\infty} \varphi(x+n) = \lim_{N\to\infty} \sum_{-N}^{N} \hat{\varphi}(k)\, e^{2\pi i k x}$, each $x \in \mathbb{R}$.

* The TRICK will be very similar to pp. 3 (bottom) – 4 (line 8) of Lec 11.

Since $\varphi \in L_1(\mathbb{R})$, clearly $\liminf_{x \to \infty} |\varphi(x)| = 0$. Since $\varphi' \in L_1(\mathbb{R})$, $\varphi(y)$ is uniformly Cauchy as $y \to \infty$. Hence $\lim_{x \to \infty} \varphi(x) = 0$. The case $x \to -\infty$ is similar. $\Rightarrow$ (a) OK

Notice that $\int_0^1 \left(\sum_{-\infty}^{\infty} |\varphi(x+n)| \right) dx < \infty$. Hence $\sum_{-\infty}^{\infty} \varphi(x+n)$ conv <u>absolutely</u> almost everywhere on $[0,1]$. E.g., at point x_0. Consider any $x_1 \in [0,1]$ with, say, $x_1 > x_0$. (The case $x_1 < x_0$ will be similar.) For $N \geqq M$ large, observe that:

$$\sum_M^N |\varphi(x_1+n)| = \sum_M^N \left| \varphi(x_0+n) + \int_{x_0}^{x_1} \varphi'(v+n)\,dv \right|$$

$$\leqq \sum_M^N |\varphi(x_0+n)| + \sum_M^N \int_{x_0+n}^{x_1+n} |\varphi'(w)|\,dw$$

$$\left\{ \frac{\text{but the intervals } [n, n+1] \text{ are non-overlapping}}{\text{and } x_j + n \in [n, n+1]} \right\}$$

$$\leqq \sum_M^N |\varphi(x_0+n)| + \int_M^{\infty} |\varphi'(w)|\,dw .$$

Negative N and M are treated similarly; the "new" w-integral will be

$$\int_{-\infty}^{N+1} |\varphi'(w)|\,dw .$$

This proves (b) on $K = [0,1]$. By virtue of (a), we then get (b) on a general K.

To prove (c), a standard translation shows that $x = 0$ wlog. Let M be big. We have:

$\boxed{\text{by E-M}}$

$$\frac{1}{2}\varphi(-M) + \sum_{|n|<M} \varphi(n) + \frac{1}{2}\varphi(M) = \int_{-M}^{M} \varphi(x)\,dx + \int_{-M}^{M} \varphi'(x)\beta(x)\,dx$$

$$\left\{ \beta(x) = x - [\![x]\!] - \frac{1}{2} \right\}.$$

Let $M \to \infty$ remembering that $|\beta(x)| \lesssim \frac{1}{2}$. Get:

$$\sum_{-\infty}^{\infty} \varphi(n) = \hat{\varphi}(0) + \int_{-\infty}^{\infty} \varphi'(x)\beta(x)\,dx.$$

Let $s_N(x)$ be the usual partial sum for $\beta(x)$; recall the bounded convergence properties of s_N. Hence,

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} |\varphi'(x)|\,|\beta(x) - s_N(x)|\,dx = 0$$

$$\int_{-\infty}^{\infty} \varphi'(x)\beta(x)\,dx = \lim_{N\to\infty} \sum_{k=1}^{N} \int_{-\infty}^{\infty} \varphi'(x)\frac{\sin(2\pi kx)}{-\pi k}\,dx.$$

When $k \geqq 1$, we immediately check via (a) and integ-by-parts that

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi k x)}{-\pi k}\, d\varphi(x) = \frac{1}{\pi k}\int_{-\infty}^{\infty} \varphi(x)(2\pi k)\cos(2\pi k x)\, dx$$

$$= \int_{-\infty}^{\infty} \varphi(x)\left[e^{2\pi i k x} + e^{-2\pi i k x}\right] dx$$

$$= \hat{\varphi}(k) + \hat{\varphi}(-k)\,.$$

Hence:

$$\sum_{-\infty}^{\infty} \varphi(n) = \hat{\varphi}(0) + \lim_{N\to\infty}\left\{\sum_{1\le|l|\le N} \hat{\varphi}(l)\right\}$$

$$= \lim_{N\to\infty} \sum_{-N}^{N} \hat{\varphi}(l)\,. \qquad \blacksquare$$

10. (Lec 16, p.7) Once the 2nd box was obtained on p.7, had I not been in a rush, it would have been useful to stop for a moment and obtain

$$\boxed{\frac{\zeta'(0)}{\zeta(0)} = \ln(2\pi)}$$

by letting $s\to 0$. See Lec 18 (41) (42) and item #4 above.

Notice incidentally that

$$\operatorname*{Res}_{s=0}\left[\frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\right] = \left(-\frac{\zeta'(0)}{\zeta(0)}\right)x\,.$$

for p.1 in Lec 16

Part II (Lecs. 17–30)

11. (Lec 17+18, pp. 28–35) A slight improvement can be made on pp. 28(top) – 30(bottom) with very little effort. Recall that $x \geq 1+\delta_0$, $1<c\leq 2$, $T \geq 3$ as on pp. 10+11(top). Consider those ρ for which

$$T-1 \leq \gamma < T.$$

On p. 28 (bot), it is preferable to replace $\mathcal{C}$ by a new path $\tilde{\mathcal{C}}_\rho$ which changes with ρ.

$\boxed{\rho = \beta + i\gamma}$

$$\tilde{\mathcal{C}}_\rho = \mathcal{C}_1 \cup \mathcal{C}_2$$

$\mathcal{C}_1$: semicircular part $\qquad$ $\mathcal{C}_2$: linear part

Keep $0 < \varepsilon \leq \frac{c-1}{4} \leq \frac{1}{4}$. Notice that

$$\beta + \varepsilon \leq 1 + \varepsilon < 1 + \frac{c-1}{2} = \frac{c+1}{2} < c.$$

On $\mathcal{C}_1$, clearly $|s-\rho| \geqq |s-(\beta+iT)| = \varepsilon$ by elementary geometry. On $\mathcal{C}_2$, clearly $|s-\rho| \geqq |\sigma-\beta|$.

Observe that:

$$\left| \frac{1}{2\pi i} \int_{\mathcal{C}_1} \frac{x^s}{s(s-\rho)}\, ds \right| \leqq \frac{1}{2\pi} \int_{\mathcal{C}_1} \frac{x^\sigma}{|s||s-\rho|} |ds|$$

$$\leqq \frac{1}{2\pi} \int_{\mathcal{C}_1} \frac{x^\sigma}{(T-1)\,\varepsilon} |ds|$$

$$\leqq \frac{1}{2} \frac{x^{\beta+\varepsilon}}{(T-1)} = O\left(\frac{1}{T}\right) x^c .$$

At the same time,

$$\left| \frac{1}{2\pi i} \int_{\mathcal{C}_2} \frac{x^s}{s(s-\rho)}\, ds \right| \leqq \frac{1}{2\pi} \int_{\mathcal{C}_2} \frac{x^\sigma}{(T-1)|\sigma-\beta|}\, d\sigma$$

$$\leqq \frac{1}{2\pi\varepsilon(T-1)} \int_{\mathcal{C}_2} x^\sigma d\sigma$$

$$\leqq \frac{1}{2\pi\varepsilon(T-1)} \int_{-\infty}^{c} x^\sigma d\sigma \qquad (x>1)$$

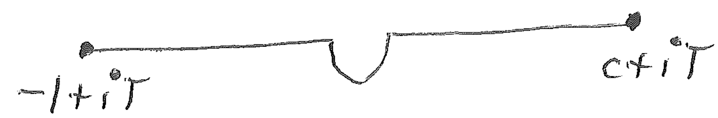
$$= \frac{1}{2\pi\varepsilon(T-1)} \frac{x^c}{\ln x}$$

$$= O\left(\frac{1}{T}\right) \frac{x^c}{\varepsilon \ln x} .$$

Accordingly,

$$\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}_\rho} \frac{x^s}{s(s-\rho)}\,ds = O\left(\frac{1}{T}\right) x^c \left[1 + \frac{1}{\varepsilon \ln x}\right].$$

The case $T < \gamma \leq T+1$ is addressed similarly via

$\tilde{\mathcal{C}}_\rho$ 

As such, p. 30 line 4 now becomes

$$O\left(\frac{\ln T}{T}\right) x^c \left[1 + \frac{1}{\varepsilon \ln x}\right]$$

(<u>provided</u> ε is kept independent of ρ);

line 7 becomes

$$O\left(\frac{\ln T}{T}\right) x^c \left[1 + 1 + \frac{1}{\varepsilon \ln x}\right];$$

and, lastly, 30 (bottom) becomes

$$O\left(\frac{\ln T}{T}\right) x^c \left[1 + \frac{1}{\varepsilon \ln x}\right].$$

On pp. 31–32, we can now make the replacement

$$O\left(\frac{\ln T}{T}\right)\frac{x^c}{c-1} \hookrightarrow O\left(\frac{\ln T}{T}\right)x^c\left[1+\frac{1}{\varepsilon \ln x}\right].$$

Note that the term $O\left(\frac{x^c \ln x}{T(c-1)}\right)$ is still present: minimization of this term leads to

$$c \approx 1+\frac{b}{\ln x} \qquad (b \approx \text{tiny constant}).$$

As far as ε goes, it is $\overset{(\text{clearly})}{\vee}$ natural to put

$$\varepsilon \approx \frac{c-1}{4} = \frac{b}{4 \ln x}.$$

The error term on p. 33 (bottom) thus becomes

$$O\left(\frac{x \ln T}{T}\right) + O\left(\frac{x \ln^2 x}{T}\right) + O(\ln x)\min\left\{1, \frac{x}{T\langle x\rangle}\right\}.$$

The implied constants will depend solely on δ_0.

On p. 35, in the statement of the explicit formula, we finally reach:

$$\boxed{\begin{aligned}\psi^*(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln(1-x^{-2}) \\ + O\left(\frac{x \ln^2 x}{T}\right) + O\left(\frac{x \ln T}{T}\right) \\ + O(\ln x) \min\left\{1, \frac{x}{T\langle x\rangle}\right\}.\end{aligned}}$$

This is the improved version that was tacitly referred to in the footnote on p. 37.

12. (Lec 19+20, p. 25) Concerning Prop 3 ($\sum \frac{\mu(n)}{n} = 0$) and the Euler product developments for $\zeta(s)$ and $1/\zeta(s)$, see:
(Lec 6, 3)

L. Euler, Opera Omnia, Ser. I

products	vol. 8 pp. 286 (§269), 288 (§274), 300 (1748)
sum = 0	vol. 8 291, 307 for $\mu(n)$, $\lambda(n)$, resp.;
products	vol. 14 pp. 230–231 and, à la logs, 243–244 (1737)
sum = 0	vol. 14 241–242 for $\lambda(n)$ with methodology; compare, however, 227–229 (1737).

{vol. 8 = Introductio in analysin infinitorum}

13. (Lec 19+20, p. 28) Concerning item $\boxed{D}$: during the lecture, I misstated the relation to $\sum \frac{\mu(n)\ln n}{n} = -1$ (i.e. p. 27 box). I wanted the implication for this aspect to "go" only one way.

This point is now correct in the revised pdf for Lec 19+20.

Incidentally: note that if $\sum \frac{\mu(n)\ln n}{n}$ converges, its value must be -1 thanks to p. 27 (bot) and Lec 21, p. 11 (Fact 2b). Similarly for $\sum \frac{\mu(n)}{n}$ and 0.

14. (Lec 24, p. 18) The current fraction [after much technical effort] is $\frac{13}{84} = .15476^+$. Note that $\frac{1}{7} = .14285^+$.

In 2005, the fraction was $\frac{32}{205} = .15609^+$.

15. (Lec 24, pp. 16–18) As the [upper] bound for $\mu(\sigma)$ gradually improves, it is only natural to wonder what can be obtained via Perron's formula (Lec 19, p. 4) in a variety of problems utilizing just a crude absolute value technique over a rectangle – akin to what we did in Lec 19, p. 18 ff with $M(x)$.

My original thought was to give another homework problem or two touching on this matter; alas, time (and endurance?) constraints intervened.

In the for what it's worth "department", I'll now scratch the surface on this topic by sketching what happens for

$$\sum_{n \leq x} d(n) \quad .$$

Here, of course, we have $f(s) = \zeta(s)^2 = \sum_1^\infty \frac{d(n)}{n^s}$. Nothing is lost by taking x à la Lec 19, pp. 14–15. Nothing is lost by taking x to have form $N + \frac{1}{2}$. For Lec 19, p. 4, we want:

$$\left\{ \begin{array}{l} a_n = d(n), \ \alpha = 2, \ \Phi(v) = \mathcal{M} v^\varepsilon, \ c = 1 + \frac{1}{\ln x} \\ \qquad x = \text{big} \end{array} \right\} .$$

Get:

$$\sum_{n \leq x} d(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta(w)^2 \frac{x^w}{w} dw + O\left[\frac{x \ln^2 x}{T}\right]$$

$$+ O\left[\frac{x^{1+\varepsilon} \ln x}{T}\right] + O\left[\frac{x^{1+\varepsilon}}{T}\right] .$$

We'll write this with a minor abuse of language in the equivalent form

$$(\star) \qquad \sum_{n \leq x} d(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta(w)^2 \frac{x^w}{w}\, dw + O\left[\frac{x^{1+\varepsilon}}{T}\right].$$

The residue at $w=1$ was computed earlier as

$$x \ln x + (2\gamma - 1)x$$

in Lec 21, p. 9. It seems reasonable to now select any $\lambda \in (0,2]$ and push $\mathrm{Re}(w) = c$ over to $\mathrm{Re}(w) = 1-\lambda$. If $\lambda = 1$, we make a minor indentation at $w = 0$.

Prior to continuing, we recall that the fcn $\mu(\sigma)$ satisfies $\mu(\sigma) = \mu(1-\sigma) + \frac{1}{2} - \sigma$, which can be rewritten as

$$(\star\star) \qquad \mu(\sigma) + \frac{1}{2}\sigma = \mu(1-\sigma) + \frac{1}{2}(1-\sigma).$$

Put:

$$k(\sigma) = \begin{cases} \frac{1}{2} - \sigma, & \sigma \leq 0 \\ \frac{1-\sigma}{2}, & 0 \leq \sigma \leq 1 \\ 0, & \sigma \geq 1 \end{cases}, \qquad \nu(\sigma) = \begin{cases} \frac{1}{2} - \sigma, & \sigma \leq \frac{1}{2} \\ 0, & \sigma \geq \frac{1}{2} \end{cases}.$$

The fcns k and ν both satisfy $(\star\star)$; it is obvious that $k(\sigma) \geq \nu(\sigma)$. One <u>hopes</u> that $\mu(\sigma) = \nu(\sigma)$.

Let $A > 0$ satisfy

$$|\zeta(1-\lambda+it)| = O(1)(1+|t|)^{A}.$$

To avoid trivia in (★) when $c \hookrightarrow 1-\lambda$, we insist that

$$x^{1-\lambda}T^{2A} \lesssim x, \quad \underline{\text{i.e.}} \quad T^{2A} \lesssim x^{\lambda}.$$

In short order, by employing convexity àla pp. 8+12 in Lec 24, estimate (★) transforms into

$$\sum_{n \lesssim x} d(n) = O(T^{2A}x^{1-\lambda}) + \overbrace{O\left(T^{\varepsilon}\frac{x}{T}\right)}^{\text{horiz. contr.}}$$

$$+ O(x^{\frac{1}{10}}) + x\ln x + (2\gamma-1)x$$

$$+ O\left(\frac{x^{1+\varepsilon}}{T}\right),$$

<u>at least</u> when $x > x_0(A,\varepsilon,\lambda)$. [$x_0 = \exp\left(\frac{2A}{\varepsilon\lambda}\right)$ is fine.] The term $O(x^{\frac{1}{10}})$ is present whenever $\lambda \in [\frac{9}{10}, 2]$ in order to accommodate <u>either</u> the residue at $w=0$ or an indentation made to avoid any issues at $w=0$.

To optimize, we imagine ε as 0 and set

$$x^{1-\lambda}T^{2A} = \frac{x}{T} .$$

Thus

$$\boxed{T = x^{\frac{\lambda}{2A+1}}} ,$$

which then leads to a collective error term of

$$O(x^{\frac{1}{10}}) + O\left[x^{\frac{2A+(1-\lambda)}{2A+1}+2\varepsilon}\right] .$$

We remark here that $T \leq x^{2}$ <u>and</u> <u>that</u> $\frac{\lambda}{2A+1} < \frac{\lambda}{2A}$, this last relation showing that T is admissible.

Because of the 2ε, one is now free to substitute any number $\geq \mu(1-\lambda)$ for A and still have a true result (i.e., <u>valid</u> remainder term).

We'll go with $A = k(1-\lambda)$. Accordingly,

$$0 < \lambda < 1 \Rightarrow A = \frac{\lambda}{2} \Rightarrow O\left[x^{\frac{1}{1+\lambda}+2\varepsilon}\right] ;$$

$$1 \leq \lambda \leq 2 \Rightarrow A = \lambda - \frac{1}{2} \Rightarrow O\left[x^{\frac{1}{2}+2\varepsilon}\right] .$$

The <u>optimal</u> estimate for $\sum_{n \leq x} d(n)$ with this [crude] Perron-type technique is therefore:

$$(\star\star\star) \qquad \sum_{n \leq x} d(n) = x \ln x + (2\gamma - 1)x + O\left(x^{\frac{1}{2} + 2\varepsilon}\right)$$

attained basically for any $\lambda \in [1, 2]$.

Under the Lindelöf Hypothesis, one can take $A = v(1-\lambda)$. This produces

$$0 < \lambda < \tfrac{1}{2} \Rightarrow A = 0 \Rightarrow O\left[x^{1-\lambda+2\varepsilon}\right]$$

$$\tfrac{1}{2} \leq \lambda \leq 2 \Rightarrow A = \lambda - \tfrac{1}{2} \Rightarrow O\left[x^{\frac{1}{2}+2\varepsilon}\right] .$$

In other words: we <u>still</u> get $(\star\star\star)$, only beginning already at $\lambda = \frac{1}{2}$.

One can SUMMARIZE by saying that a crude application of Perron's method essentially leads to <u>just</u>

$$\sum_{n \leq x} d(n) = x \ln x + (2\gamma - 1)x + O(\sqrt{x}),$$

a result which was proved earlier, indeed, almost effortlessly, in Lec <u>21</u>, pages 2 + 3.

To obtain a remainder term of size $O(x^{1/3})$ [which is the <u>classical</u> <u>nontrivial</u> estimate], it is basically necessary to exploit the functional equation along $\mathrm{Re}(w) = 1 - \lambda$ with $\lambda > 1$ and then use techniques resembling those in Lec 22, p. 15 (Lemma IV) and Lec 23, pp. 11–13. See Titchmarsh, <u>Theory of $\zeta(s)$</u>, § 12.2 with $k = 2$.

Compare: Landau, <u>Vorlesungen</u>, Sätze 508 + 509 (where an important, much more <u>intrinsic</u> method is used).

Out of curiosity, it is natural to wonder what taking $f(s) = \zeta(s)^1$ produces with this crude Perron-type technique. One would <u>hope</u> that $[\![x]\!]$ could be estimated reasonably accurately!

One has

$$\left\{ \begin{matrix} a_n = 1, \ q = 1, \ \Phi(v) = 1, \ c = 1 + \dfrac{1}{\ln x} \\ x = \text{big} \end{matrix} \right\}.$$

It is convenient to keep $\lambda \in (0, M]$ with the tacit restriction that $|\lambda - 1| > 10^{-3}$ (say). Here M = some large integer.

The earlier procedure is now easily mimicked. One insists that $T^A \lesssim x^\lambda$ (at the very least). The collective error term is easily seen to be

$$O(1) + O(T^A x^{1-\lambda}) + O\left(T^\varepsilon \frac{x}{T}\right) + O\left(\frac{x^{1+\varepsilon}}{T}\right).$$

One optimizes with $T = x^{\frac{\lambda}{A+1}}$. We have $T \lesssim x^M$. This leads to

$$O(1) + O\left[x^{\frac{A+(1-\lambda)}{A+1} + M\varepsilon}\right].$$

We have:

$$0 < \lambda < 1 \Rightarrow k(1-\lambda) = \frac{\lambda}{2} \Rightarrow O\left[x^{\frac{2-\lambda}{2+\lambda} + M\varepsilon}\right]$$

$$1 < \lambda \lesssim M \Rightarrow k(1-\lambda) = \lambda - \frac{1}{2} \Rightarrow O\left[x^{\frac{1}{2\lambda+1} + M\varepsilon}\right];$$

$$0 < \lambda < \frac{1}{2} \Rightarrow \nu(1-\lambda) = 0 \Rightarrow O\left[x^{1-\lambda+M\varepsilon}\right]$$

$$\frac{1}{2} \lesssim \lambda \lesssim M \Rightarrow \nu(1-\lambda) = \lambda - \frac{1}{2} \Rightarrow O\left[x^{\frac{1}{2\lambda+1} + M\varepsilon}\right].$$

In each instance, as λ grows, the exponent <u>decreases</u>. Since ε is arbitrary, we get

$$[\![x]\!] = x + O\left(x^{\frac{1}{2M+1} + \frac{\varepsilon}{2}}\right).$$

Letting M grow, it emerges that

$$[\![x]\!] = x + O(x^\varepsilon).$$

$\boxed{\text{Not bad!}}$

16. (Lec 25, p. 20 last line) The proof can be seen in Selberg's *Collected Papers*, vol. 2, p. 225.

17. (Lec 26, p. 11) A very nice complex variable proof of this THEOREM is given in Titchmarsh, *Theory of $\zeta(s)$*, § 4.14.

18. (Lec 28, p. 9) In Newman's General Thm, it goes virtually without saying that $g(ir)$ can be addressed simply by setting $s = \sigma + ir$.

19. (Lec 29, p. 29) In regard to the use of other kernel functions $k(w)$, it is worth mentioning that an interesting *variant* of the 1936 Ingham method is pushed through in Montgomery and Vaughan, *Multiplicative Number Theory*, vol. I, pages 477–479, 482 (bottom) – 483 (top).
Cf. also: Atle Selberg Archive, Hong Kong Lecture Series, 1998, Lec 4, pp. 2–10.
http://publications.ias.edu/selberg/section/2479

20. (Lec 30, p. 3) Prior to beginning the proof of THM, it would have been wise to step back and draw attention to an immediate Corollary first; viz.,

Let $q \geqq 1$. **If** $\sum_1^\infty \frac{\mu(n)(\ln n)^q}{n}$ converges, then $\sum_1^\infty \frac{\mu(n)}{n}$ also converges and its value must be 0; as such, one will again get (i) + (ii) in an elementary fashion.

PF

That $\sum_{n=1}^\infty \frac{\mu(n)}{n}$ converges is self-evident by Dirichlet's test; see Lec 7, p. 1 (bottom). By item #13 above (i.e., Fact 2(b) in Lec 21), the summation's value is immediately ascertained to be 0. ∎

21. (Lec 30, p. 3, proof of THM) The (following Landau's Handbuch) proof clearly possesses a certain beauty. After finishing it, however, one is left wondering how the "q-condition" of item #20 fits into the overall scheme.

This issue was ultimately clarified by Kienast in Math. Annalen 95 (1925) 427-445. See also Landau, Handbuch der Primzahlen, CHELSEA EDITION, Appendix (by P. Bateman), p. 941 § 159.

A bit of "diagram-chasing" is necessary in order to unravel Kienast's paper. Let $\sim$ mean "elementarily equivalent"; let

$$[M+k] \quad \text{mean} \quad M(x) = o(1)\frac{x}{(\ln x)^k}$$

$$[\psi+k] \quad \text{mean} \quad \psi(x) - x = o(1)\frac{x}{(\ln x)^k}$$

$$g_k \quad \text{mean} \quad \sum_1^\infty \frac{\mu(n)(\ln n)^k}{n} \quad \text{converges.}$$

(Here $k \geqq 0$.) Note that $A \sim B$ and $B \sim C \Rightarrow A \sim C$.

<u>Lemma 1</u> (baby calculus – very useful)

Let $\varphi \in C'[a,b]$ be positive and monotonic. Let F be real and <u>piecewise</u> C' on $[a,b]$. Assume that $|F(x)| \leqq \Phi(x)$, where Φ is positive + monotonic + C' on $[a,b]$. We then have

$$\left|\int_a^b \varphi(x)\,dF(x)\right| \leqq 2\left[\varphi(a)\Phi(a) + \varphi(b)\Phi(b)\right] + \left|\int_a^b \varphi(x)\,d\Phi(x)\right| .$$

<u>Pf</u>

The fcn F has only a finite number of <u>actual</u> discontinuities. Because of the <u>Lipschitz condition</u> on the "chunks" of the rest of the graph, $F(x)$ has bounded total variation on $[a,b]$. Any such fcn is expressible as the difference of two monotonic increasing $\varphi_j(x)$. (The R–S integral of $\varphi\, dF$ is thus fine.)

We now exploit integ-by-parts twice.

$$\left| \int_a^b \varphi(x)\, dF(x) \right| = \left| \varphi(b) F(b) - \varphi(a) F(a) - \int_a^b F(x) \varphi'(x)\, dx \right|$$

$$\leqq \varphi(b) \Phi(b) + \varphi(a) \Phi(a)$$

$$+ \left| \int_a^b \Phi(x) \varphi'(x)\, dx \right|$$

{ this is correct when the sign of φ' is fixed }

$$= \varphi(b) \Phi(b) + \varphi(a) \Phi(a)$$

$$+ \left| \Phi(b)\varphi(b) - \Phi(a)\varphi(a) - \int_a^b \varphi(x)\, d\Phi(x) \right|$$

$$\leqq 2\varphi(b) \Phi(b) + 2\varphi(a) \Phi(a)$$

$$+ \left| \int_a^b \varphi(x)\, d\Phi(x) \right| . \quad ▨$$

Lemma 2

Let $k \geqq 0$. Put $F(x) = \sum_{n \leqq x} \mu(n)(\ln n)^k$. We then have

$$M(x) = \frac{o(x)}{(\ln x)^k} \iff F(x) = o(x)$$

elementarily [i.e. via a sequence of elementary techniques].

Pf

$k = 0$ is trivial, so take $k \geqq 1$ wlog.

Assume first that $F(x) = o(x)$. Keep T large.
Notice that

$$M(T) = O(1) + \int_2^T (\ln x)^{-k} dF(x)$$

$$= O(1) + O_\varepsilon(1) + \int_{T_\varepsilon}^T (\ln x)^{-k} dF(x)$$

$$\left\{ \frac{\text{apply Lemma 1 with } \Phi(x) = \varepsilon x}{\text{let } |\Theta| \leqq 1 \text{ as usual}} \right\}$$

$$= O_\varepsilon(1) + 2\Theta\left[(\ln T)^{-k} \varepsilon T + O_\varepsilon(1) + \int_{T_\varepsilon}^T (\ln x)^{-k} d(\varepsilon x) \right]$$

$$= O_\varepsilon(1) + 2\Theta \frac{\varepsilon T}{(\ln T)^k} + 2\Theta\varepsilon \int_2^T \frac{dx}{(\ln x)^k}$$

familiar

$$= O_\varepsilon(1) + 2\Theta \frac{\varepsilon T}{(\ln T)^k} + 2\Theta\varepsilon \frac{O(T)}{(\ln T)^k}$$

$$= O_\varepsilon(1) + \varepsilon\, O(1) \frac{T}{(\ln T)^k} .$$

Hence $M(T) = o(1) \frac{T}{(\ln T)^k}$. $\boxed{OK}$

Next, suppose that $M(x) = \frac{o(x)}{(\ln x)^k}$. Keep T large. Notice that

$$F(T) = O(1) + \int_2^T (\ln x)^k \, dM(x)$$

$$= O(1) + O_\varepsilon(1) + \int_{T_\varepsilon}^T (\ln x)^k \, dM(x)$$

$$\left\{ \frac{\text{apply lemma 1 with } \Phi(x) = \varepsilon \frac{x}{(\ln x)^k}}{\text{let } |\Theta| \leq 1 \text{ as usual}} \right\}$$

$$= O_\varepsilon(1) + 2\Theta\Big[(\ln T)^k \varepsilon \frac{T}{(\ln T)^k} + O_\varepsilon(1) + \int_{T_\varepsilon}^T (\ln x)^k d\Big(\frac{\varepsilon x}{(\ln x)^k}\Big) \Big]$$

$$= O_\varepsilon(1) + 2\Theta\varepsilon T + 2\Theta\varepsilon \int_{\exp(k)}^T (\ln x)^k \Big[\frac{(\ln x)^k - k(\ln x)^{k-1}}{(\ln x)^{2k}} \Big] dx$$

$$= O_\varepsilon(1) + 2\Theta\varepsilon T + 2\Theta\varepsilon \int_{\exp(k)}^{T} \left[1 - \frac{k}{\ln x}\right] dx$$

$$= O_\varepsilon(1) + 4\Theta\varepsilon T .$$

Hence $F(T) = o(T)$. $\blacksquare$

Going back (now) to $[M+k]$, $[\Psi+k]$, g_k, one observes that — in his paper — Kienast either recalls or proves:

Satz 1. $[M+0] \sim [\Psi+0] \sim g_0$;

Satz 3. $[\Psi+k] \sim g_k$, all $k \geq 0$;

Lemma 2 (above) + Satz 10. Anytime $[\Psi+k]$ is true, we have $[M+(k+1)] \sim g_{k+1}$.

The following assertion is now the key!

<u>CLAIM</u>: we have

$$[M+q] \sim g_q \sim [\Psi+q] \quad \text{for each } q \geq 0.$$

Proof

Suppose not. Let k be the smallest case where the proposed 3-way relation is FALSE. By Satz 1, $k \geq 1$.

Suppose $[M+k]$ holds. Clearly $[M+(k-1)]$ holds. But 3-way relation $[M+(k-1)] \sim g_{k-1} \sim [\Psi+(k-1)]$ is TRUE. Hence we get $[\Psi+(k-1)]$ elementarily. By Satz 10, the truth of $[M+k] \Rightarrow$ that of g_K elementarily. So we can write $[M+k] \overset{e}{\Rightarrow} g_K$, the "e" meaning elementary.

Suppose next that g_K holds. By Satz 3, we get $[\Psi+k]$ in an elementary fashion. So, $g_K \overset{e}{\Rightarrow} [\Psi+k]$.

Finally, suppose $[\Psi+k]$ holds. By Satz 3, g_K holds elementarily. Trivially, of course, $[\Psi+(k-1)]$ holds. By Satz 10, we then have $g_K \Longrightarrow [M+k]$ elementarily. So, $[\Psi+k] \overset{e}{\Longrightarrow} [M+k]$.

All told, we have seen $[M+k] \overset{e}{\Rightarrow} g_K \overset{e}{\Longrightarrow} [\Psi+k] \overset{e}{\Rightarrow} [M+k]$. This contradicts the definition of k.

The CLAIM is thus proved. ∎

22. (Lec 30, p. 3, "already knew") Both here and in regard to Lec 2, p. 20 line 2, it is thought-provoking to have a look at Euler [1737], Opera Omnia, Ser. I, Vol. 14, pp. 242 (thm 19, second sentence) and 243~244. Cf. also item #12 above.

Euler's assertion that $\sum p^{-1-\varepsilon} = \ln\left(\frac{C}{\varepsilon} + O(1)\right)$ is but 1 or 2 steps away from heuristically concluding that $T(e^u) \sim \ln u$, wherein

$$T(x) \equiv \sum_{p \leq x} \frac{1}{p},$$

— and, then, via an analogous reasoning, engendering/sparking the suspicion that [in all likelihood] $\pi(x)$ must be roughly of size $\int_2^x \frac{dt}{\ln t} \sim \frac{x}{\ln x}$. Alas, none of this is said there explicitly.

In modern parlance, the $T(e^u)$ deduction (based strictly on real $\varepsilon > 0$) is a standard example of a Karamata-type Tauberian theorem; in this regard, cf. also Titchmarsh, Theory of $\zeta(s)$, (7.12.1)~(7.12.5).*
See Ingham, p. 10, for an Euler-style proof that $T(e^u) > \ln u - \frac{1}{2}$; note too p. 11 (bot, "infinitely fewer than the integers").

* and Hardy, *Divergent Series*, theorem 108

For some additional perspective on both aspects of this matter, T and π, see Edwards, Riemann's Zeta Fcn, pp. 1-2.

☆ ☆ ☆

"2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, ...

14.134725^+, 21.022039^+, 25.010857^+, ...

$$\pi(x) \sim \frac{x}{\log x}$$

AND ALL THAT !! "

COURSE ANNOUNCEMENT – MATH 8280 – SPRING 2016
(Topics in Number Theory)
AN INTRODUCTION TO ANALYTIC NUMBER THEORY
Instructor: D. A. Hejhal

Beginning in the 19th century, it began to be realized (by Riemann, among others) that certain fundamental questions involving the ordinary integers, more specifically the primes, were amenable to study by bringing to bear on them methods of analysis, especially *complex* analysis.

In very loose terms, *analytic number theory* is that part of number theory whose results are obtained principally with the aid of constructs and techniques having at least one foot in some aspect of (either classical or modern) analysis. Today, for instance, in addition to complex, both harmonic and spectral analysis have begun to be used.

The purpose of this course, which should probably carry a number closer to 8001 (8009 would be apt!) is to offer students conversant in the standard advanced undergraduate courses in real, complex, and Fourier analysis, plus a bit of modern algebra, a kind of "gentle" introduction to analytic number theory, by coming in chiefly from the *multiplicative* side — that is to say, primes and constructs like the Riemann zeta function, $\zeta(s)$.

Analytic number theory is a subject steeped in history. One of its main theorems is the celebrated Prime Number Theorem from 1896 (and its counterpart for primes in arithmetic progressions). Several approaches to the PNT are now known. One of the best ways of getting a feel for analytic number theory, and what makes it tick, is to simply make a careful study of the various approaches to the PNT.

The basic plan of the first 2/3 of the course is to do exactly this – taking the time, where need be, to develop some interesting cognate material involving, e.g., aspects of complex (and real!) analysis pertinent for $\zeta(s)$ and the study of its zeros. Connections with the Riemann Hypothesis and (so-called) "explicit formula" for the prime counting function $\pi(x)$ will arise here.

Following that, as time permits, a few topics further afield (but still relatively gentle) will touched on. One possibility: some "nitty-gritty" numerical calculation of a few Riemann zeros and related zeros of $L(s, \chi)$, where χ is a multiplicative character.

The course format will primarily be lectures, guided in part by the classic books of Ingham and Davenport. Some unpublished course notes by A. Selberg and a recent AMS volume by Iwaniec/Kowalski will also prove useful.

To facilitate fixing a class time, any students interested in taking this course (or simply desiring further information) should contact the instructor in the very near future. {Email: hejhal@math.umn.edu}

Students interested in the course, but lacking some of the prerequisites, may, *after* a discussion with the instructor, be allowed to join the class and receive Math 5990 credit for it. (A slightly modified syllabus would then apply; early contact with the instructor is again encouraged.)