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Pages 440, 441, 442; 444.

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2. Transverse Oscillation of a Flexible String. Consider an elastic string under tension, as shown in Fig. 3. For the present the string is assumed to be of infinite length or, at any rate, so long that the effect of the end supports can be neglected for the portion of string under observation. Also, the string is flexible, so that there is no resistance to bending. This means that the tension force is directed tangentially to the curve formed by the string at any instant.

A complete analysis of the problem of a vibrating string requires knowledge of the nonlinear mechanics of deformable media and is somewhat out of place here. Hence, instead of discussing the string from the point of view of nonlinear mechanics and elasticity theory, we shall assume a particular type of motion in which the string lies in a vertical plane at all times and the motion in that plane is straight up and down, not sideways. The behavior is described by giving the vertical displacement $u(x,t)$ at a distance x from some suitable origin at time t. It will be found that this assumption of *transverse oscillation* is consistent with Newton's laws and leads to a differential equation with many desirable properties. The

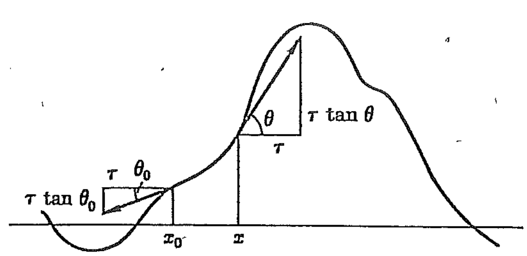

FIGURE 3

method is applicable to the general problem in which both transverse and longitudinal oscillations are present, though this aspect of the subject is not discussed here.

When the string is at rest in the horizontal position the density is denoted by $\rho = \rho(x)$, so that $\rho\, dx$ gives the differential of mass between x and $x + dx$. By hypothesis there is no sideways movement; hence the mass originally between x and $x + dx$ remains between x and $x + dx$ throughout the motion. Thus, $\rho\, dx$ describes this mass for all t, even though ρ is independent of t. The corresponding differential of momentum is $u_t \rho\, dx$, since $u_t \equiv \partial u/\partial t$ is the vertical velocity, and the total momentum of the part of the string between x_0 and x is therefore

$$M = \int_{x_0}^{x} \rho u_t \, dx. \tag{2-1}$$

We shall obtain the equation of motion by applying Newton's law

$$\frac{d}{dt}\,(\text{momentum}) = \text{force}$$

to the portion of the string between x_0 and x. If there is enough continuity so that (2-1) can be differentiated under the integral sign, the rate of change of momentum is

$$\frac{\partial M}{\partial t} = \int_{x_0}^{x} \rho u_{tt}\, dx \tag{2-2}$$

and it remains only to compute the force.

For the particular type of motion being considered here the horizontal component of the tensile force has a simpler behavior than the tension $\bar{\tau}$ itself. We denote this horizontal component by $\tau = \tau(x,t)$. Since there is no horizontal acceleration of the string between x_0 and x, the horizontal components of tensile force at the two ends must balance. We conclude that $\tau(x,t) = \tau(x_0,t)$ or, in other words, *τ is independent of x.*

According to Fig. 3 the vertical component of the tensile force is $\tau \tan\theta$, where θ is the angle between the tangent to the string and the x axis. Since $\tan\theta$ is the slope of the curve, $\tan\theta = du/dx$ at any given value of t. More explicitly,

$$\tan\theta = \frac{\partial u}{\partial x} \equiv u_x.$$

The net vertical component of force due to tension is therefore

$$\tau u_x \Big|_{x_0}^{x} = \int_{x_0}^{x} \frac{\partial}{\partial x}(\tau u_x)\, dx = \int_{x_0}^{x} \tau u_{xx}\, dx$$

where the second equality follows from the fact that τ is independent of x.

If in addition to the tensile force there is a distributed load $F_1 = F_1(x,t)$ the corresponding vertical force on the segment of string between x_0 and x is

$$\int_{x_0}^{x} F_1 \, dx.$$

The result of equating the rate of change of momentum (2-2) with the total force as given above is

$$\int_{x_0}^{x} \rho u_{tt}\, dx = \int_{x_0}^{x} \tau u_{xx}\, dx + \int_{x_0}^{x} F_1\, dx.$$

Differentiation with respect to the upper limit x now produces the desired differential equation,

$$\rho u_{tt} = \tau u_{xx} + F_1$$

$$\left\{ \begin{matrix} \rho(x) \\ \tau(t) \end{matrix} \right\}$$

provided the integrands are continuous. This in turn may be written

$$u_{tt} = a^2 u_{xx} + F(x,t) \qquad \text{where } a = \sqrt{\frac{\tau}{\rho}} \qquad F = \frac{F_1}{\rho} \tag{2-3}$$

According to our analysis the horizontal density function ρ depends on x alone, not on t, and the horizontal tensile force τ depends on t alone, not on x. For small oscillations of a uniform string under great tension both τ and ρ are practically constant, and this case is considered in the sequel. We refer to the constant τ as the *tension* and to ρ as the *density*, since these quantities actually are the tension and density when the string is at rest. Further discussion can be found in the problems at the end of this section.

When the force function $F(x,t)$ is zero, the vibrations of the string are termed *free vibrations*. By (2-3) the equation for free vibrations is

$$u_{tt} = a^2 u_{xx} \tag{2-4}$$

and hence the solution has the form (1-8). According to the discussion in Sec. 1, the motion can always be regarded as a superposition of two waves moving with velocity

$$a = \sqrt{\frac{\tau}{\rho}} \tag{2-5}$$

in opposite directions. Later we shall determine the precise form of these waves by considering the initial state of the string, that is, the state at $t = 0$, together with the conditions at the end points, $x = 0$ and $x = l$.

3. Let $\bar{\tau}$ denote the actual tension of the string, measured tangentially, and let $\bar{\rho}$ denote the actual linear density referred to the arc length along the curved string. Show that $\bar{\tau}$ and $\bar{\rho}$ are related to the parameters τ and ρ of the text by

$$\bar{\tau} = \tau(1 + u_x^2)^{1/2} \qquad \bar{\rho} = \rho(1 + u_x^2)^{-1/2}.$$

Hence $\bar{\tau}\bar{\rho} = \tau\rho$. Also $\tau \doteq \bar{\tau}$ and $\rho \doteq \bar{\rho}$ if $\frac{1}{2}u_x^2$ is small compared with unity. *Hint:* $\bar{\tau} = \tau \sec \theta$, and $\bar{\rho}\, ds = \rho\, dx$, where s is the arc along the string.

4. In deriving the equation for vibration of a string stretched between fixed supports it is often assumed that the actual tension $\bar{\tau}$ of Prob. 3 is constant, as well as that the motion is vertical. Show that these assumptions are mutually contradictory. *Hint:* Since the motion is vertical, τ depends on t alone, not on x. Hence Prob. 3 shows that u_x is independent of x. If the displacement is 0 at the fixed ends, we get $u \equiv 0$, which contradicts the fact that the string is vibrating.

5. In deriving the equation for vibration of a string stretched between two fixed supports, it is often assumed that the actual density $\bar{\rho}$ of Prob. 3 is constant. Show that the string cannot oscillate under this assumption. *Hint:* The length must increase if it oscillates, but the total mass between supports does not change.