Another standard derivation

Derivation of the Wave Equation. In this section we will derive the wave equation in one space dimension as it applies to the transverse vibrations of an elastic string, or cable; the elastic string may be thought of as a violin string, a guy wire, or possibly an electric power line. The same equation, however, with the variables properly interpreted, occurs in many other wave problems having only one significant space variable. The wave equation for an elastic string was first derived in 1746 by D'Alembert.

Consider a perfectly flexible elastic string stretched tightly between supports fixed at the same horizontal level; see Figure 10.16a. Let the x axis lie along the string with the end points located at $x = 0$ and $x = l$. If the string is set in motion at some initial time $t = 0$ (by plucking, for example) and is thereafter left undisturbed, it will vibrate freely in a vertical plane provided damping effects, such as air

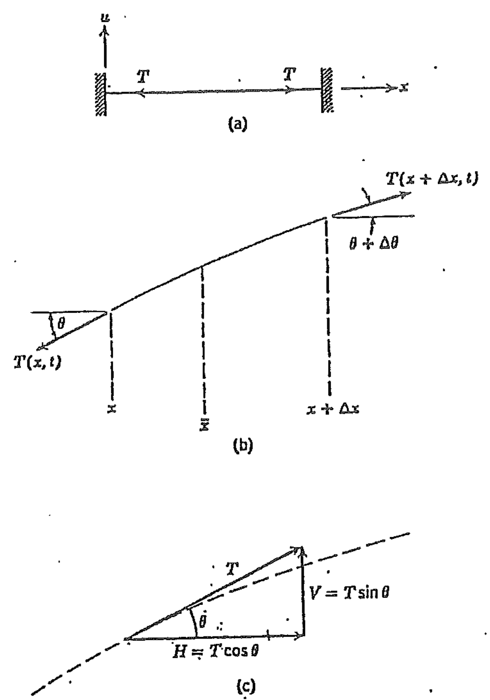

FIGURE 10.16

resistance, are neglected. To determine the differential equation governing this motion we will consider the forces acting on a small element of the string of length Δx lying between the points x and $x + \Delta x$; see Figure 10.16b. We assume that each point on the string moves solely on a vertical line, and will denote by $u(x, t)$ the vertical displacement of the point x at the time t. Let the tension in the string, which always acts in the tangential direction, be denoted by $T(x, t)$, and let ρ denote the mass per unit length of the string.

Newton's law, as it applies to the element Δx of the string, states that the net external force, due to the tension at the ends of the element, must be equal to the product of the mass of the element and the acceleration of its mass center. Since there is no horizontal acceleration, the horizontal components must satisfy

$$T(x + \Delta x, t)\cos(\theta + \Delta\theta) - T(x, t)\cos\theta = 0. \tag{1}$$

Denoting the horizontal component of the tension (see Figure 10.16c) by H, Eq. (1) states that H is independent of x. On the other hand, the vertical components satisfy

$$T(x + \Delta x, t)\sin(\theta + \Delta\theta) - T(x, t)\sin\theta = \rho\,\Delta x\, u_{tt}(\bar{x}, t), \tag{2}$$

where $\bar{x}$ is the coordinate of the center of gravity of the element of the string under consideration. Clearly $\bar{x}$ lies in the interval $x < \bar{x} < x + \Delta x$. The weight of the string, which acts vertically downward, is assumed to be negligible, and has been neglected in Eq. (2).

If the vertical component of T is denoted by V, then Eq. (2) can be written as

$$\frac{V(x + \Delta x, t) - V(x, t)}{\Delta x} = \rho u_{tt}(\bar{x}, t).$$

Passing to the limit as $\Delta x \to 0$ gives

$$V_x(x, t) = \rho u_{tt}(x, t). \tag{3}$$

To express Eq. (3) entirely in terms of u we note that

$$V(x, t) = H(t)\tan\theta = H(t)u_x(x, t).$$

Hence Eq. (3) becomes

$$(Hu_x)_x = \rho u_{tt},$$

or, since H is independent of x,

$$Hu_{xx} = \rho u_{tt}. \tag{4}$$

It is customary to write Eq. (4) in the form

$$a^2 u_{xx} = u_{tt}, \tag{5}$$

where

$$a^2 = \frac{H}{\rho}. \tag{6}$$

In this derivation it is possible for H to depend on t, and for ρ to depend on both x and t; in this event a^2 would be a function of both x and t. We will assume, however, that a^2 is a constant. Equation (5) is called the wave equation for one space dimension. Since H has the dimension of force, and ρ that of mass/length, it follows that the constant a has the dimension of velocity. It is possible to identify a as the velocity with which a small disturbance (wave) moves along the string. According to Eq. (6) the wave velocity a varies directly with the tension in the string, but inversely with the density of the string material. These facts are in agreement with experience.

Elem. Diff. Eqs. and B.V. Problems

From:

Boyce & DePrima's book

2nd edition, pp. 481-483

In this derivation, note that

$\rho = \rho(x)$
$H = H(t)$ are perfectly OK!

This would give:

$$u_{tt} = \frac{H(t)}{\rho(x)} u_{xx}.$$