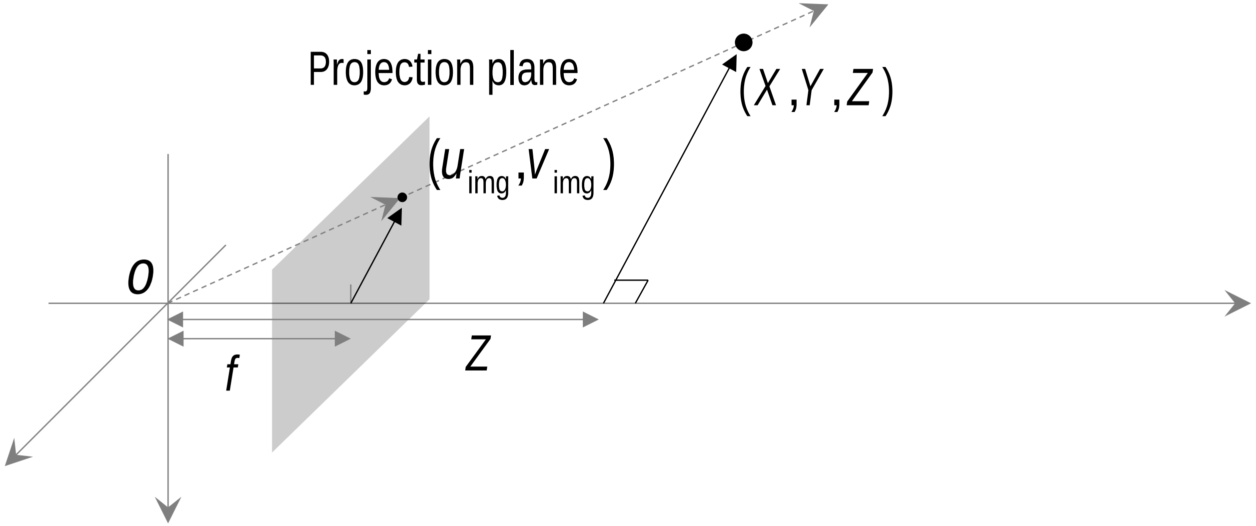
Camera Projection Model

3D Point Projection (Pixel Space)

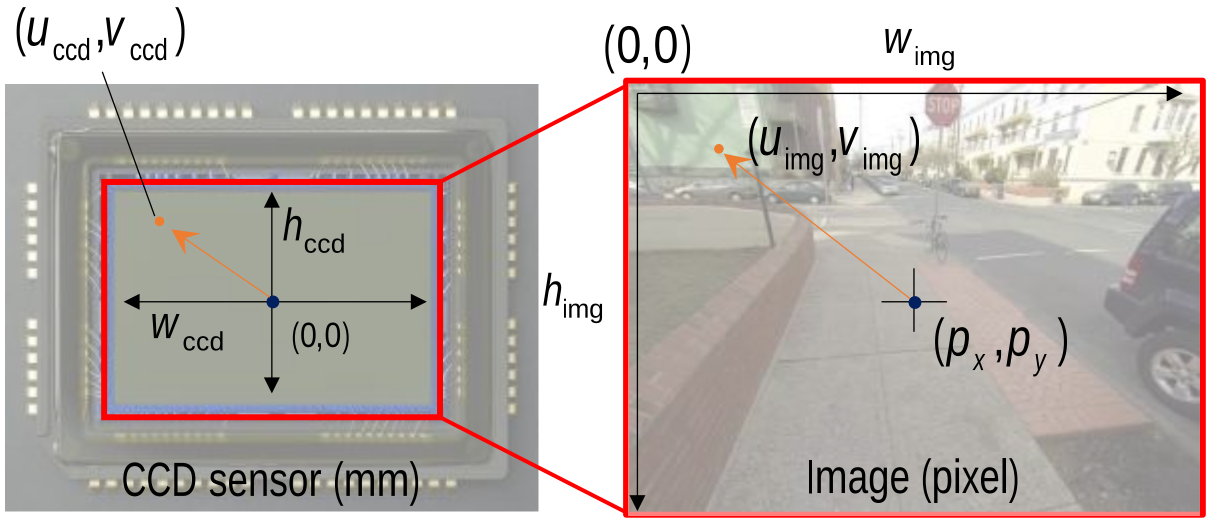


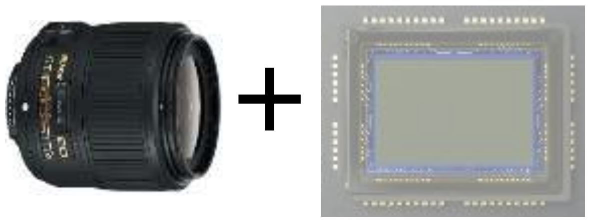
$$u_{img} = f\frac{X}{Z} + p_x \longrightarrow Zu_{img} = fX + p_x Z$$

$$v_{img} = f\frac{Y}{Z} + p_y \longrightarrow Zv_{img} = fY + p_y Z$$

Pixel space   Metric space

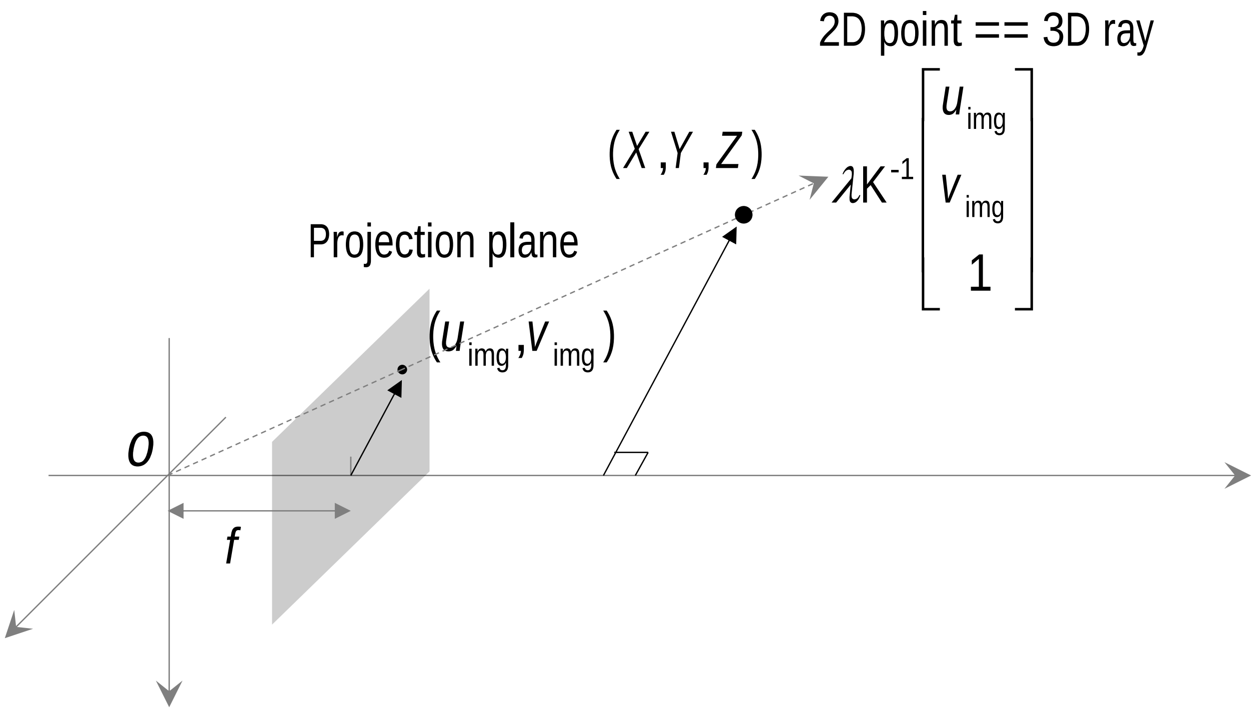
$$\lambda \begin{bmatrix} u_{img} \\ v_{img} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$





Camera intrinsic parameter
: metric space to pixel space

2D Inverse Projection



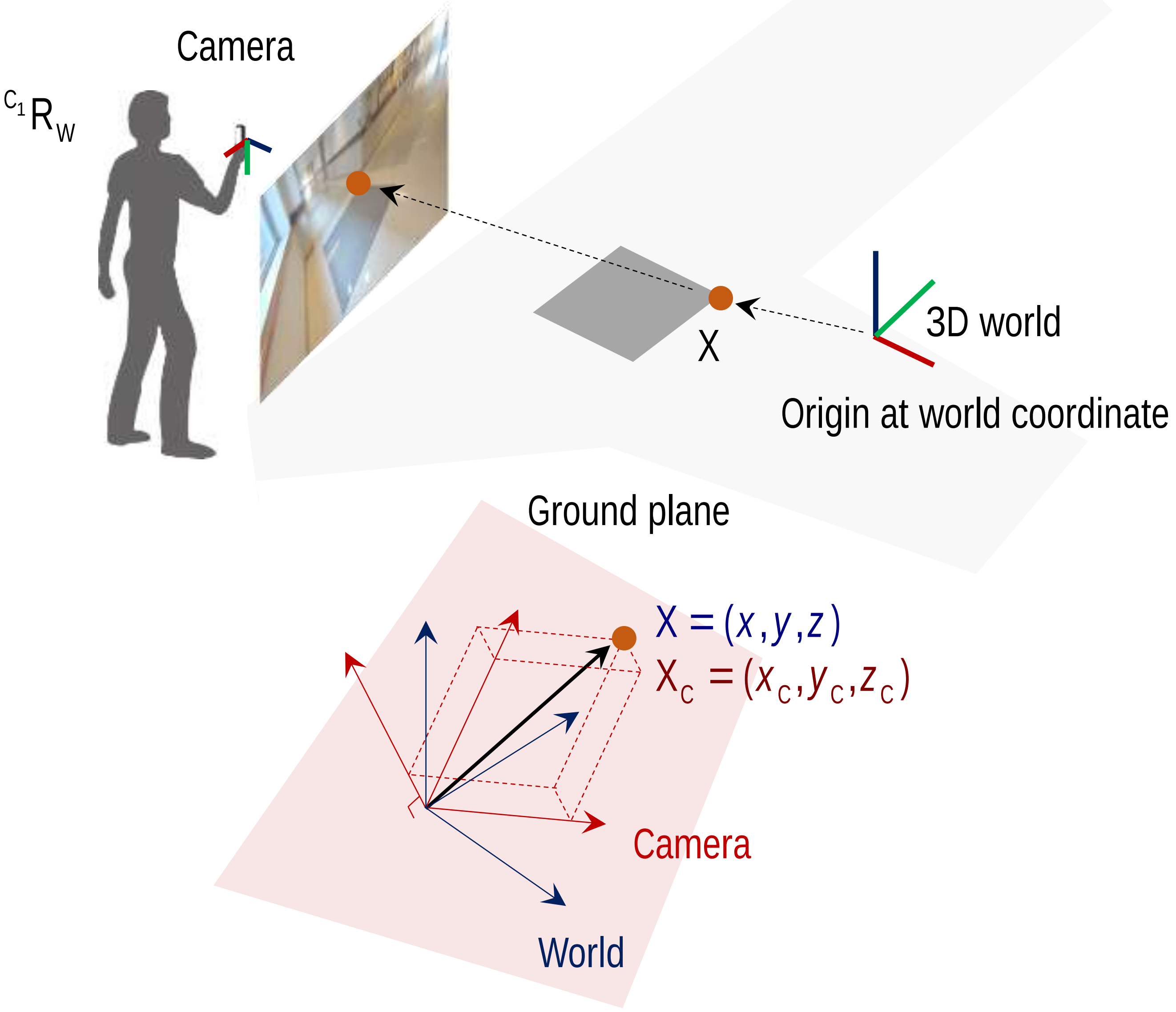
Pixel space   Metric space

$$\lambda \begin{bmatrix} u_{img} \\ v_{img} \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f\,K & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\underbrace{\lambda K^{-1} \begin{bmatrix} u_{img} \\ v_{img} \\ 1 \end{bmatrix}}_{\text{3D ray}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

The 3D point must lie in
the *3D ray* passing through the origin and 2D image point.

Coordinate Transform (Rotation)



Coordinate transformation from world to camera:

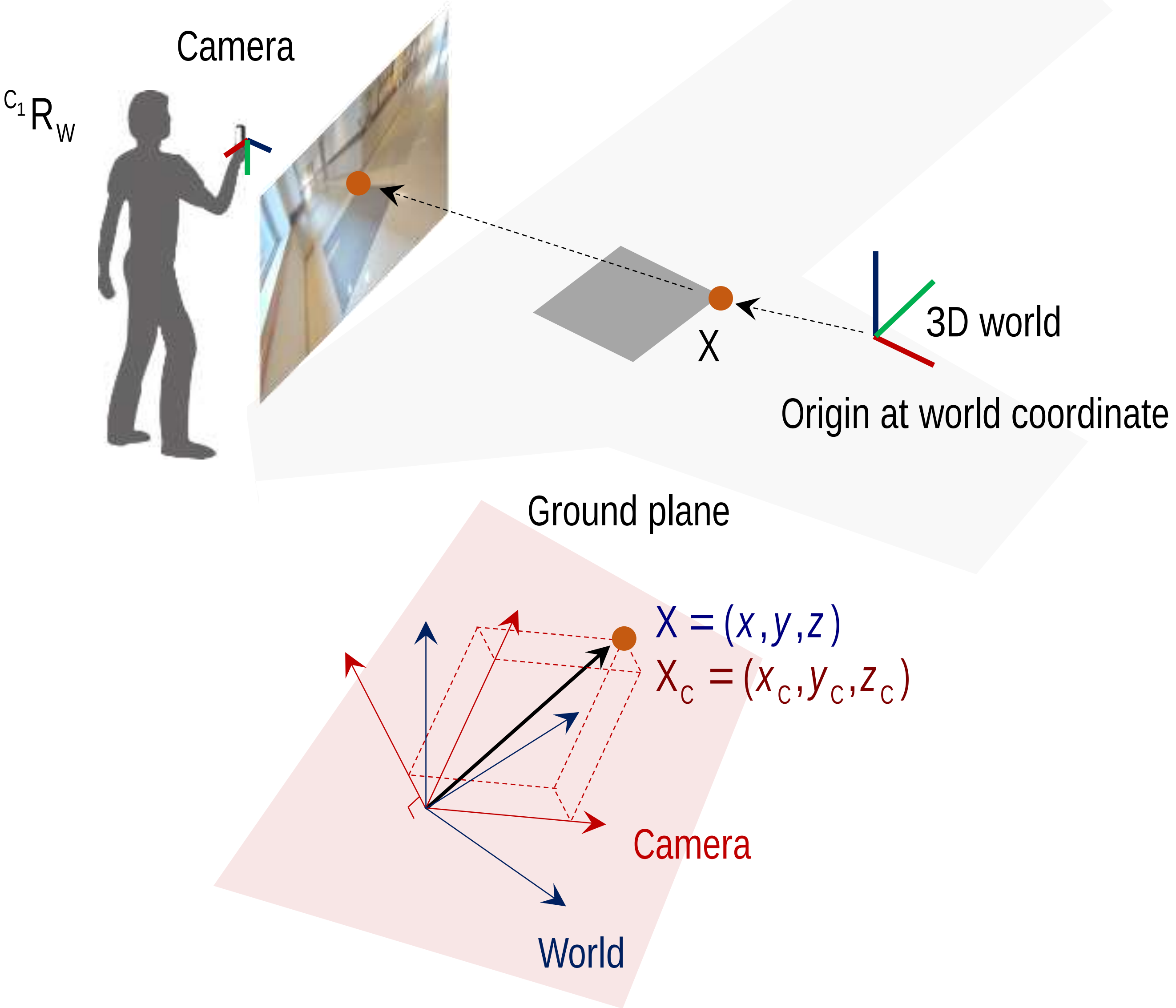
$$X_C = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} X = {}^{C}R_W X$$

Degree of freedom?

$${}^{C}R_W \in SO(3)$$

- Orthogonal matrix $\longrightarrow$ $\left({}^{C}R_W\right)^T \left({}^{C}R_W\right) = I_3, \quad \det\left({}^{C}R_W\right) = 1$

Coordinate Transform (Rotation)

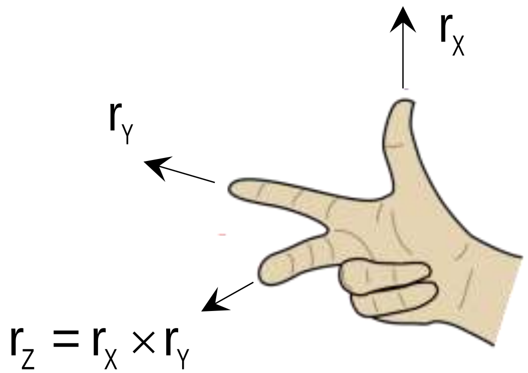


Coordinate transformation from world to camera:

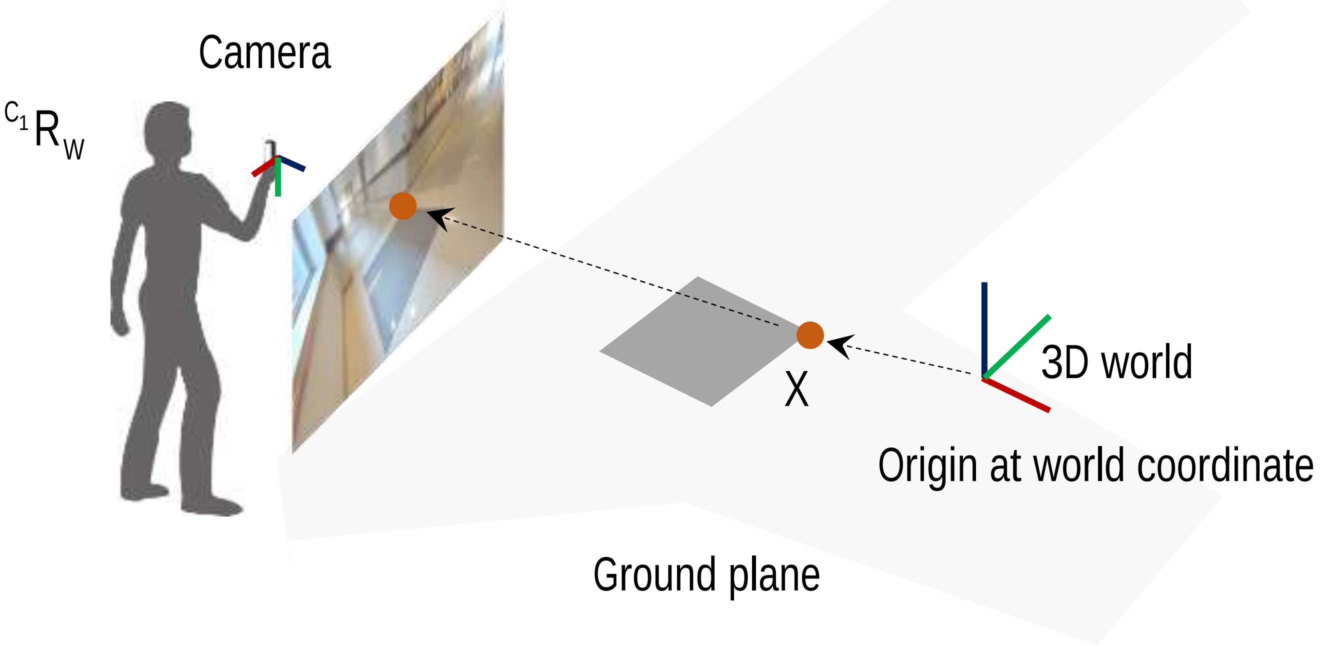
$$X_C = \begin{array}{c} r_X \\ r_Y \\ r_Z \end{array} \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} X = {}^{C}R_W X$$

$${}^{C}R_W \in SO(3)$$

- Orthogonal matrix $\longrightarrow \left({}^{C}R_W\right)^T \left({}^{C}R_W\right) = I_3, \quad \det\left({}^{C}R_W\right) = 1$
- Right hand rule



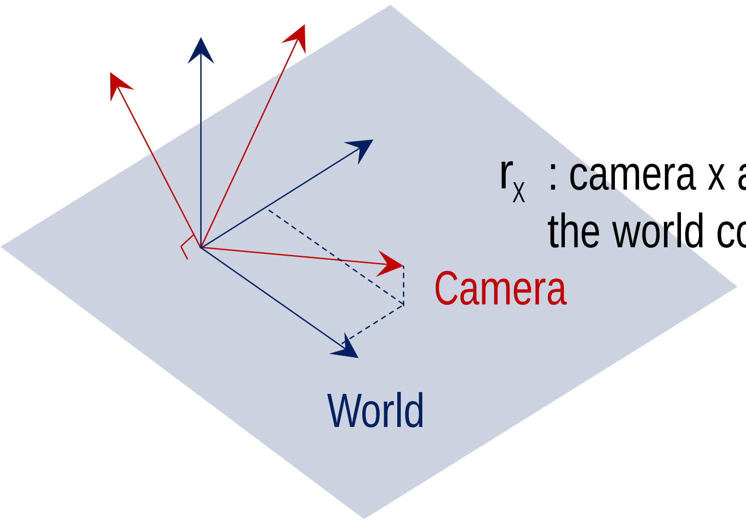
Coordinate Transform (Rotation)



Coordinate transformation from world to camera:

$$\mathbf{X}_C = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} \mathbf{X} = {}^{C}R_W \mathbf{X}$$

$$\begin{bmatrix} \mathbf{r}_X \\ \mathbf{r}_Y \\ \mathbf{r}_Z \end{bmatrix}$$



$\mathbf{r}_X$: camera x axis seen from the world coord.

$${}^{C}R_W \in SO(3)$$

- Orthogonal matrix $\longrightarrow$ $\left({}^{C}R_W\right)^T\left({}^{C}R_W\right) = I_3, \quad \det\left({}^{C}R_W\right) = 1$
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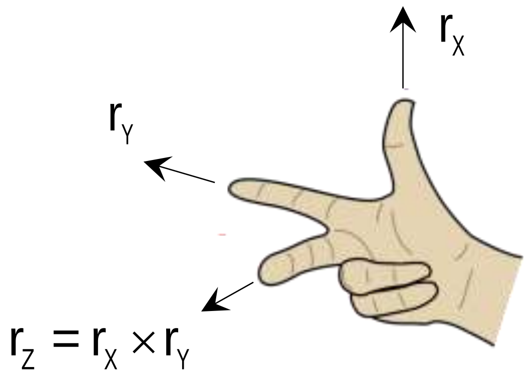
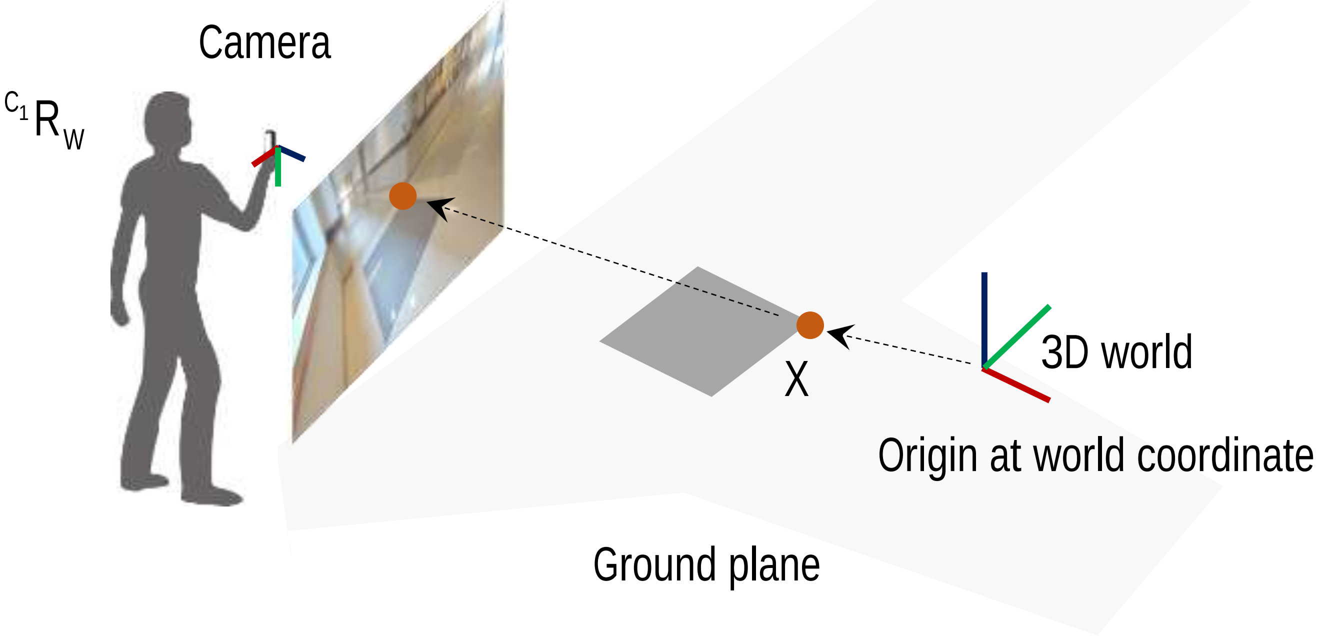


Coordinate Transform (Rotation)



Coordinate transformation from world to camera:

$$X_C = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} X = {}^{C}R_W X$$

$${}^{C}R_W \in SO(3)$$

- Orthogonal matrix $\longrightarrow \left({}^{C}R_W\right)^T \left({}^{C}R_W\right) = I_3, \quad \det\left({}^{C}R_W\right) = 1$
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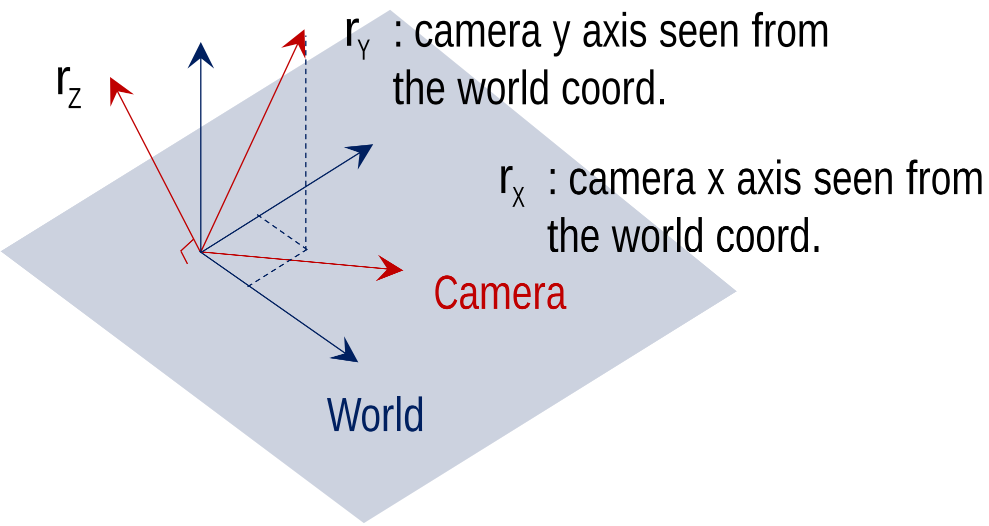
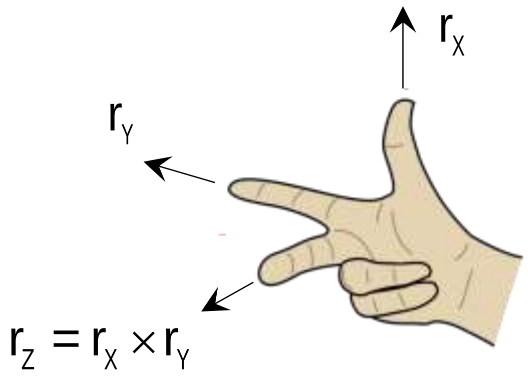
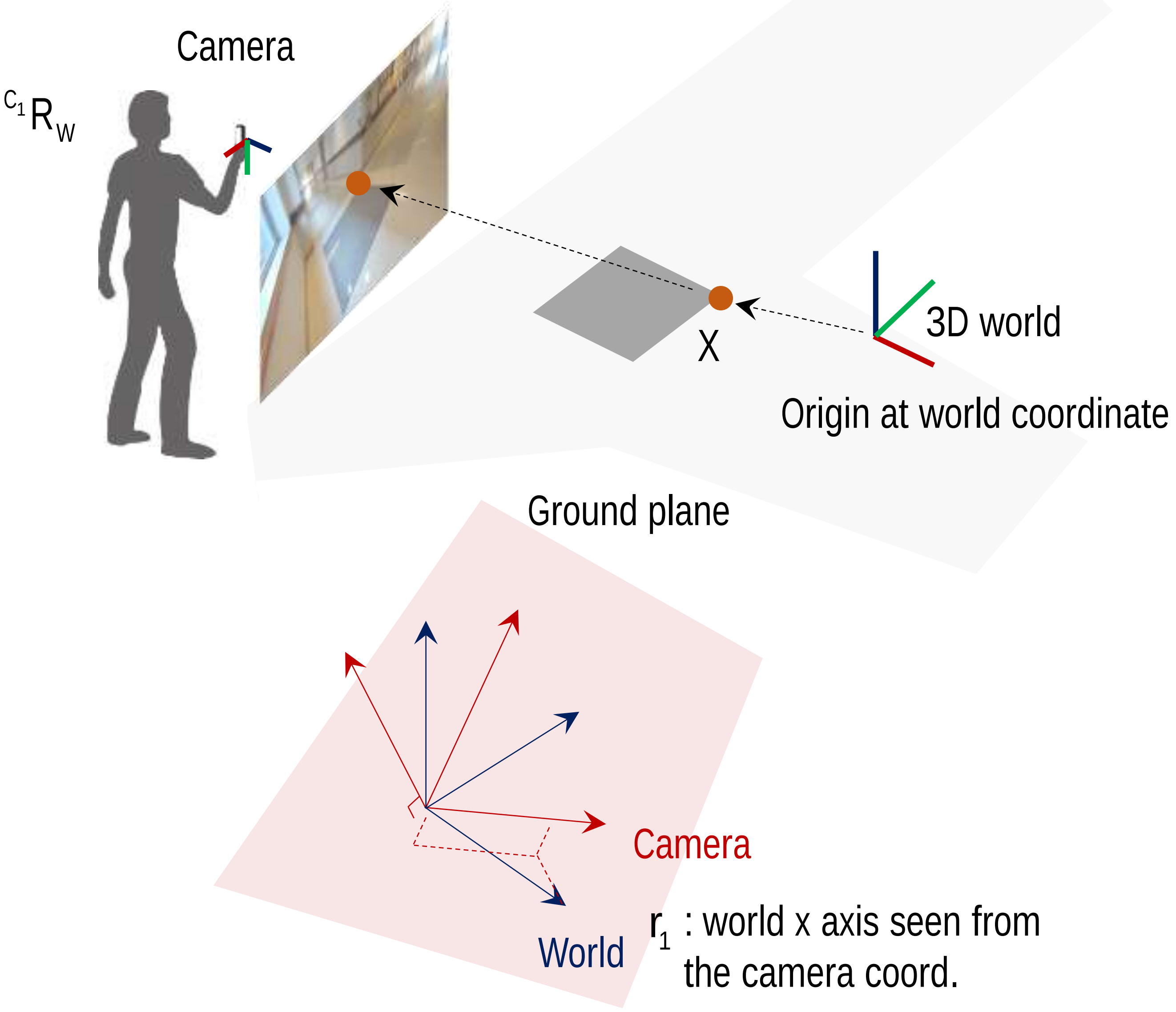




Coordinate Transform (Rotation)

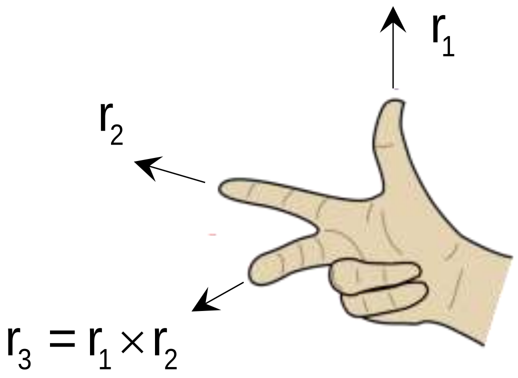


Coordinate transformation from world to camera:

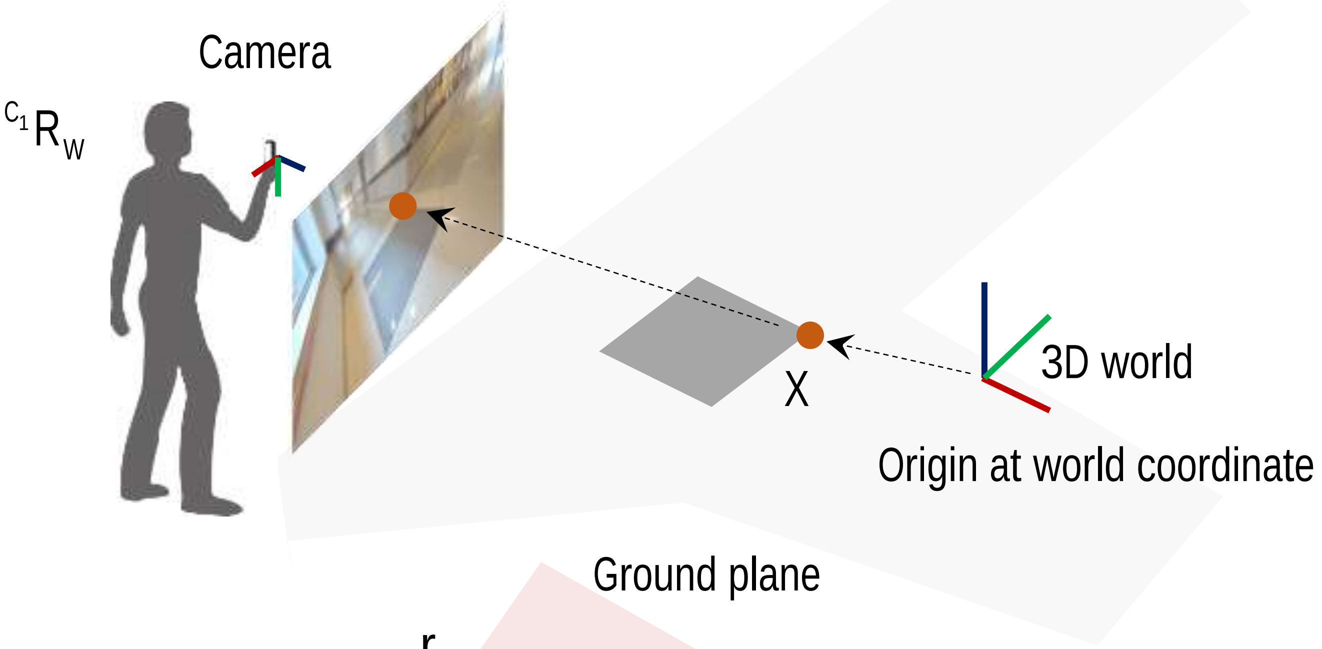
$$\begin{array}{cc} & \begin{array}{ccc} r_1 & r_2 & r_3 \end{array} \\ X_C = & \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} X = {}^{C}R_W X \end{array}$$

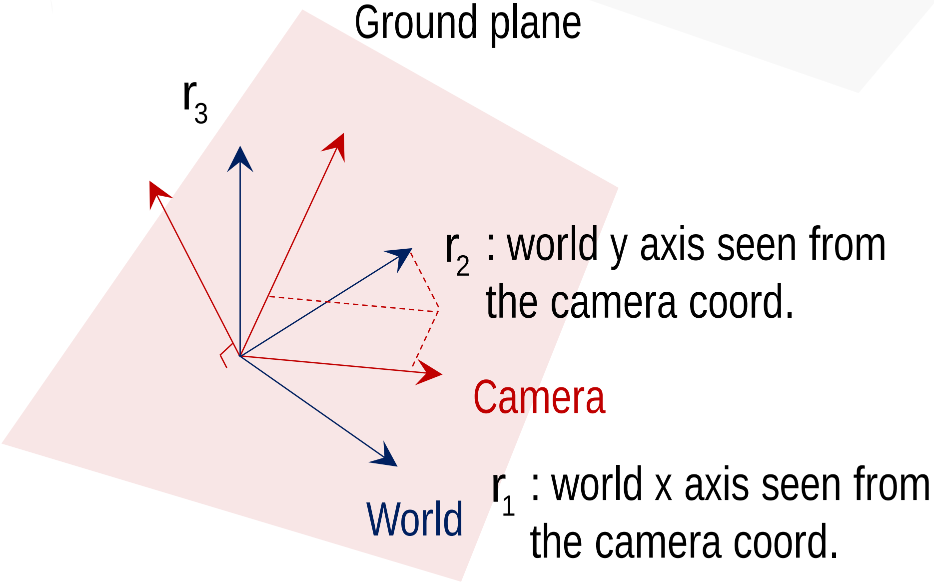
$^{C}R_W \in SO(3)$

- Orthogonal matrix $\longrightarrow \left({}^{C}R_W\right)^T \left({}^{C}R_W\right) = I_3, \quad \det\left({}^{C}R_W\right) = 1$
- Right hand rule



Coordinate Transform (Rotation)





Coordinate transformation from world to camera:

$$X_C = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} X = {}^{C}R_W X$$

$${}^{C}R_W \in SO(3)$$

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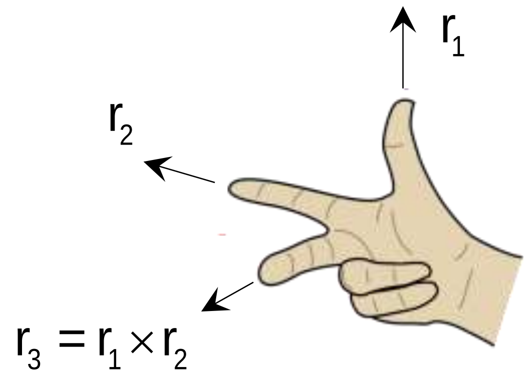
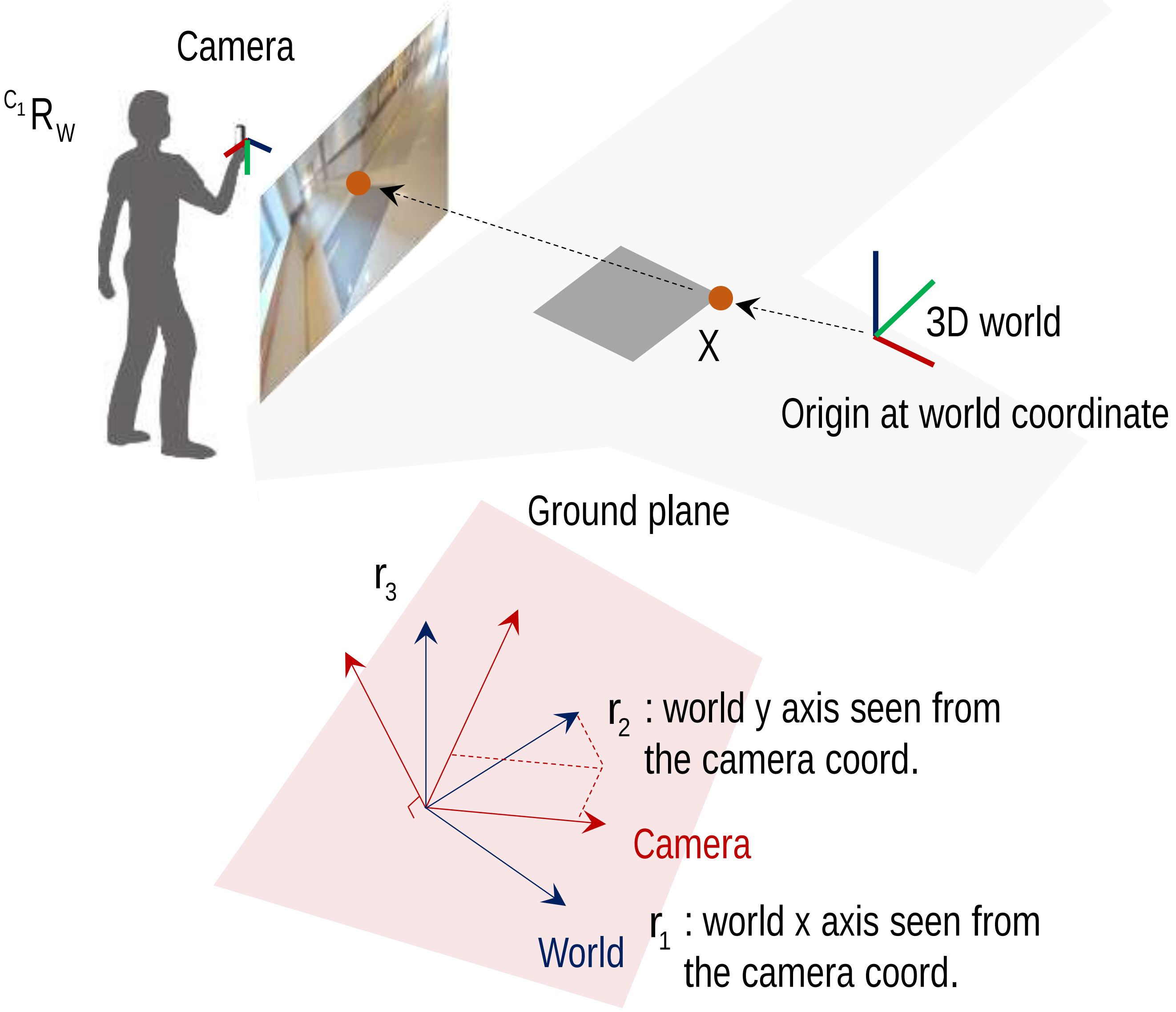


Camera Projection (Pure Rotation)



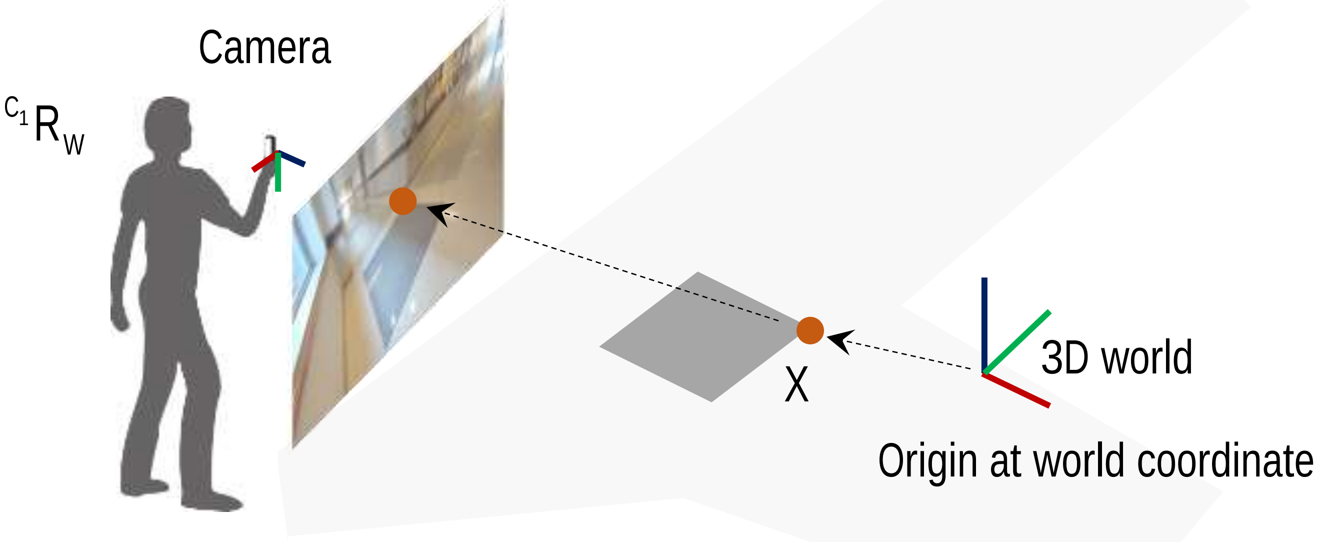
Coordinate transformation from world to camera:

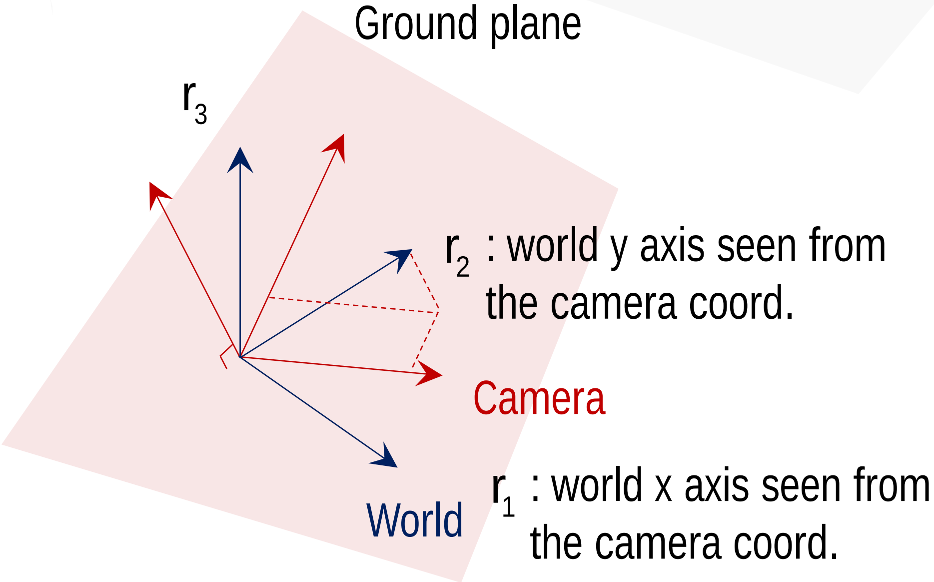
$$X_C = \begin{array}{ccc} r_1 & r_2 & r_3 \end{array} \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} X = {}^{C}R_W X$$

Camera projection of world point:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{K} \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix}$$

Camera Projection (Pure Rotation)





Coordinate transformation from world to camera:

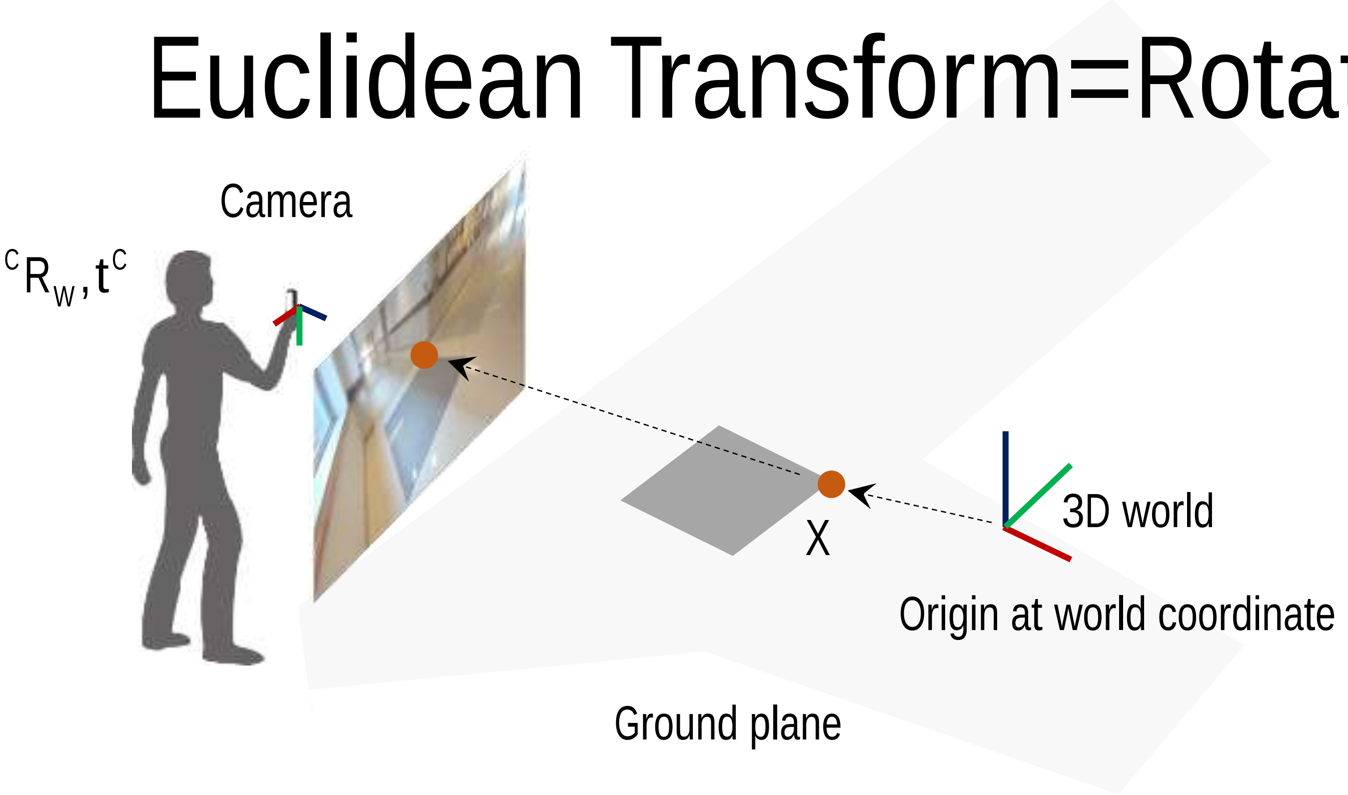
$$X_C = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} X = {}^{C}R_W X, \qquad \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix}$$

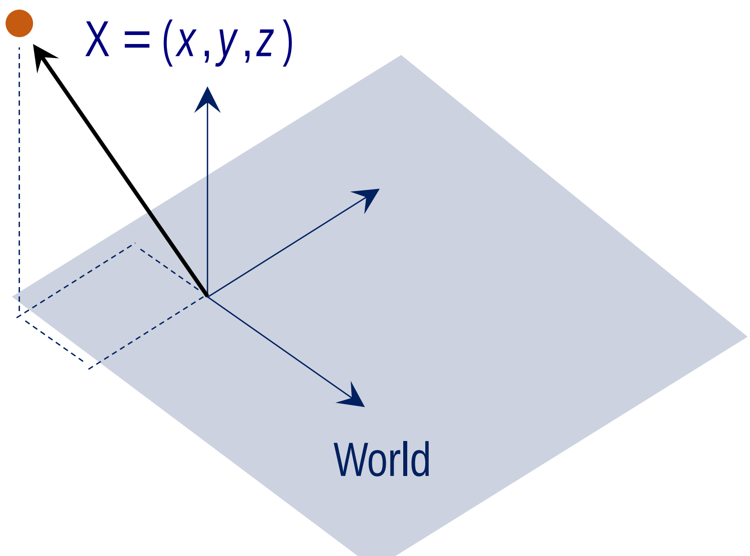
Camera projection of world point:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{K} \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix}$$

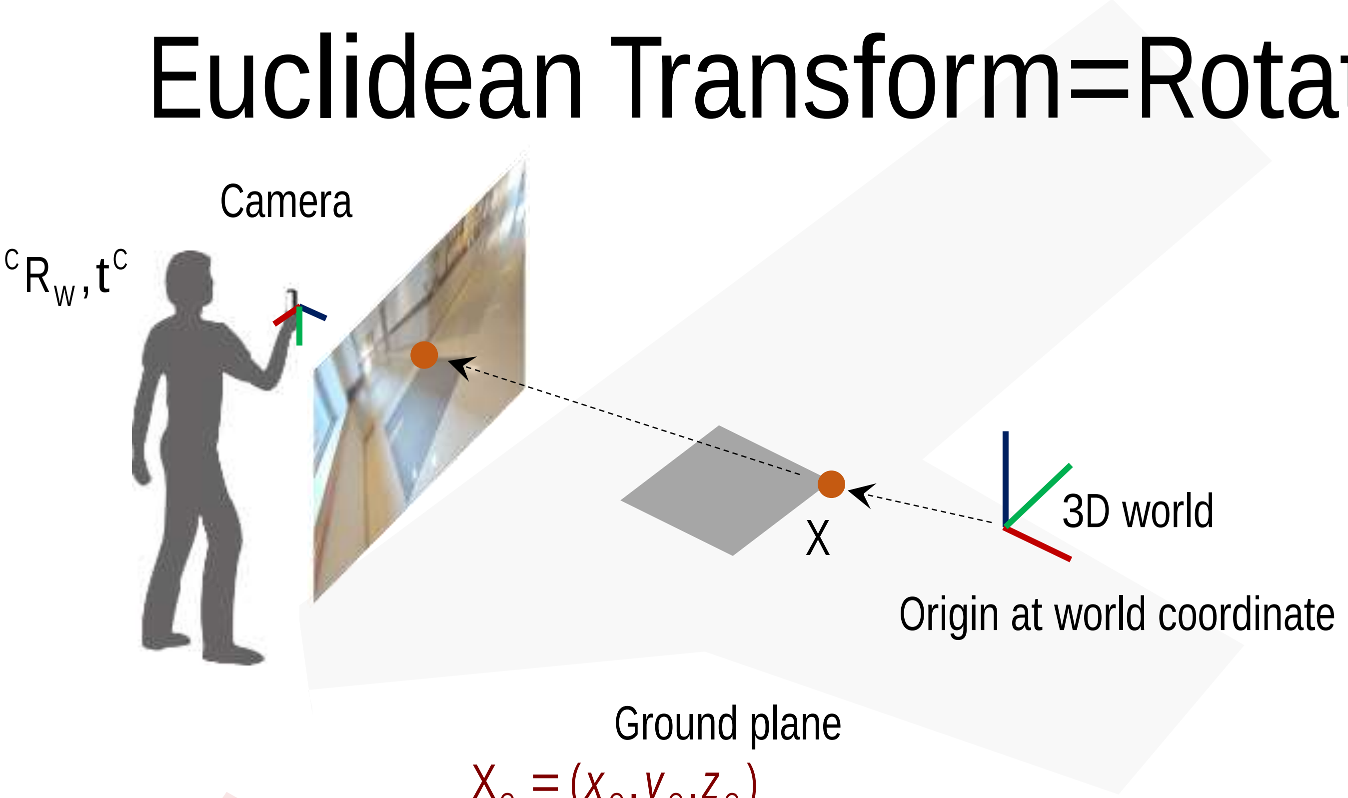
$$= \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{K} \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

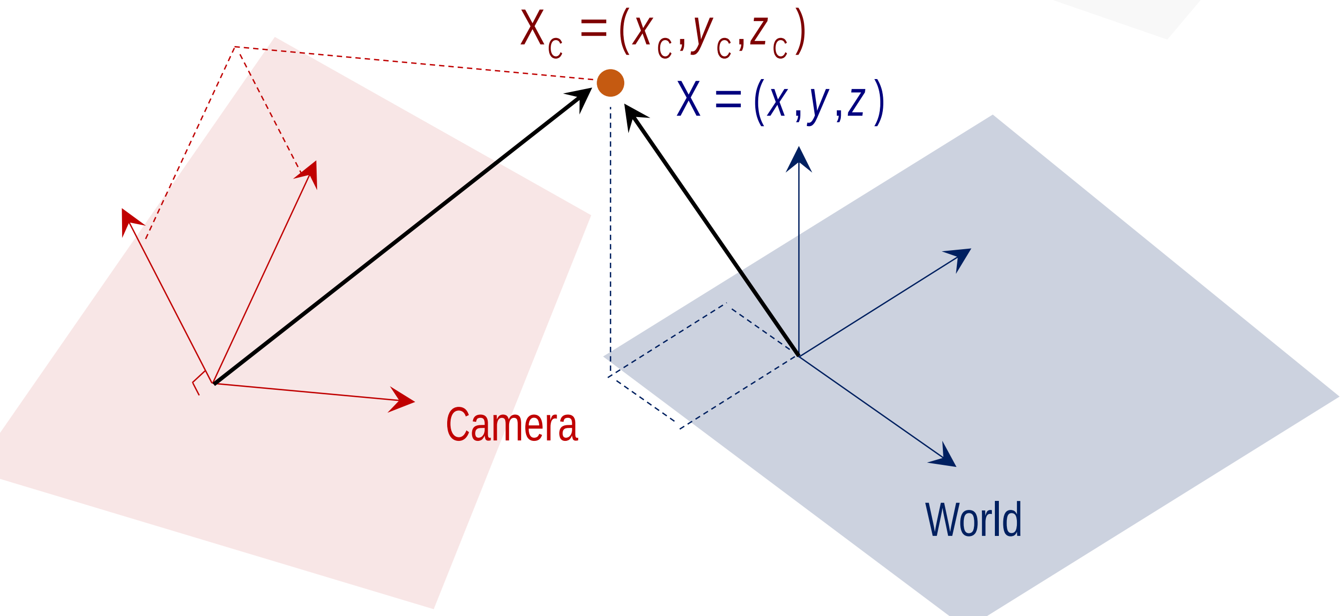
Euclidean Transform=Rotation+Translation



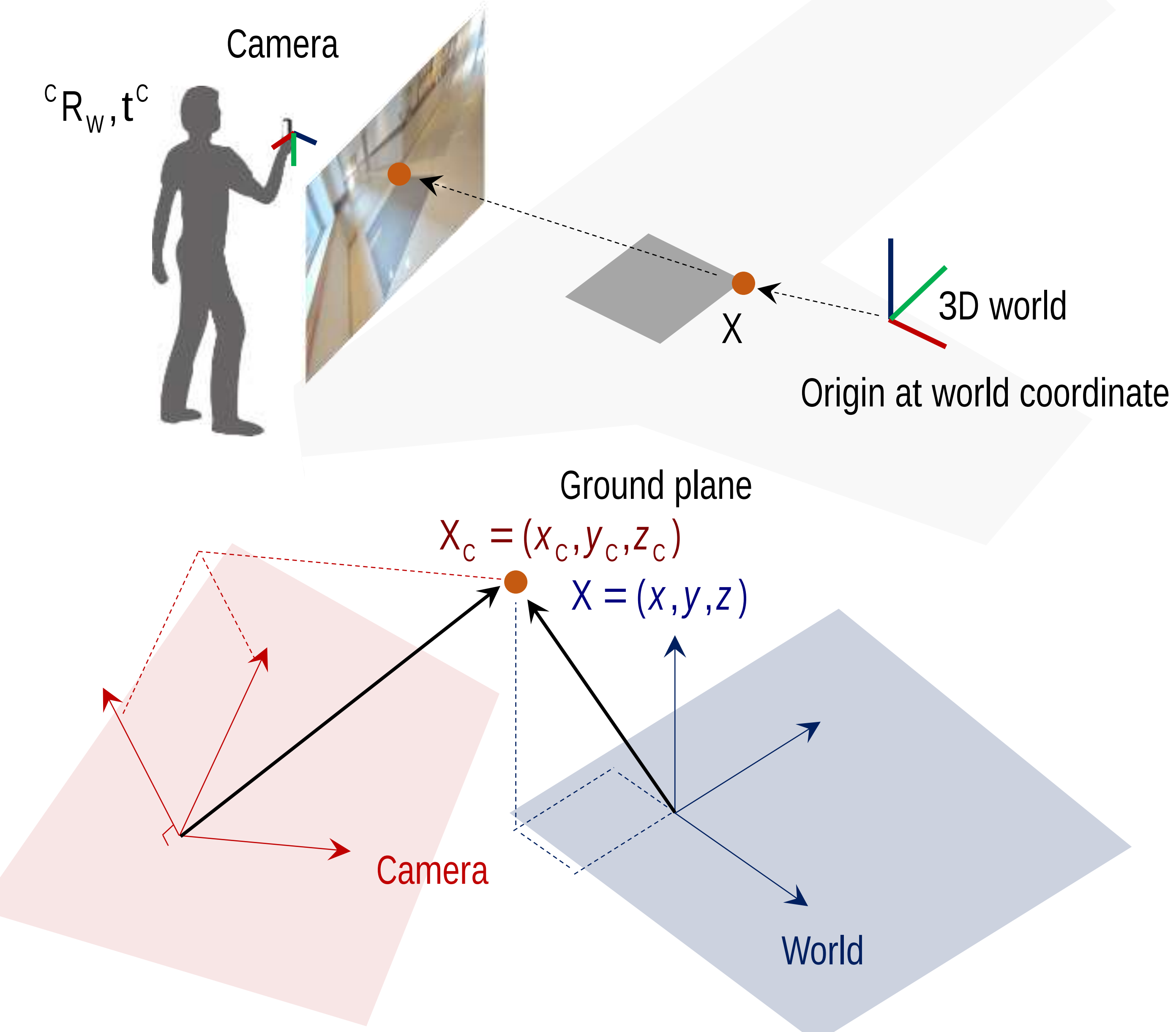


Euclidean Transform=Rotation+Translation

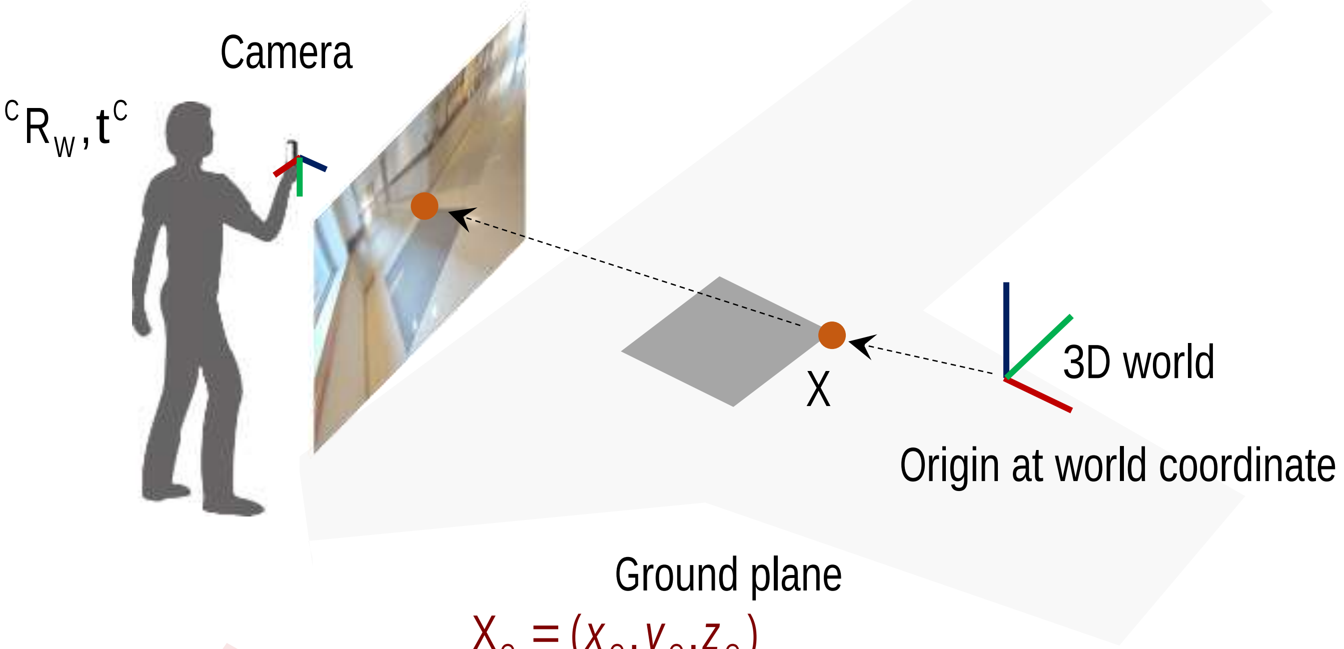




Euclidean Transform=Rotation+Translation



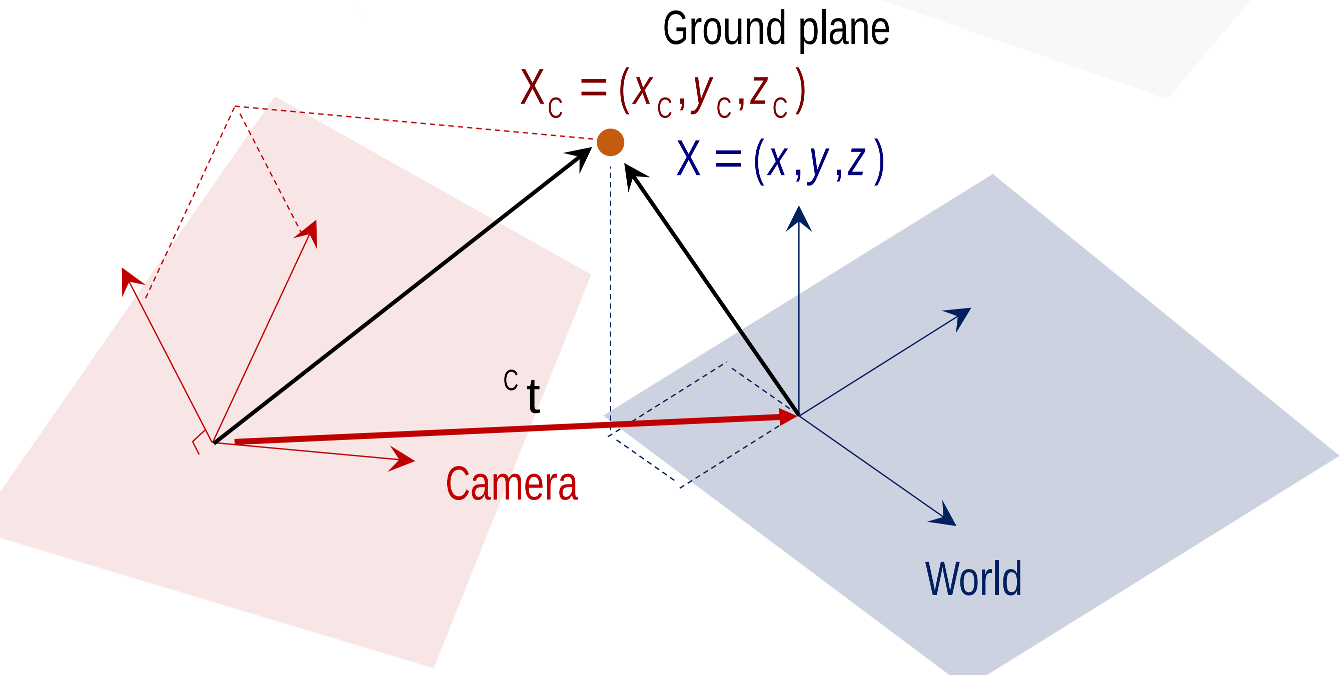
Euclidean Transform=Rotation+Translation



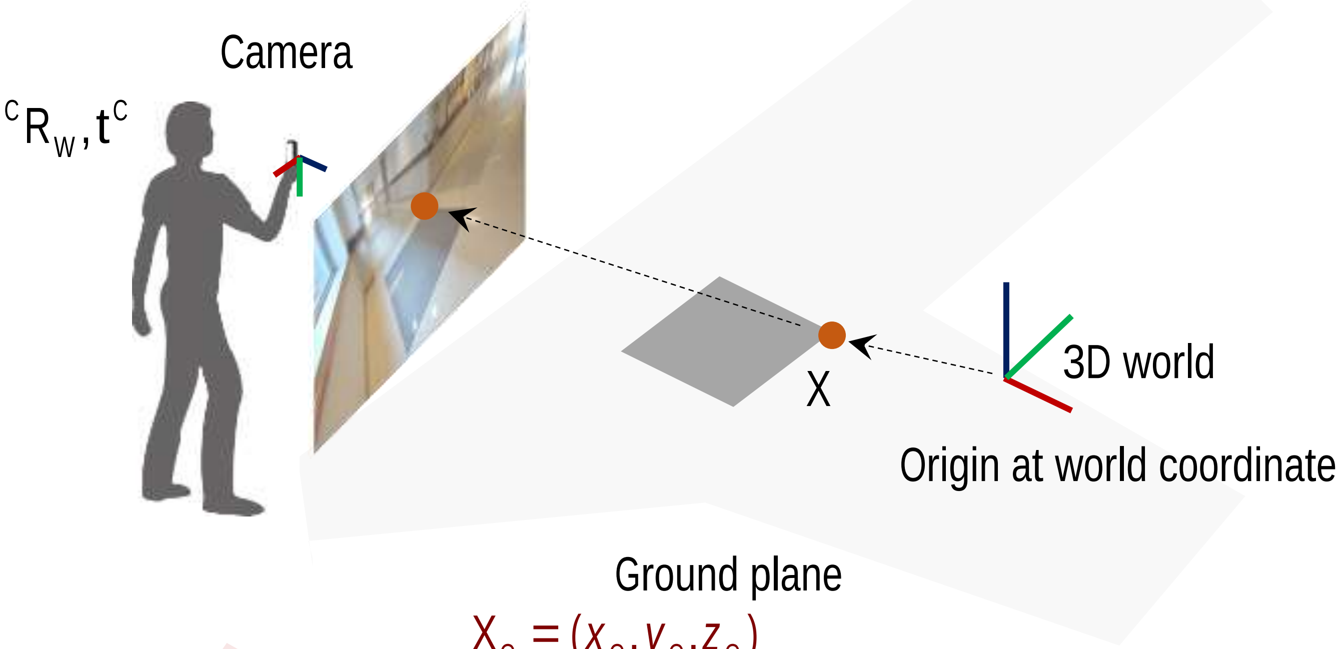
Coordinate transformation from world to camera:

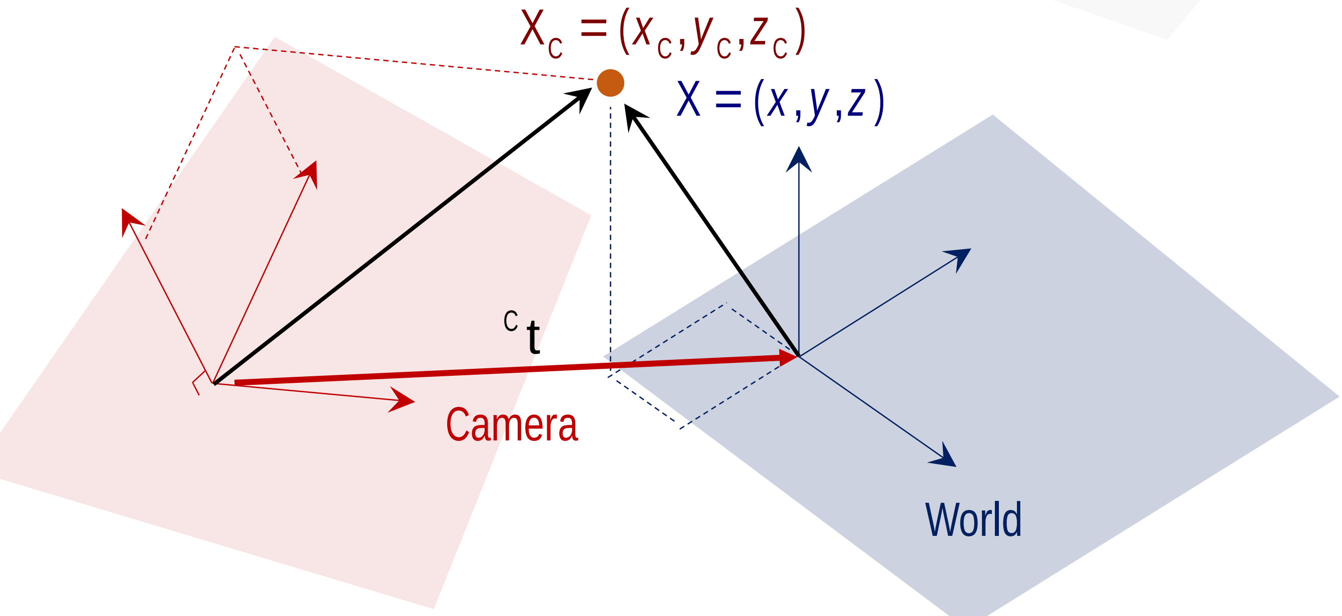
$$X_C = {}^{C}R_W X + {}^{C}t$$

where ${}^{C}t$ is the world orgin seen from camera.



Euclidean Transform=Rotation+Translation



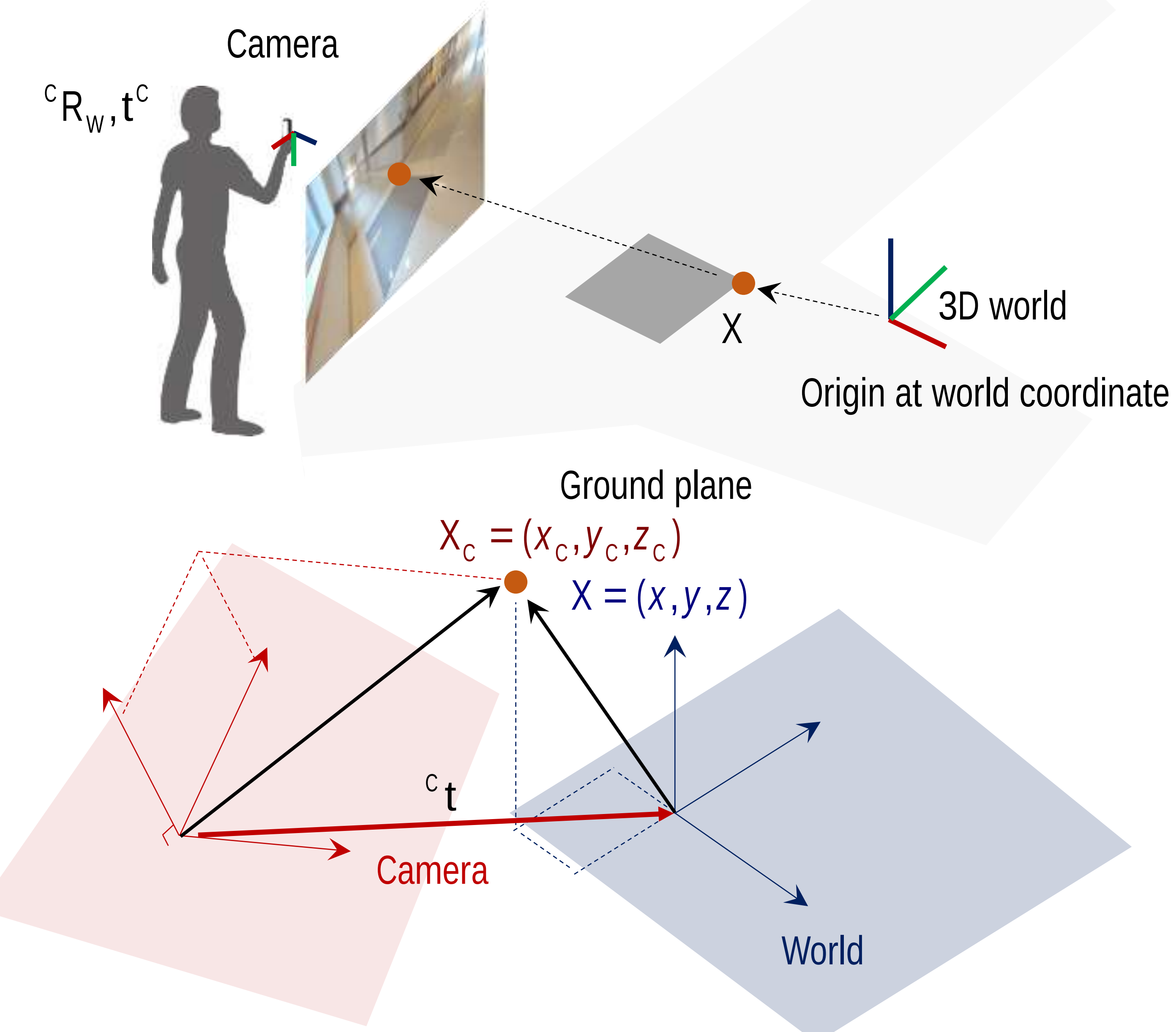


Coordinate transformation from world to camera:

$$X_C = {}^{C}R_W X + {}^{C}t = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} & t_x \\ r_{y1} & r_{y2} & r_{y3} & t_y \\ r_{z1} & r_{z2} & r_{z3} & t_z \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

where ${}^{C}t$ is the world orgin seen from camera.

Geometric Interpretation



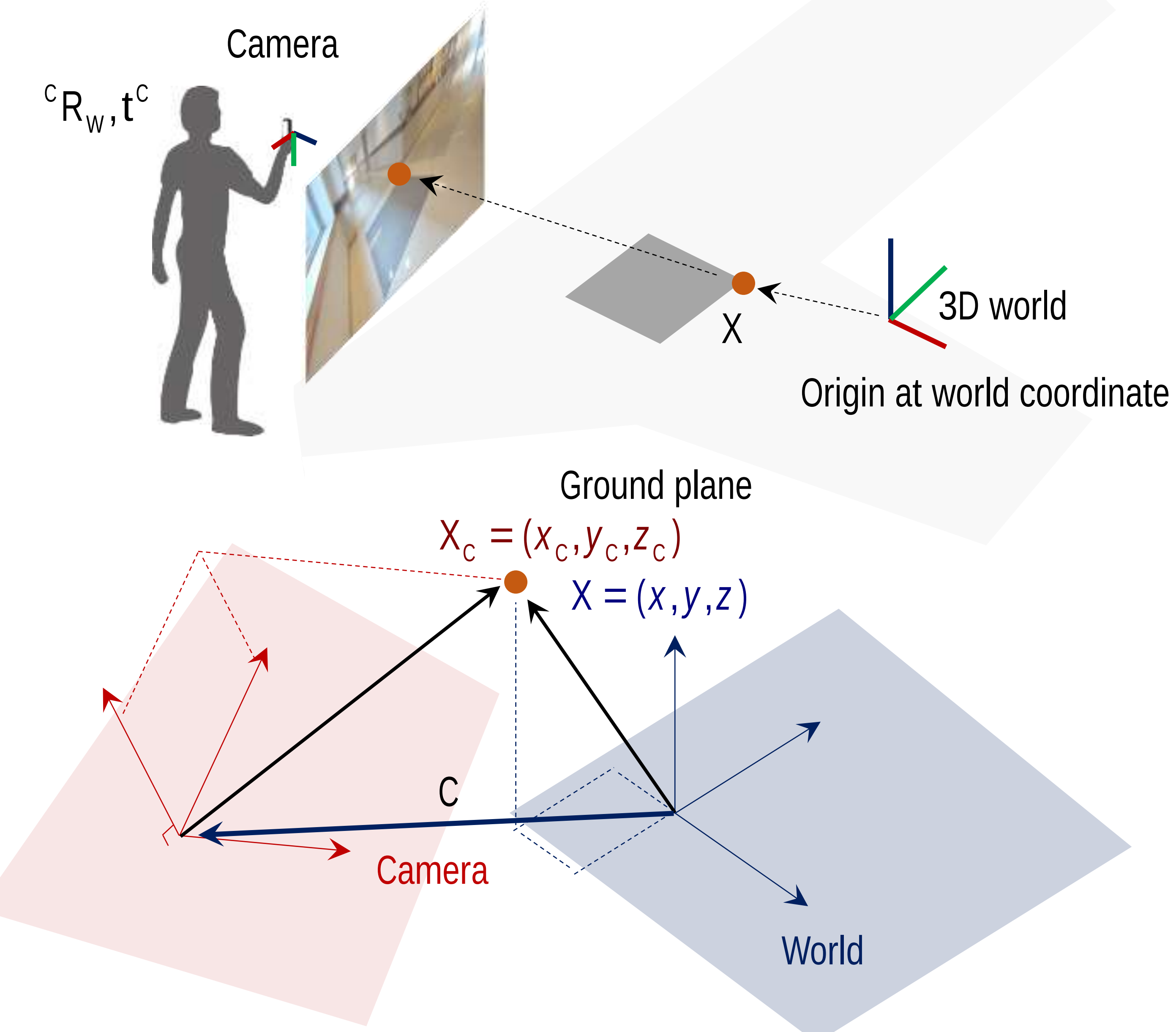
Coordinate transformation from world to camera:

$$X_C = {}^{C}R_W X + {}^{C}t = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} & t_x \\ r_{y1} & r_{y2} & r_{y3} & t_y \\ r_{z1} & r_{z2} & r_{z3} & t_z \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

where ${}^{C}t$ is the world orgin seen from camera.

Rotate and then, translate.

Geometric Interpretation



Coordinate transformation from world to camera:

$$X_C = {}^C R_W X + {}^C t = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} & t_x \\ r_{y1} & r_{y2} & r_{y3} & t_y \\ r_{z1} & r_{z2} & r_{z3} & t_z \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

where ${}^C t$ is the world orgin seen from camera.

Rotate and then, translate.

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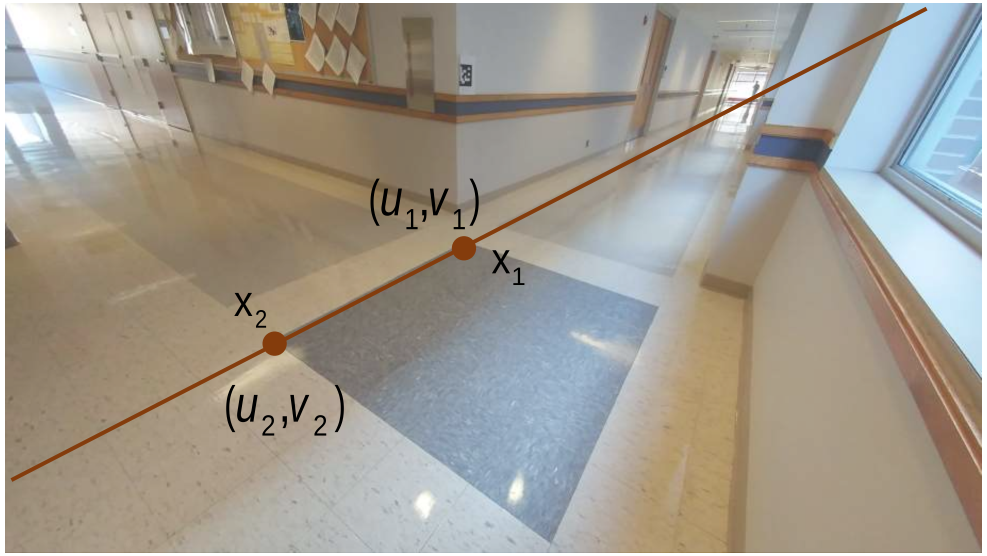
cf) Translate and then, rotate.

$$X_C = {}^C R_W (X - C) = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} \begin{bmatrix} 1 & & & -C_x \\ & 1 & & -C_y \\ & & 1 & -C_z \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

where C is the camera location seen from world.

Projective Line

Point-Point in Image



A 2D line passing through two 2D points:

$$au_1 + bv_1 + c = 0 \qquad au_2 + bv_2 + c = 0$$

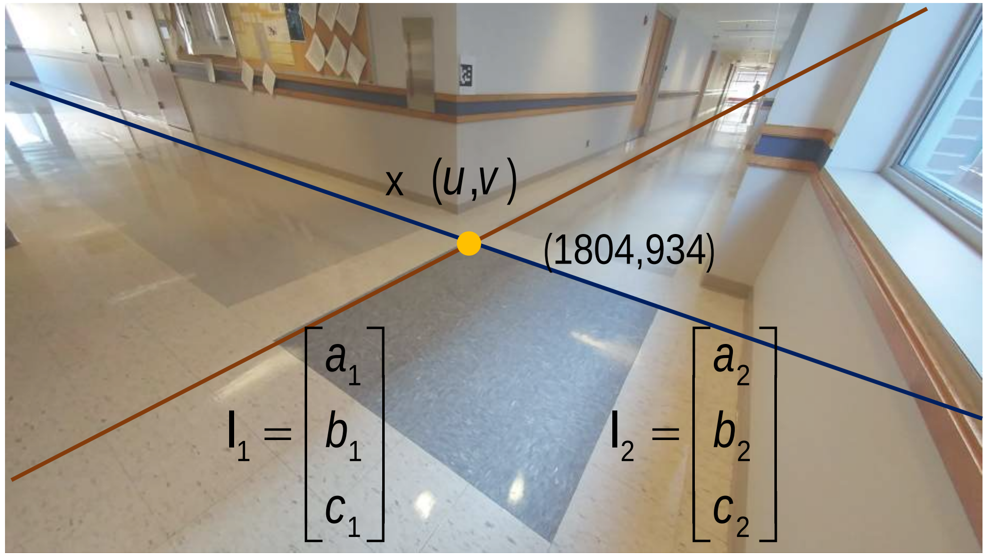
$$\mathrm{x}_1^{\mathsf{T}}\mathrm{l} = 0 \qquad \mathrm{x}_2^{\mathsf{T}}\mathrm{l} = 0$$

where $\mathrm{x}_1 = \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix} \quad \mathrm{x}_2 = \begin{bmatrix} u_2 \\ v_2 \\ 1 \end{bmatrix} \quad \mathrm{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\rightarrow \begin{bmatrix} \mathrm{x}_1^{\mathsf{T}} \\ \mathrm{x}_2^{\mathsf{T}} \end{bmatrix} \mathrm{l} = 0$$

$$\underbrace{\mathrm{A}}_{2\mathrm{x}3} \, \mathrm{l} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \mathrm{l} = \mathrm{null}\left(\mathrm{A}\right) \quad \text{or} \quad \mathrm{l} = \mathrm{x}_1 \times \mathrm{x}_2$$

Line-Line in Image



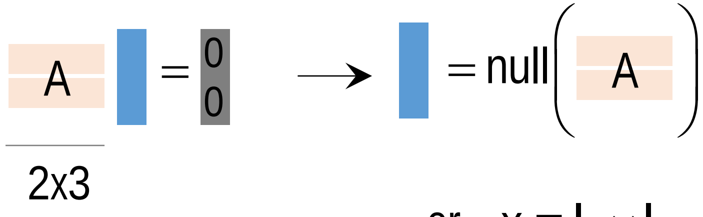
Two 2D lines in an image intersect at a 2D point:

$$a_1 u + b_1 v + c_1 = 0 \qquad a_2 u + b_2 v + c_2 = 0$$

$$\mathbf{l}_1^{\mathsf{T}} \mathbf{x} = 0 \qquad \mathbf{l}_2^{\mathsf{T}} \mathbf{x} = 0$$

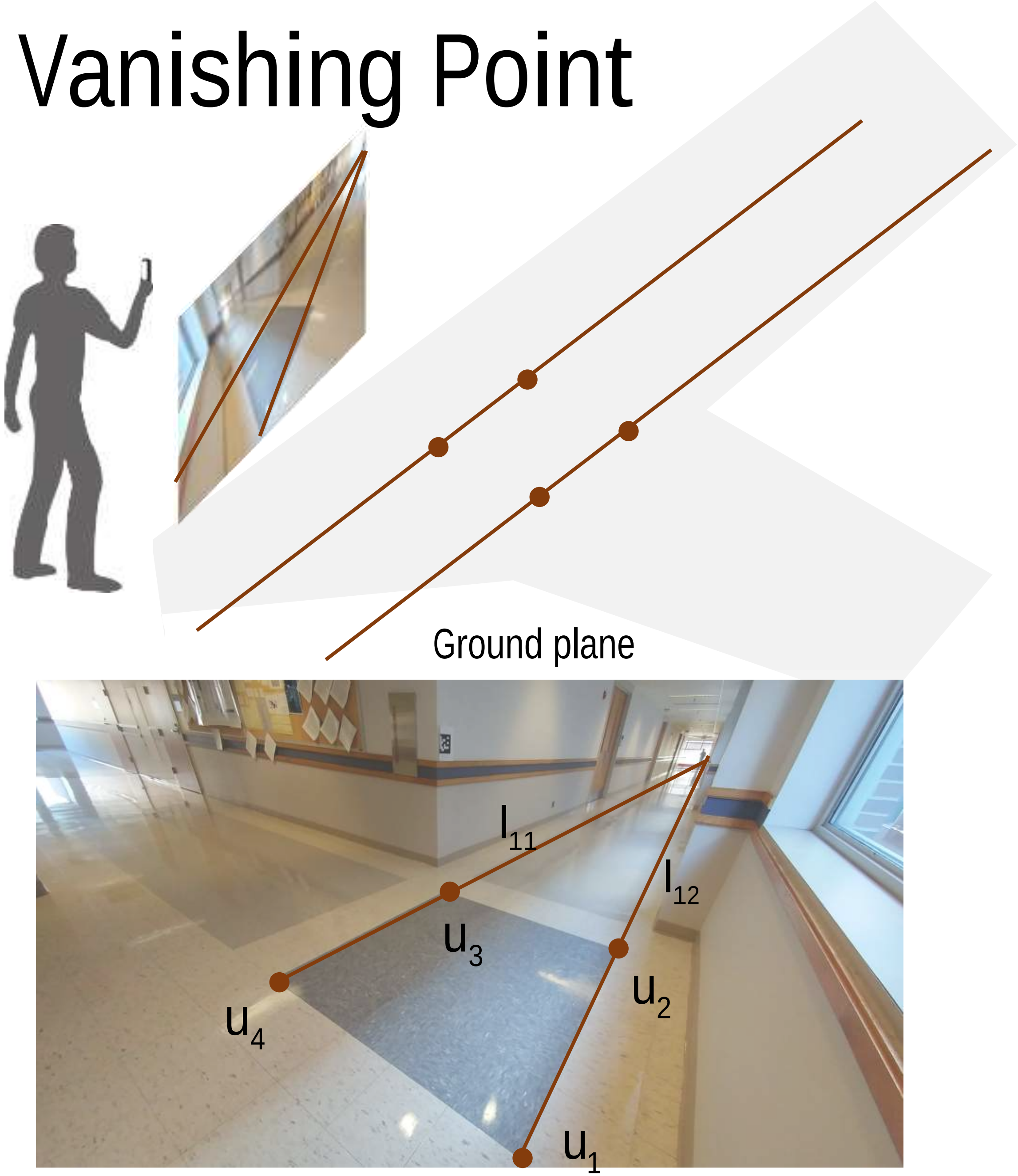
where $\mathbf{x} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad \mathbf{l}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \quad \mathbf{l}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} \mathbf{l}_1^{\mathsf{T}} \\ \mathbf{l}_2^{\mathsf{T}} \end{bmatrix} \mathbf{x} = 0$$



or $\quad \mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$

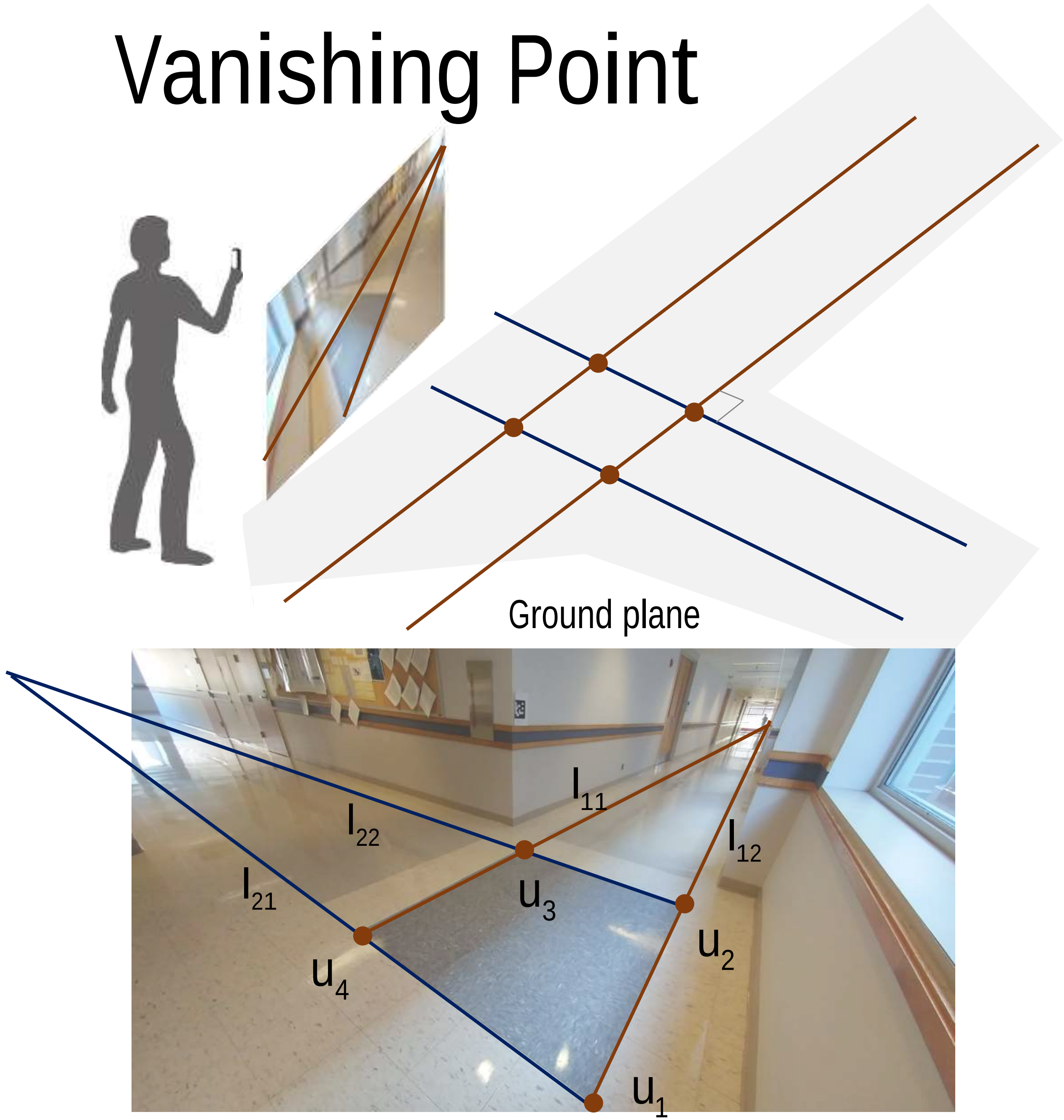
Vanishing Point



Parallel lines:

$$l_{11} = u_4 \times u_3 \qquad l_{12} = u_1 \times u_2$$

Vanishing Point

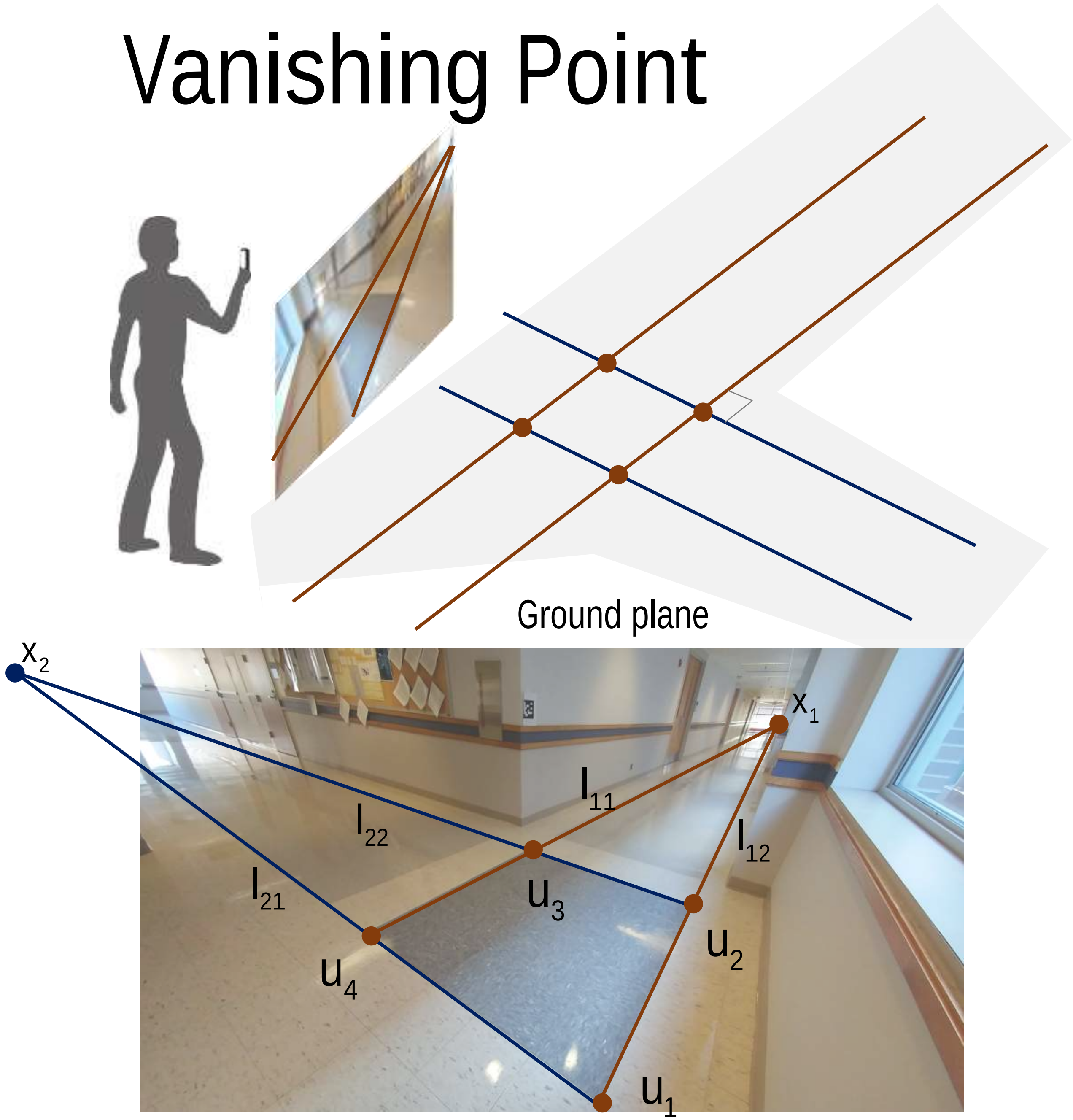


Parallel lines:

$$l_{11} = u_4 \times u_3 \qquad l_{12} = u_1 \times u_2$$

$$l_{21} = u_4 \times u_1 \qquad l_{22} = u_3 \times u_4$$

Vanishing Point



Parallel lines:

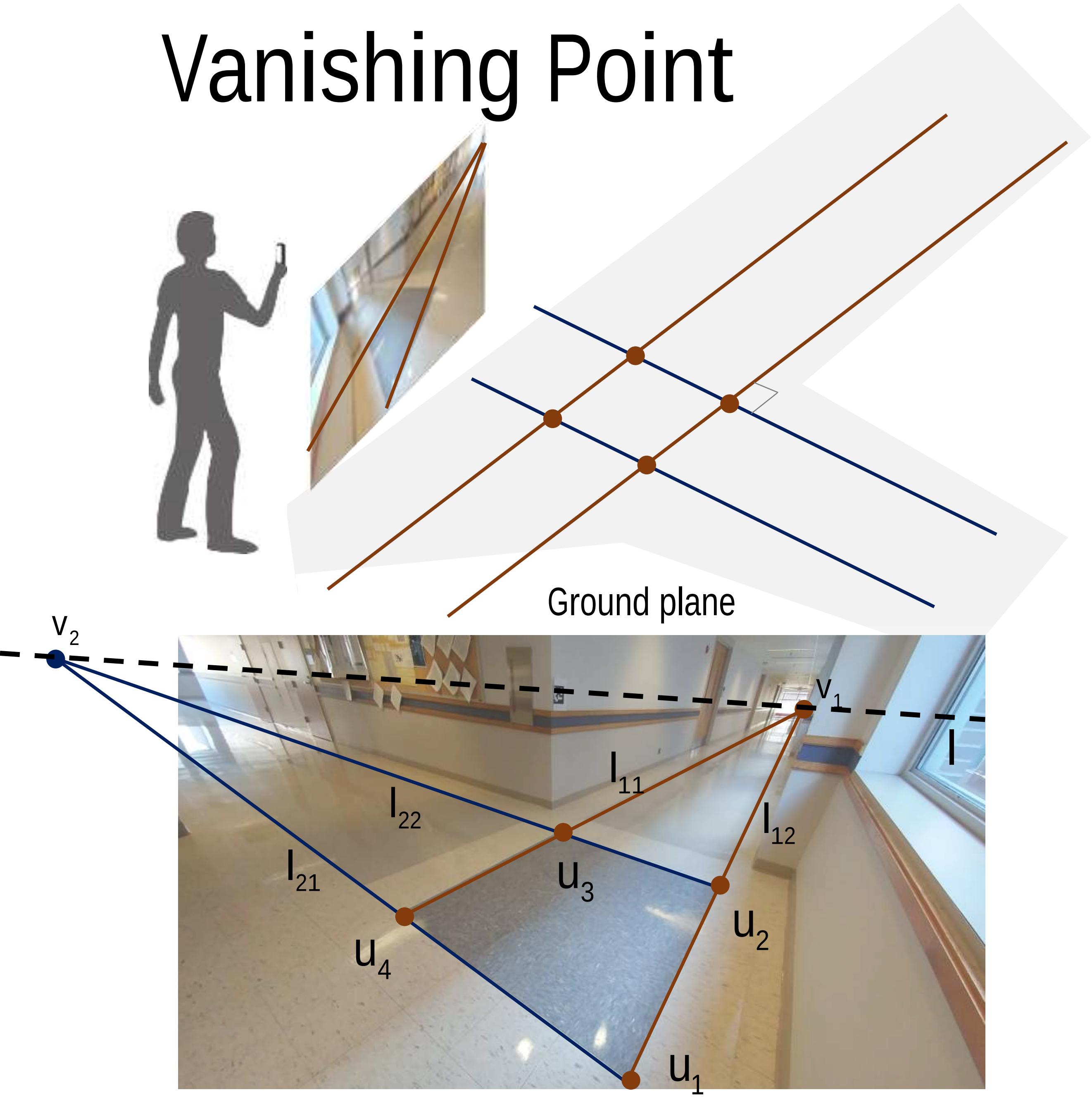
$$l_{11} = u_4 \times u_3 \qquad l_{12} = u_1 \times u_2$$

$$l_{21} = u_4 \times u_1 \qquad l_{22} = u_3 \times u_4$$

Vanishing points:

$$x_1 = l_{11} \times l_{12} \qquad x_2 = l_{21} \times l_{22}$$

Vanishing Point



Parallel lines:

$$l_{11} = u_4 \times u_3 \qquad l_{12} = u_1 \times u_2$$

$$l_{21} = u_4 \times u_1 \qquad l_{22} = u_3 \times u_4$$

Vanishing points:

$$v_1 = l_{11} \times l_{12} \qquad v_2 = l_{21} \times l_{22}$$

Vanishing line:

$$l = v_1 \times v_2$$

Where was I (how high)?

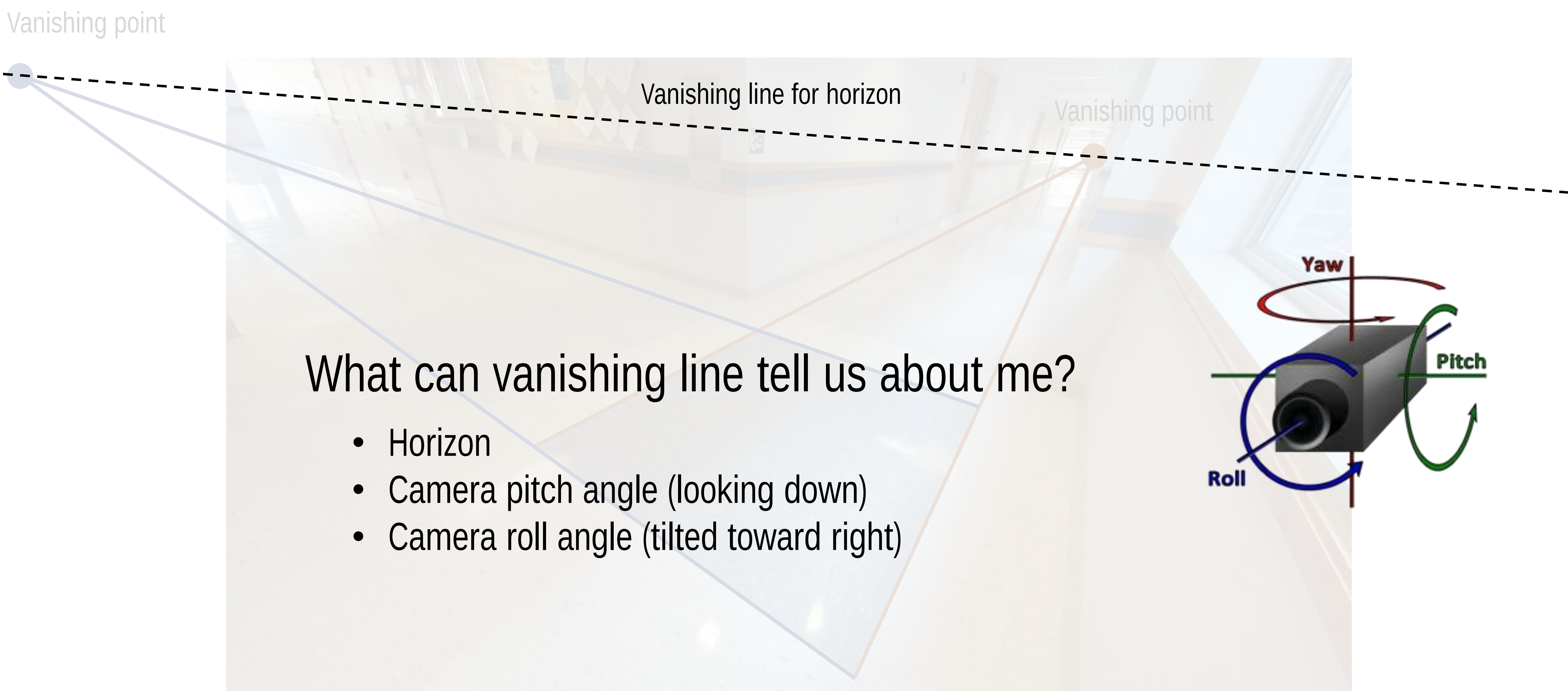
Taken from my hotel room (6^{th} floor)

Taken from beach

Vanishing point

Multiple vanishing point

Vanishing line: Horizon

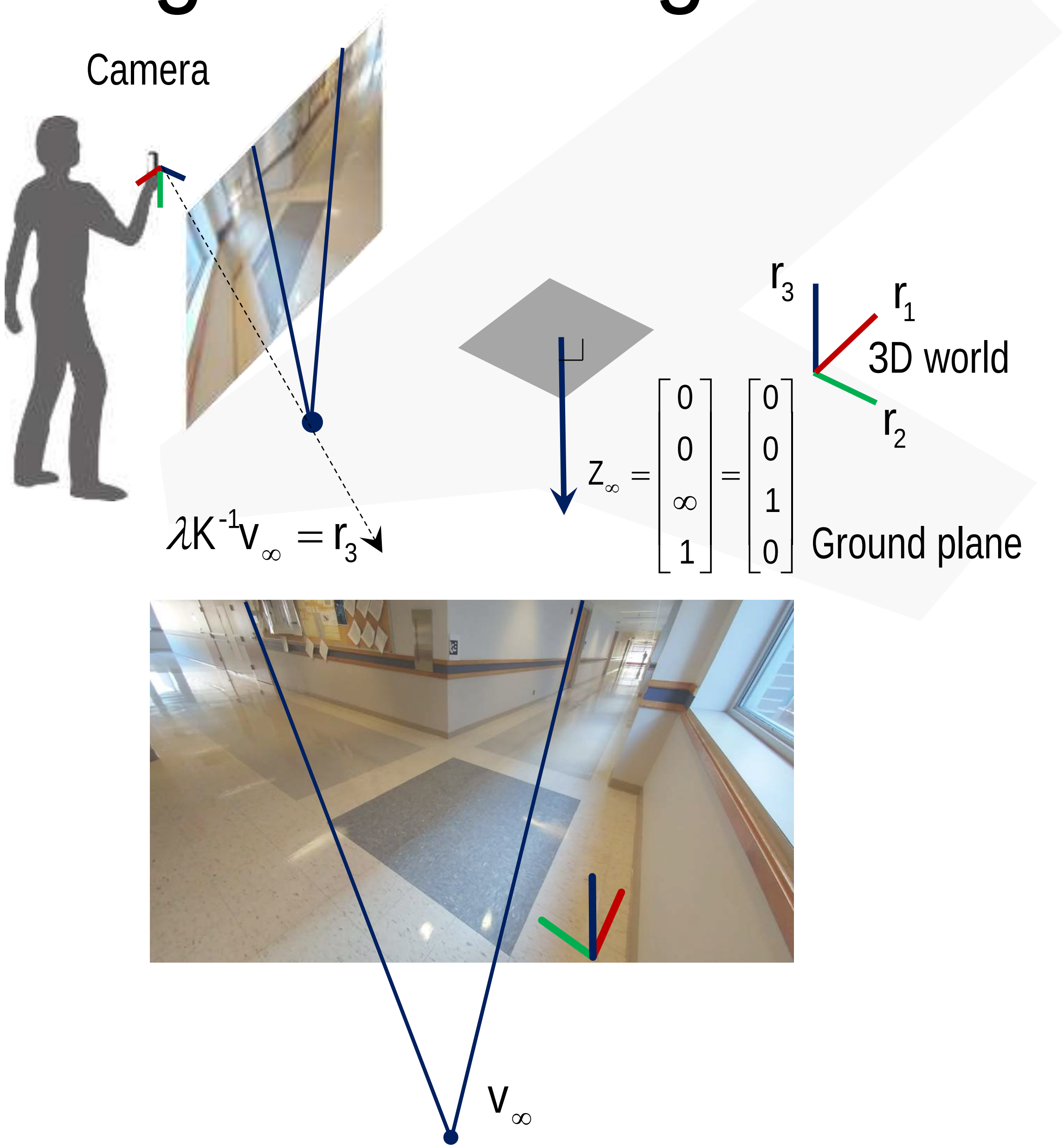


What can vanishing line tell us about me?

- Horizon
- Camera pitch angle (looking down)
- Camera roll angle (tilted toward right)

Keller Hall

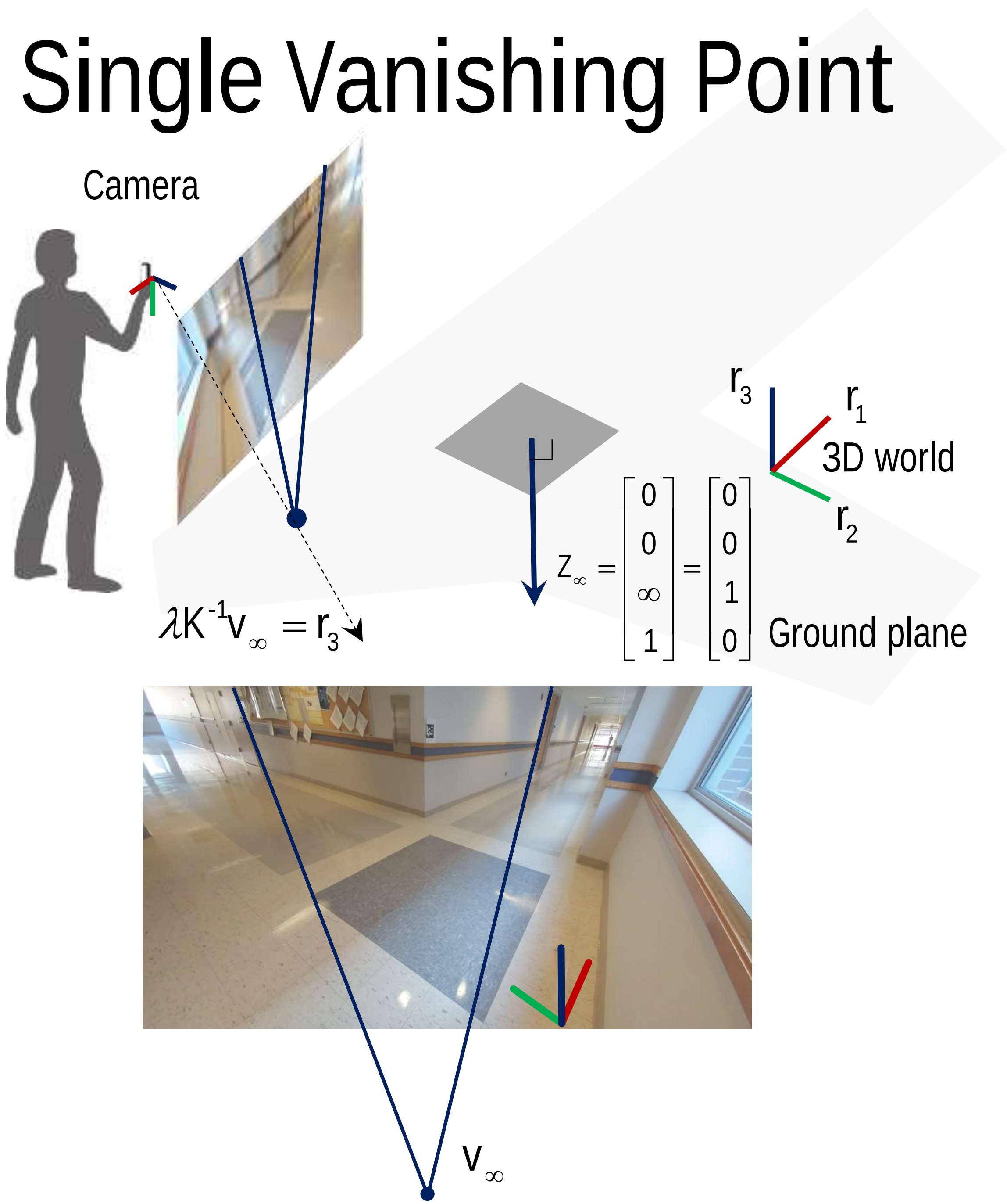
Single Vanishing Point



$$\lambda \mathrm{v}_{\infty} = \mathrm{K}\begin{bmatrix} \mathrm{r}_1 & \mathrm{r}_2 & \mathrm{r}_3 & \mathrm{t}_{\mathrm{W}}^{\mathrm{C}} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda \mathrm{v}_{\infty} = \mathrm{K}\mathrm{r}_3$$

Single Vanishing Point



$$\lambda v_\infty = K \begin{bmatrix} r_1 & r_2 & r_3 & t_W^C \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

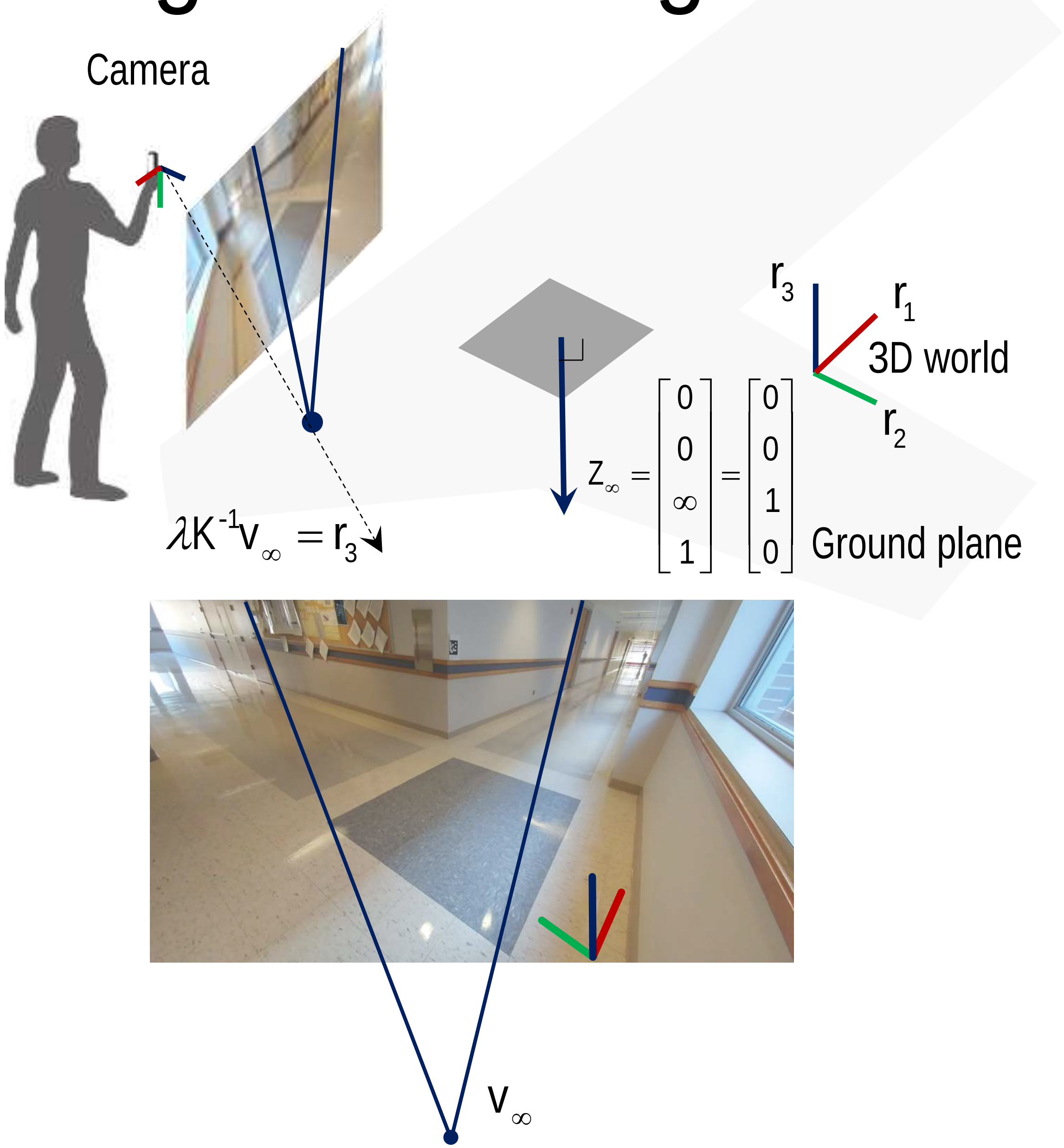
$$\lambda v_\infty = K r_3$$

$$\longrightarrow \quad r_3 = \frac{K^{-1} v_\infty}{\left\| K^{-1} v_\infty \right\|}$$

because r_3 is a unit vector.

Z vanishing point tells us about the surface normal of the ground plane

Single Vanishing Point



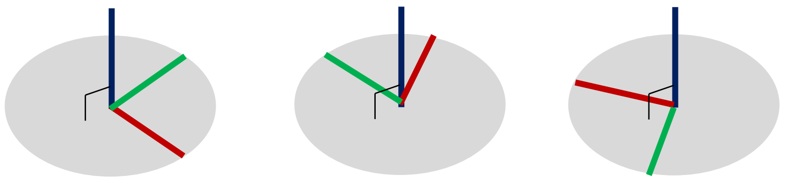
$$\lambda v_\infty = K \begin{bmatrix} r_1 & r_2 & r_3 & t_W^C \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda v_\infty = K r_3$$

$$\longrightarrow \quad r_3 = \frac{K^{-1} v_\infty}{\left\| K^{-1} v_\infty \right\|}$$ because r_3 is a unit vector.

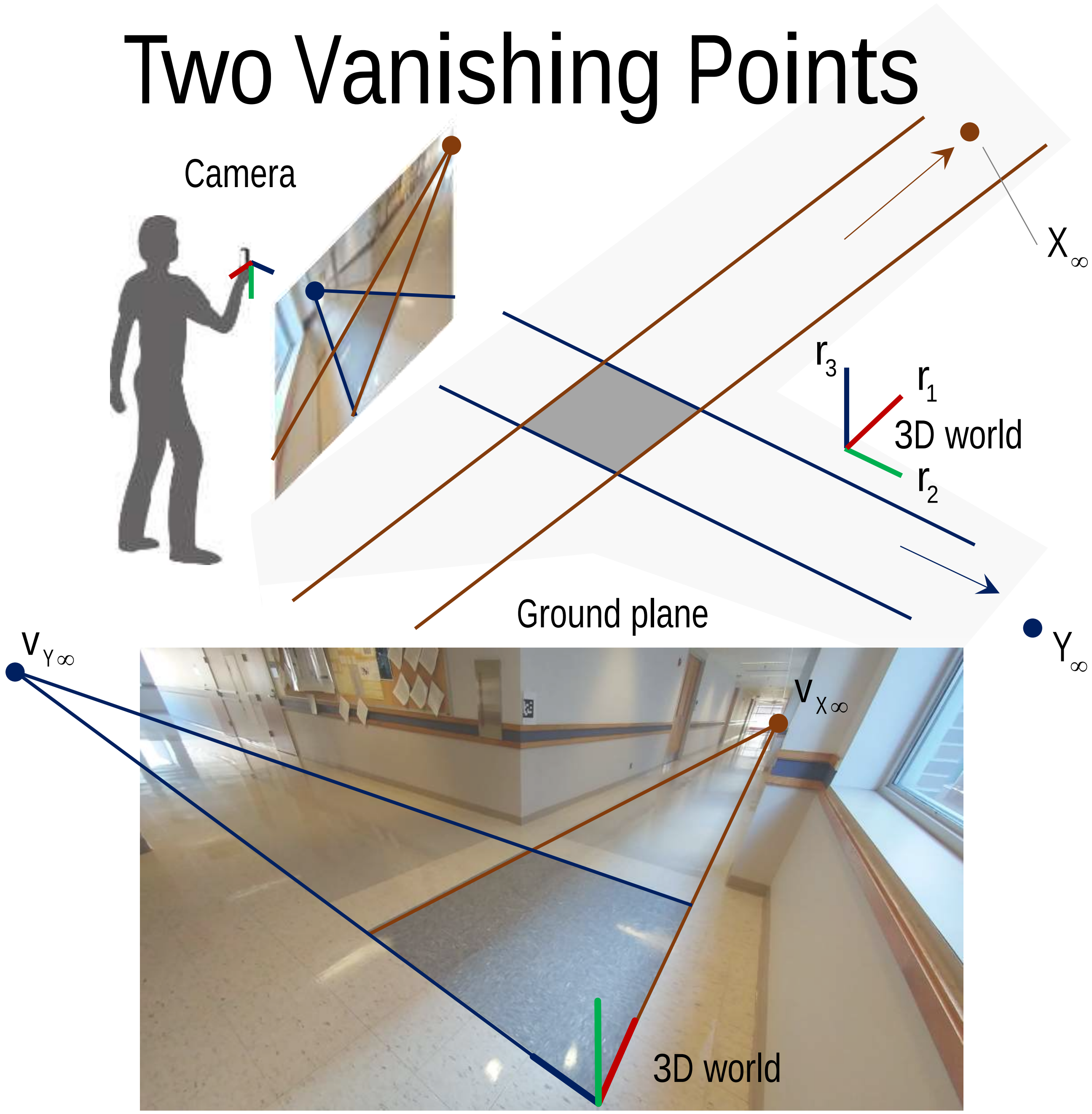
Z vanishing point tells us about the surface normal of the ground plane

Rotation ambiguity



Ground plane

Two Vanishing Points

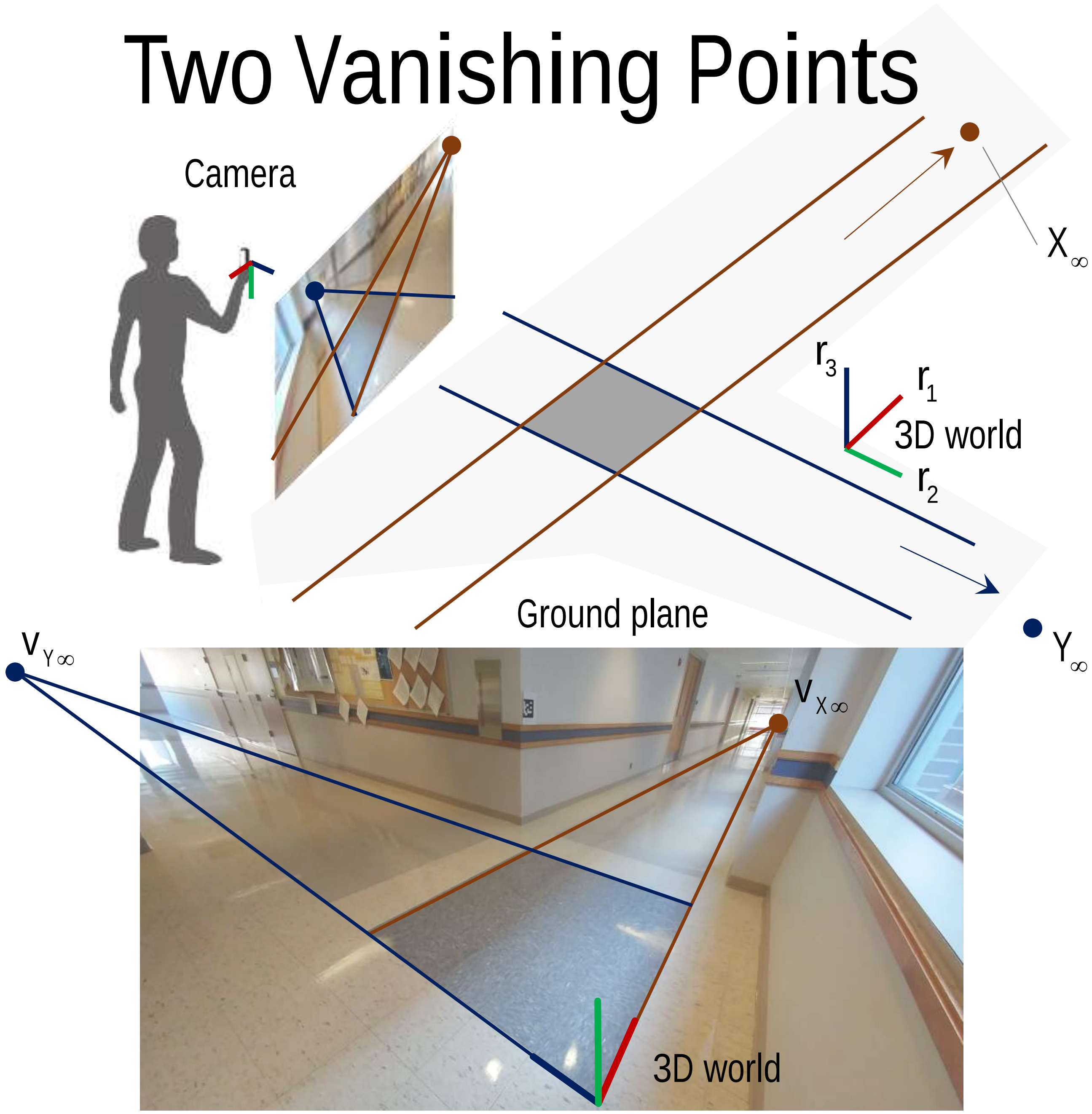


$$X_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Y_\infty = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \lambda v_{X\infty} &= K\begin{bmatrix} r_1 & r_2 & r_3 & t_W^C \end{bmatrix} X_\infty \\ &= K r_1 \end{aligned}$$

Two Vanishing Points



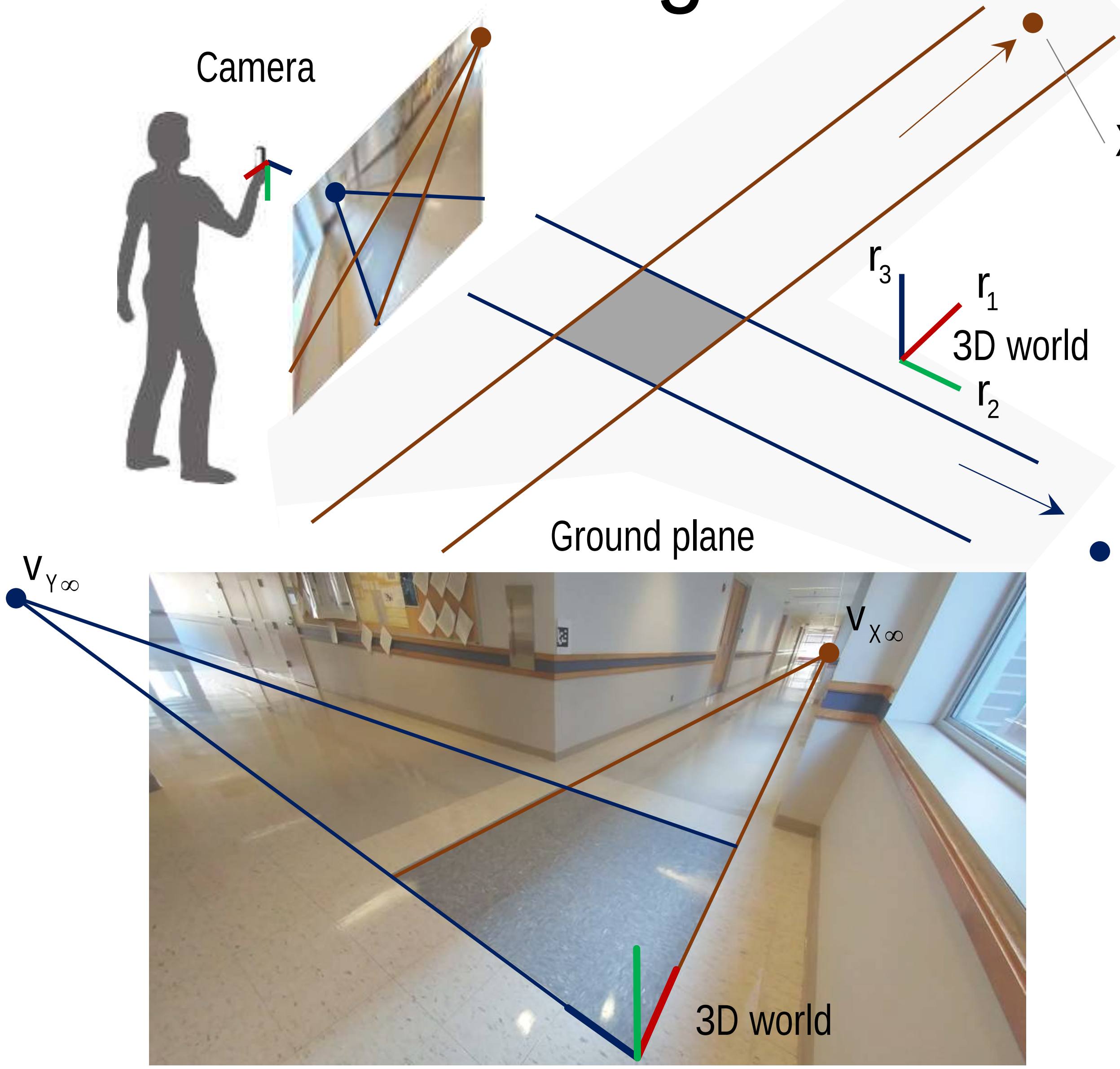
$$X_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Y_\infty = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda v_{X\infty} = K\begin{bmatrix} r_1 & r_2 & r_3 & t_W^C \end{bmatrix} X_\infty$$
$$= Kr_1$$

$$\lambda v_{Y\infty} = K\begin{bmatrix} r_1 & r_2 & r_3 & t_W^C \end{bmatrix} Y_\infty$$
$$= Kr_2$$

Two Vanishing Points



$$X_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Y_\infty = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda v_{X\infty} = K\begin{bmatrix} r_1 & r_2 & r_3 & t_W^C \end{bmatrix} X_\infty$$
$$= Kr_1$$

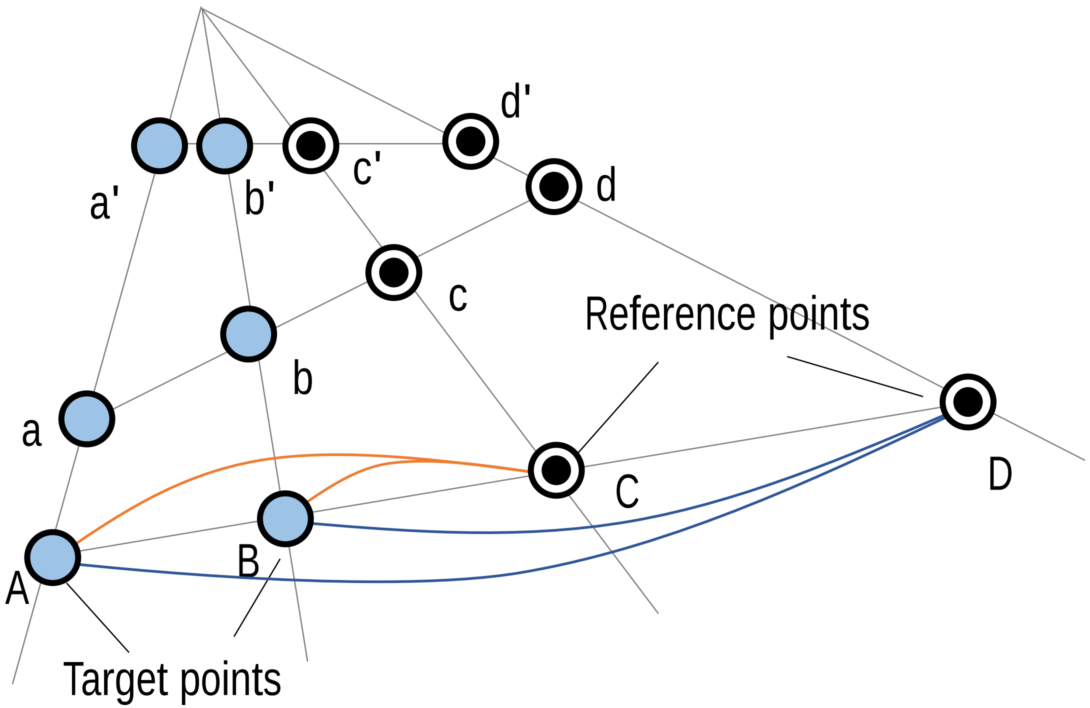
$$\lambda v_{Y\infty} = K\begin{bmatrix} r_1 & r_2 & r_3 & t_W^C \end{bmatrix} Y_\infty$$
$$= Kr_2$$

$$\rightarrow \quad r_1 = \frac{K^{-1} v_{X\infty}}{\left\| K^{-1} v_{X\infty} \right\|}, \quad r_2 = \frac{K^{-1} v_{Y\infty}}{\left\| K^{-1} v_{Y\infty} \right\|}$$

$r_3 = r_1 \times r_2$: Orthogonality constraint

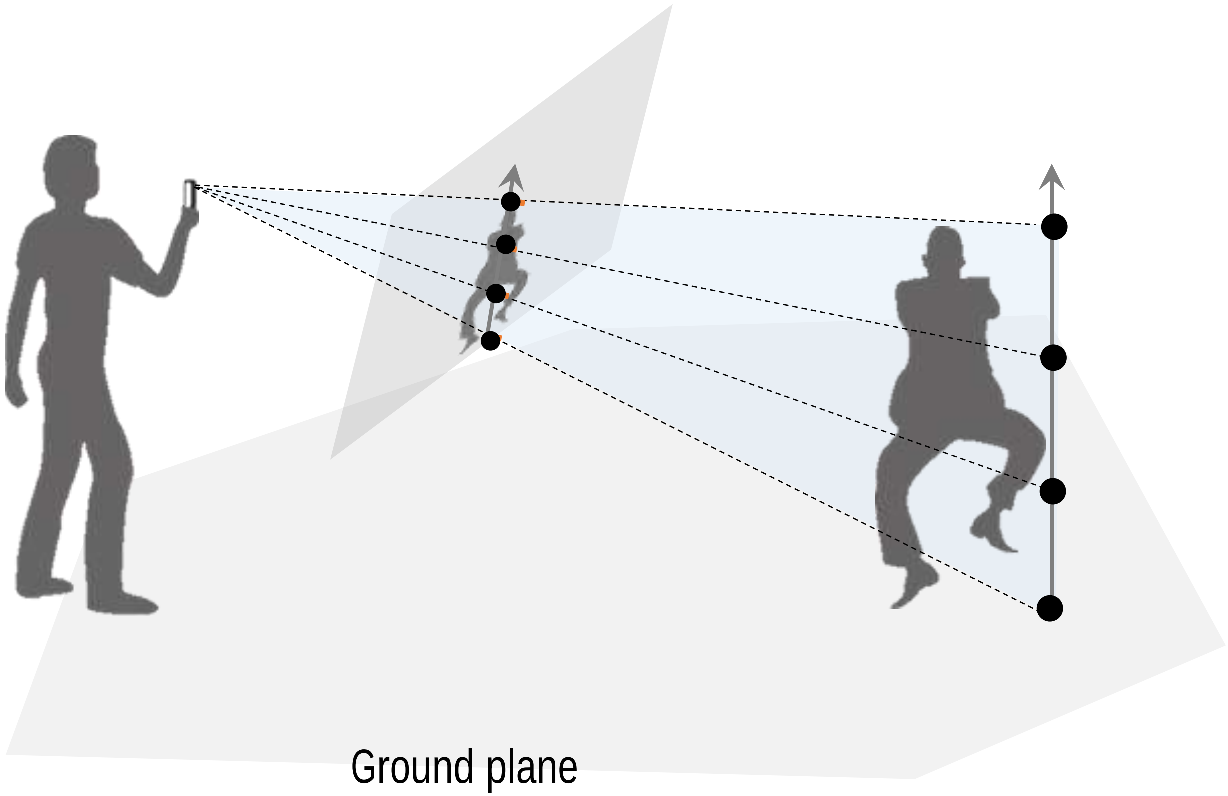
Single View Metrology

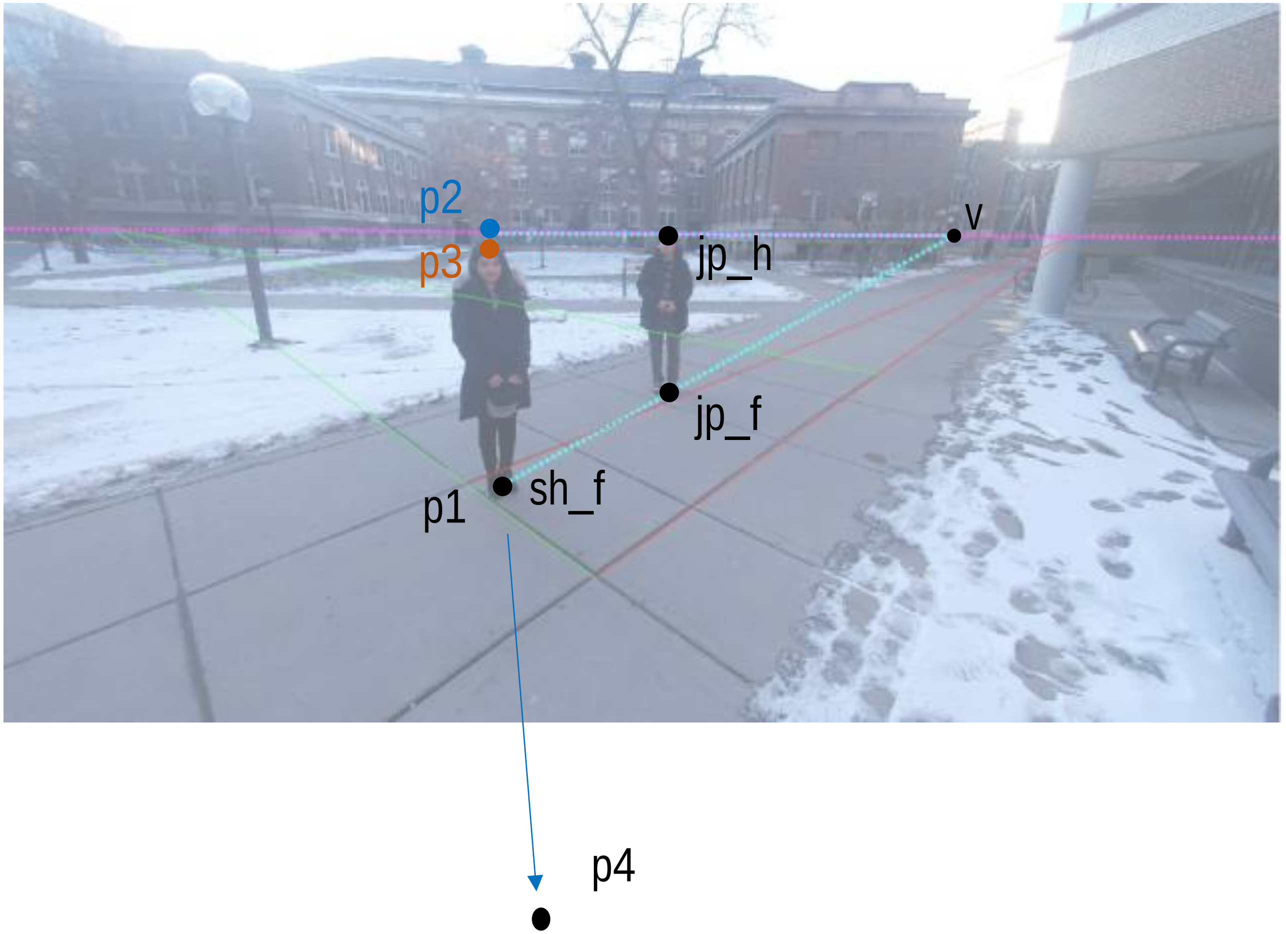
Cross Ratio



$$\frac{\overline{AC}}{\overline{BC}}\,\frac{\overline{BD}}{\overline{AD}} = \frac{\overline{ac}}{\overline{bc}}\,\frac{\overline{bd}}{\overline{ad}} = \frac{\overline{a'c'}}{\overline{b'c'}}\,\frac{\overline{b'd'}}{\overline{a'd'}}$$

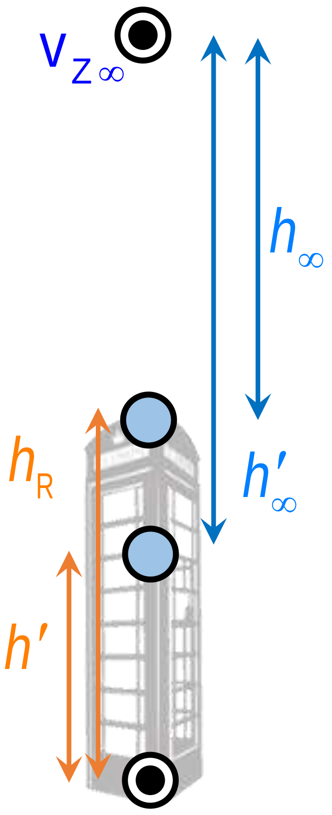
Cross ratio (perspective transformation invariant)






```
h_prime = norm(p1-p2);
h_R = norm(p1-p3);
h_prime_inf = norm(p4-p2);
h_inf = norm(p4-p3);

H = H_R * h_prime * h_prime_inf / h_R / h_inf
```



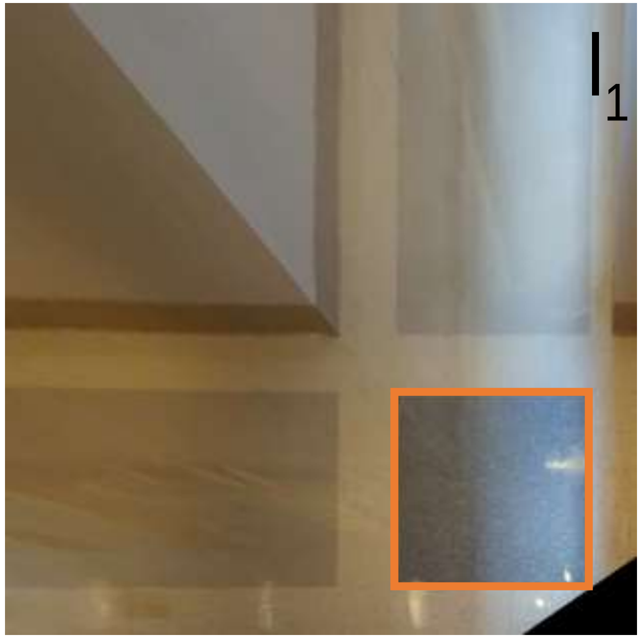
H = 1.6779	Ground truth: 1.7m

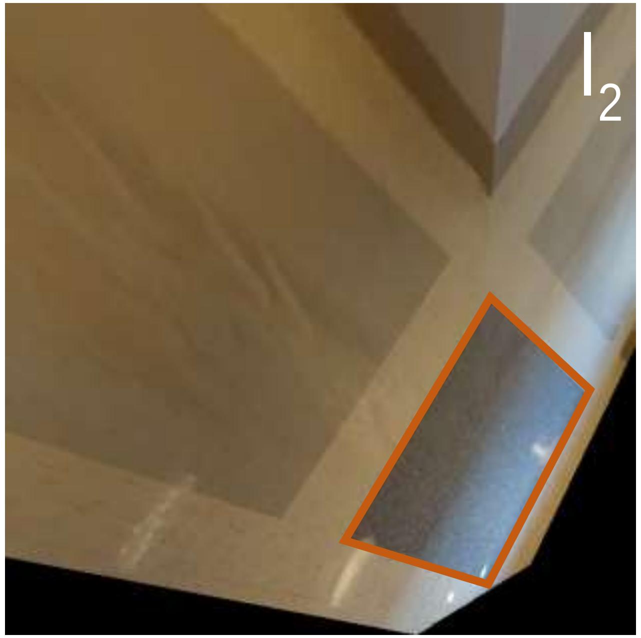
$$H = H_R \frac{h' h_\infty}{h_R h'_\infty}$$

ComputeHeightFromCrossRatio.m

Homography

Perspective Transform (Homography)

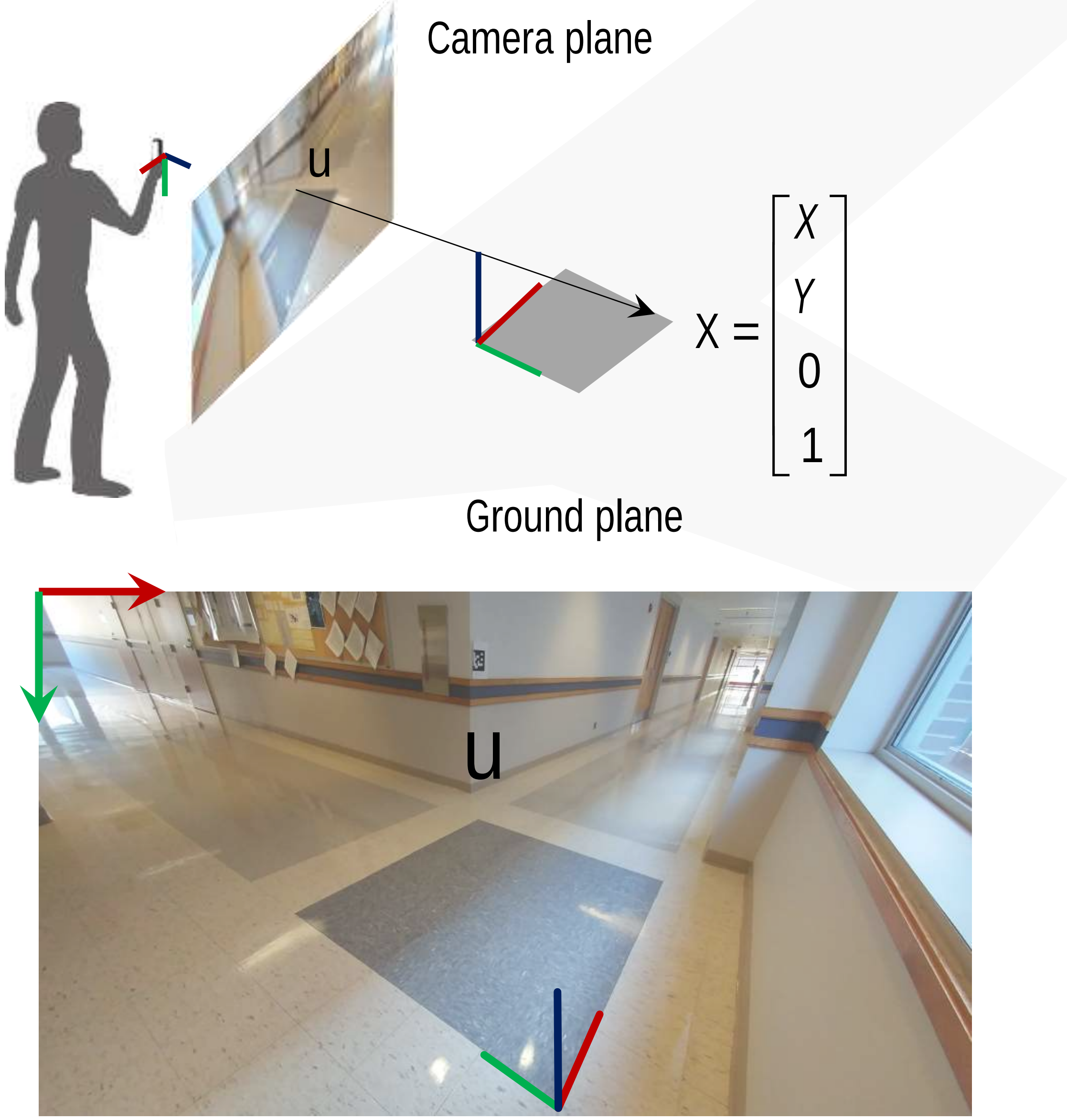




$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix} = \mathrm{H} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

: General form of plane to plane linear mapping

Perspective Transform (Homography)



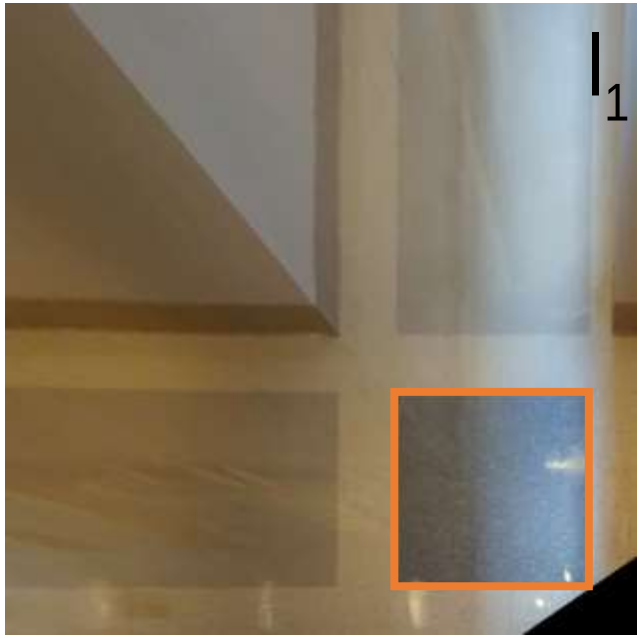
$$\underbrace{\lambda u}_{\text{Camera plane}} = K\begin{bmatrix} R & t \end{bmatrix}\underbrace{X}_{\text{Ground plane}}$$

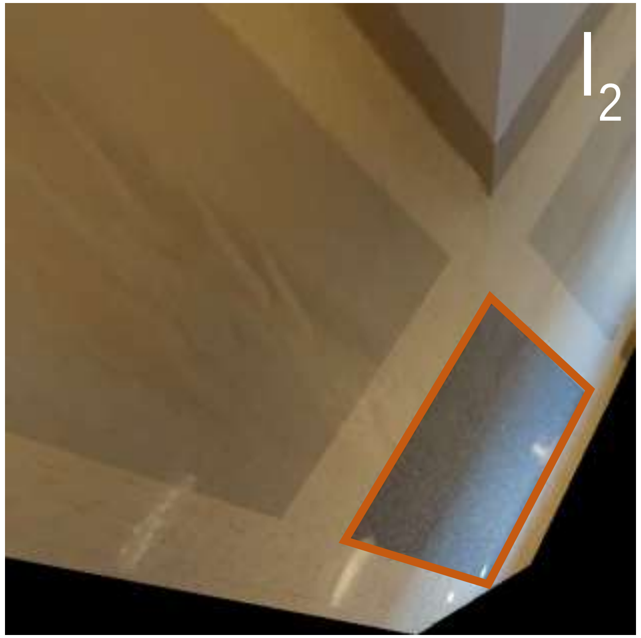
$$\longrightarrow \lambda u = K\begin{bmatrix} r_1 & r_2 & r_3 & t \end{bmatrix}\begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}$$

$$\longrightarrow \lambda u = K\begin{bmatrix} r_1 & r_2 & t \end{bmatrix}\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$\longrightarrow \lambda\begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix} = H\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Perspective Transform (Homography)

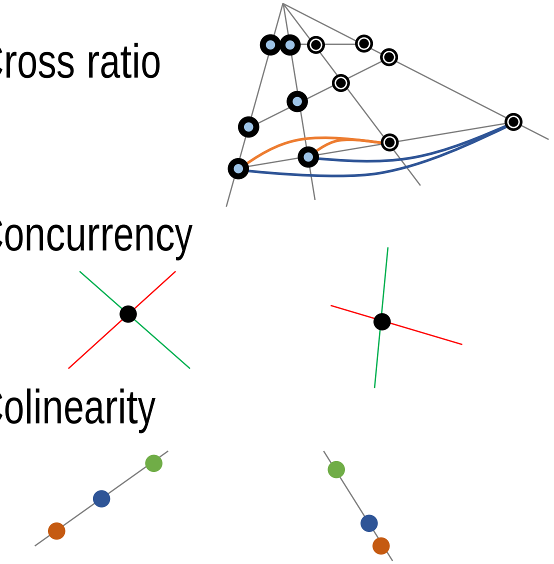




$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix} = \mathrm{H} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Invariant properties

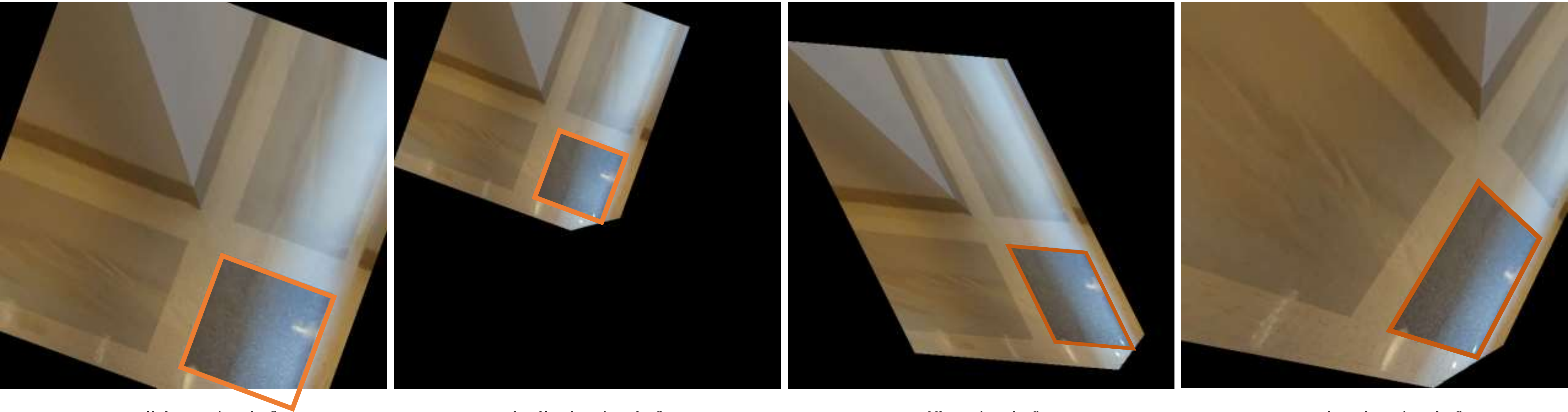
- Cross ratio
- Concurrency
- Colinearity



Degree of freedom

8 (9 variables – 1 scale)

Hierarchy of Transformations



Euclidean (3 dof)
- Length
- Angle
- Area

$$\begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ & & 1 \end{bmatrix}$$

Similarity (4 dof)
- Length ratio
- Angle

$$\begin{bmatrix} \alpha\cos\theta & -\alpha\sin\theta & t_x \\ \alpha\sin\theta & \alpha\cos\theta & t_y \\ & & 1 \end{bmatrix}$$

Affine (6 dof)
- Parallelism
- Ratio of area
- Ratio of length

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Projective (8 dof)
- Cross ratio
- Concurrency
- Colinearity

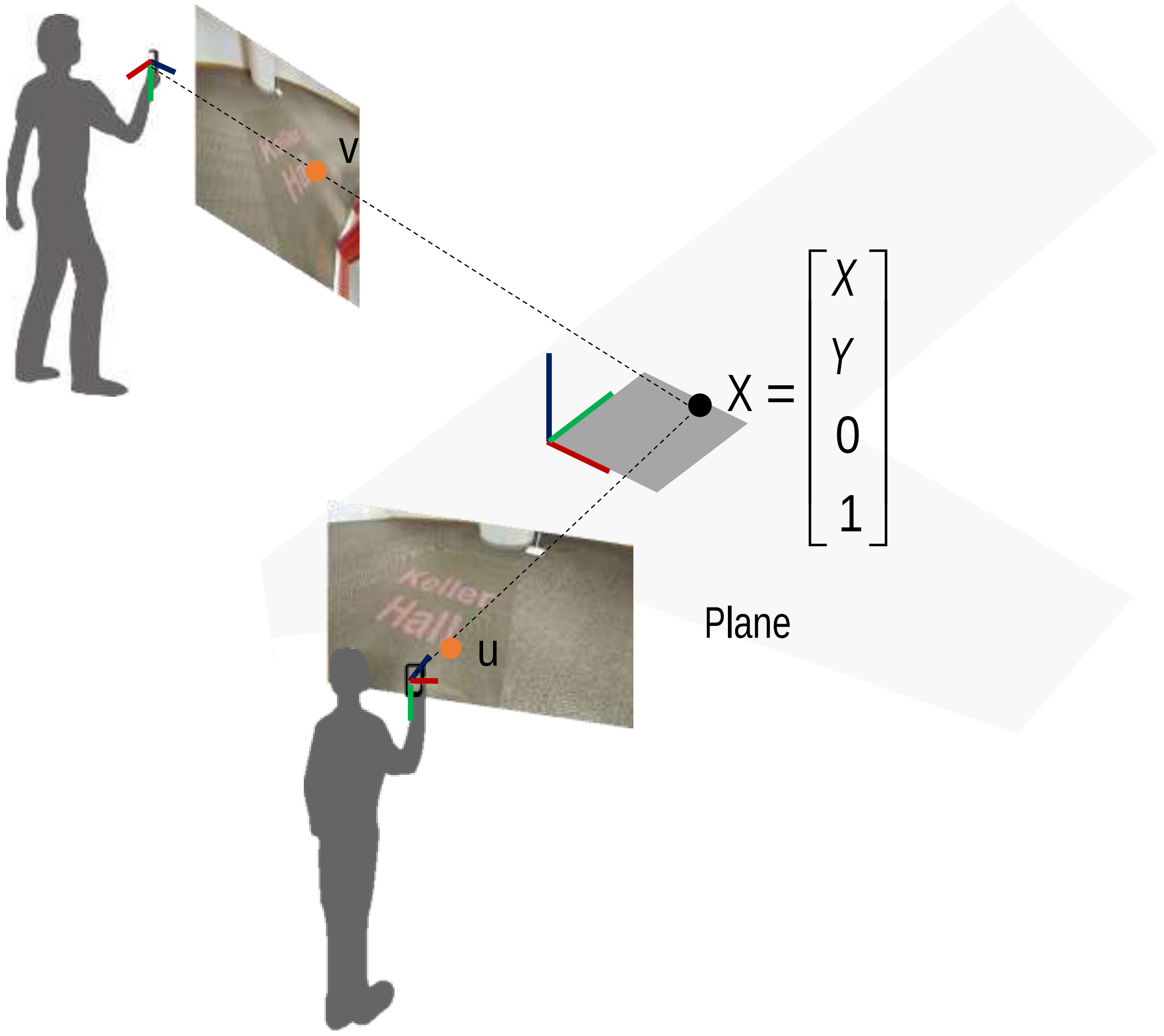
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix}$$

Image Transform via Plane

Keller entrance left

Keller entrance right

Image Transform via 3D Plane



$$\lambda \mathbf{u} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}\mathbf{X}$$

$$\longrightarrow \lambda \mathbf{u} = \mathbf{K}\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t} \end{bmatrix}\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$\mu \mathbf{v} = \mathbf{K}'\begin{bmatrix} \mathbf{r}_1' & \mathbf{r}_2' & \mathbf{t}' \end{bmatrix}\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

How are two image coordinates (u,v) related?

$$\longrightarrow \lambda \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t} \end{bmatrix}^{-1}\mathbf{K}^{-1}\mathbf{u} = \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \mu \begin{bmatrix} \mathbf{r}_1' & \mathbf{r}_2' & \mathbf{t}' \end{bmatrix}^{-1}\mathbf{K}'^{-1}\mathbf{v}$$

$$\alpha \mathbf{v} = \mathbf{K}'\begin{bmatrix} \mathbf{r}_1' & \mathbf{r}_2' & \mathbf{t}' \end{bmatrix}\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t} \end{bmatrix}^{-1}\mathbf{K}^{-1}\mathbf{u}$$

$$\alpha \mathbf{v} = \mathbf{H}\mathbf{u}$$

Image Transform by Pure 3D Rotation

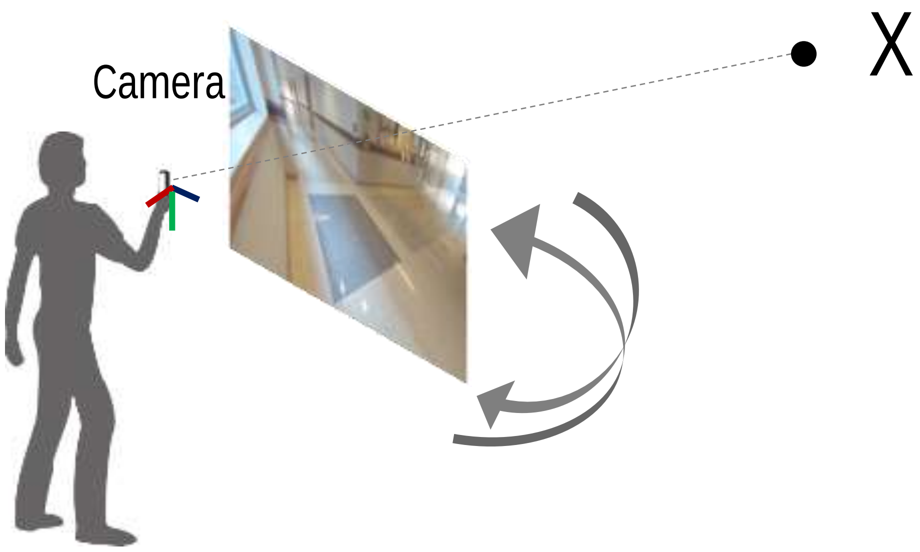
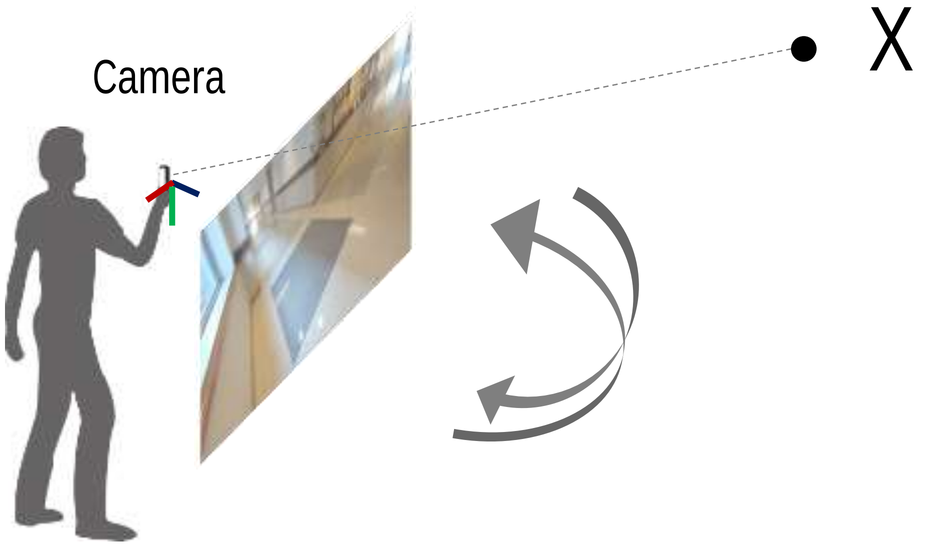


Image Transform by Pure 3D Rotation



$$\lambda_1 u = KX$$

$$\lambda_2 v = KRX$$

$$\longrightarrow X = \lambda_1 K^{-1} u = \lambda_2 R^T K^{-1} v$$

$$\longrightarrow \lambda v = KRK^{-1} u$$

$$\longrightarrow H = KRK^{-1}$$

$$\longrightarrow R = K^{-1} H K$$

Homography Computation

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\longrightarrow \begin{array}{l} h_{11}u_x + h_{12}u_y + h_{13} + h_{31}u_x v_x + h_{32}u_y v_x + h_{33}v_x = 0 \\ h_{21}u_x + h_{22}u_y + h_{23} + h_{31}u_x v_y + h_{32}u_y v_y + h_{33}v_y = 0 \end{array}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Unknowns: $h_{11}, \cdots, h_{33}$

Equations: 2 per correspondence

Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\longrightarrow \begin{aligned} h_{11}u_x + h_{12}u_y + h_{13} - h_{31}u_x v_x - h_{32}u_y v_x - h_{33}v_x = 0 \\ h_{21}u_x + h_{22}u_y + h_{23} - h_{31}u_x v_y - h_{32}u_y v_y - h_{33}v_y = 0 \end{aligned}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\rightarrow \quad h_{11}u_x + h_{12}u_y + h_{13} - h_{31}u_x v_x - h_{32}u_y v_x - h_{33}v_x = 0$$
$$h_{21}u_x + h_{22}u_y + h_{23} - h_{31}u_x v_y - h_{32}u_y v_y - h_{33}v_y = 0$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\rightarrow \quad \underbrace{\begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix}}_{\mathbf{A}\ (2\text{x}9)} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

Homography Computation

How many correspondences are needed?

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

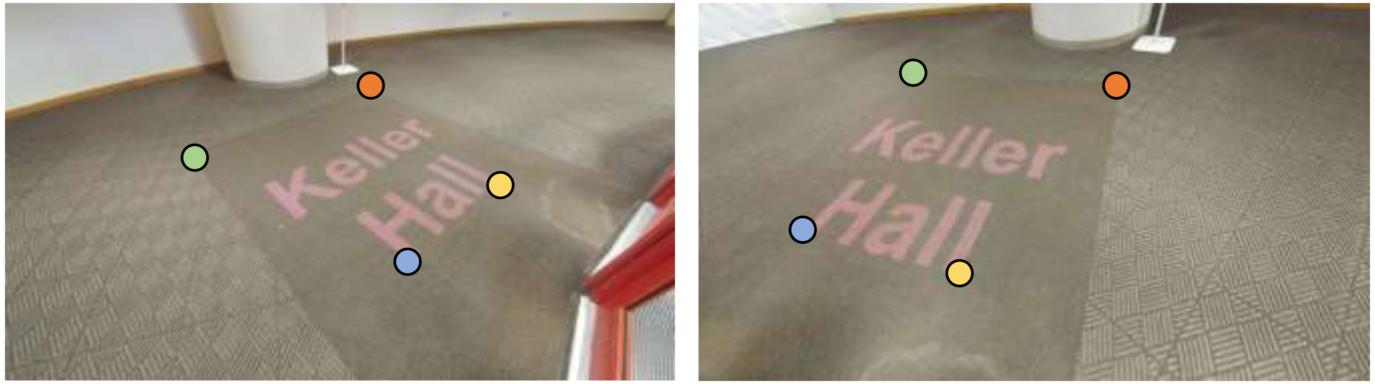
$$\underbrace{\begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0}$$

$$A \; x = 0$$

2x9

Homography Computation

How many correspondences are needed? 4



$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

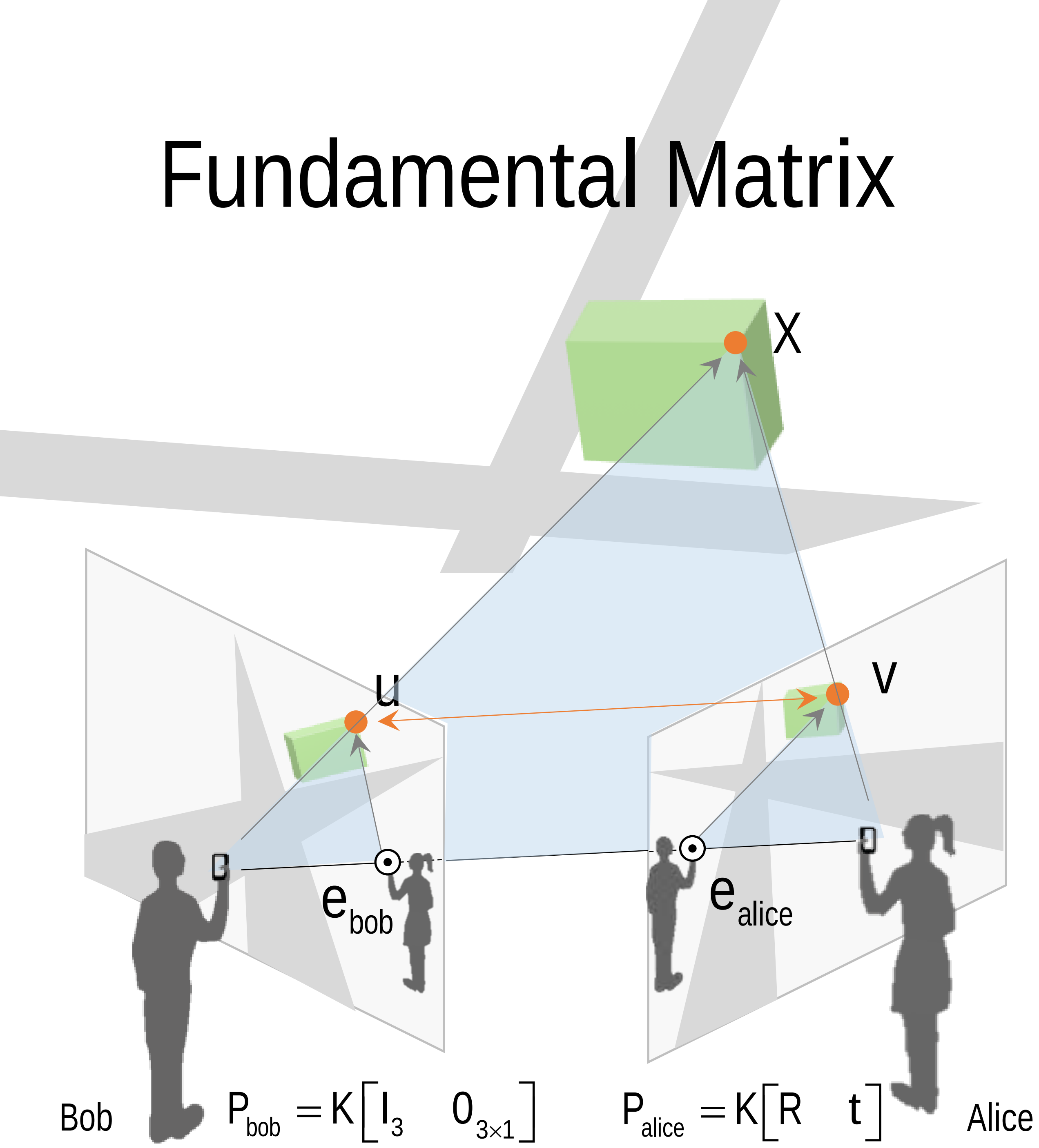
$$\begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{A}_{8\text{x}9} \, x = 0$$

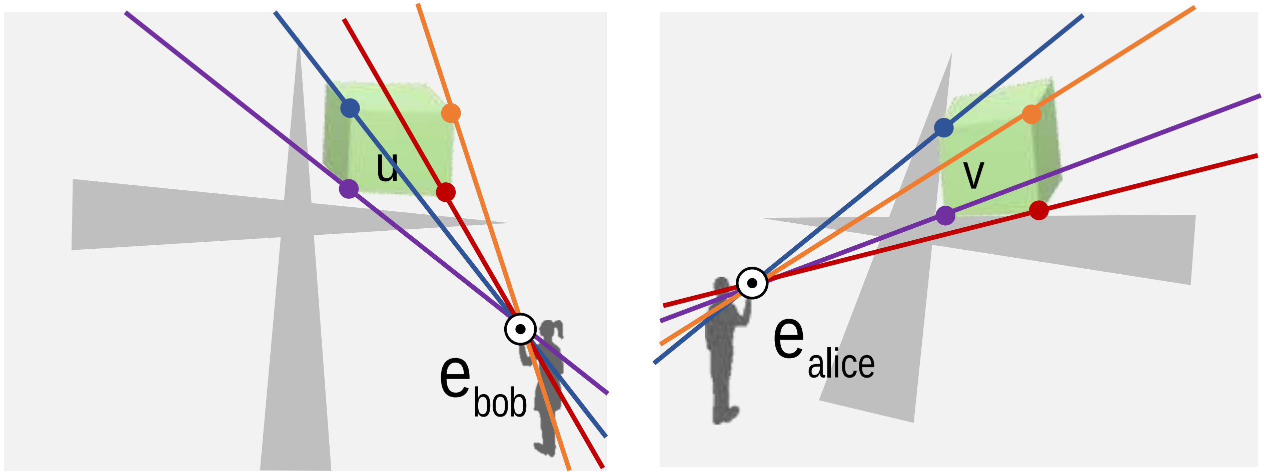
$$x = V_{:end} = \mathrm{null}(A)$$

Epipolar Geometry

Fundamental Matrix



$P_{bob} = K[I_3 \quad 0_{3\times1}]$ $P_{alice} = K[R \quad t]$



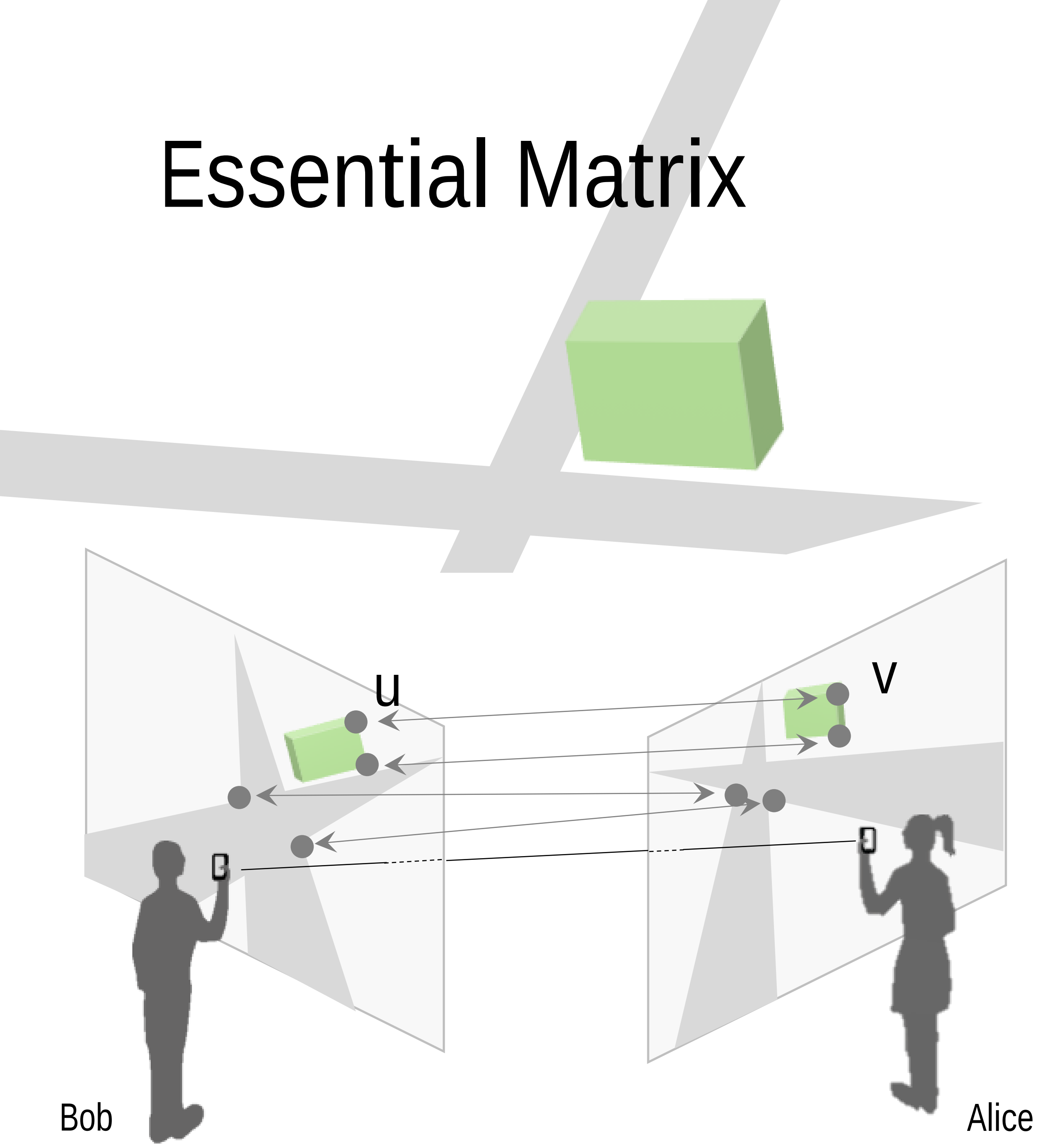
Bob's image Alice's image

Properties of Fundamental Matrix

- Transpose: if F is for P_{bob}, P_{alice}, then F^T is for P_{alice}, P_{bob}.
- Epipolar line: $l_u = Fu \quad l_v = F^T v$
- Epipole: $Fe_{bob} = 0 \quad F^T e_{alice} = 0$
- rank(F)=2: degree of freedom 9 (3x3 matrix)-1 (scale)-1 (rank)=7

$$F = K^{-T} \underbrace{[t]_\times R K^{-1}}_{\text{rank 2 matrix}}$$

Essential Matrix



Essential Matrix:

$$F = F(R, t)$$

$$= K^{-T}[t]_{\times} RK^{-1} = K^{-T}EK^{-1}$$

$$\rightarrow \underbrace{E = K^{T}FK}_{\text{Calibrated fundamental matrix}} \qquad \text{where} \quad E = [t]_{\times} R$$

Property of essential matrix:

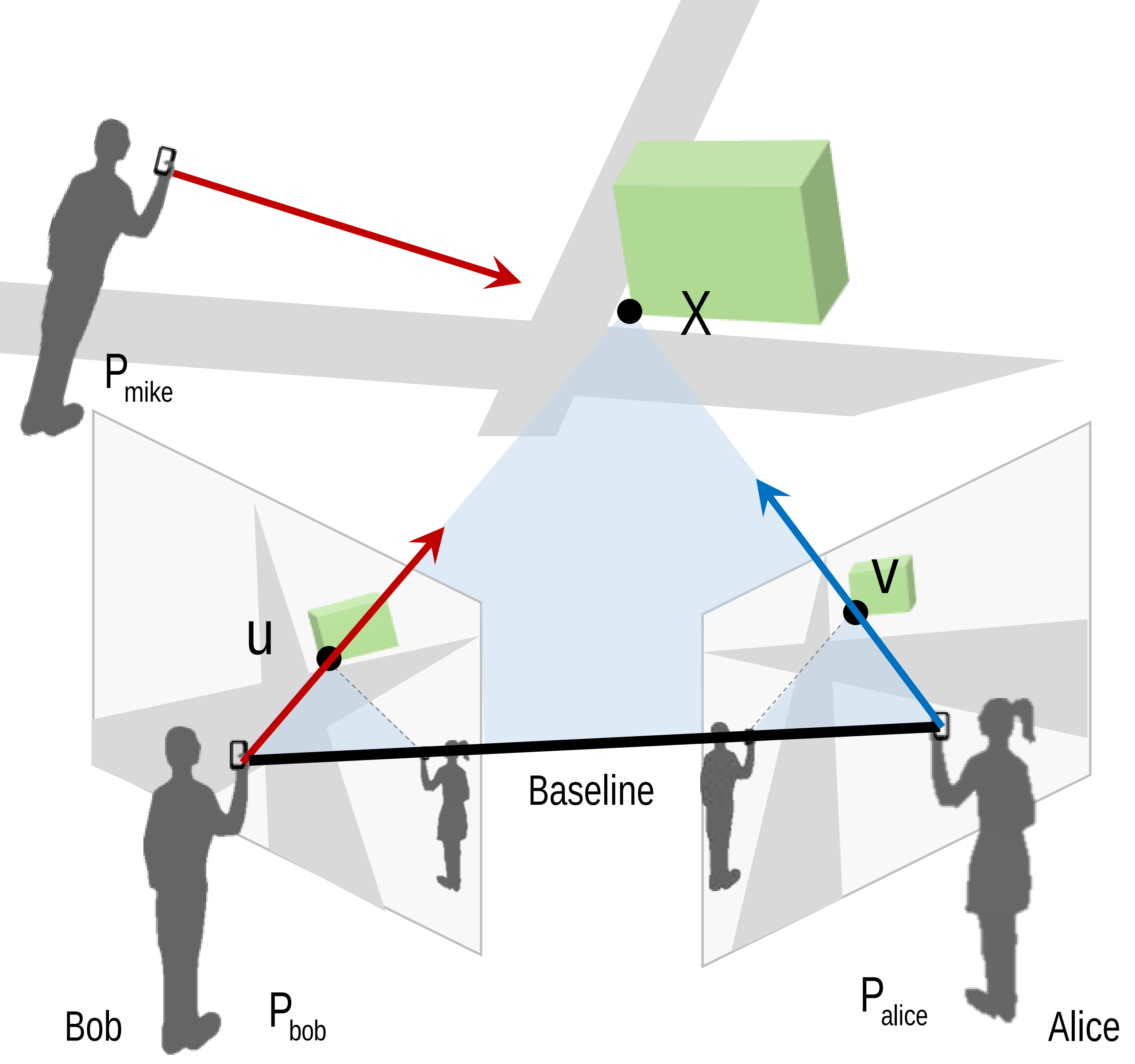
$$E = UDV^{T} = U\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}V^{T}$$

HW #4 RANSAC Fundamental matrix

Algorithm 1 GetInliersRANSAC

$n \leftarrow 0$
for $i = 1 : M$ **do**
  Choose 8 correspondences, $\mathbf{u}_r$ and $\mathbf{v}_r$, randomly from $\mathbf{u}$ and $\mathbf{v}$.
  $\mathbf{F}_r$ `= ComputeFundamentalMatrix(`$\mathbf{u}_r$`, `$\mathbf{v}_r$`)`
  Compute the number of inliers, n_r, with respect to `F`.
  if $n_r > n$ **then**
    $n \leftarrow n_r$
    $\mathbf{F} = \mathbf{F}_r$
  end if
end for

Triangulation



General camera pose

$$\lambda_1 \begin{bmatrix}u\\1\end{bmatrix} = P_{bob}\begin{bmatrix}X\\1\end{bmatrix}$$

Two 3D vectors are parallel.

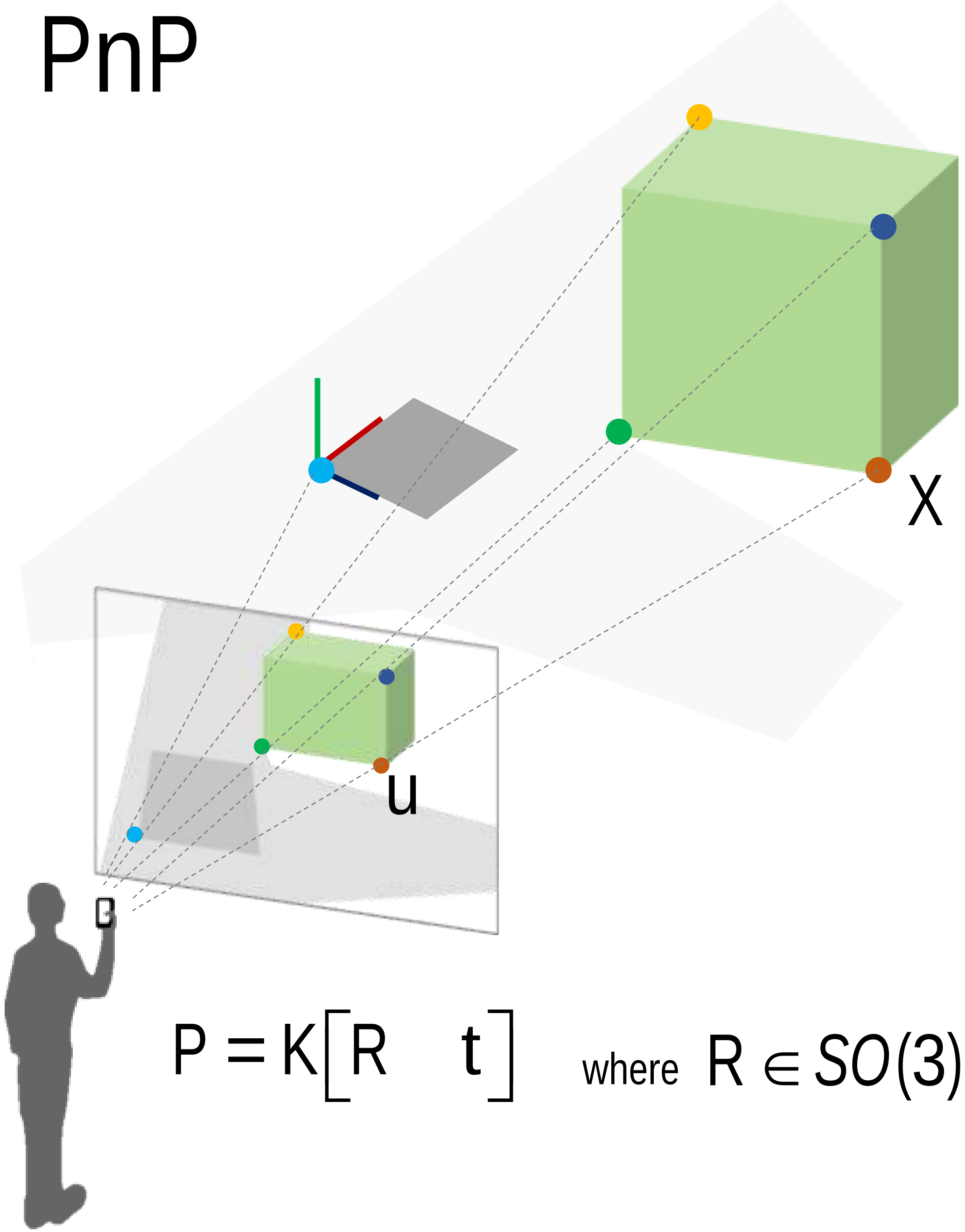
$$\rightarrow \begin{bmatrix}u\\1\end{bmatrix}\times P_{bob}\begin{bmatrix}X\\1\end{bmatrix}=0$$

$$\rightarrow \begin{array}{c}\begin{bmatrix}u\\1\end{bmatrix}_\times P_{bob}\\ \begin{bmatrix}v\\1\end{bmatrix}_\times P_{alice}\\ \begin{bmatrix}w\\1\end{bmatrix}_\times P_{mike}\end{array}\begin{bmatrix}X\\1\end{bmatrix}=0$$

: Knowns

: Unknowns

PnP



$$P = K\begin{bmatrix} R & t \end{bmatrix} \quad \text{where } R \in SO(3)$$

3D-2D correspondence: $u \leftrightarrow X$

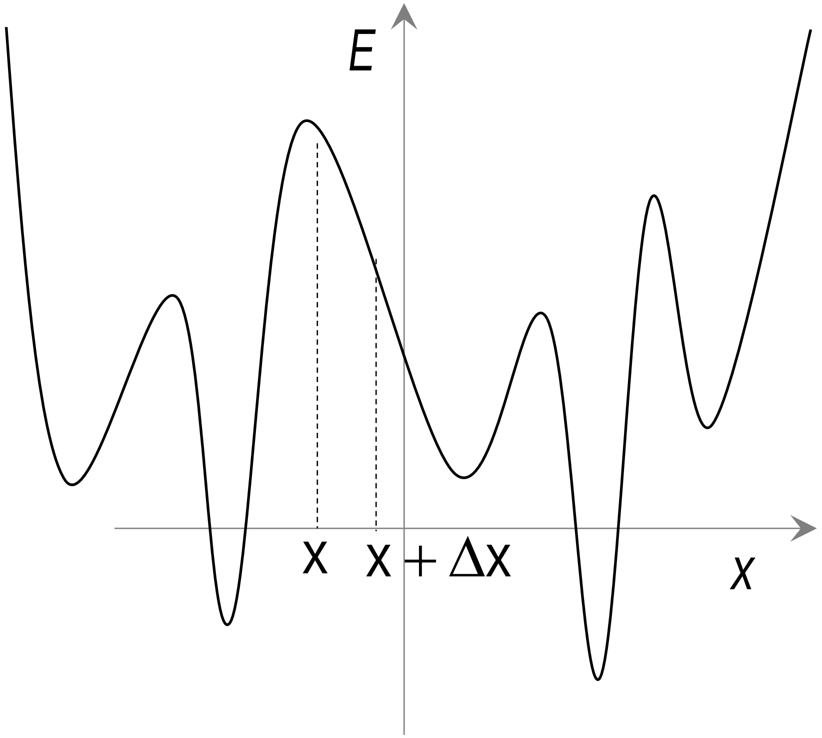
$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & & & & & -u_1^x X & -u_1^x Y & -u_1^x Z & -u_1^x \\ & & & & X_1 & Y_1 & Z_1 & 1 & -u_1^y X & -u_1^y Y & -u_1^y Z & -u_1^y \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_m & Y_m & Z_m & 1 & & & & & -u_m^x X & -u_m^x Y & -u_m^x Z & -u_m^x \\ & & & & X_m & Y_m & Z_m & 1 & -u_m^y X & -u_m^y Y & -u_m^y Z & -u_m^y \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$2m$x12

Bundle Adjustment

Nonlinear System



Find x such that the following equation is satisfied:

$$\frac{\partial f(x)}{\partial x}^{T} f(x) = \frac{\partial f(x)}{\partial x}^{T} b \qquad \text{How?}$$

Strategy: Given x, move Δx such that $E(x+\Delta x) \leq E(x)$

Taylor expansion:

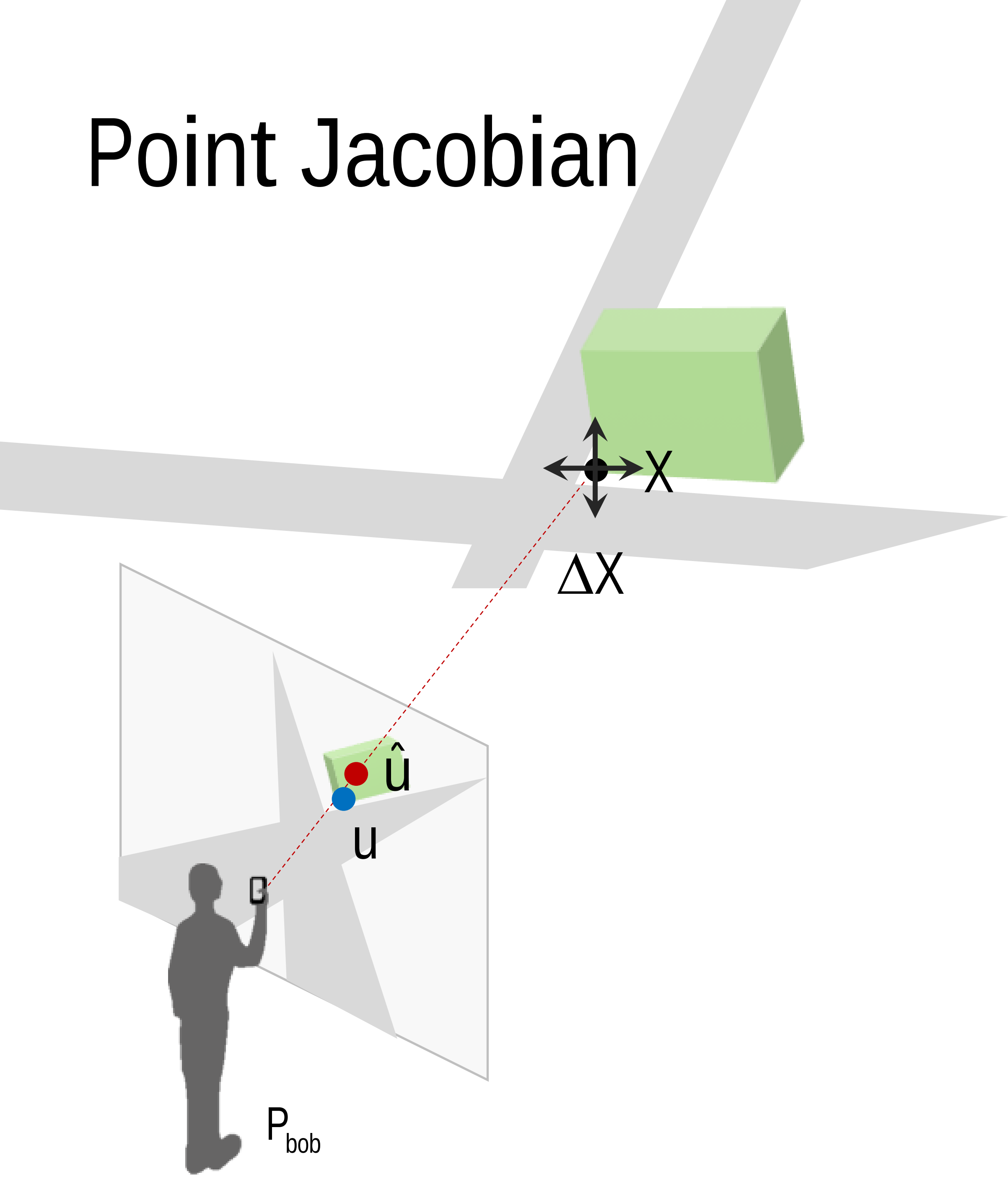
$$f(x+\Delta x) = f(x) + \frac{\partial f(x)}{\partial x}\Delta x + \text{H.O.T.}$$

$$\rightarrow \frac{\partial f(x)}{\partial x}^{T}\left(f(x) + \frac{\partial f(x)}{\partial x}\Delta x\right) = \frac{\partial f(x)}{\partial x}^{T} b$$

$$\rightarrow \frac{\partial f(x)}{\partial x}^{T}\frac{\partial f(x)}{\partial x}\Delta x = \frac{\partial f(x)}{\partial x}^{T}\left(b - f(x)\right)$$

Cf.) $x = \left[A^{T} A\right]^{-1} A^{T} b$

Point Jacobian



Black: given variables

Red: unknowns

$$E_{geom} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = KR(X - C)$$

$$\rightarrow \frac{\partial}{\partial X}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = KR$$

$$f(X) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} \rightarrow \frac{\partial f(X)}{\partial X} = \frac{\partial}{\partial X}\begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \dfrac{w\frac{\partial u}{\partial X} - u\frac{\partial w}{\partial X}}{w^2} \\ \dfrac{v\frac{\partial u}{\partial X} - v\frac{\partial w}{\partial X}}{w^2} \end{bmatrix}$$

$$\Delta x = \left(\frac{\partial f(x)}{\partial x}^{T}\frac{\partial f(x)}{\partial x}\right)^{-1}\frac{\partial f(x)}{\partial x}^{T}\left(b - f(x)\right)$$

Algorithm 3 Nonlinear Point Refinement

$\mathbf{b} = \begin{bmatrix} \mathbf{u}_1^\mathsf{T} & \mathbf{u}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$

for $j = 1 : \text{nIters}$ **do**

  Build point Jacobian, $\frac{\partial f(\mathbf{X})_j}{\partial \mathbf{X}}$.

  Compute $f(\mathbf{X})$.

  $\Delta \mathbf{X} = \left(\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}}^\mathsf{T} \frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} + \boxed{\lambda \mathbf{I}} \right)^{-1} \frac{\partial f(\mathbf{X})}{\partial \mathbf{X}}^\mathsf{T} (\mathbf{b} - f(\mathbf{X}))$

  $\mathbf{X} = \mathbf{X} + \Delta \mathbf{X}$

end for

$$\frac{\partial f(X)}{\partial X} = \frac{\partial}{\partial X}\begin{bmatrix} \dfrac{u}{w} \\ \dfrac{v}{w} \end{bmatrix} = \begin{bmatrix} \dfrac{w\dfrac{\partial u}{\partial X} - u\dfrac{\partial w}{\partial X}}{w^2} \\ \dfrac{v\dfrac{\partial u}{\partial X} - v\dfrac{\partial w}{\partial X}}{w^2} \end{bmatrix}$$

$$\Delta x = \left(\frac{\partial f(x)}{\partial x}^\mathsf{T} \frac{\partial f(x)}{\partial x} \right)^{-1} \frac{\partial f(x)}{\partial x}^\mathsf{T} \left(b - f(x) \right)$$

Damping factor (Levenberg-Marquardt algorithm)

$$\Delta x = \left(\frac{\partial f(x)}{\partial x}^\mathsf{T} \frac{\partial f(x)}{\partial x} + \lambda I \right)^{-1} \frac{\partial f(x)}{\partial x}^\mathsf{T} \left(b - f(x) \right)$$

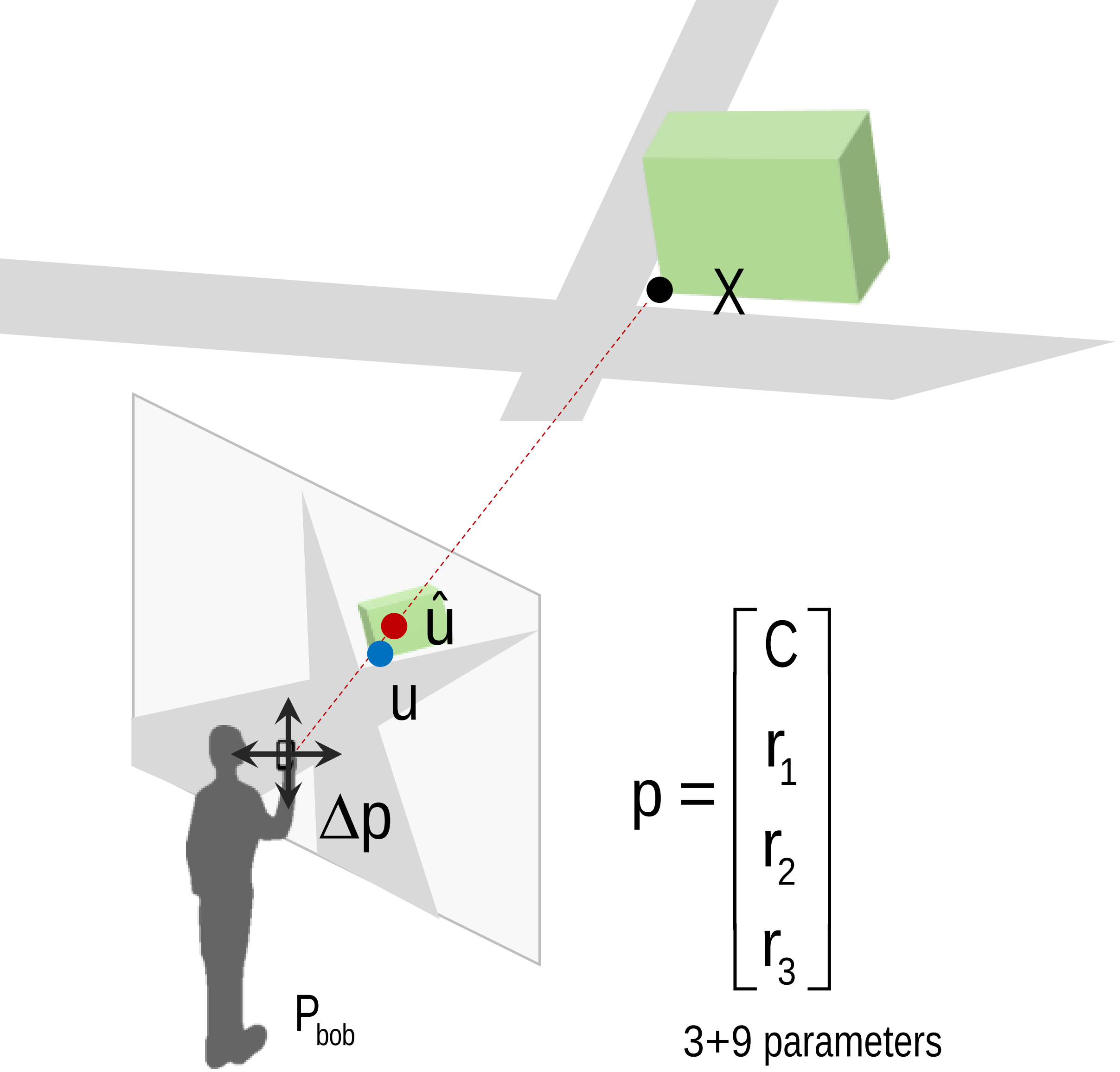
Camera Jacobian

Black: given variables
Red: unknowns

$$E_{geom} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = KR(X - C)$$

$$\rightarrow \quad \frac{\partial}{\partial C}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = -KR$$

$$\rightarrow \quad \frac{\partial}{\partial R}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} K_{11}(X-C) & 0_{1\times 3} & K_{13}(X-C) \\ 0_{1\times 3} & K_{22}(X-C) & K_{23}(X-C) \\ 0_{1\times 3} & 0_{1\times 3} & (X-C) \end{bmatrix}$$



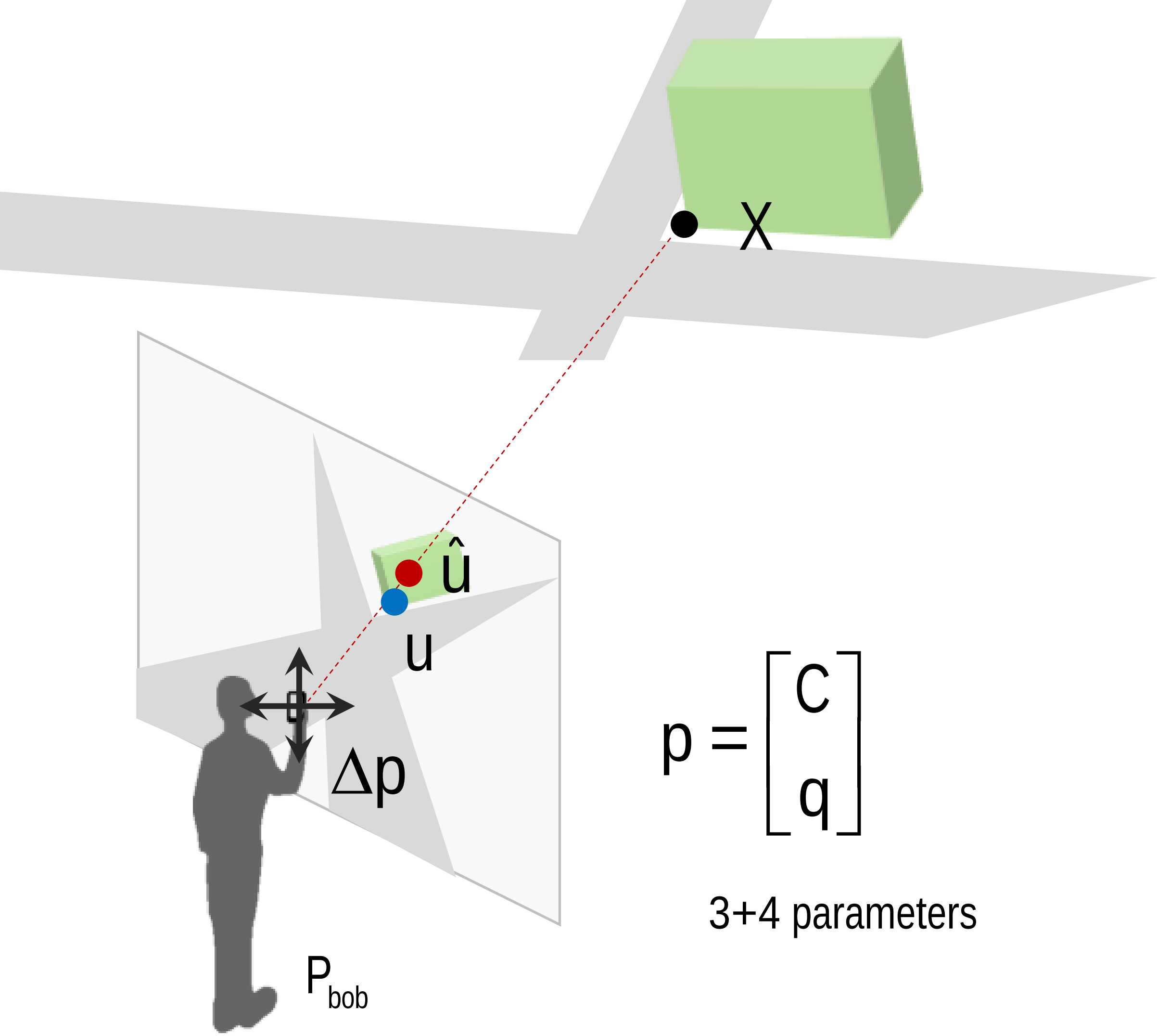
$$p = \begin{bmatrix} C \\ r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

3+9 parameters

$$\Delta p = \left(\frac{\partial f(p)^T}{\partial p}\frac{\partial f(p)}{\partial p}\right)^{-1}\frac{\partial f(p)^T}{\partial p}\left(b - f(p)\right)$$

Camera Jacobian

Black: given variables
Red: unknowns



$$p = \begin{bmatrix} C \\ q \end{bmatrix}$$

3+4 parameters

$$E_{geom} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = KR(X - C)$$

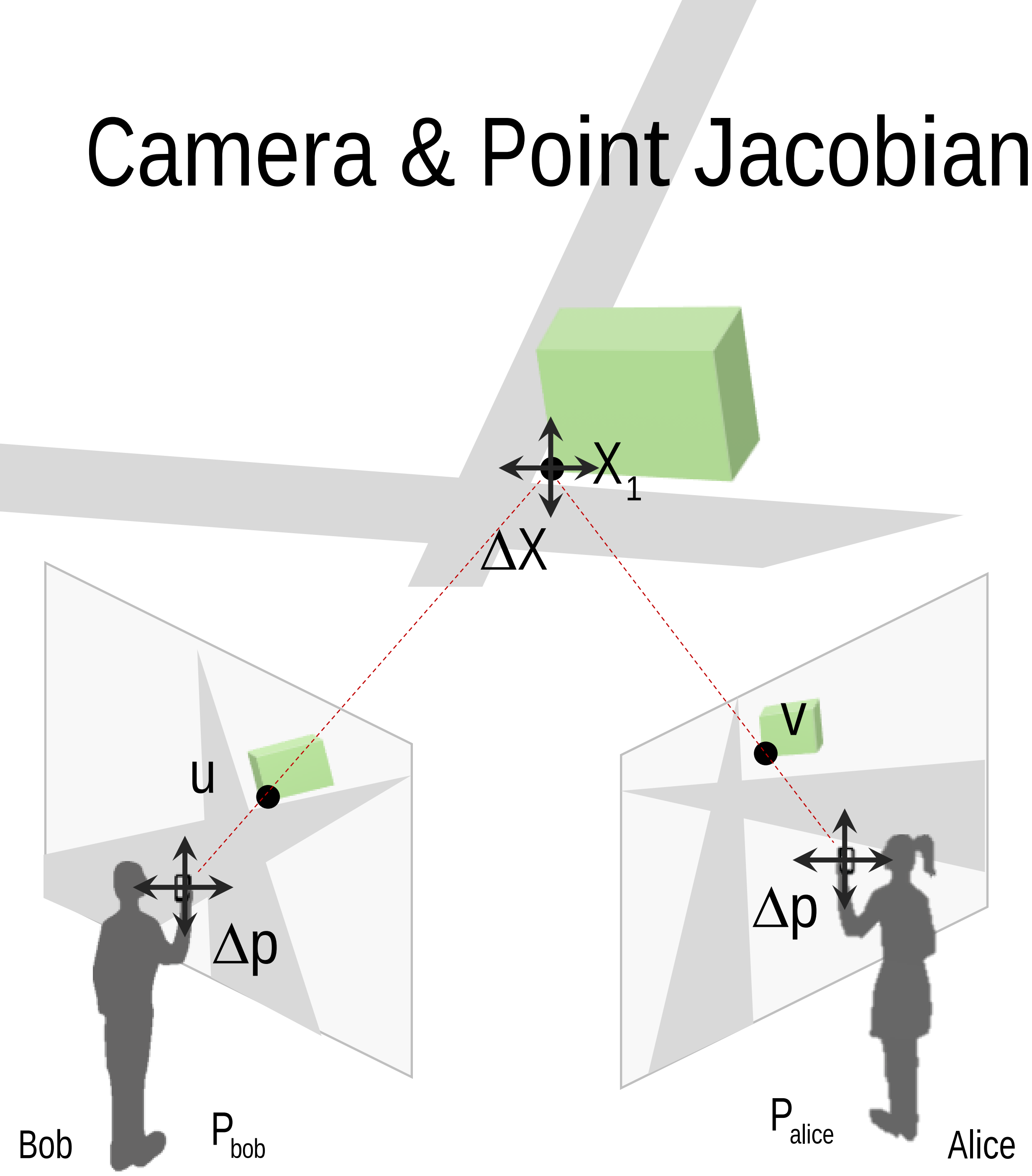
$$\rightarrow \quad \frac{\partial}{\partial C}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = -KR$$

$$\rightarrow \quad \frac{\partial}{\partial R}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} K_{11}(X-C) & 0_{1\times3} & K_{13}(X-C) \\ 0_{1\times3} & K_{22}(X-C) & K_{23}(X-C) \\ 0_{1\times3} & 0_{1\times3} & (X-C) \end{bmatrix}$$

$$\rightarrow \quad \frac{\partial}{\partial q}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{\partial}{\partial R}\begin{bmatrix} u \\ v \\ w \end{bmatrix} \underbrace{\frac{\partial R}{\partial q}}_{\text{Quaternion jacobian}} \quad \text{: Chain rule}$$

Black: given variables
Red: unknowns

Camera & Point Jacobian



$$J_{ij} = \begin{bmatrix} \dfrac{\partial f(p_j, X_i)}{\partial p_j} & \dfrac{\partial f(p_j, X_i)}{\partial X_i} \end{bmatrix} = \begin{bmatrix} J_{pij} & J_{Xij} \end{bmatrix}$$

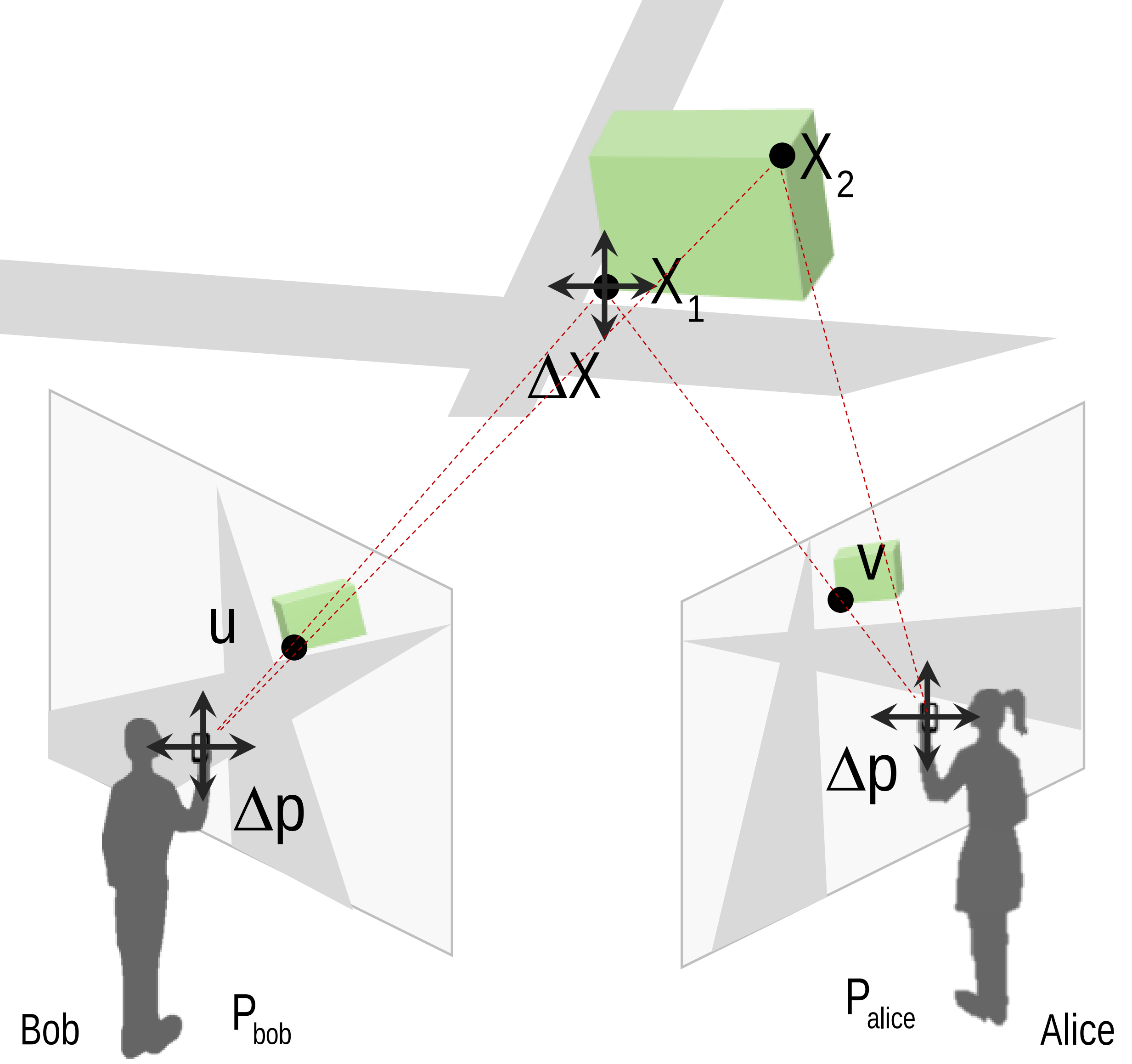
$$J = \begin{matrix} J_{p1,bob} & 0_{2\times 7} & J_{X1,bob} \\ 0_{2\times 7} & J_{p1,alice} & J_{X1,alice} \end{matrix}$$

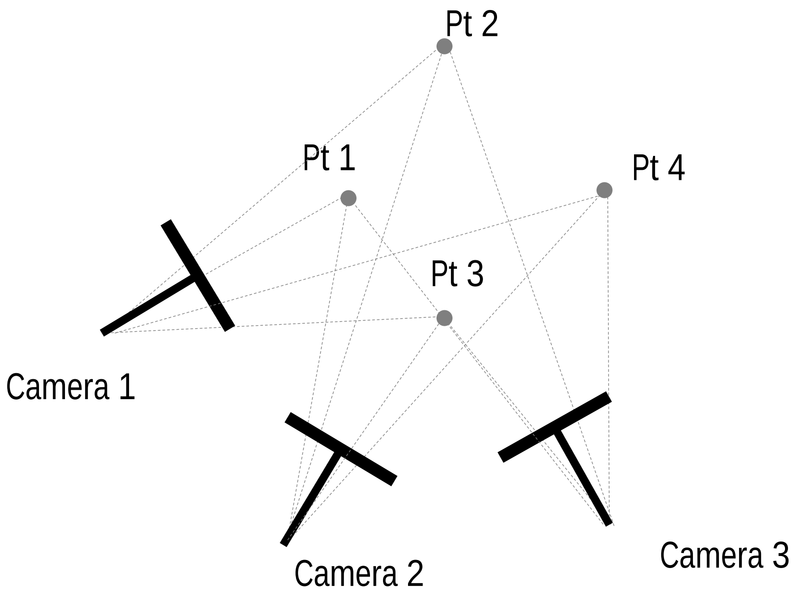
Black: given variables
Red: unknowns

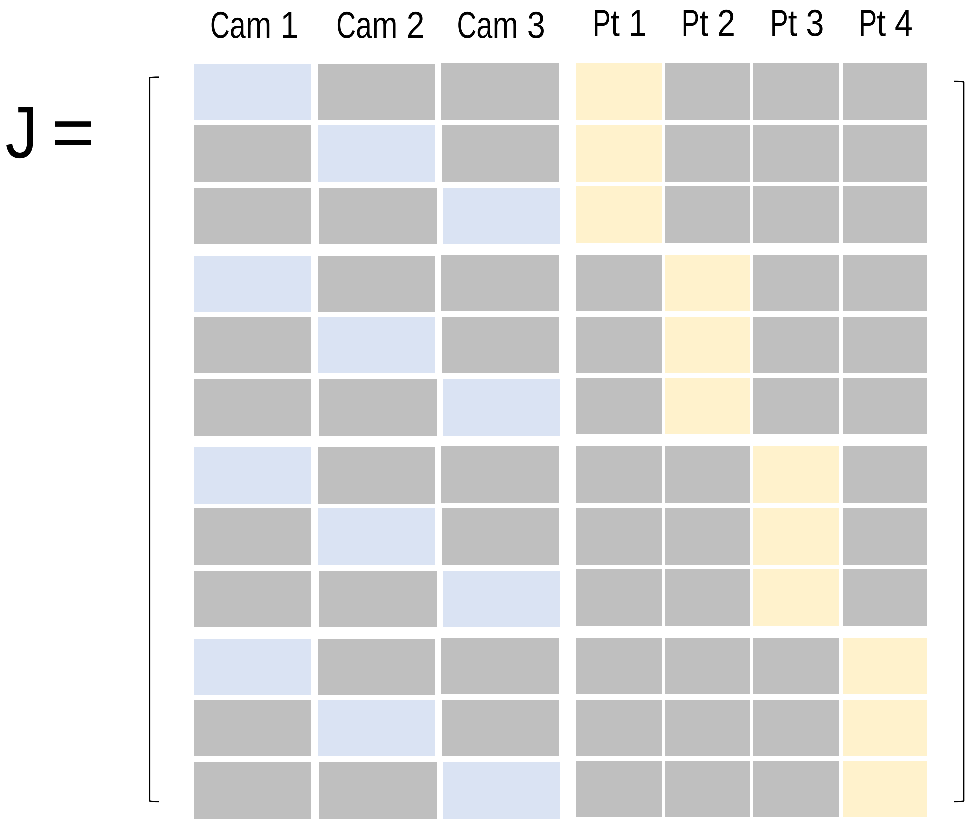
Camera & Point Jacobian

$$J_{ij} = \begin{bmatrix} \dfrac{\partial f(p_j, X_i)}{\partial p_j} & \dfrac{\partial f(p_j, X_i)}{\partial X_i} \end{bmatrix} = \begin{bmatrix} J_{pij} & J_{Xij} \end{bmatrix}$$

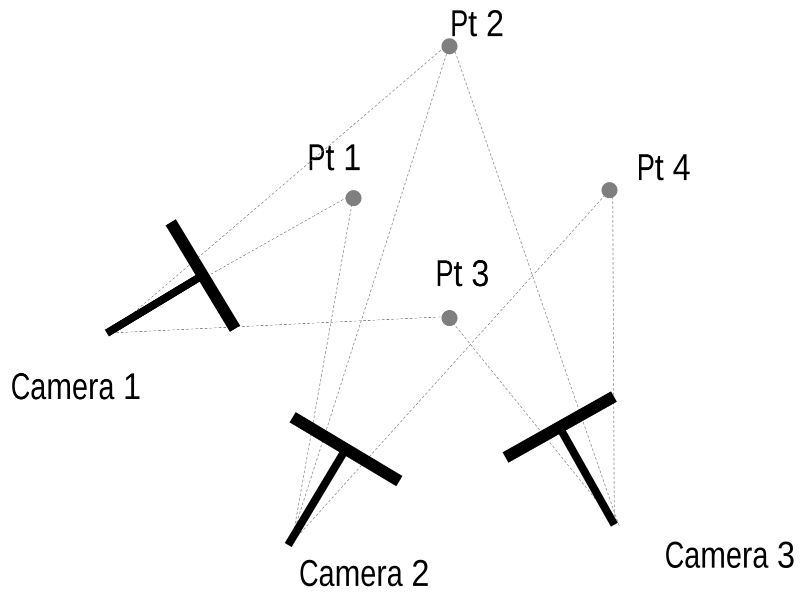
$$J = \begin{matrix} J_{p1,bob} & 0_{2\times7} & J_{X1,bob} & 0_{2\times3} \\ 0_{2\times7} & J_{p1,alice} & J_{X1,alice} & 0_{2\times3} \\ J_{p2,bob} & 0_{2\times7} & 0_{2\times3} & J_{X2,bob} \\ 0_{2\times7} & J_{p2,alice} & 0_{2\times3} & J_{X2,alice} \end{matrix}$$

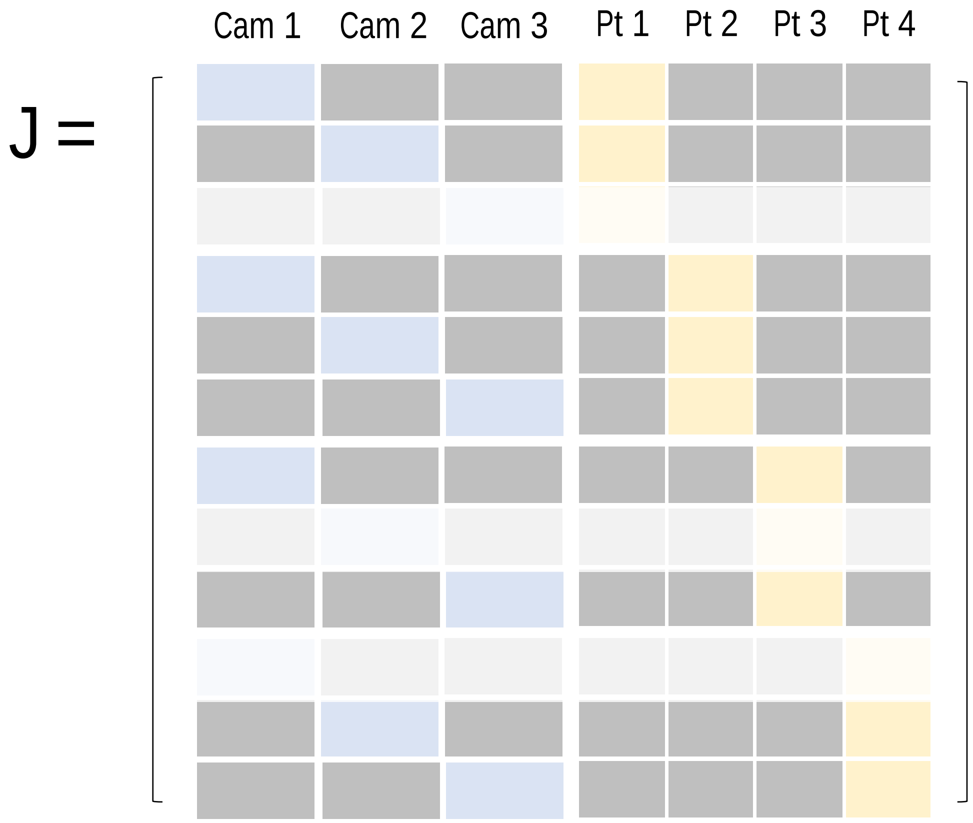






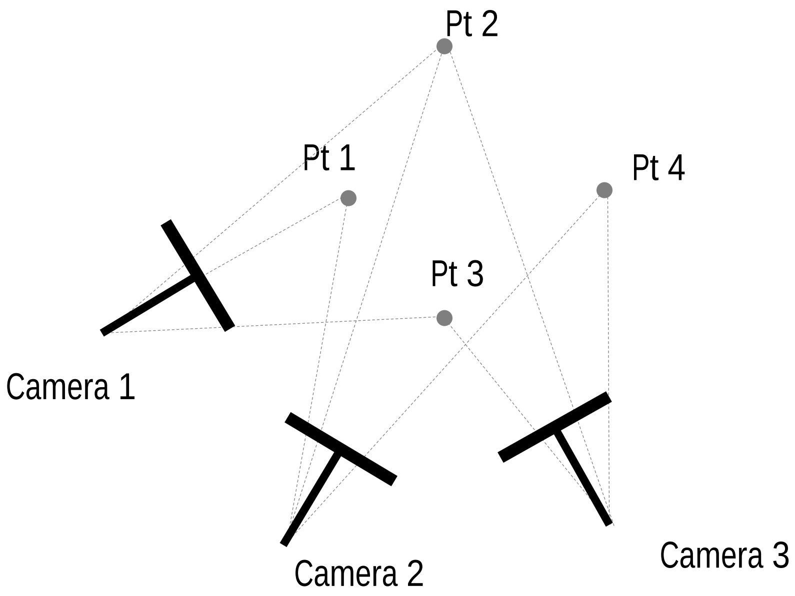
# of unknowns: 3x7+4x3
# of projections: 3x4

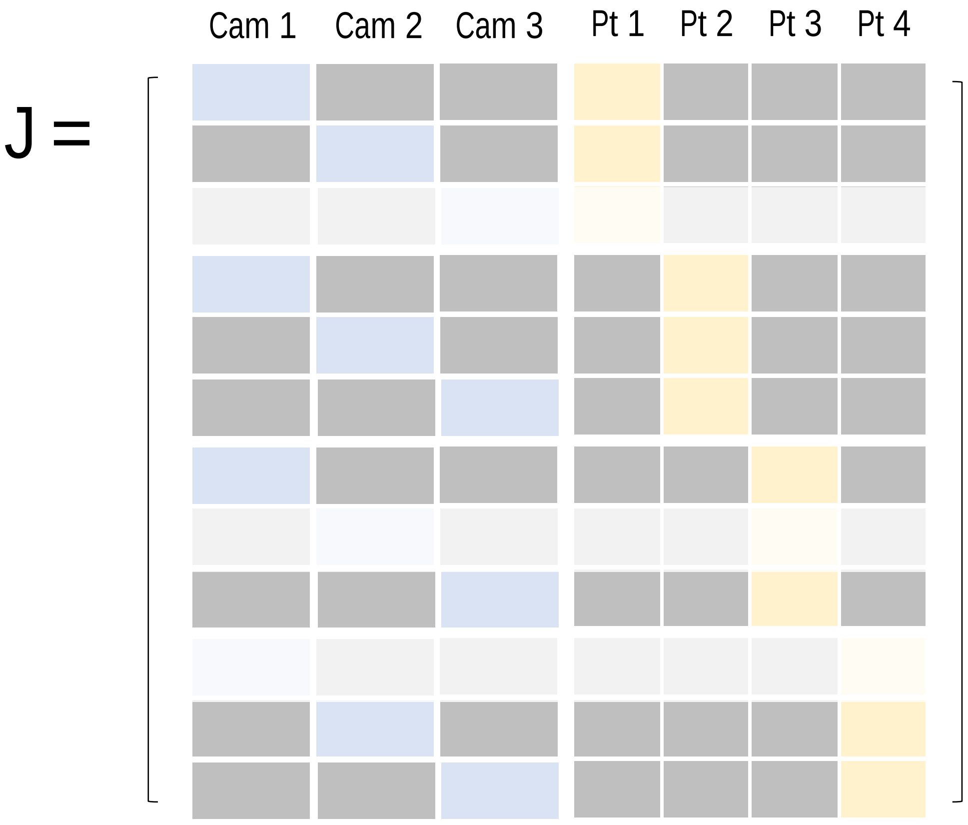




# of unknowns: 3x7+4x3

# of projections: 9 (not all points are visible from cameras)



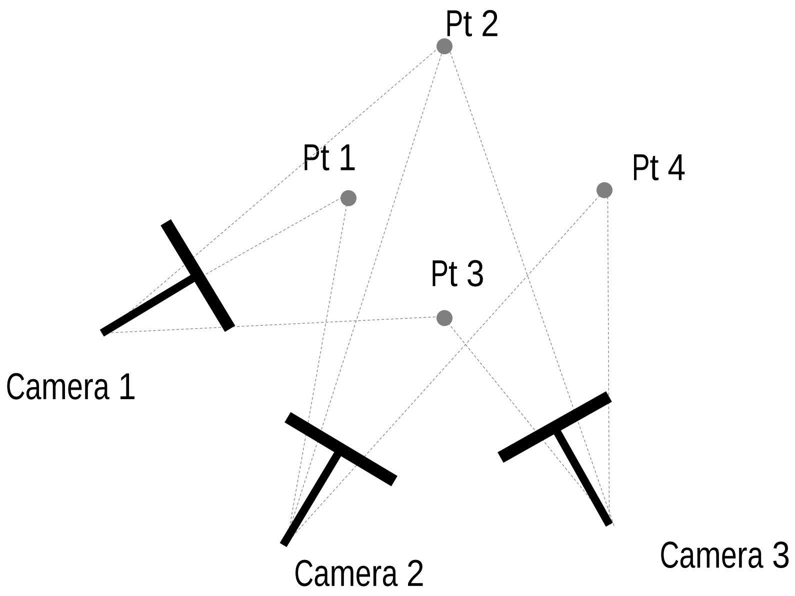


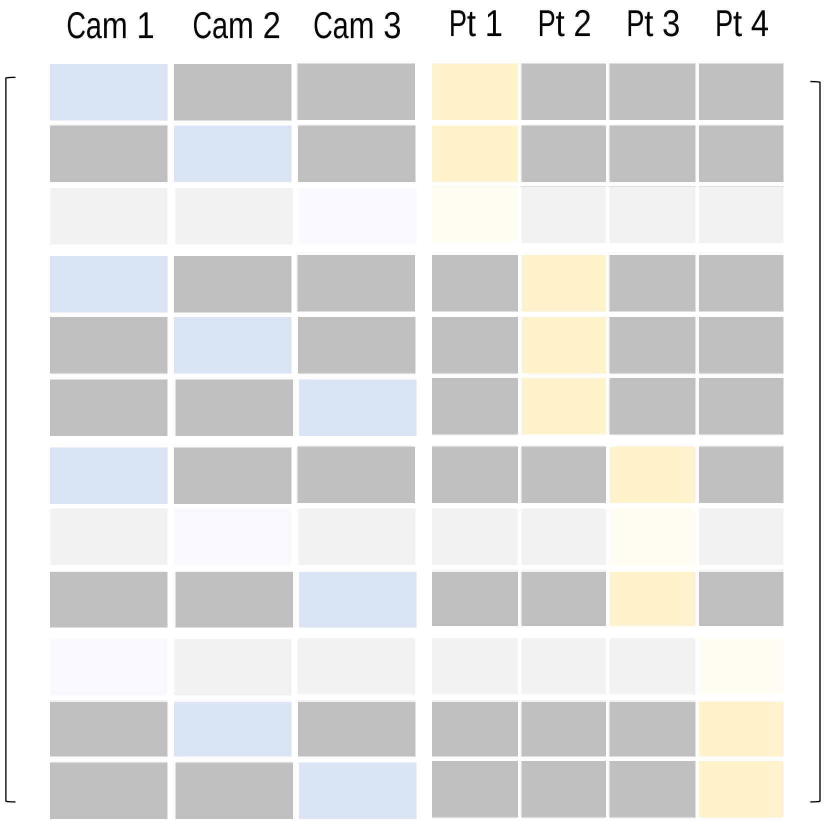
# of unknowns: 3x7+4x3

# of projections: 9 (not all points are visible from cameras)

$$\Delta x = \left(J^T J\right)^{-1} J^T \left(b - f(x)\right)$$

size of $J^T J$: 24x24



$J =$ 

# of unknowns: 3x7+4x3

# of projections: 9 (not all points are visible from cameras)

$$\Delta x = \underline{\left(J^T J\right)^{-1}} J^T \left(b - f(x)\right)$$

Main computational bottle neck

size of $J^T J$: 24x24