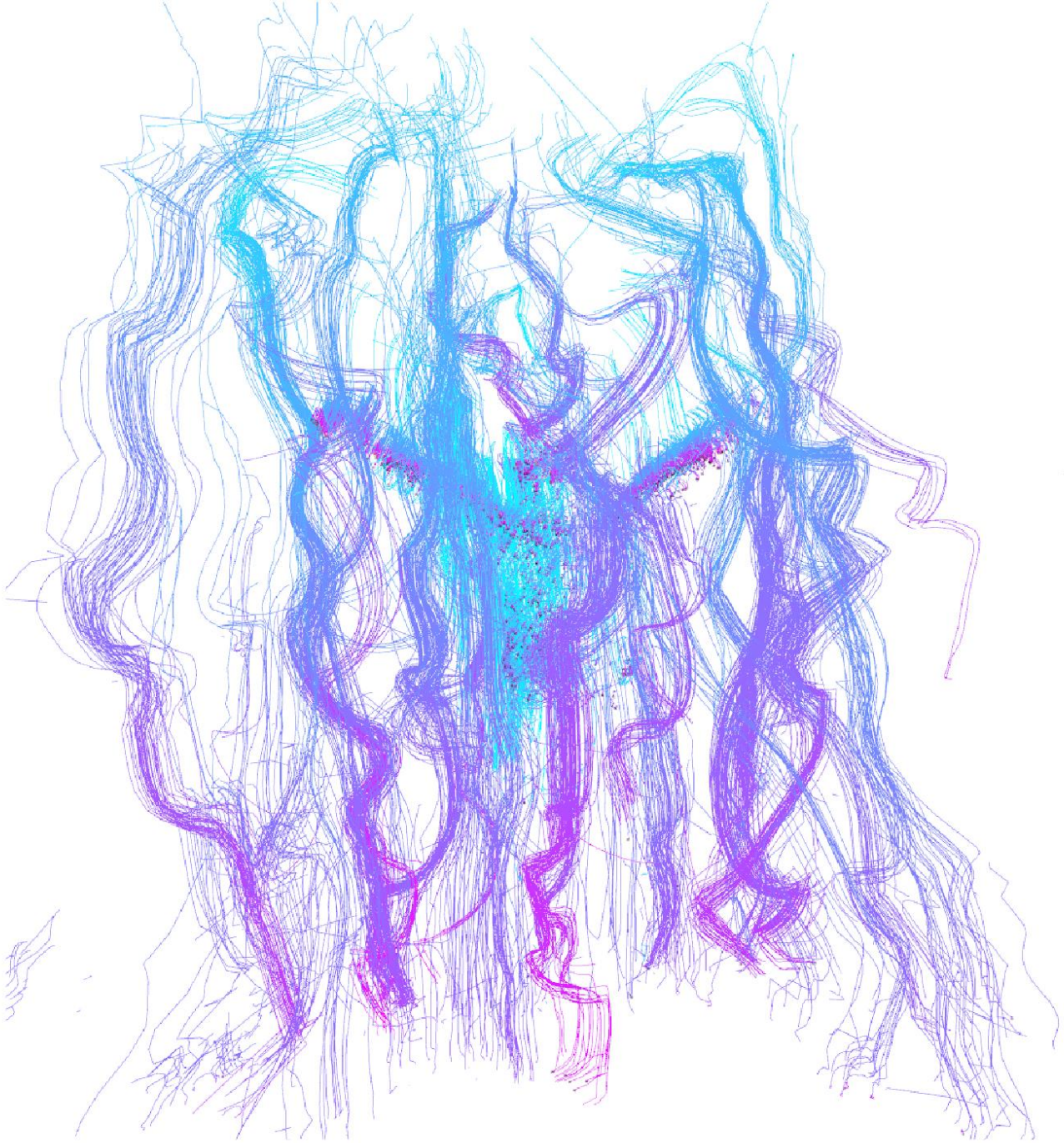
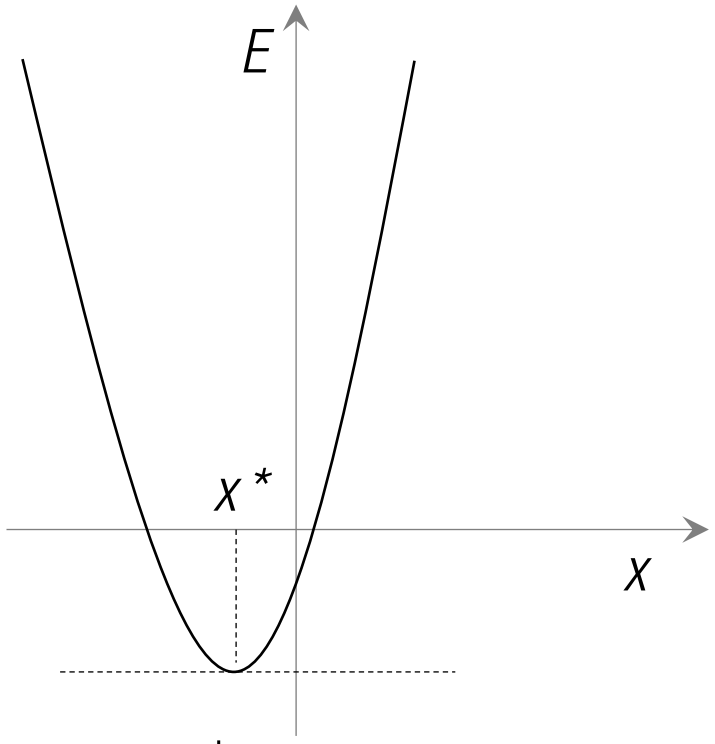


# Nonlinear Estimation

# Announcement

- Anushree will present the paper
- HW #5 due by Today
- HW #6 will be out by Tomorrow night.

# Linear System Recap



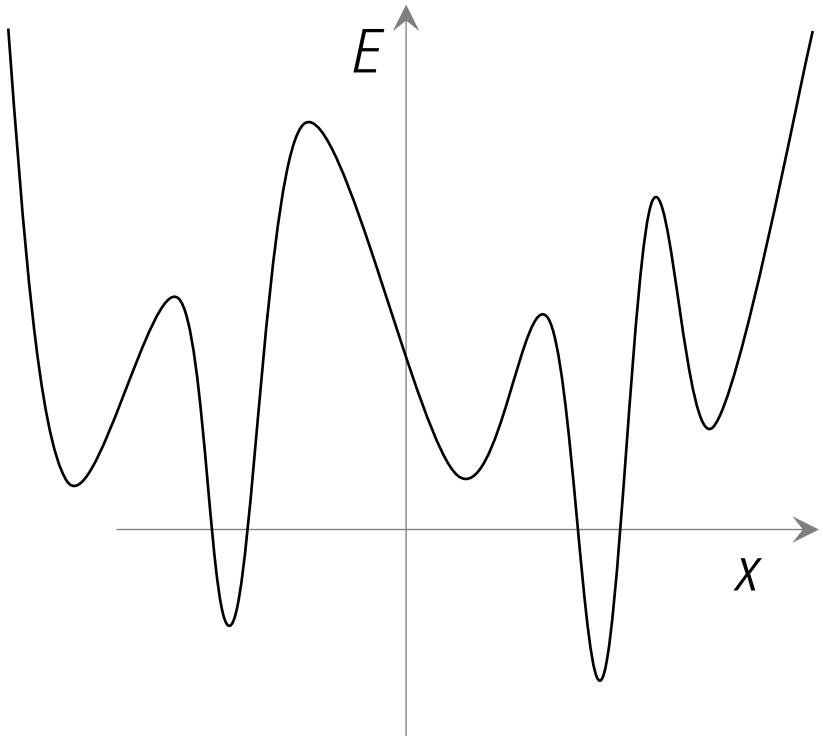

$$\left.\frac{dE}{dx}\right|_{x=x^*} = 0$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{x} = \left[ \mathbf{A}^T \mathbf{A} \right]^{-1} \mathbf{A}^T \mathbf{b}$$

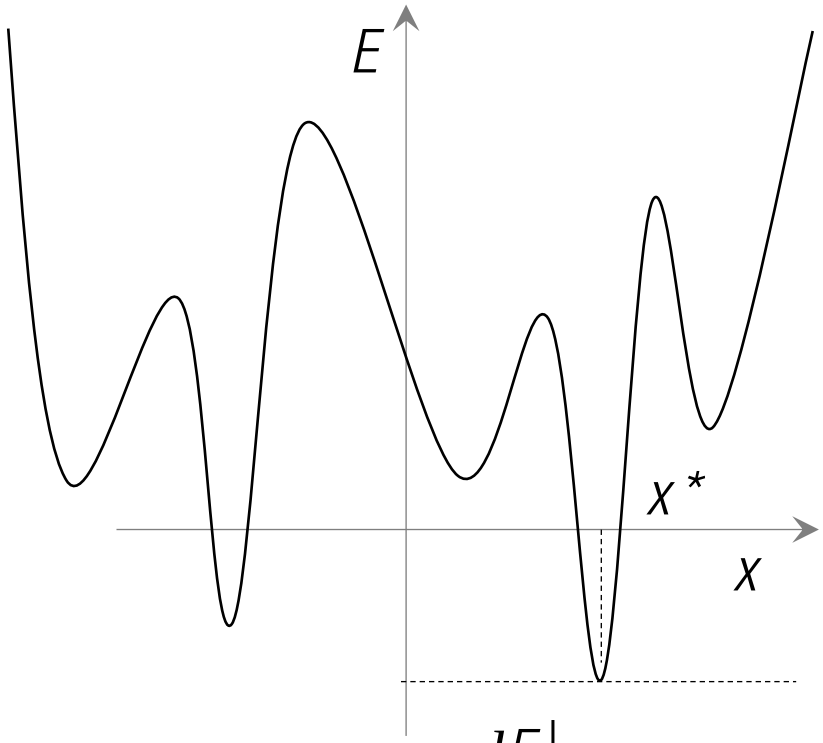
- Has the global solution
- Has the closed form solution (non-iterative solve)
- Is solved efficiently ($O(n^2)$)
- Does not require an initialization

# Nonlinear System




$$f(\mathbf{x}) = \mathbf{b}$$

# Nonlinear System




$$\left.\frac{dE}{dx}\right|_{x=x^*} = 0$$

$$f(\mathbf{x}) = \mathbf{b}$$

$$E = \left\| f(\mathbf{x}) - \mathbf{b} \right\|^2$$

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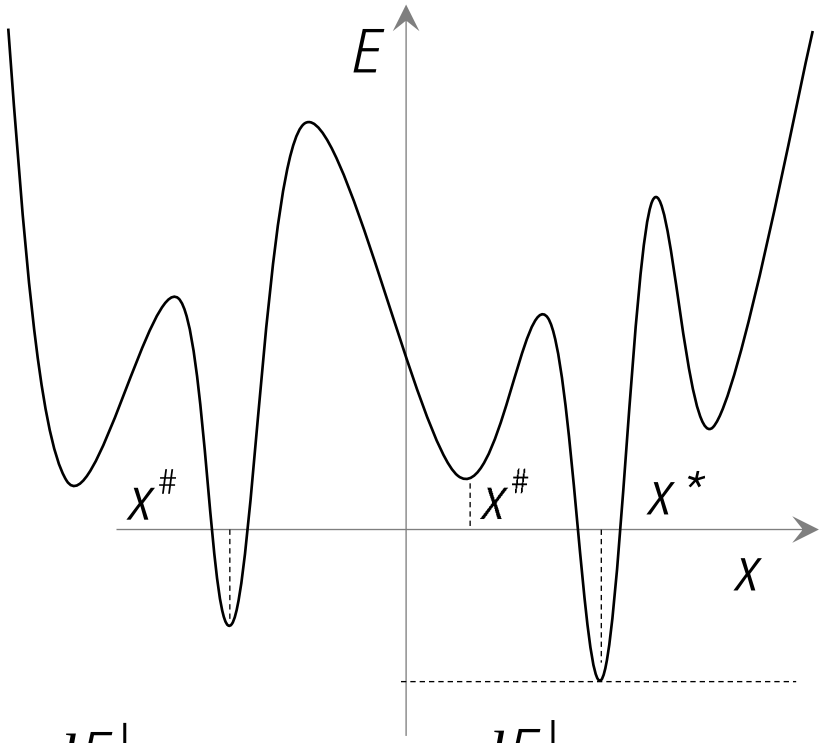


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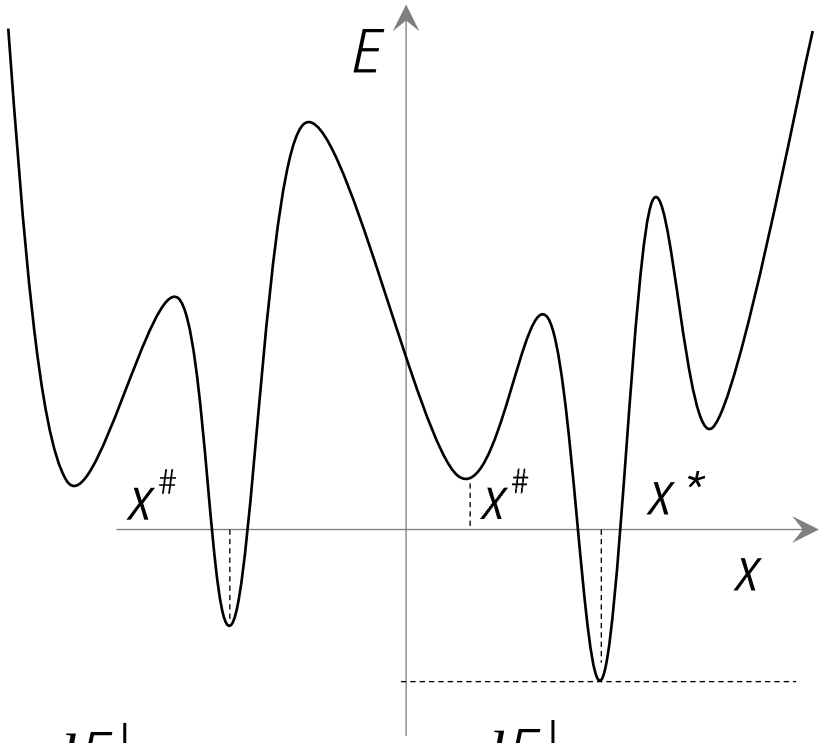


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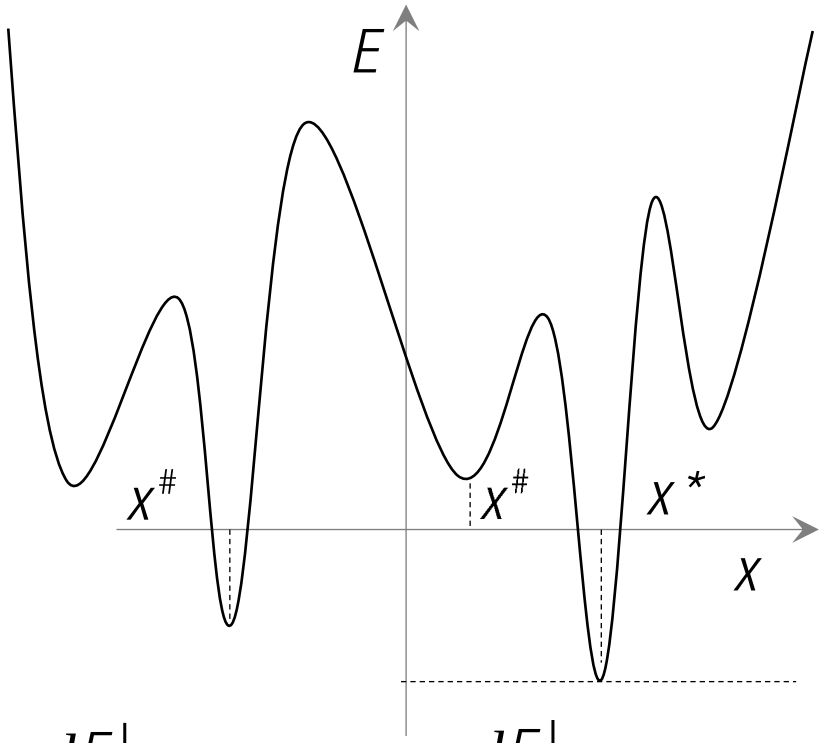


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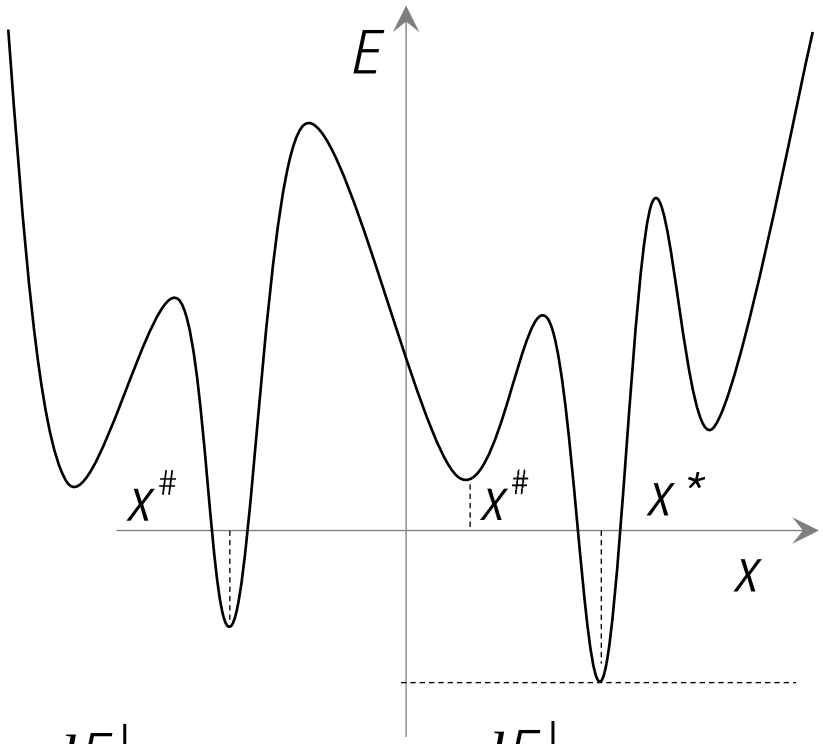


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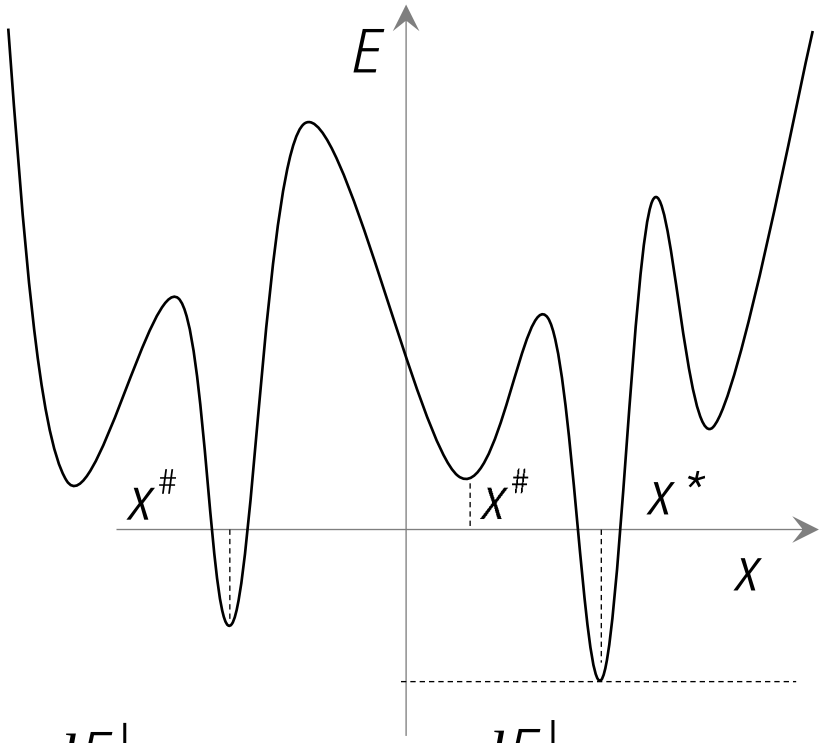


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Find **x** such that the following equation is satisfied:

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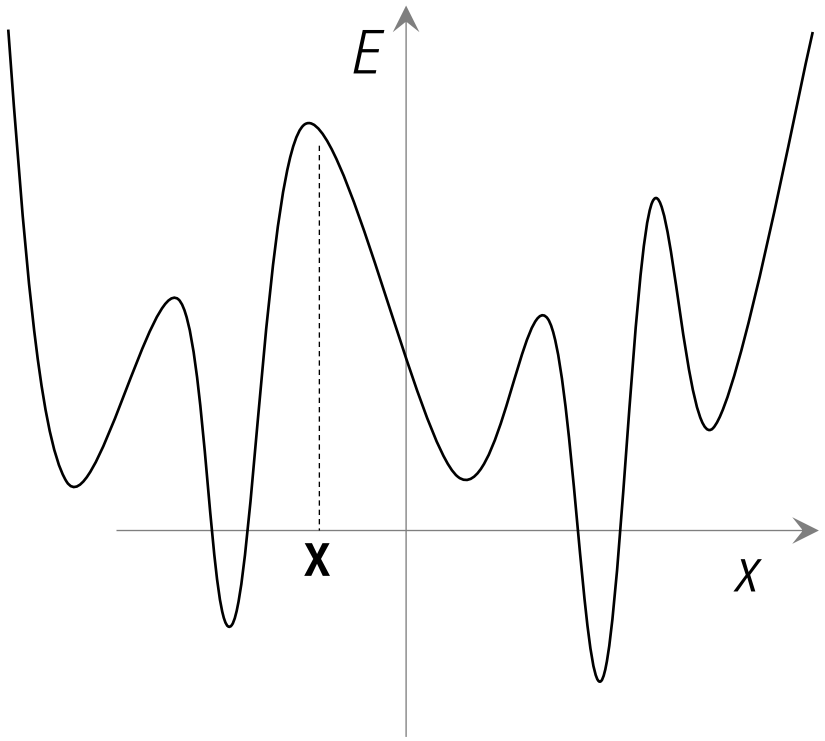
How?

# Nonlinear System

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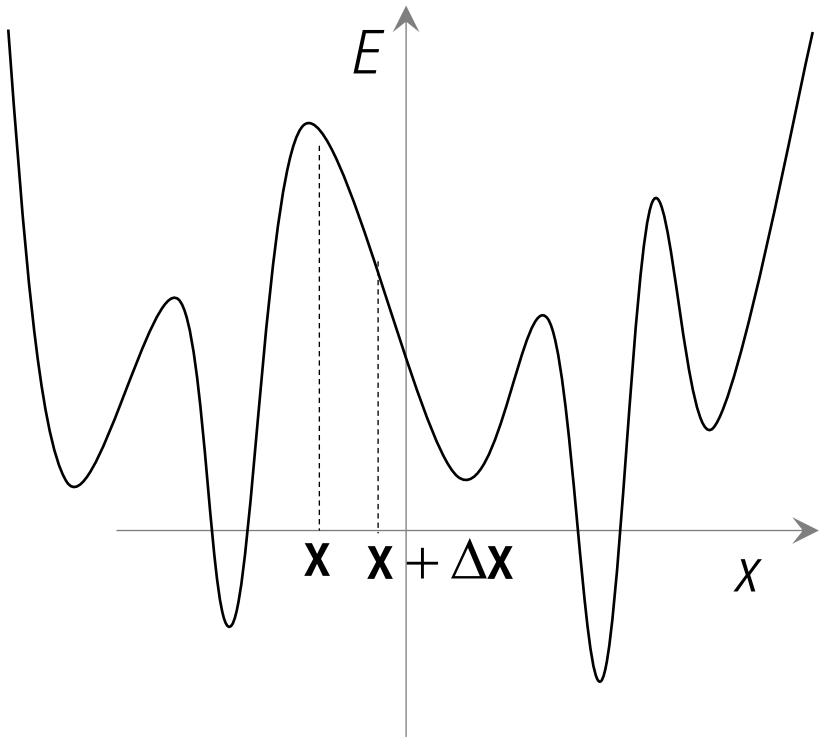
# Nonlinear System

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Strategy: Given **x,** move $\Delta\mathbf{x}$ such that $E(\mathbf{x}+\Delta\mathbf{x}) \leq E(\mathbf{x})$

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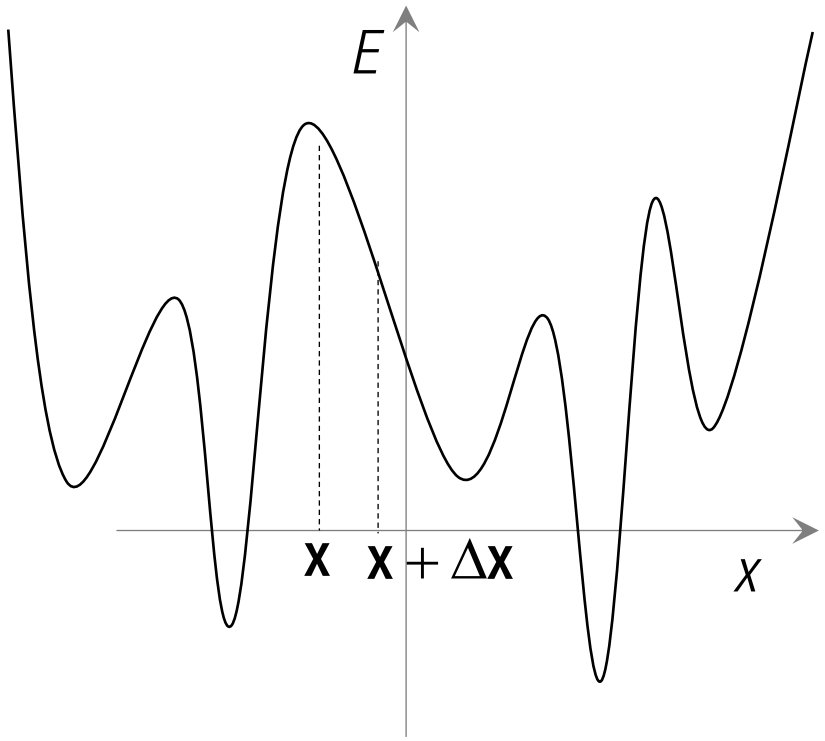


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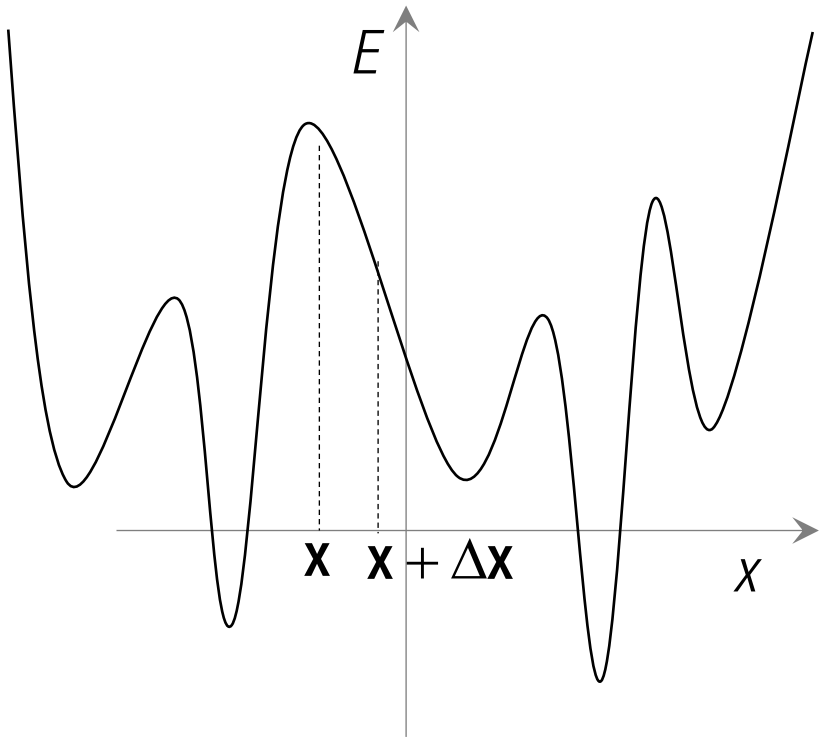


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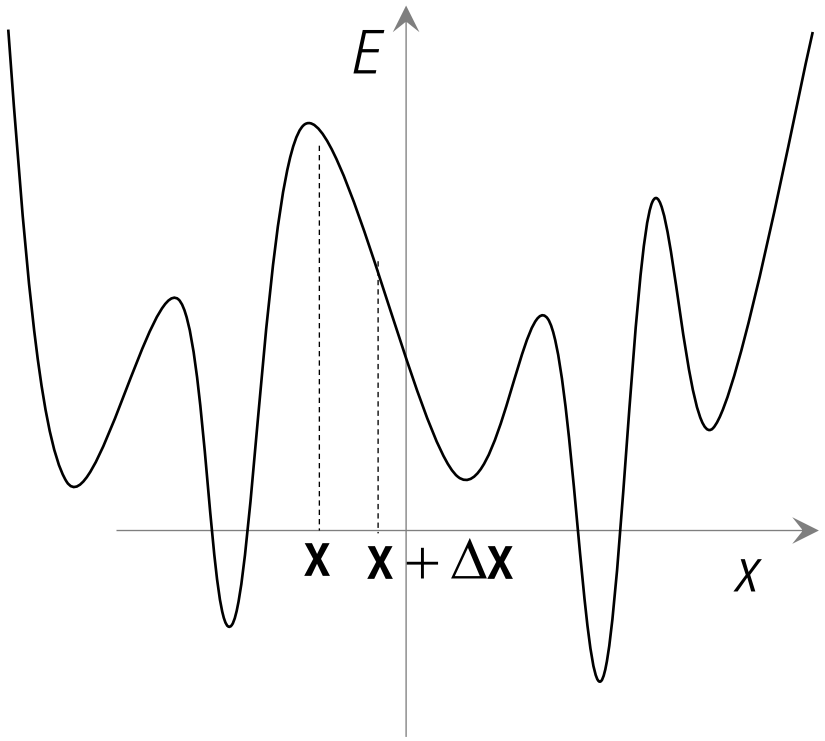


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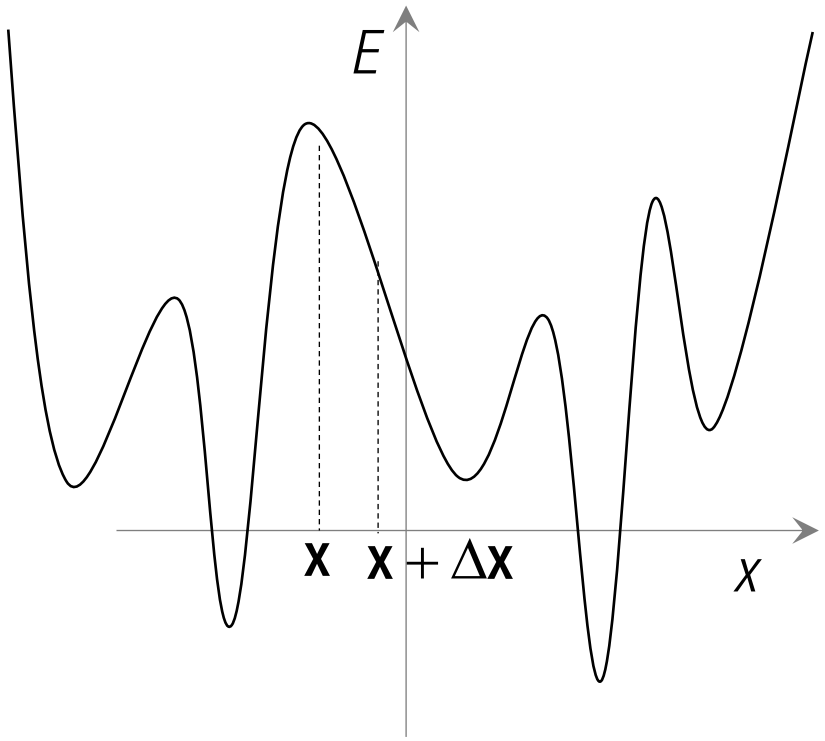


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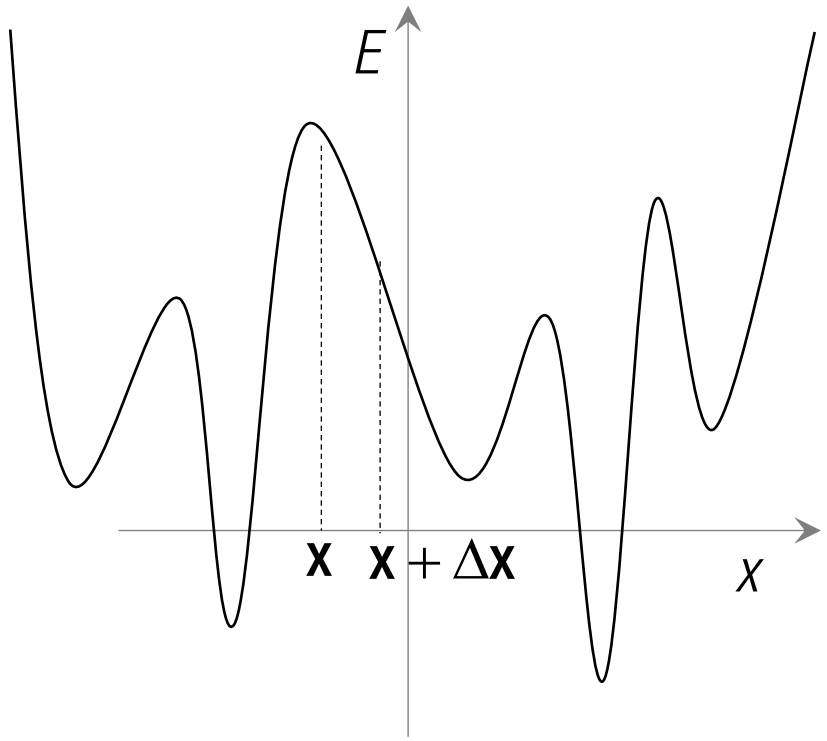


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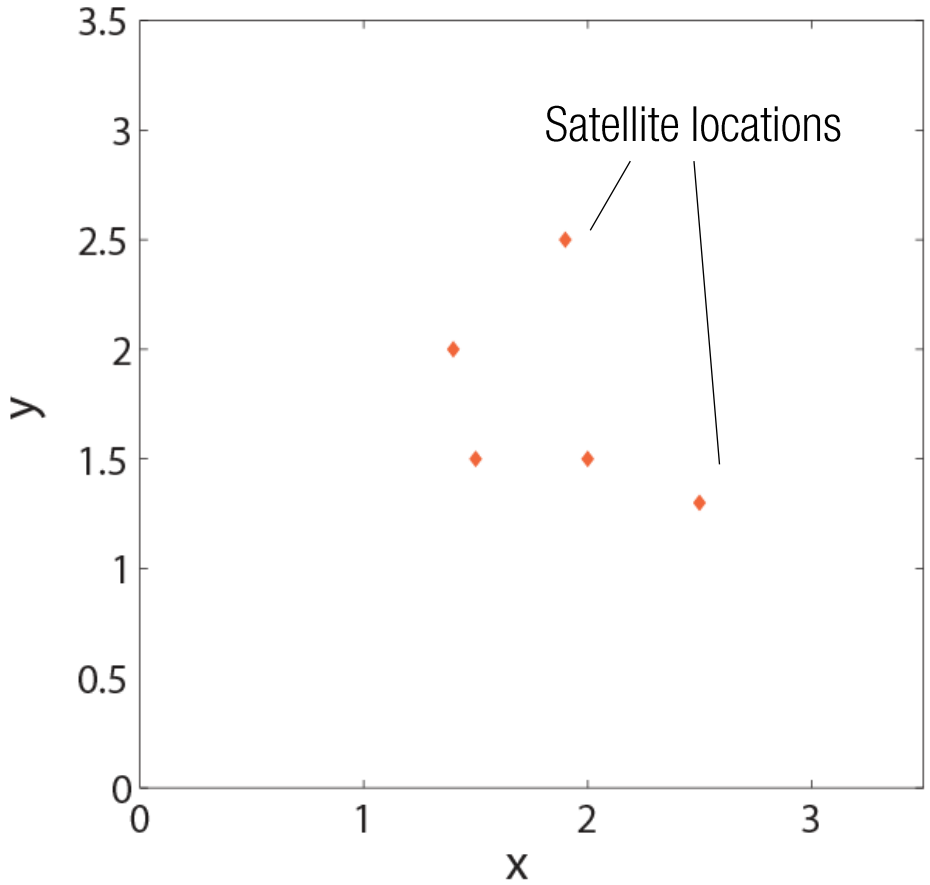
Cf.) $\mathbf{x} = \left[ \mathbf{A}^{\mathsf{T}} \mathbf{A} \right]^{-1} \mathbf{A}^{\mathsf{T}} \mathbf{b}$

# Where am I?: GPS Localization

Example:
Localization using range data from beacons



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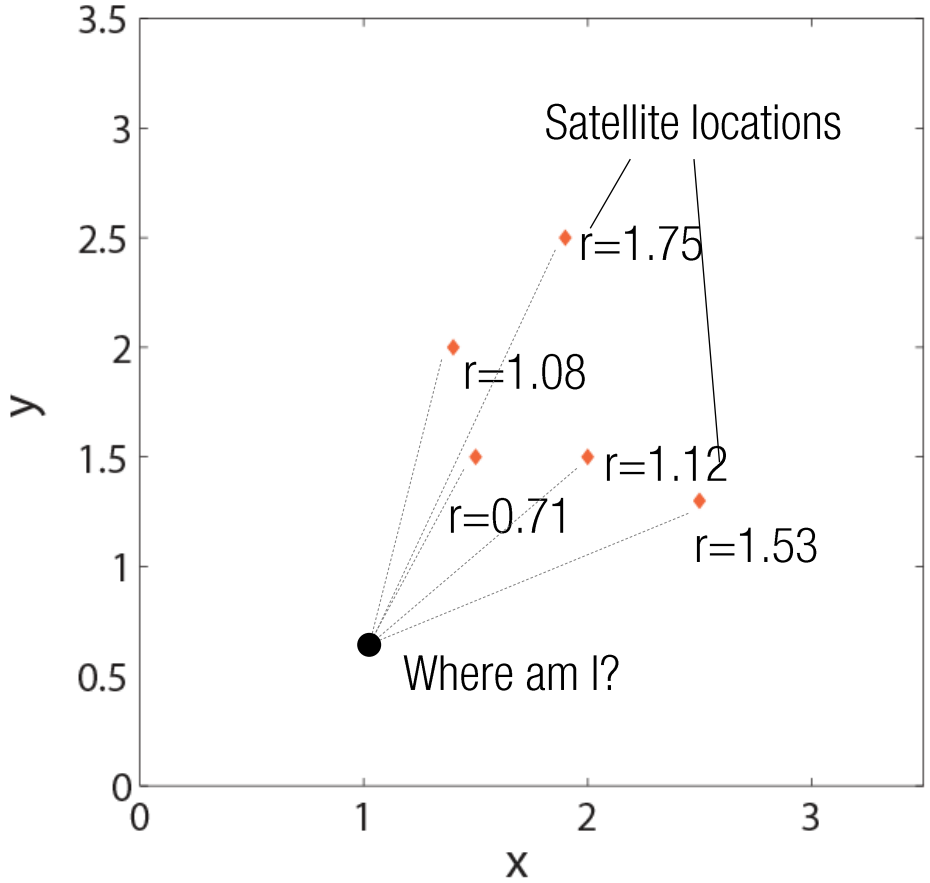

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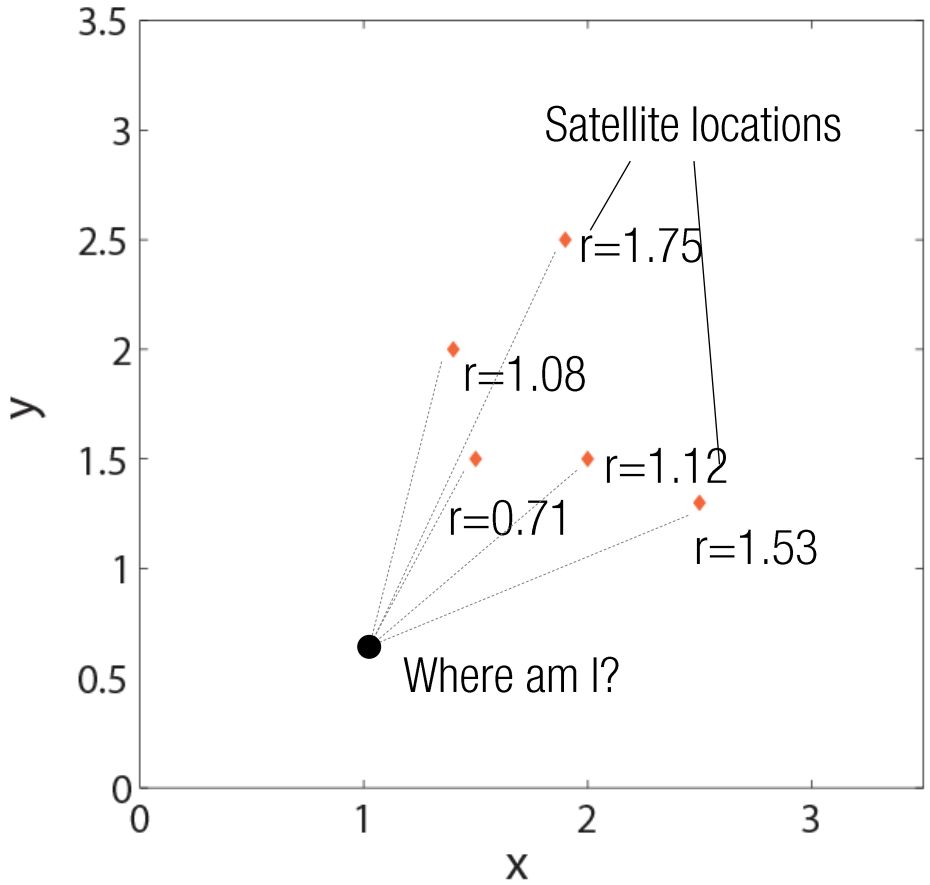


$E =$

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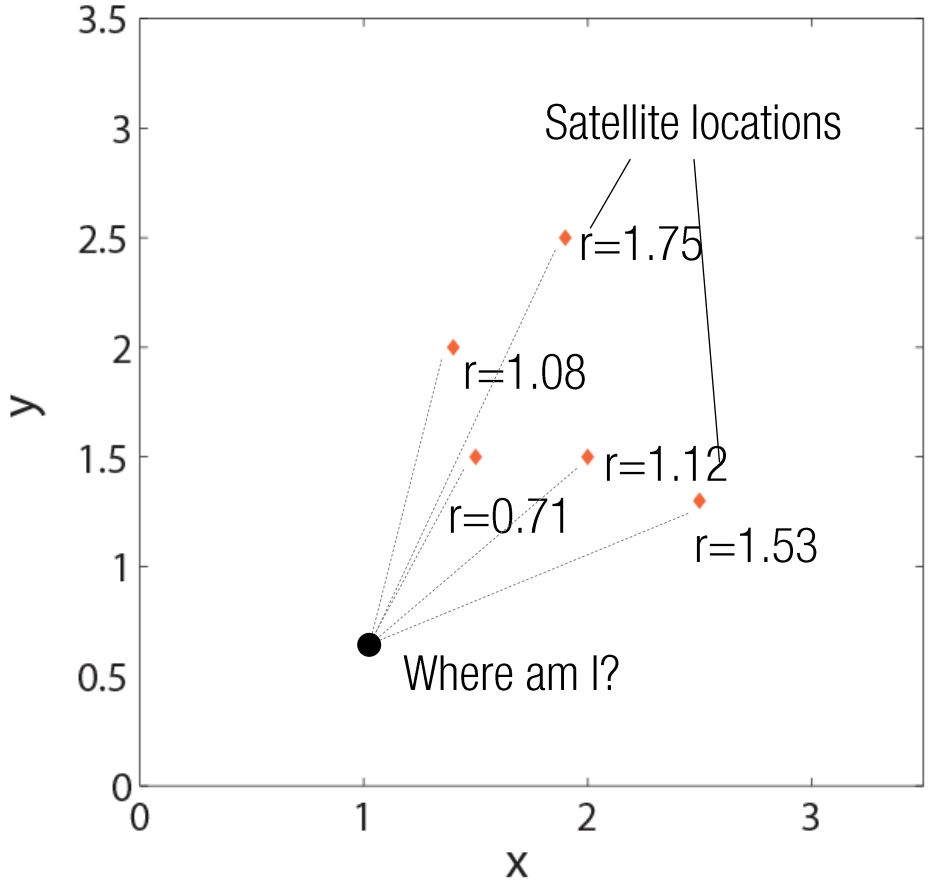


$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

# Where am I?: GPS Localization

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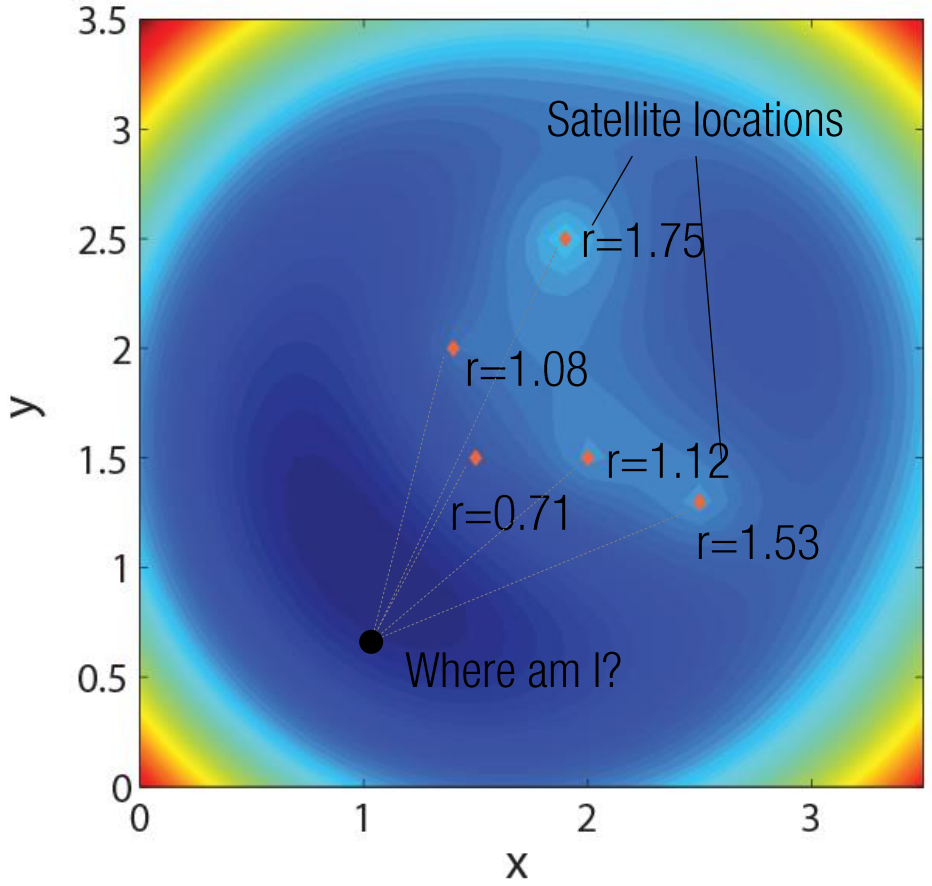


$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \mathbf{f}(\mathbf{x}) \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \mathbf{b} \\ r_5 \end{bmatrix} \right\|^2$$

# Where am I?: GPS Localization

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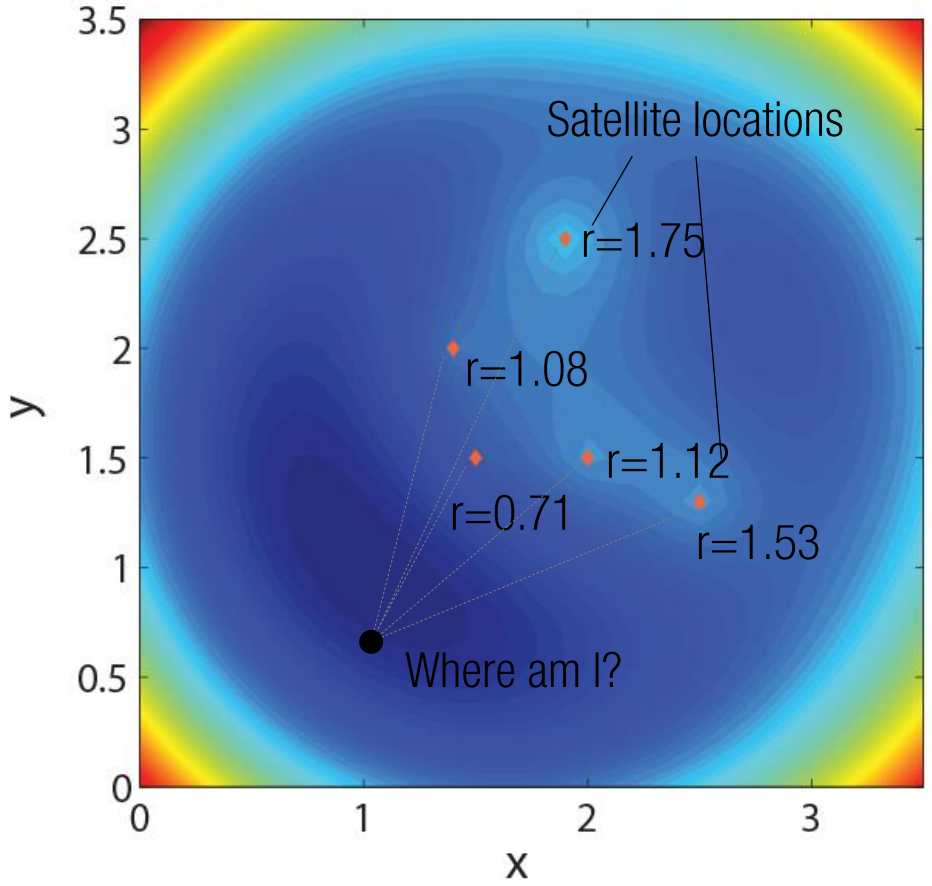


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$$\mathbf{f}(\mathbf{x})$$

$$\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} = \frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

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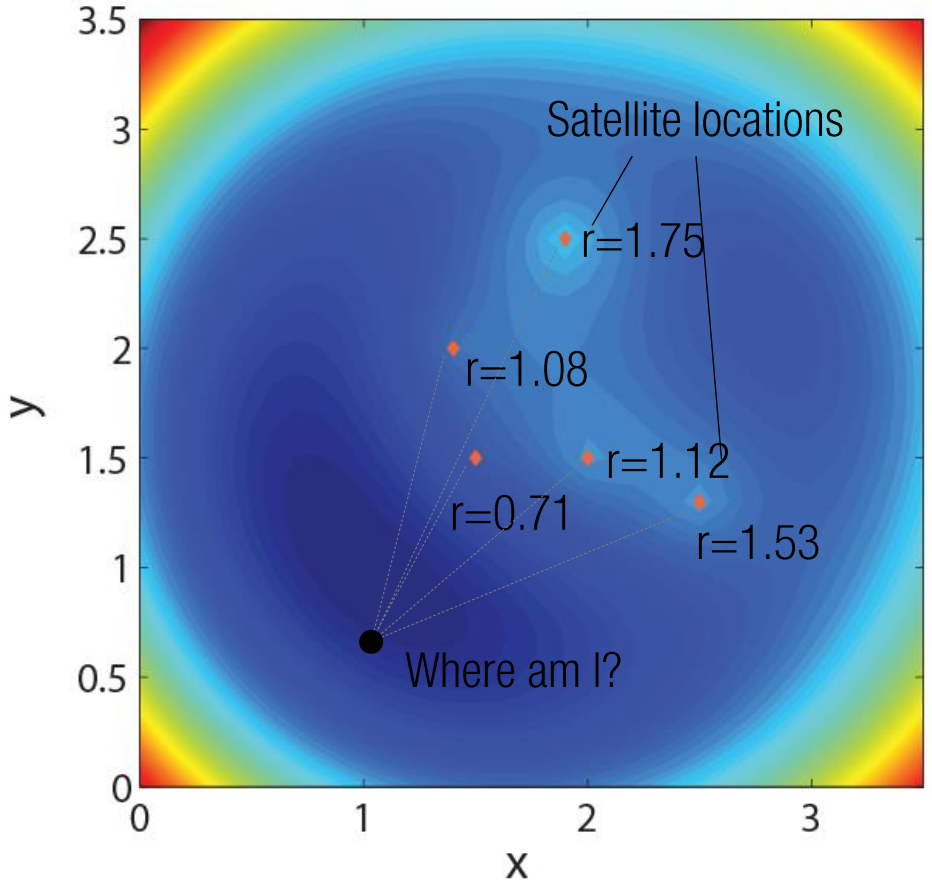
# Where am I?: GPS Localization

Example:
Localization using range data from beacons

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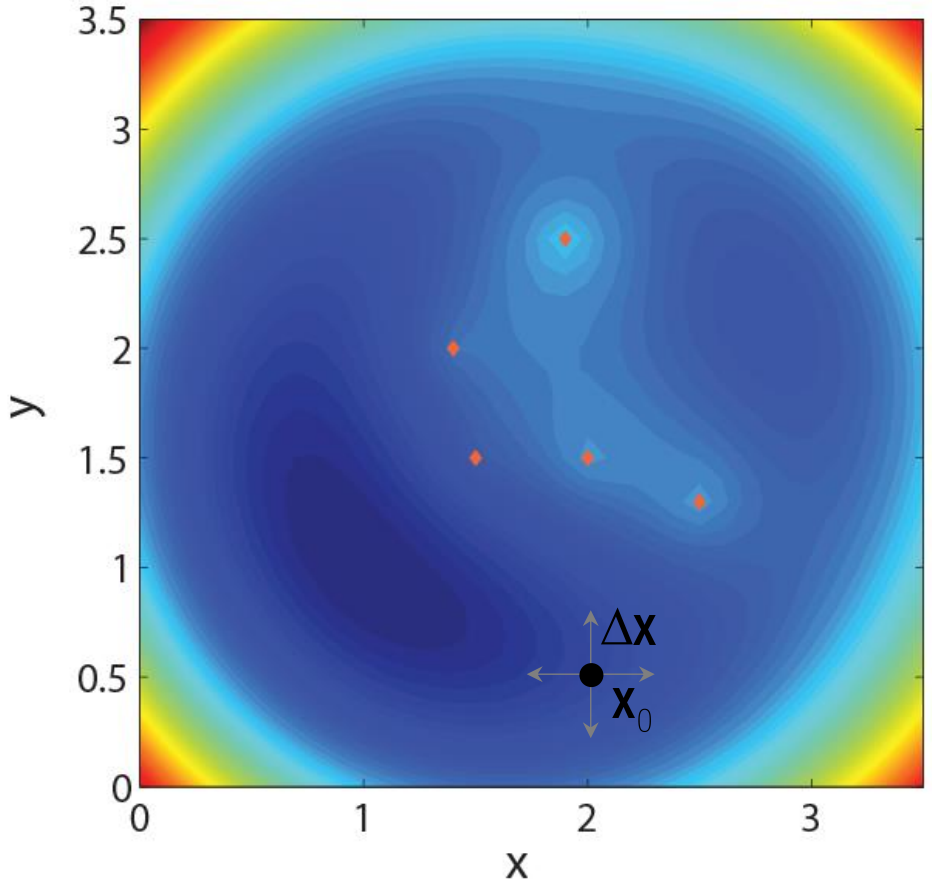
$$\frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} = \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} -\frac{u_1 - x}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} & -\frac{v_1 - y}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} \\ -\frac{u_5 - x}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} & -\frac{v_5 - y}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} \end{bmatrix}$$

# Where am I?: GPS Localization

Example:
Localization using range data from beacons

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$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

$$\Delta\mathbf{x} = \left( \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

# Where am I?: GPS Localization

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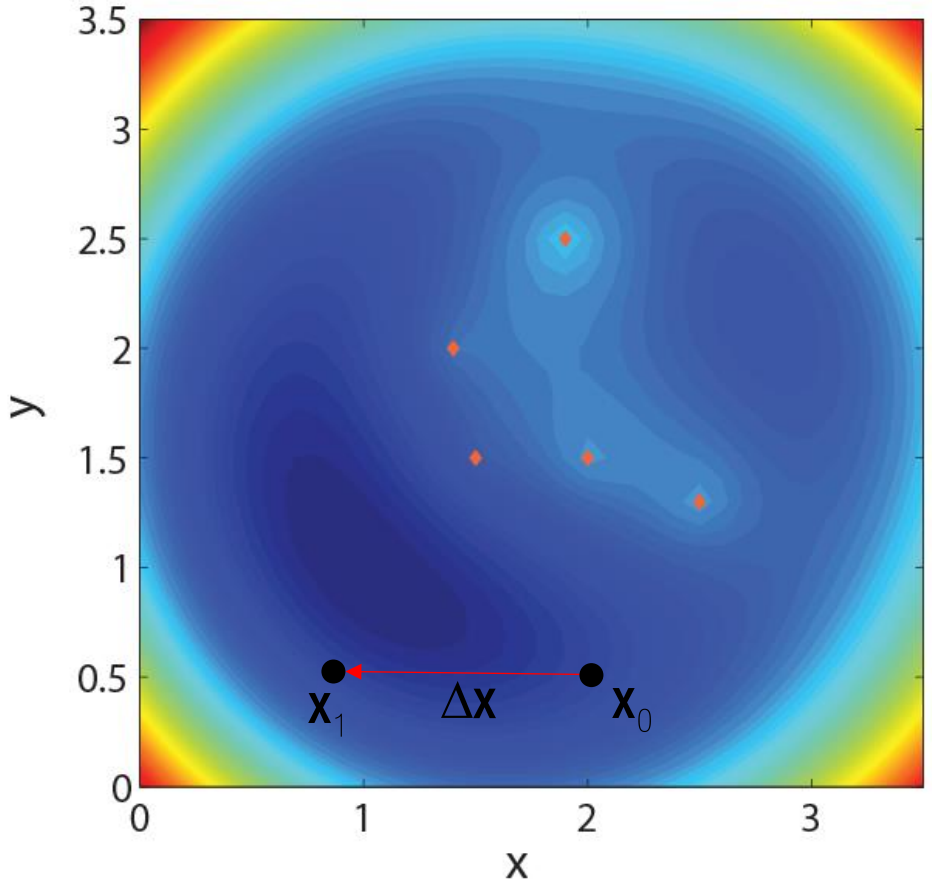


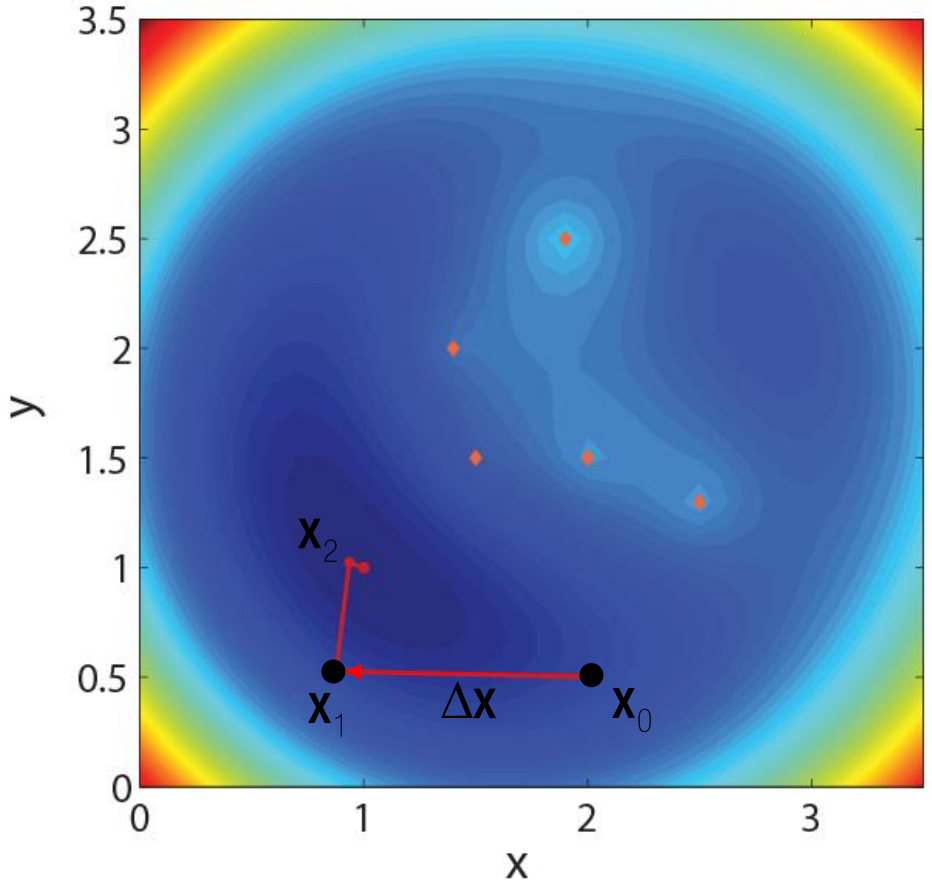

$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \mathbf{f}(\mathbf{x}) \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \mathbf{b} \\ r_5 \end{bmatrix} \right\|^2$$

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

$$\Delta \mathbf{x} = \left( \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

# Where am I?: GPS Localization

Example:
Localization using range data from beacons



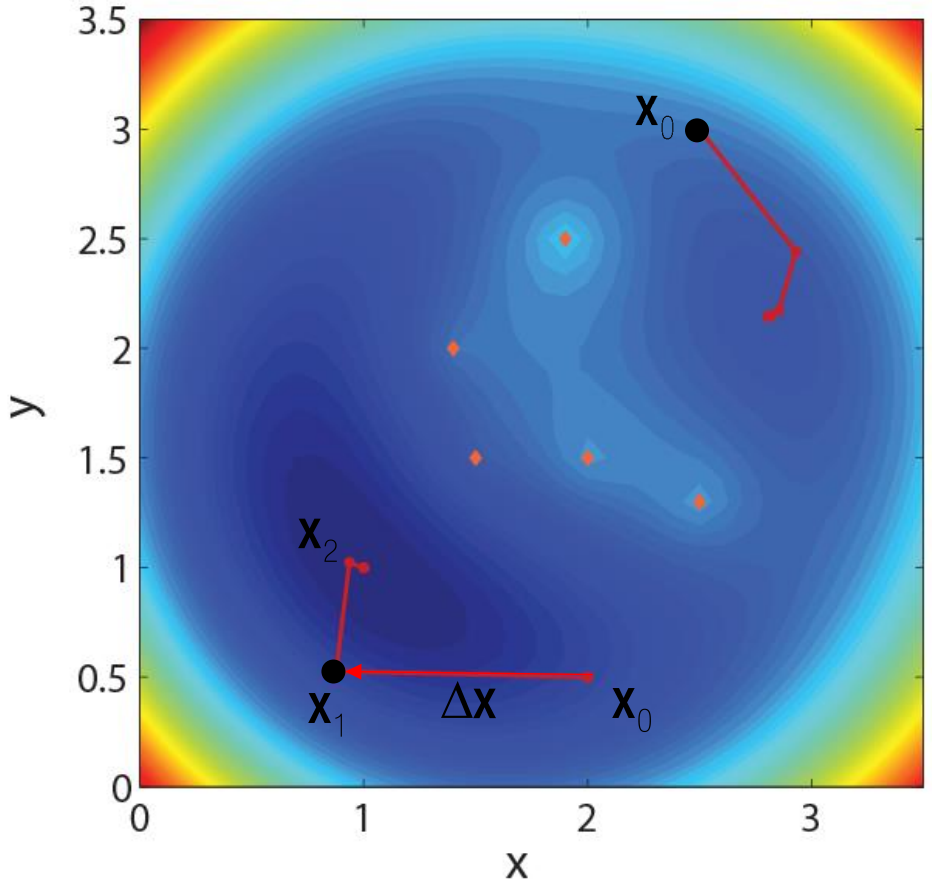

$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

$$\frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} = \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

$$\Delta\mathbf{x} = \left( \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

# Where am I?: GPS Localization

Example:
Localization using range data from beacons



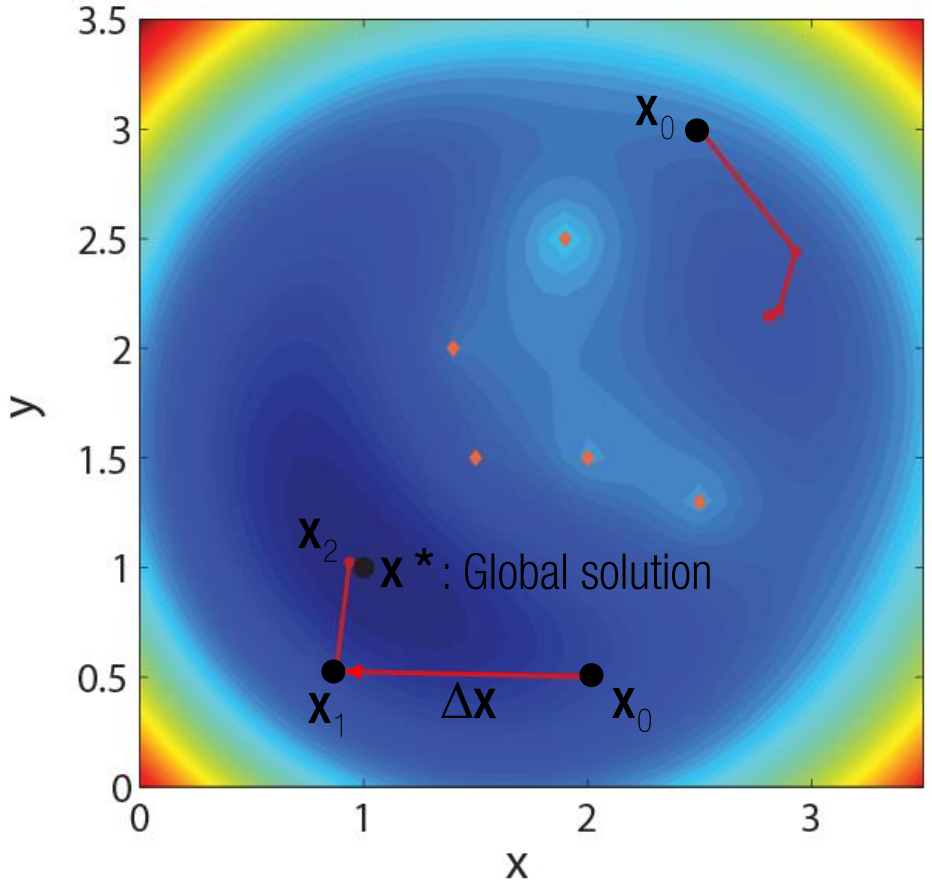

$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \mathbf{f}(\mathbf{x}) \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \mathbf{b} \\ r_5 \end{bmatrix} \right\|^2$$

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

$$\Delta \mathbf{x} = \left( \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

# Where am I?: GPS Localization

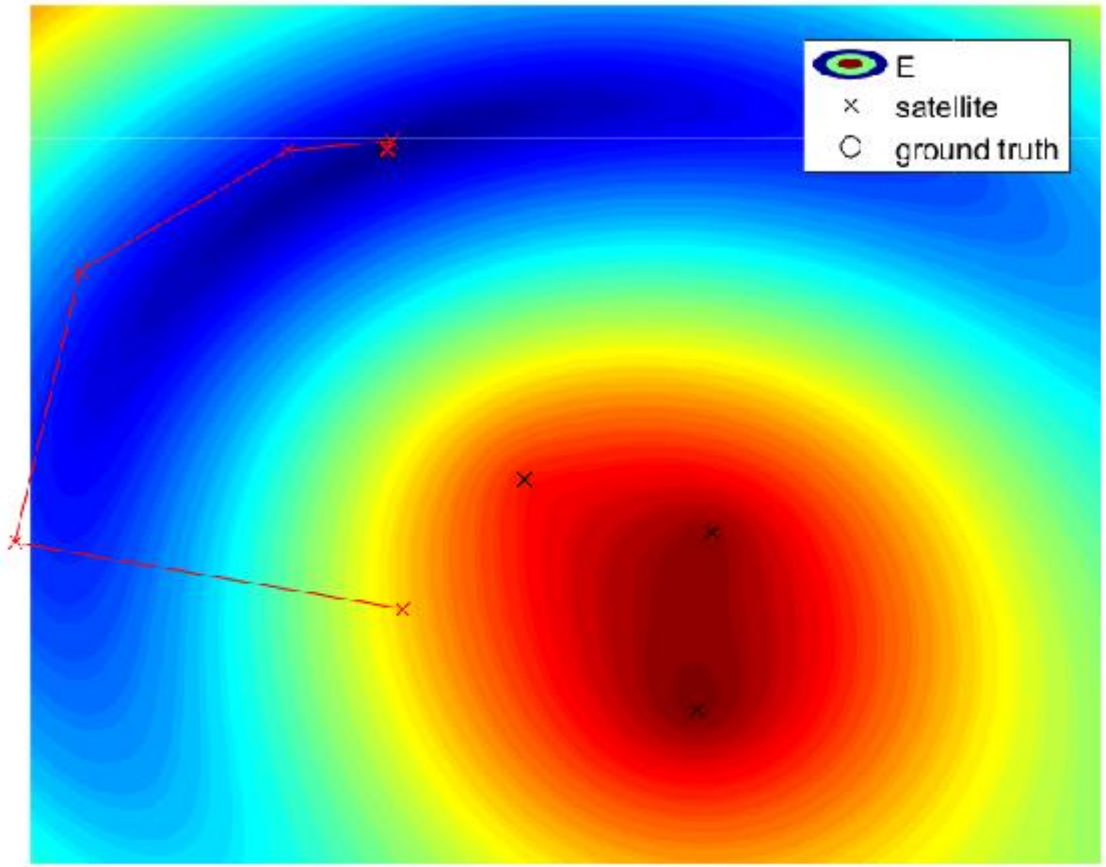
Example:
Localization using range data from beacons




$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \mathbf{f}(\mathbf{x}) \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \mathbf{b} \\ r_5 \end{bmatrix} \right\|^2$$

$$\frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} = \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

$$\Delta \mathbf{x} = \left( \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

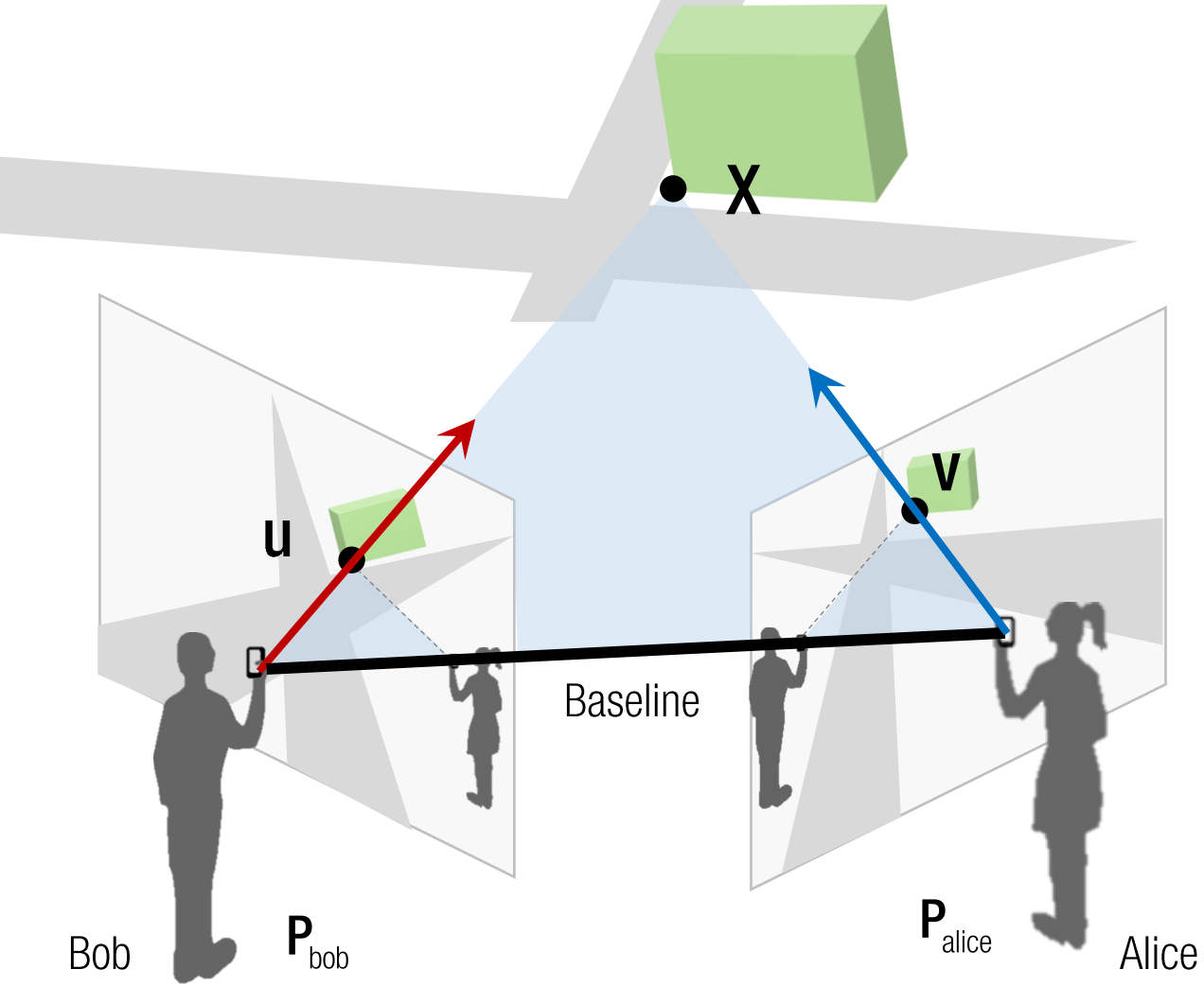



## Beacon.m

```
u = 2*(rand(3,2)-0.5);
x = [-0.5 1];

for i = 1 : size(u,1)
   d(i,1) = norm(x-u(i,:));
end

[x_grid, y_grid] = meshgrid(-1.5:0.01:1.5, -1.5:0.01:1.5);

E = zeros(size(x_grid));
for i = 1 : size(u,1)
   E = E + (sqrt((x_grid-u(i,1)).^2 +(y_grid-u(i,2)).^2)-d(i)).^2 ;
end
E = sqrt(E);

x0 = 2*(rand(2,1)-0.5);
for j = 1 : 10
   J = [];
   fx = [];
   for i = 1 : size(u,1)
      denom = norm(u(i,:)-x0');
      J = [J; (u(i,:)-x0')/denom];
      fx = [fx; norm(u(i,:)-x0')];
   end
   b = d;
   delta_x = -inv(J'*J)*J'*(b-fx);
   x0 = x0 + delta_x;
end
```

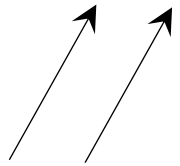
# Triangulation Refinement

# Recall: Triangulation




General camera pose

$$\lambda_1 \begin{bmatrix} u \\ 1 \end{bmatrix} = P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix}$$



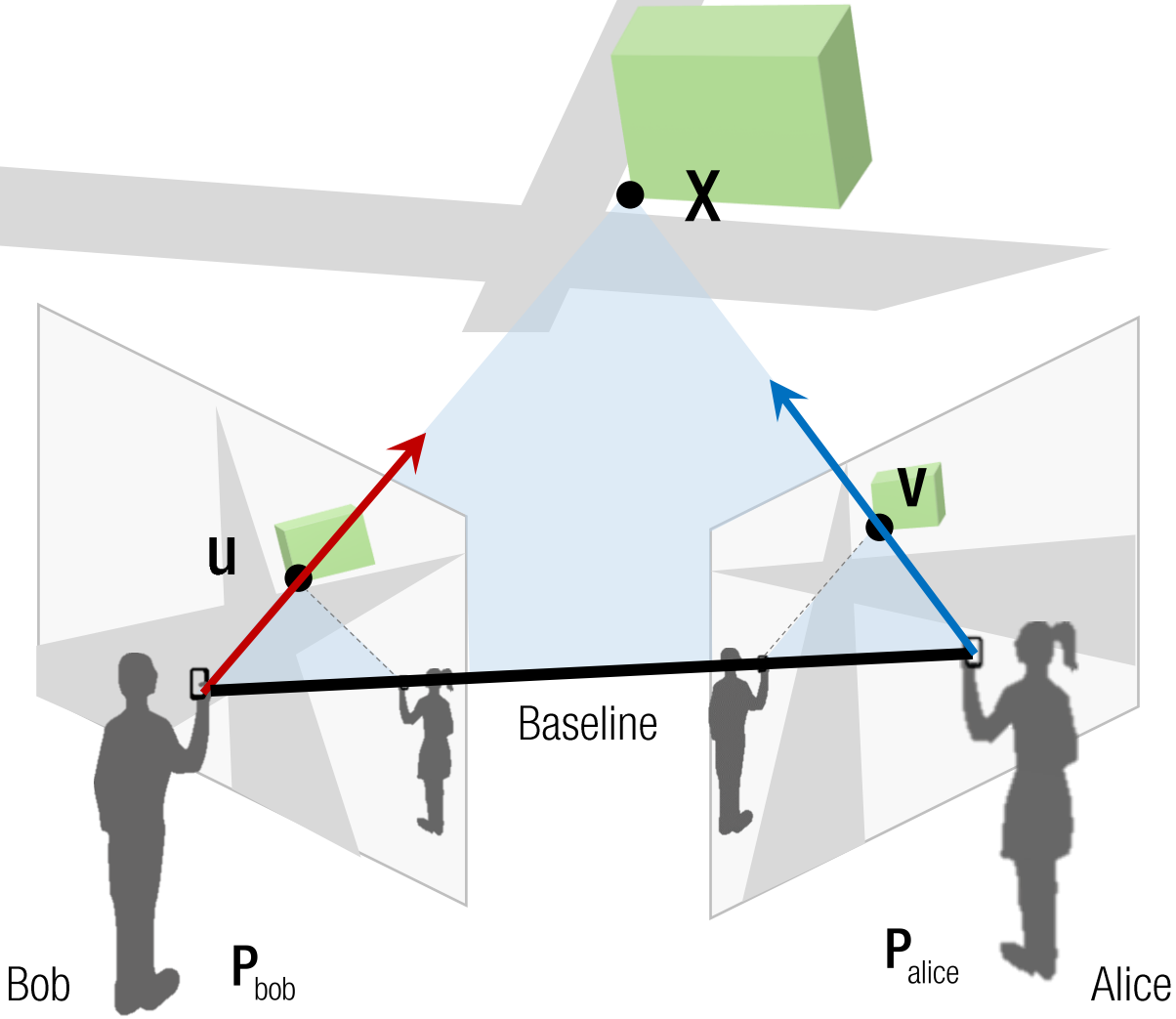
Two 3D vectors are parallel.

$$\rightarrow \begin{bmatrix} u \\ 1 \end{bmatrix} \times P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} u \\ 1 \end{bmatrix}_{\times} P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix} = 0$$

Skew-symmetric matrix

# Recall: Triangulation




General camera pose

$$\lambda_1 \begin{bmatrix} u \\ 1 \end{bmatrix} = P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

Two 3D vectors are parallel.

$$\rightarrow \begin{bmatrix} u \\ 1 \end{bmatrix} \times P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix} = 0$$

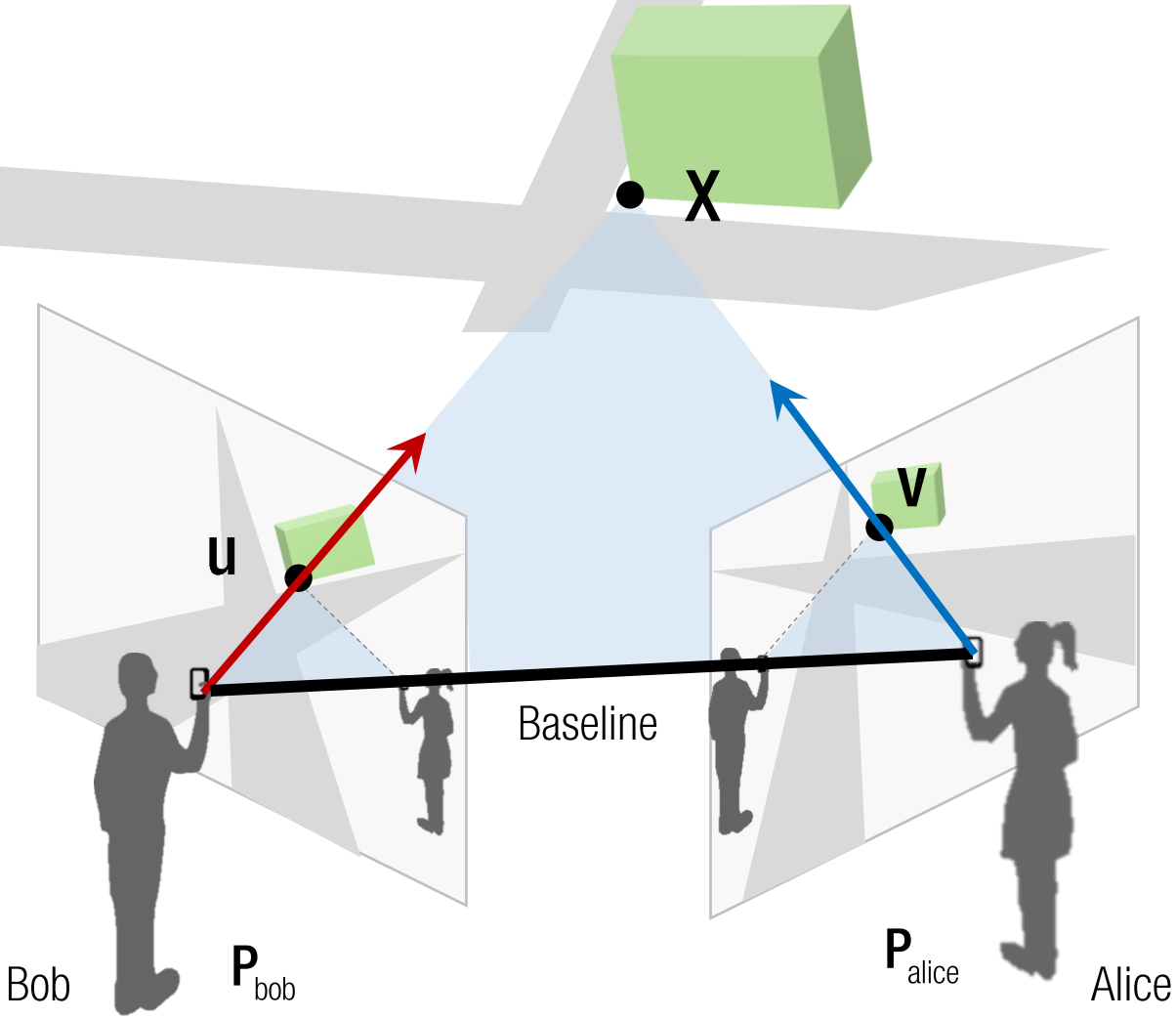
$$\rightarrow \begin{bmatrix} u \\ 1 \end{bmatrix}_{\times} P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix} = 0$$

Skew-symmetric matrix

: Knowns

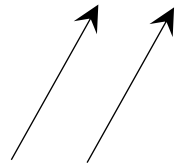
: Unknowns

# Recall: Triangulation




General camera pose

$$\lambda_1 \begin{bmatrix} u \\ 1 \end{bmatrix} = P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix}$$



Two 3D vectors are parallel.

$$\rightarrow \begin{bmatrix} u \\ 1 \end{bmatrix} \times P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} u \\ 1 \end{bmatrix}_{\times} P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix} = 0$$

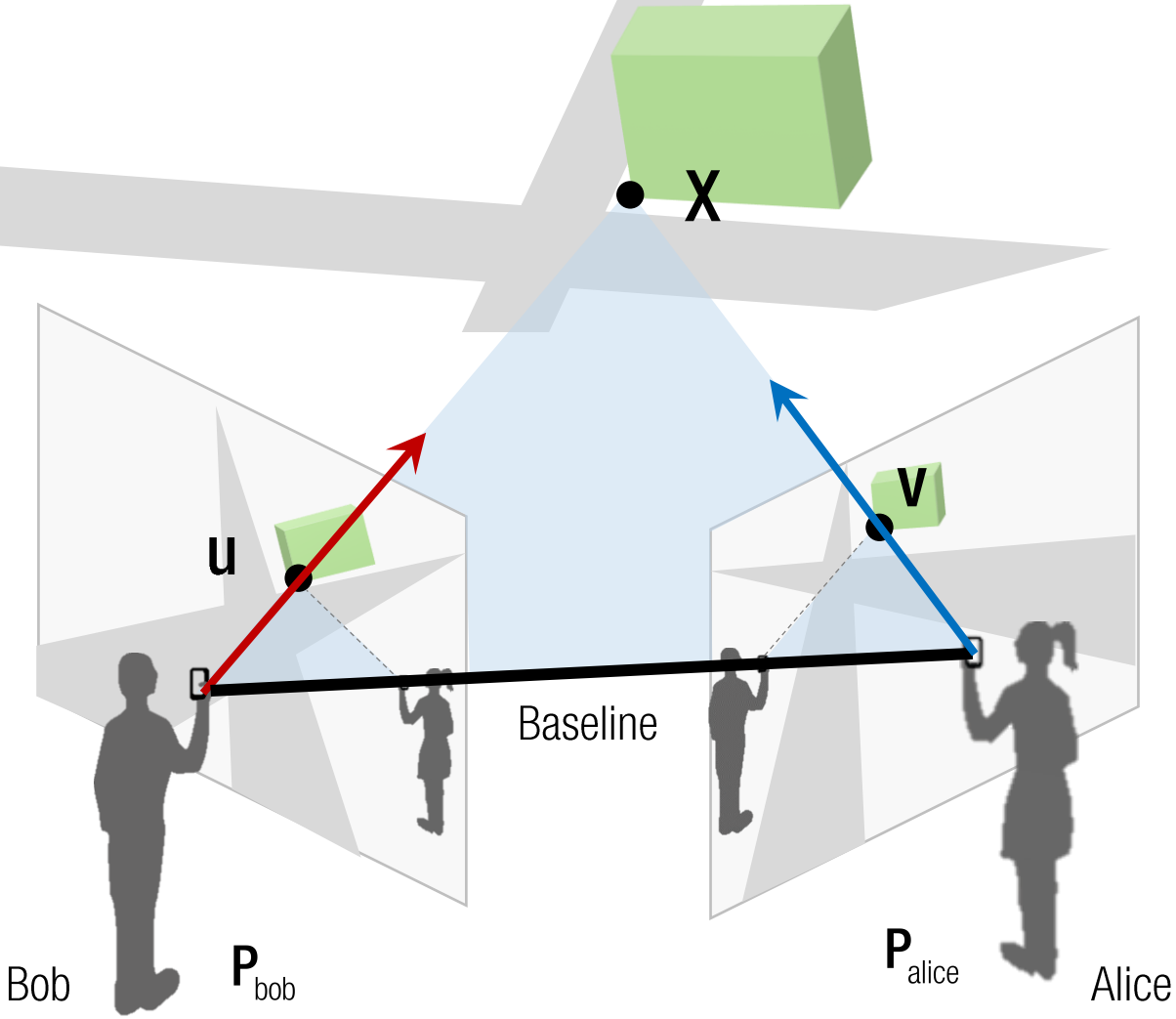
3x4

: Knowns

: Unknowns

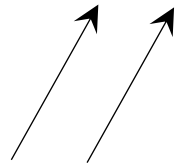
Can we solve for **X**? (single view reconstruction)
Why not?

# Recall: Triangulation




General camera pose

$$\lambda_1 \begin{bmatrix} u \\ 1 \end{bmatrix} = P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix}$$



Two 3D vectors are parallel.

$$\longrightarrow \begin{bmatrix} u \\ 1 \end{bmatrix} \times P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix} = 0$$

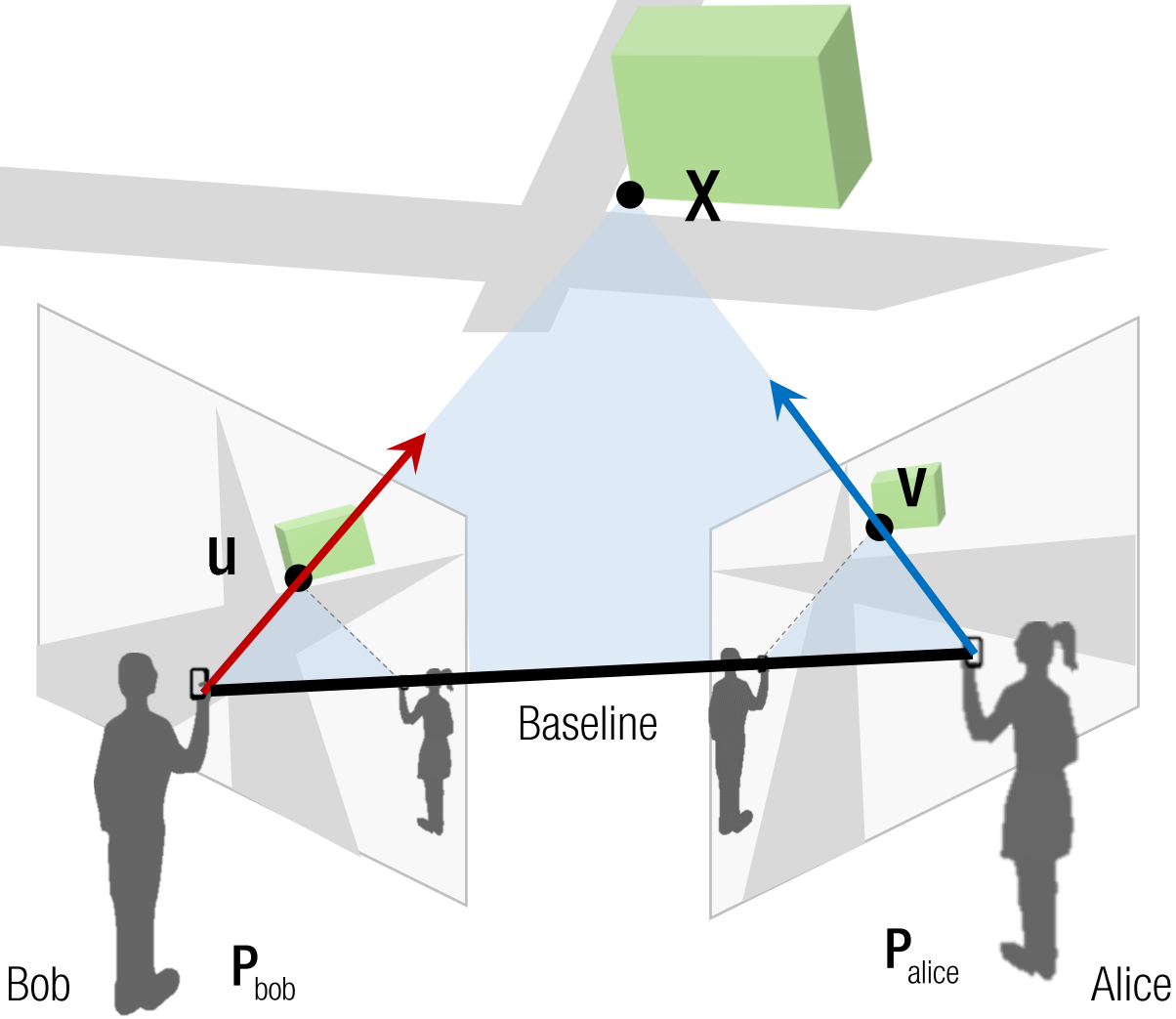
$$\longrightarrow \begin{bmatrix} u \\ 1 \end{bmatrix}_{\times} P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix} = 0$$

2x4

: Knowns

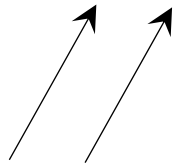
: Unknowns

# Recall: Triangulation




General camera pose

$$\lambda_1 \begin{bmatrix} u \\ 1 \end{bmatrix} = P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix}$$



Two 3D vectors are parallel.

$$\longrightarrow \begin{bmatrix} u \\ 1 \end{bmatrix} \times P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix} = 0$$

$$\longrightarrow \begin{bmatrix} u \\ 1 \end{bmatrix}_{\times} P_{bob} \begin{bmatrix} X \\ 1 \end{bmatrix} = 0$$

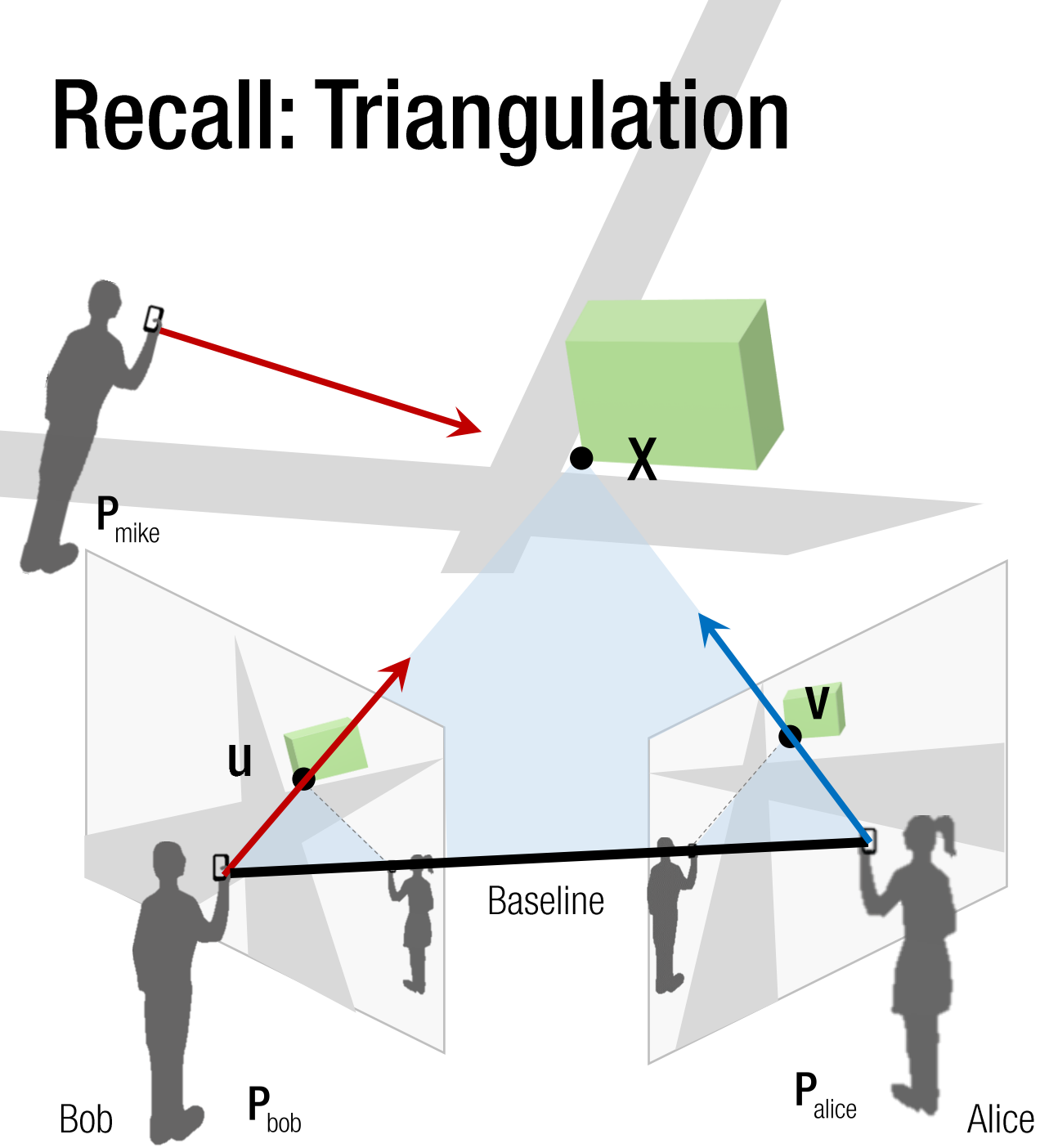
$$\begin{bmatrix} v \\ 1 \end{bmatrix}_{\times} P_{alice}$$

4x4

: Knowns

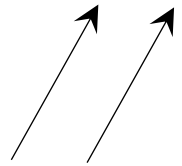
: Unknowns

# Recall: Triangulation




General camera pose

$$\lambda_1 \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} = \mathbf{P}_{bob} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$



Two 3D vectors are parallel.

$$\longrightarrow \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} \times \mathbf{P}_{bob} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{0}$$

$$\longrightarrow \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix}_{\times} \mathbf{P}_{bob} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{0}$$

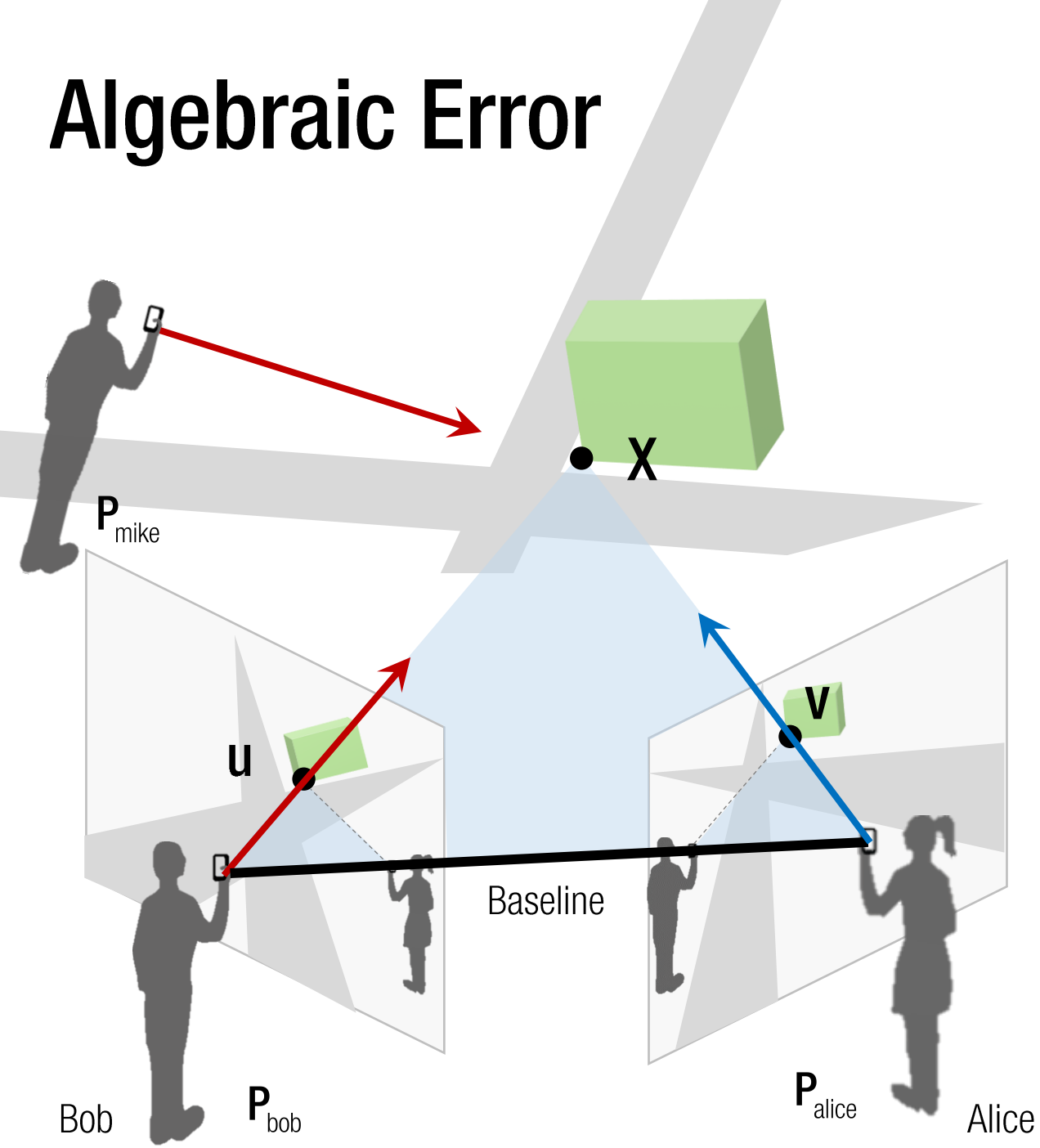
$$\begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}_{\times} \mathbf{P}_{alice}$$

$$\begin{bmatrix} \mathbf{w} \\ 1 \end{bmatrix}_{\times} \mathbf{P}_{mike}$$

: Knowns

: Unknowns

# Algebraic Error




$$E_{alge} = \| \mathbf{A}\ \mathbf{x} - \mathbf{b} \|^2$$

Algebraic error does not have geometric meaning.

# Recall: Geometric Verification via Reprojection Error

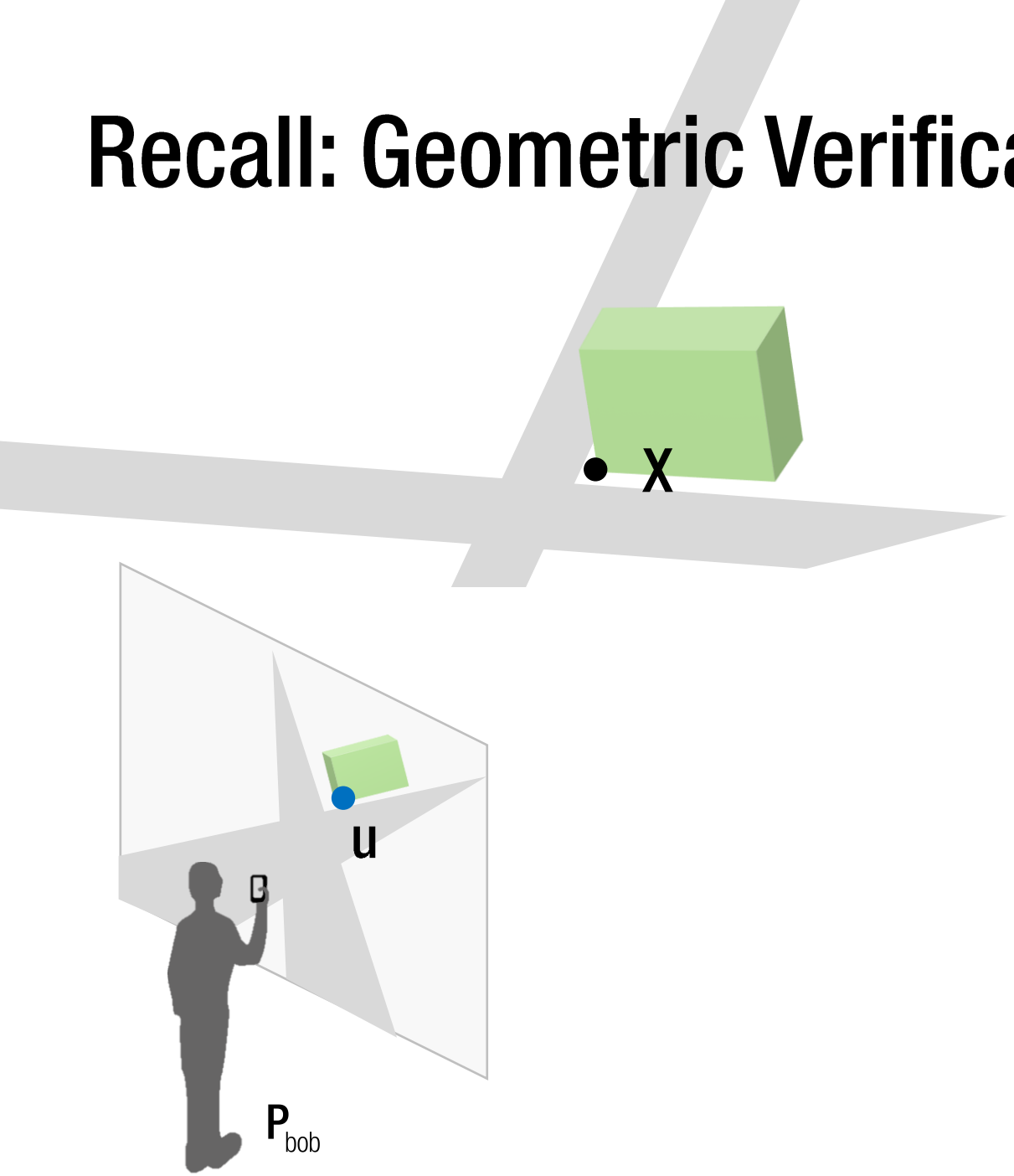



Image feature measurement, e.g. SIFT detection:

u

# Recall: Geometric Verification via Reprojection Error

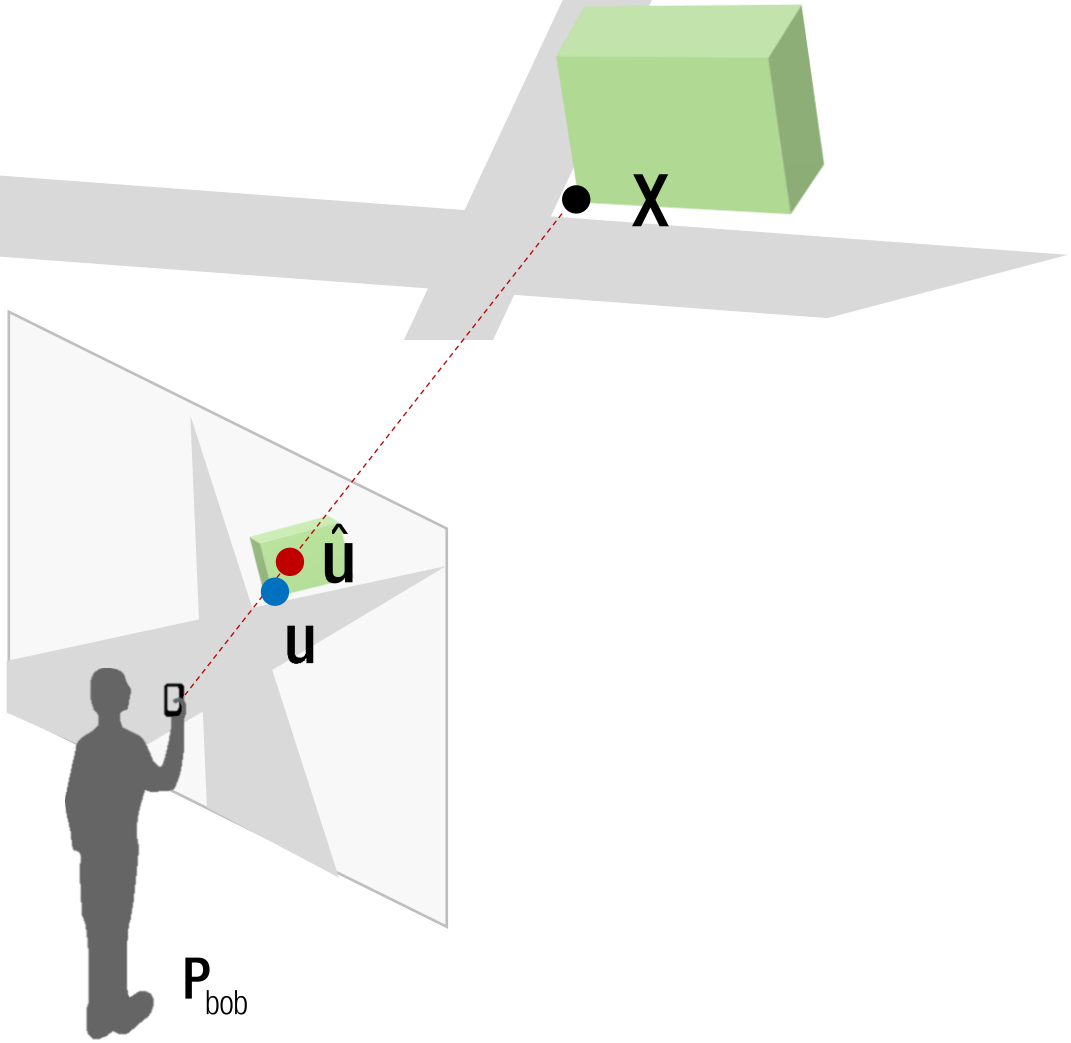



Image feature measurement, e.g. SIFT detection:

$\mathbf{u}$

3D point projection, or reprojection:

$\lambda \hat{\mathbf{u}} = \mathbf{PX}$

# Recall: Geometric Verification via Reprojection Error

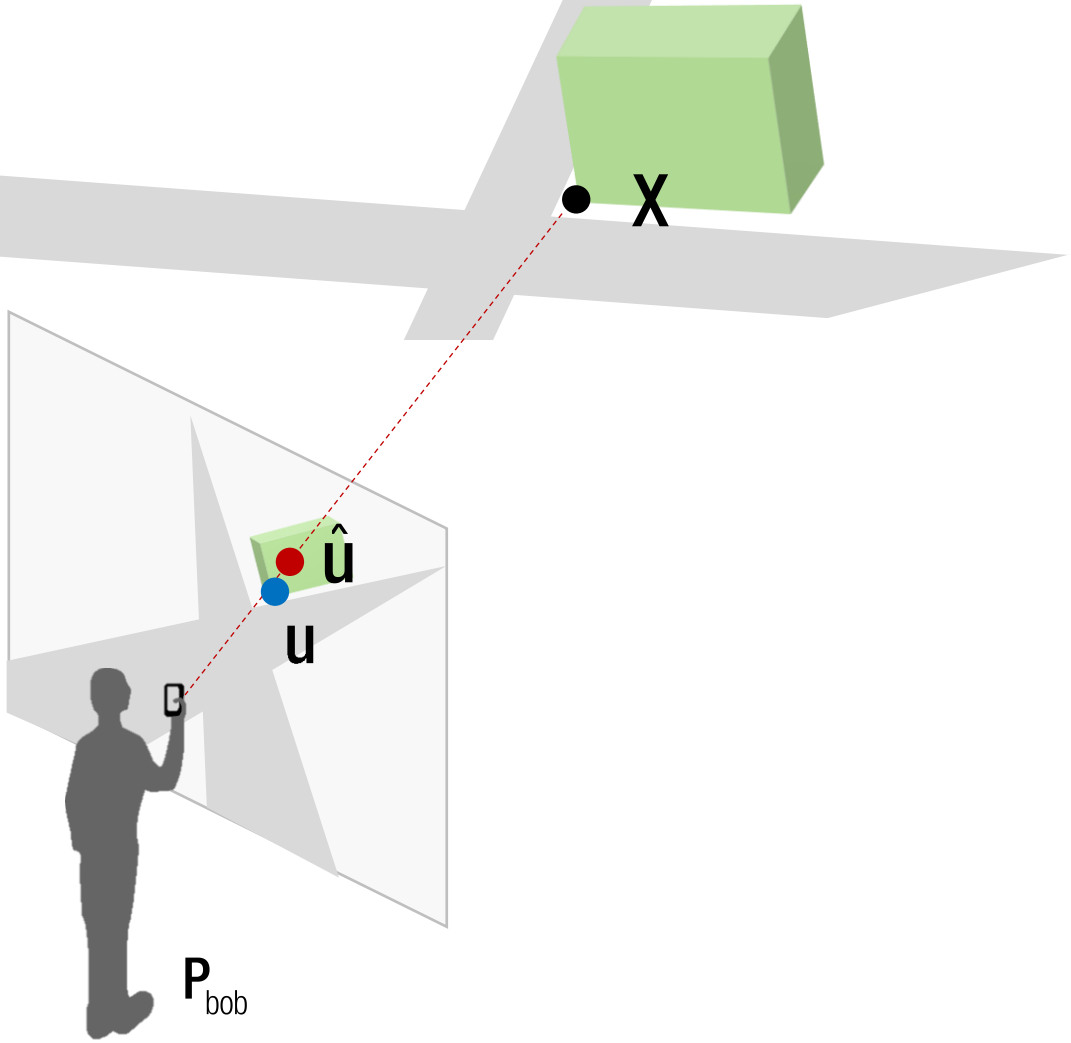



Image feature measurement, e.g. SIFT detection:

$\mathbf{u}$

3D point projection, or reprojection:

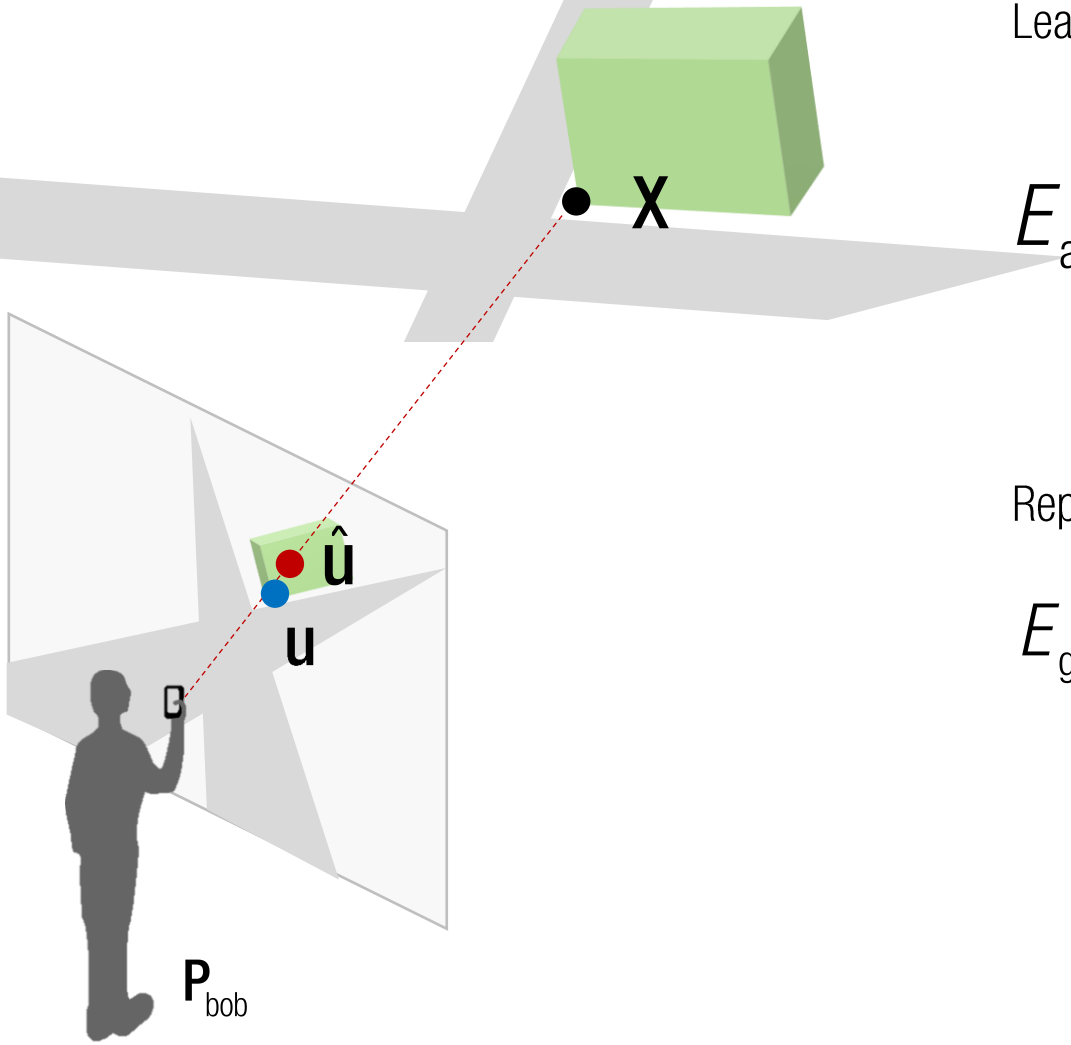
$$\lambda \hat{\mathbf{u}} = \mathbf{P}\mathbf{X}$$

Reprojection error (geometric error):

$$E_{\text{geom}} = \left\| \hat{\mathbf{u}} - \mathbf{u} \right\|^2$$

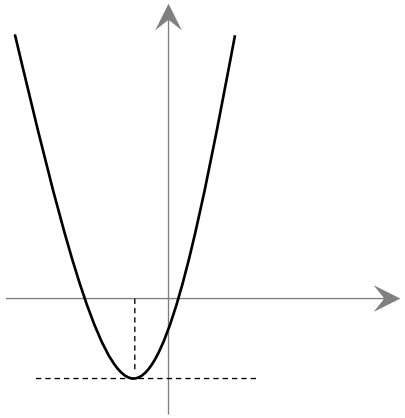
$$= \left( \frac{\mathbf{P}_1\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - \mathbf{u}_1 \right)^2 + \left( \frac{\mathbf{P}_2\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - \mathbf{u}_2 \right)^2$$

# Algebraic vs. Geometric error




Least squares solution (algebraic error):

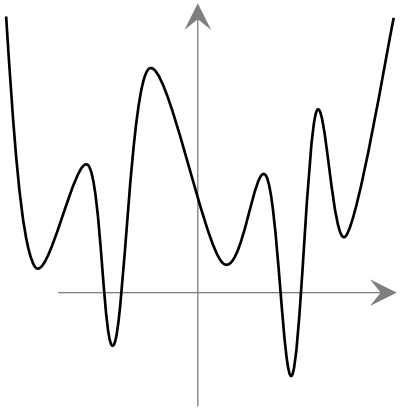
$$E_{\text{alge}} = \| \mathbf{A}\ \mathbf{x} - \mathbf{b} \|^2$$



Reprojection error (geometric error):

$$E_{\text{geom}} = \|\hat{\mathbf{u}} - \mathbf{u}\|^2$$

$$= \left(\frac{\mathbf{P}_1\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - \mathbf{u}_1\right)^2 + \left(\frac{\mathbf{P}_2\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - \mathbf{u}_2\right)^2$$

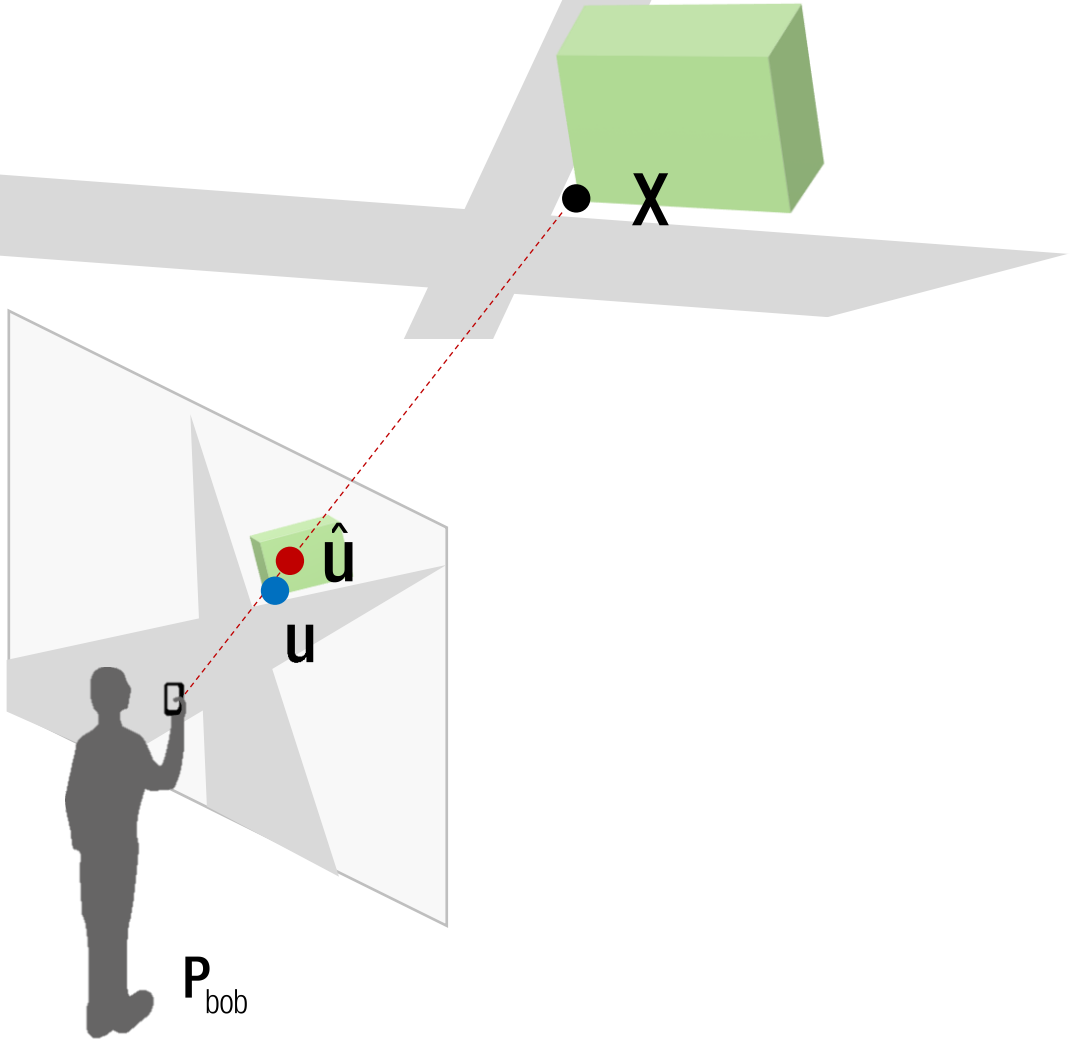


# Point Jacobian

Black: given variables
Red: unknowns

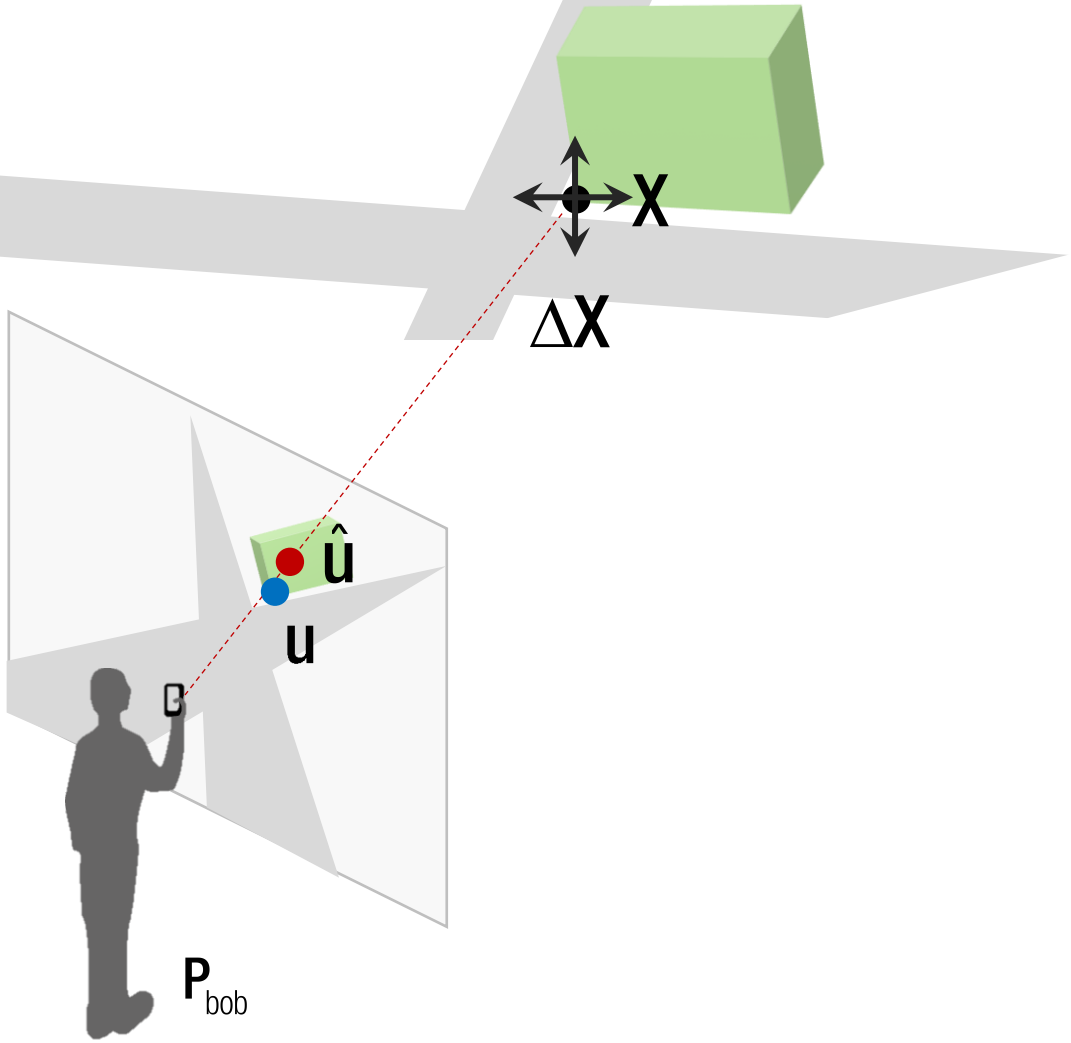
$$E_{\text{geom}} = \left\| \hat{\mathbf{u}} - \mathbf{u} \right\|^2$$

$$= \left( \frac{\mathbf{P}_1 \mathbf{X}}{\mathbf{P}_3 \mathbf{X}} - x \right)^2 + \left( \frac{\mathbf{P}_2 \mathbf{X}}{\mathbf{P}_3 \mathbf{X}} - y \right)^2$$



Black: given variables
Red: unknowns

# Point Jacobian




$$E_{geom} = \|\hat{\mathbf{u}} - \mathbf{u}\|^2$$

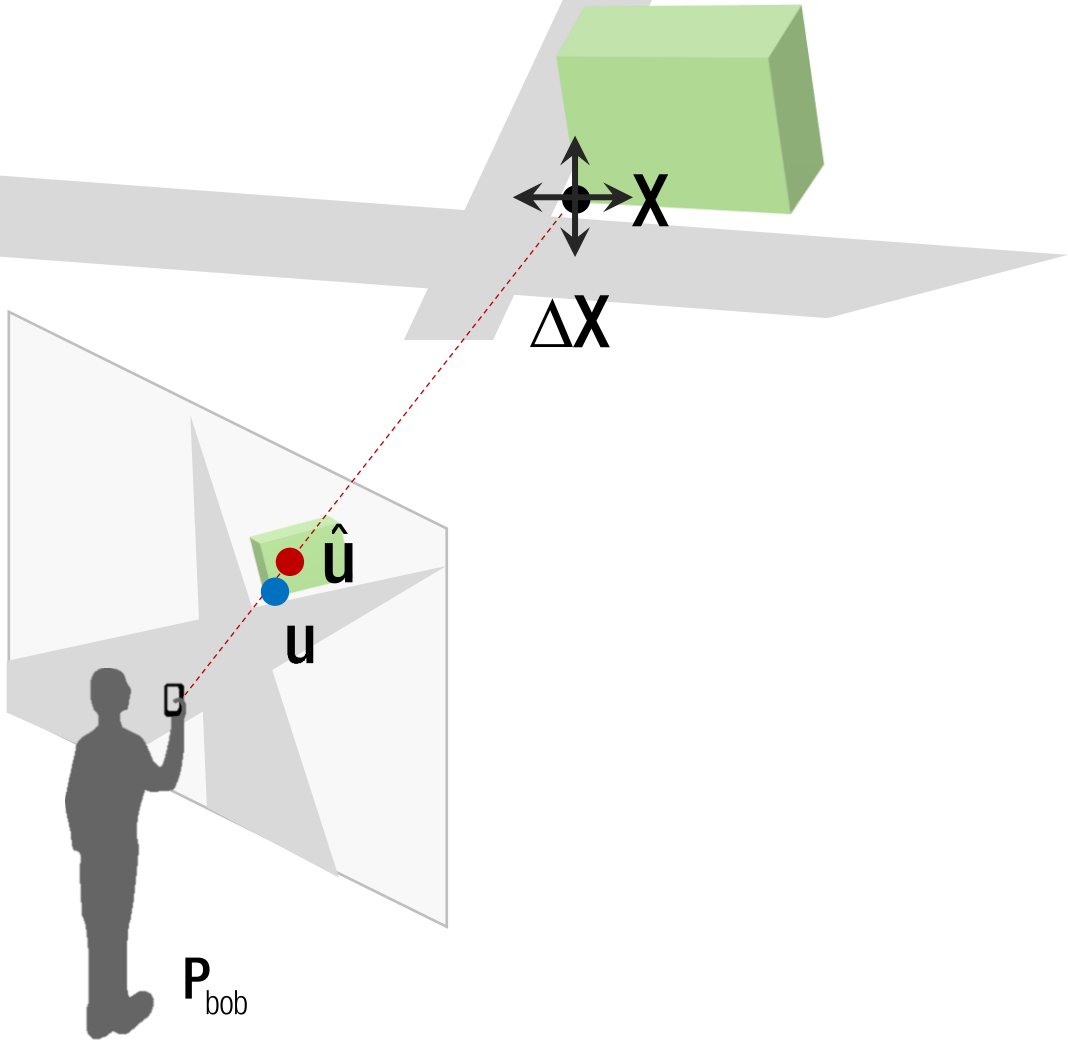
$$= \left(\frac{\mathbf{P}_1\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - x\right)^2 + \left(\frac{\mathbf{P}_2\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - y\right)^2$$

$$\Delta\mathbf{x} = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

# Point Jacobian

Black: given variables
Red: unknowns

$$E_{\text{geom}} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}(\mathbf{X} - \mathbf{C})$$

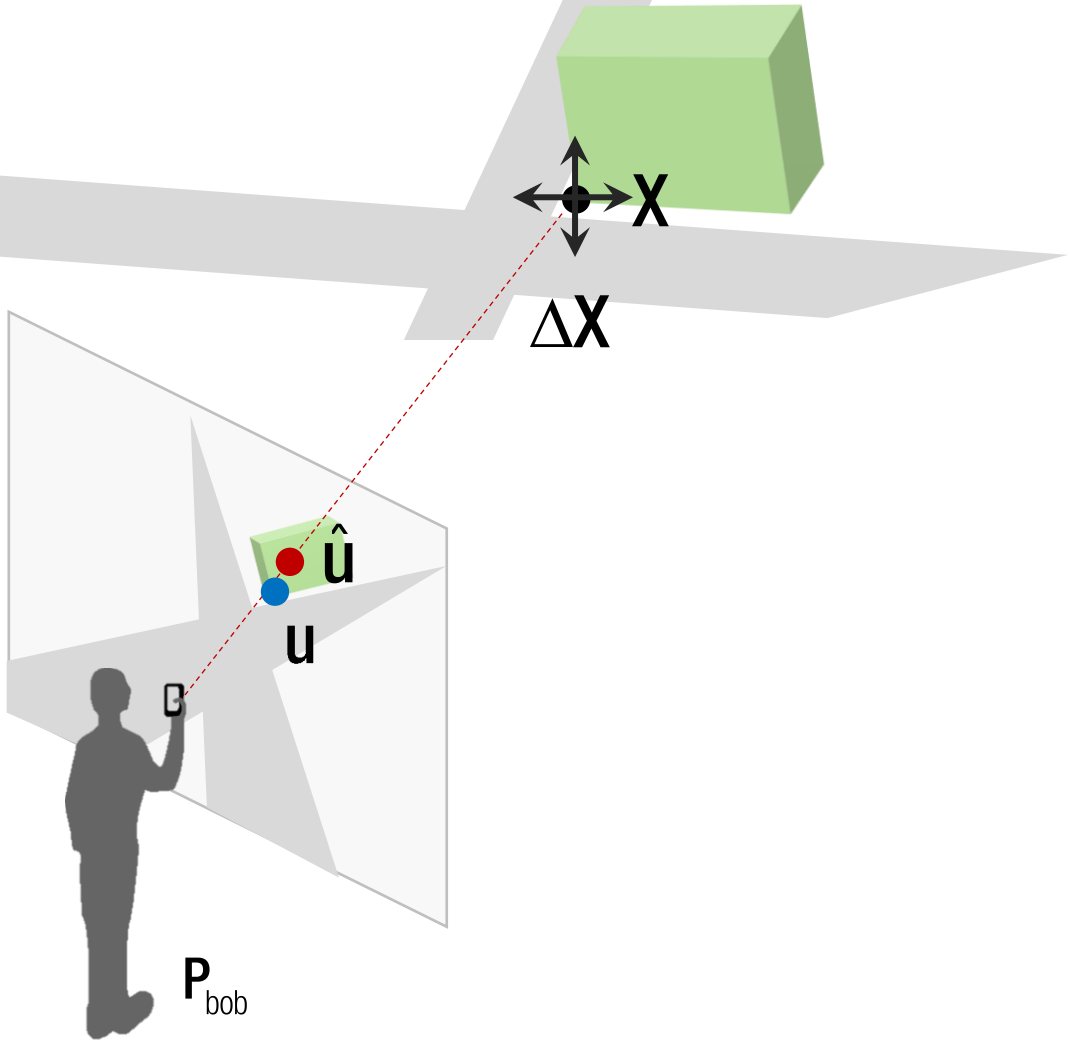



$$\Delta\mathbf{x} = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

# Point Jacobian

Black: given variables
Red: unknowns

$$E_{\text{geom}} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}(\mathbf{X} - \mathbf{C})$$



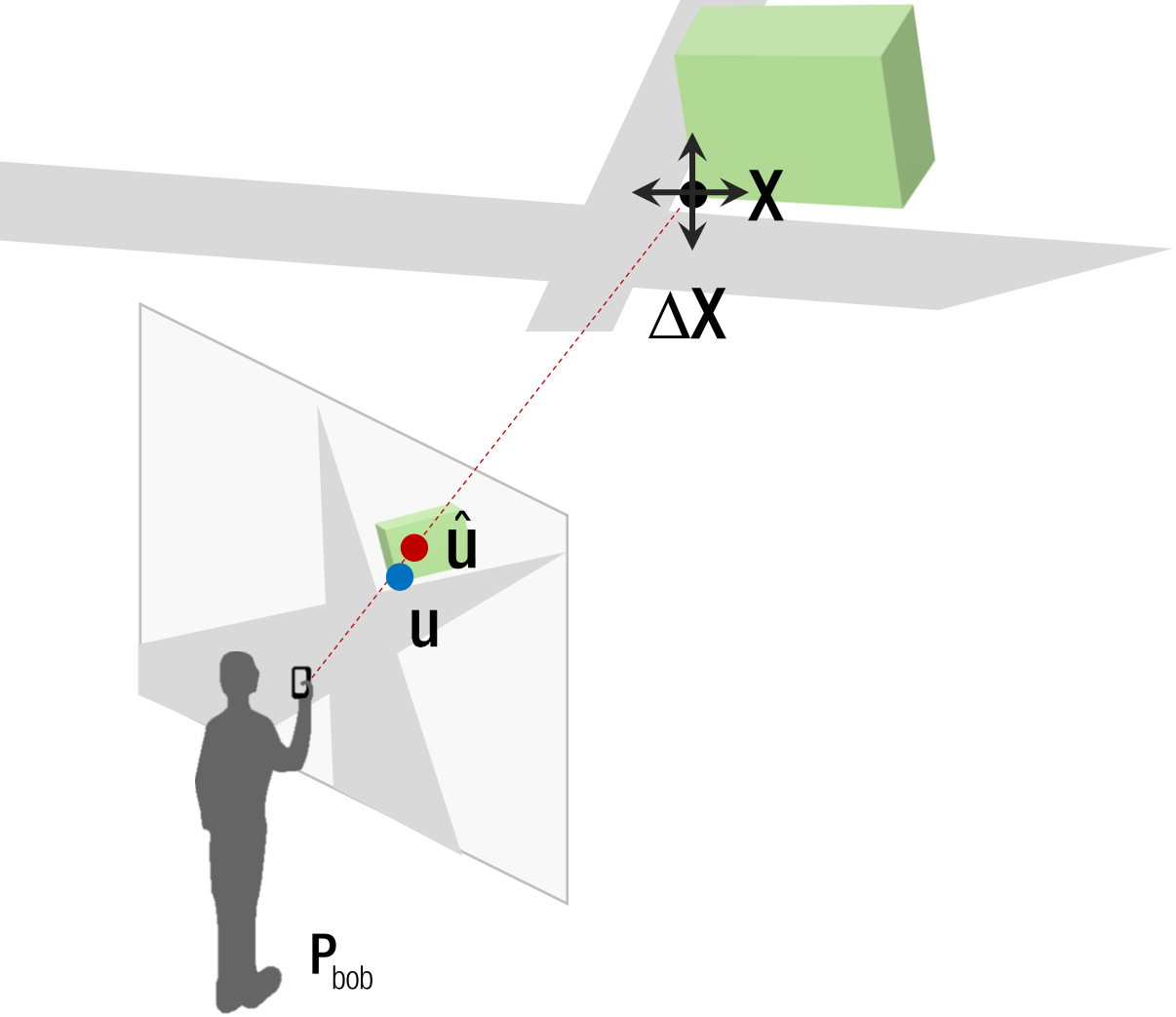

$$f(\mathbf{X}) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix}$$

$$\Delta\mathbf{x} = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

# Point Jacobian

Black: given variables
Red: unknowns

$$E_{geom} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}(\mathbf{X} - \mathbf{C})$$

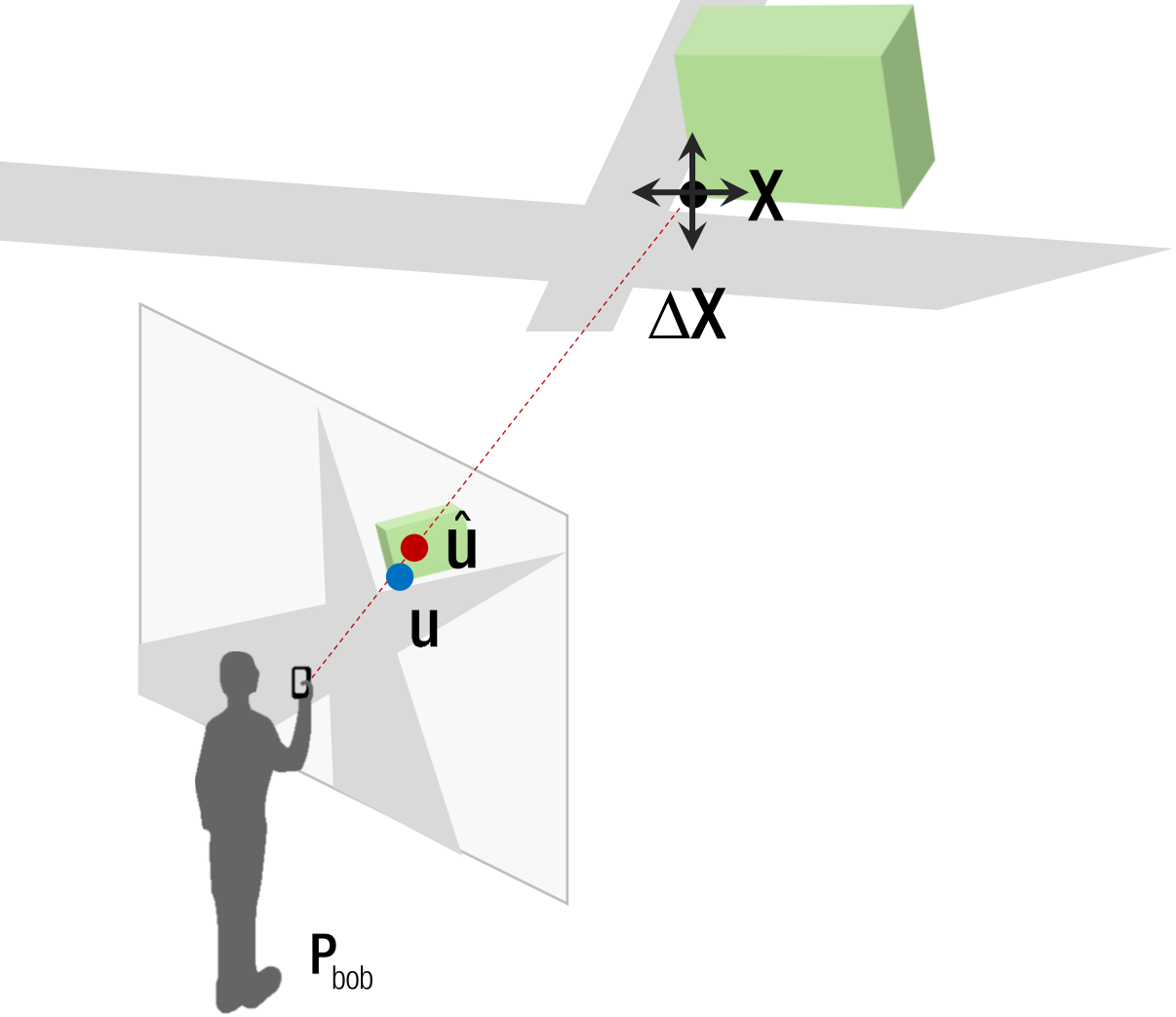



$$f(\mathbf{X}) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} \longrightarrow \frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \frac{w\frac{\partial u}{\partial \mathbf{X}} - u\frac{\partial w}{\partial \mathbf{X}}}{w^2} \\ \frac{v\frac{\partial u}{\partial \mathbf{X}} - v\frac{\partial w}{\partial \mathbf{X}}}{w^2} \end{bmatrix}$$

$$\Delta \mathbf{x} = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

# Point Jacobian

Black: given variables
Red: unknowns




$$E_{geom} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}(\mathbf{X} - \mathbf{C})$$

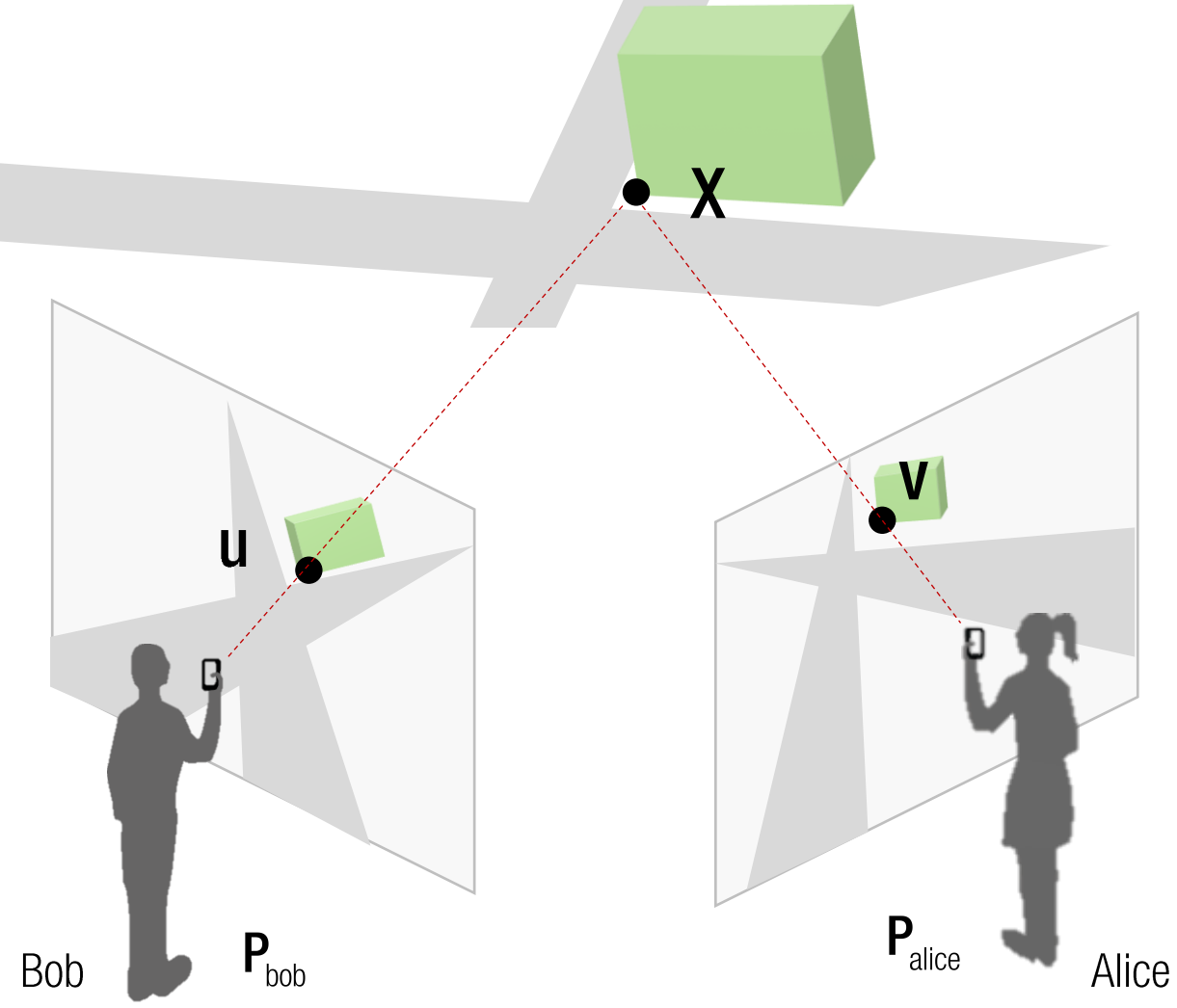
$$\rightarrow \frac{\partial}{\partial \mathbf{X}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}$$

$$f(\mathbf{X}) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} \quad \rightarrow \quad \frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \frac{w\frac{\partial u}{\partial \mathbf{X}} - u\frac{\partial w}{\partial \mathbf{X}}}{w^2} \\ \frac{v\frac{\partial u}{\partial \mathbf{X}} - v\frac{\partial w}{\partial \mathbf{X}}}{w^2} \end{bmatrix}$$

$$\Delta \mathbf{x} = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

# Point Jacobian

Black: given variables
Red: unknowns



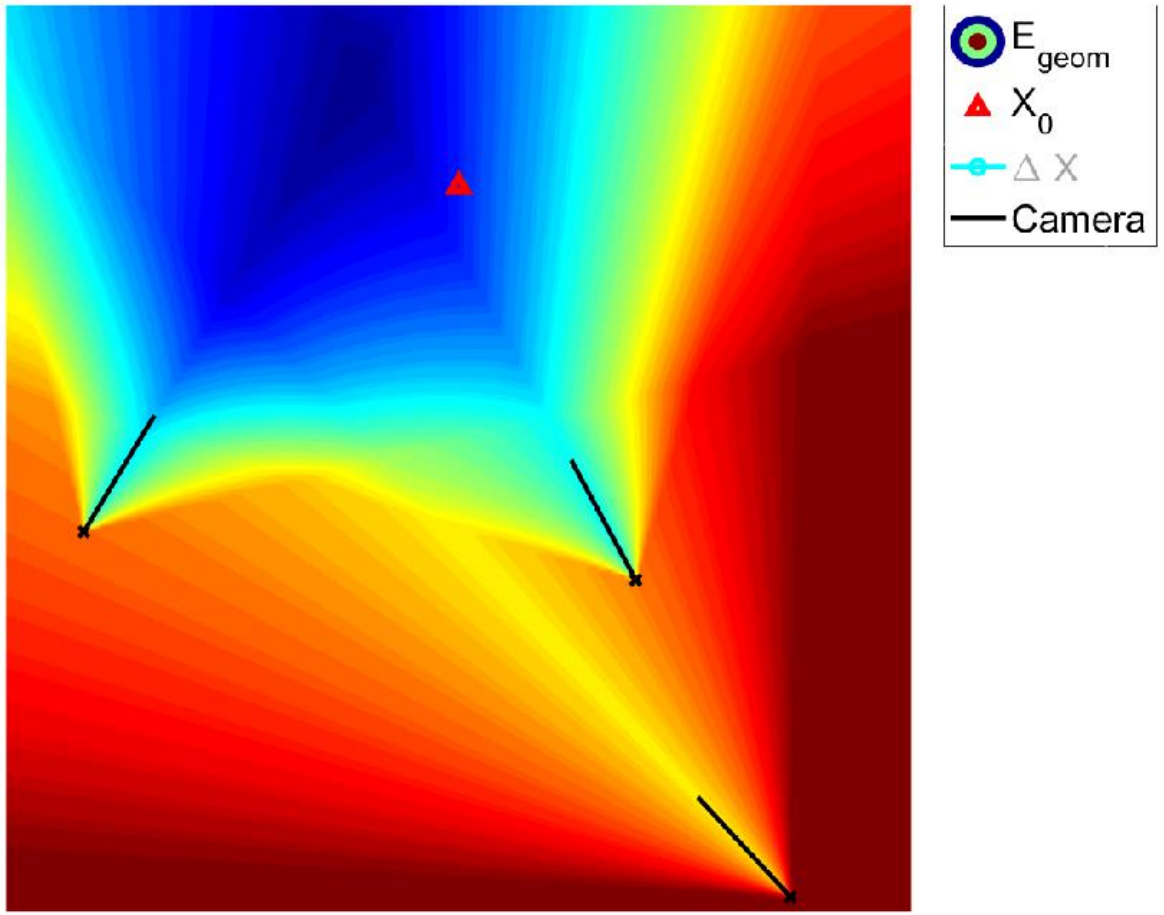

$$E_{geom} = \left\| \begin{bmatrix} u_{bob}/w_{bob} \\ v_{bob}/w_{bob} \\ u_{alice}/w_{alice} \\ v_{alice}/w_{alice} \end{bmatrix} - \begin{bmatrix} x_{bob} \\ y_{bob} \\ x_{alice} \\ y_{alice} \end{bmatrix} \right\|^2$$

$$\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \frac{w\frac{\partial u}{\partial \mathbf{X}} - u\frac{\partial w}{\partial \mathbf{X}}}{w^2} \\ \frac{v\frac{\partial u}{\partial \mathbf{X}} - v\frac{\partial w}{\partial \mathbf{X}}}{w^2} \end{bmatrix}$$

$$\Delta \mathbf{x} = \left( \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

Black: given variables
Red: unknowns

# Example: 1D Camera Triangulation

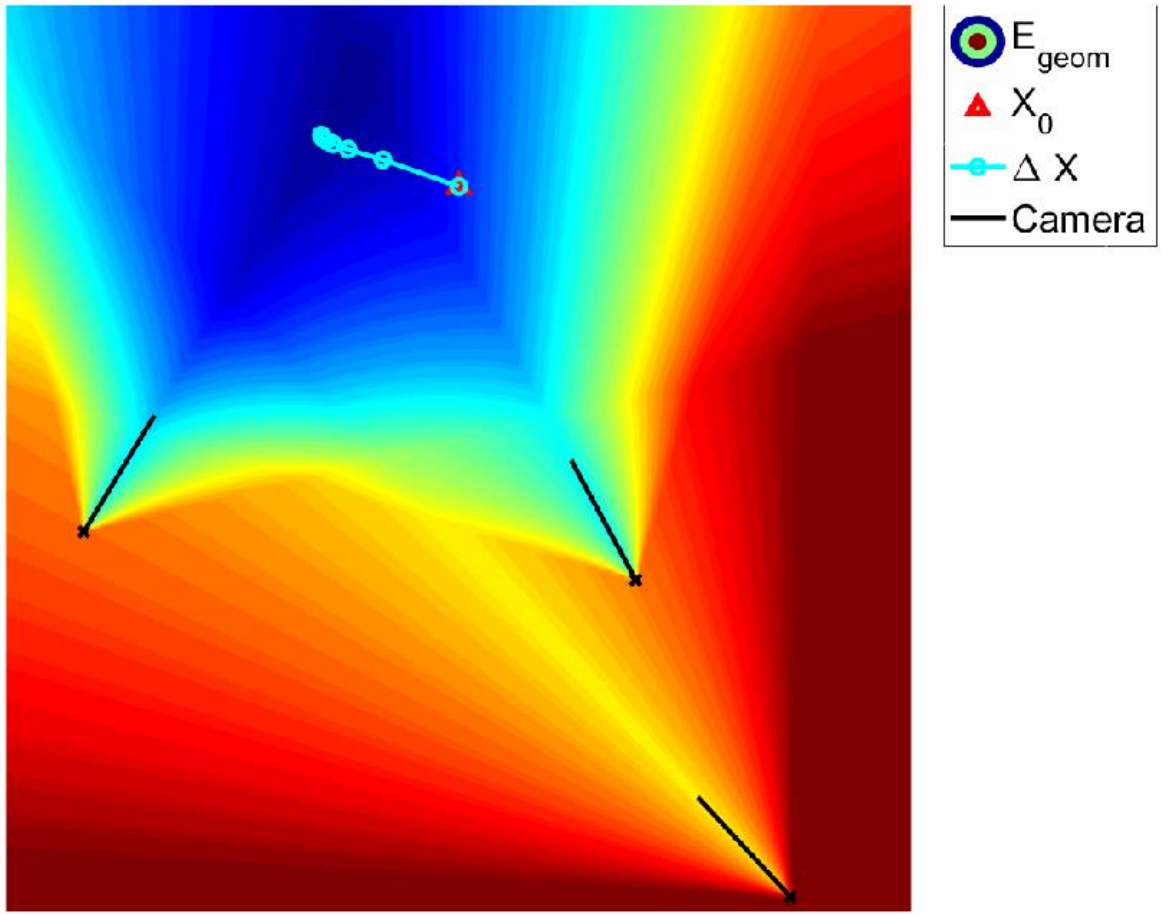



```
function df_dx = JacobianX_1D(K, R,  C,  X)

x = K * R * (X-C);

u = x(1);
w = x(2);

del = K * R;
du_dc = del(1,:);
dw_dc = del(2,:);

df_dx = [(w*du_dc-u*dw_dc)/w^2];
```

$$\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} = \begin{bmatrix} \dfrac{w\dfrac{\partial u}{\partial \mathbf{X}} - u\dfrac{\partial w}{\partial \mathbf{X}}}{w^2} \\ \dfrac{v\dfrac{\partial u}{\partial \mathbf{X}} - v\dfrac{\partial w}{\partial \mathbf{X}}}{w^2} \end{bmatrix}$$

Black: given variables
Red: unknowns

# Example: 1D Camera Triangulation



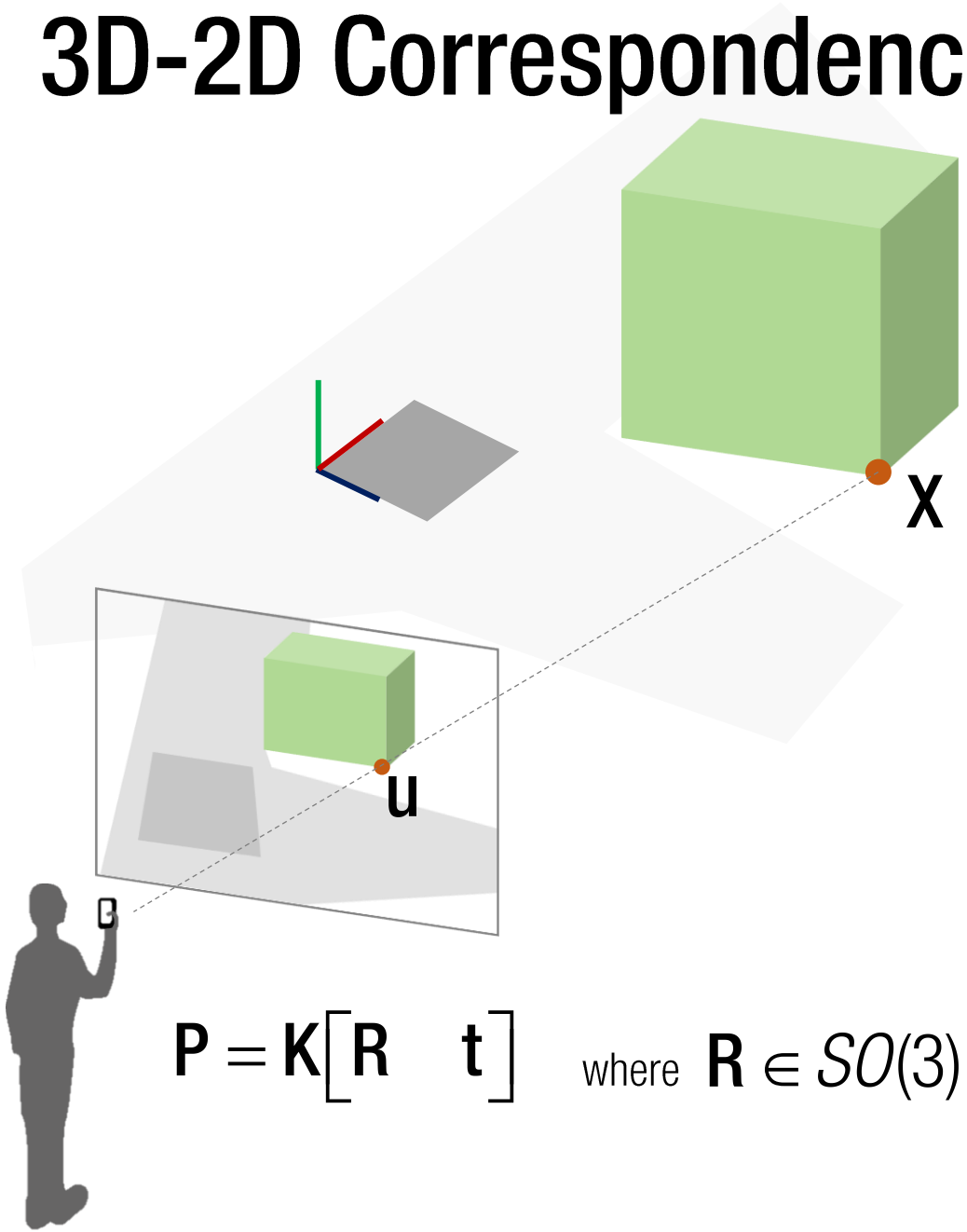

NonlinearTriangulation1D.m

```
for j = 1 : 10
   df_dX = [];
   delta_b = [];
   for i = 1 : size(c,2)
      df_dX = [df_dX; JacobianX(eye(2), R{i}, c(:,i), x)];
      u = R{i} * (x-c(:,i));
      delta_b = [delta_b; -u(1)/u(2)];
   end

   jacobian = df_dX;

   norm(delta_b)

   delta_x = inv(jacobian'*jacobian)*jacobian'*delta_b;
   x = x + delta_x;
   X(:,j+1) = x;
end
```

$$\Delta\mathbf{x} = \left( \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

# PnP Refinement

# 3D-2D Correspondence



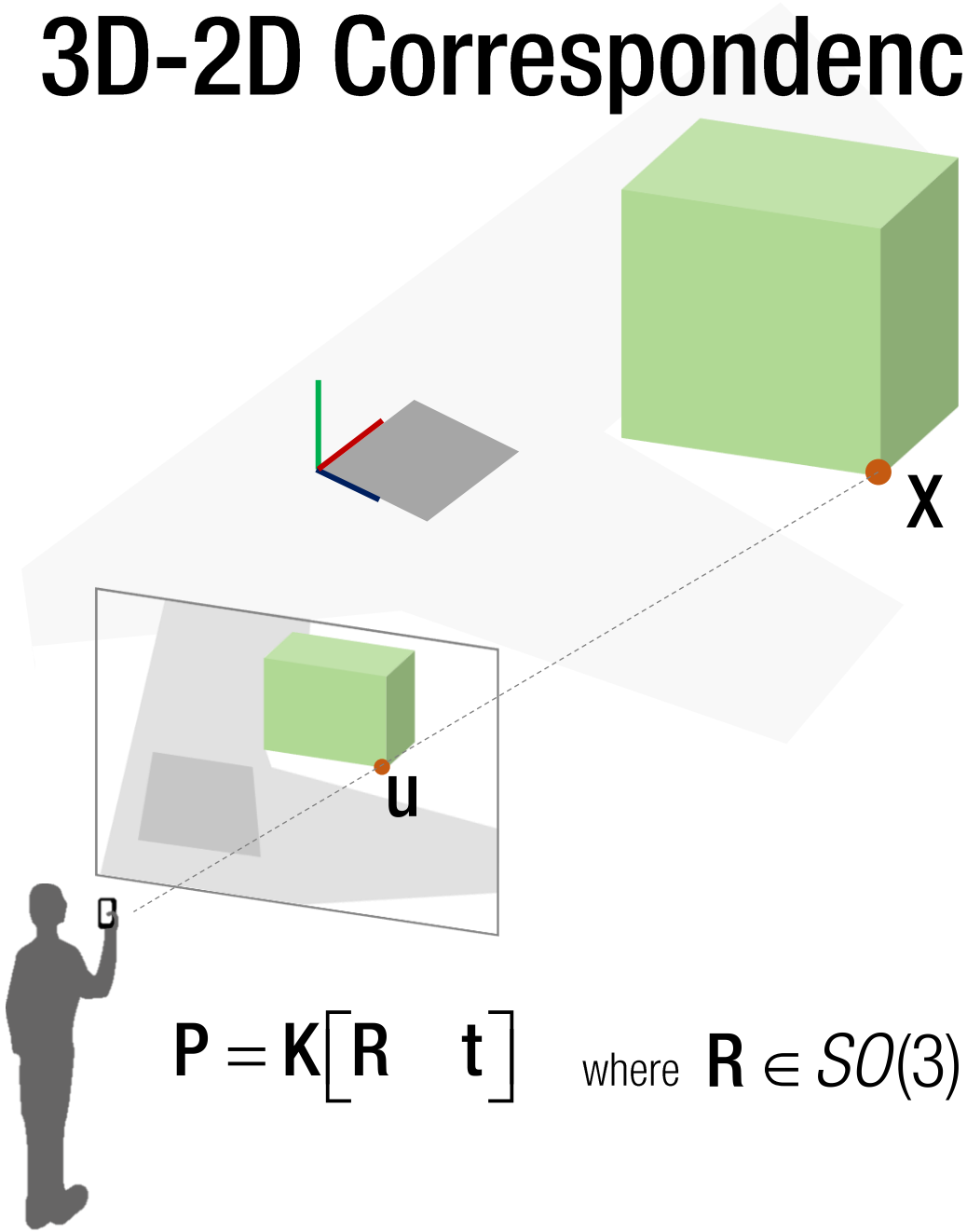

3D-2D correspondence: $\mathbf{u} \leftrightarrow \mathbf{X}$

$$\lambda \mathbf{u} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

# 3D-2D Correspondence

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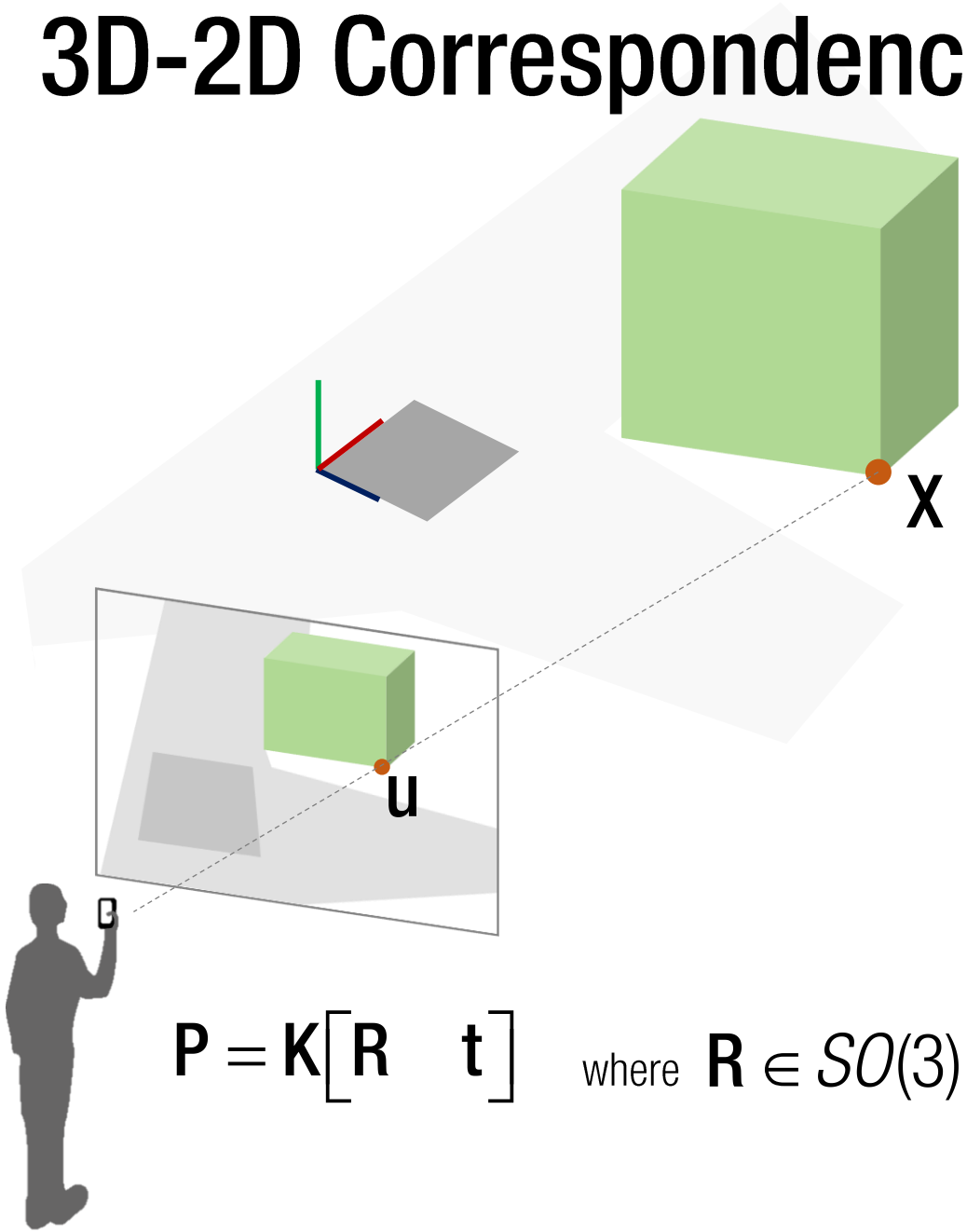
Linear in camera matrix

$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

# 3D-2D Correspondence




3D-2D correspondence: $\mathbf{u} \leftrightarrow \mathbf{X}$

$$\lambda \mathbf{u} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

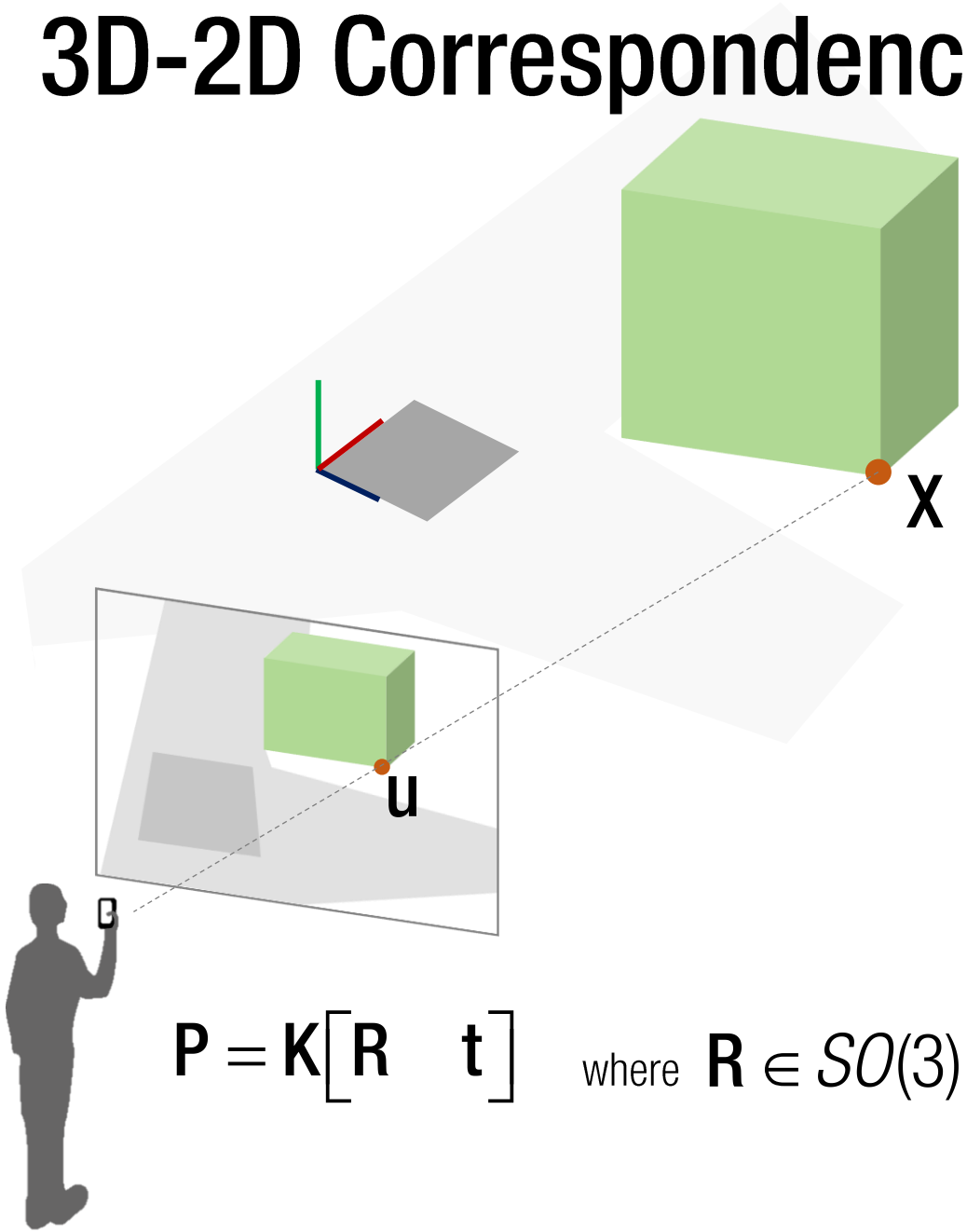
$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Linear in camera matrix

$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

# 3D-2D Correspondence

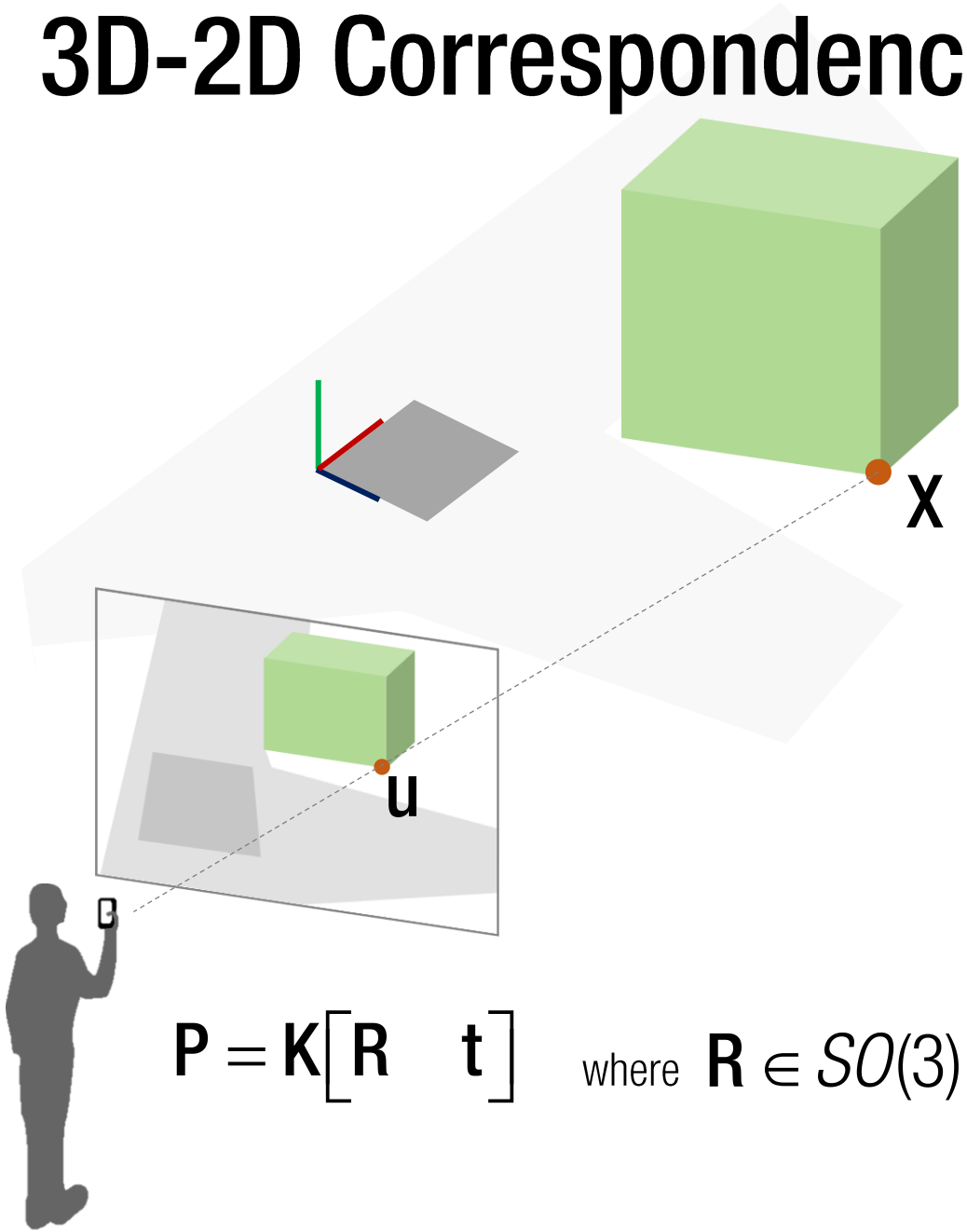



3D-2D correspondence: $\mathbf{u} \leftrightarrow \mathbf{X}$

$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

# 3D-2D Correspondence




$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where} \quad \mathbf{R} \in SO(3)$$

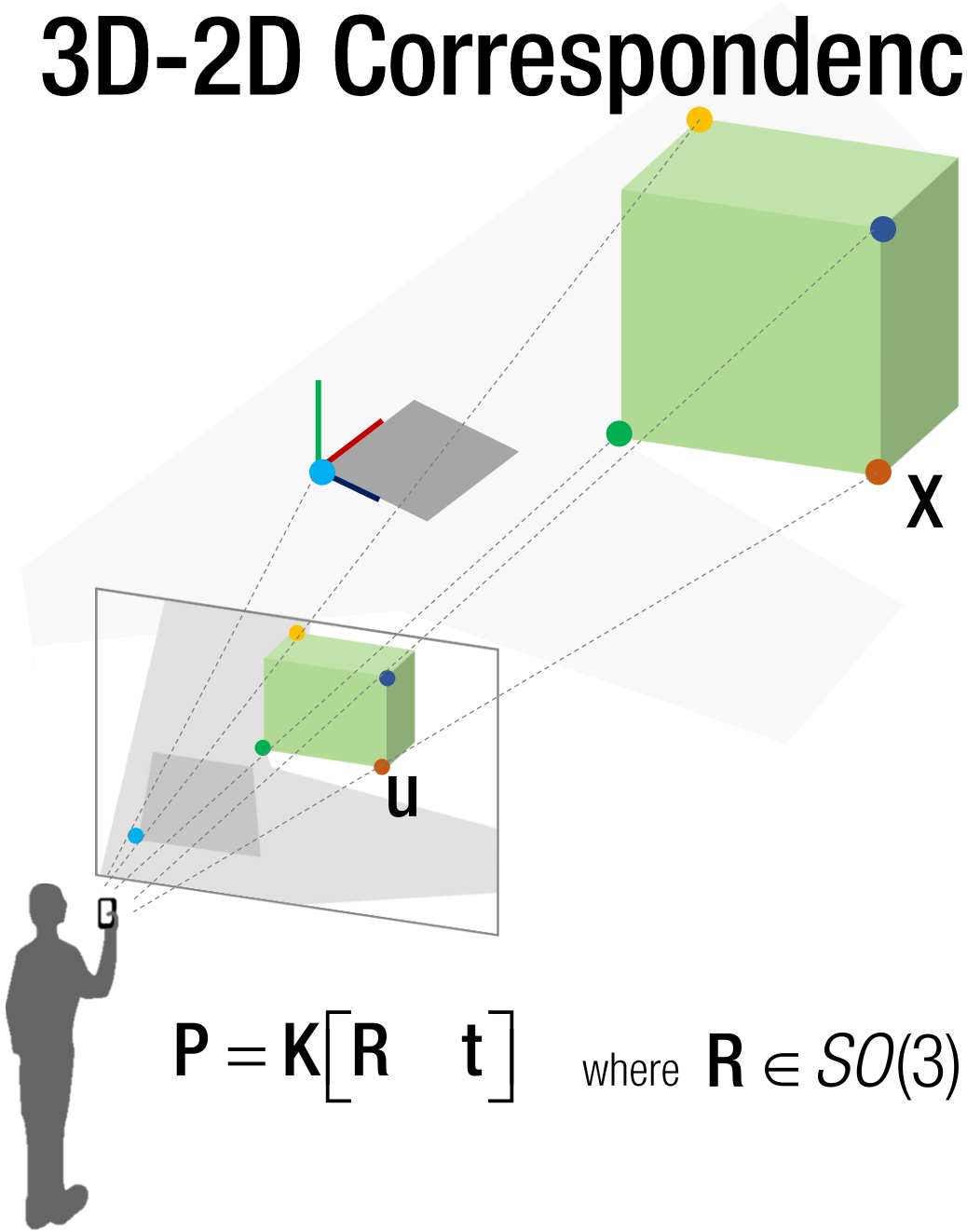
3D-2D correspondence: $\mathbf{u} \leftrightarrow \mathbf{X}$

$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

$$\begin{bmatrix} X & Y & Z & 1 & & & & -u^xX & -u^xY & -u^xZ & -u^x \\ & & & & X & Y & Z & 1 & -u^yX & -u^yY & -u^yZ & -u^y \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2x12

# 3D-2D Correspondence




3D-2D correspondence: $\mathbf{u} \leftrightarrow \mathbf{X}$

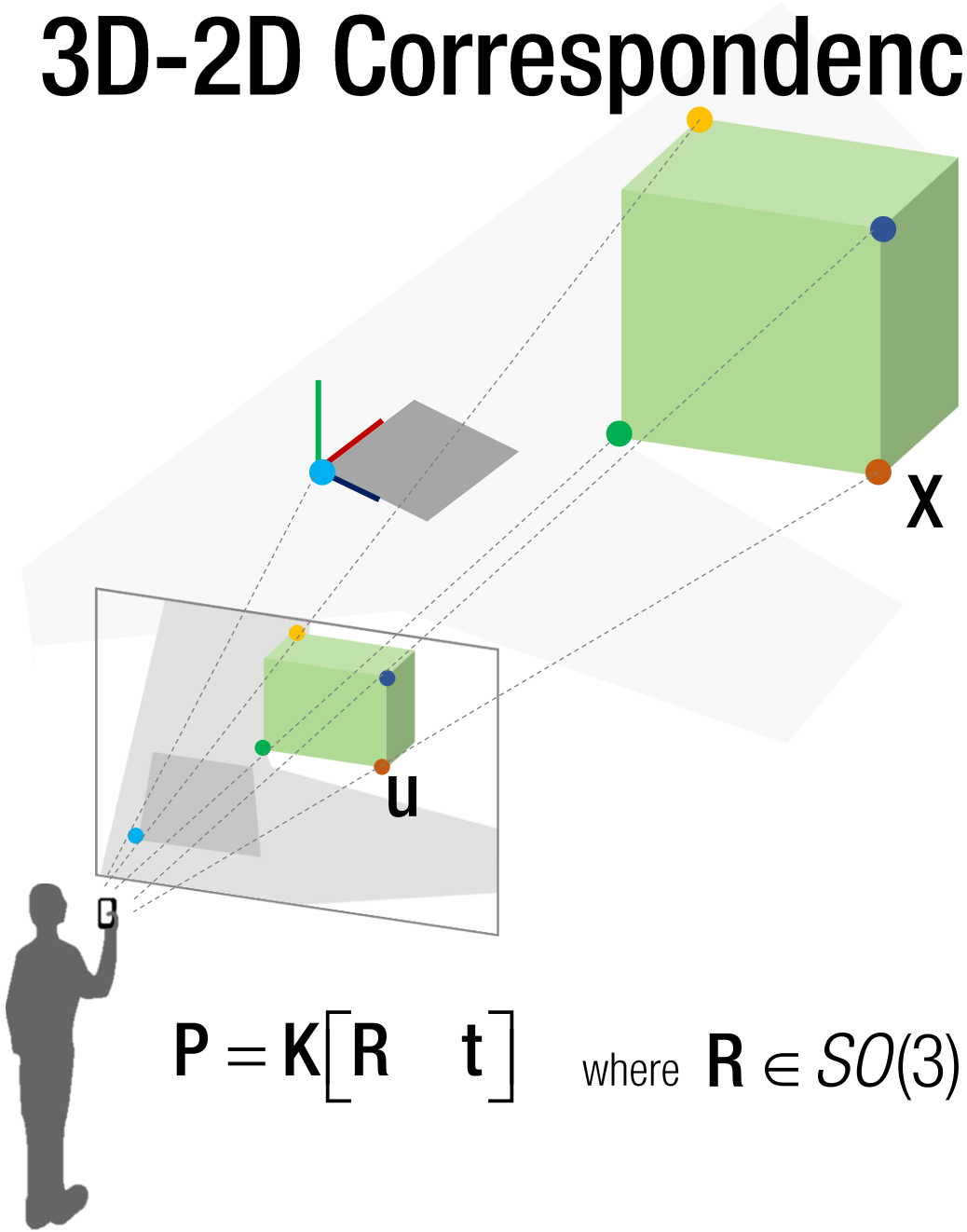
$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & & & & & -u_1^x X & -u_1^x Y & -u_1^x Z & -u_1^x \\ & & & & X_1 & Y_1 & Z_1 & 1 & -u_1^y X & -u_1^y Y & -u_1^y Z & -u_1^y \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_m & Y_m & Z_m & 1 & & & & & -u_m^x X & -u_m^x Y & -u_m^x Z & -u_m^x \\ & & & & X_m & Y_m & Z_m & 1 & -u_m^y X & -u_m^y Y & -u_m^y Z & -u_m^y \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$2m \times 12$

$$\mathbf{P} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \quad \text{where} \quad \mathbf{R} \in SO(3)$$

# 3D-2D Correspondence




3D-2D correspondence: $\mathbf{u} \leftrightarrow \mathbf{X}$

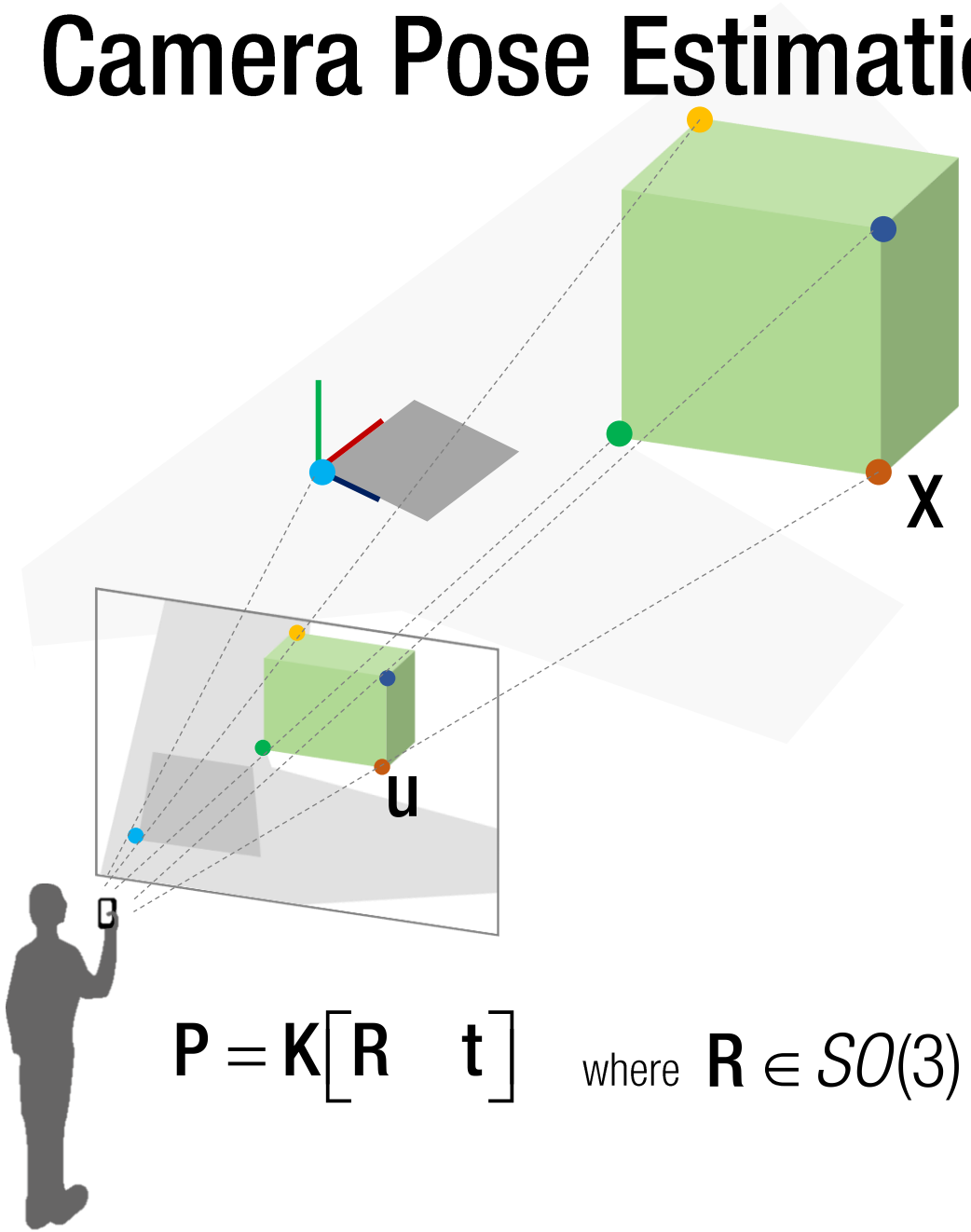
$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & & & & -u_1^x X & -u_1^x Y & -u_1^x Z & -u_1^x \\ & & & & X_1 & Y_1 & Z_1 & 1 & -u_1^y X & -u_1^y Y & -u_1^y Z & -u_1^y \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ X_m & Y_m & Z_m & 1 & & & & -u_m^x X & -u_m^x Y & -u_m^x Z & -u_m^x \\ & & & & X_m & Y_m & Z_m & 1 & -u_m^y X & -u_m^y Y & -u_m^y Z & -u_m^y \end{bmatrix} \mathbf{A} \quad \mathbf{x} \quad \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$2m \times 12$

$$\mathbf{P} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \quad \text{where} \quad \mathbf{R} \in SO(3)$$

# Camera Pose Estimation




$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

If $\mathbf{K}$ is given,

$$\mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \gamma\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4\end{bmatrix}$$

$$\longrightarrow \gamma\mathbf{R} = \mathbf{K}^{-1}\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix}$$

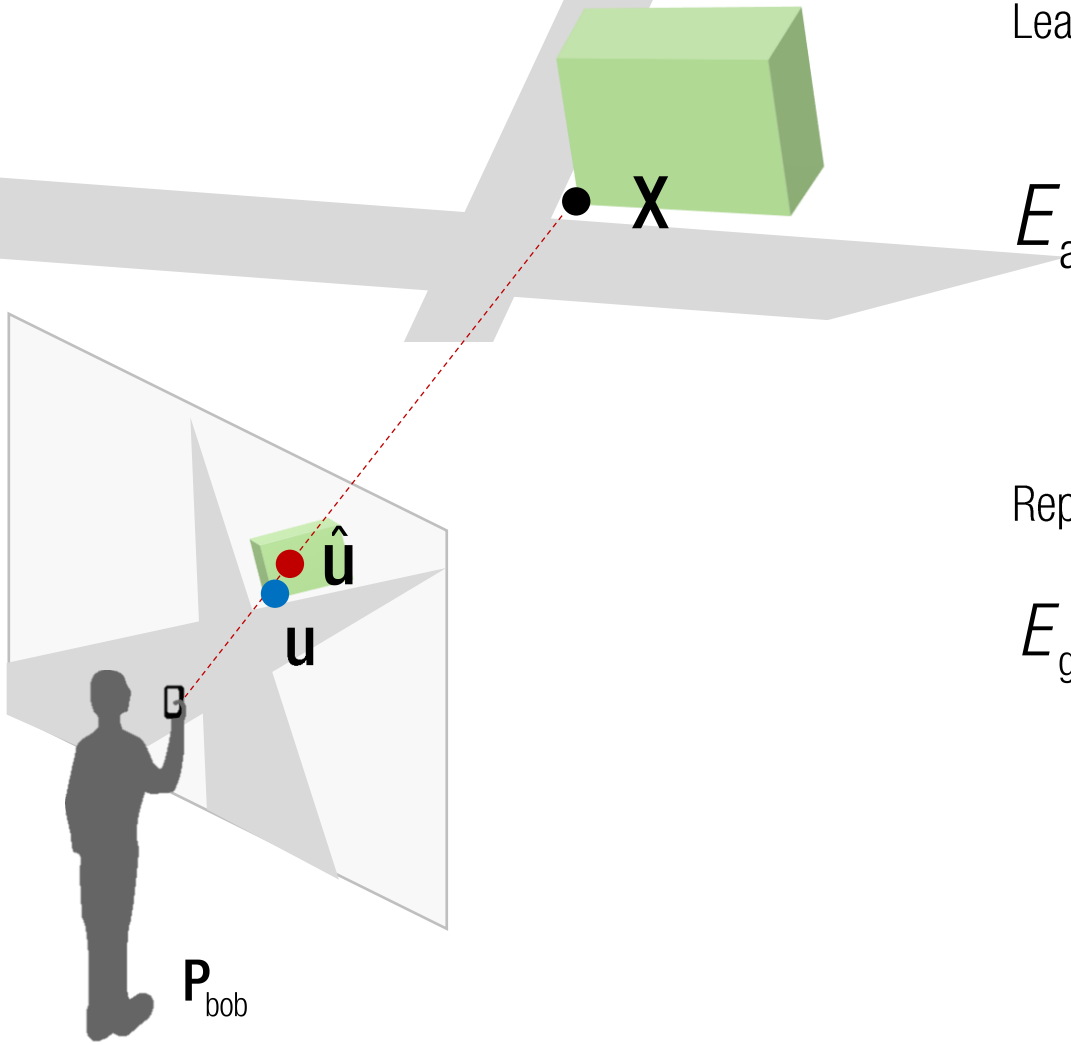
$$\mathbf{K}^{-1}\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix} = \mathbf{U}\begin{bmatrix} d_{11} & & \\ & d_{22} & \\ & & d_{33}\end{bmatrix}\mathbf{V}^{\mathsf{T}}$$

$$\longrightarrow \gamma \approx d_{11}$$

$$\mathbf{R} = \mathbf{U}\mathbf{V}^{\mathsf{T}} \quad : \text{SVD cleanup}$$

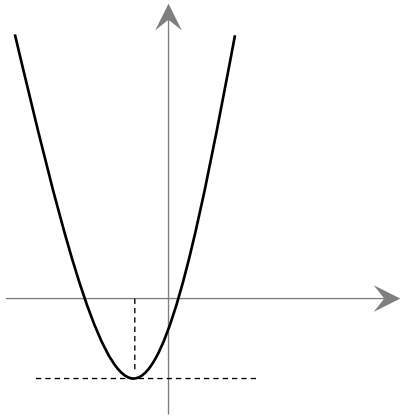
$$\longrightarrow \mathbf{t} = \frac{\mathbf{K}^{-1}\mathbf{p}_4}{d_{11}} \quad : \text{Translation and scale recovery}$$

# Algebraic vs. Geometric error




Least squares solution (algebraic error):

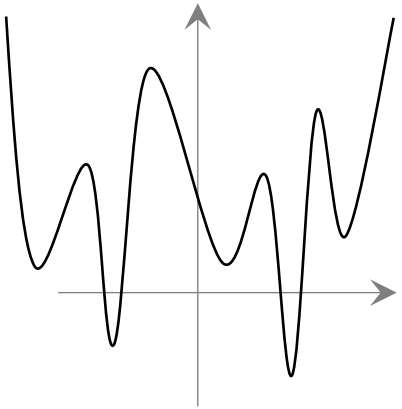
$$E_{\text{alge}} = \| \mathbf{A}\ \mathbf{x} - \mathbf{b} \|^2$$



Reprojection error (geometric error):

$$E_{\text{geom}} = \|\hat{\mathbf{u}} - \mathbf{u}\|^2$$

$$= \left(\frac{\mathbf{P}_1\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - \mathbf{u}_1\right)^2 + \left(\frac{\mathbf{P}_2\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - \mathbf{u}_2\right)^2$$

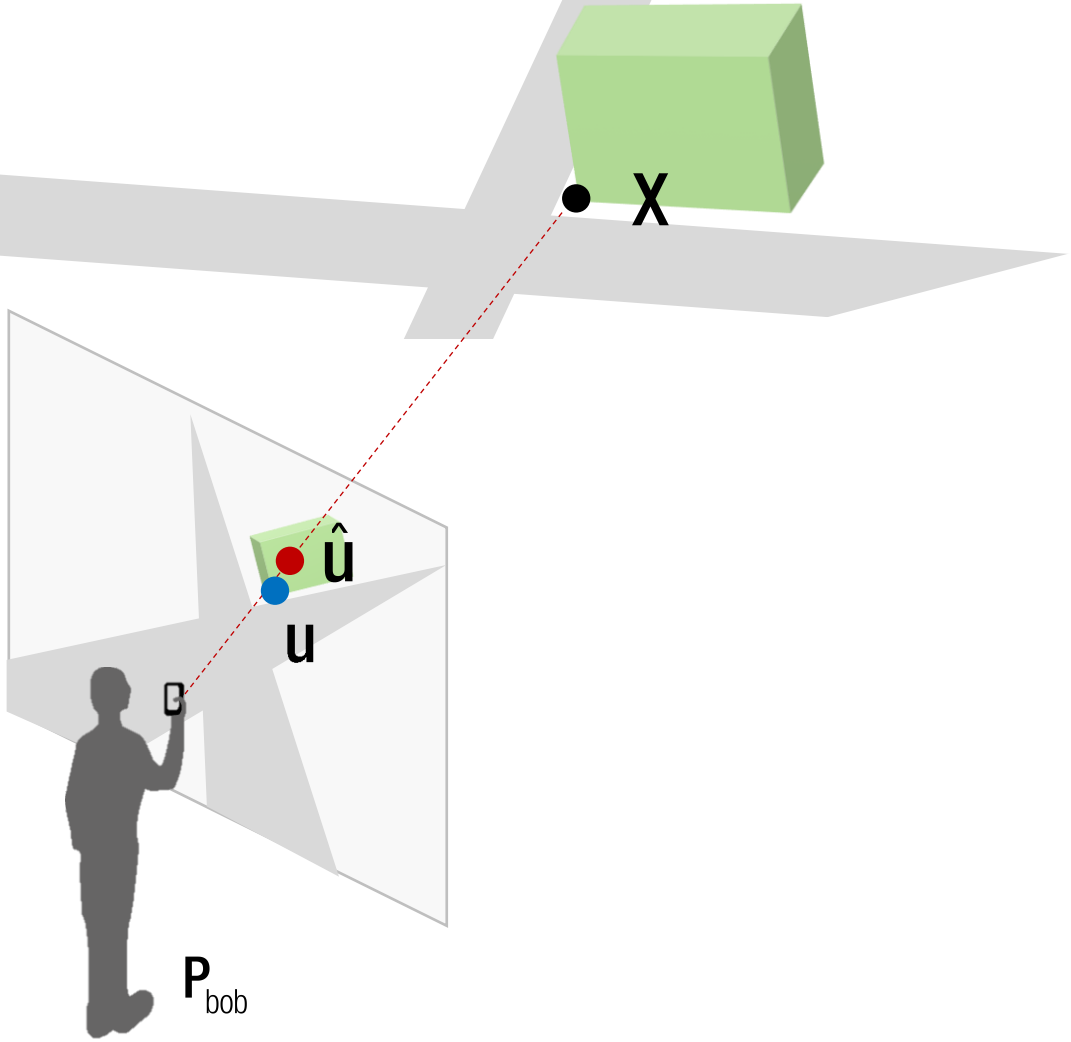


# Point Jacobian

Black: given variables
Red: unknowns

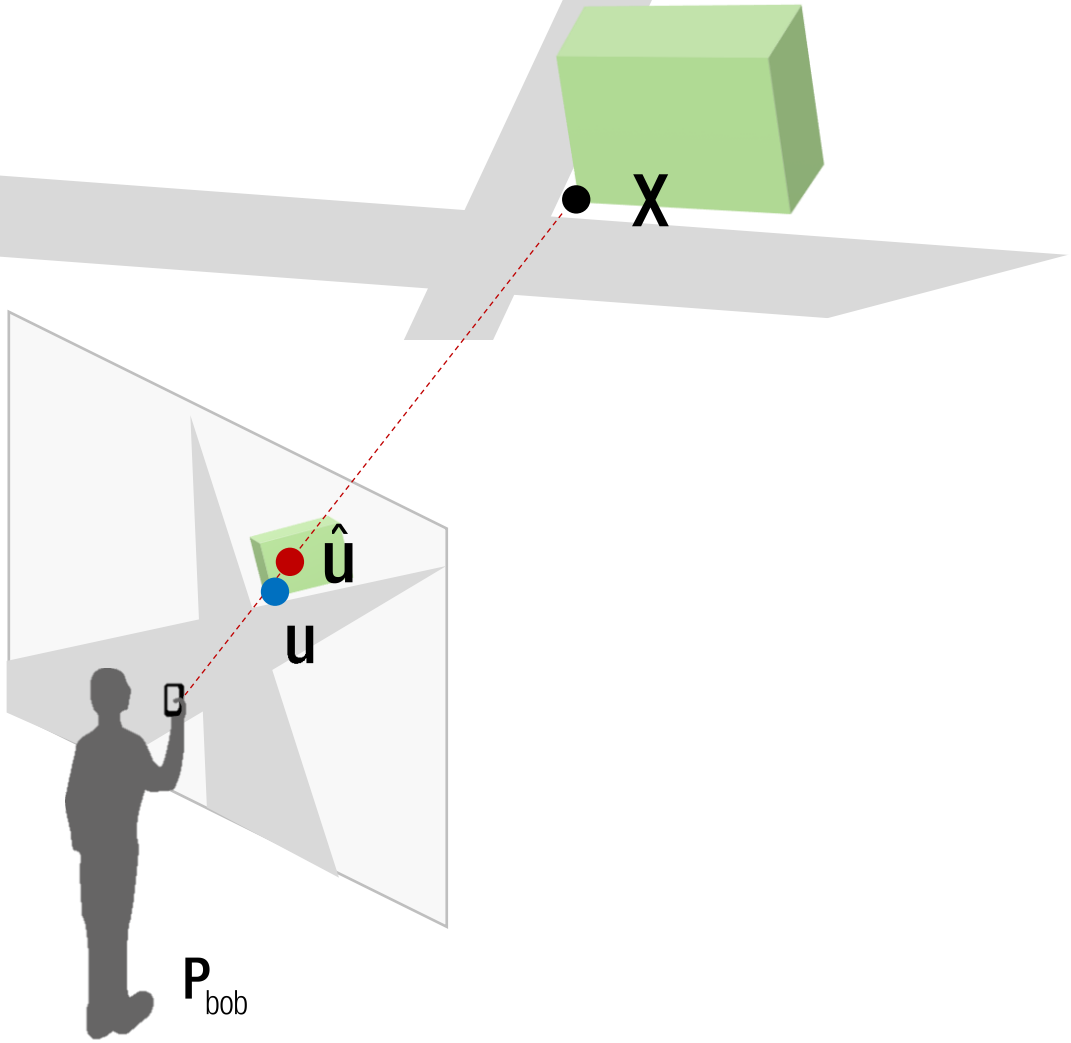
$$E_{\text{geom}} = \left\| \hat{\mathbf{u}} - \mathbf{u} \right\|^2$$

$$= \left( \frac{\mathbf{P}_1 \mathbf{X}}{\mathbf{P}_3 \mathbf{X}} - x \right)^2 + \left( \frac{\mathbf{P}_2 \mathbf{X}}{\mathbf{P}_3 \mathbf{X}} - y \right)^2$$



Black: given variables
Red: unknowns

# Camera Jacobian

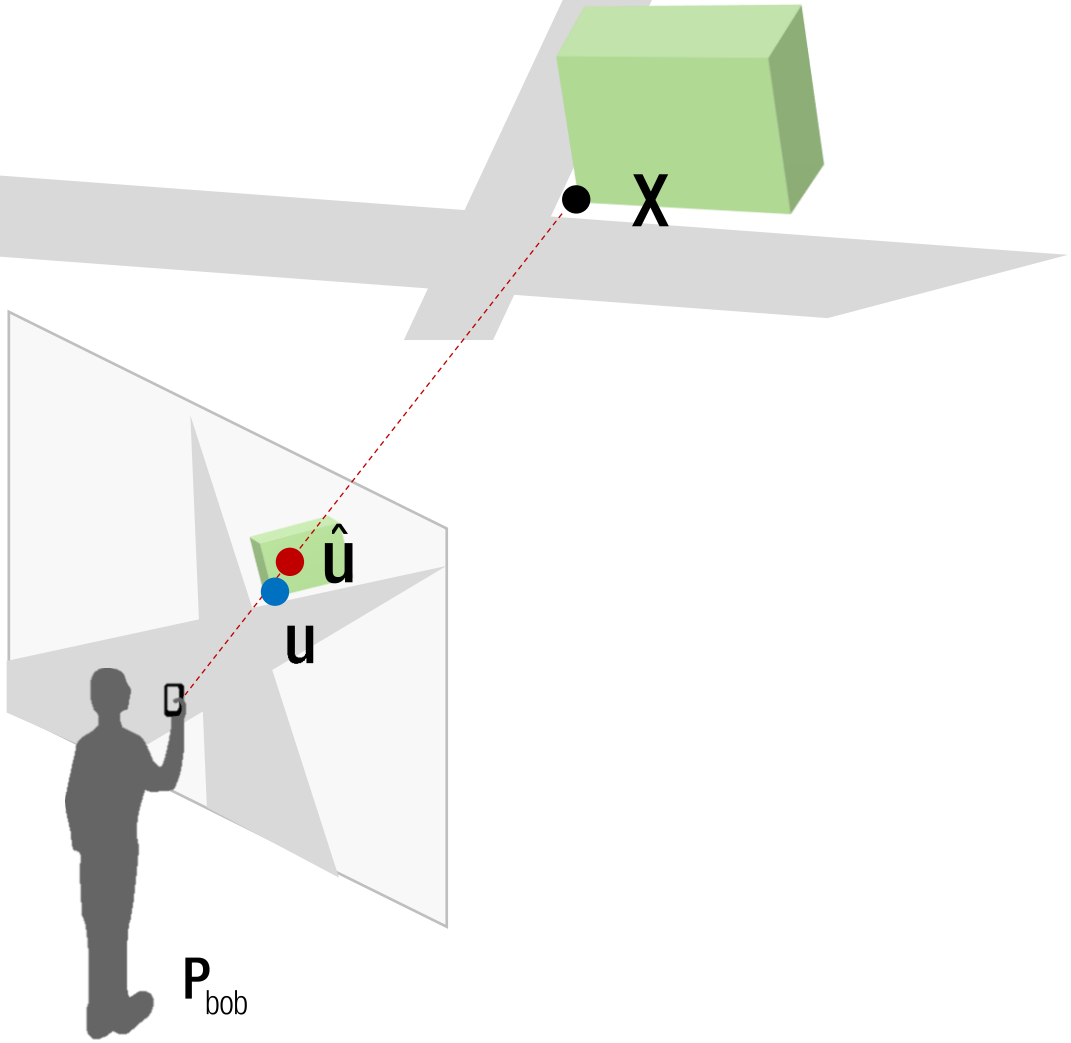



$$E_{\text{geom}} = \|\hat{\mathbf{u}} - \mathbf{u}\|^2$$

$$= \left(\frac{\mathbf{P}_1\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - x\right)^2 + \left(\frac{\mathbf{P}_2\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - y\right)^2$$

Black: given variables
Red: unknowns

# Camera Jacobian

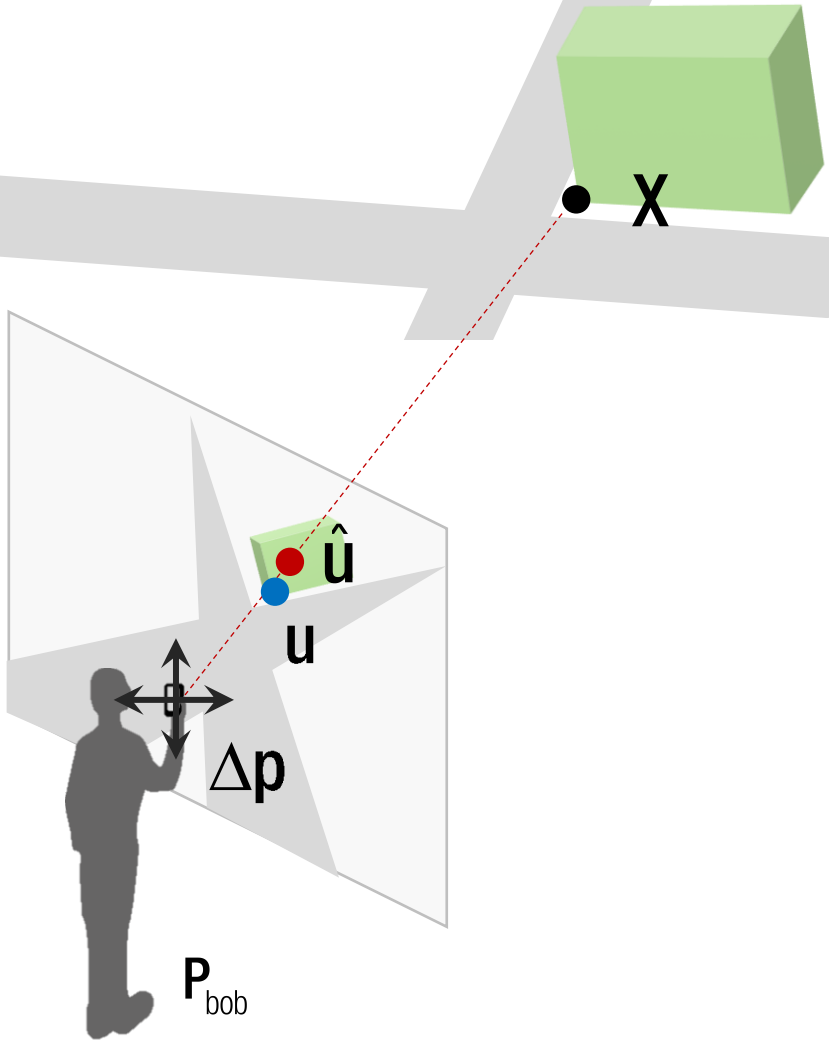



$$E_{\text{geom}} = \|\hat{\mathbf{u}} - \mathbf{u}\|^2$$

$$= \left(\frac{\mathbf{P}_1\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - x\right)^2 + \left(\frac{\mathbf{P}_2\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - y\right)^2$$

Black: given variables
Red: unknowns

# Camera Jacobian




$$E_{geom} = \|\hat{\mathbf{u}} - \mathbf{u}\|^2$$

$$= \left(\frac{\mathbf{P}_1\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - x\right)^2 + \left(\frac{\mathbf{P}_2\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - y\right)^2$$

$$\Delta\mathbf{p} = \left(\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}^{\mathsf{T}} \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}\right)^{-1} \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{p})\right)$$

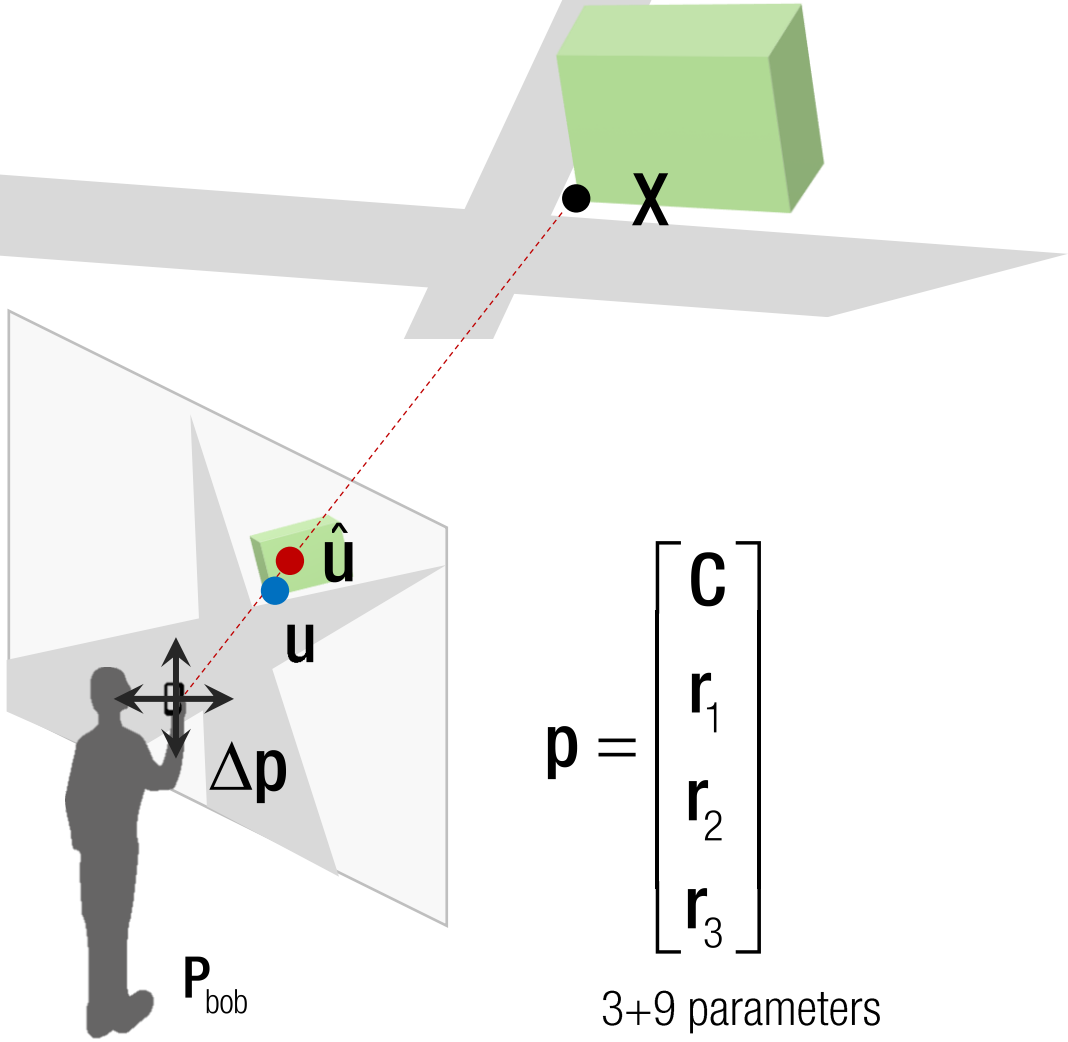
# Camera Jacobian

Black: given variables
Red: unknowns

$$E_{\text{geom}} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}(\mathbf{X} - \mathbf{C})$$

$$\rightarrow \quad \frac{\partial}{\partial \mathbf{C}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = -\mathbf{KR}$$

$$\rightarrow \quad \frac{\partial}{\partial \mathbf{R}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11}(\mathbf{X} - \mathbf{C}) & \mathbf{0}_{1\times3} & \mathbf{K}_{13}(\mathbf{X} - \mathbf{C}) \\ \mathbf{0}_{1\times3} & \mathbf{K}_{22}(\mathbf{X} - \mathbf{C}) & \mathbf{K}_{23}(\mathbf{X} - \mathbf{C}) \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & (\mathbf{X} - \mathbf{C}) \end{bmatrix}$$



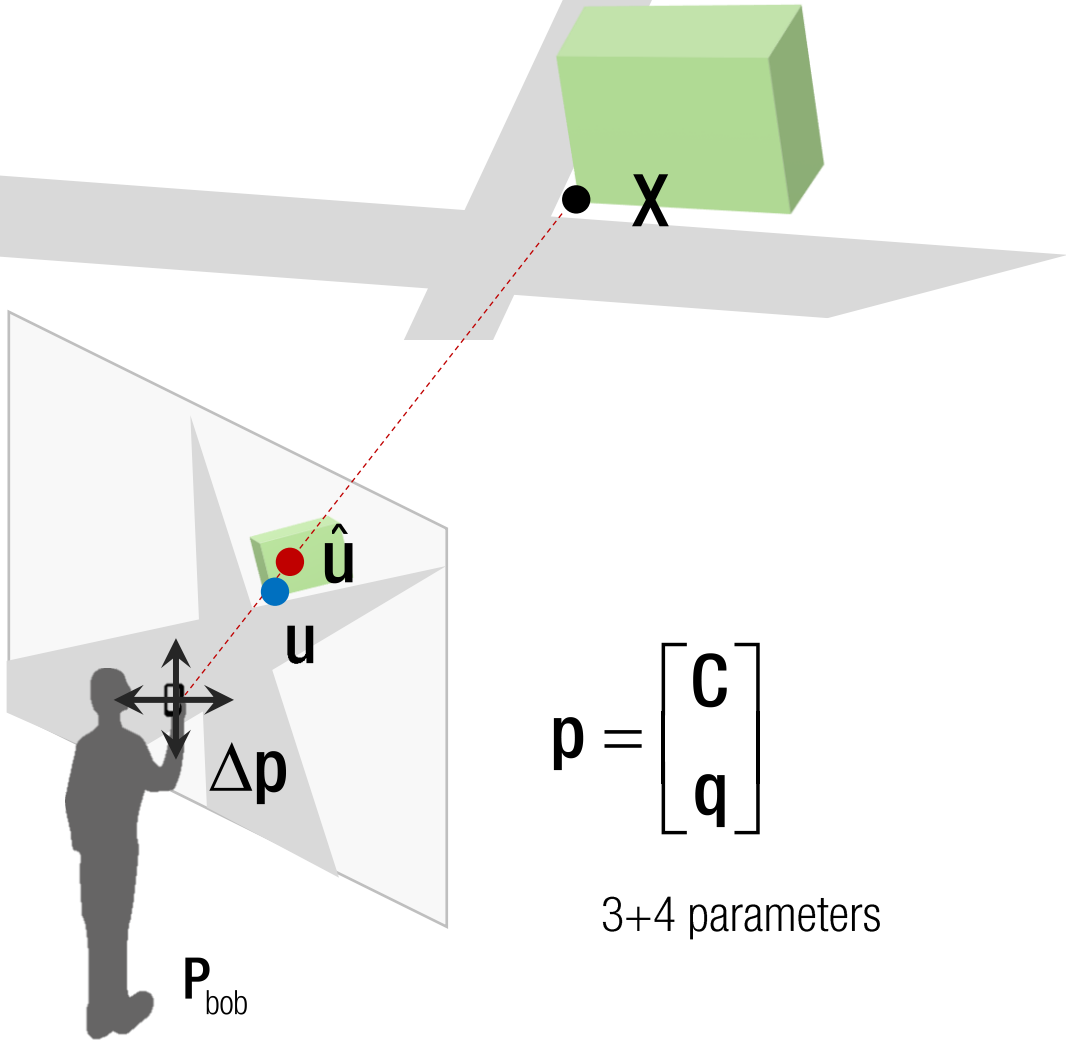

$$\mathbf{p} = \begin{bmatrix} \mathbf{C} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

3+9 parameters

$$\Delta\mathbf{p} = \left(\frac{\partial f(\mathbf{p})^T}{\partial \mathbf{p}} \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}\right)^{-1} \frac{\partial f(\mathbf{p})^T}{\partial \mathbf{p}} \left(\mathbf{b} - f(\mathbf{p})\right)$$

# Camera Jacobian

Black: given variables
Red: unknowns




$$E_{geom} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}(\mathbf{X} - \mathbf{C})$$

$$\rightarrow \quad \frac{\partial}{\partial \mathbf{C}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = -\mathbf{KR}$$

$$\rightarrow \quad \frac{\partial}{\partial \mathbf{R}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11}(\mathbf{X} - \mathbf{C}) & \mathbf{0}_{1\times3} & \mathbf{K}_{13}(\mathbf{X} - \mathbf{C}) \\ \mathbf{0}_{1\times3} & \mathbf{K}_{22}(\mathbf{X} - \mathbf{C}) & \mathbf{K}_{23}(\mathbf{X} - \mathbf{C}) \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & (\mathbf{X} - \mathbf{C}) \end{bmatrix}$$

$$\rightarrow \quad \frac{\partial}{\partial \mathbf{q}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{\partial}{\partial \mathbf{R}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \frac{\partial \mathbf{R}}{\partial \mathbf{q}} \quad : \text{Chain rule}$$

$$\mathbf{p} = \begin{bmatrix} \mathbf{C} \\ \mathbf{q} \end{bmatrix}$$

3+4 parameters

Quaternion jacobian

# Quaternion Jacobian

$$\mathbf{R} = \begin{bmatrix} 1-2q_z^2-2q_y^2 & -2q_zq_w+2q_yq_x & 2q_yq_w+2q_zq_x \\ 2q_xq_y+2q_wq_z & 1-2q_z^2-2q_x^2 & 2q_zq_y-2q_xq_w \\ 2q_xq_z-2q_wq_y & 2q_yq_z+2q_wq_x & 1-2q_y^2-2q_x^2 \end{bmatrix} \quad \text{where} \quad \mathbf{q} = \begin{bmatrix} q_w & q_x & q_y & q_z \end{bmatrix}^T$$

---

$$\underset{9x4}{\frac{\partial \mathbf{R}}{\partial \mathbf{q}}} = \begin{bmatrix} \frac{\partial r_{11}}{\partial \mathbf{q}} \\ \frac{\partial r_{12}}{\partial \mathbf{q}} \\ \vdots \\ \frac{\partial r_{33}}{\partial \mathbf{q}} \end{bmatrix}$$

where

$$\frac{\partial \mathbf{R}_{11}}{\partial \mathbf{q}} = \begin{bmatrix} 0 & -4q_y & -4q_z & 0 \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{12}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_y & 2q_x & -2q_w & -2q_z \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{13}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_z & 2q_w & 2q_x & 2q_y \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{21}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_y & 2q_x & 2q_w & 2q_z \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{22}}{\partial \mathbf{q}} = \begin{bmatrix} -4q_x & 0 & -4q_z & 0 \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{23}}{\partial \mathbf{q}} = \begin{bmatrix} -2q_w & 2q_z & 2q_y & 2q_x \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{31}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_z & -2q_w & 2q_x & -2q_y \end{bmatrix}$$

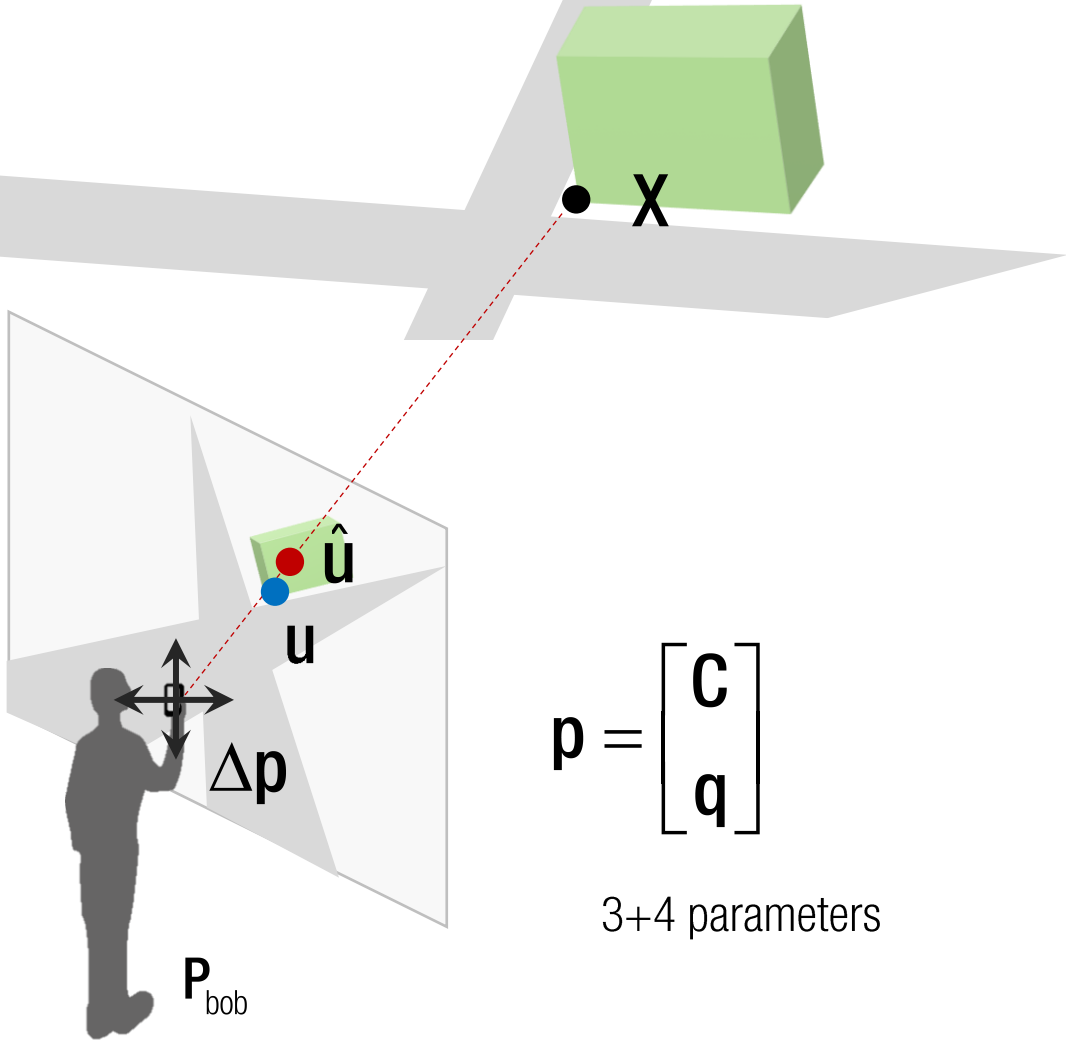
$$\frac{\partial \mathbf{R}_{33}}{\partial \mathbf{q}} = \begin{bmatrix} -4q_x & -4q_y & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{32}}{\partial \mathbf{q}} = \begin{bmatrix} 2q_w & 2q_z & 2q_y & 2q_x \end{bmatrix}$$

# Camera Jacobian

Black: given variables
Red: unknowns

$$E_{\text{geom}} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}(\mathbf{X} - \mathbf{C})$$




$$\mathbf{p} = \begin{bmatrix} \mathbf{C} \\ \mathbf{q} \end{bmatrix}$$

3+4 parameters

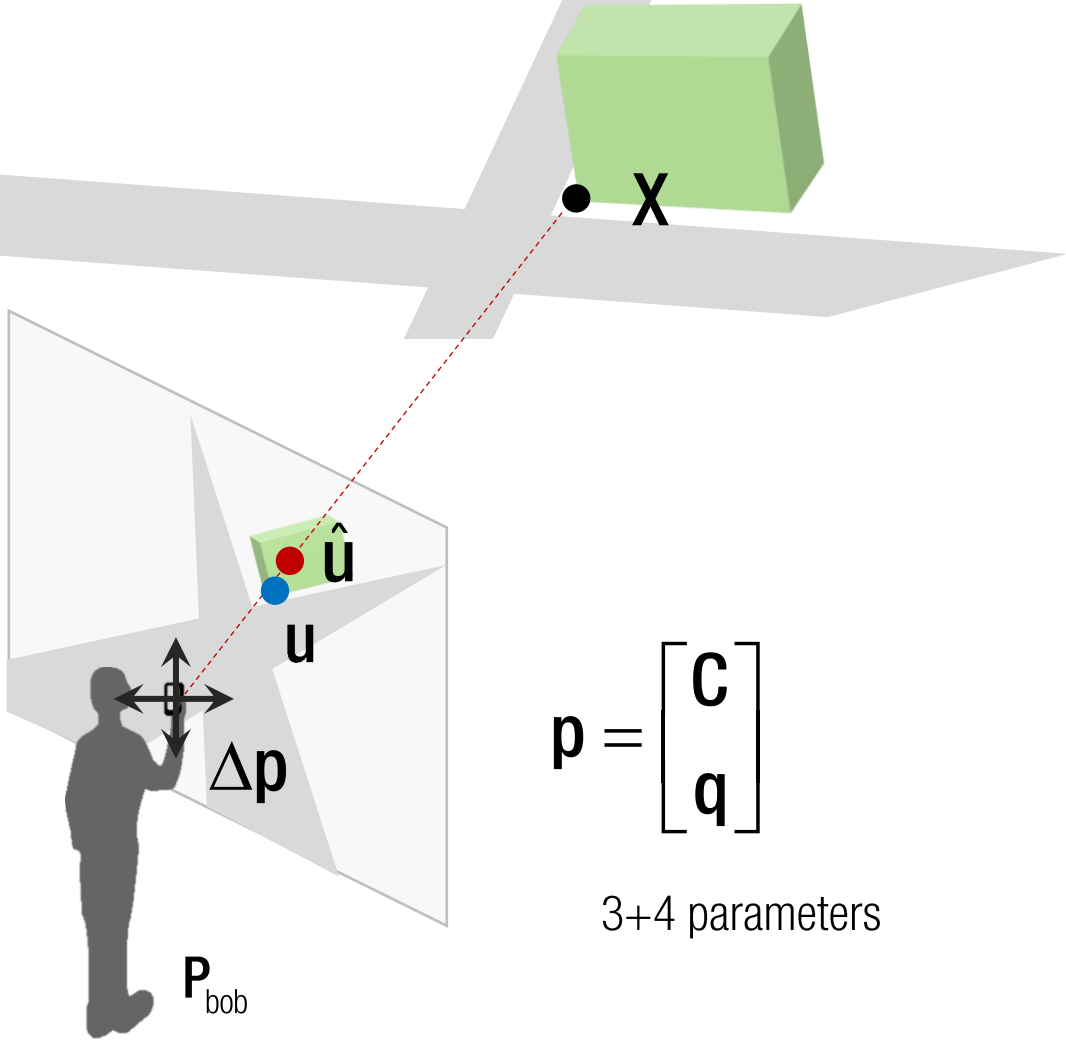
$$f(\mathbf{p}) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} \quad \rightarrow \quad \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \frac{w\frac{\partial u}{\partial \mathbf{p}} - u\frac{\partial w}{\partial \mathbf{p}}}{w^2} \\ \frac{v\frac{\partial u}{\partial \mathbf{p}} - v\frac{\partial w}{\partial \mathbf{p}}}{w^2} \end{bmatrix}$$

# Camera Jacobian

Black: given variables
Red: unknowns

$$E_{geom} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}(\mathbf{X} - \mathbf{C})$$

$$\rightarrow \quad \frac{\partial}{\partial \mathbf{C}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = -\mathbf{KR} \qquad \frac{\partial}{\partial \mathbf{q}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{\partial}{\partial \mathbf{R}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \frac{\partial \mathbf{R}}{\partial \mathbf{q}}$$



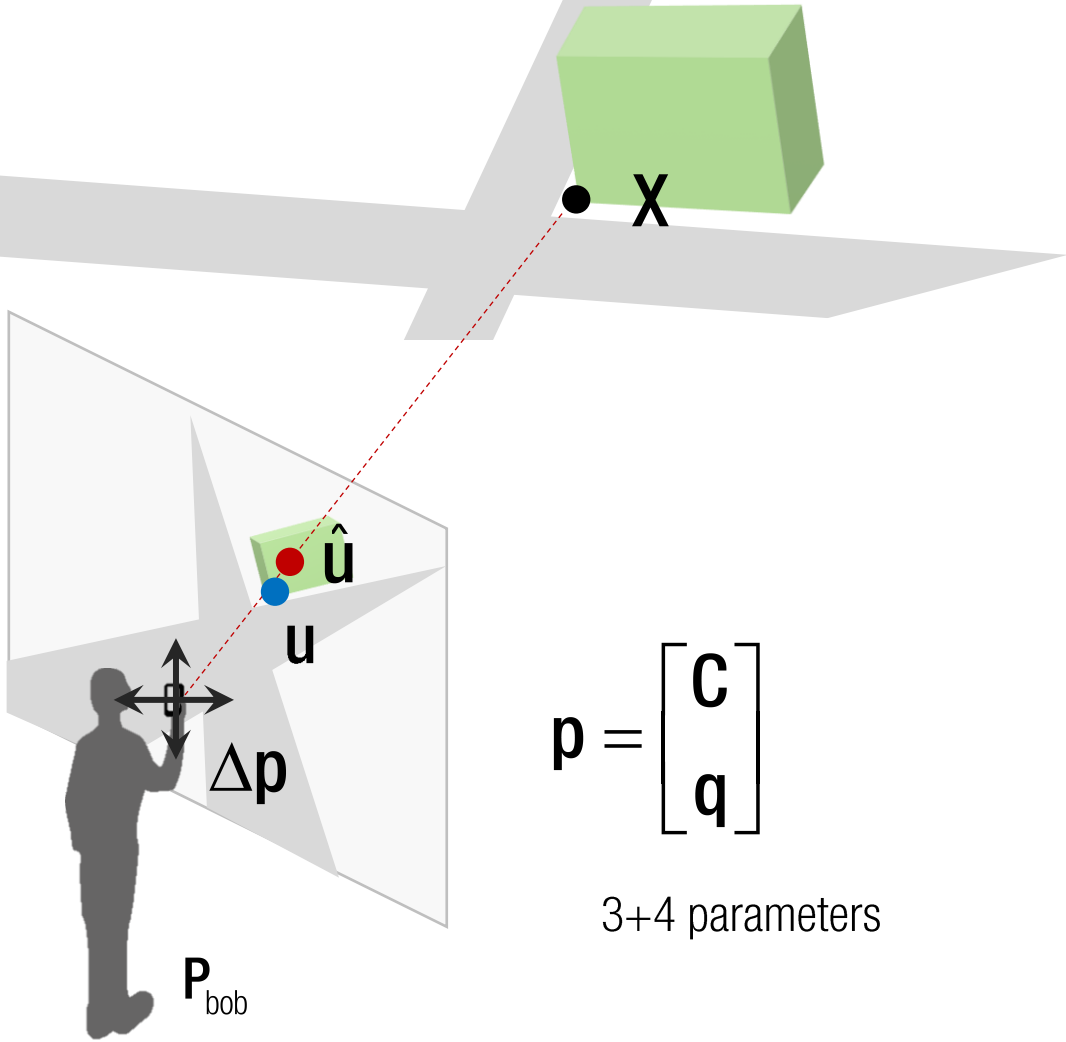

$$\mathbf{p} = \begin{bmatrix} \mathbf{C} \\ \mathbf{q} \end{bmatrix}$$

3+4 parameters

$$f(\mathbf{p}) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} \quad \rightarrow \quad \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \frac{w\frac{\partial u}{\partial \mathbf{p}} - u\frac{\partial w}{\partial \mathbf{p}}}{w^2} \\ \frac{v\frac{\partial u}{\partial \mathbf{p}} - v\frac{\partial w}{\partial \mathbf{p}}}{w^2} \end{bmatrix}$$

# Camera Jacobian

Black: given variables
Red: unknowns




$$E_{\text{geom}} = \left(\frac{u}{w} - x\right)^2 + \left(\frac{v}{w} - y\right)^2 \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{KR}(\mathbf{X} - \mathbf{C})$$

$$\rightarrow \quad \frac{\partial}{\partial \mathbf{C}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = -\mathbf{KR} \qquad \frac{\partial}{\partial \mathbf{q}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{\partial}{\partial \mathbf{R}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \frac{\partial \mathbf{R}}{\partial \mathbf{q}}$$

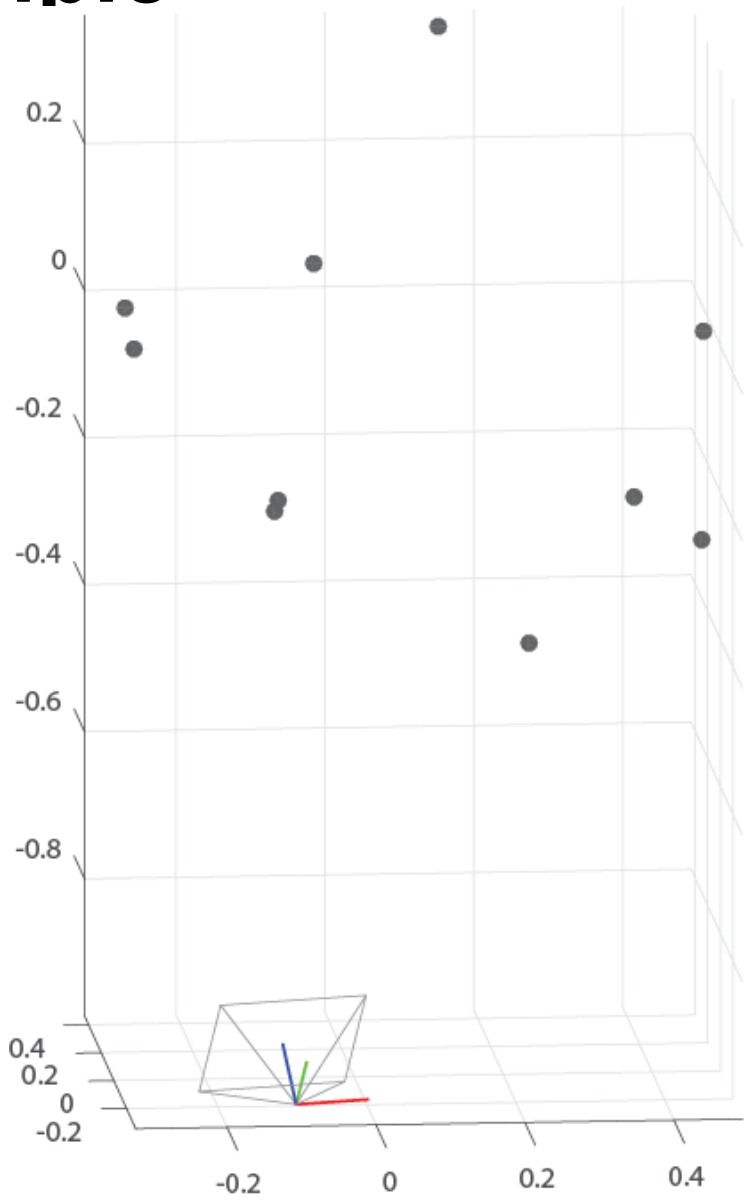
$$\rightarrow \quad \frac{\partial}{\partial \mathbf{p}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \left[ \frac{\partial}{\partial \mathbf{C}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \frac{\partial}{\partial \mathbf{q}} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right]$$

$$\mathbf{p} = \begin{bmatrix} \mathbf{C} \\ \mathbf{q} \end{bmatrix}$$

3+4 parameters

$$f(\mathbf{p}) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} \quad \rightarrow \quad \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \frac{w \frac{\partial u}{\partial \mathbf{p}} - u \frac{\partial w}{\partial \mathbf{p}}}{w^2} \\ \frac{v \frac{\partial u}{\partial \mathbf{p}} - v \frac{\partial w}{\partial \mathbf{p}}}{w^2} \end{bmatrix}$$

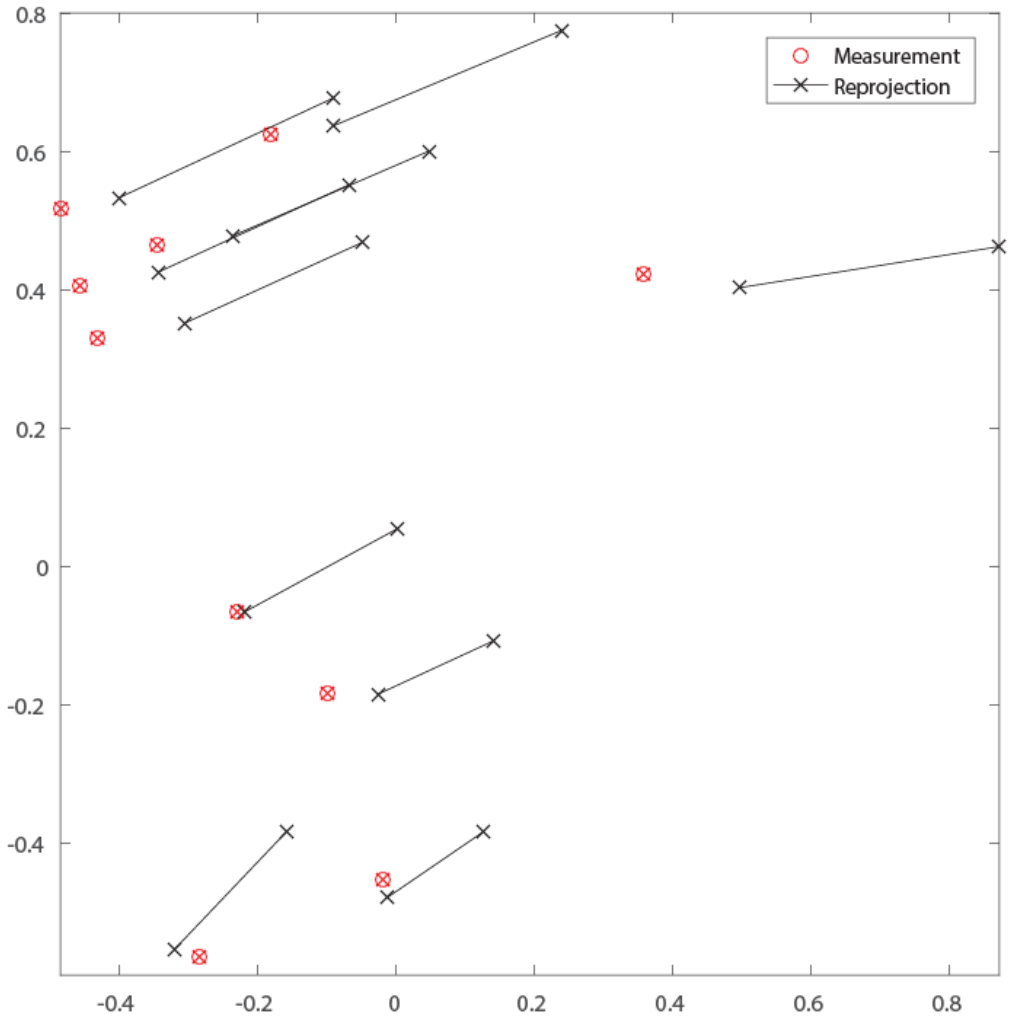
# Example




```
u = K*R*[eye(3) -C]*[X'; ones(1,nPoints)];
u = [u(1,:)./u(3,:); u(2,:)./u(3,:)];

x = [C; q];
for j = 1 : 40
    R1 = Quaternion2Rotation(x(4:7));
    C1 = x(1:3);

    df_dc = [];
    df_dR = [];
    for k = 1 : nPoints
        df_dc = [df_dc; JacobianC(K, R1, C1, X(k,:)')];
        df_dR = [df_dR; JacobianR(K, R1, C1, X(k,:)')*JacobianQ(x(4:7))];
    end

    u1 = K*R1*[eye(3) -C1]*[X'; ones(1,nPoints)];
    u1 = [u1(1,:)./u1(3,:); u1(2,:)./u1(3,:)];

    jacobian = [df_dc df_dR];
    delta_b = u(:)-u1(:);

    delta_x = inv(jacobian'*jacobian)*jacobian'*delta_b;
    x = x + delta_x;
    x(4:7) = x(4:7)/norm(x(4:7));
end
```

$$\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}}\begin{bmatrix}\frac{u}{w}\\ \frac{v}{w}\end{bmatrix} = \begin{bmatrix}\frac{w\frac{\partial u}{\partial \mathbf{p}} - u\frac{\partial w}{\partial \mathbf{p}}}{w^2}\\ \frac{v\frac{\partial u}{\partial \mathbf{p}} - v\frac{\partial w}{\partial \mathbf{p}}}{w^2}\end{bmatrix}$$

# Example

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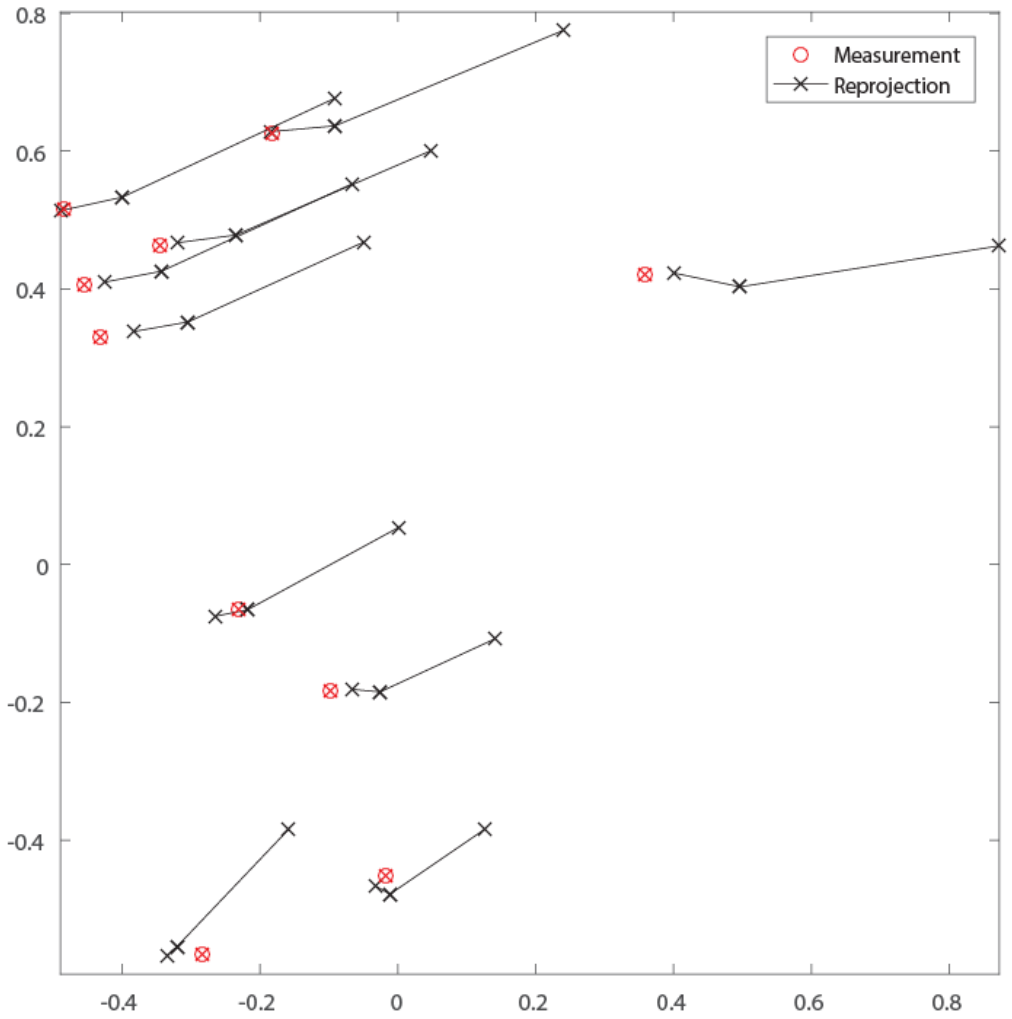
# Example

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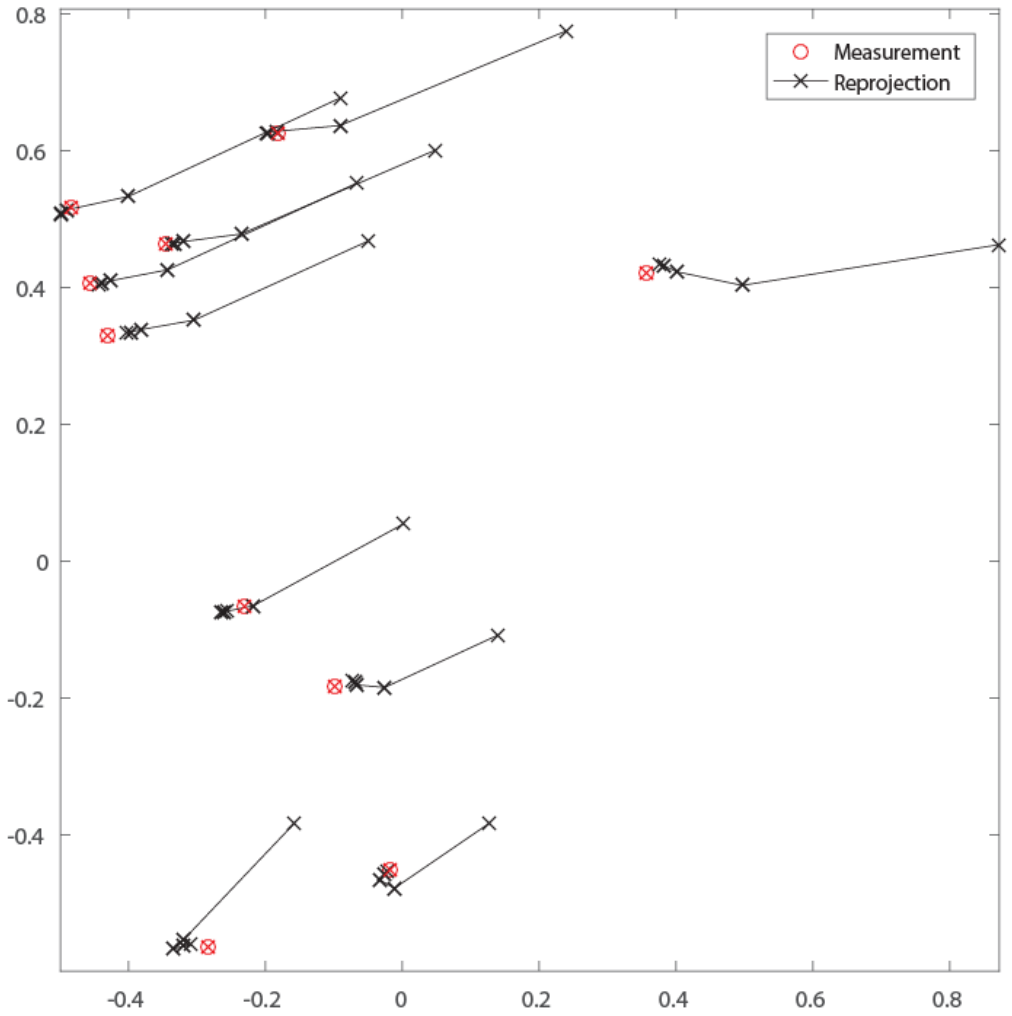
# Example

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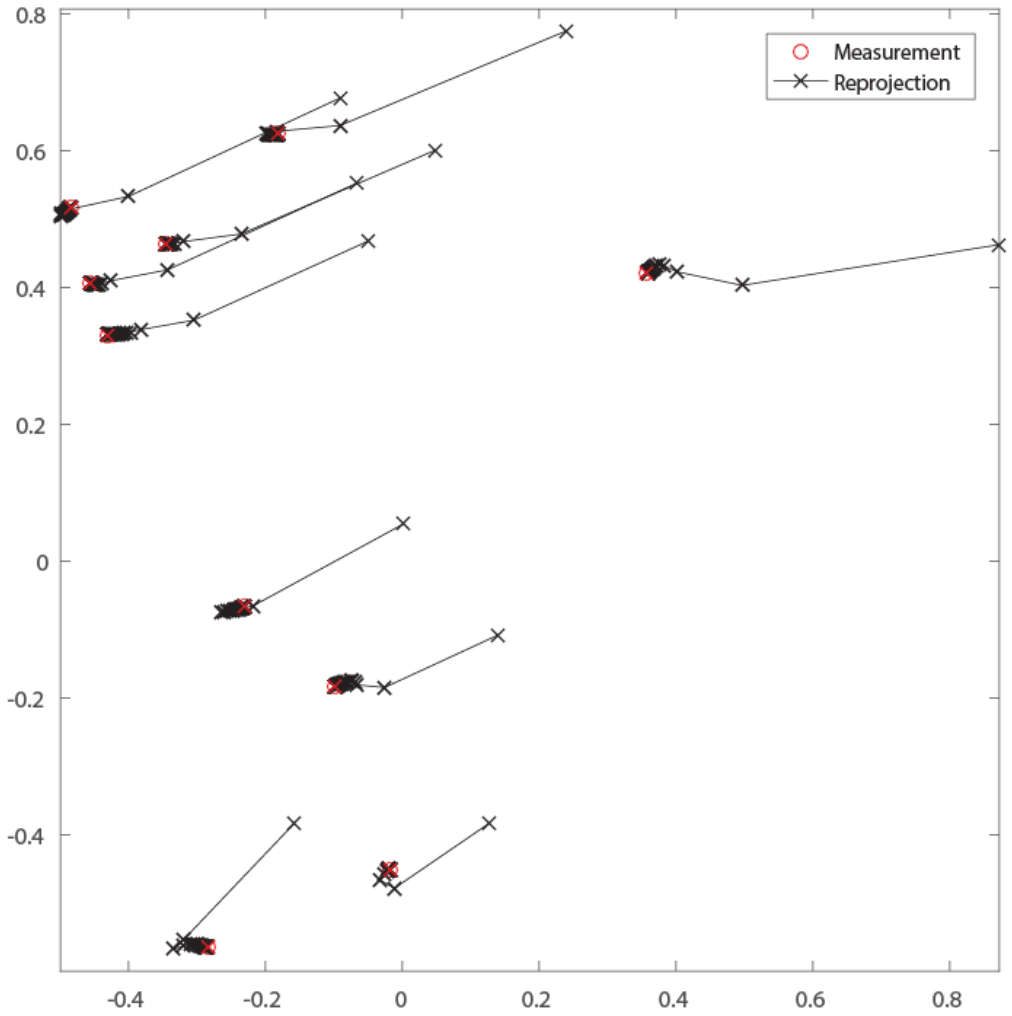
# Example

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    end

    u1 = K*R1*[eye(3) -C1]*[X'; ones(1,nPoints)];
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$$\Delta\mathbf{p} = \left(\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}^{\mathsf{T}}\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}\right)^{-1}\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}^{\mathsf{T}}\left(\mathbf{b} - f(\mathbf{p})\right)$$