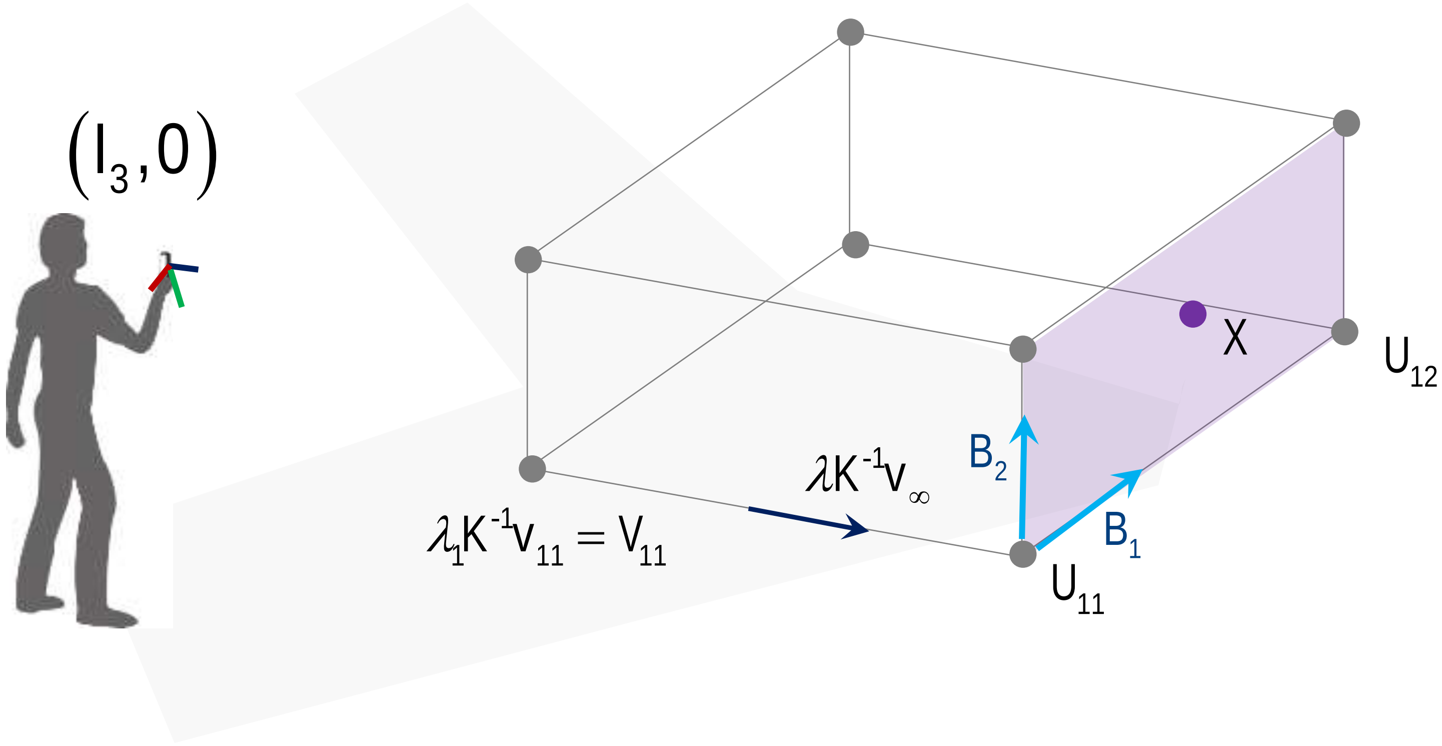
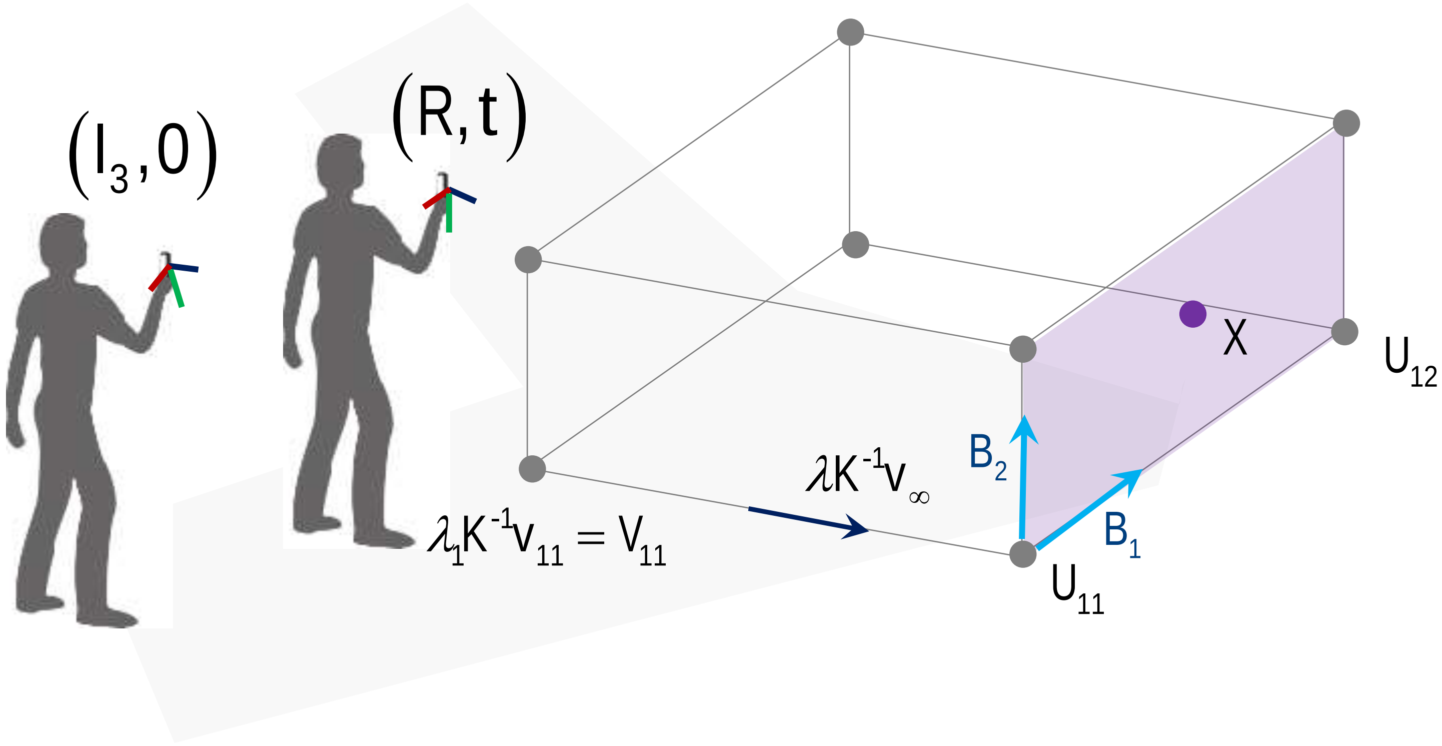
Spatial Rotation

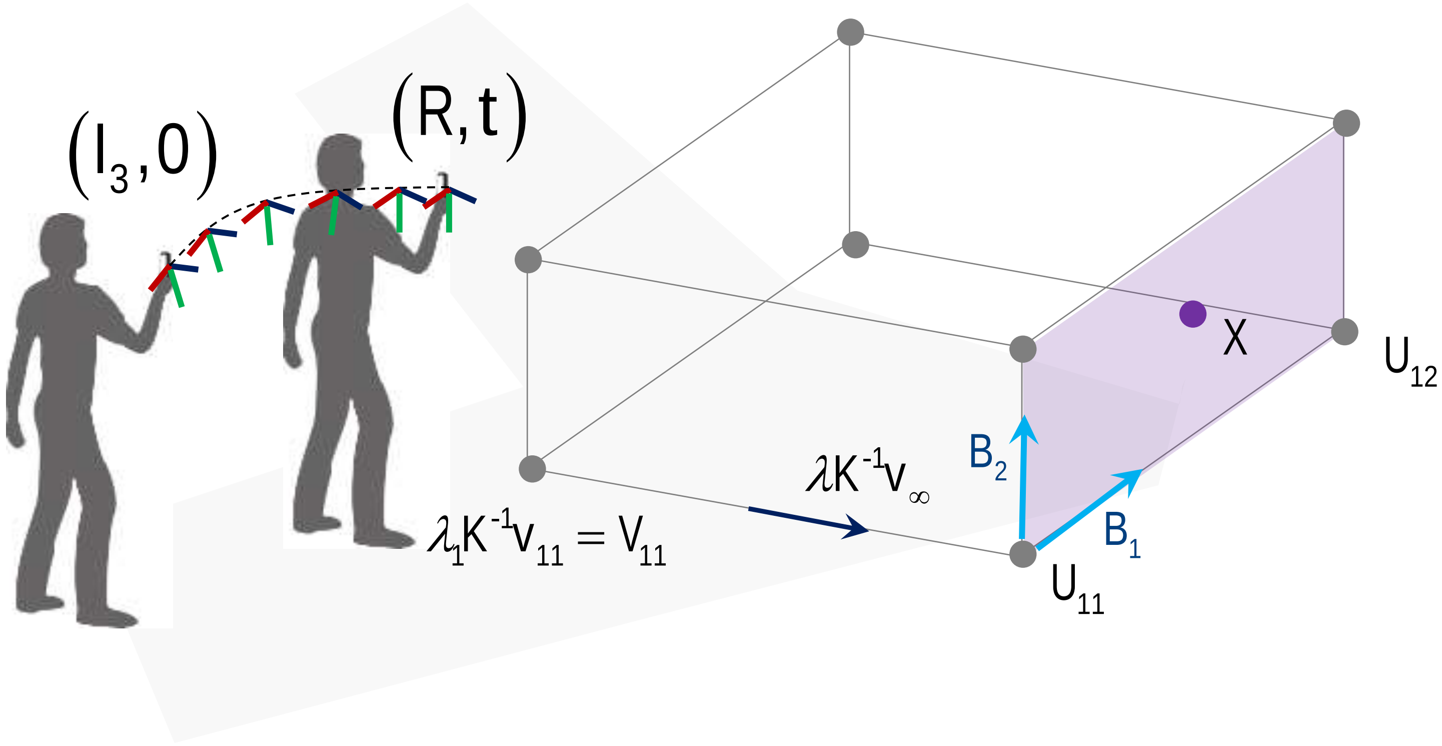
Interpolation of Transformation



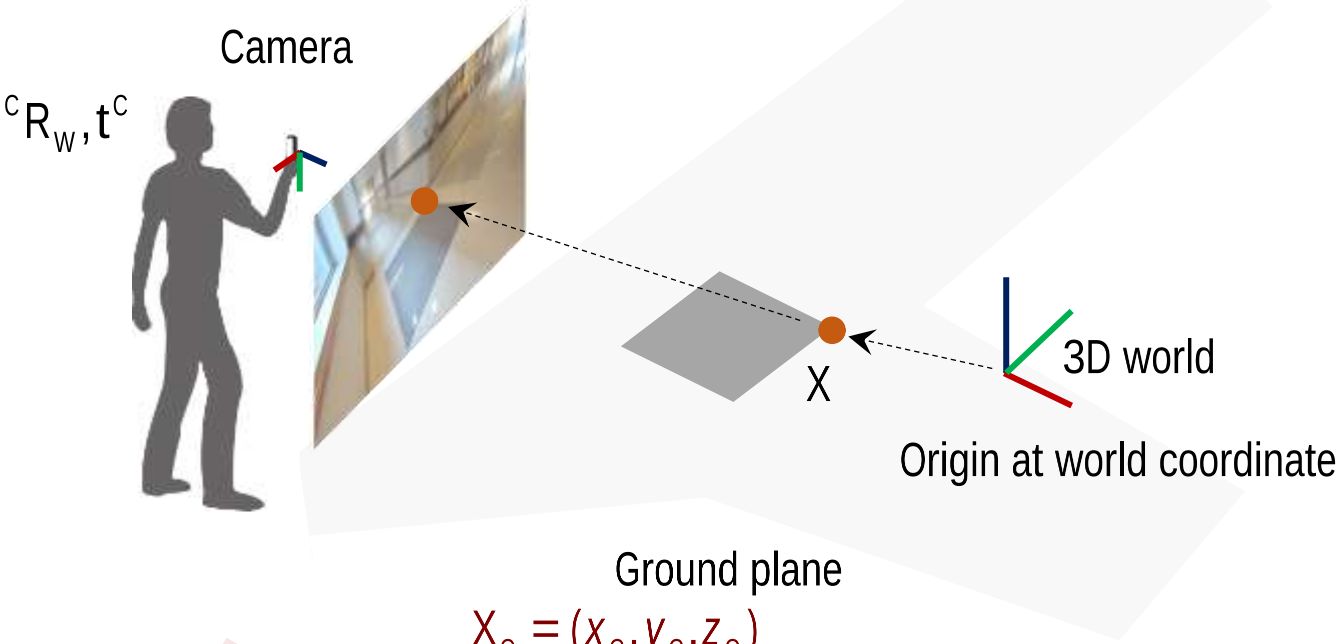
Interpolation of Transformation

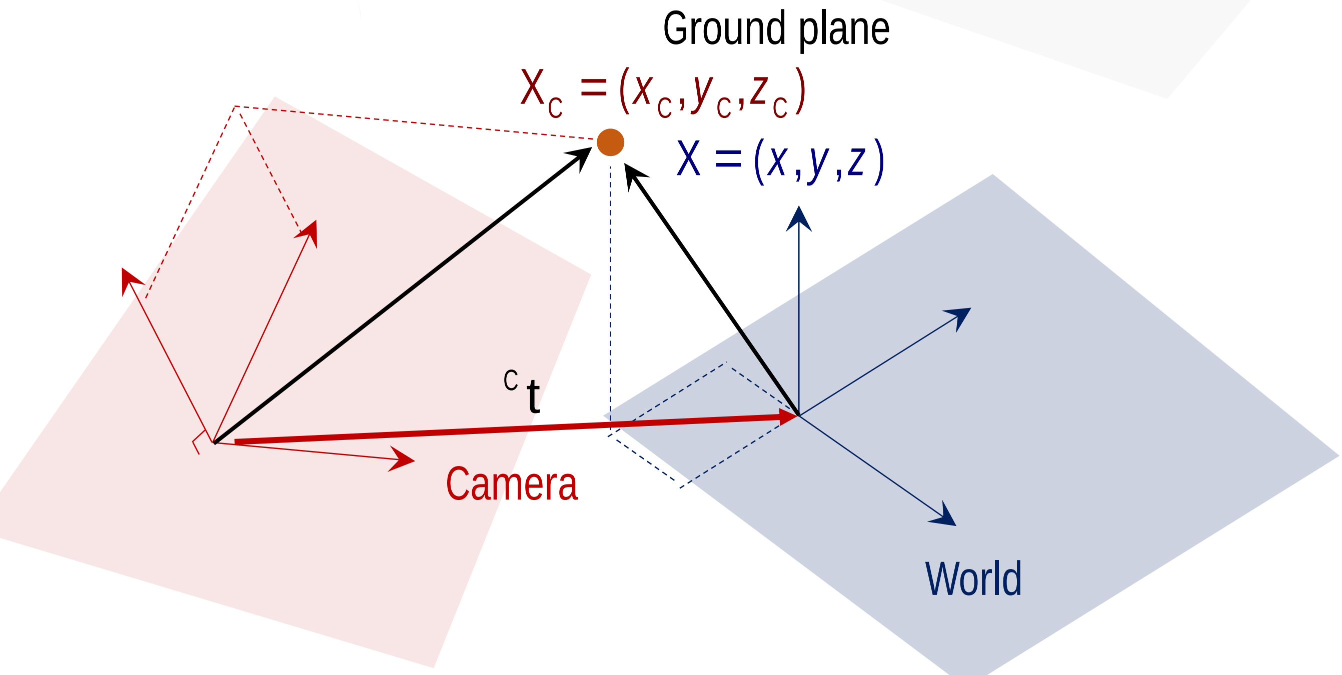


Interpolation of Transformation



Recall: Rotate and then, Translate



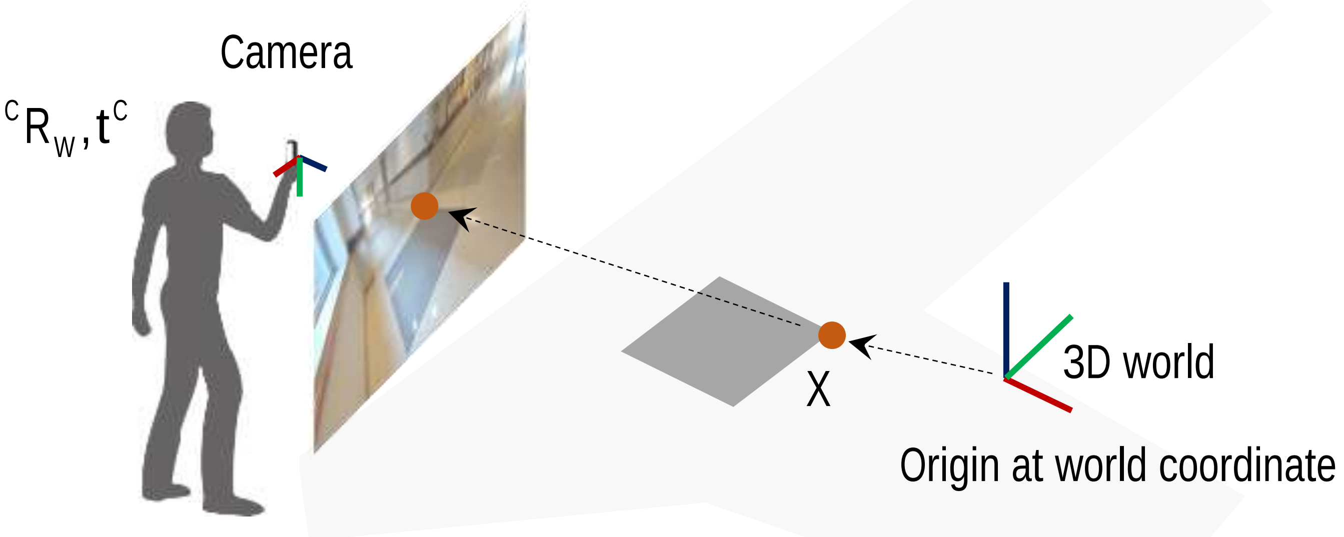


$$X_C = {}^{C}R_W X + {}^{C}t = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} & t_x \\ r_{y1} & r_{y2} & r_{y3} & t_y \\ r_{z1} & r_{z2} & r_{z3} & t_z \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

where ${}^{C}t$ is translation from world to camera seen from camera.

Rotate and then, translate.

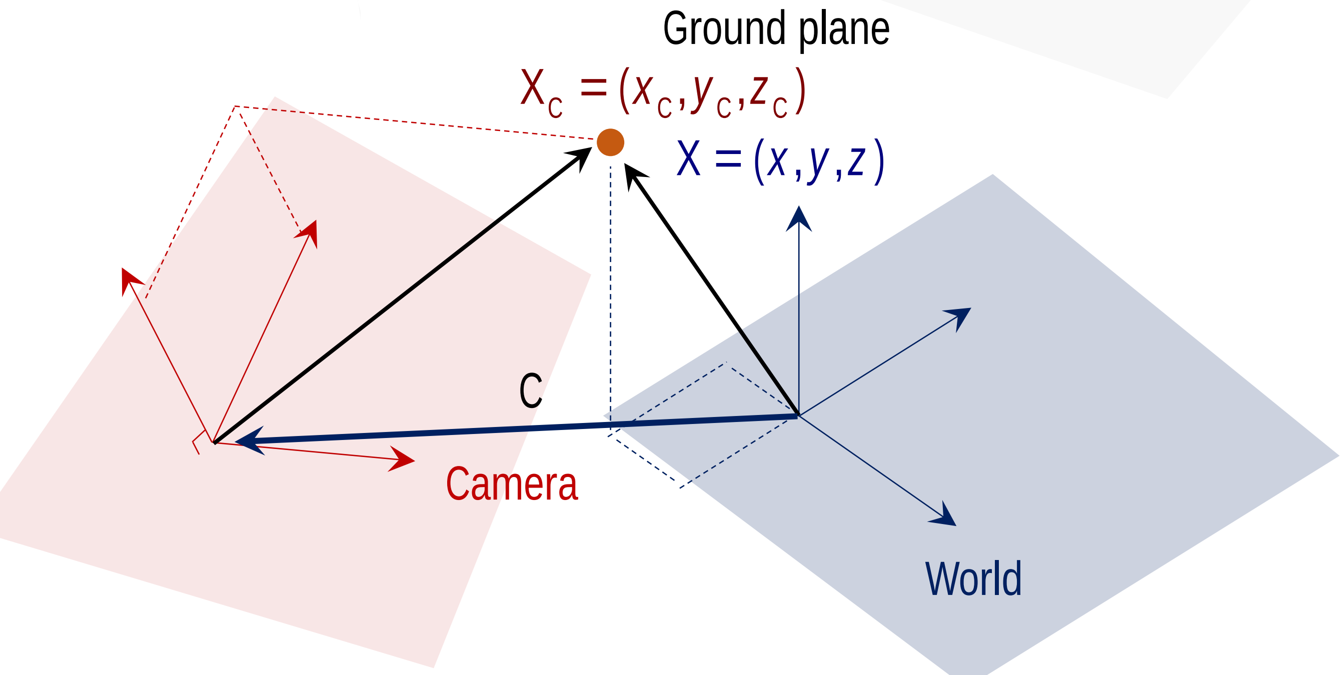
Recall: Translate and the, Rotate



$$X_C = {}^{C}R_W X + {}^{C}t = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} & t_x \\ r_{y1} & r_{y2} & r_{y3} & t_y \\ r_{z1} & r_{z2} & r_{z3} & t_z \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

where ${}^{C}t$ is translation from world to camera seen from camera.

Rotate and then, translate.



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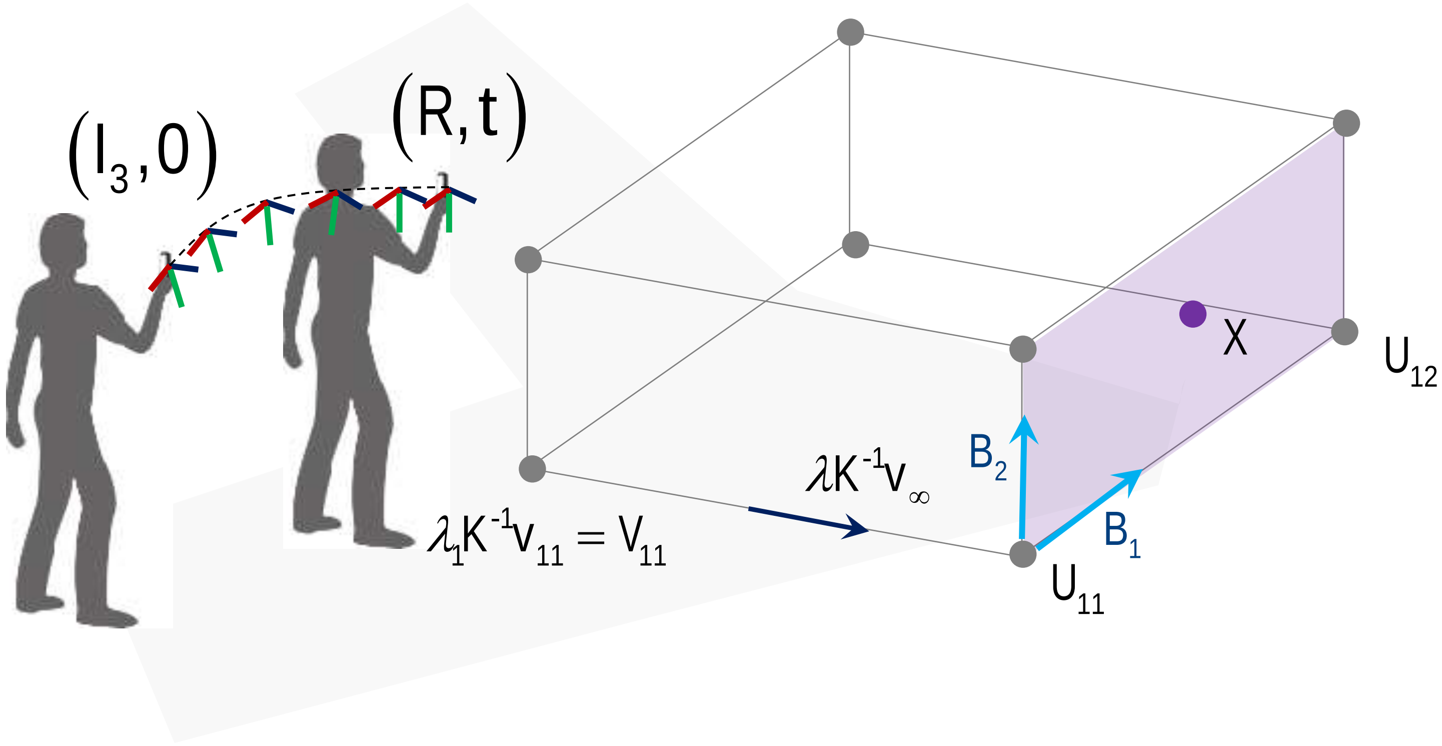
cf) Translate and then, rotate.

$$X_C = {}^{C}R_W (X - C) = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} \begin{bmatrix} 1 & & & -C_x \\ & 1 & & -C_y \\ & & 1 & -C_z \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

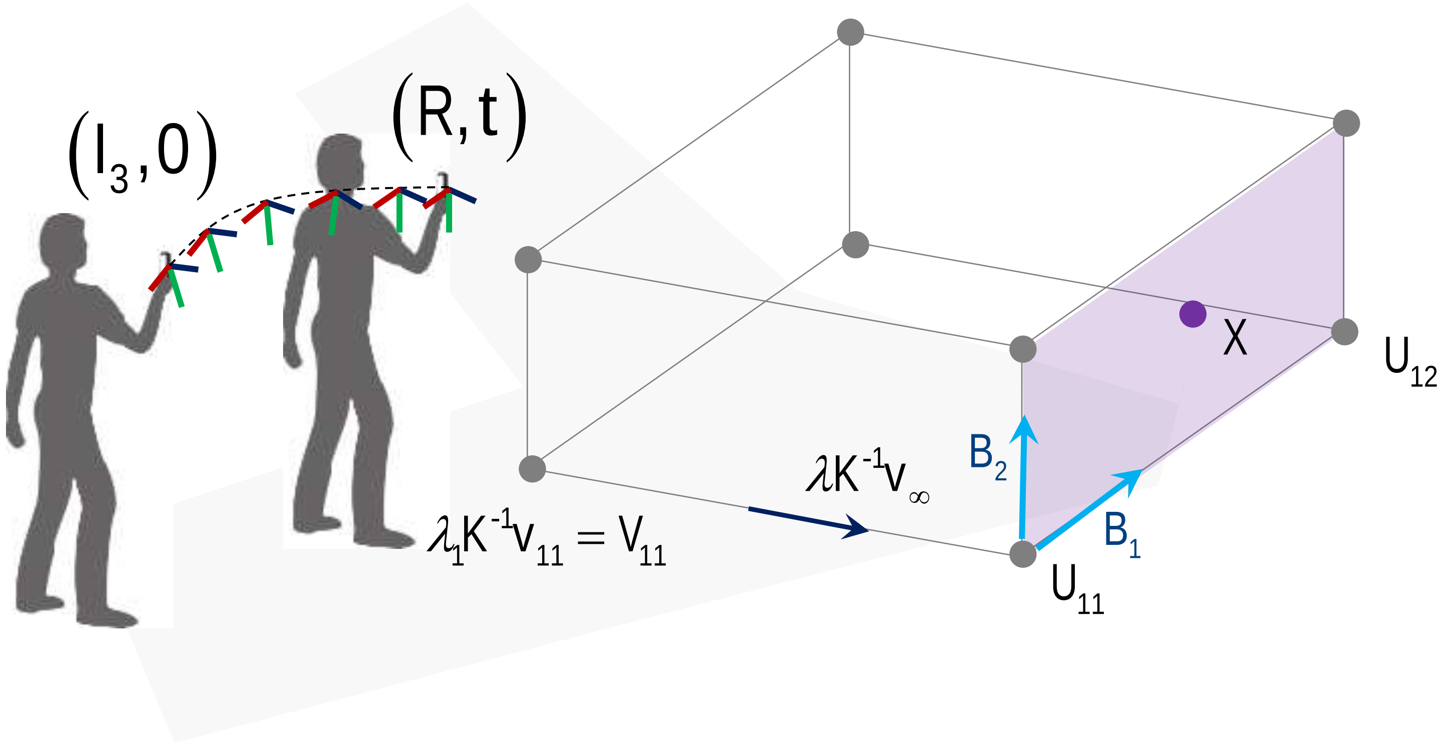
where C is translation from world to camera seen from world.

Interpolation of Translation



$$\lambda \tilde{u} = K \begin{bmatrix} R & t \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = KR \begin{bmatrix} I_3 & -C \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Interpolation of Translation



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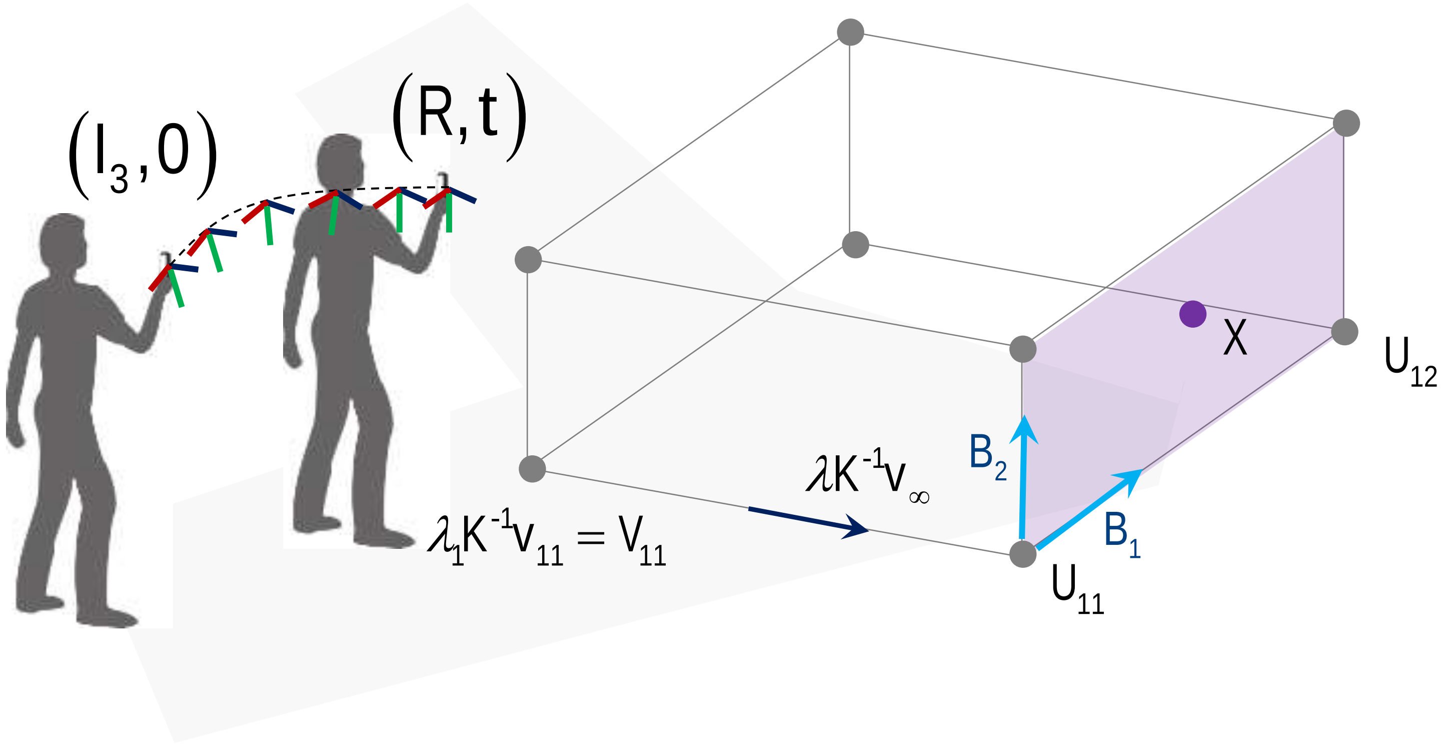
Rot. ➔ Trans. Trans. ➔ Rot.

Translation is independent on rotation.

How to interpolate translation?

$$C_1 = \begin{bmatrix} C_1^x \\ C_1^y \\ C_1^z \end{bmatrix} \rightarrow C_2 = \begin{bmatrix} C_2^x \\ C_2^y \\ C_2^z \end{bmatrix}$$

Interpolation of Translation



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Rot. ➔ Trans. Trans. ➔ Rot.

Translation is independent on rotation.

How to interpolate translation?

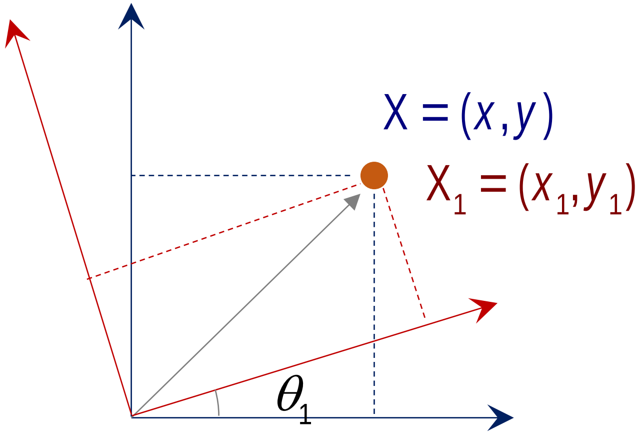
$$C_1 = \begin{bmatrix} C_1^x \\ C_1^y \\ C_1^z \end{bmatrix} \rightarrow C_2 = \begin{bmatrix} C_2^x \\ C_2^y \\ C_2^z \end{bmatrix}$$

Interpolated camera center:

$$C_i = wC_1 + (1-w)C_2 \qquad w \in [0,1]$$

Interpolation of Rotation

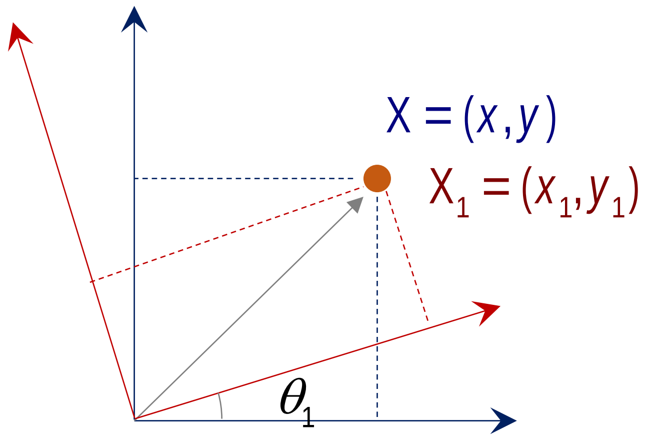
2D coordinate transform:



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \qquad \begin{bmatrix} x \\ y \end{bmatrix}$$

Interpolation of Rotation

2D coordinate transform:

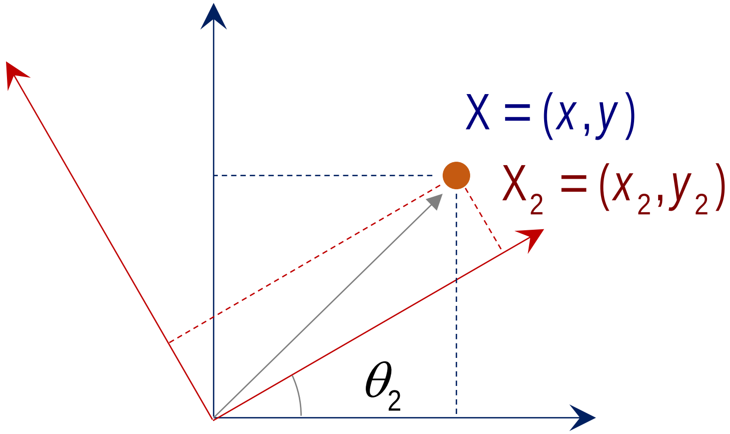


$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det\left(\begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \right) = \cos^2\theta_1 + \sin^2\theta_1 = 1$$

Interpolation of Rotation

2D coordinate transform:



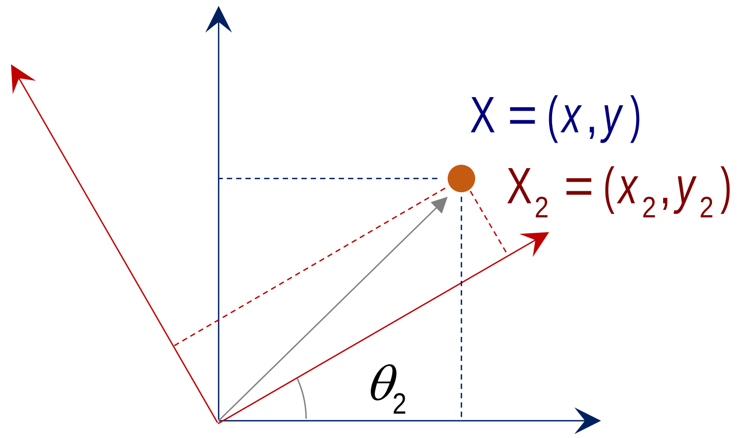
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Interpolation of Rotation

2D coordinate transform:



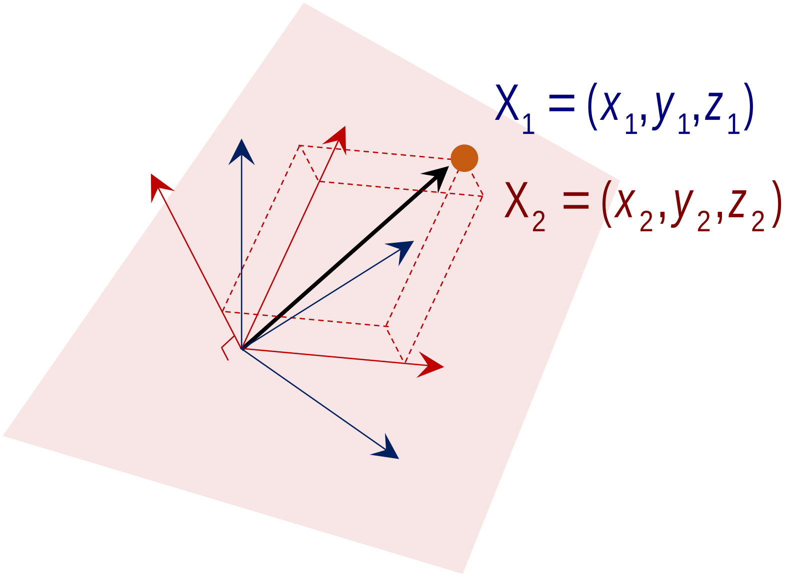
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\theta = w\theta_1 + (1-w)\theta_2$$

$$w \in [0,1]$$

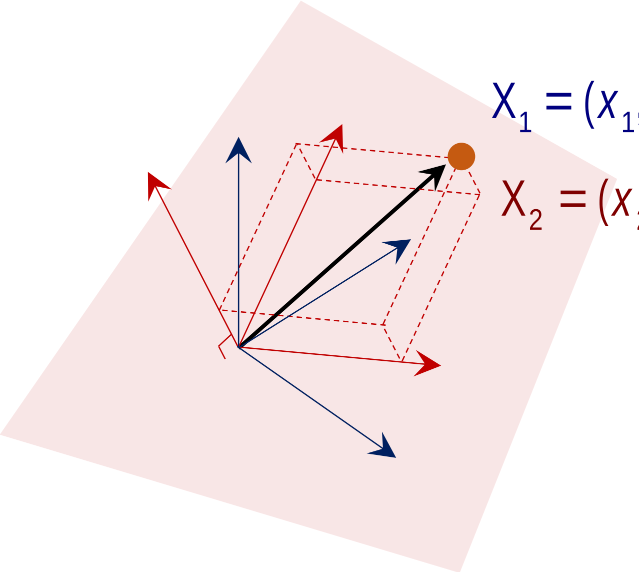
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Interpolation of Rotation in 3D



$$X_2 = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} X = R_1 X_1$$

Interpolation of Rotation in 3D



$$X_2 = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} \\ r_{y1} & r_{y2} & r_{y3} \\ r_{z1} & r_{z2} & r_{z3} \end{bmatrix} X = R_1 X_1$$

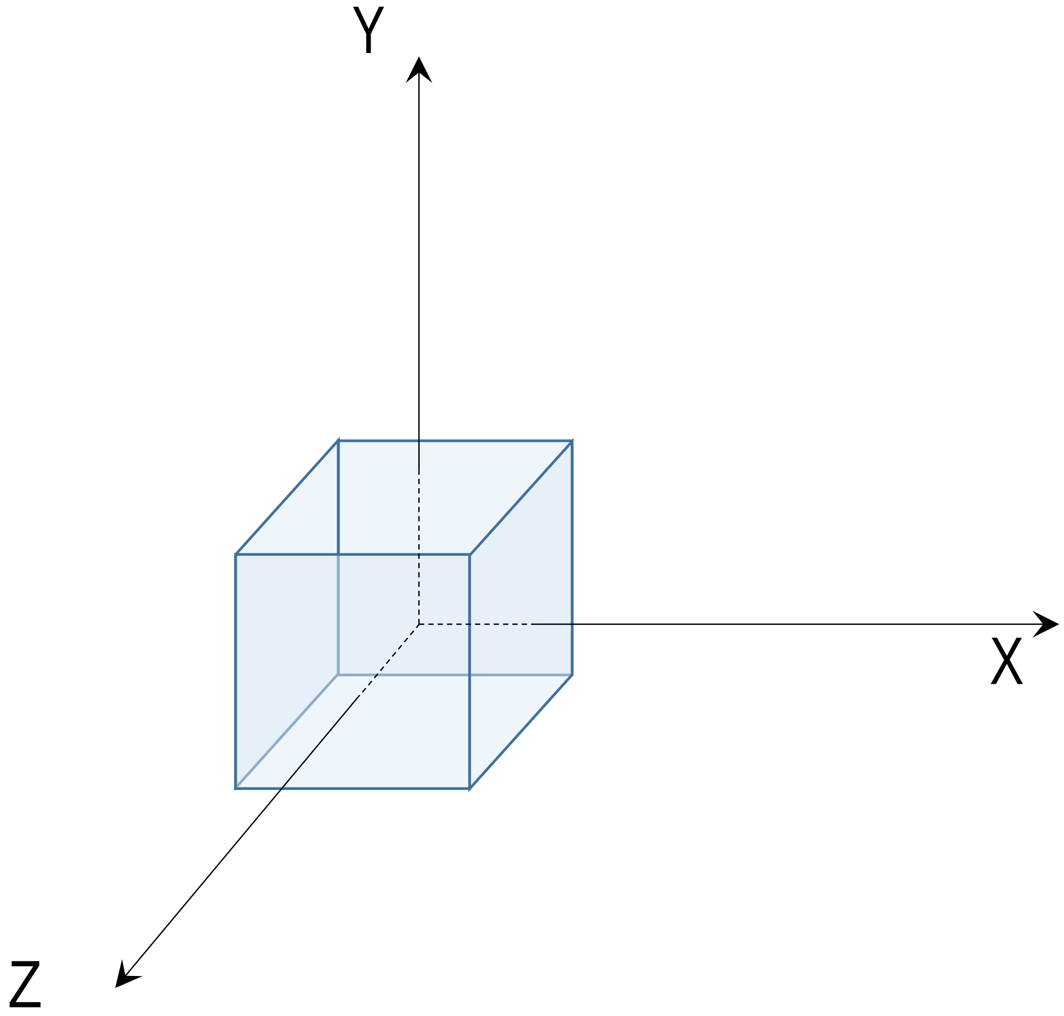
How to interpolate between two coordinates?

$$R_1 \rightarrow R_2$$

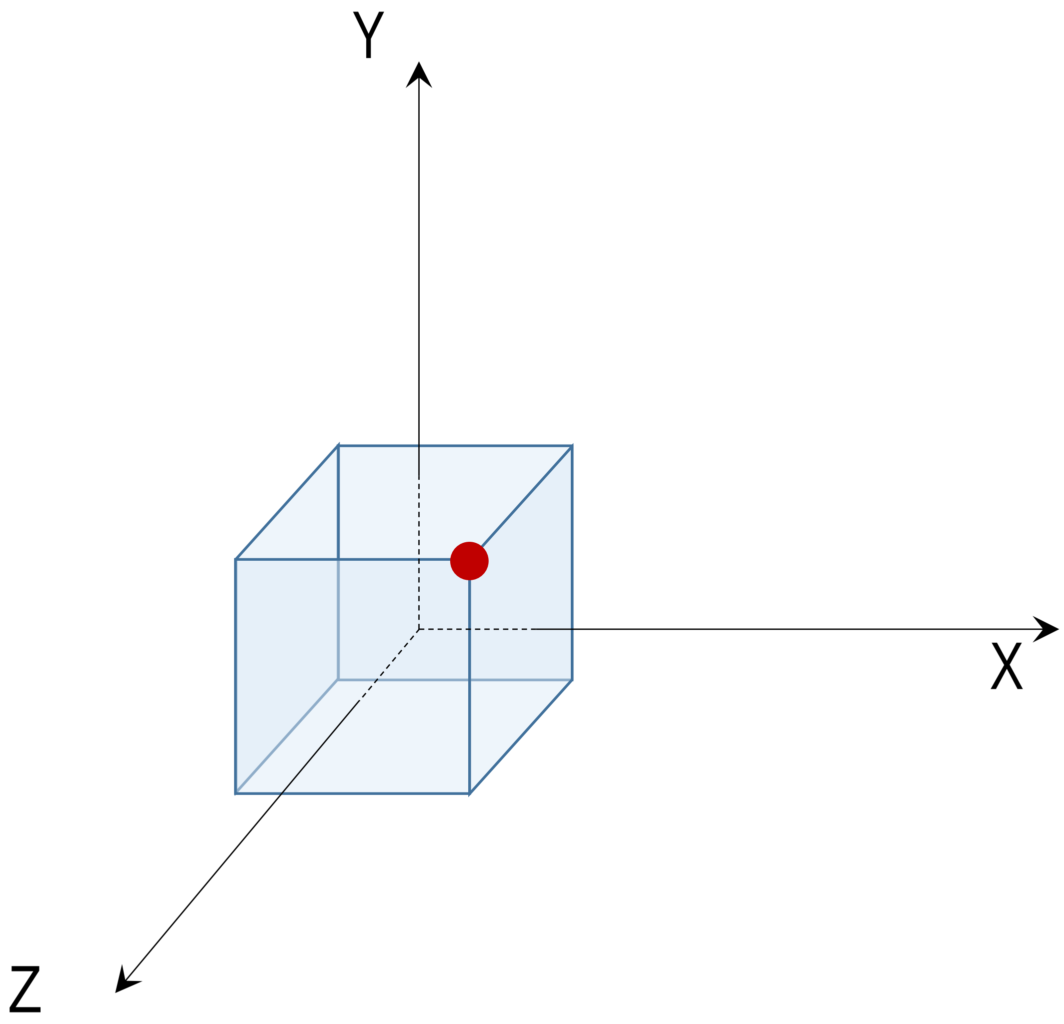
dof: 3

of parameters: 9

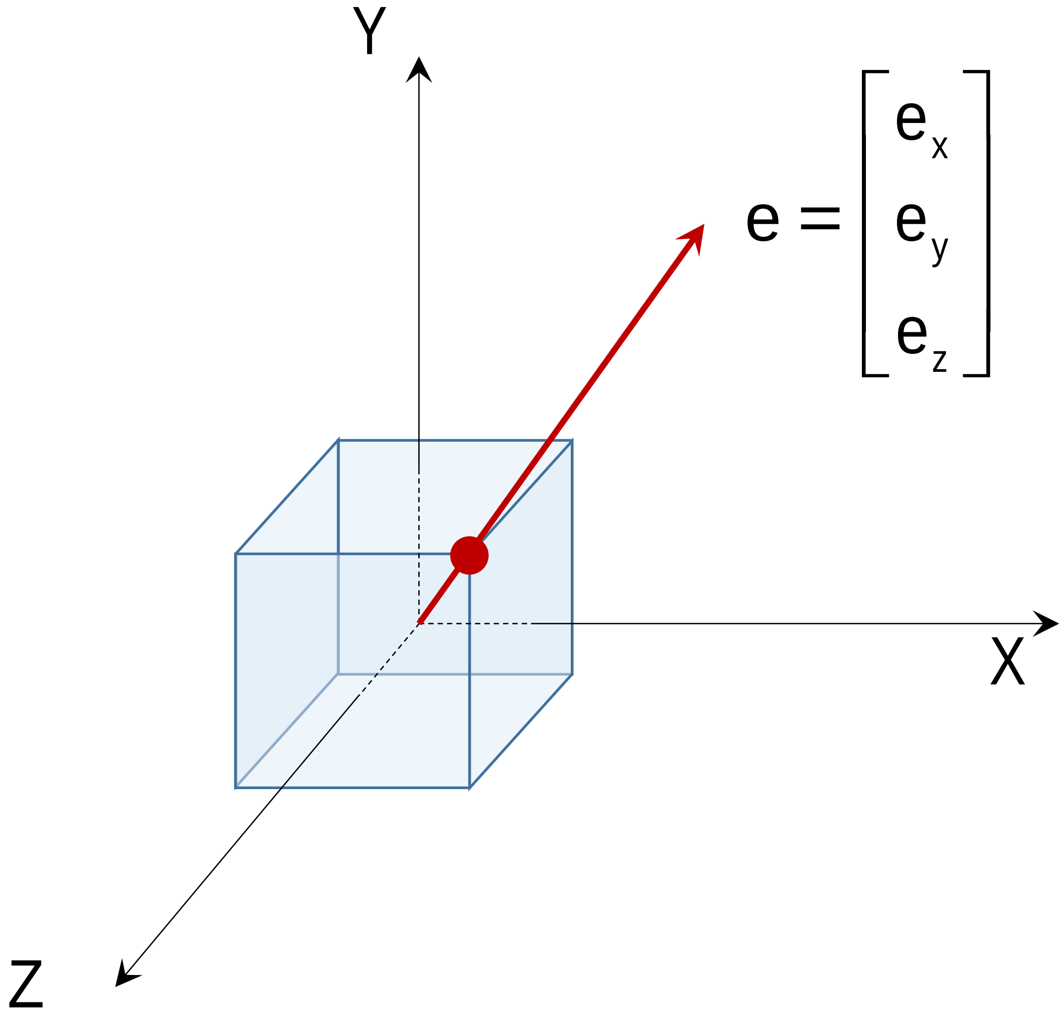
Axis Angle Representation



Axis Angle Representation

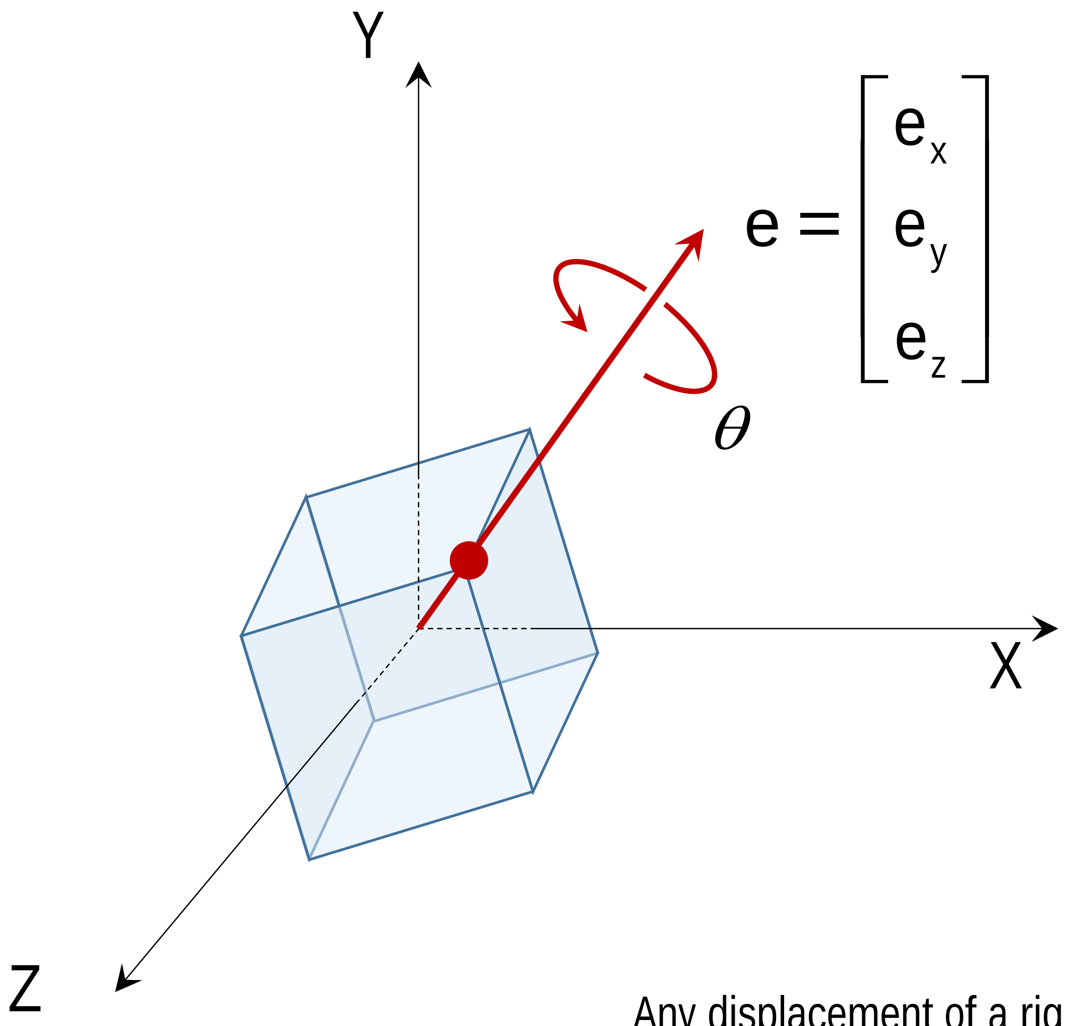


Axis Angle Representation



$$e = \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}$$

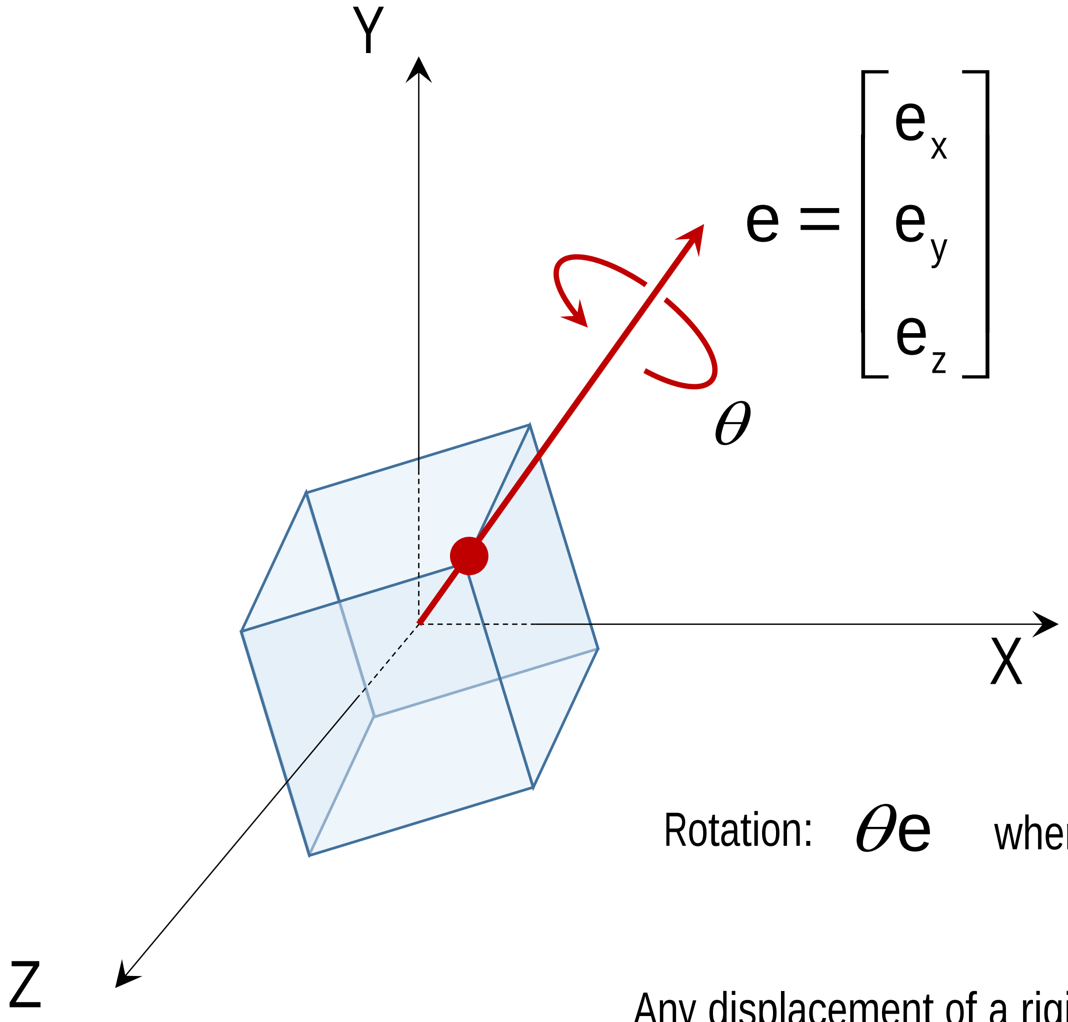
Axis Angle Representation



Euler's theorem

Any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point.

Axis Angle Representation



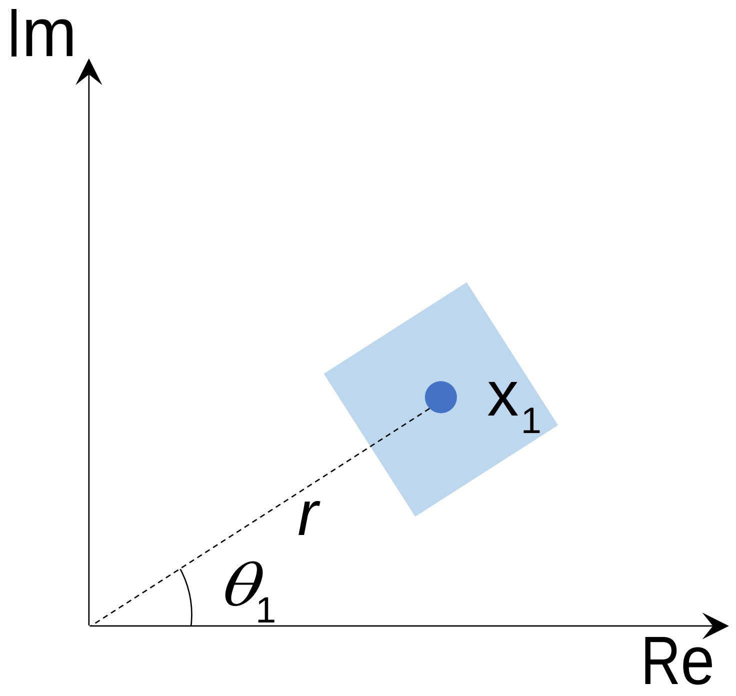
$$e = \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}$$

Rotation: θe where $\|e\| = 1$

Euler's theorem

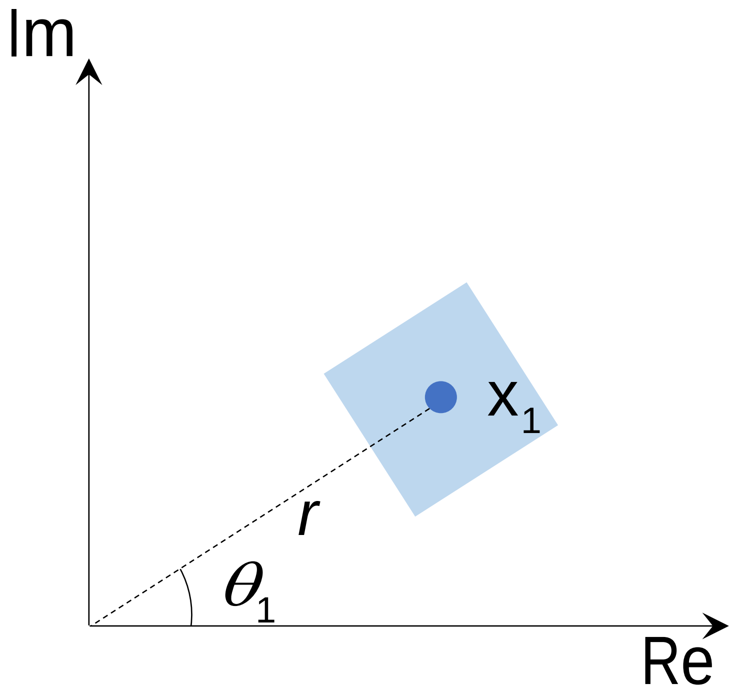
Any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point.

2D Exponential Map (Euler's Formula)



$x_1 =$

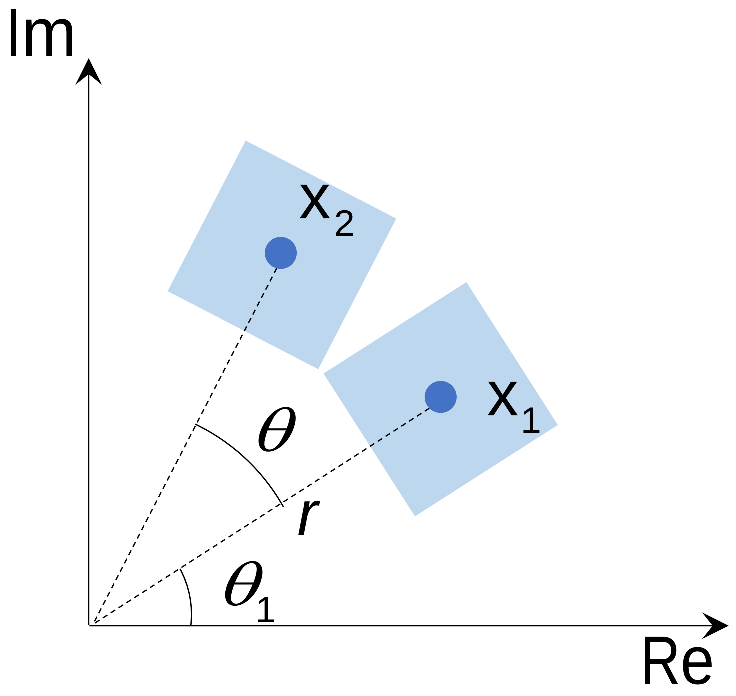
2D Exponential Map (Euler's Formula)



$$x_1 = r\exp(i\theta_1) = r\left(\cos\theta_1 + i\sin\theta_1\right)$$

Ref) $$\exp(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

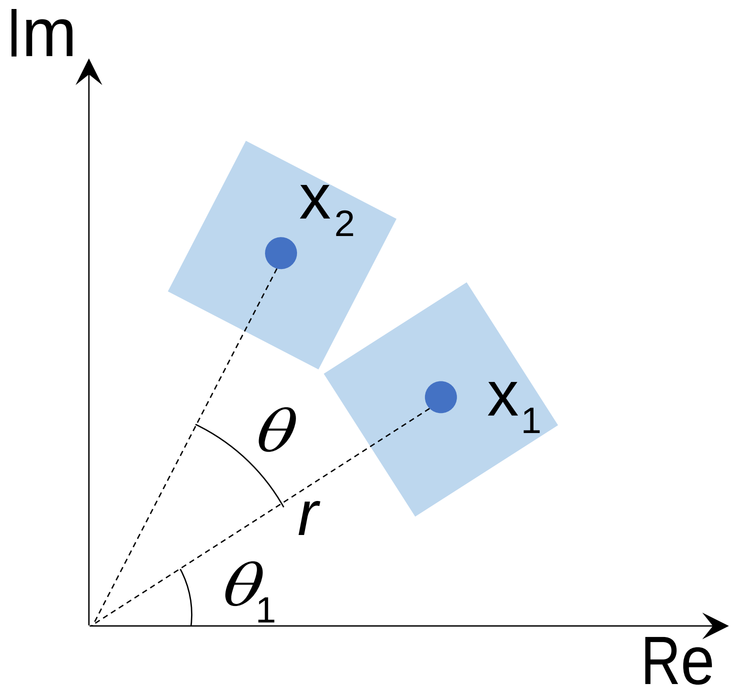
2D Exponential Map (Euler's Formula)



$$x_1 = r\exp(i\theta_1) = r\left(\cos\theta_1 + i\sin\theta_1\right)$$

x_2

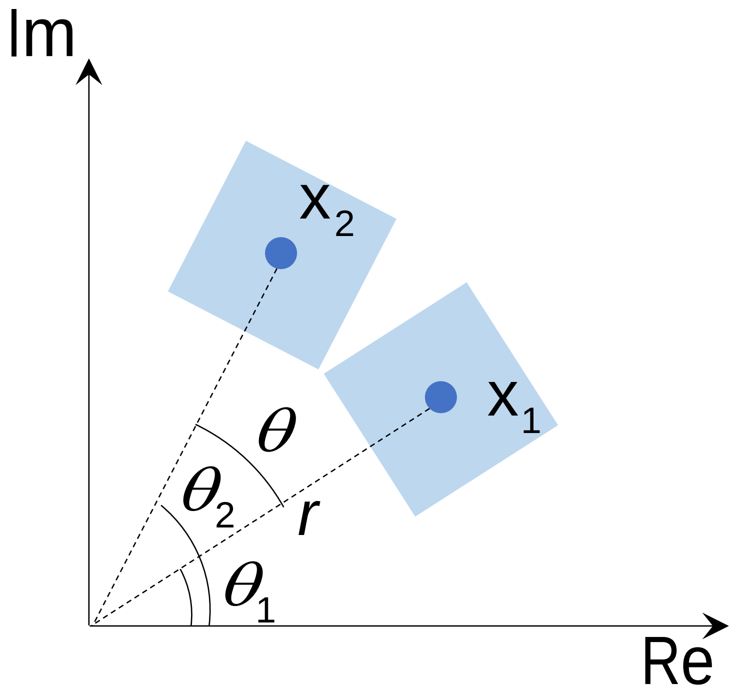
2D Exponential **Map (Euler's Formula)**



$$x_1 = r\exp(i\theta_1) = r\left(\cos\theta_1 + i\sin\theta_1\right)$$

$$\begin{aligned} x_2 &= \exp(i\theta)x_1 = r\left(\cos\theta + i\sin\theta\right)\left(\cos\theta_1 + i\sin\theta_1\right) \\ &= r\left(\cos\theta\cos\theta_1 - \sin\theta\sin\theta_1 + i\left(\cos\theta\sin\theta_1 + \sin\theta\cos\theta_1\right)\right) \\ &= r\left(\cos(\theta+\theta_1) + i\sin(\theta+\theta_1)\right) \end{aligned}$$

2D Exponential Map (Euler's Formula)

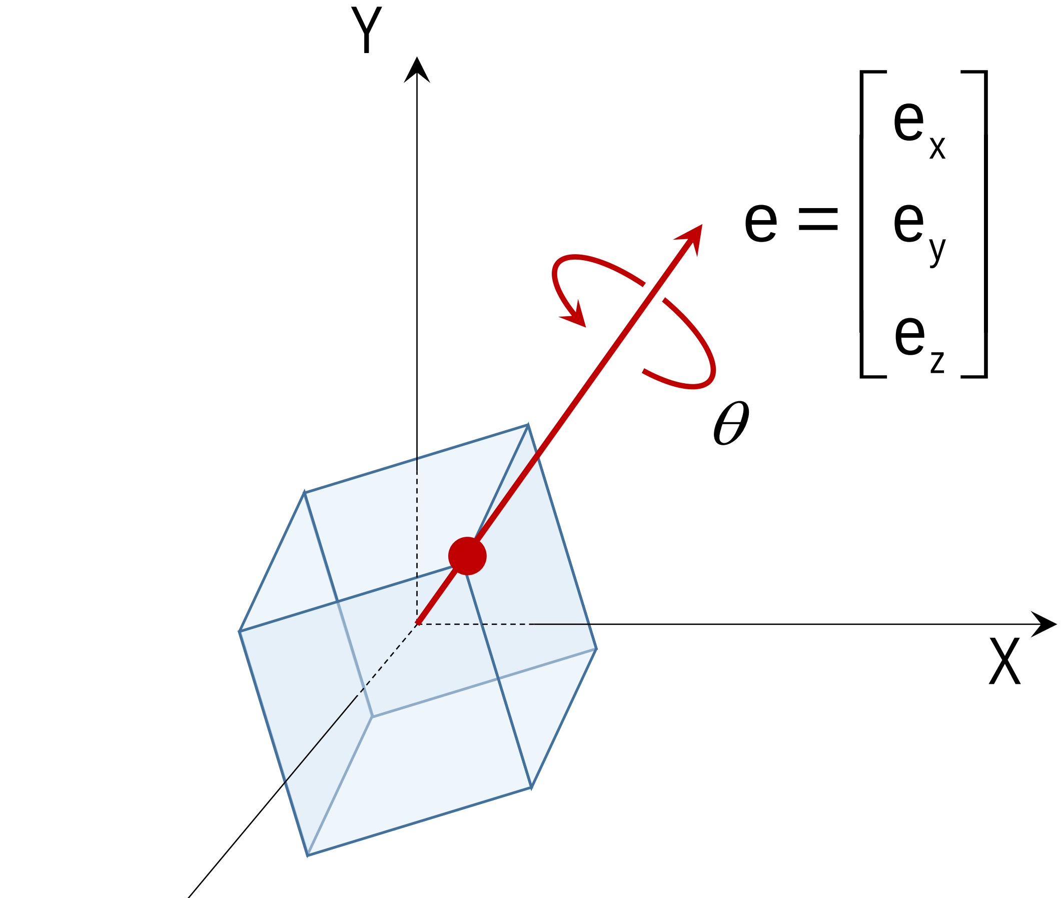


$$x_1 = r\exp(i\theta_1) = r\left(\cos\theta_1 + i\sin\theta_1\right)$$

$$\begin{aligned}
x_2 &= \exp(i\theta)x_1 = r\left(\cos\theta + i\sin\theta\right)\left(\cos\theta_1 + i\sin\theta_1\right) \\
&= r\left(\cos\theta\cos\theta_1 - \sin\theta\sin\theta_1 + i\left(\cos\theta\sin\theta_1 + \sin\theta\cos\theta_1\right)\right) \\
&= r\left(\cos(\theta+\theta_1) + i\sin(\theta+\theta_1)\right) \\
&= r\left(\cos\theta_2 + i\sin\theta_2\right)
\end{aligned}$$

$$\theta_2 = \theta_1 + \theta$$

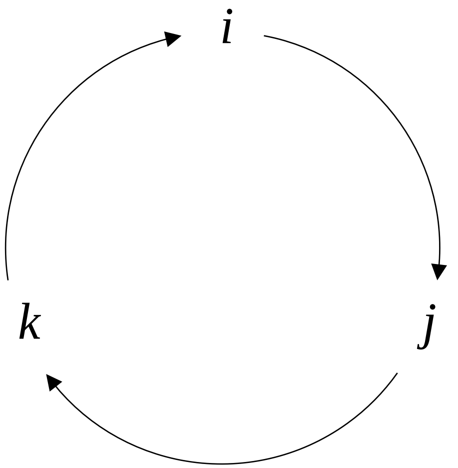
3D Exponential Map: Quaternion



Rotation: $\theta \mathrm{e}$ where $\|\mathrm{e}\| = 1$

$$\exp\left(\frac{\theta}{2}\mathrm{e}\right) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(i\,\mathrm{e}_x + j\,\mathrm{e}_y + k\,\mathrm{e}_z\right)$$

$$i^2 = j^2 = k^2 = ijk = -1$$

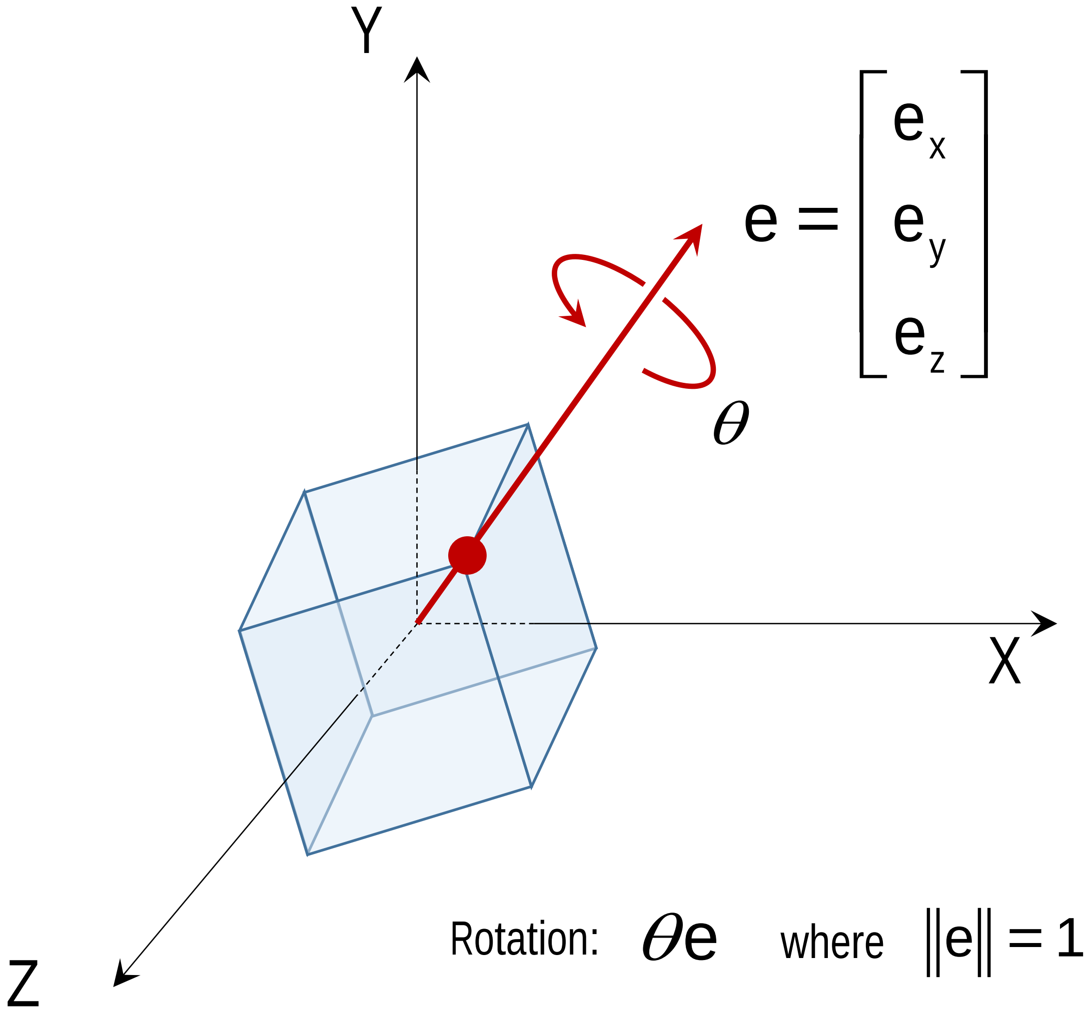


$$ij = k$$

$$jk = i$$

$$ki = j$$

Exercise

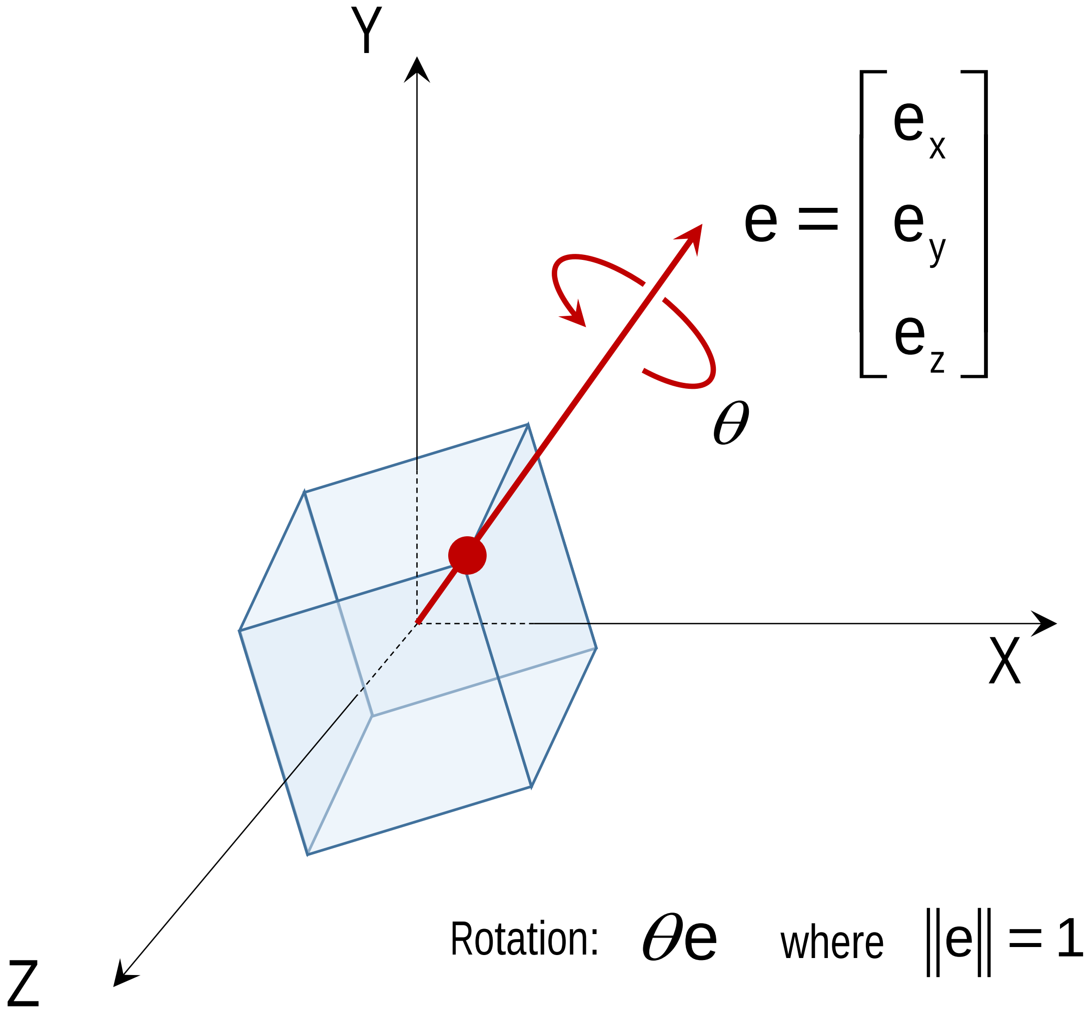


Rotation: θe where $\|e\| = 1$

$$\exp\left(\frac{\theta}{2} e\right) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(i e_x + j e_y + k e_z\right)$$

Find a quaternion q such that it describes a rotation of 60 degrees about the axis a=[3, 4, 0].

Exercise



$$e = \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}$$

θ

Rotation: $\theta \mathrm{e}$ where $\|\mathrm{e}\| = 1$

$$\exp\left(\frac{\theta}{2}\mathrm{e}\right) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(i\,e_x + j\,e_y + k\,e_z\right)$$

Find a quaternion q such that it describes a rotation of 60 degrees about the axis a=[3, 4, 0].

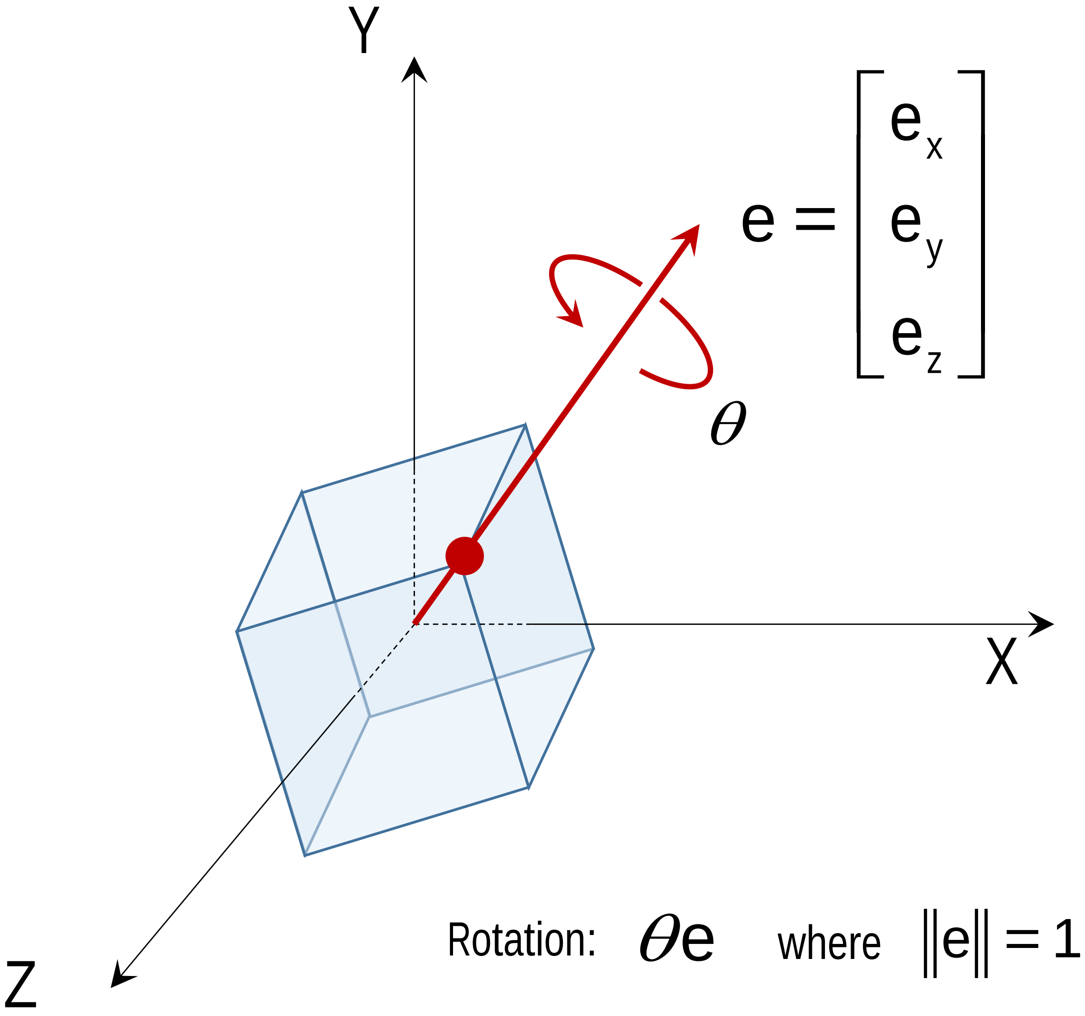
$$\mathrm{e} = \mathrm{a}/\|\mathrm{a}\| = \underbrace{i\frac{3}{5} + j\frac{4}{5} + k\frac{0}{5}}_{\text{Unit vector}}$$

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(i\frac{3}{5} + j\frac{4}{5} + k\frac{0}{5}\right) \qquad \theta = \frac{\pi}{3}$$

$$= \cos\frac{\pi}{3\cdot 2} + \sin\frac{\pi}{3\cdot 2}\left(i\frac{3}{5} + j\frac{4}{5} + k\frac{0}{5}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(i\frac{3}{5} + j\frac{4}{5} + k\frac{0}{5}\right)$$

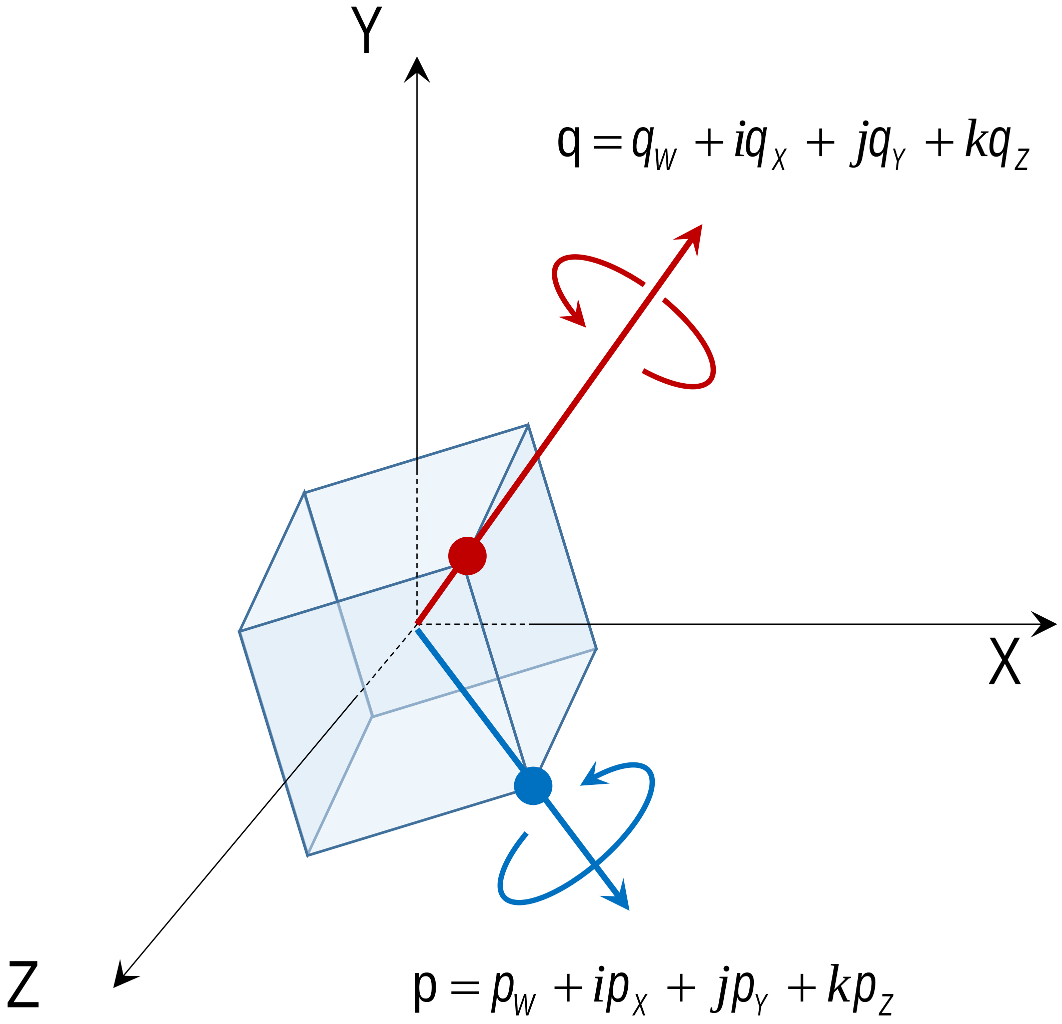
3D Exponential Map: Quaternion



Rotation: θe where $\|e\| = 1$


Broom Bridge, Dublin, Ireland

Quaternion Product

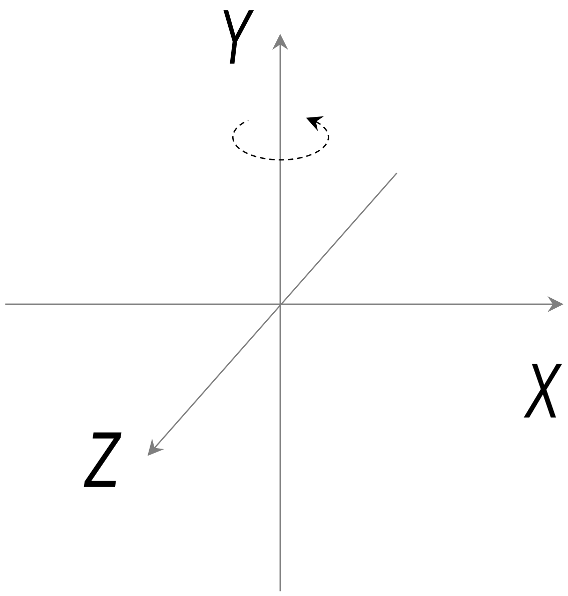


Rotate q and then, p:

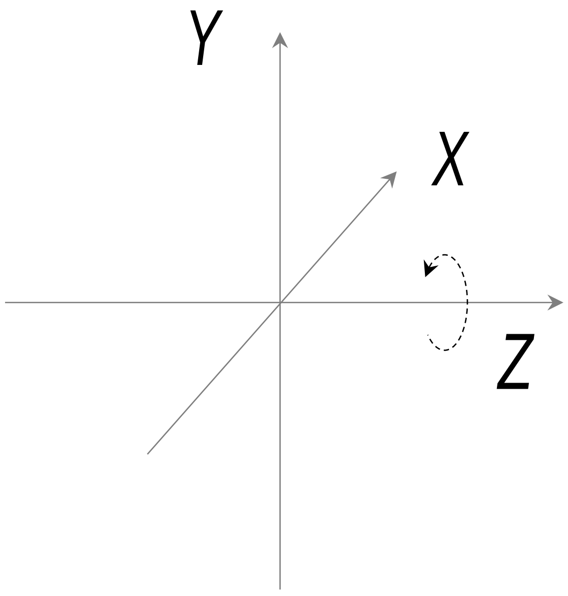
$$
\begin{aligned}
qp &= (q_W + iq_X + jq_Y + kq_Z)(p_W + ip_X + jp_Y + kp_Z) \\
&= (q_W p_W - q_X p_X - q_Y p_Y - q_Z p_Z) + i(q_W p_X + q_X p_W + q_Y p_Z - q_Z p_Y) \\
&\quad + j(q_W p_Y - q_X p_Z + q_Y p_W + q_Z p_X) + k(q_W p_Z + q_X p_Y - q_Y p_X + q_Z p_W) \\
&= (q_W p_W - \hat{q} \cdot \hat{p}) + (q_W \hat{p} + p_W \hat{q} + \hat{q} \times \hat{p})
\end{aligned}
$$

where $\hat{q} = iq_X + jq_Y + kq_Z$

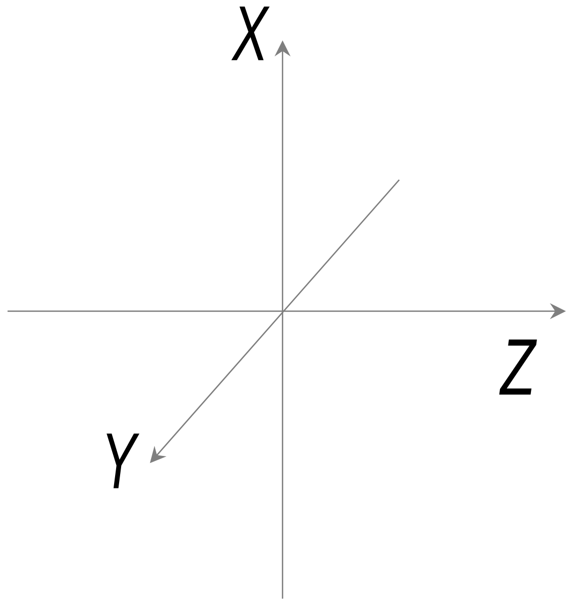
Quaternion Product Example



Rotating 90 degrees about *Y* axis.



Rotating 90 degrees about *Z* axis.



$$q_1 = \cos\frac{\pi/2}{2} + j\sin\frac{\pi/2}{2}$$

$$q_2 = \cos\frac{\pi/2}{2} + k\sin\frac{\pi/2}{2}$$

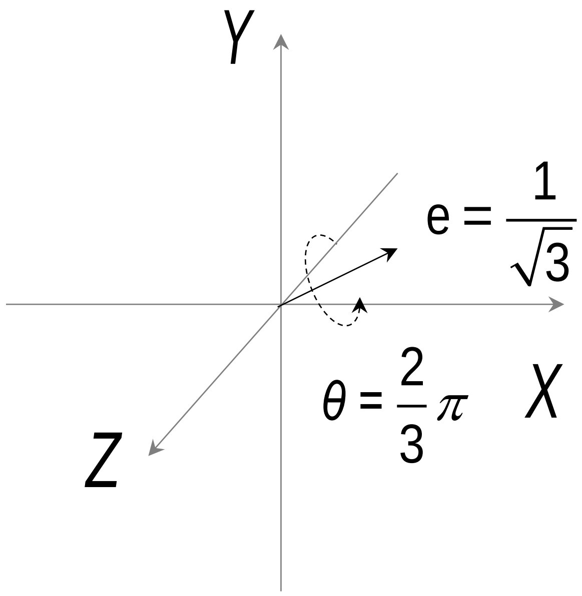
$$q_{12} = q_1 q_2 \qquad ?$$

Quaternion Product Example

$$\mathrm{qp} = \left(q_w p_w - \hat{\mathrm{q}} \cdot \hat{\mathrm{p}}\right) + \left(q_w \hat{\mathrm{p}} + p_w \hat{\mathrm{q}} + \hat{\mathrm{q}} \times \hat{\mathrm{p}}\right)$$

$$\mathrm{q}_1 = \cos\frac{\pi/2}{2} + j\sin\frac{\pi/2}{2} \qquad \mathrm{q}_2 = \cos\frac{\pi/2}{2} + k\sin\frac{\pi/2}{2}$$

Quaternion Product Example



$$e = \frac{1}{\sqrt{3}}(i + j + k)$$

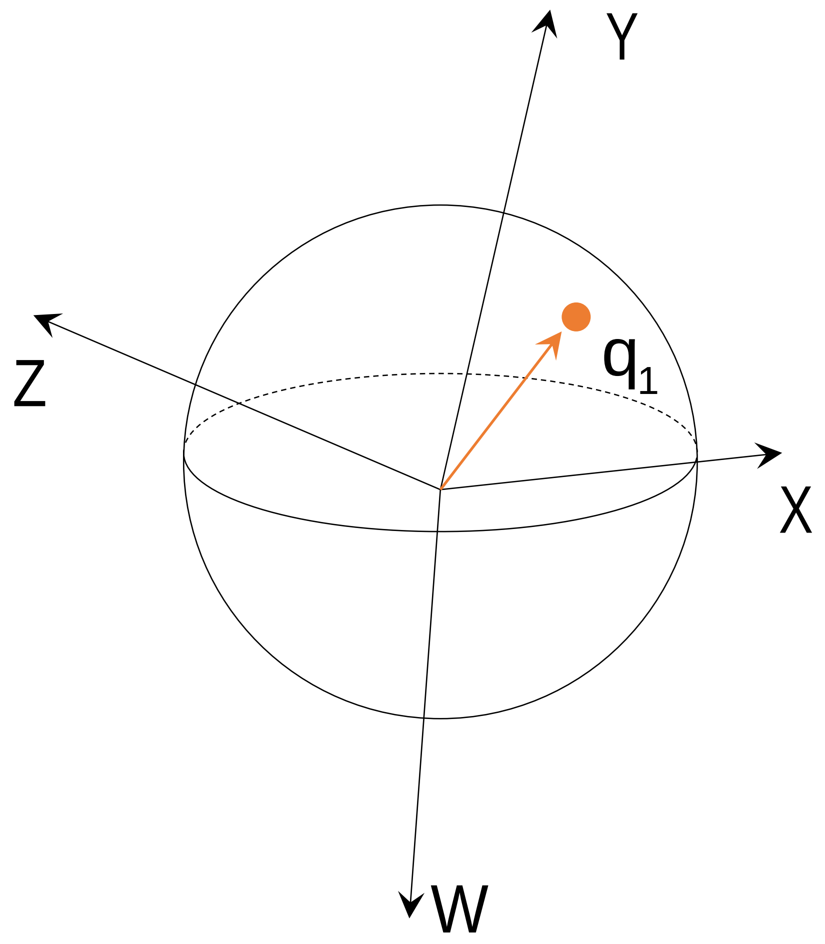
$$\theta = \frac{2}{3}\pi$$

$$qp = \left(q_w p_w - \hat{q}\cdot\hat{p}\right) + \left(q_w \hat{p} + p_w \hat{q} + \hat{q}\times\hat{p}\right)$$

$$q_1 = \cos\frac{\pi/2}{2} + j\sin\frac{\pi/2}{2} \qquad q_2 = \cos\frac{\pi/2}{2} + k\sin\frac{\pi/2}{2}$$

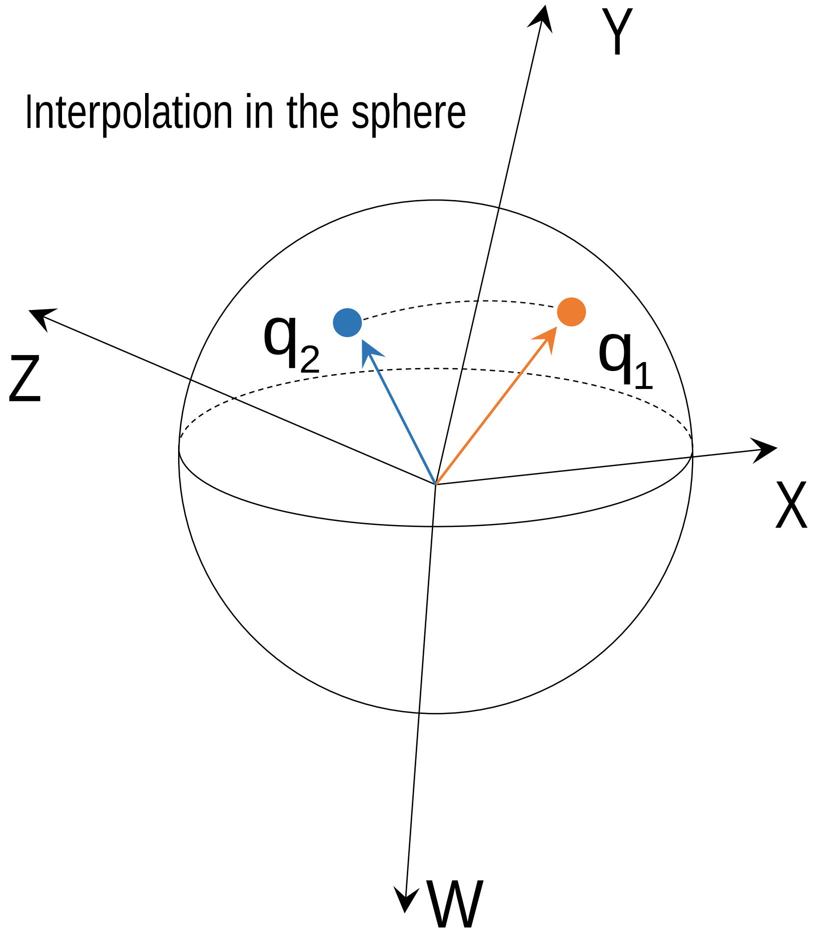
$$\begin{aligned}
q_{12} &= q_1 q_2 \\
&= \left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} + j\,\frac{\sqrt{2}}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k \\
&= \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k \\
&= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k\right)
\end{aligned}$$

Quaternion in 4D Sphere



$$q = q_W + iq_X + jq_Y + kq_Z = \begin{bmatrix} q_W \\ q_X \\ q_Y \\ q_Z \end{bmatrix} \qquad q_W^2 + q_X^2 + q_Y^2 + q_Z^2 = 1$$

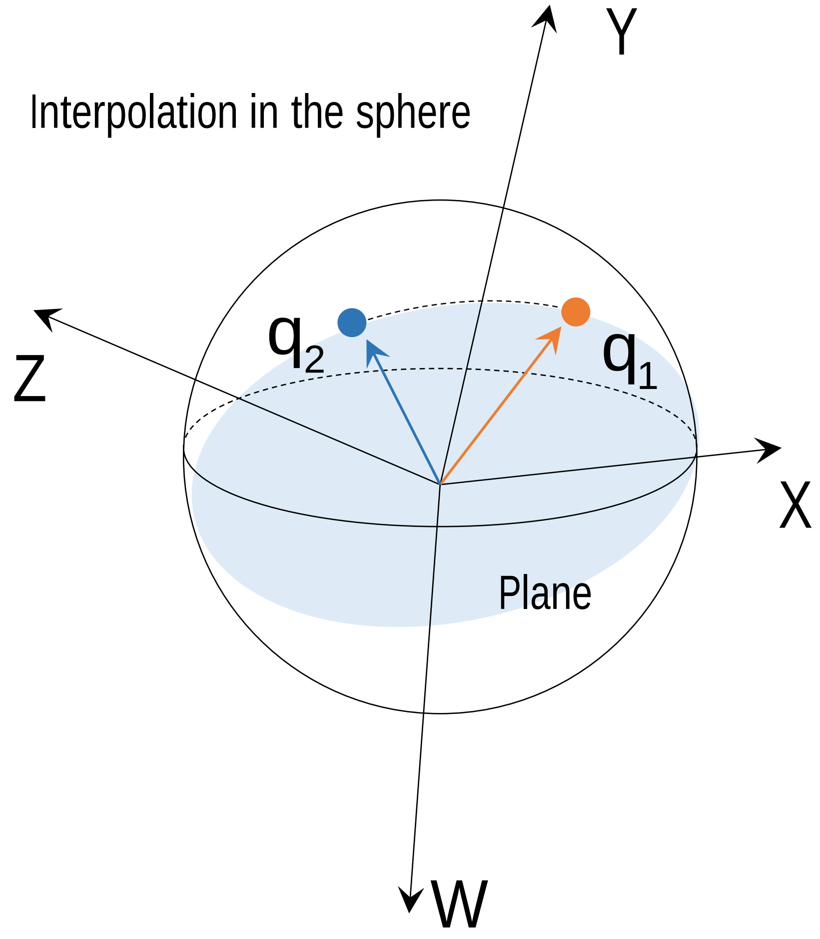
Quaternion in 4D Sphere



$$q = q_W + iq_X + jq_Y + kq_Z = \begin{bmatrix} q_W \\ q_X \\ q_Y \\ q_Z \end{bmatrix}$$

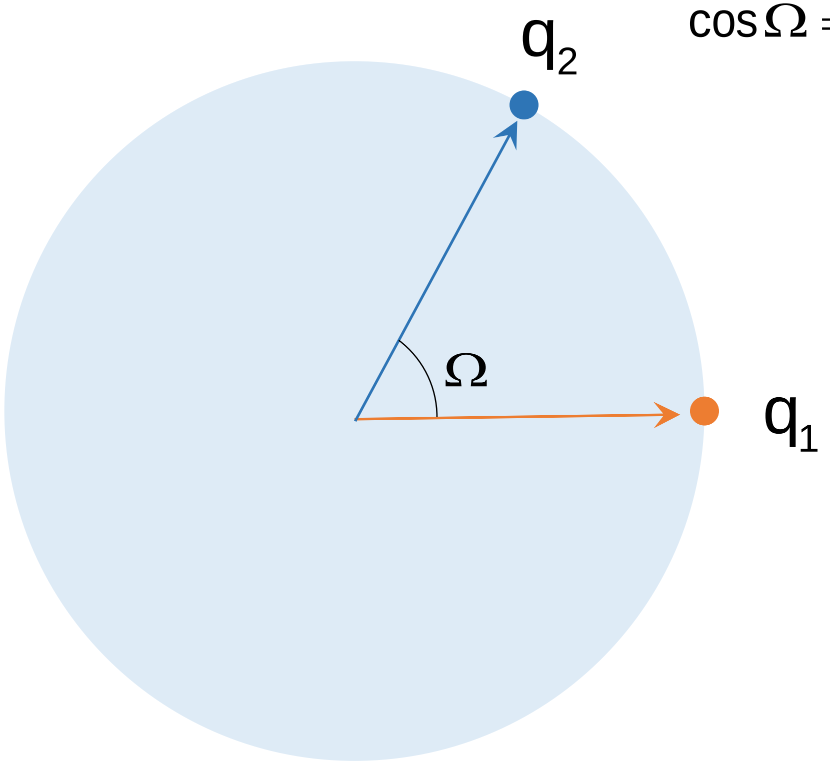
$$q_W^2 + q_X^2 + q_Y^2 + q_Z^2 = 1$$

Quaternion in 4D Sphere



$$q = q_W + iq_X + jq_Y + kq_Z = \begin{bmatrix} q_W \\ q_X \\ q_Y \\ q_Z \end{bmatrix} \qquad q_W^2 + q_X^2 + q_Y^2 + q_Z^2 = 1$$

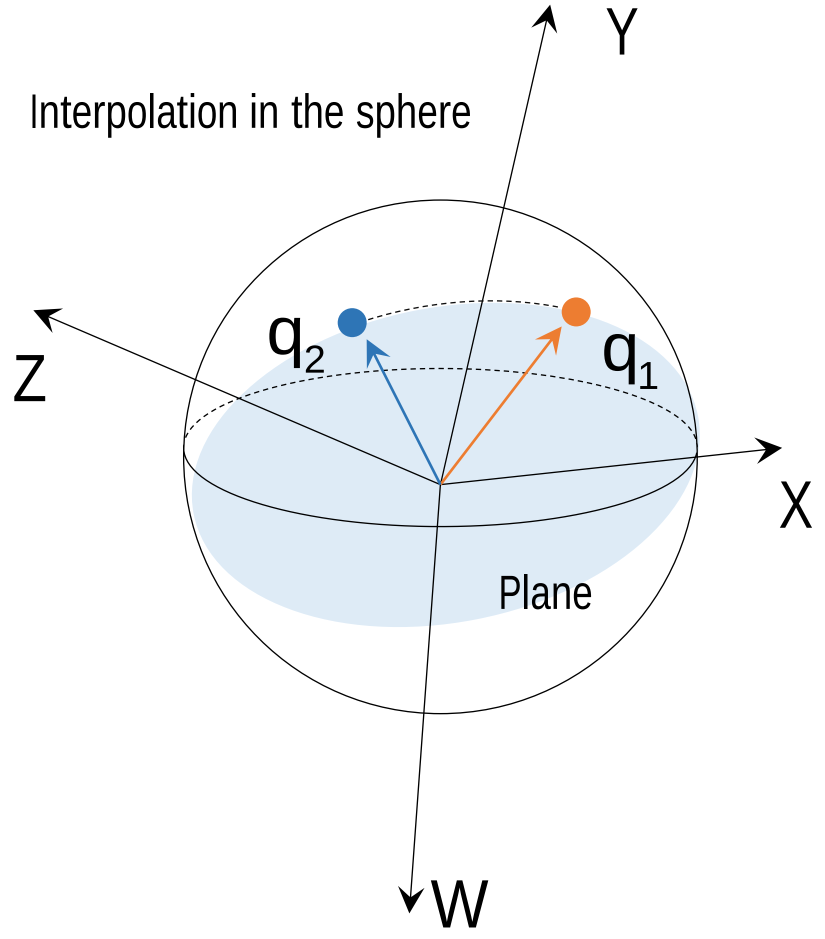
$$\cos\Omega = q_1 \cdot q_2$$



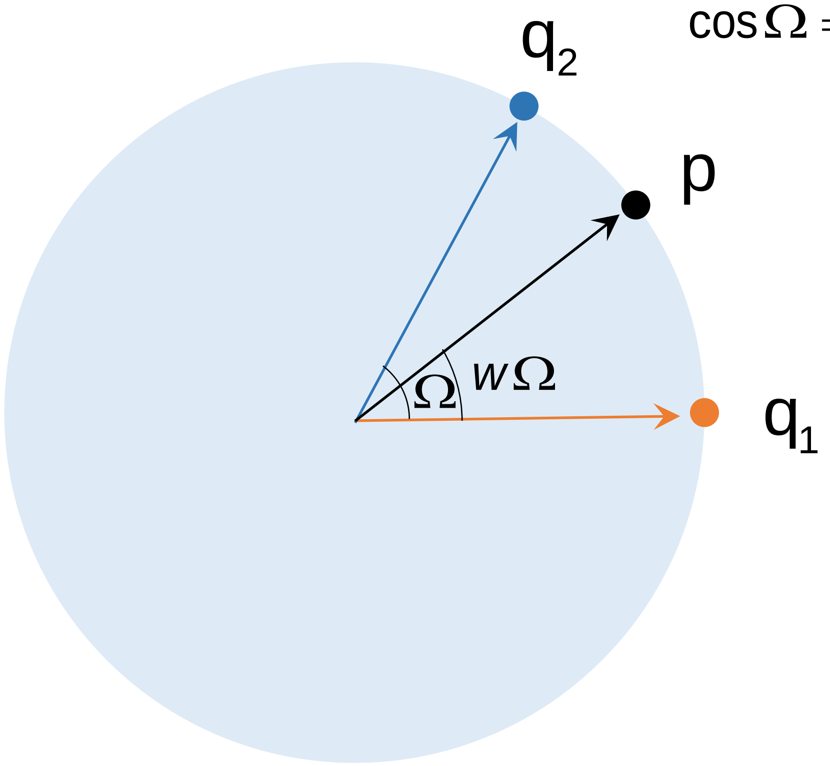
Quaternion Interpolation

$$q = q_W + iq_X + jq_Y + kq_Z = \begin{bmatrix} q_W \\ q_X \\ q_Y \\ q_Z \end{bmatrix} \qquad q_W^2 + q_X^2 + q_Y^2 + q_Z^2 = 1$$

Interpolation in the sphere

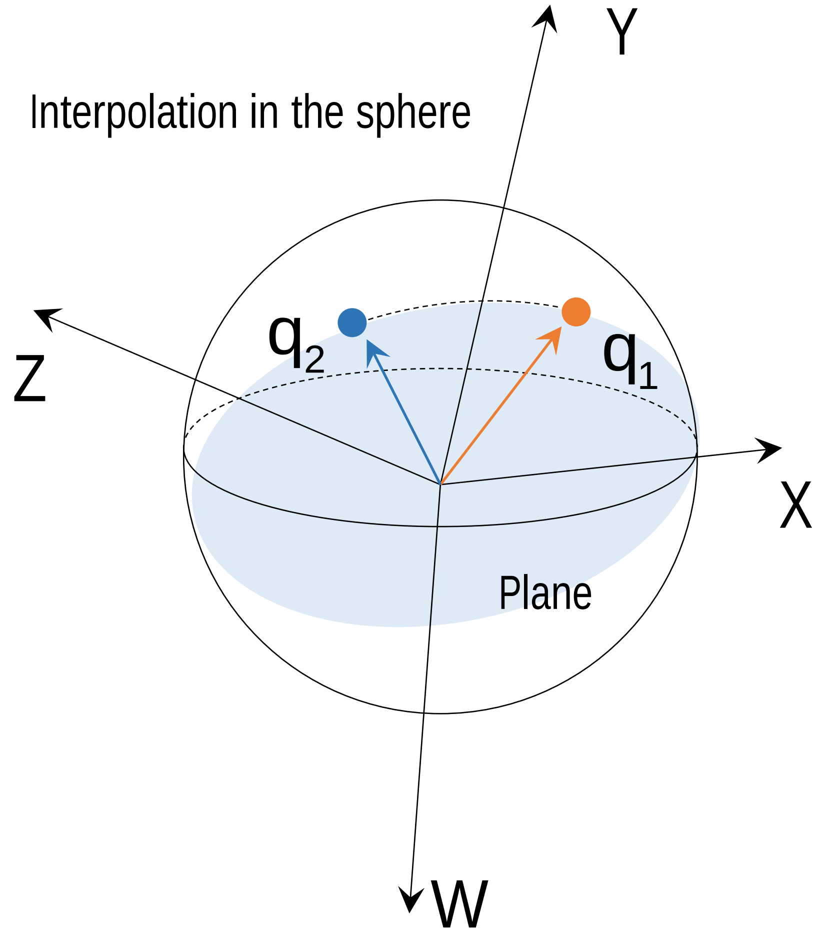


$$\cos\Omega = q_1 \cdot q_2$$

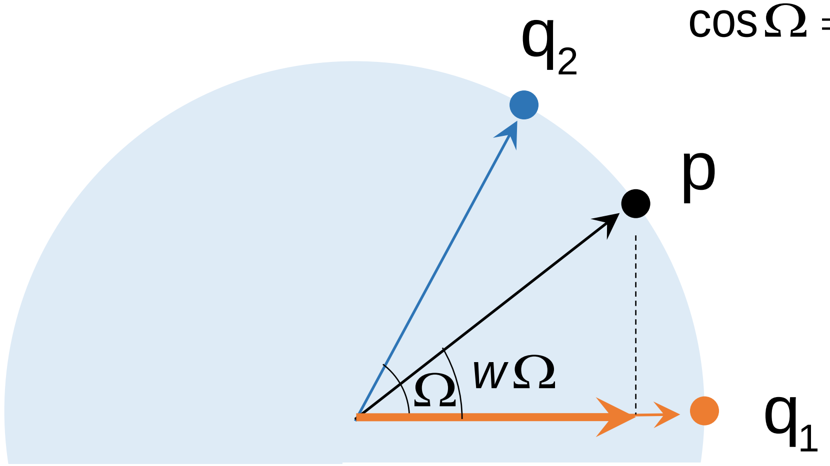


Quaternion Interpolation

$$q = q_W + iq_X + jq_Y + kq_Z = \begin{bmatrix} q_W \\ q_X \\ q_Y \\ q_Z \end{bmatrix} \qquad q_W^2 + q_X^2 + q_Y^2 + q_Z^2 = 1$$

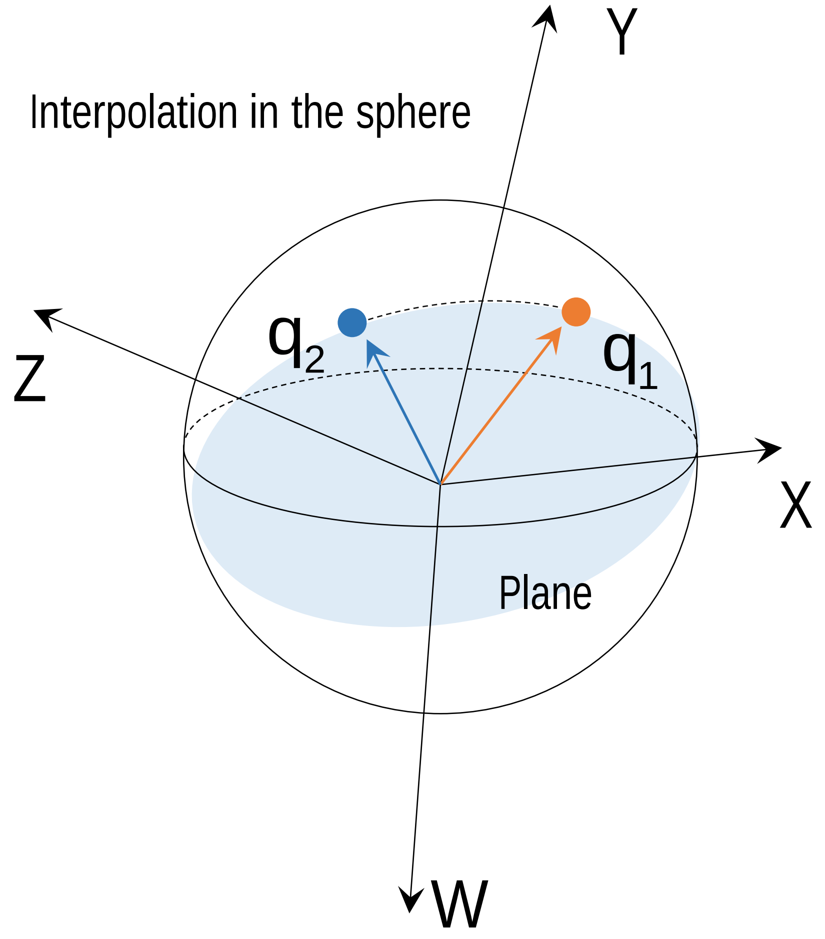


$$\cos\Omega = q_1 \cdot q_2$$



$$p = q_1 \cos w\Omega$$

Quaternion Interpolation



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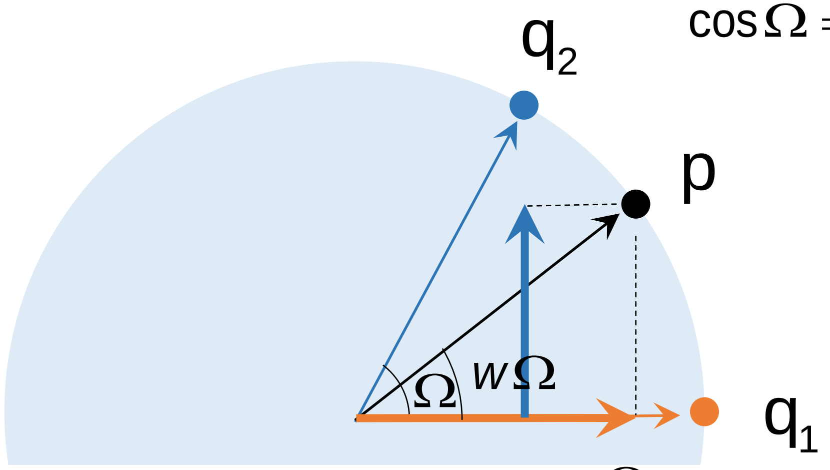
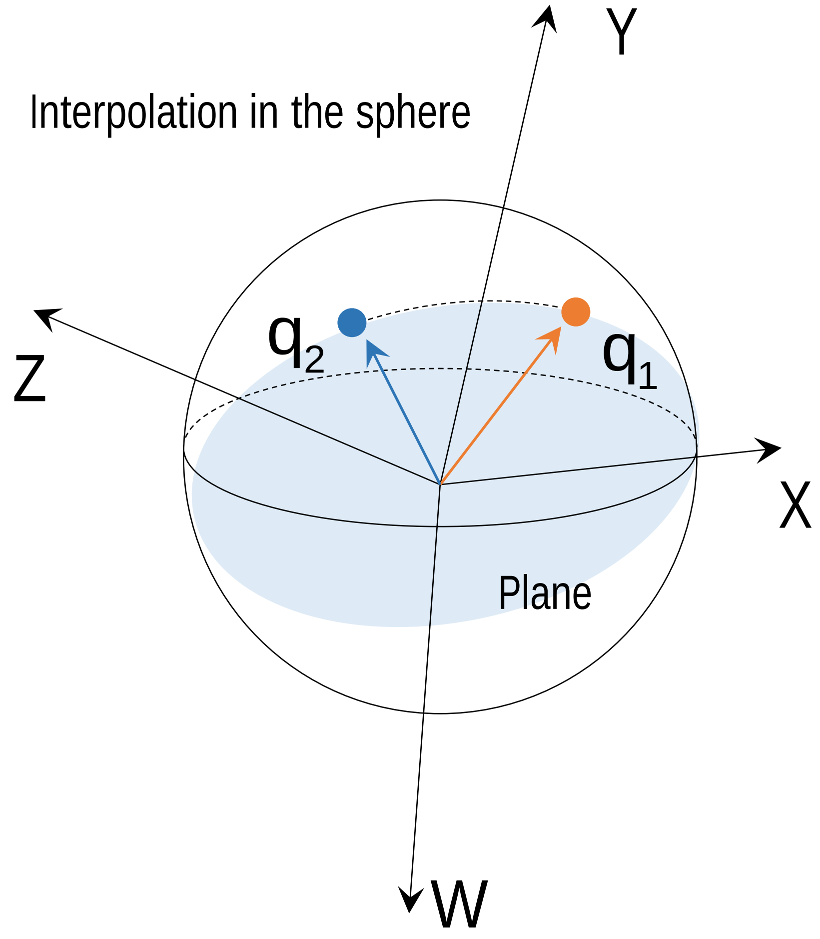


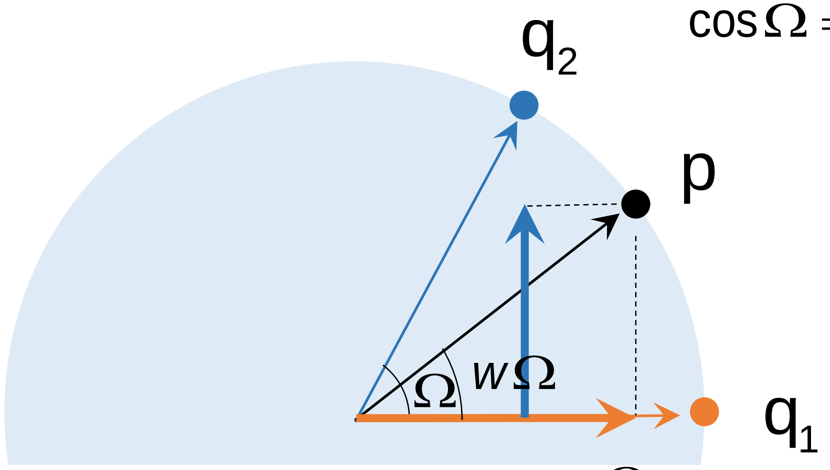
$$p = q_1 \cos w\Omega + \frac{q_2 - q_1 \cos\Omega}{\sin\Omega} \sin w\Omega$$

Quaternion Interpolation



$$q = q_W + iq_X + jq_Y + kq_Z = \begin{bmatrix} q_W \\ q_X \\ q_Y \\ q_Z \end{bmatrix} \qquad q_W^2 + q_X^2 + q_Y^2 + q_Z^2 = 1$$

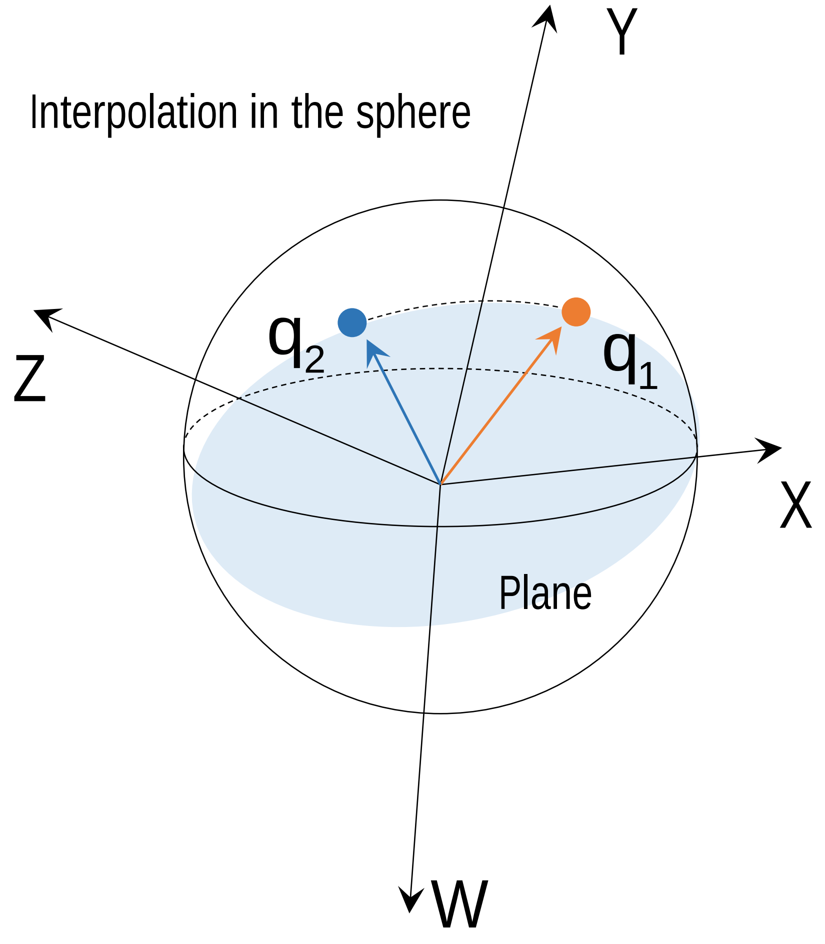
$$\cos\Omega = q_1 \cdot q_2$$



$$p = q_1 \cos w\Omega + \frac{q_2 - q_1 \cos\Omega}{\sin\Omega} \sin w\Omega$$

$$= \frac{q_1(\sin\Omega \cos w\Omega - \cos\Omega \sin w\Omega) + q_2 \sin w\Omega}{\sin\Omega}$$

Quaternion Interpolation



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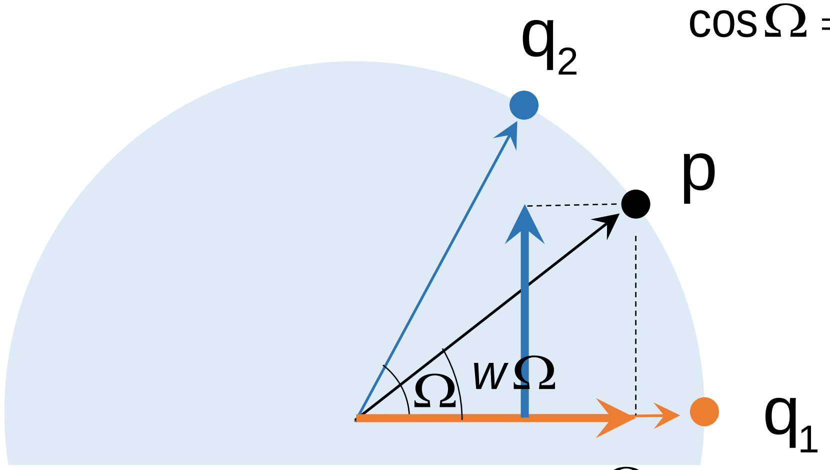
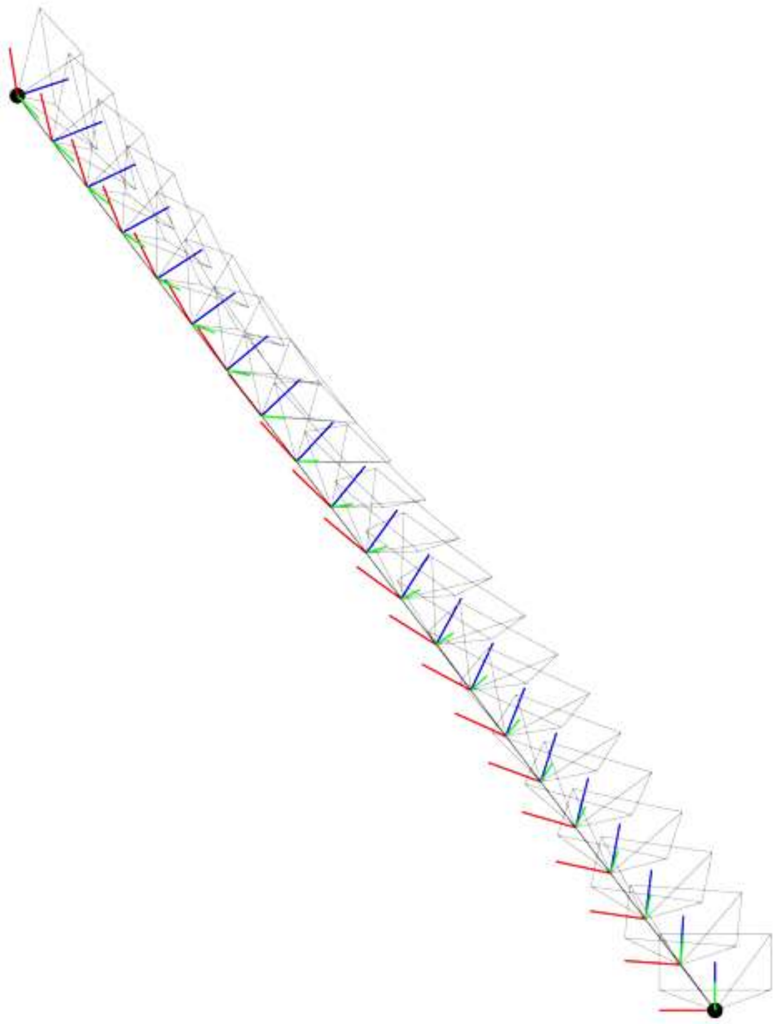
$$\cos\Omega = q_1 \cdot q_2$$

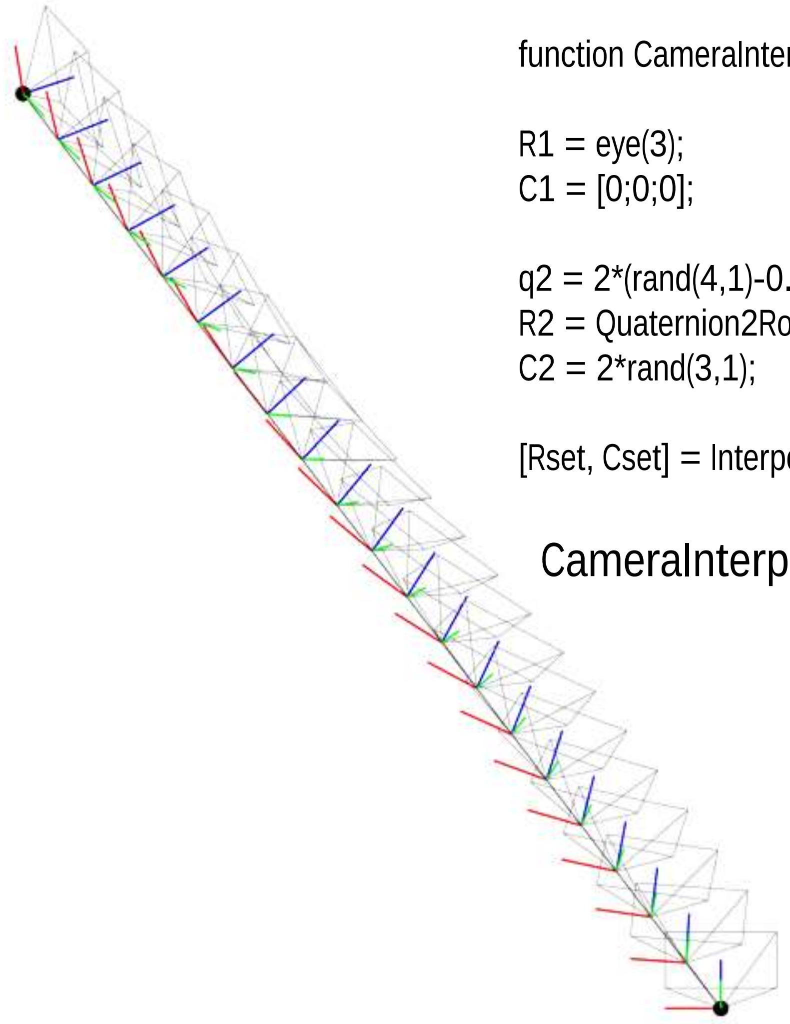


$$p = q_1 \cos w\Omega + \frac{q_2 - q_1 \cos\Omega}{\sin\Omega} \sin w\Omega$$

$$= \frac{q_1\left(\sin\Omega \cos w\Omega - \cos\Omega \sin w\Omega\right) + q_2 \sin w\Omega}{\sin\Omega}$$

$$= \frac{q_1 \sin(1\text{-}w)\Omega + q_2 \sin w\Omega}{\sin\Omega}$$





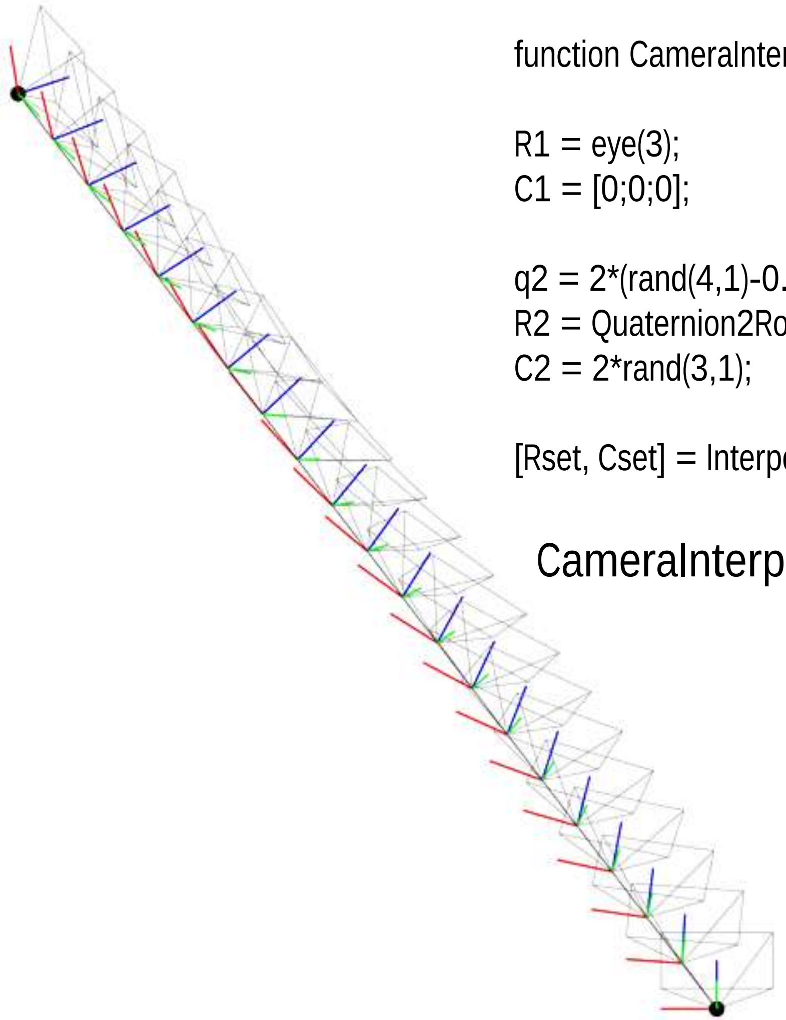
```
function CameraInterpolation

R1 = eye(3);
C1 = [0;0;0];

q2 = 2*(rand(4,1)-0.5);
R2 = Quaternion2Rotation(q2);
C2 = 2*rand(3,1);

[Rset, Cset] = InterpolateCoordinate(R1, C1, R2, C2, 20);
```

CameraInterpolation.m



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[Rset, Cset] = InterpolateCoordinate(R1, C1, R2, C2, 20);
```

CameraInterpolation.m

```
function [Rset, Cset] = InterpolateCoordinate(R1, C1, R2, C2, n)

Cx = linspace(C1(1), C2(1), n+1);
Cy = linspace(C1(2), C2(2), n+1);
Cz = linspace(C1(3), C2(3), n+1);

Cset = [Cx; Cy; Cz];

w = 0 : 1/n : 1;
q1 = Rotation2Quaternion(R1);
q2 = Rotation2Quaternion(R2);

omega = acos(q1'*q2);
for i = 1 : length(w)
    q = sin(omega*(1-w(i)))/sin(omega) * q1 + sin(omega*w(i))/sin(omega) * q2;
    Rset{i} = Quaternion2Rotation(q);
end
```

InterpolateCoordinate.m

View Interpolation

Looking left

Looking right

View Interpolation (HW #3)

Single View Review (Where am I?)