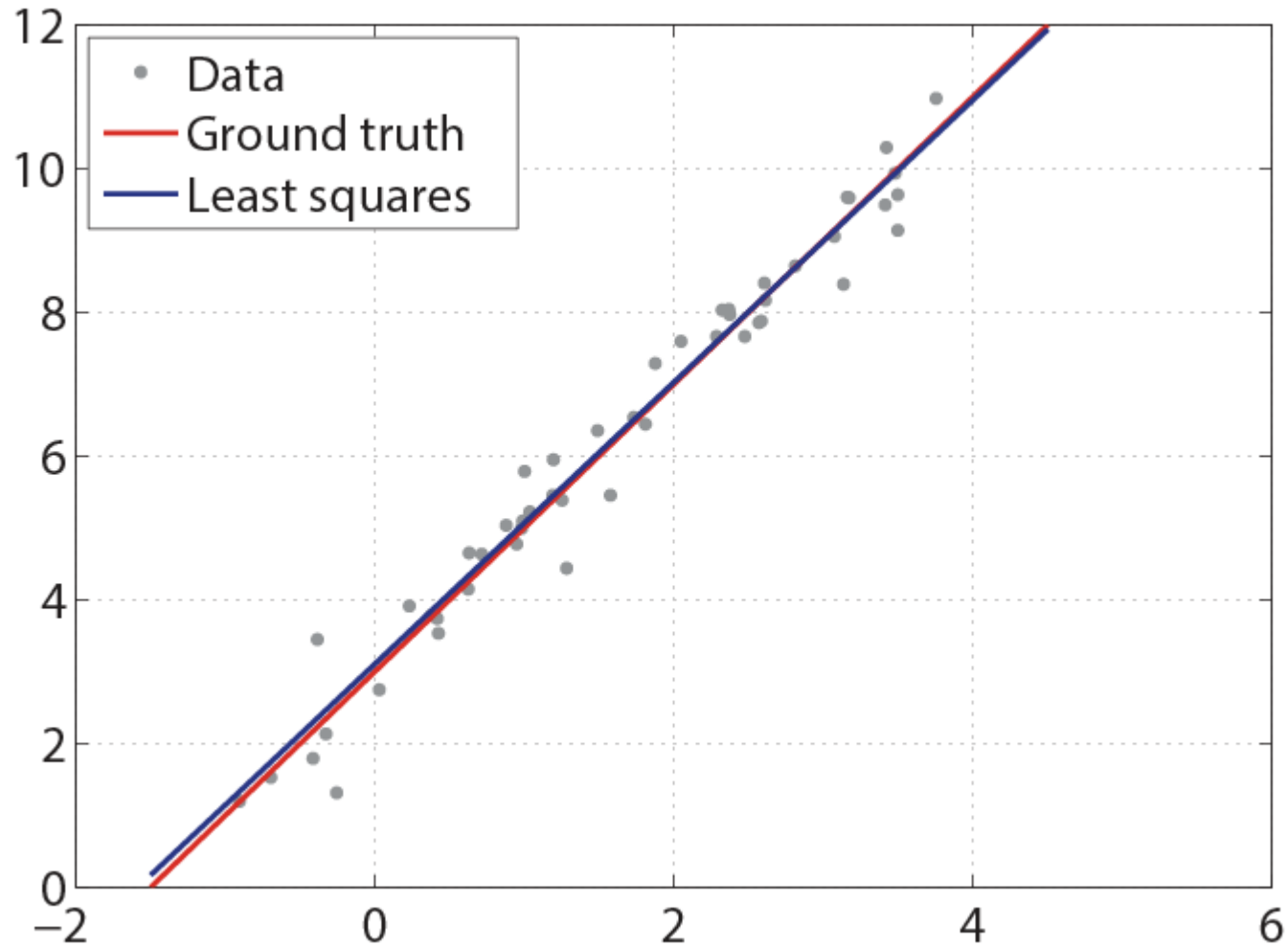


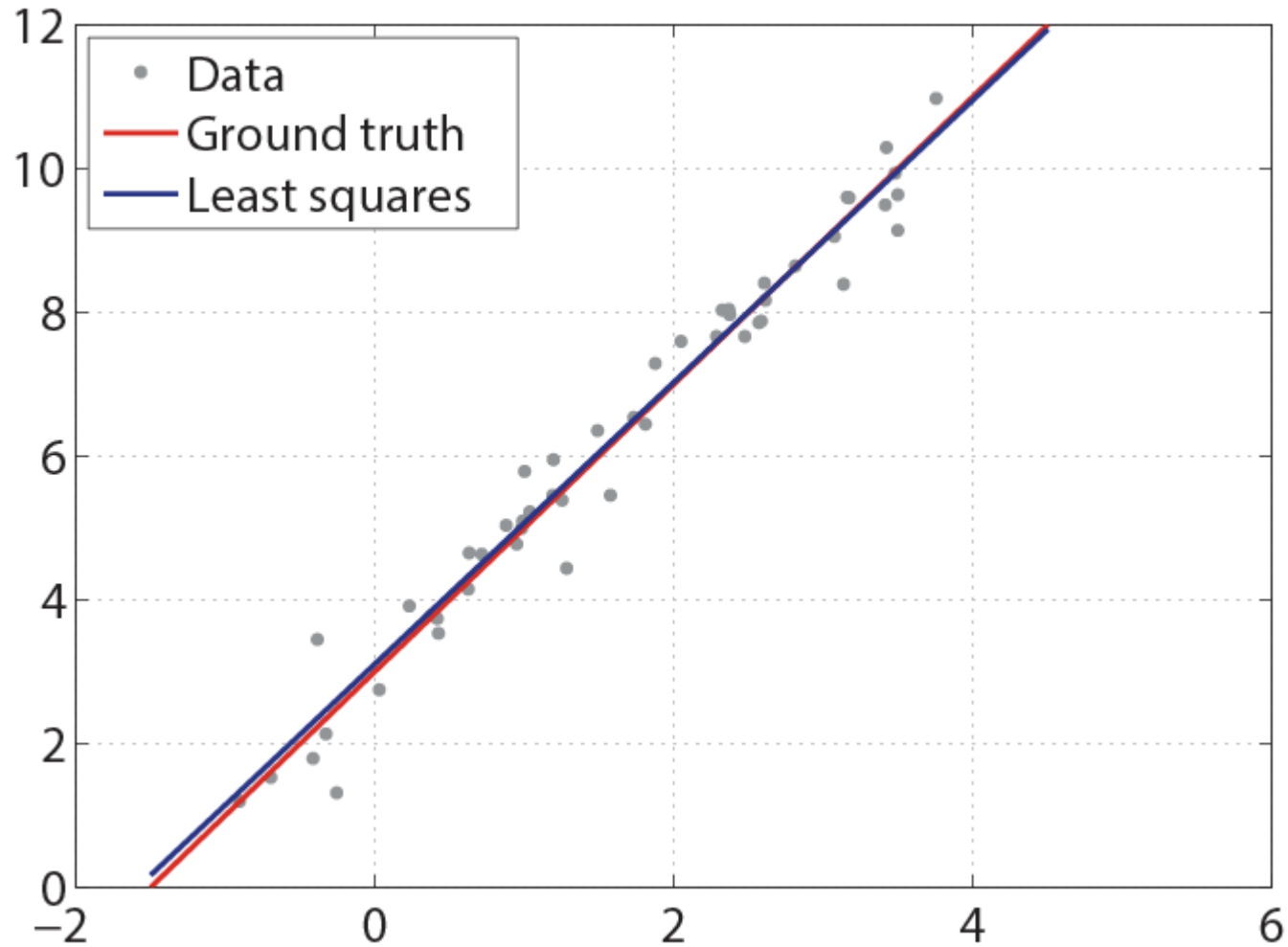
Nonlinear Estimation

Recall: Line Fitting ($Ax=b$)



$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

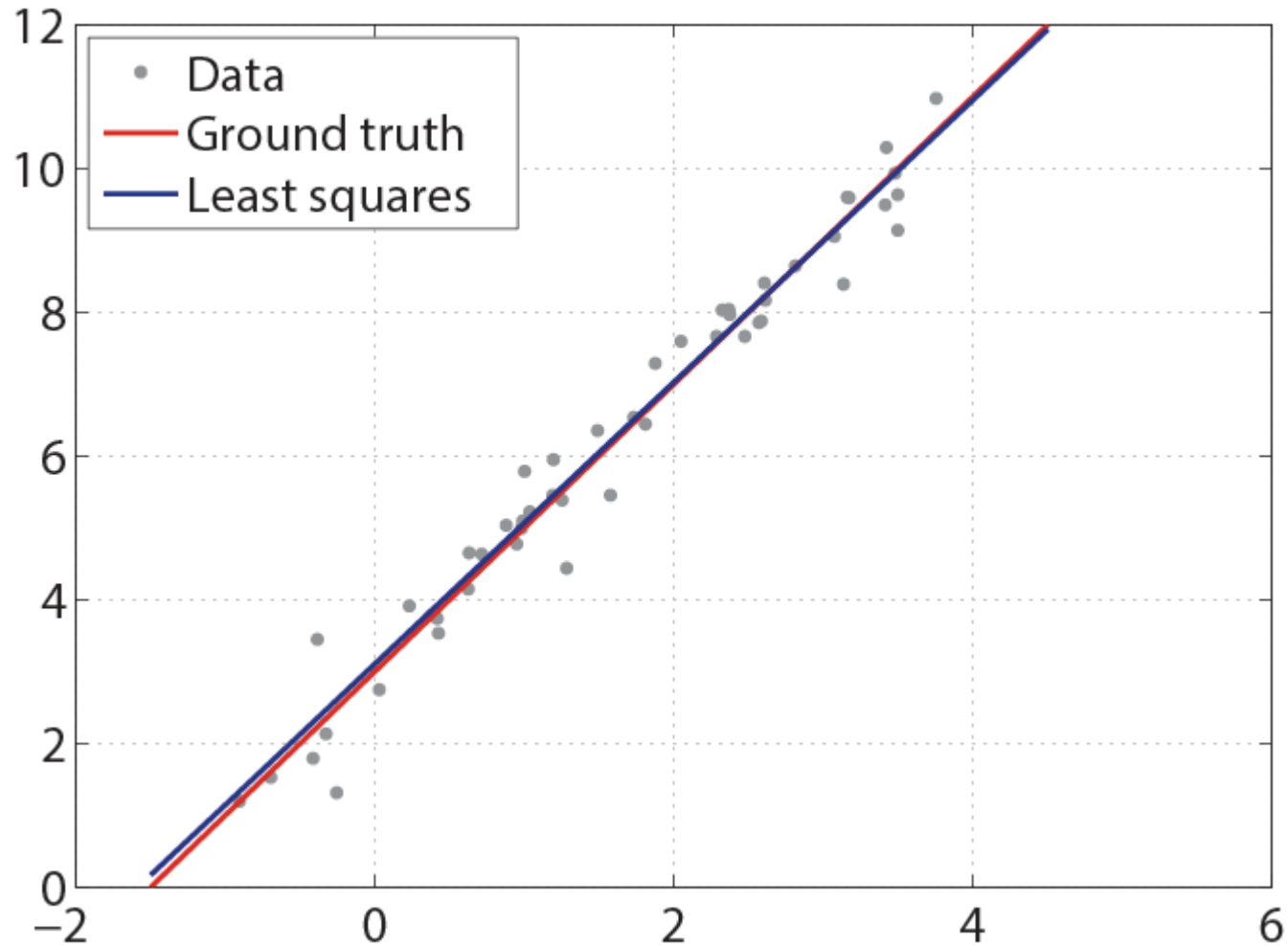
Recall: Line Fitting ($Ax=b$)



$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$A^T A x = A^T b$$

Recall: Line Fitting ($Ax=b$)



$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The diagram shows a matrix A with n rows and 2 columns. The first column contains values u_1, u_2, \dots, u_n and the second column contains ones. This matrix is multiplied by a vector x with elements m and d . The result is approximately equal to a vector b with elements v_1, v_2, \dots, v_n .

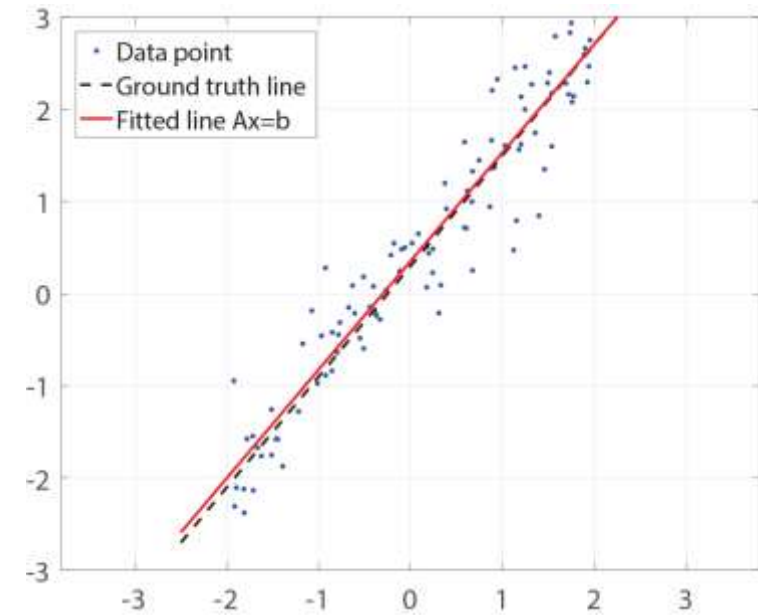
$$A^T A x = A^T b$$

$$x = \left[A^T A \right]^{-1} A^T b$$

Recall: Line Fitting ($Ax=b$)

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \mathbf{x} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Error:

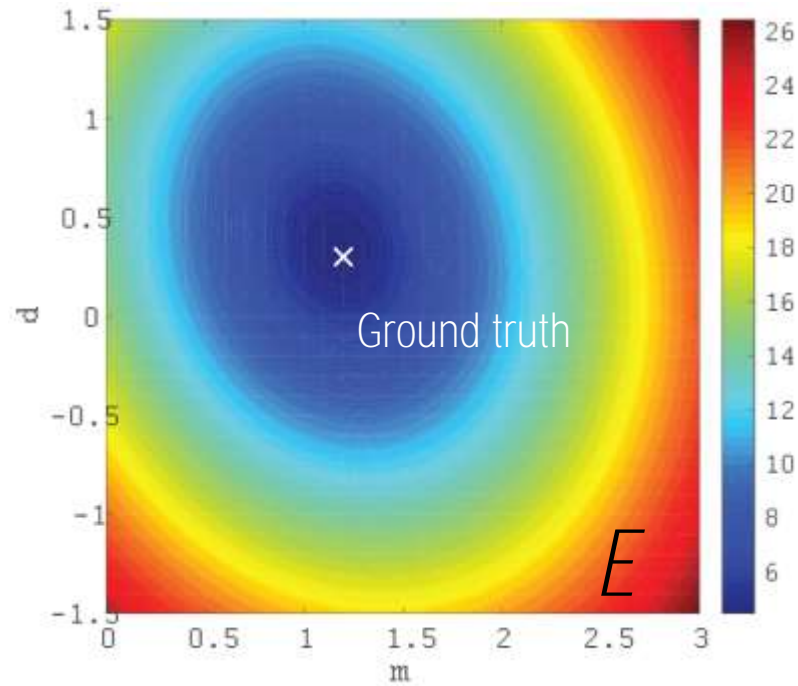
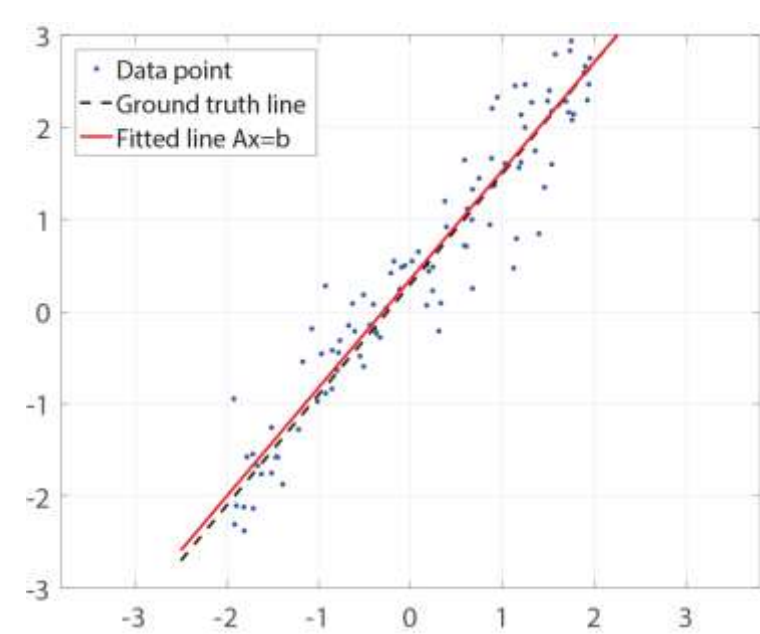


Recall: Line Fitting ($Ax=b$)

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Error:

$$E = \left\| \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|^2$$

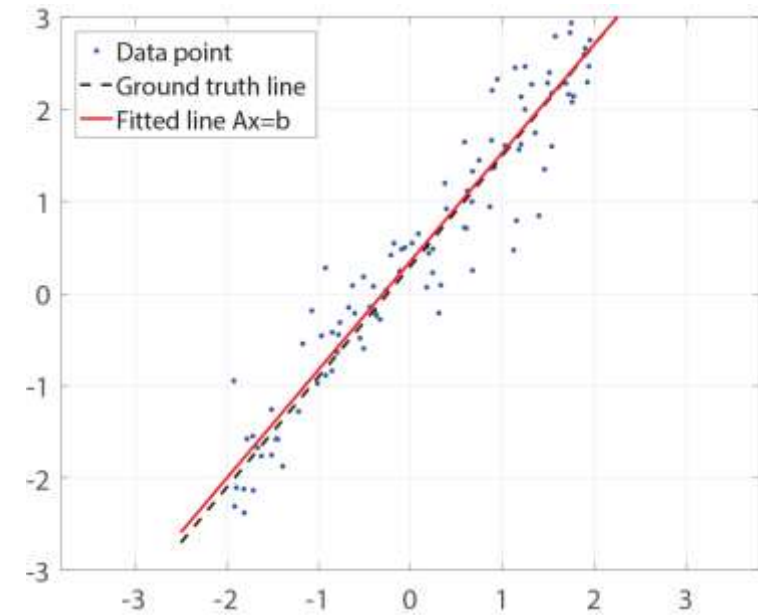


Recall: Line Fitting

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

Error:

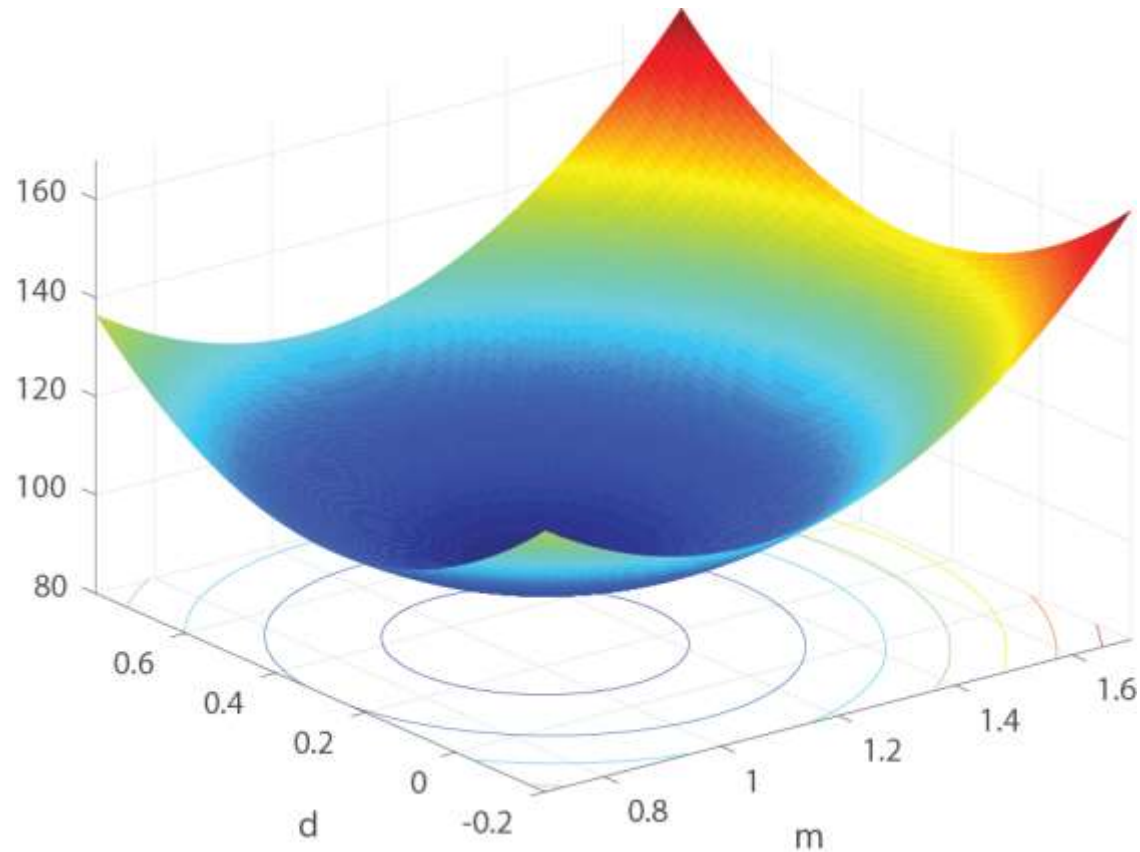
$$E = \left\| \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|^2$$



Recall: Line Fitting

Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

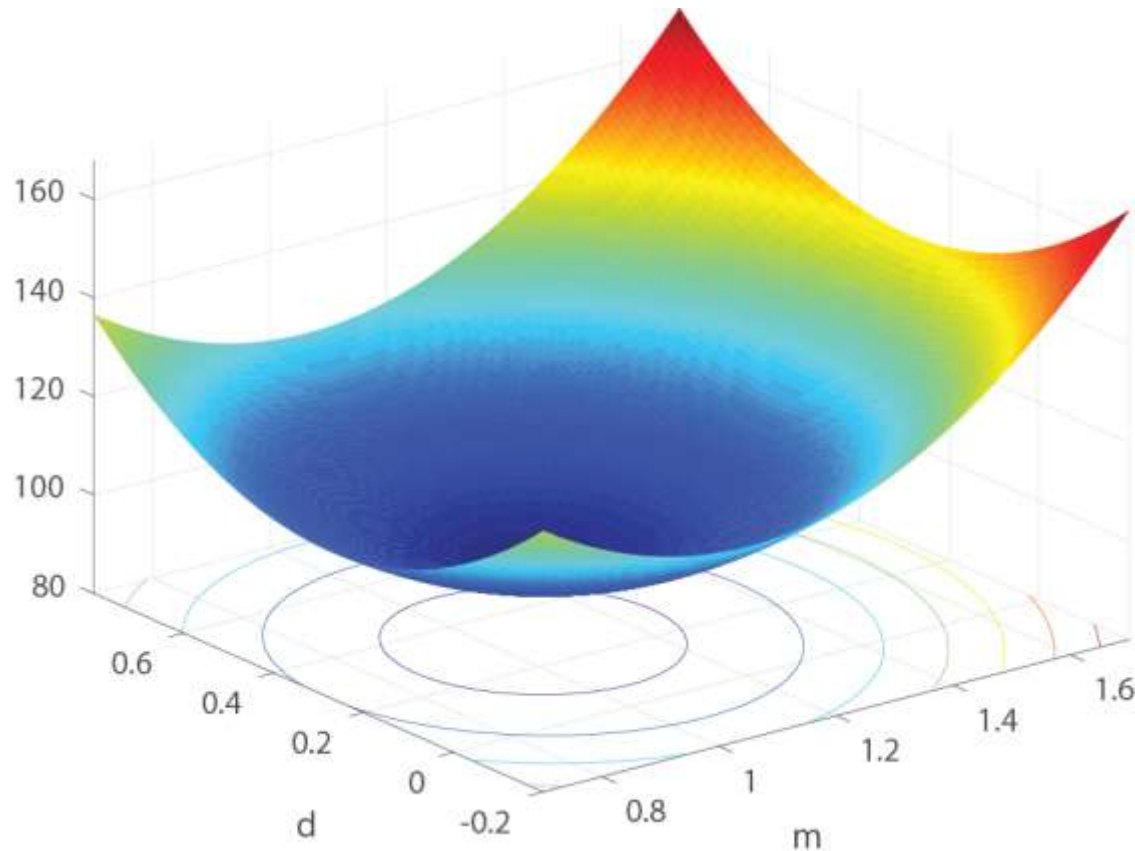
Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$



Recall: Line Fitting

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$\text{Optimal point: } \frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$$



$$V_1 \approx mu_1 + d$$

$$V_2 \approx mu_2 + d$$

$$V_n \approx mu_n + d$$

$$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$

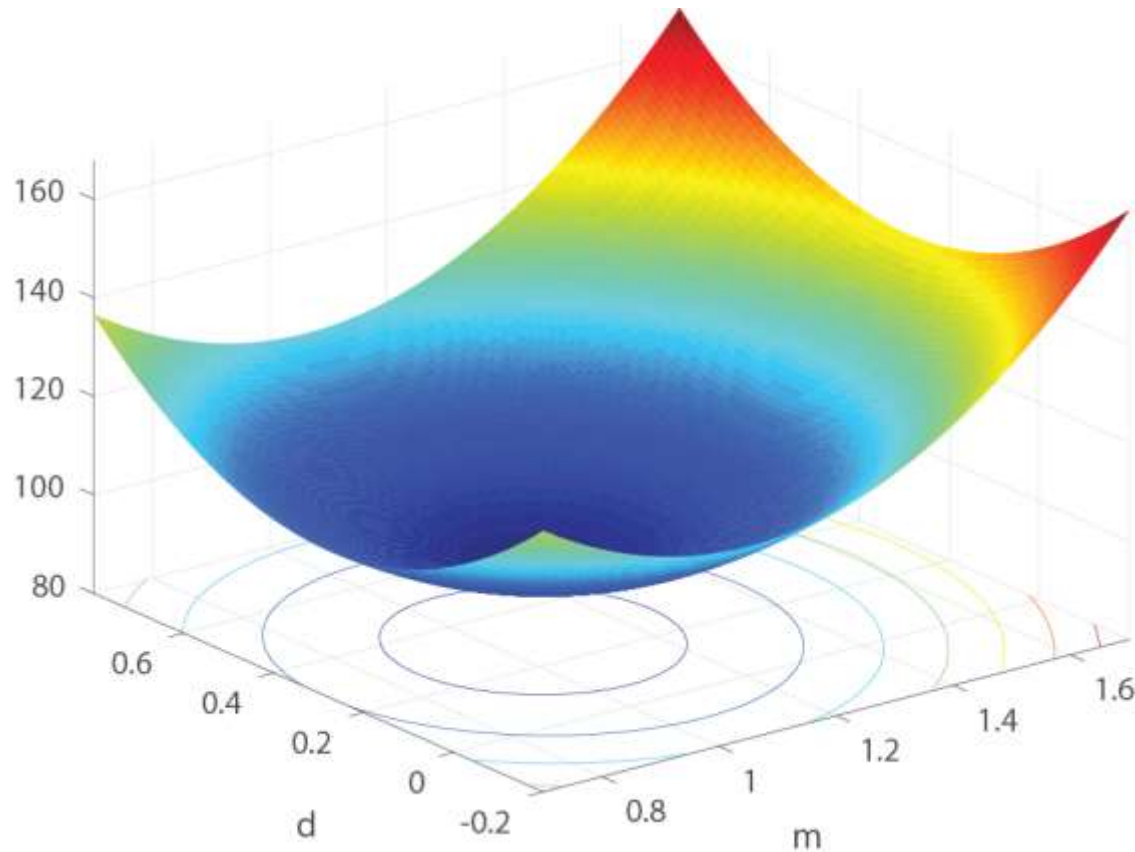
$$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$

$$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

Recall: Line Fitting

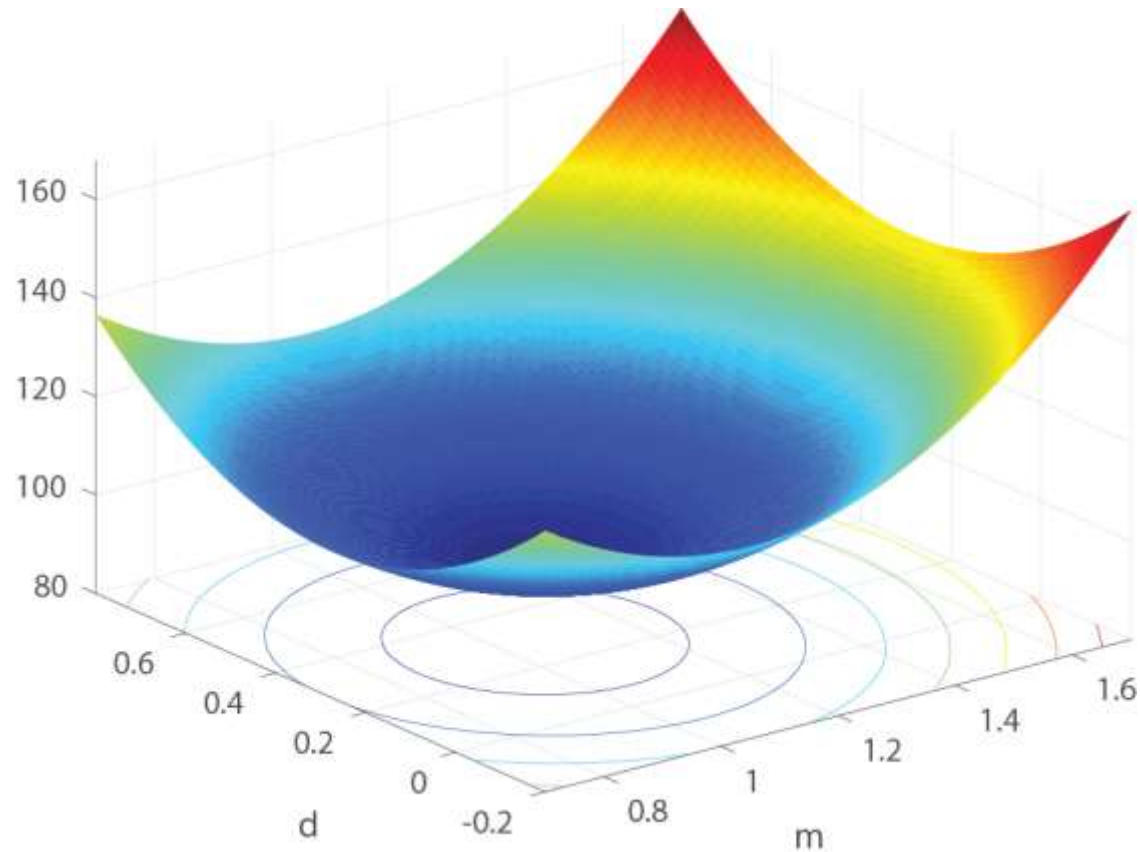
$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$\text{Optimal point: } \frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$$



$$\begin{aligned} V_1 &\approx mu_1 + d && \begin{bmatrix} u_1 & 1 \\ m \\ d \end{bmatrix} \approx v_1 \\ V_2 &\approx mu_2 + d && \begin{bmatrix} u_2 & 1 \\ m \\ d \end{bmatrix} \approx v_2 \\ &&& \vdots \\ V_n &\approx mu_n + d && \begin{bmatrix} u_n & 1 \\ m \\ d \end{bmatrix} \approx v_n \end{aligned} \quad \longrightarrow \quad \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Recall: Line Fitting



$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$\text{Optimal point: } \frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$$

$$\begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned} \quad \rightarrow \quad \begin{aligned} \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} &\approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \end{aligned} \quad \rightarrow \quad \begin{aligned} \underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_A \begin{bmatrix} m \\ d \end{bmatrix} &\approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \end{aligned}$$

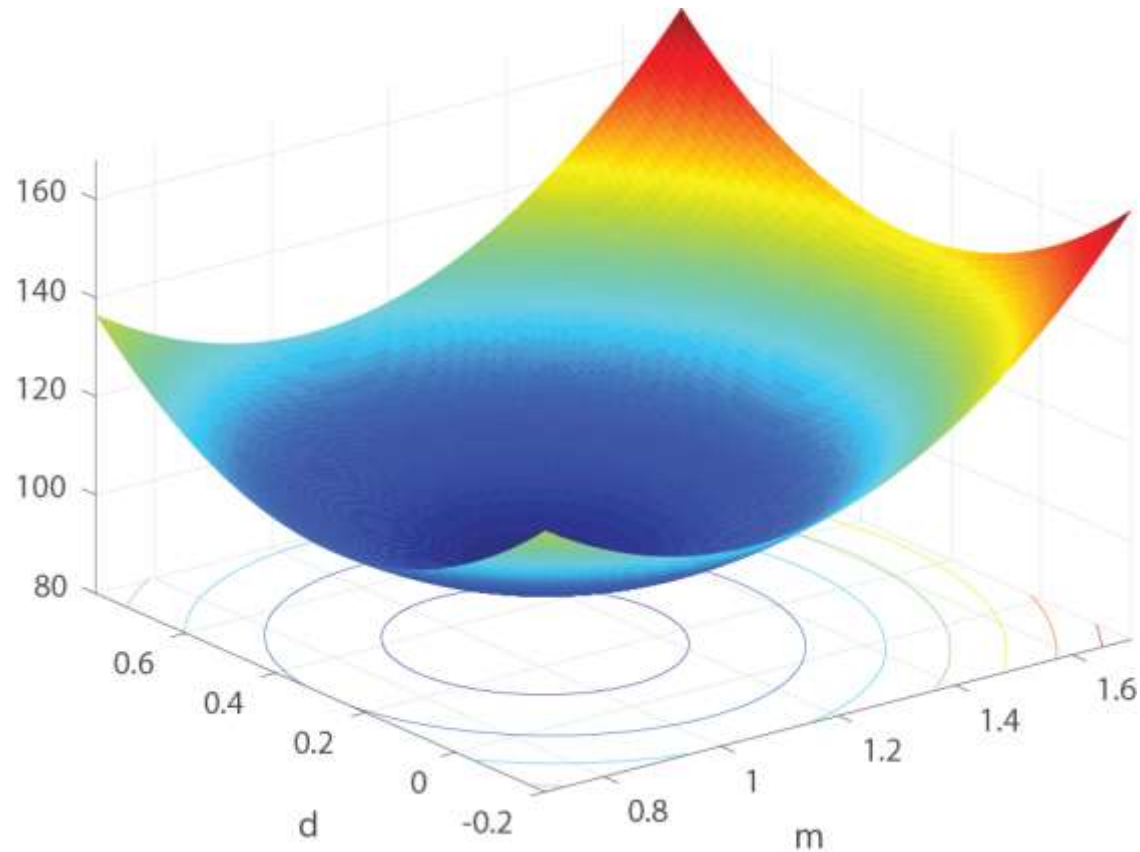
We can't invert A.

Recall: Line Fitting

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$



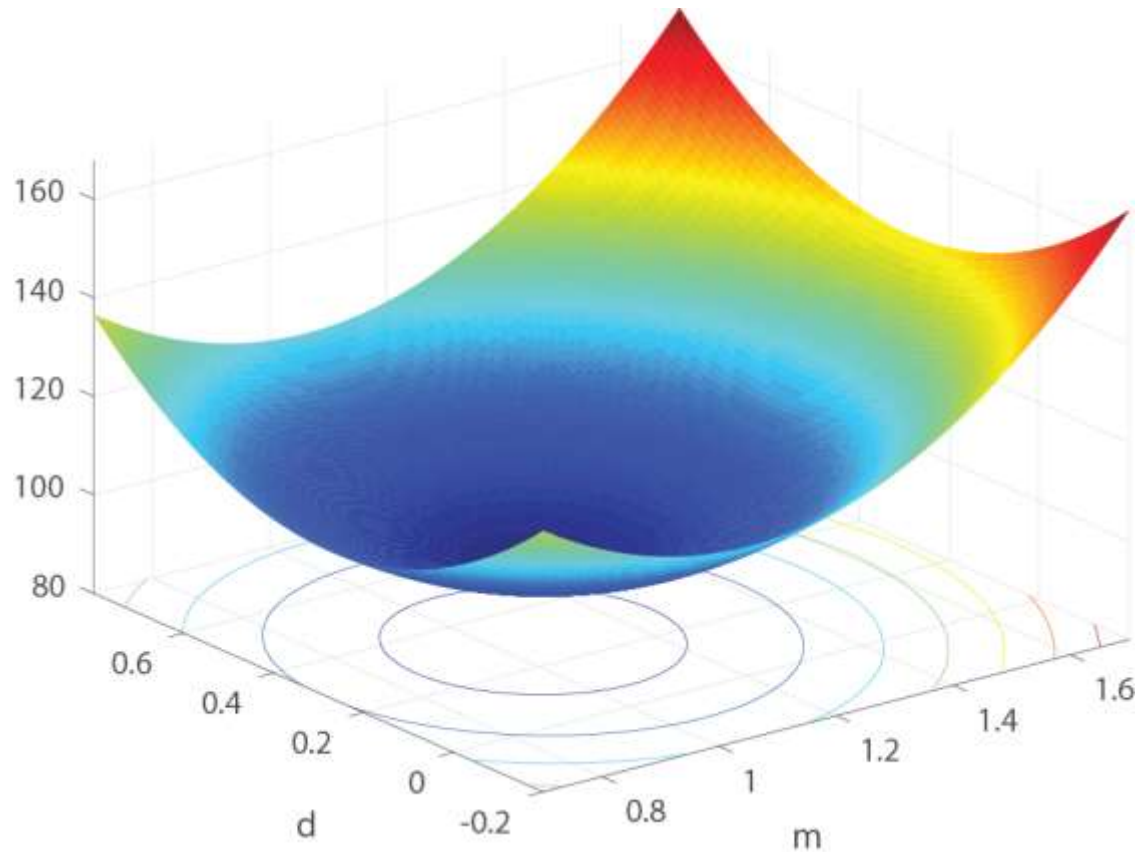
Recall: Line Fitting

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$$
$$= x^T A^T Ax - 2x^T A^T b + b^T b$$

$$\frac{\partial E}{\partial x} = \quad ?$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$



Recall: Line Fitting

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

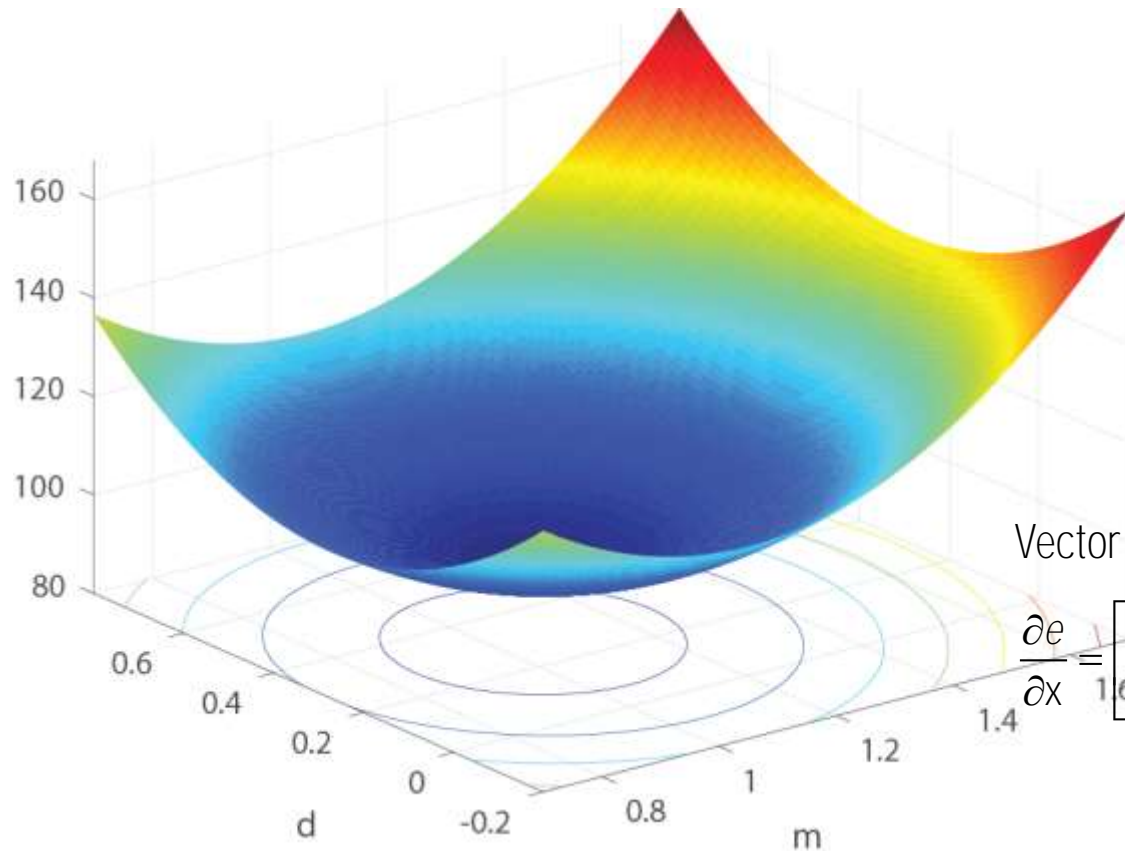
$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2 \\ = x^T A^T A x - 2x^T A^T b + b^T b$$

$$\frac{\partial E}{\partial x} = ?$$

Vector derivative:

$$\frac{\partial e}{\partial x} = \left[\frac{\partial e}{\partial x_1} \quad \dots \quad \frac{\partial e}{\partial x_n} \right]$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$



Recall: Line Fitting

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$$

$$= x^T A^T A x - 2x^T A^T b + b^T b$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

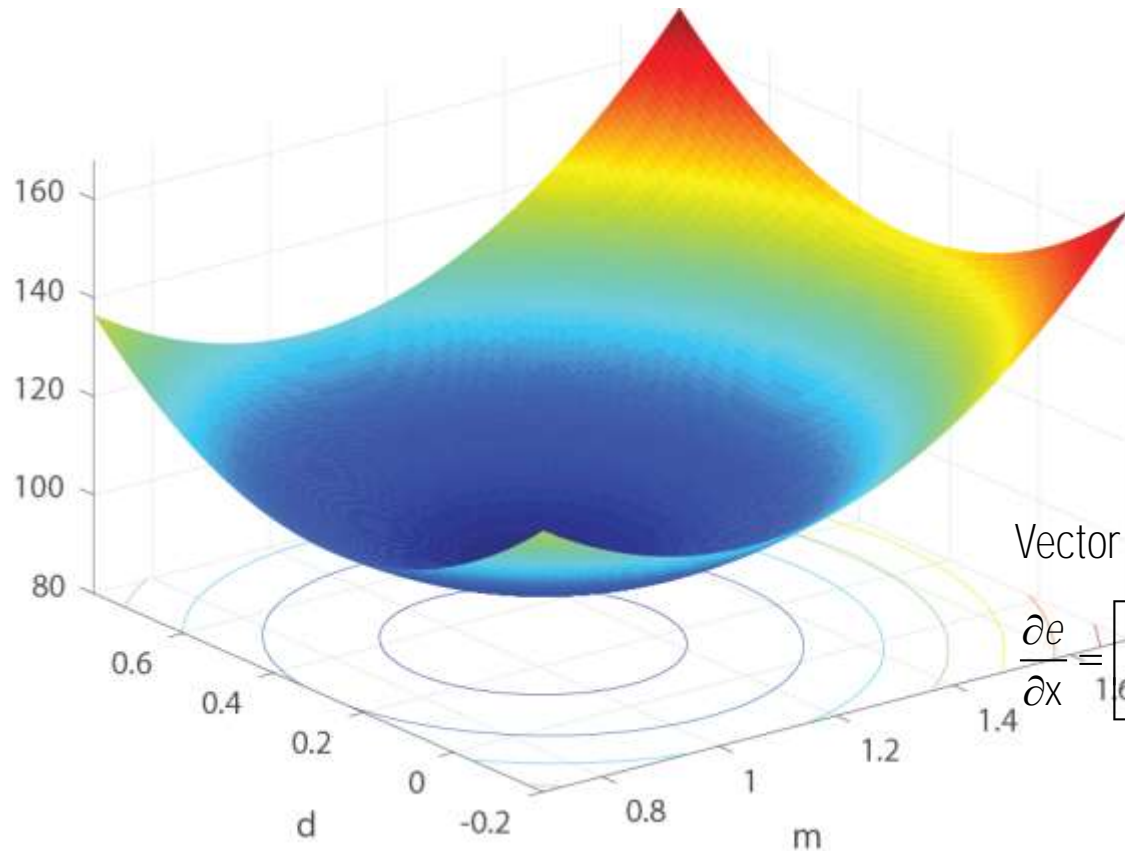
$$\frac{\partial E}{\partial x} = ?$$

Vector derivative:

$$\frac{\partial e}{\partial x} = \left[\frac{\partial e}{\partial x_1} \quad \dots \quad \frac{\partial e}{\partial x_n} \right]$$

$$\text{Ex) } e = c^T x = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\frac{\partial c^T x}{\partial x} =$$



Recall: Line Fitting

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$$

$$= x^T A^T A x - 2x^T A^T b + b^T b$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

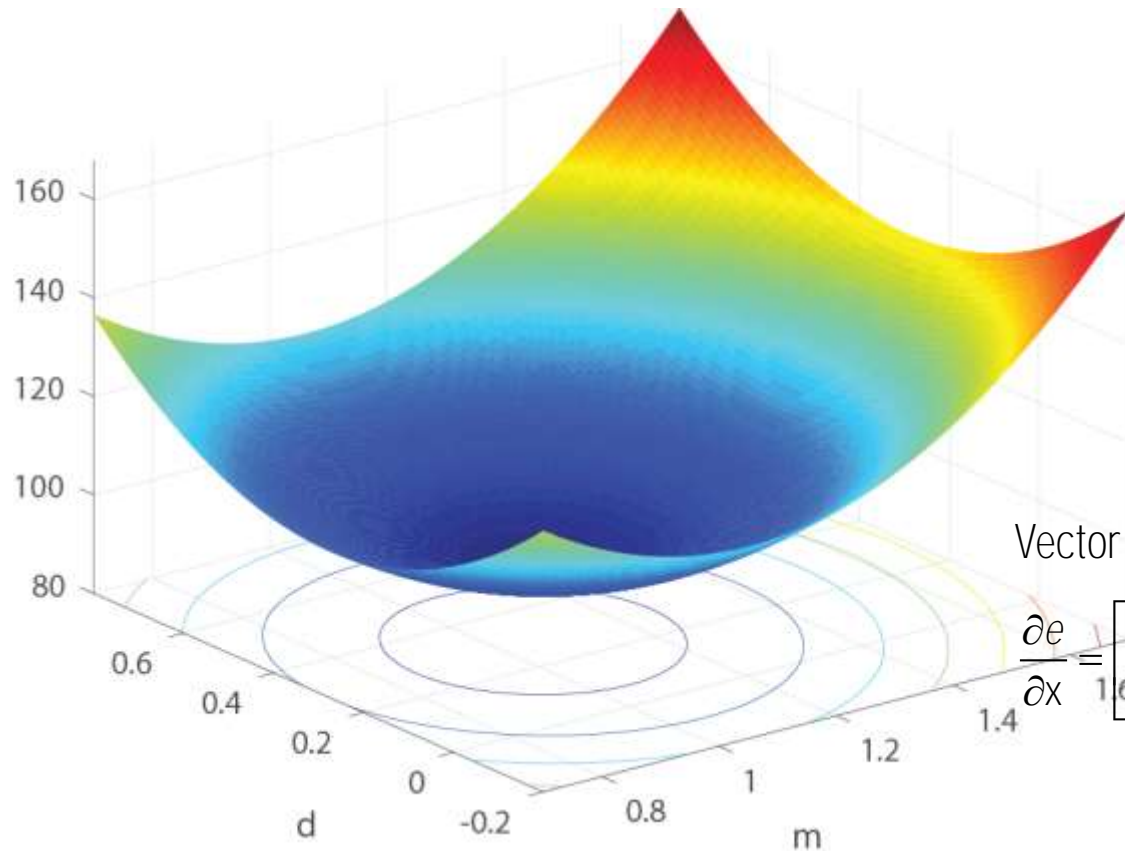
$$\frac{\partial E}{\partial x} = ?$$

Vector derivative:

$$\frac{\partial e}{\partial x} = \left[\frac{\partial e}{\partial x_1} \quad \dots \quad \frac{\partial e}{\partial x_n} \right]$$

$$\text{Ex) } e = c^T x = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\frac{\partial c^T x}{\partial x} = \frac{\partial}{\partial x} (c_1 x_1 + \dots + c_n x_n)$$



Recall: Line Fitting

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$$

$$= x^T A^T A x - 2x^T A^T b + b^T b$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\frac{\partial E}{\partial x} = ?$$

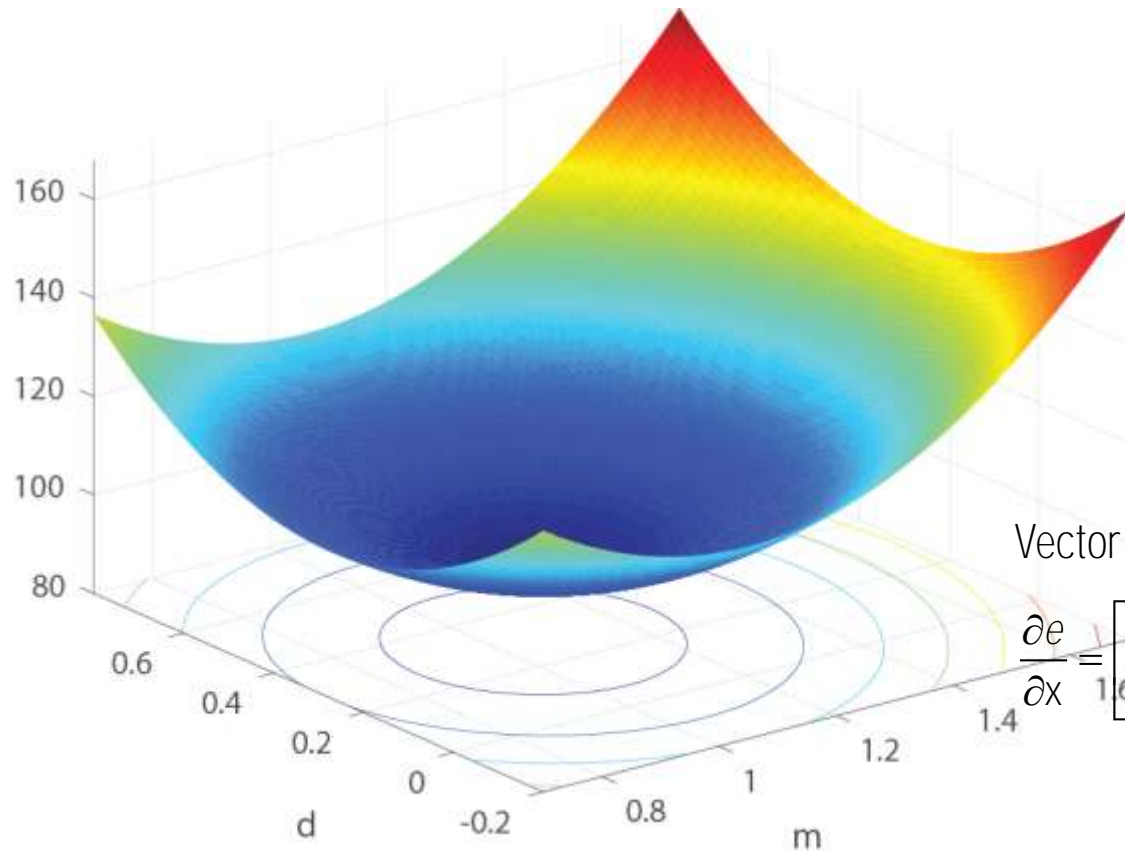
Vector derivative:

$$\frac{\partial e}{\partial x} = \left[\frac{\partial e}{\partial x_1} \quad \dots \quad \frac{\partial e}{\partial x_n} \right]$$

$$\text{Ex) } e = c^T x = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\frac{\partial c^T x}{\partial x} = \frac{\partial}{\partial x} (c_1 x_1 + \dots + c_n x_n)$$

$$= \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}$$



Recall: Line Fitting

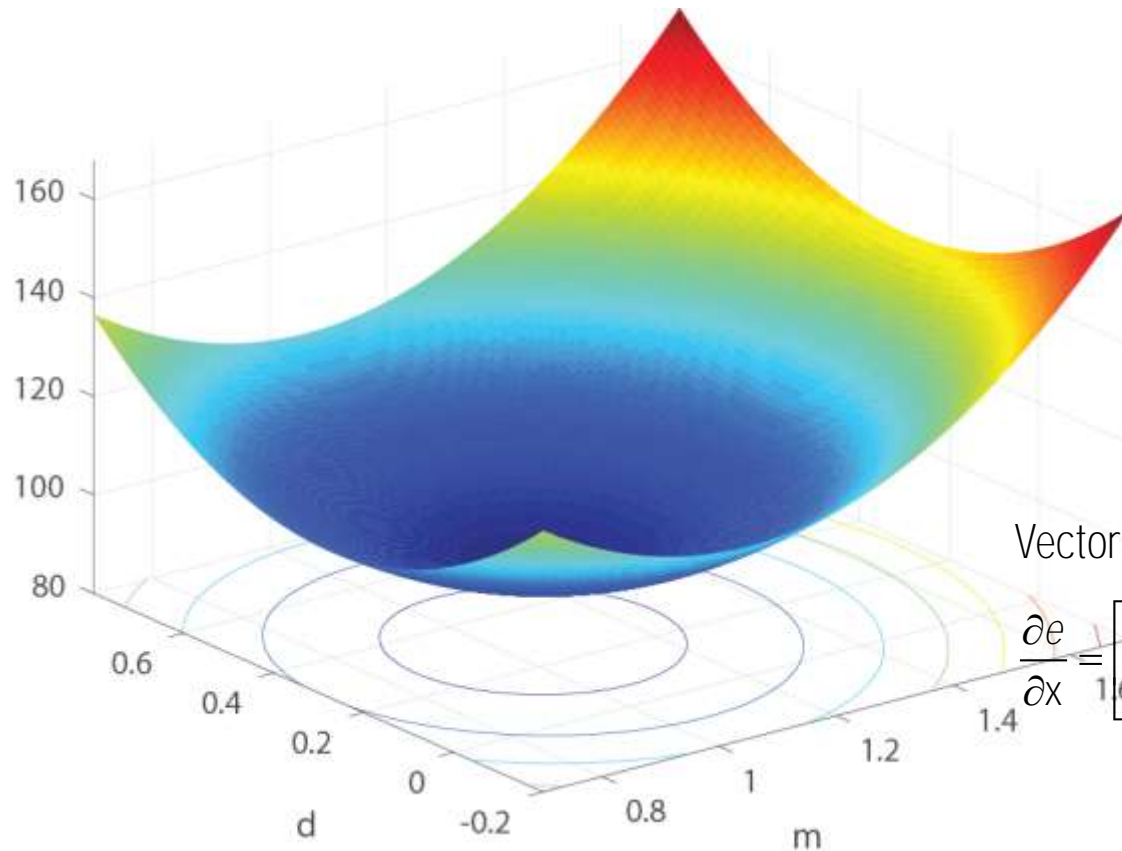
$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$$

$$= x^T A^T Ax - 2x^T A^T b + b^T b$$

$$\frac{\partial E}{\partial x} = 2A^T Ax - 2A^T b = 0$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$



Vector derivative:

$$\frac{\partial e}{\partial x} = \left[\frac{\partial e}{\partial x_1} \quad \dots \quad \frac{\partial e}{\partial x_n} \right]$$

$$\text{Ex) } e = c^T x = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\frac{\partial c^T x}{\partial x} = \frac{\partial}{\partial x} (c_1 x_1 + \dots + c_n x_n)$$

$$= \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}$$

Recall: Line Fitting

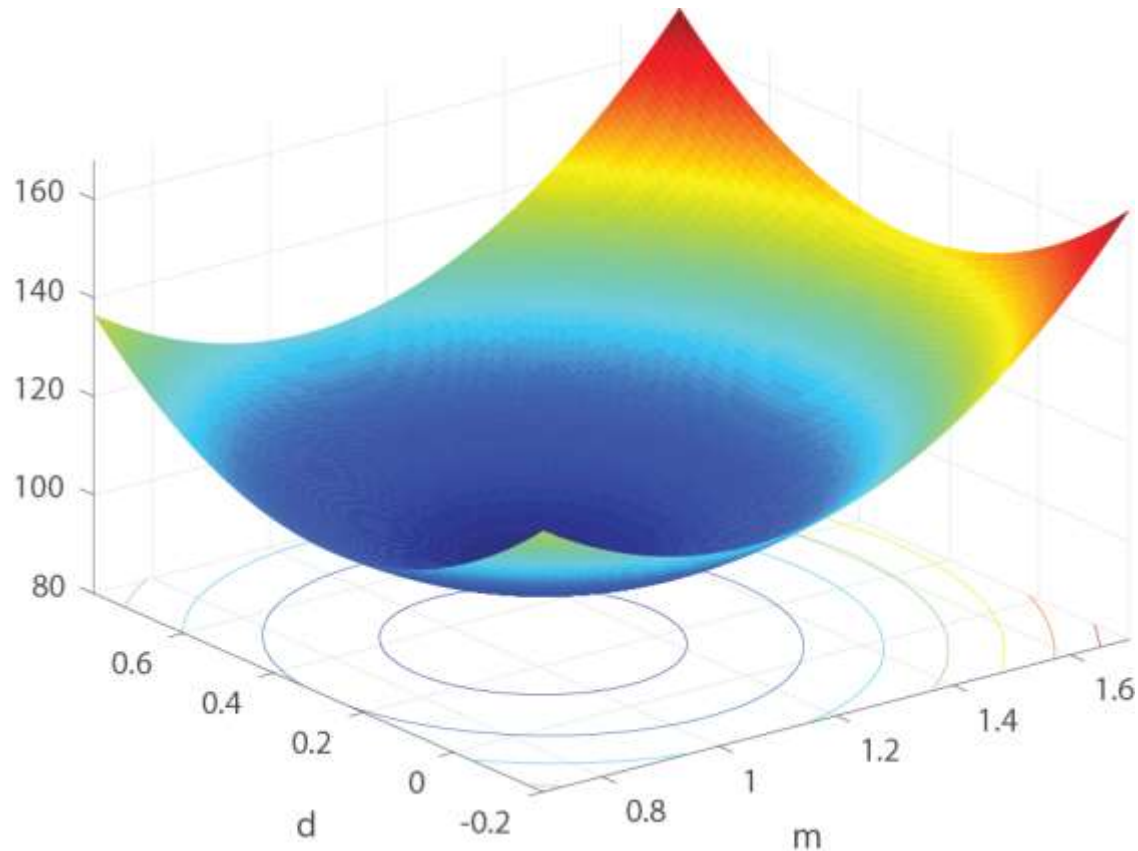
$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2 \\ = x^T A^T Ax - 2x^T A^T b + b^T b$$

$$\frac{\partial E}{\partial x} = 2A^T Ax - 2A^T b = 0$$

$$\longrightarrow A^T Ax = A^T b$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$



Recall: Line Fitting

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2 \\ = x^T A^T Ax - 2x^T A^T b + b^T b$$

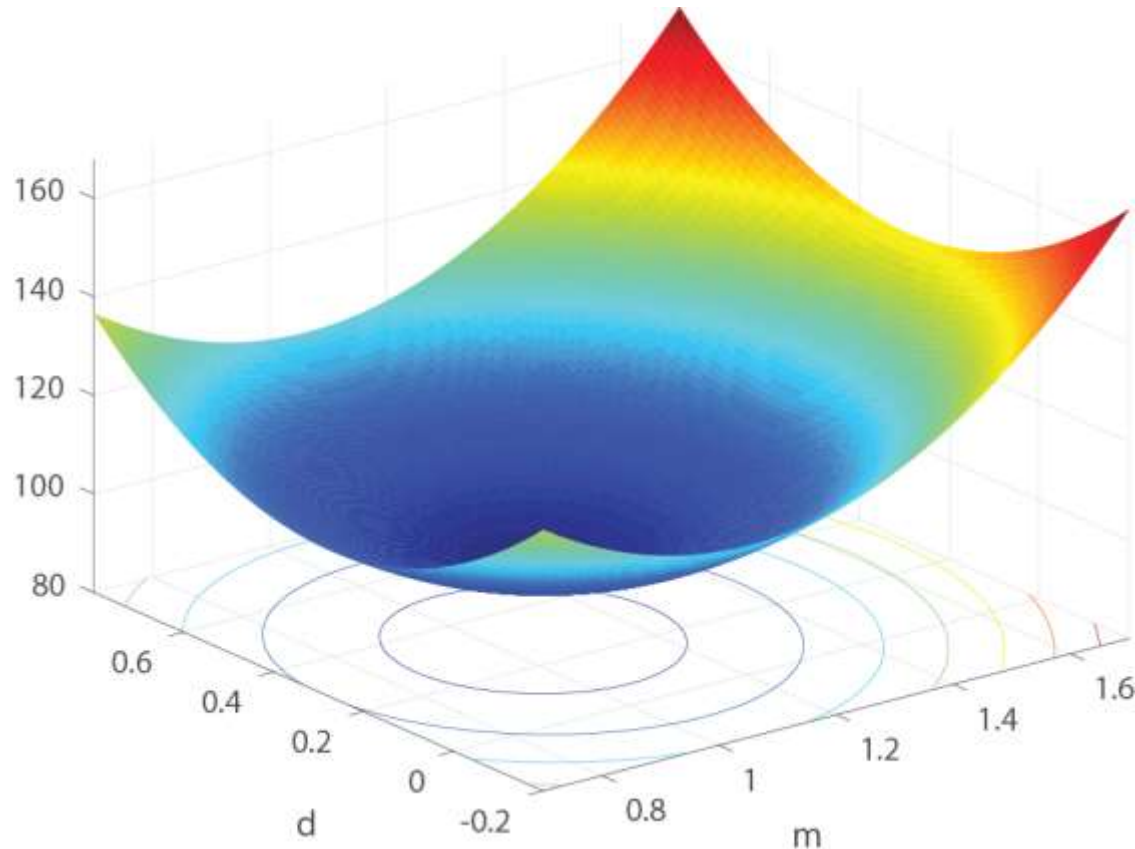
$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\frac{\partial E}{\partial x} = 2A^T Ax - 2A^T b = 0$$

$$\longrightarrow A^T Ax = A^T b$$

$$\begin{bmatrix} A^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} A^T \end{bmatrix} \begin{bmatrix} b \end{bmatrix}$$

Normal equation



Recall: Line Fitting

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2 \\ = x^T A^T A x - 2x^T A^T b + b^T b$$

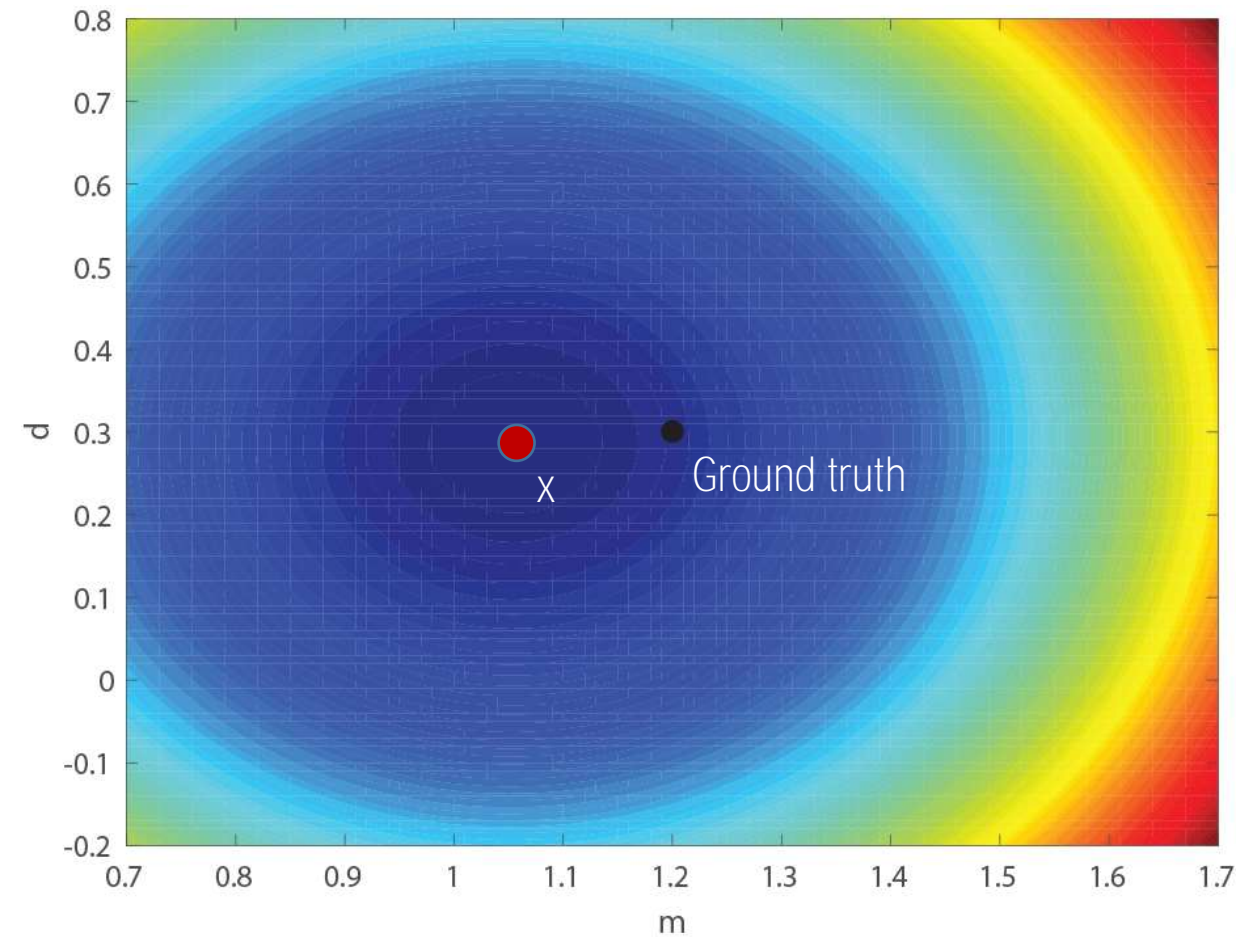
$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\frac{\partial E}{\partial x} = 2A^T Ax - 2A^T b = 0$$

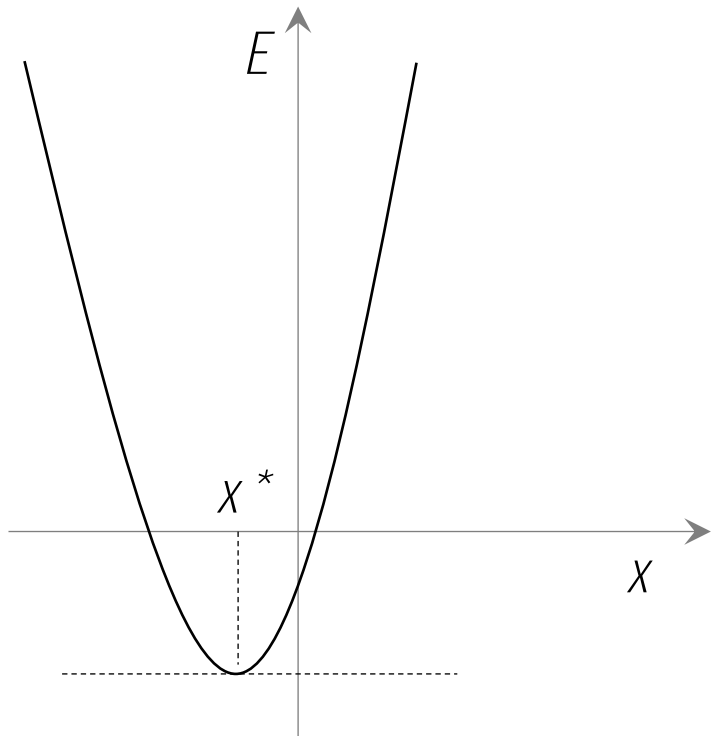
$$\longrightarrow A^T Ax = A^T b$$

$$A^T A x = A^T b$$

$$x = \left[\begin{array}{cc} A^T & A \end{array} \right]^{-1} A^T b$$



Linear System Recap



$$\left. \frac{dE}{dx} \right|_{x=x^*} = 0$$

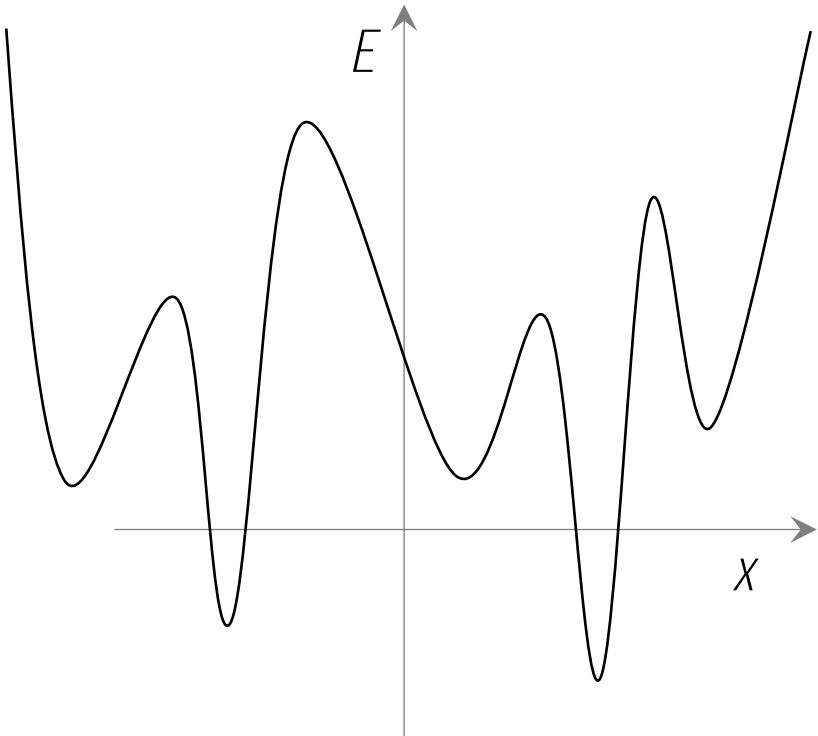
$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} m \\ d \end{bmatrix} \mathbf{x} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \mathbf{b}$$

$$\mathbf{x} = \begin{bmatrix} \mathbf{A}^\top & \mathbf{A} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}^\top & \mathbf{b} \end{bmatrix}$$

- Has the global solution
- Has the closed form solution (non-iterative solve)
- Is solved efficiently ($O(n^2)$)
- Does not require an initialization

Nonlinear System

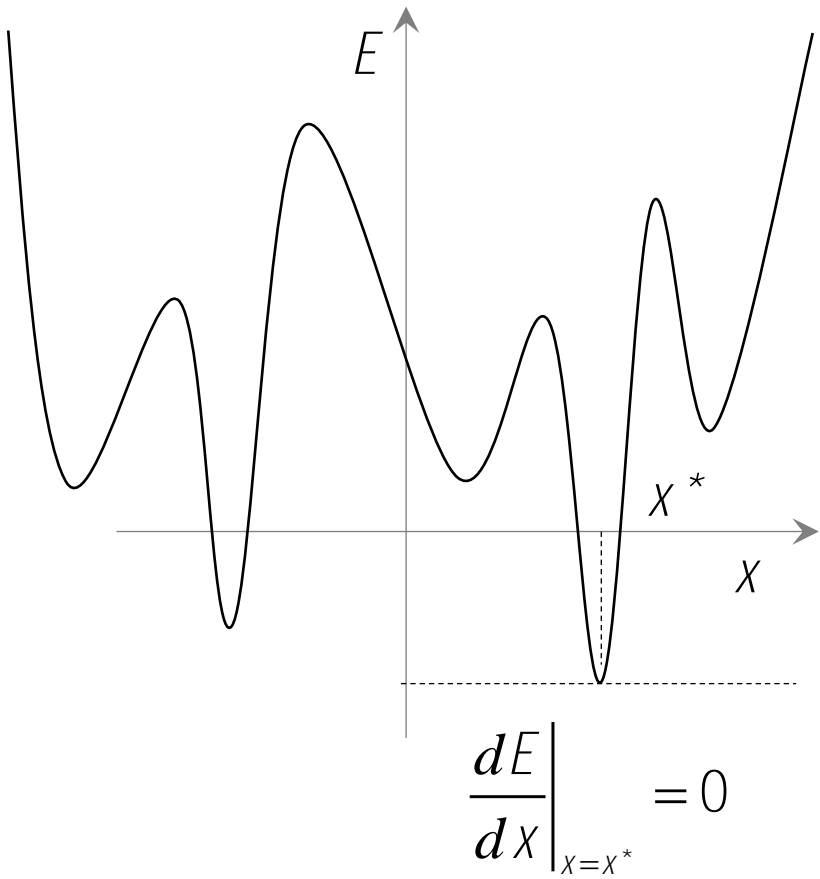
$$f(x) = b$$



Nonlinear System

$$f(x) = b$$

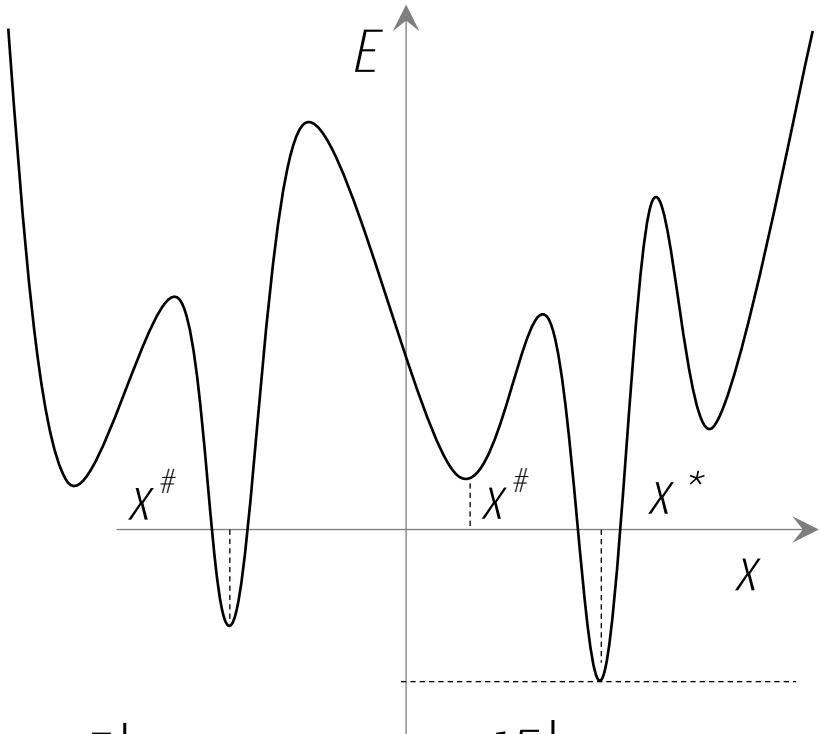
$$E = \|f(x) - b\|^2$$



Nonlinear System

$$f(x) = b$$

$$E = \|f(x) - b\|^2$$



$$\left. \frac{dE}{dx} \right|_{x=x^{\#}} = 0$$

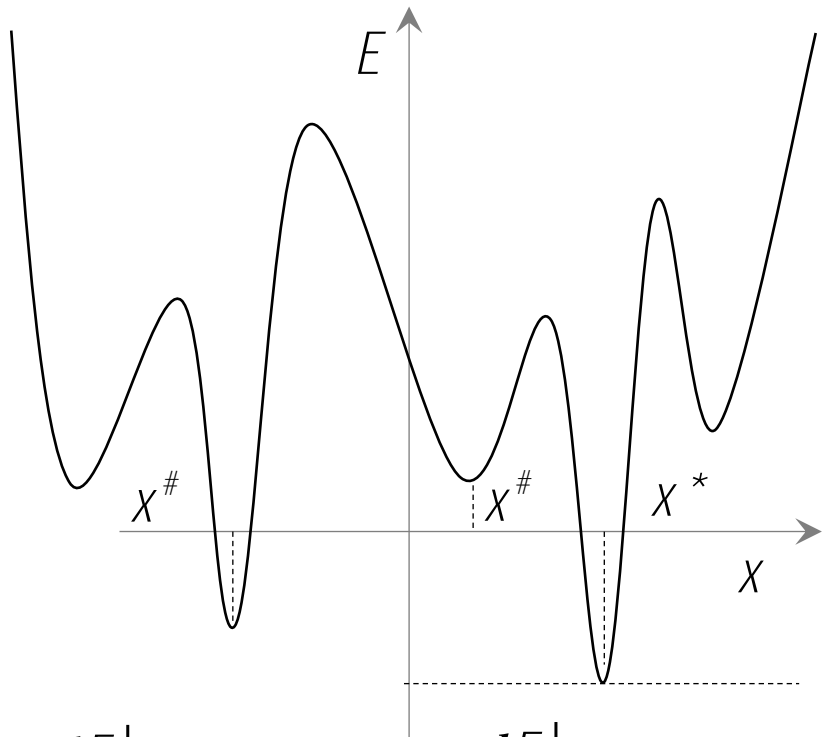
$$\left. \frac{dE}{dx} \right|_{x=x^*} = 0$$

Nonlinear System

$$f(x) = b$$

$$E = \|f(x) - b\|^2$$

$$\begin{aligned} E &= (f(x) - b)^T (f(x) - b) \\ &= f(x)^T f(x) - 2f(x)^T b + b^T b \end{aligned}$$



$$\left. \frac{dE}{dx} \right|_{x=x^\#} = 0$$

$$\left. \frac{dE}{dx} \right|_{x=x^*} = 0$$

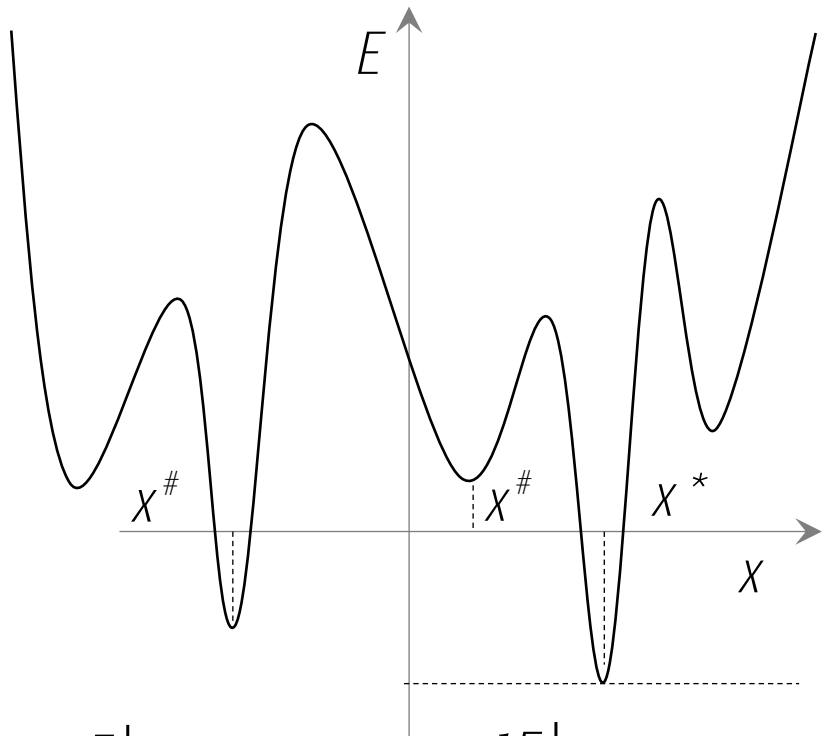
Nonlinear System

$$f(x) = b$$

$$E = \|f(x) - b\|^2$$

$$\begin{aligned} E &= (f(x) - b)^T (f(x) - b) \\ &= f(x)^T f(x) - 2f(x)^T b + b^T b \end{aligned}$$

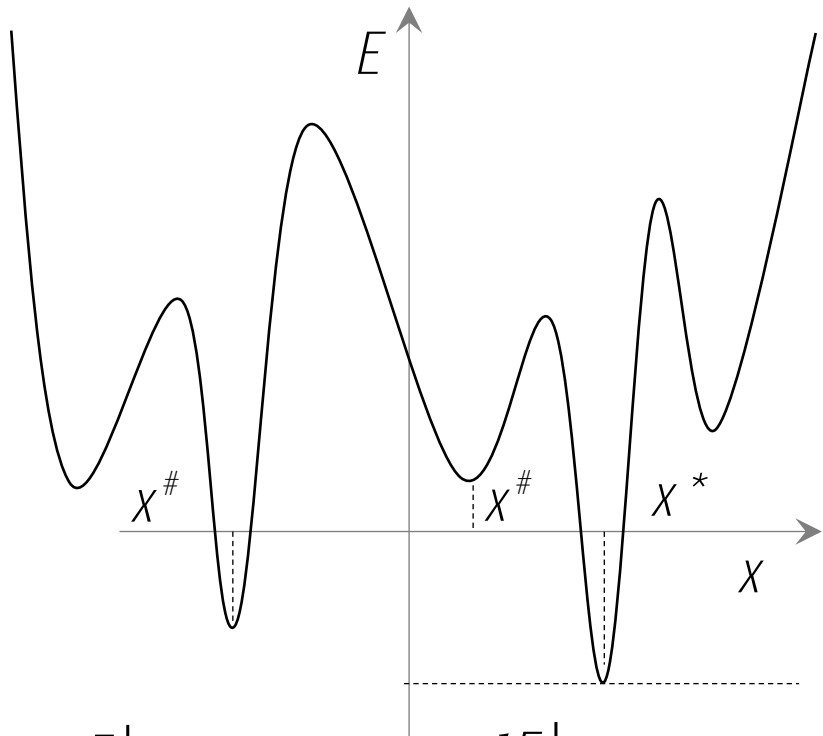
$$\frac{\partial E}{\partial x} = 2 \frac{\partial f(x)^T}{\partial x} f(x) - 2 \frac{\partial f(x)^T}{\partial x} b = 0$$



$$\left. \frac{dE}{dx} \right|_{x=x^\#} = 0$$

$$\left. \frac{dE}{dx} \right|_{x=x^*} = 0$$

Nonlinear System



$$\left. \frac{dE}{dx} \right|_{x=x^\#} = 0$$

$$\left. \frac{dE}{dx} \right|_{x=x^*} = 0$$

$$f(x) = b$$

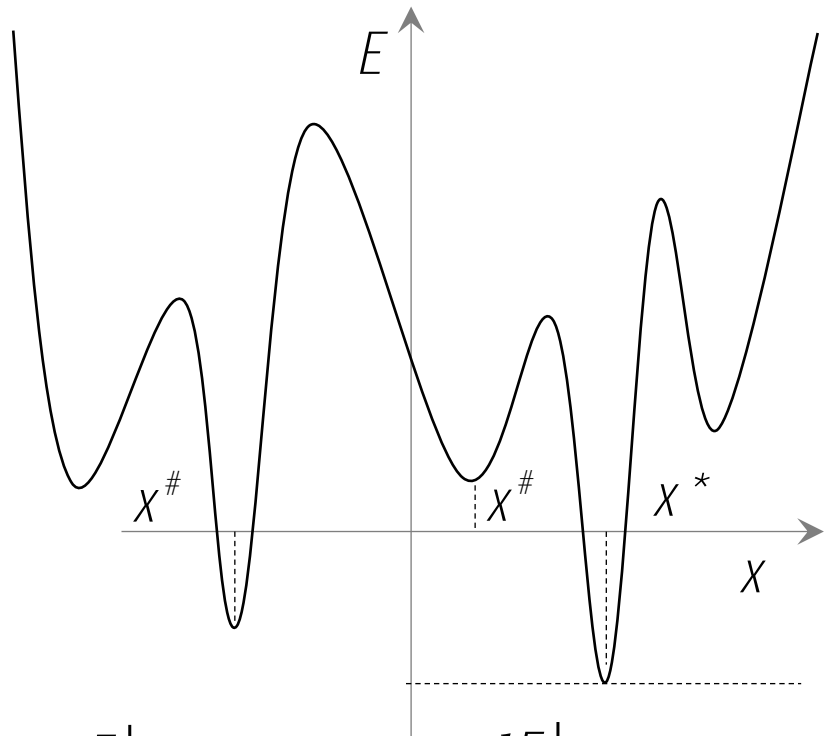
$$E = \|f(x) - b\|^2$$

$$\begin{aligned} E &= (f(x) - b)^\top (f(x) - b) \\ &= f(x)^\top f(x) - 2f(x)^\top b + b^\top b \end{aligned}$$

$$\frac{\partial E}{\partial x} = 2 \frac{\partial f(x)^\top}{\partial x} f(x) - 2 \frac{\partial f(x)^\top}{\partial x} b = 0$$

where $\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$: Jacobian

Nonlinear System



$$\left. \frac{dE}{dx} \right|_{x=x^\#} = 0$$

$$\left. \frac{dE}{dx} \right|_{x=x^*} = 0$$

$$f(x) = b$$

$$E = \|f(x) - b\|^2$$

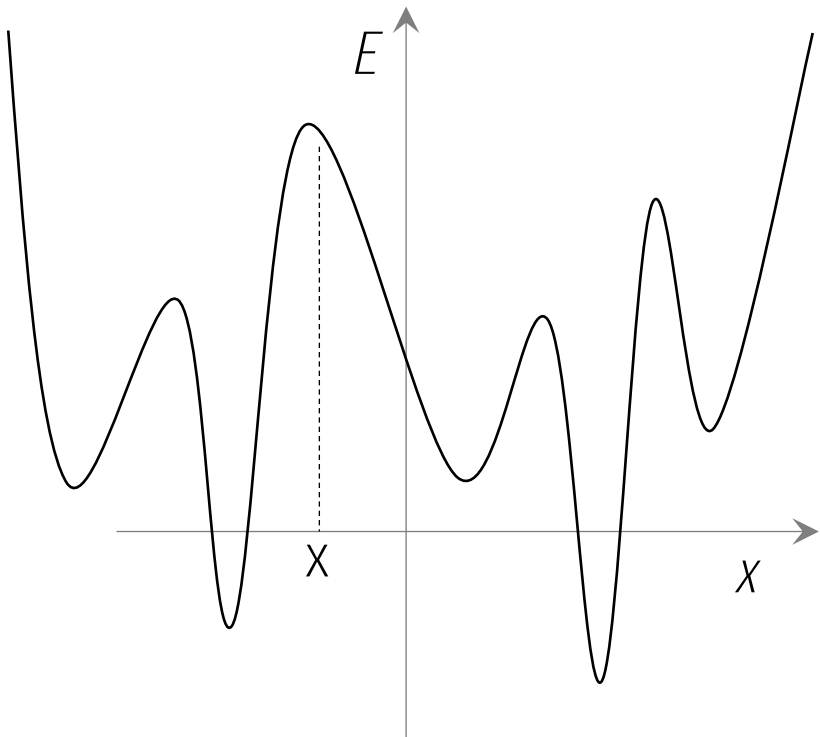
$$\begin{aligned} E &= (f(x) - b)^\top (f(x) - b) \\ &= f(x)^\top f(x) - 2f(x)^\top b + b^\top b \end{aligned}$$

$$\frac{\partial E}{\partial x} = 2 \frac{\partial f(x)^\top}{\partial x} f(x) - 2 \frac{\partial f(x)^\top}{\partial x} b = 0$$

Find x such that the following equation is satisfied:

$$\frac{\partial f(x)^\top}{\partial x} f(x) = \frac{\partial f(x)^\top}{\partial x} b \quad \text{How?}$$

Nonlinear System

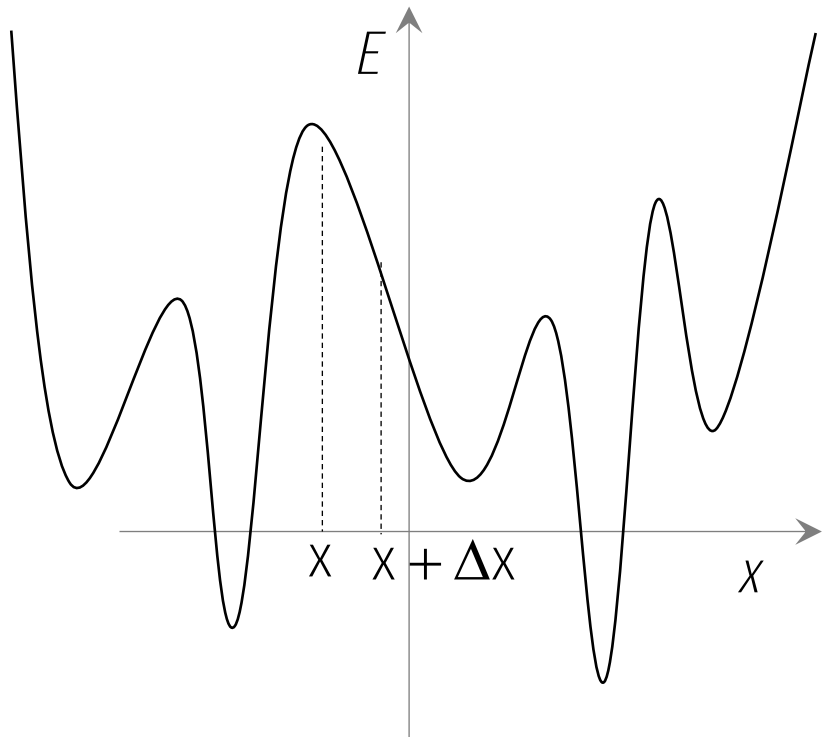


Find x such that the following equation is satisfied:

$$\frac{\partial f(x)^T}{\partial x} f(x) = \frac{\partial f(x)^T}{\partial x} b \quad \text{How?}$$

Strategy: Given x ,

Nonlinear System

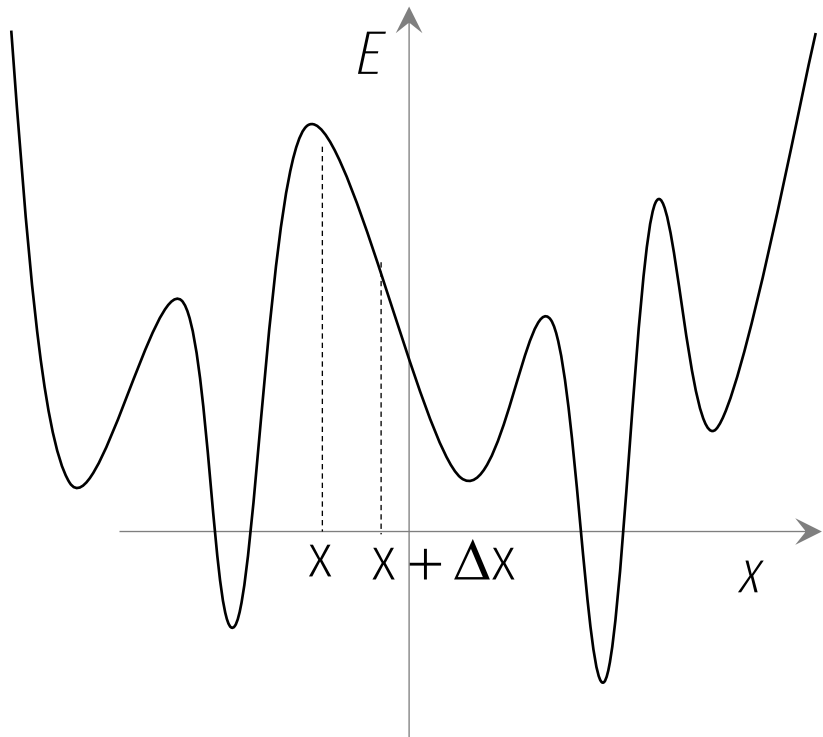


Find x such that the following equation is satisfied:

$$\frac{\partial f(x)^T}{\partial x} f(x) = \frac{\partial f(x)^T}{\partial x} b \quad \text{How?}$$

Strategy: Given x , move Δx such that $E(x + \Delta x) \leq E(x)$

Nonlinear System



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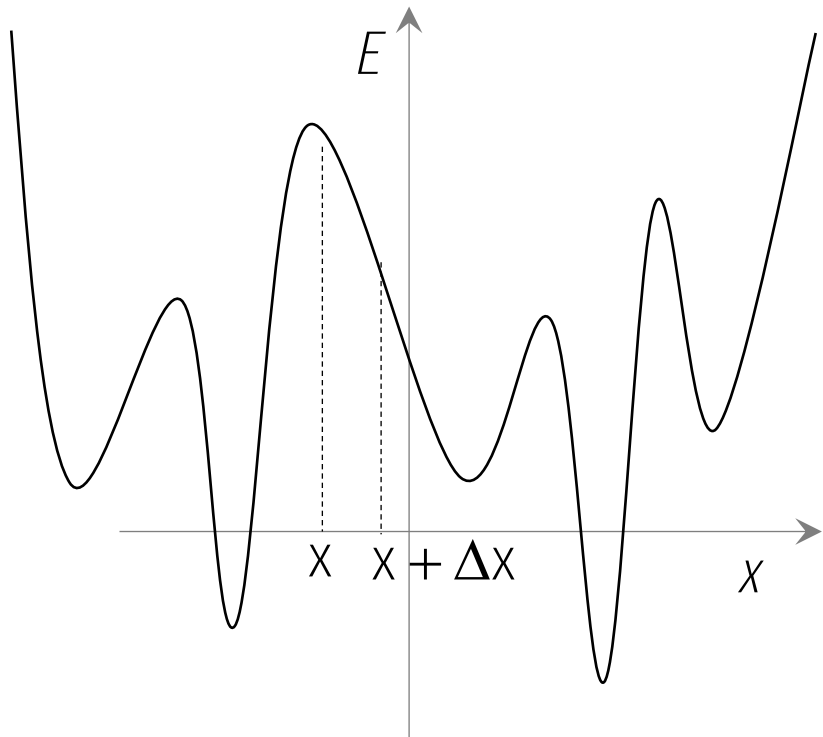
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Taylor expansion:

$$f(x + \Delta x) =$$

Nonlinear System



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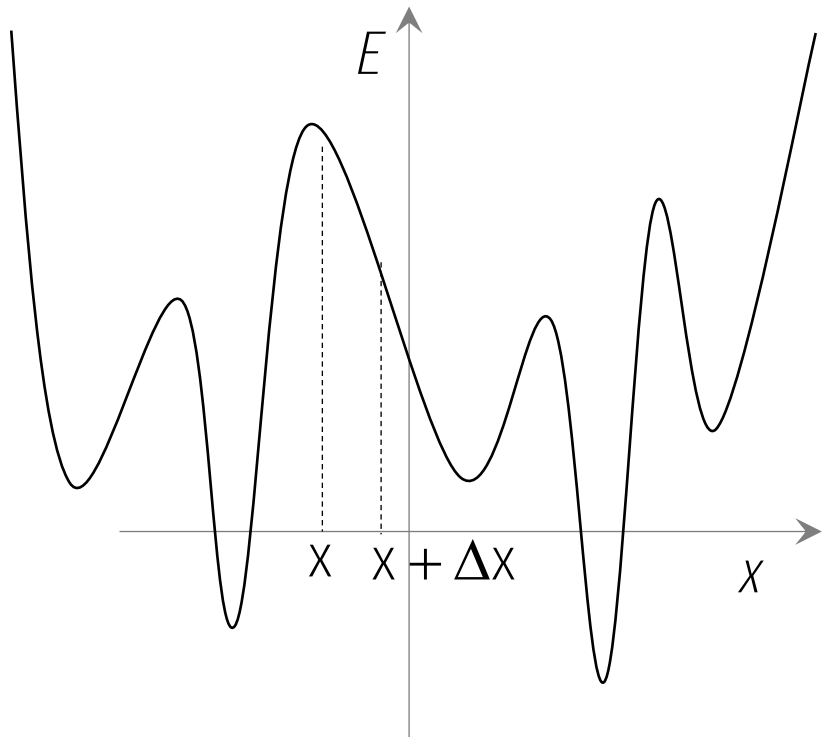
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Strategy: Given x , move Δx such that $E(x + \Delta x) \leq E(x)$

Taylor expansion:

$$f(x + \Delta x) = f(x + \Delta x) + \frac{\partial f(x)}{\partial x} \Delta x + \text{H.O.T.}$$

Nonlinear System



Find x such that the following equation is satisfied:

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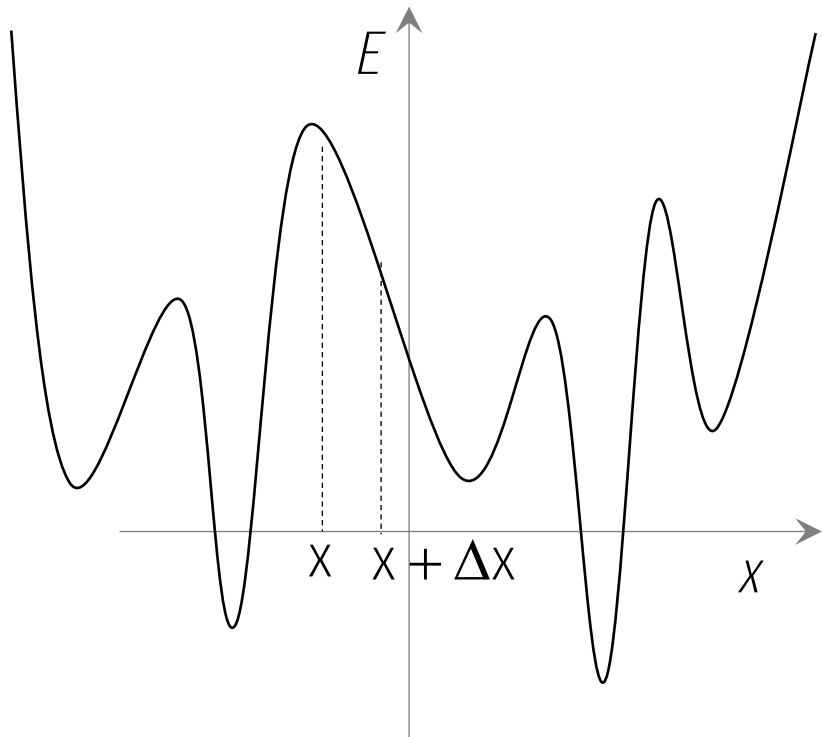
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$$\rightarrow \frac{\partial f(x)^T}{\partial x} \left(f(x) + \frac{\partial f(x)}{\partial x} \Delta x \right) = \frac{\partial f(x)^T}{\partial x} b$$

Nonlinear System



Find x such that the following equation is satisfied:

$$\frac{\partial f(x)^\top}{\partial x} f(x) = \frac{\partial f(x)^\top}{\partial x} b \quad \text{How?}$$

Strategy: Given x , move Δx such that $E(x + \Delta x) \leq E(x)$

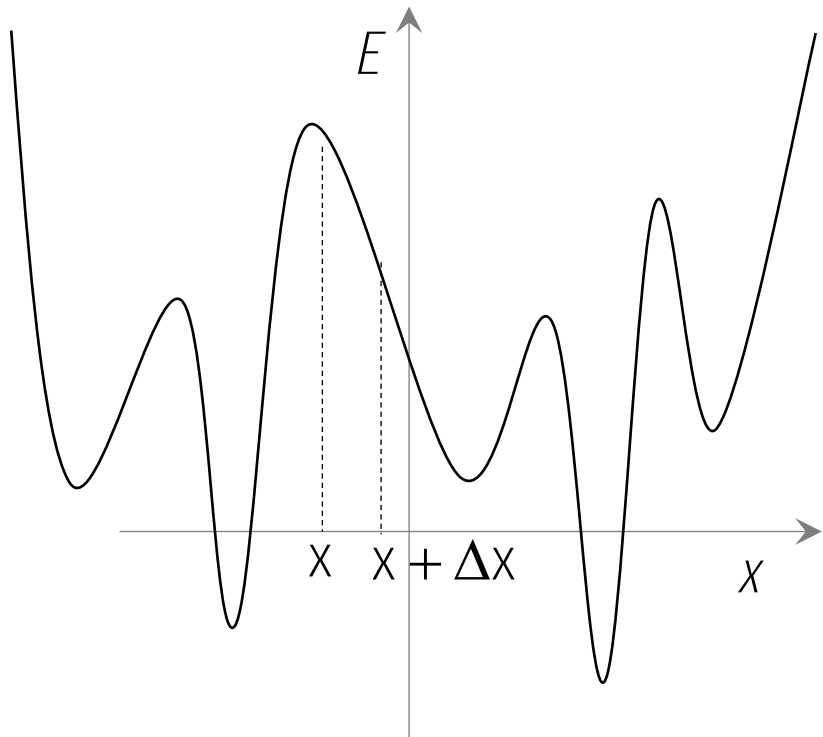
Taylor expansion:

$$f(x + \Delta x) = f(x) + \frac{\partial f(x)}{\partial x} \Delta x + \text{H.O.T.}$$

$$\rightarrow \frac{\partial f(x)^\top}{\partial x} \left(f(x) + \frac{\partial f(x)}{\partial x} \Delta x \right) = \frac{\partial f(x)^\top}{\partial x} b$$

$$\rightarrow \frac{\partial f(x)^\top}{\partial x} \frac{\partial f(x)}{\partial x} \Delta x = \frac{\partial f(x)^\top}{\partial x} (b - f(x))$$

Nonlinear System



Cf.) $x = \begin{bmatrix} A^T & A \end{bmatrix}^{-1} \begin{bmatrix} A^T & b \end{bmatrix}$

Find x such that the following equation is satisfied:

$$\frac{\partial f(x)^T}{\partial x} f(x) = \frac{\partial f(x)^T}{\partial x} b \quad \text{How?}$$

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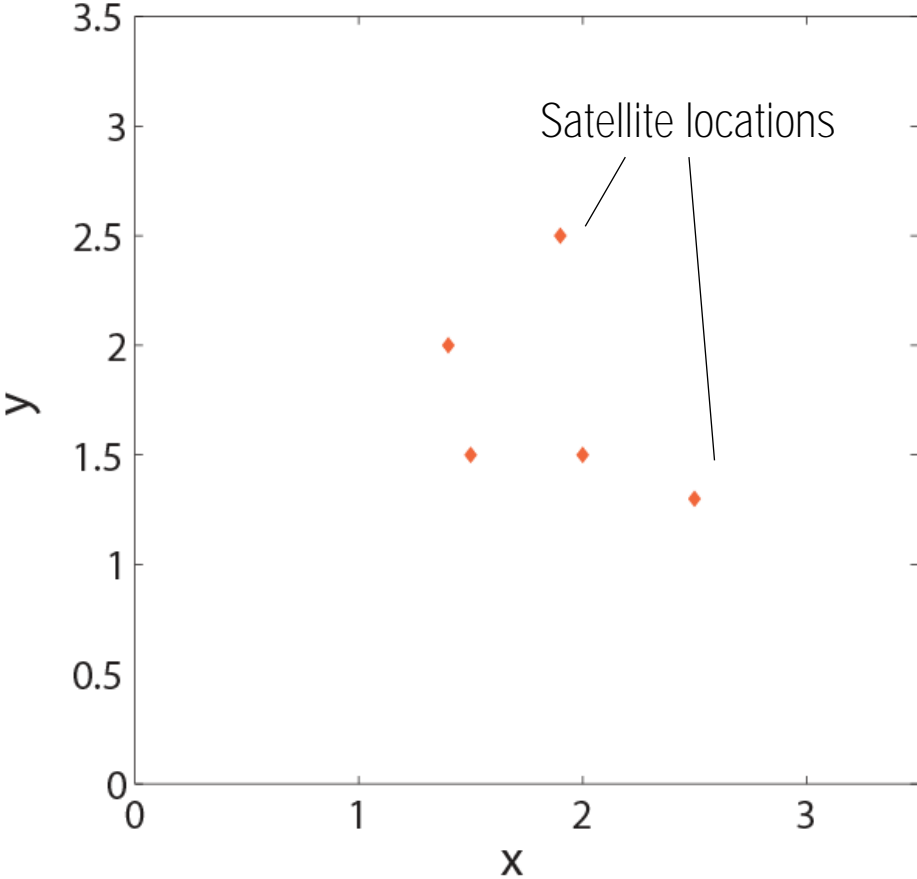
$$\rightarrow \frac{\partial f(x)^T}{\partial x} \left(f(x) + \frac{\partial f(x)}{\partial x} \Delta x \right) = \frac{\partial f(x)^T}{\partial x} b$$

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Where am I?: GPS Localization

Example:

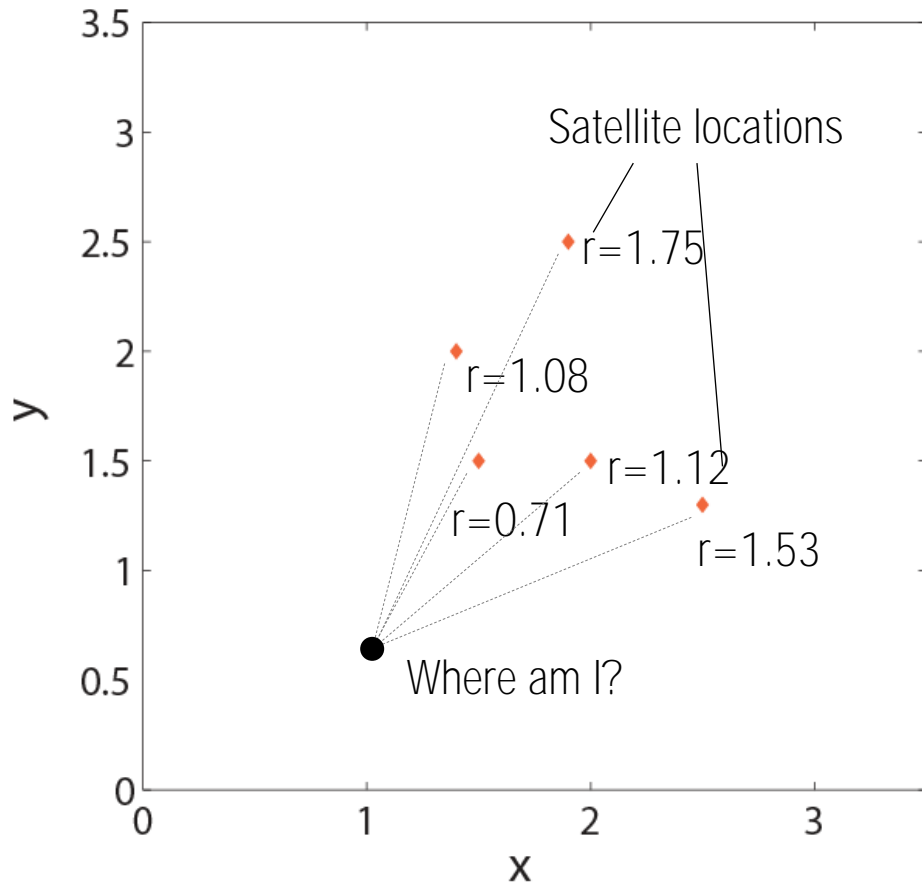
Localization using range data from beacons



Where am I?: GPS Localization

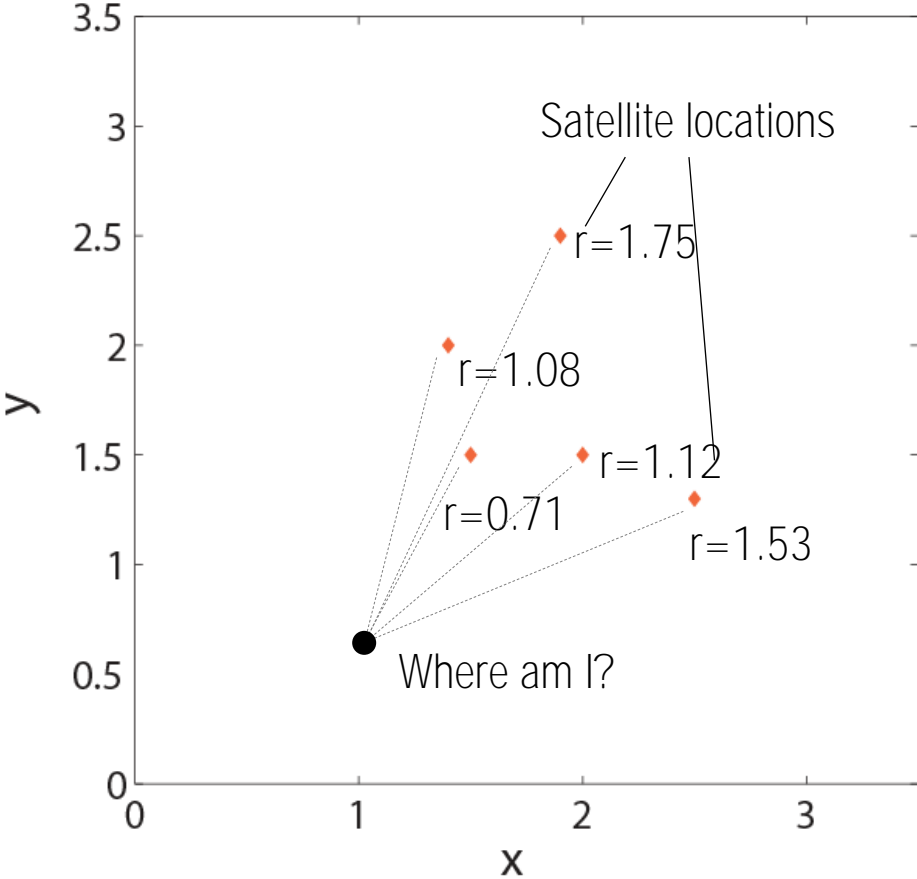
Example:

Localization using range data from beacons



Where am I?: GPS Localization

Example:
Localization using range data from beacons

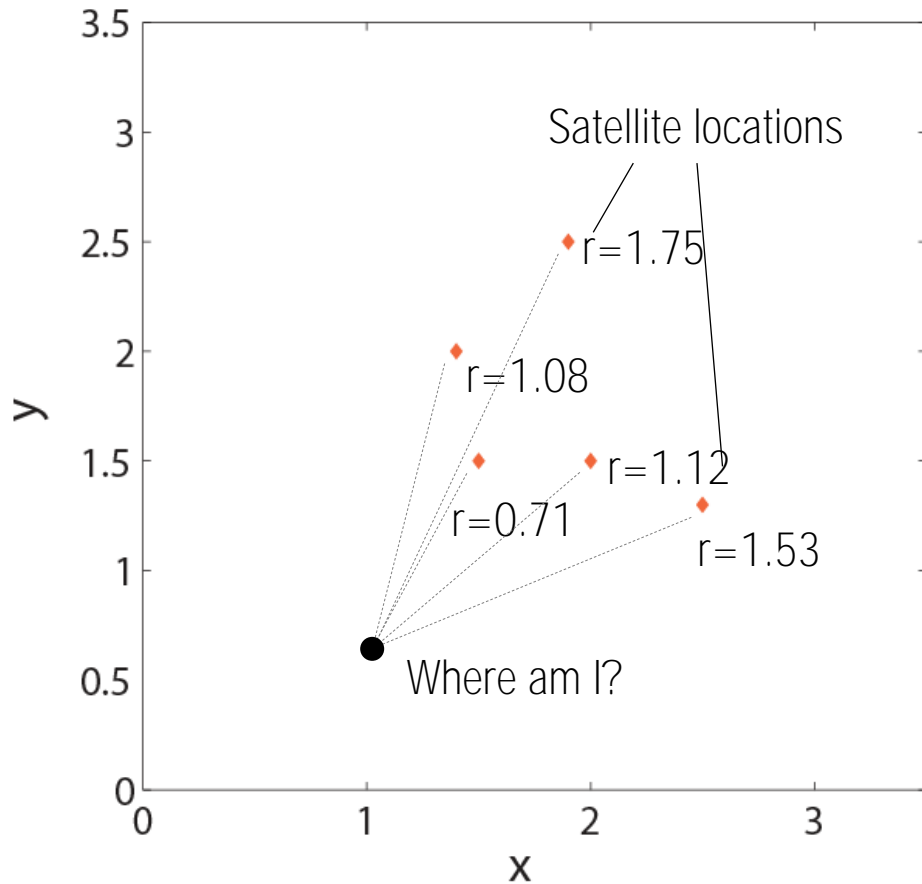


$E =$

Where am I?: GPS Localization

Example:

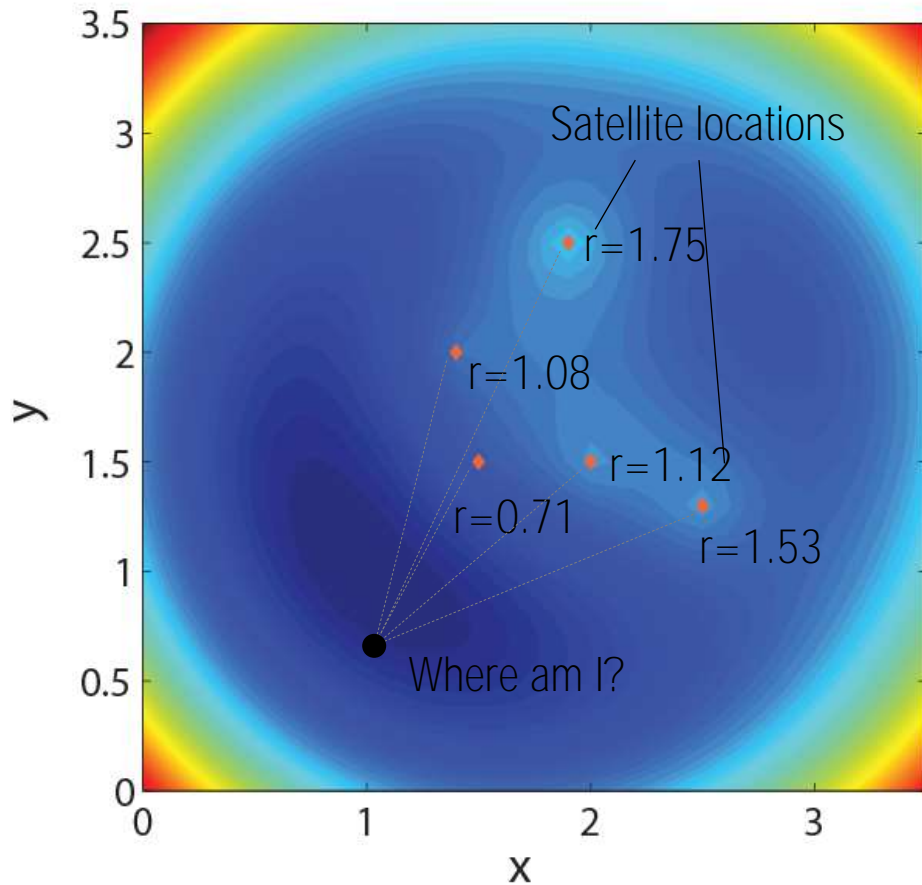
Localization using range data from beacons



$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

Where am I?: GPS Localization

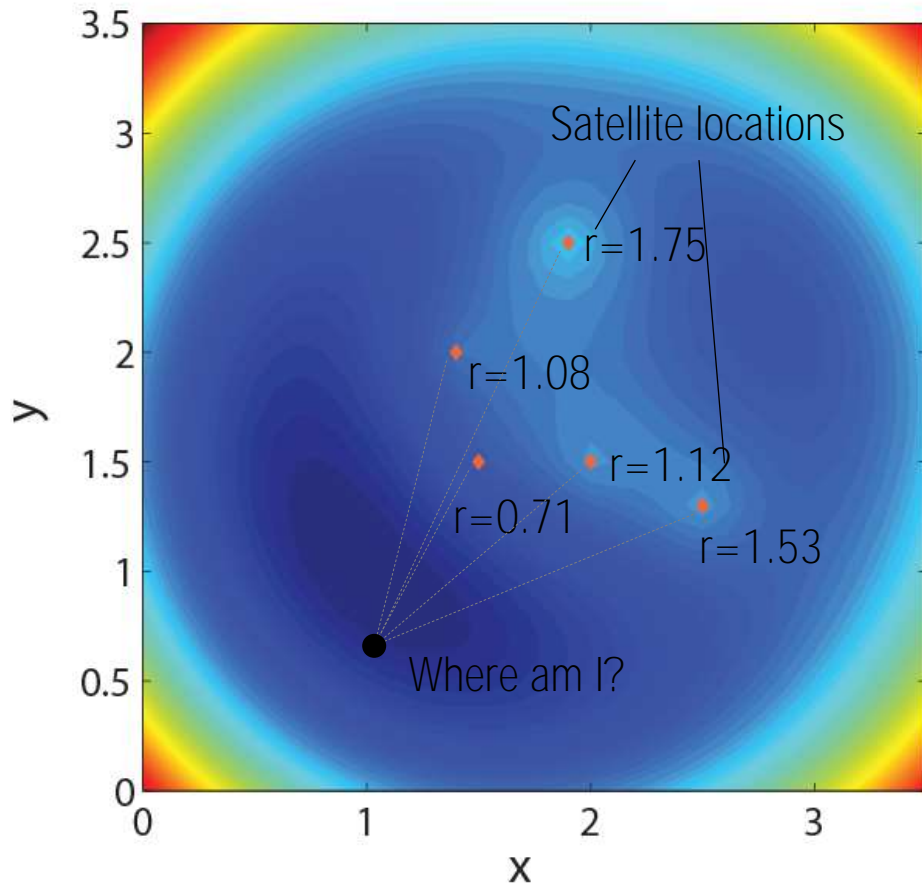
Example:
Localization using range data from beacons



$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ f(\mathbf{x}) \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

Where am I?: GPS Localization

Example:
Localization using range data from beacons



$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ f(\mathbf{x}) \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ \mathbf{b} \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

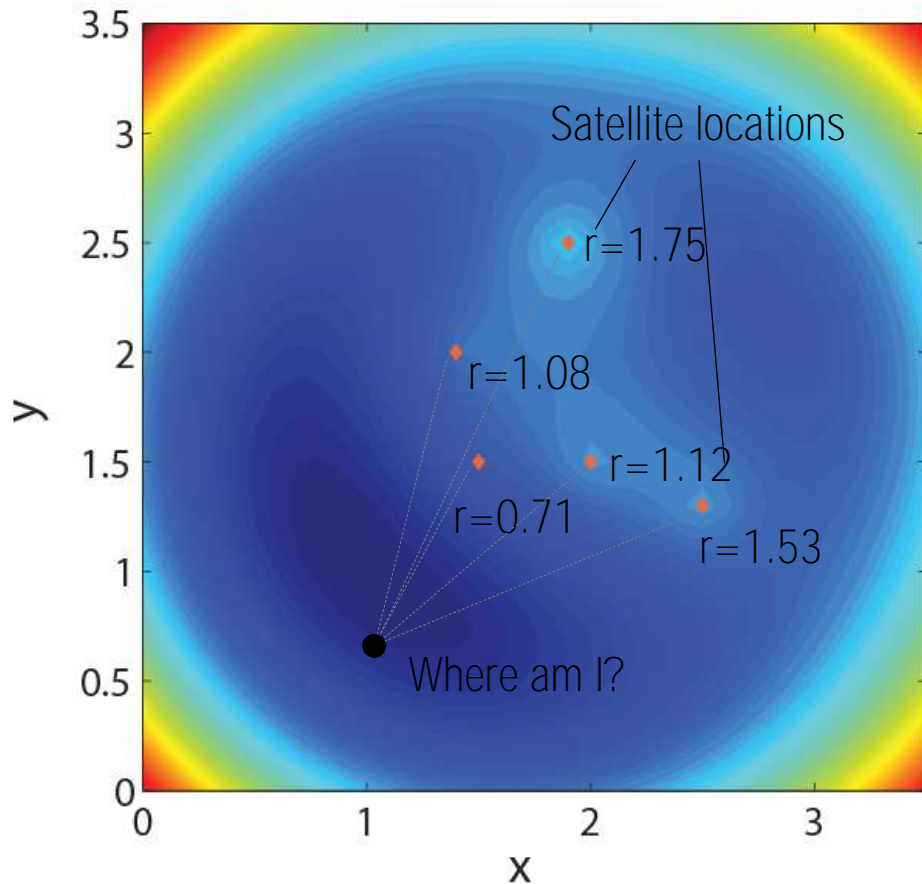
$$\frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} = \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} (\mathbf{b} - f(\mathbf{x}))$$

where $\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$: Jacobian

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Example:

Localization using range data from beacons



$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ f(\mathbf{x}) \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

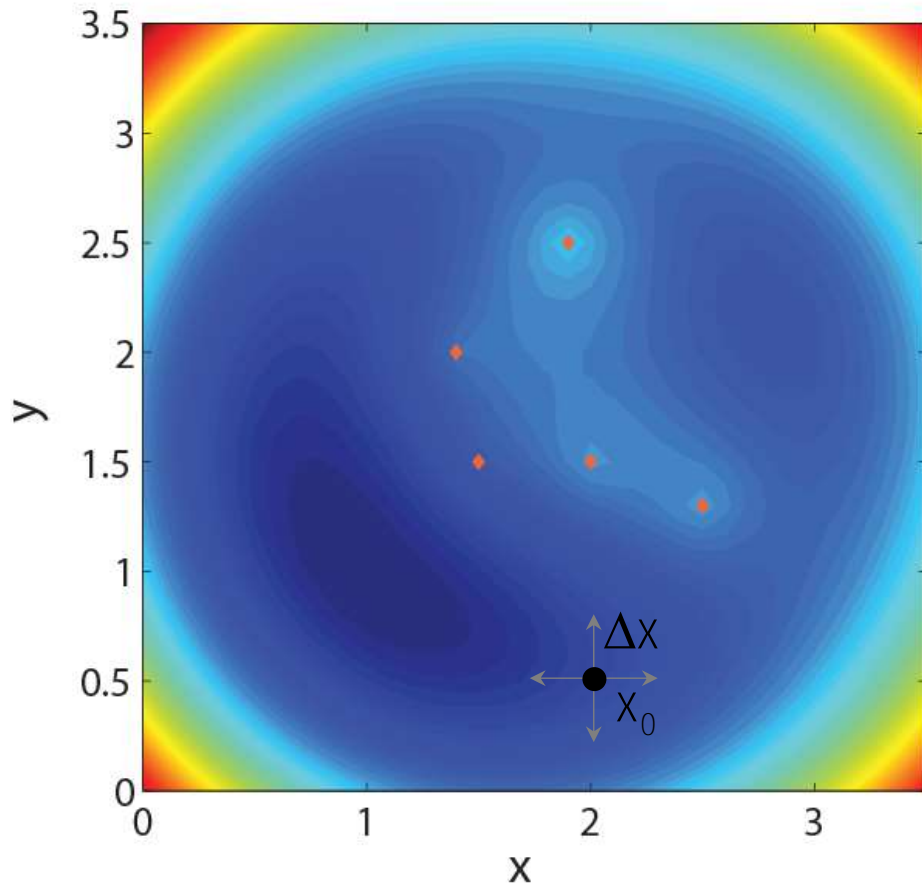
$$\frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} = \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} (\mathbf{b} - f(\mathbf{x}))$$

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} -\frac{u_1 - x}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} & -\frac{v_1 - y}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} \\ -\frac{u_5 - x}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} & -\frac{v_5 - y}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} \end{bmatrix}$$

Where am I?: GPS Localization

Example:

Localization using range data from beacons



$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ f(x) \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

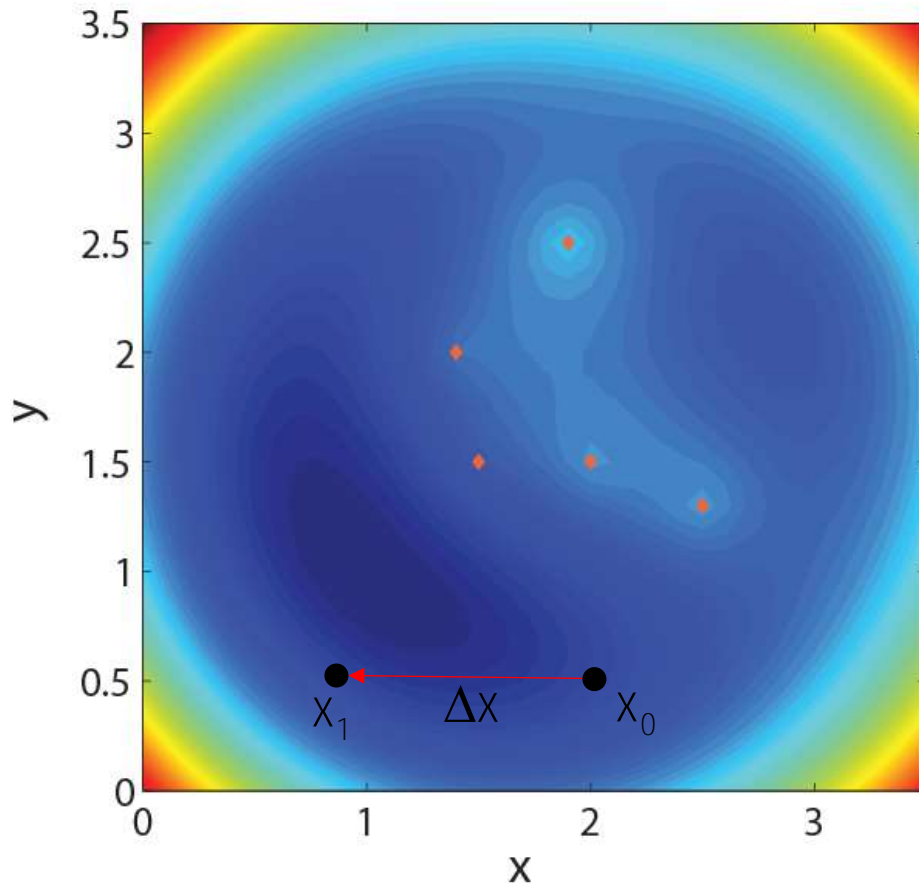
$$\frac{\partial f(x)^T}{\partial x} \frac{\partial f(x)}{\partial x} \Delta x = \frac{\partial f(x)^T}{\partial x} (b - f(x))$$

$$\Delta x = \left(\frac{\partial f(x)^T}{\partial x} \frac{\partial f(x)}{\partial x} \right)^{-1} \frac{\partial f(x)^T}{\partial x} (b - f(x))$$

Where am I?: GPS Localization

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Localization using range data from beacons



$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ f(x) \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

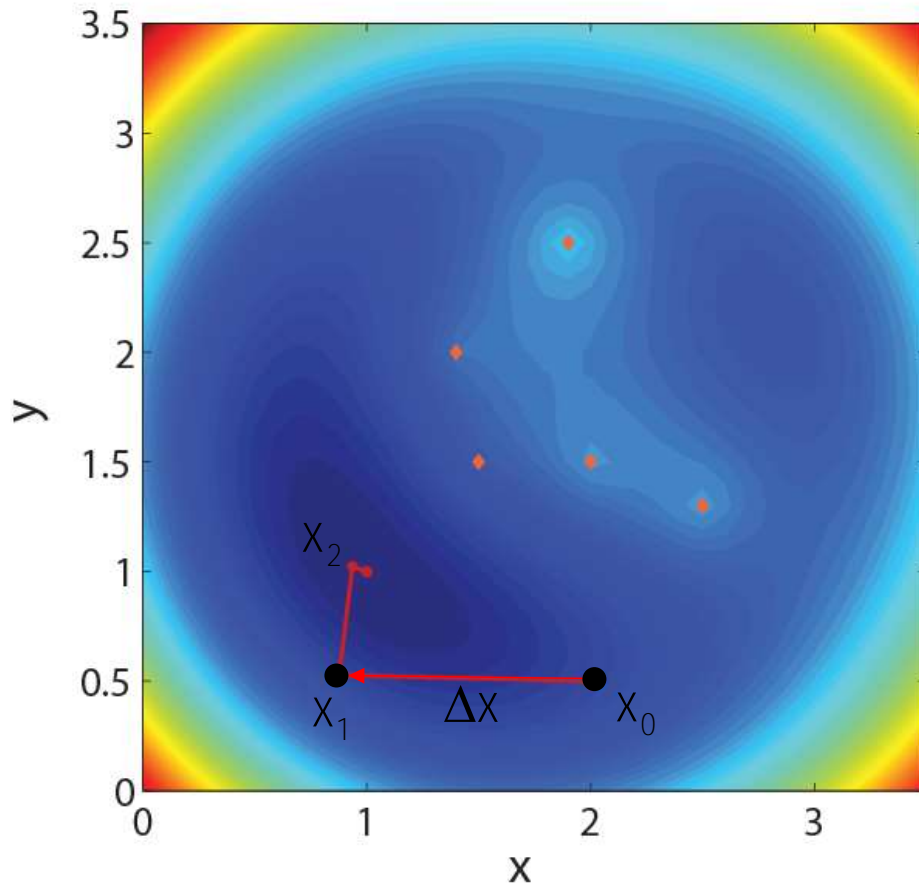
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Where am I?: GPS Localization

Example:

Localization using range data from beacons



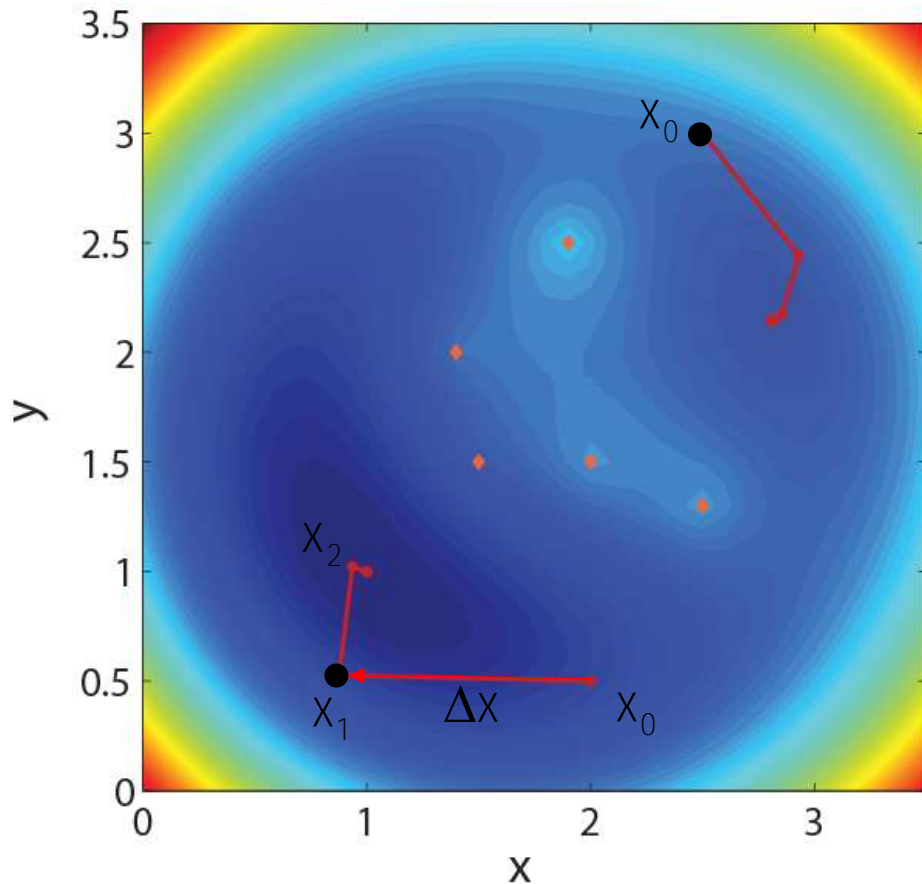
$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ f(x) \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

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Example:
Localization using range data from beacons



$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ f(x) \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

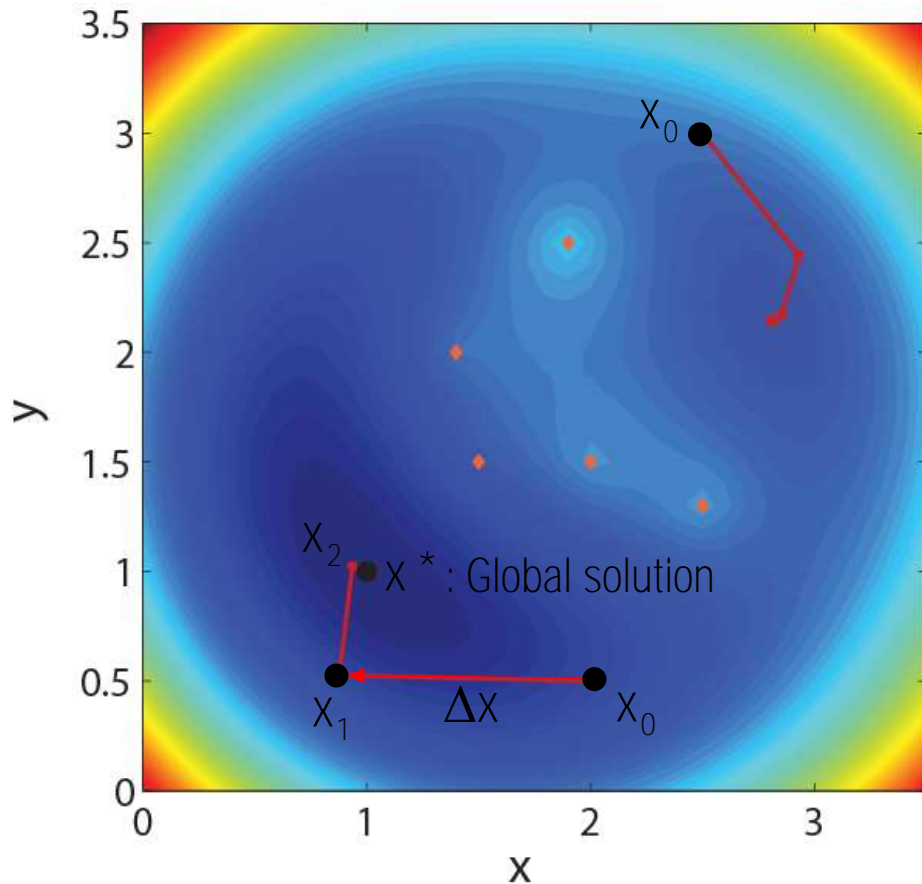
$$\frac{\partial f(x)^T}{\partial x} \frac{\partial f(x)}{\partial x} \Delta x = \frac{\partial f(x)^T}{\partial x} (b - f(x))$$

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Beacon.m

```
u = 2*(rand(3,2)-0.5);  
x = [-0.5 1];
```

```
for i = 1 : size(u,1)  
    d(i,1) = norm(x-u(i,:));  
end
```

```
[x_grid, y_grid] = meshgrid(-1.5:0.01:1.5, -1.5:0.01:1.5);
```

```
E = zeros(size(x_grid));  
for i = 1 : size(u,1)  
    E = E + (sqrt((x_grid-u(i,1)).^2 +(y_grid-u(i,2)).^2)-d(i)).^2 ;  
end  
E = sqrt(E);
```

```
x0 = 2*(rand(2,1)-0.5);  
for j = 1 : 10  
    J = [];  
    fx = [];  
    for i = 1 : size(u,1)  
        denom = norm(u(i,:)-x0');  
        J = [J; (u(i,:)-x0')/denom];  
        fx = [fx; norm(u(i,:)-x0')];  
    end  
    b = d;  
    delta_x = -inv(J'*J)*J'*(b-fx);  
    x0 = x0 + delta_x;  
end
```

