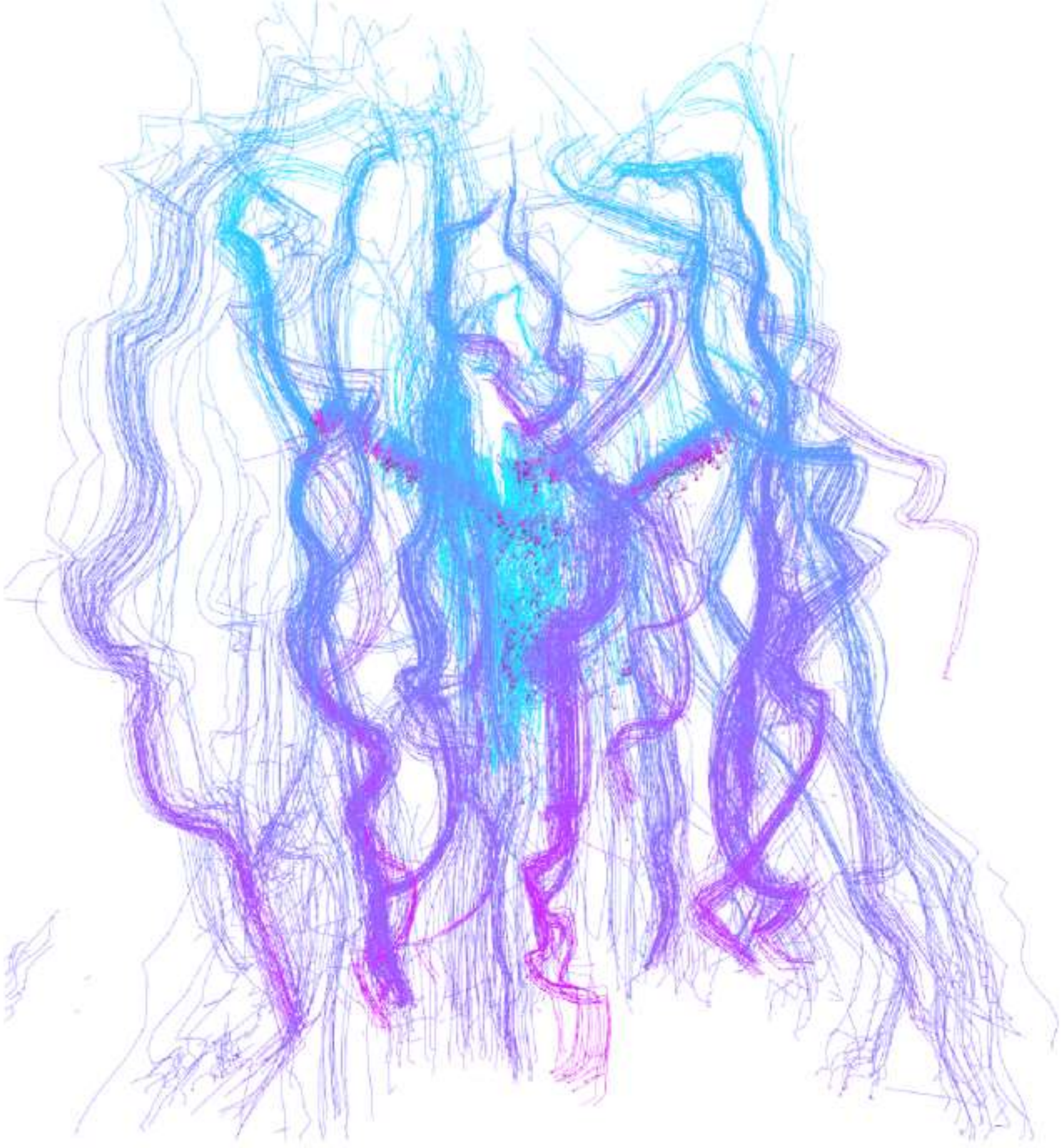
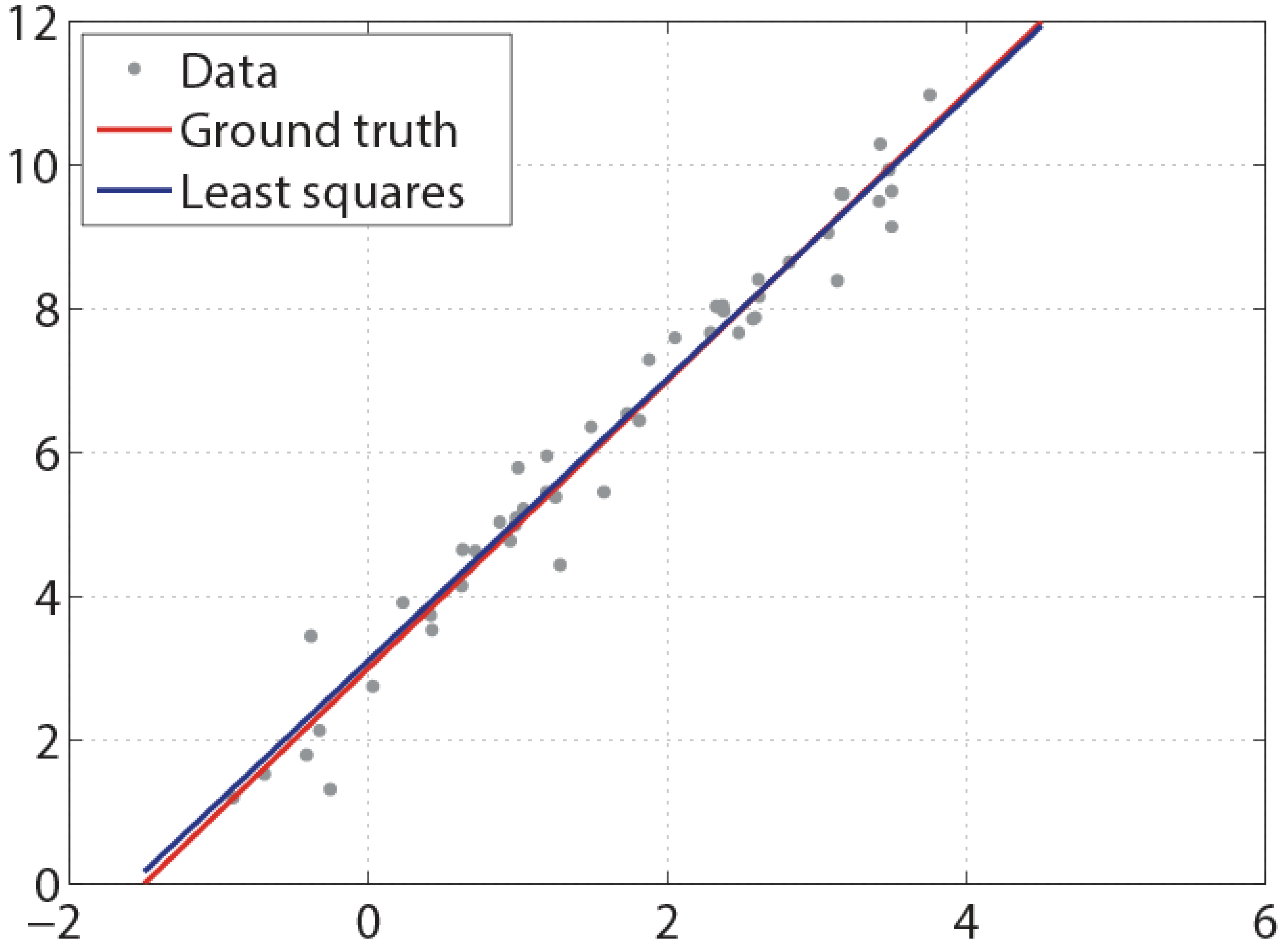
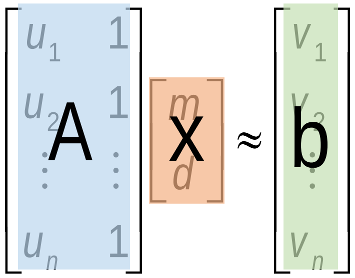


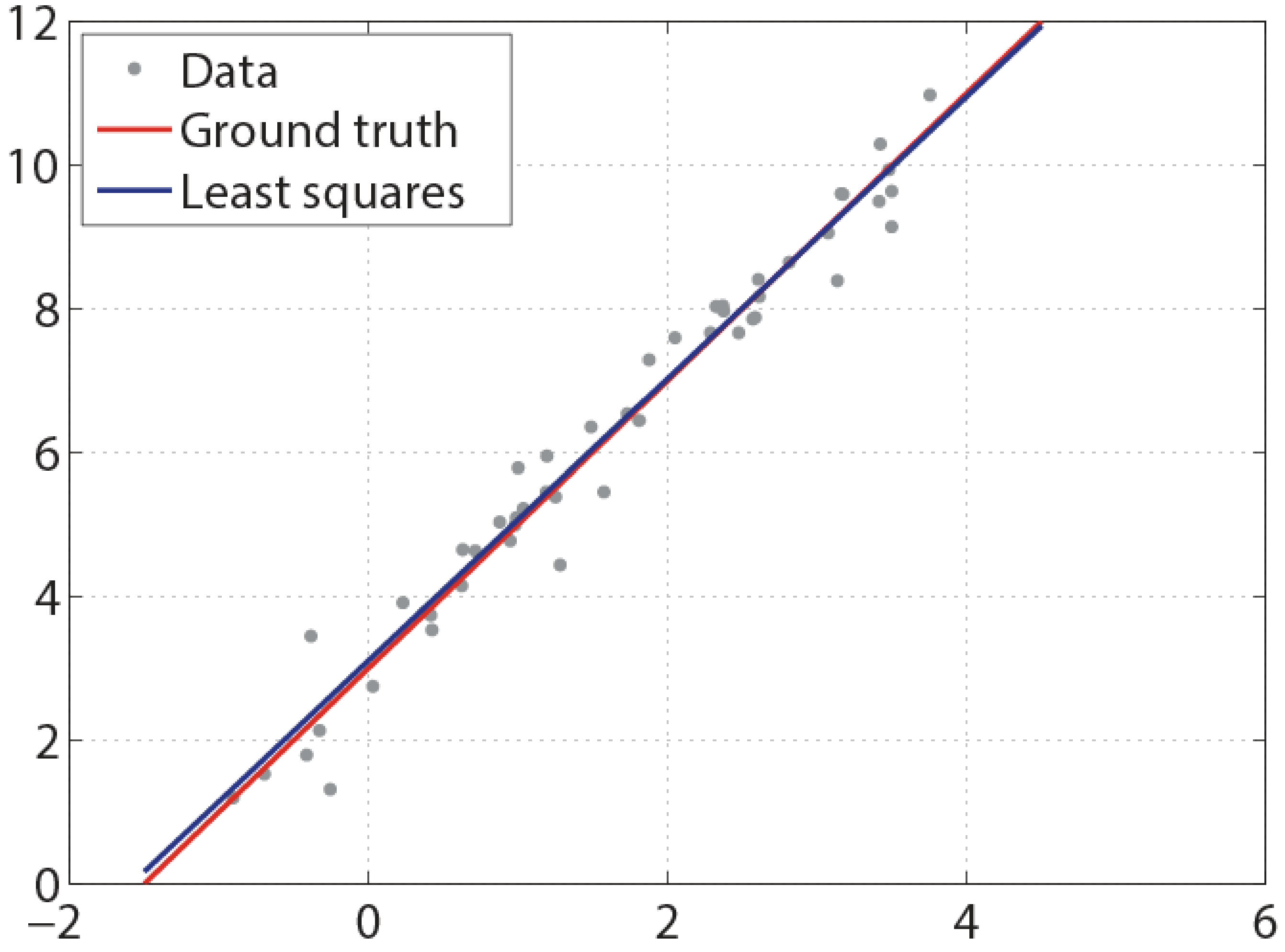
Nonlinear Estimation

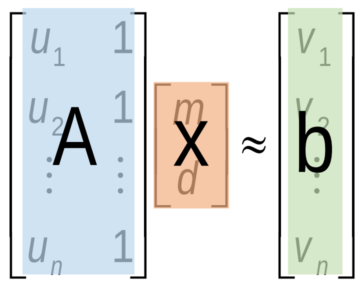
Recall: Line Fitting (Ax=b)

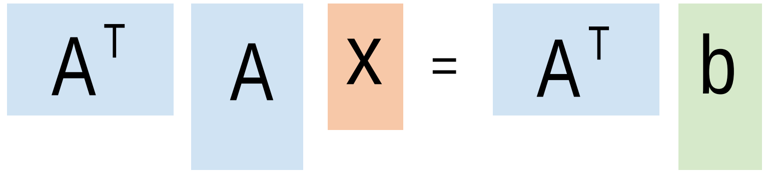




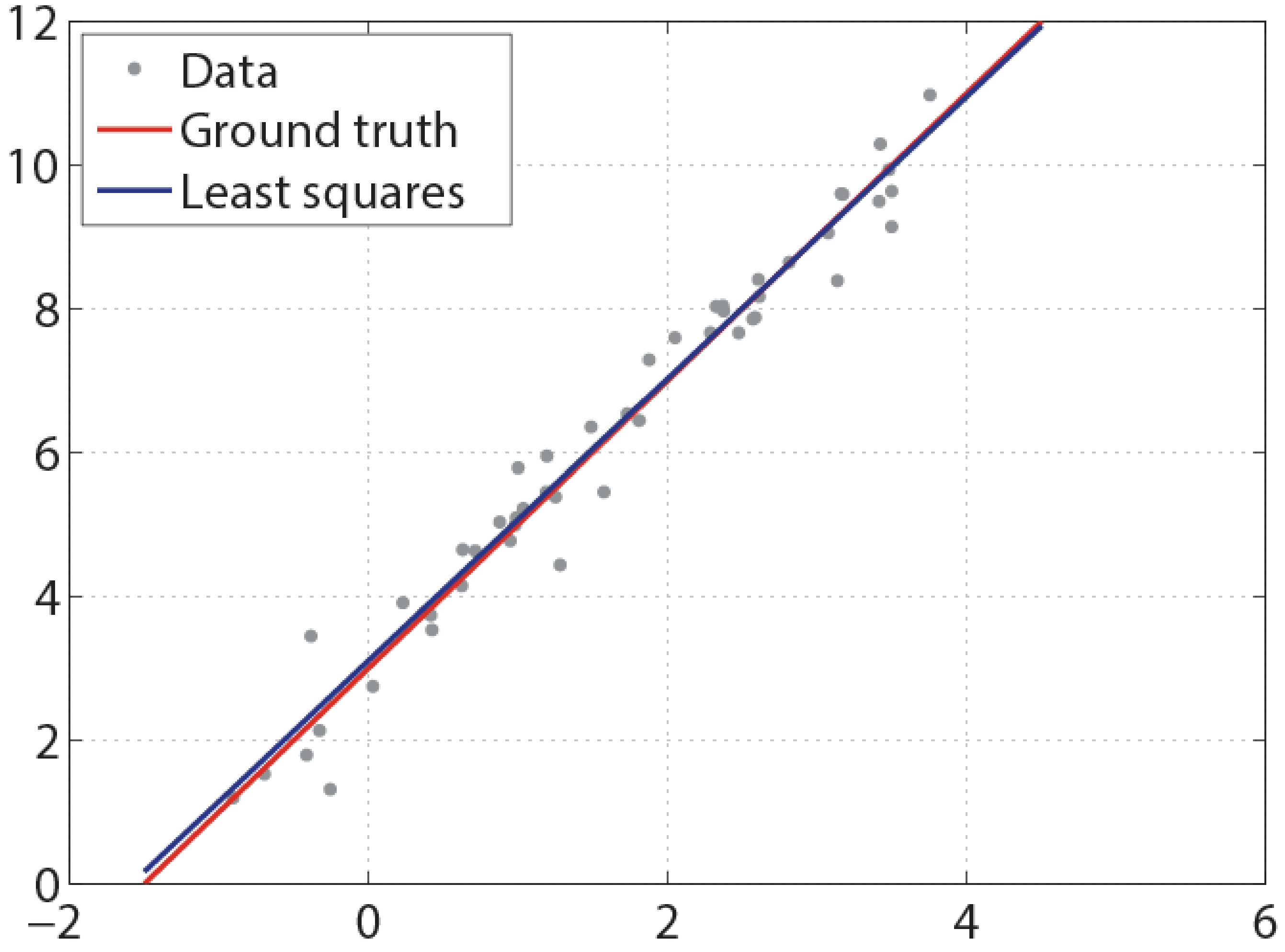
Recall: Line Fitting (Ax=b)







Recall: Line Fitting (Ax=b)



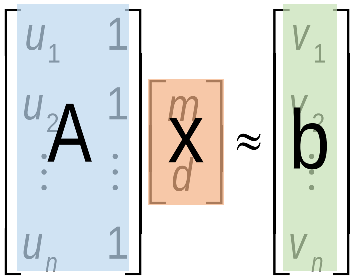
$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$A \; x \approx b$$

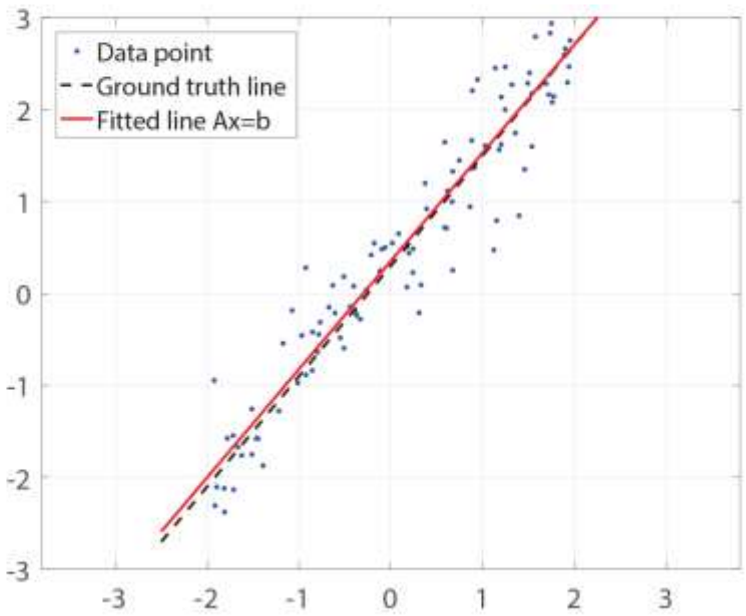
$$A^T A x = A^T b$$

$$x = \left[A^T A \right]^{-1} A^T b$$

Recall: Line Fitting (Ax=b)

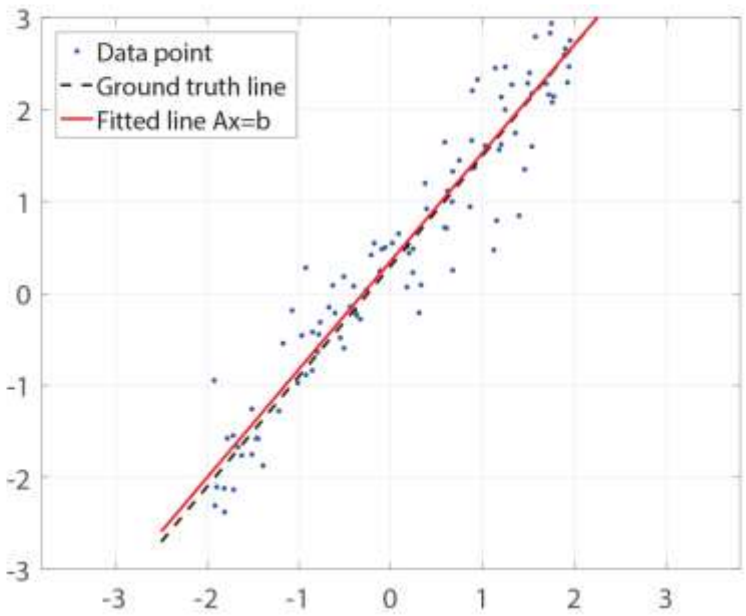


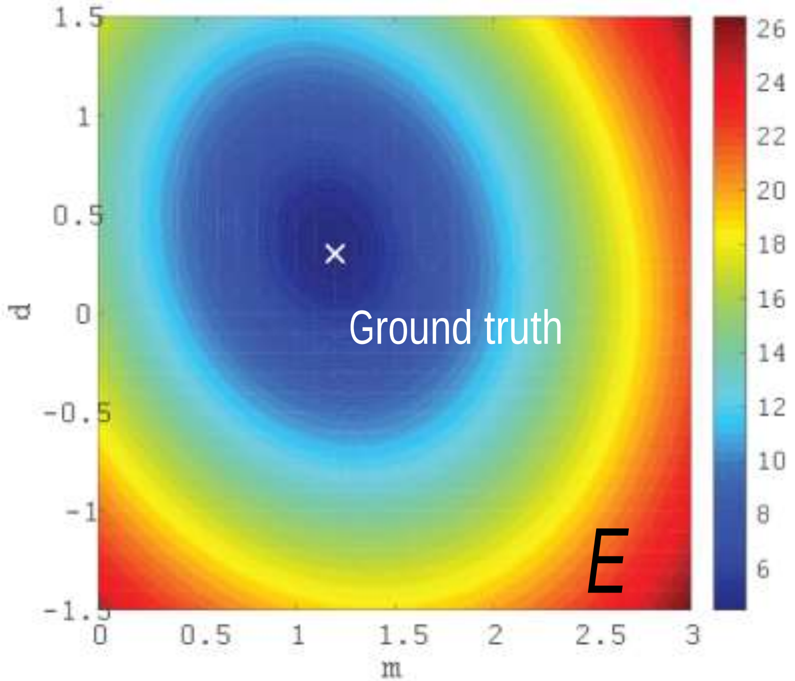
Error:



Recall: Line Fitting (Ax=b)

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$$



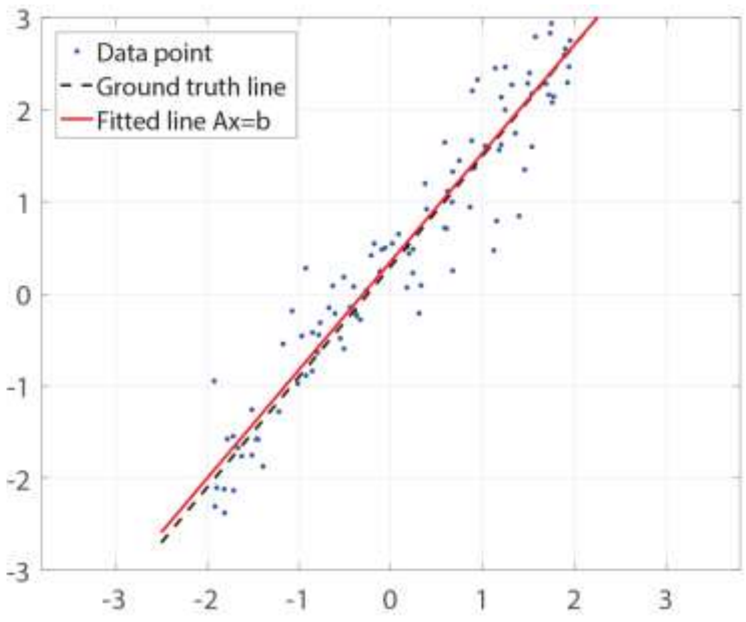


Error:

$$E = \left\| \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|^2$$

Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$



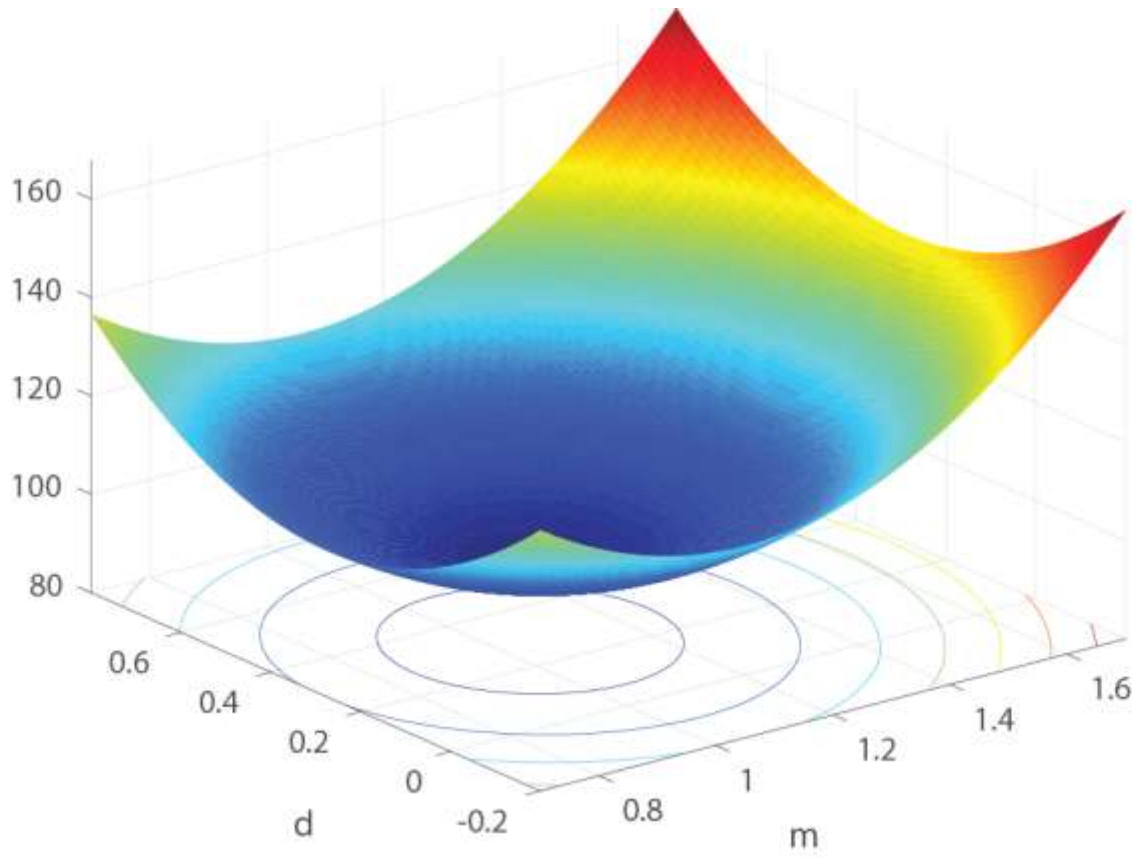
Error:

$$E = \left\| \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|^2$$

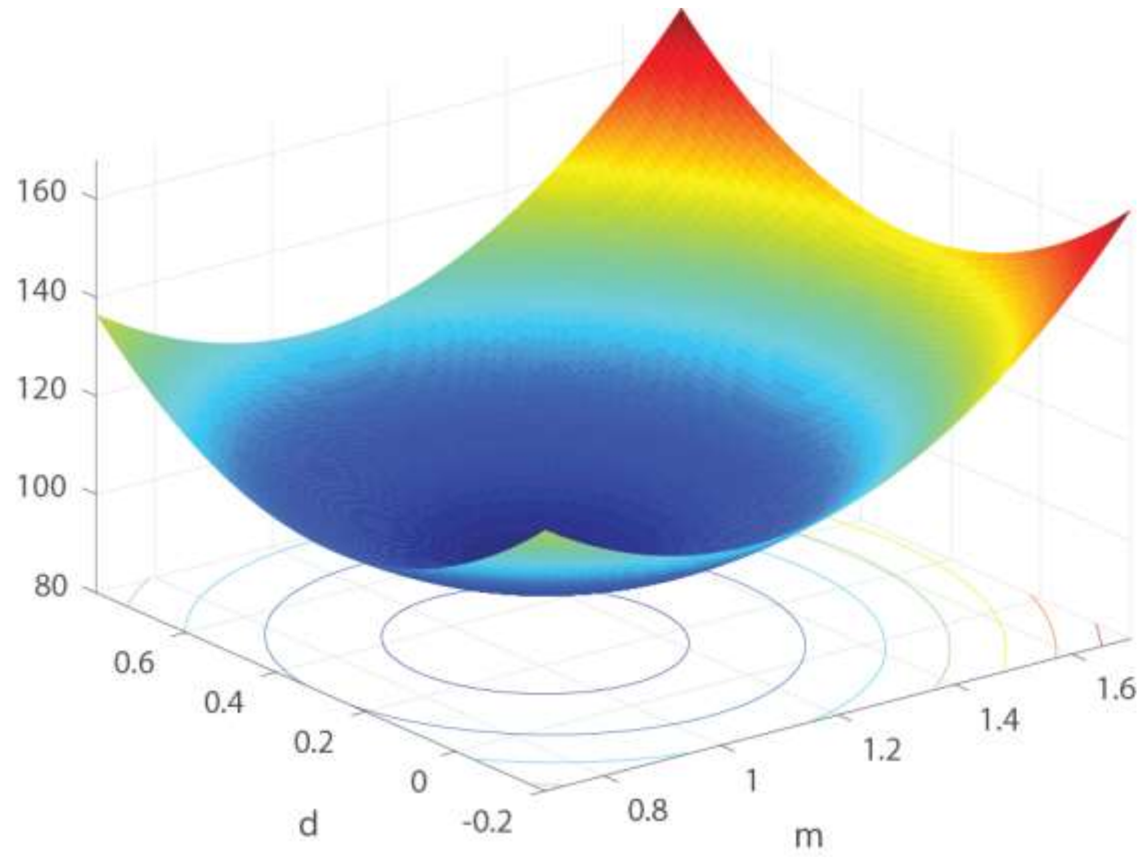
Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\dfrac{\partial E}{\partial m} = 0, \quad \dfrac{\partial E}{\partial d} = 0$



Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

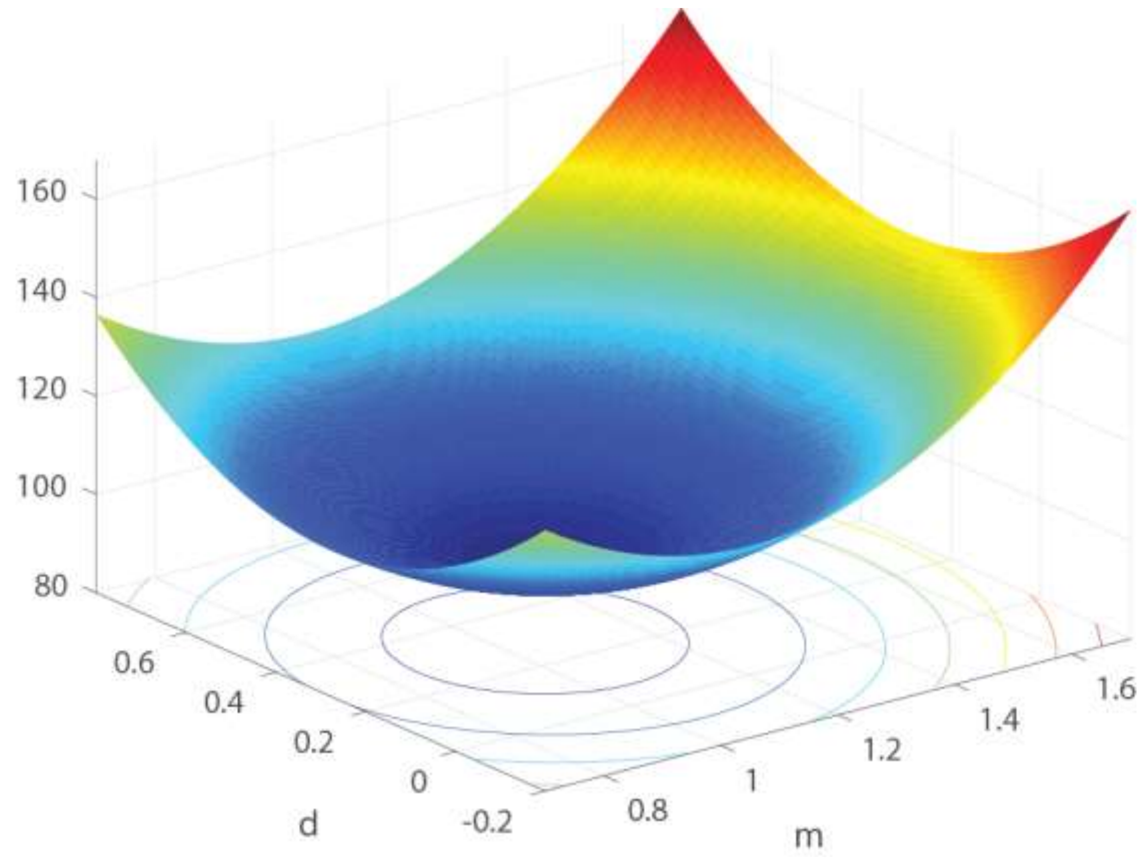
Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$v_n \approx mu_n + d$$

→

$$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$
$$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$
$$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$v_n \approx mu_n + d$$

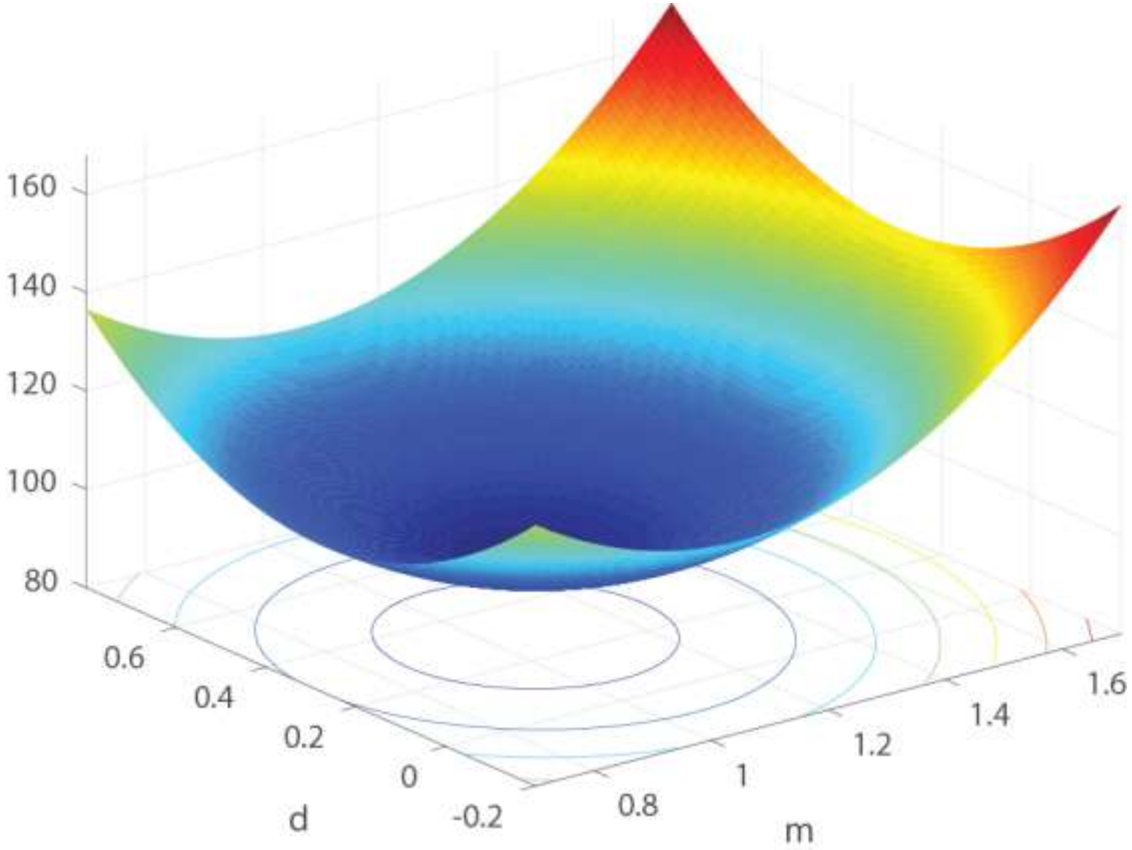
$\rightarrow$

$$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$
$$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$
$$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

$\rightarrow$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$v_1 \approx mu_1 + d$

$v_2 \approx mu_2 + d$

$v_n \approx mu_n + d$

→

$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$

$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$

$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$

→

$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$

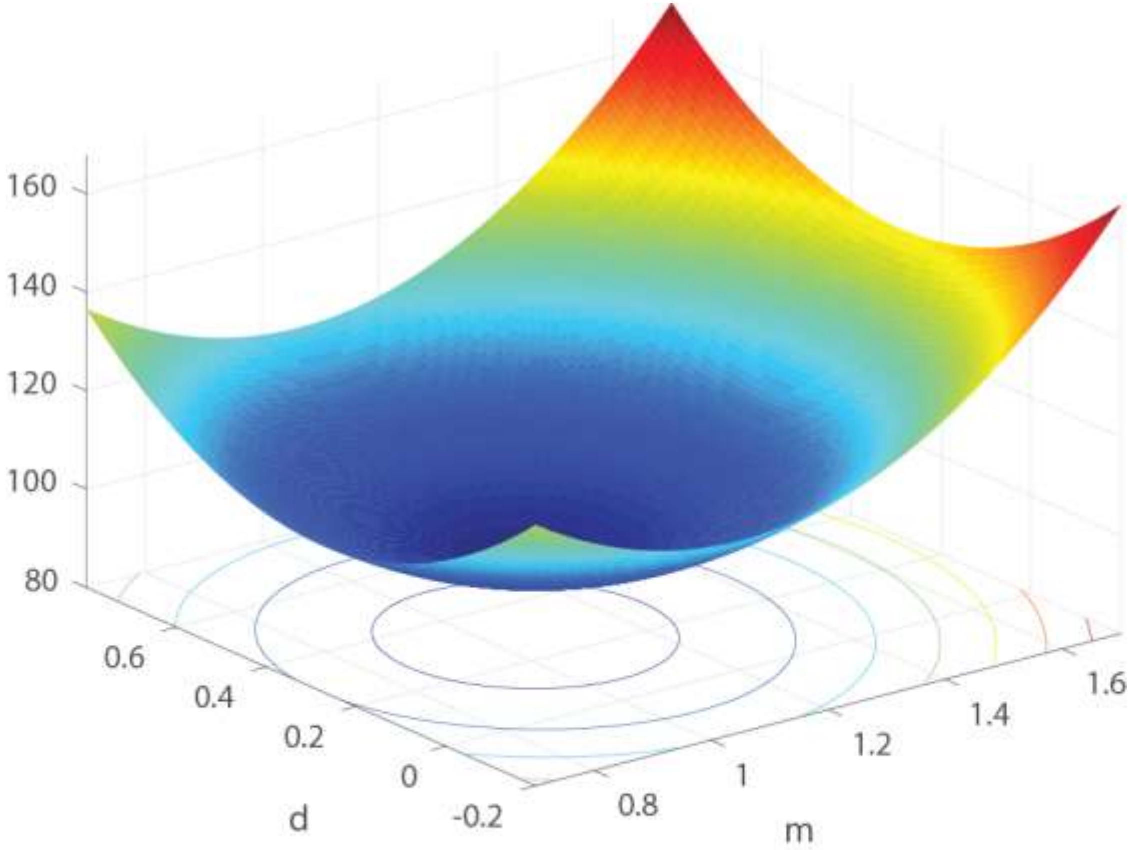
We can't invert A.

Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d) \right)^2$

$$E = (\mathrm{Ax} - \mathrm{b})^{\mathrm{T}} (\mathrm{Ax} - \mathrm{b}) = \left\| \mathrm{Ax} - \mathrm{b} \right\|^2$$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathrm{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathrm{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathrm{b}}$$



Recall: Line Fitting

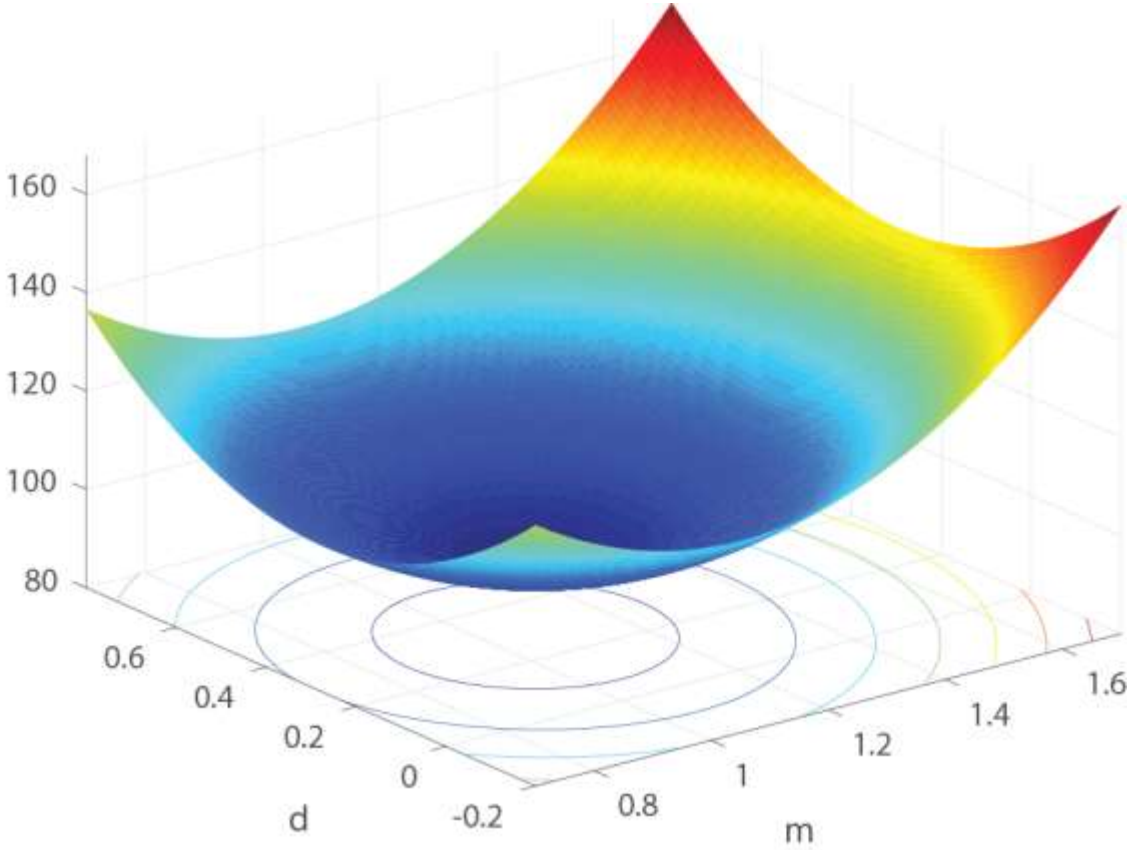
Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathrm{A} \quad \mathrm{x} \approx \mathrm{b}$$

$$E = (\mathrm{Ax} - \mathrm{b})^{\mathrm{T}} (\mathrm{Ax} - \mathrm{b}) = \|\mathrm{Ax} - \mathrm{b}\|^2$$

$$= \mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{b} + \mathrm{b}^{\mathrm{T}}\mathrm{b}$$



$$\frac{\partial E}{\partial \mathrm{x}} = \qquad ?$$

Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

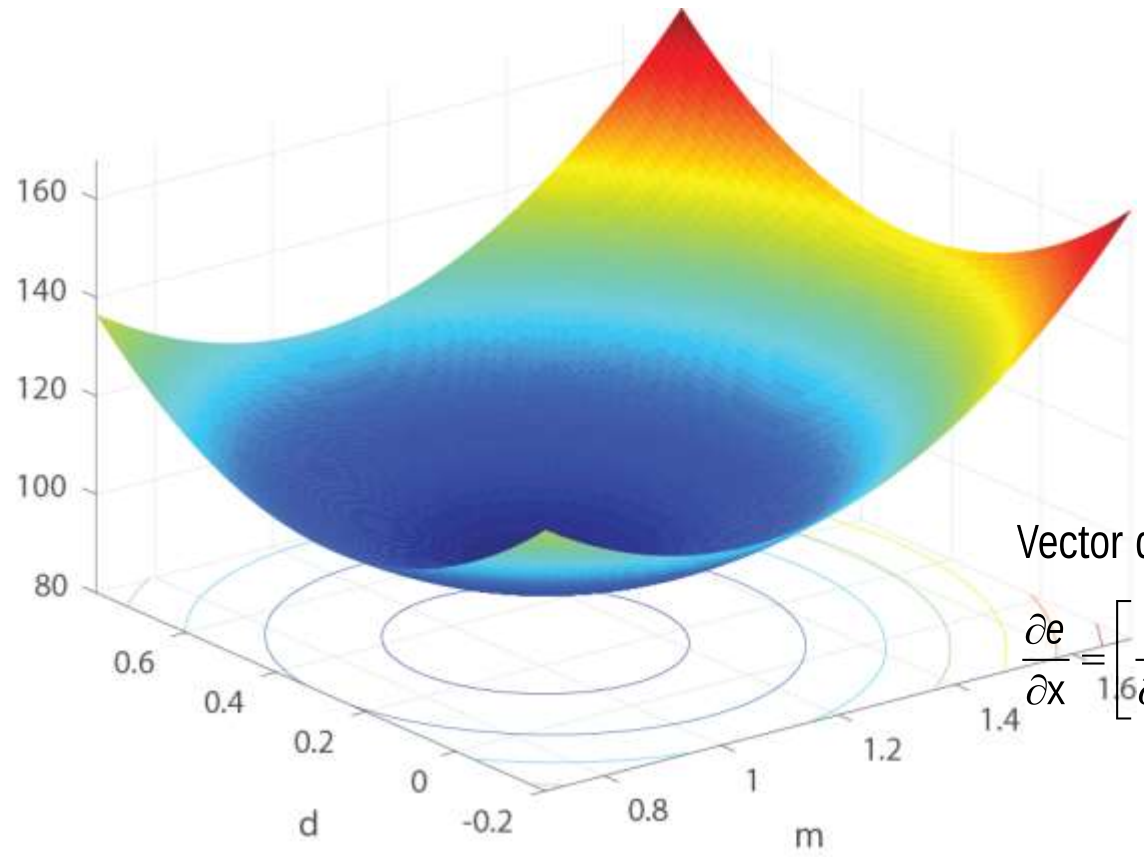
$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathrm{A} \quad \mathrm{x} \approx \mathrm{b}$$

$$E = (\mathrm{Ax} - \mathrm{b})^{\mathrm{T}} (\mathrm{Ax} - \mathrm{b}) = \|\mathrm{Ax} - \mathrm{b}\|^2$$

$$= \mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{b} + \mathrm{b}^{\mathrm{T}}\mathrm{b}$$

$$\frac{\partial E}{\partial \mathrm{x}} = \qquad ?$$



Vector derivative:

$$\frac{\partial e}{\partial \mathrm{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

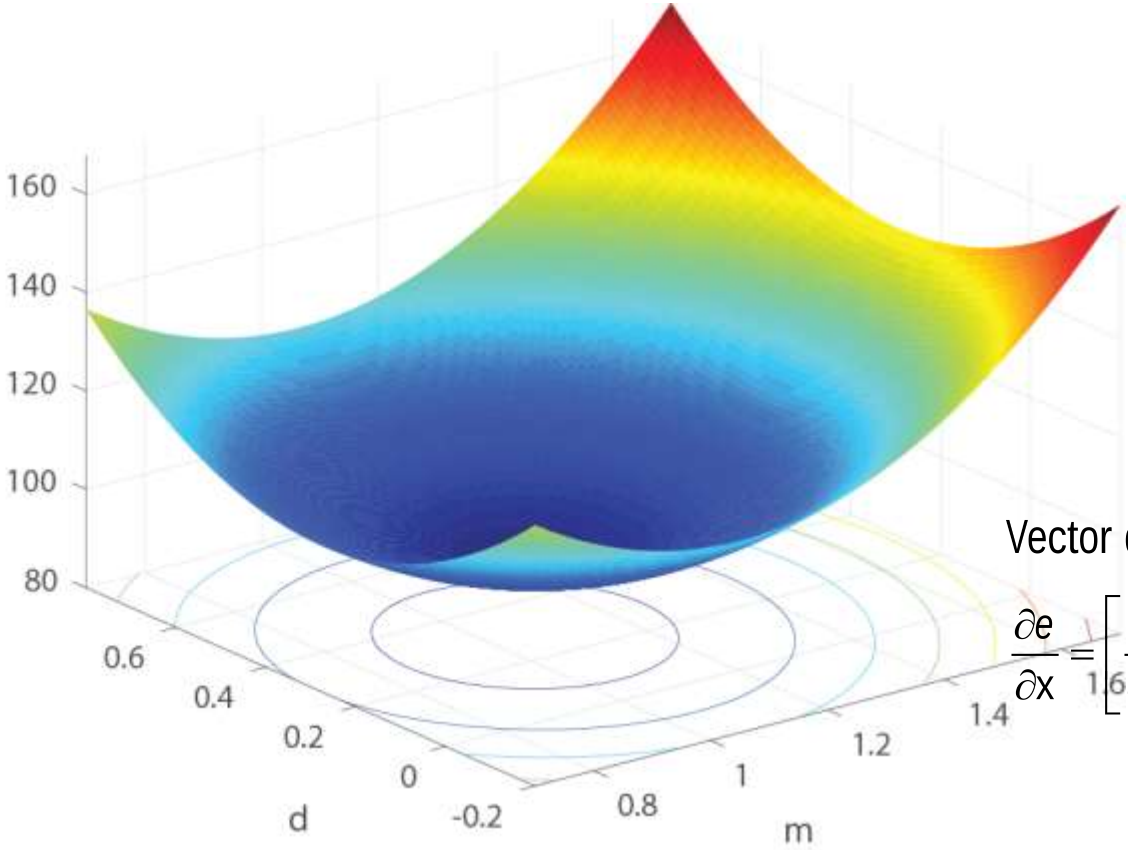
Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$$
$$= x^T A^T A x - 2x^T A^T b + b^T b$$

$$\frac{\partial E}{\partial x} = \qquad ?$$



Vector derivative:

$$\frac{\partial e}{\partial x} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = c^T x = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial c^T x}{\partial x} =$$

Recall: Line Fitting

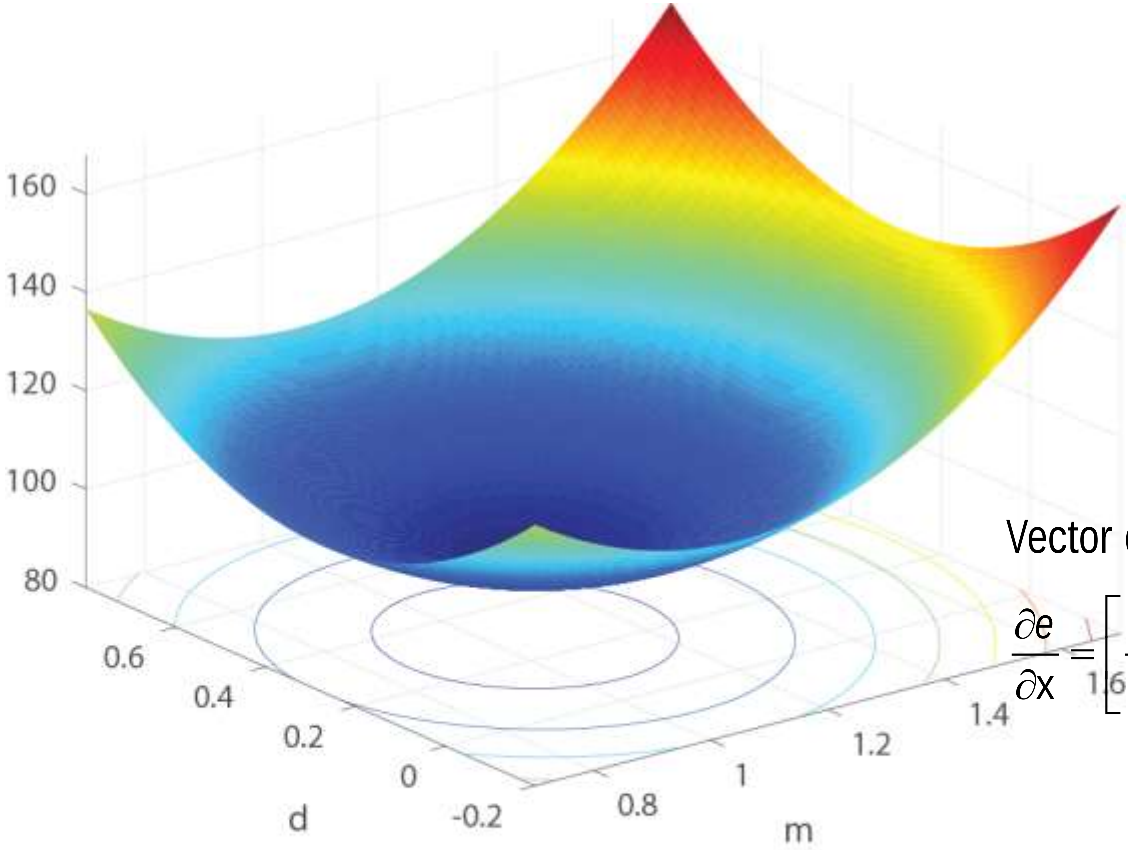
Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathrm{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathrm{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathrm{b}}$$

$$E = (\mathrm{Ax} - \mathrm{b})^{\mathrm{T}} (\mathrm{Ax} - \mathrm{b}) = \|\mathrm{Ax} - \mathrm{b}\|^2$$

$$= \mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{b} + \mathrm{b}^{\mathrm{T}}\mathrm{b}$$

$$\frac{\partial E}{\partial \mathrm{x}} = \qquad ?$$



Vector derivative:

$$\frac{\partial e}{\partial \mathrm{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathrm{c}^{\mathrm{T}}\mathrm{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathrm{c}^{\mathrm{T}}\mathrm{x}}{\partial \mathrm{x}} = \frac{\partial}{\partial \mathrm{x}} (c_1 x_1 + \cdots + c_n x_n)$$

Recall: Line Fitting

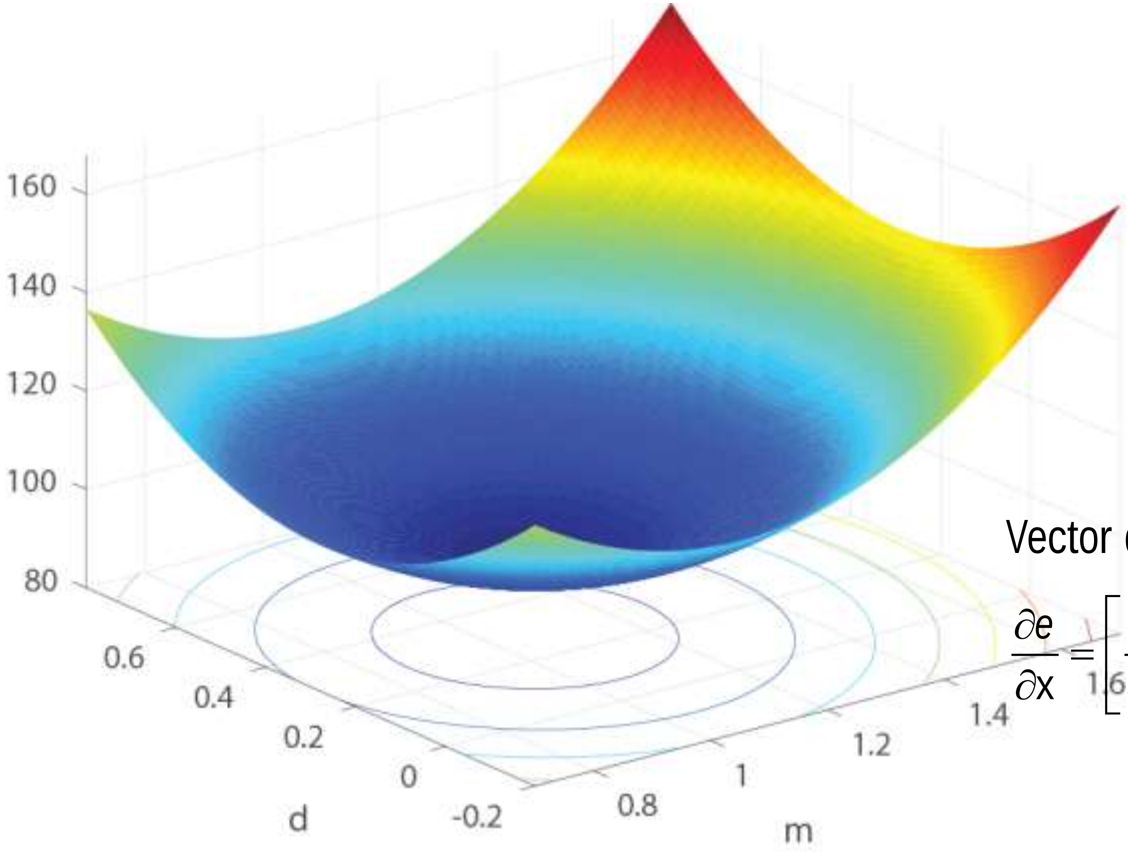
Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

A x ≈ b

$$E = (\mathrm{Ax} - \mathrm{b})^{\mathrm{T}} (\mathrm{Ax} - \mathrm{b}) = \|\mathrm{Ax} - \mathrm{b}\|^2$$
$$= \mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{b} + \mathrm{b}^{\mathrm{T}}\mathrm{b}$$

$$\frac{\partial E}{\partial \mathrm{x}} = \quad ?$$



Vector derivative:

$$\frac{\partial e}{\partial \mathrm{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathrm{c}^{\mathrm{T}}\mathrm{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathrm{c}^{\mathrm{T}}\mathrm{x}}{\partial \mathrm{x}} = \frac{\partial}{\partial \mathrm{x}}(c_1 x_1 + \cdots + c_n x_n)$$
$$= \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$$

Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} (v_i - (mu_i + d))^2$

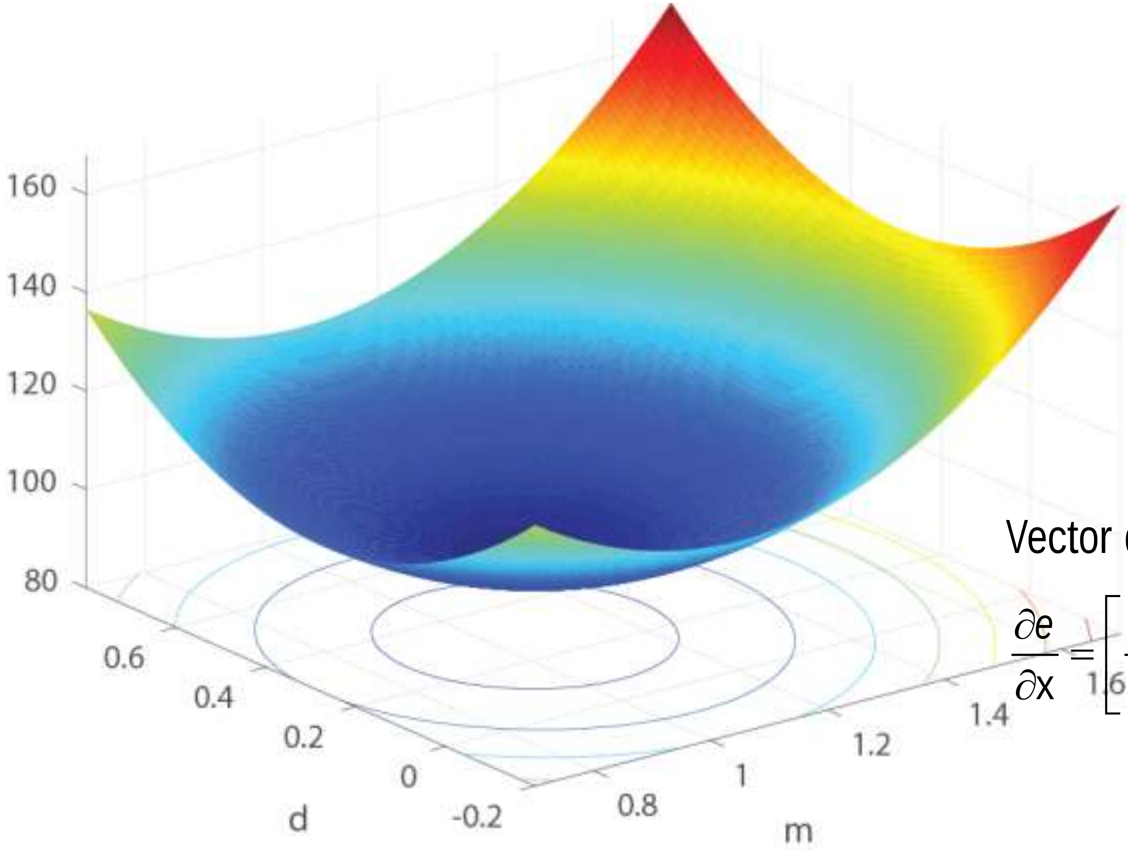
$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathrm{A}\,\mathrm{x} \approx \mathrm{b}$$

$$E = (\mathrm{Ax} - \mathrm{b})^{\mathrm{T}} (\mathrm{Ax} - \mathrm{b}) = \|\mathrm{Ax} - \mathrm{b}\|^2$$

$$= \mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{b} + \mathrm{b}^{\mathrm{T}}\mathrm{b}$$

$$\frac{\partial E}{\partial \mathrm{x}} = 2\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{A}^{\mathrm{T}}\mathrm{b} = 0$$



Vector derivative:

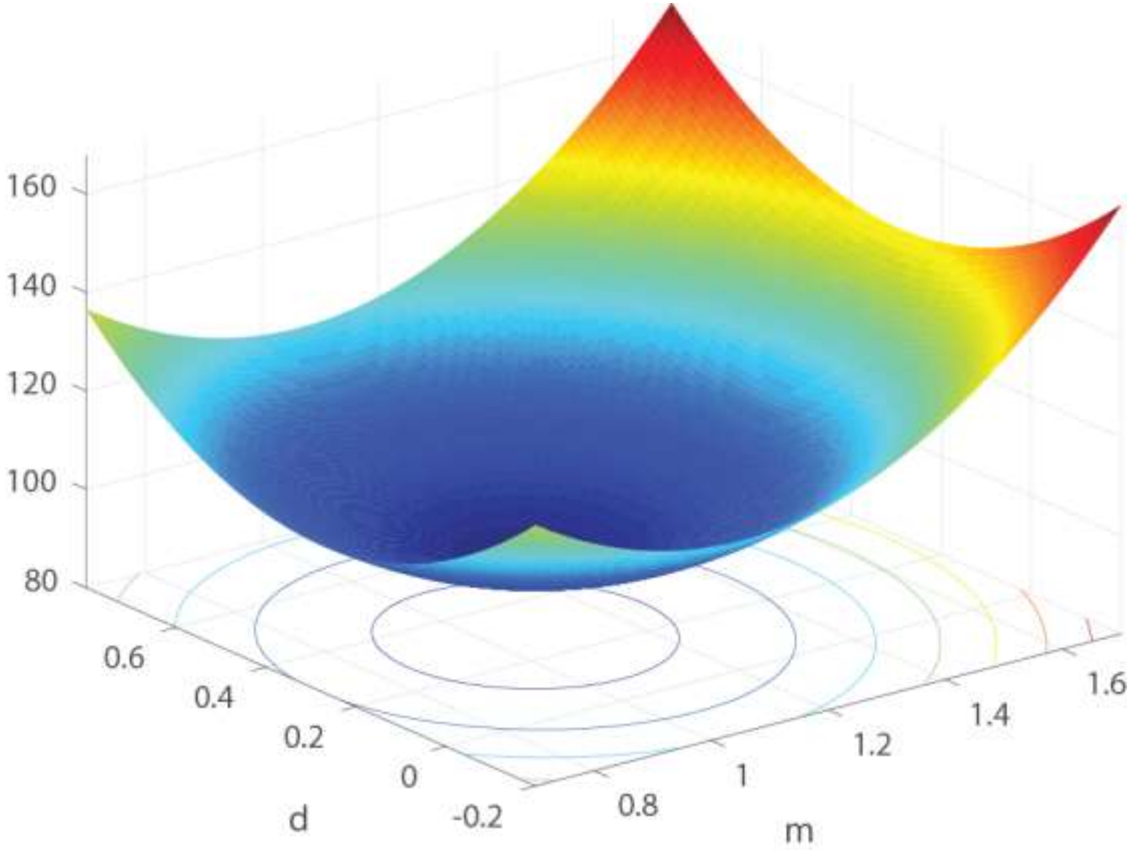
$$\frac{\partial e}{\partial \mathrm{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathrm{c}^{\mathrm{T}}\mathrm{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathrm{c}^{\mathrm{T}}\mathrm{x}}{\partial \mathrm{x}} = \frac{\partial}{\partial \mathrm{x}} (c_1 x_1 + \cdots + c_n x_n)$$

$$= \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$$

Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$$

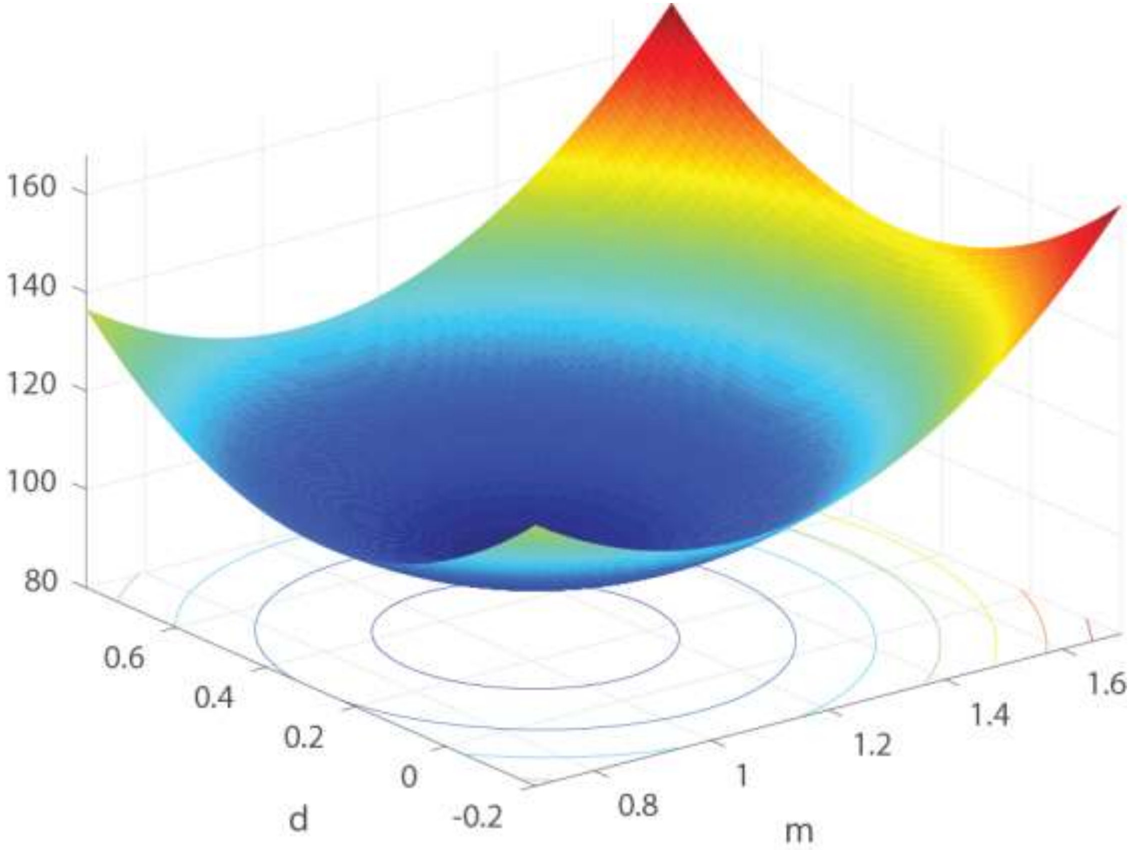
$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$$

$$= x^T A^T A x - 2x^T A^T b + b^T b$$

$$\frac{\partial E}{\partial x} = 2A^T A x - 2A^T b = 0$$

$$\longrightarrow A^T A x = A^T b$$

Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathrm{A}\,\mathrm{x} \approx \mathrm{b}$$

$$E = (\mathrm{Ax} - \mathrm{b})^{\mathrm{T}} (\mathrm{Ax} - \mathrm{b}) = \|\mathrm{Ax} - \mathrm{b}\|^2$$

$$= \mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{b} + \mathrm{b}^{\mathrm{T}}\mathrm{b}$$

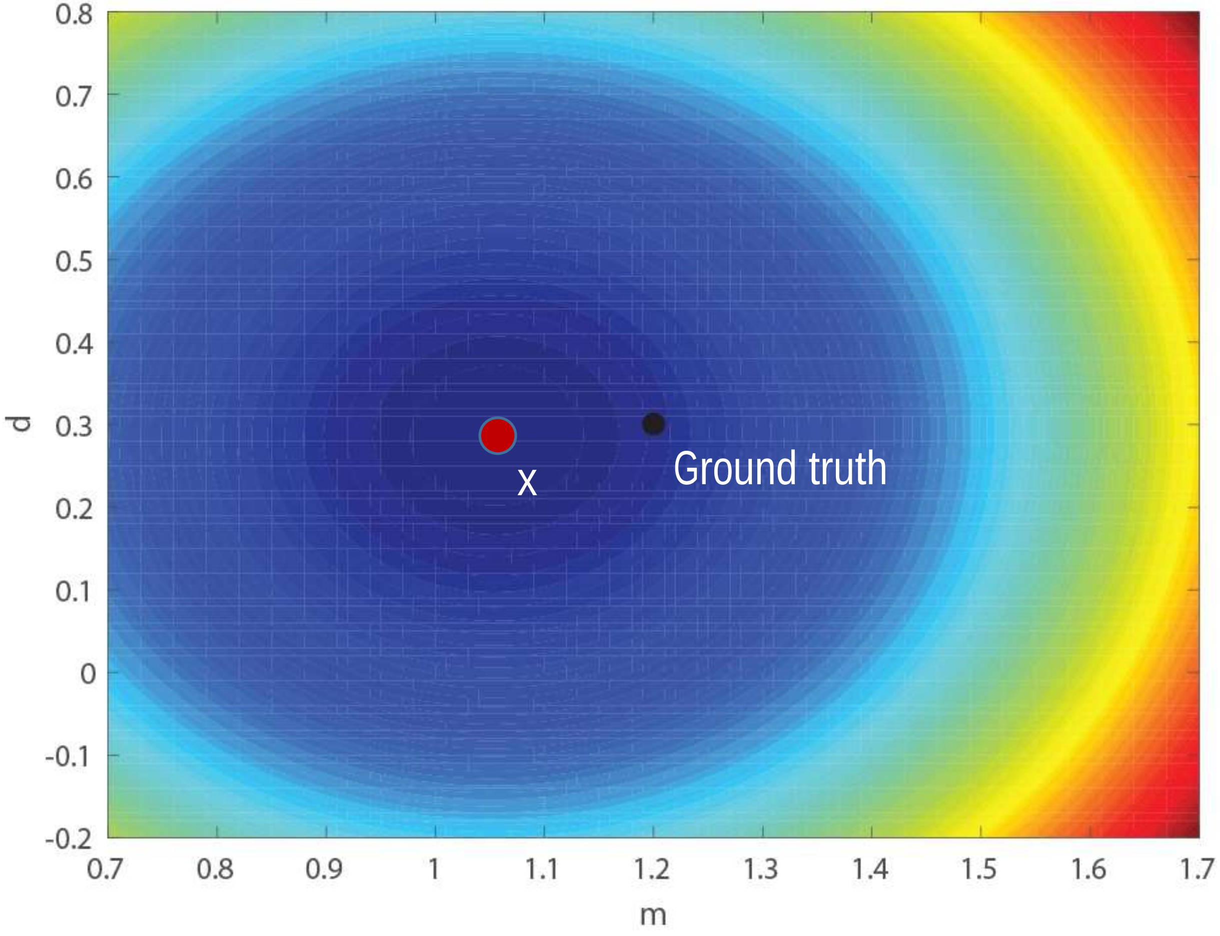
$$\frac{\partial E}{\partial \mathrm{x}} = 2\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{A}^{\mathrm{T}}\mathrm{b} = 0$$

$$\longrightarrow \mathrm{A}^{\mathrm{T}}\mathrm{Ax} = \mathrm{A}^{\mathrm{T}}\mathrm{b}$$

$$\mathrm{A}^{\mathrm{T}} \; \mathrm{A} \; \mathrm{x} = \mathrm{A}^{\mathrm{T}} \; \mathrm{b}$$

Normal equation

Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$$

$$= x^T A^T A x - 2x^T A^T b + b^T b$$

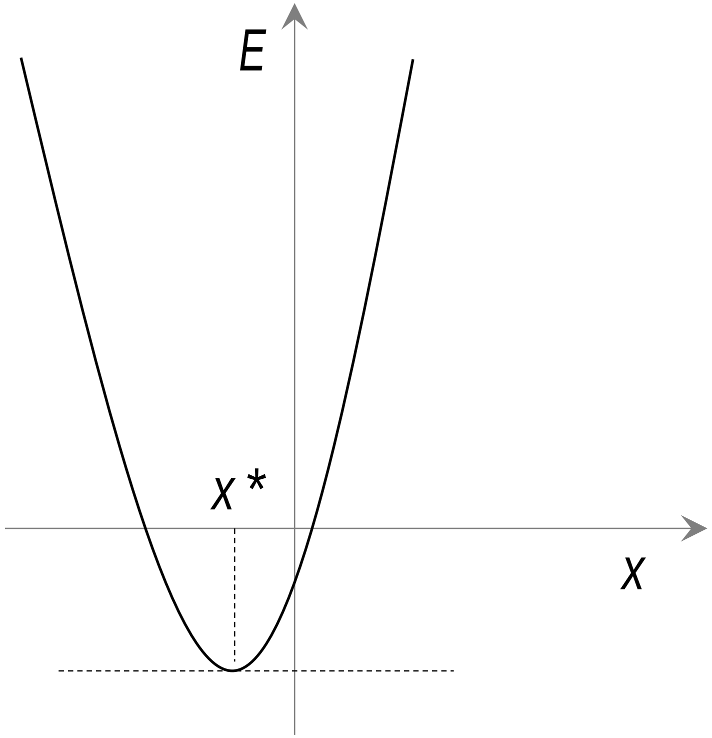
$$\frac{\partial E}{\partial x} = 2A^T A x - 2A^T b = 0$$

$$\longrightarrow A^T A x = A^T b$$

$$A^T \; A \; x = A^T \; b$$

$$x = \left[A^T \; A \right]^{-1} A^T \; b$$

Linear System Recap



$$\left.\frac{dE}{dx}\right|_{x=x^*}=0$$

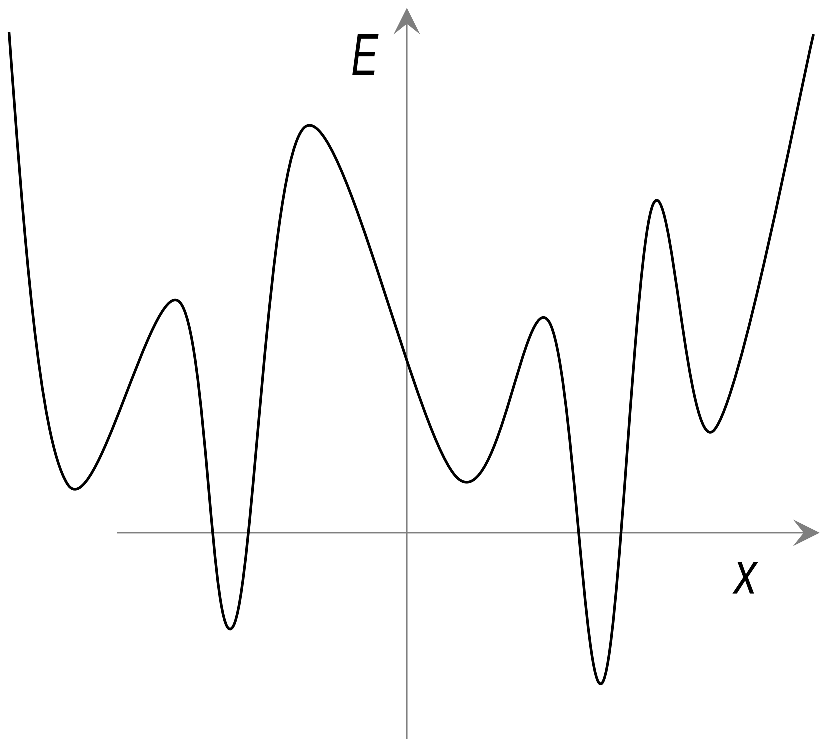
$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$$

$$x = \left[A^T A \right]^{-1} A^T b$$

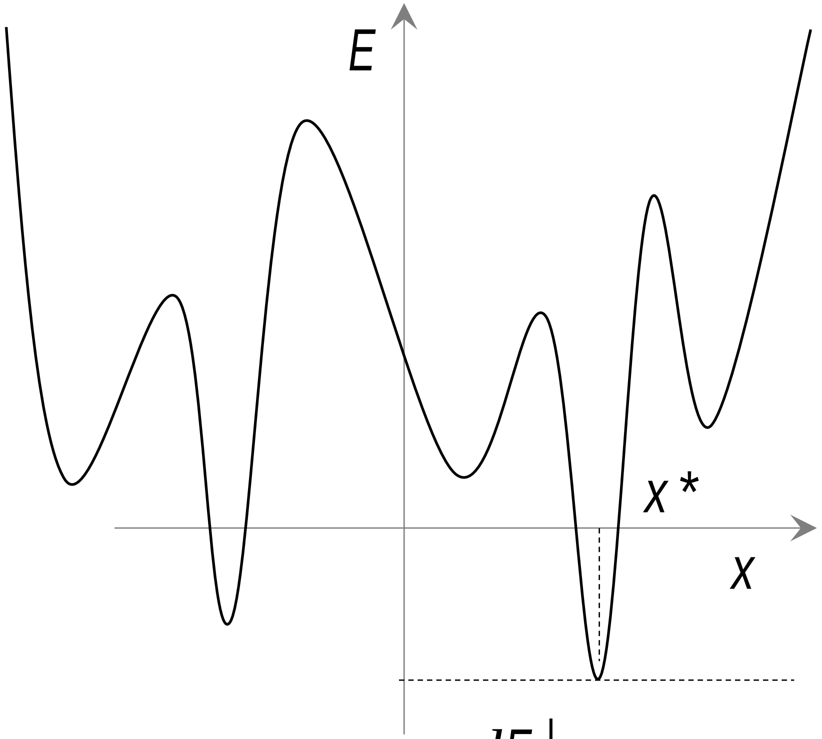
- Has the global solution
- Has the closed form solution (non-iterative solve)
- Is solved efficiently ($O(n^2)$)
- Does not require an initialization

Nonlinear System

$$f(x) = b$$



Nonlinear System

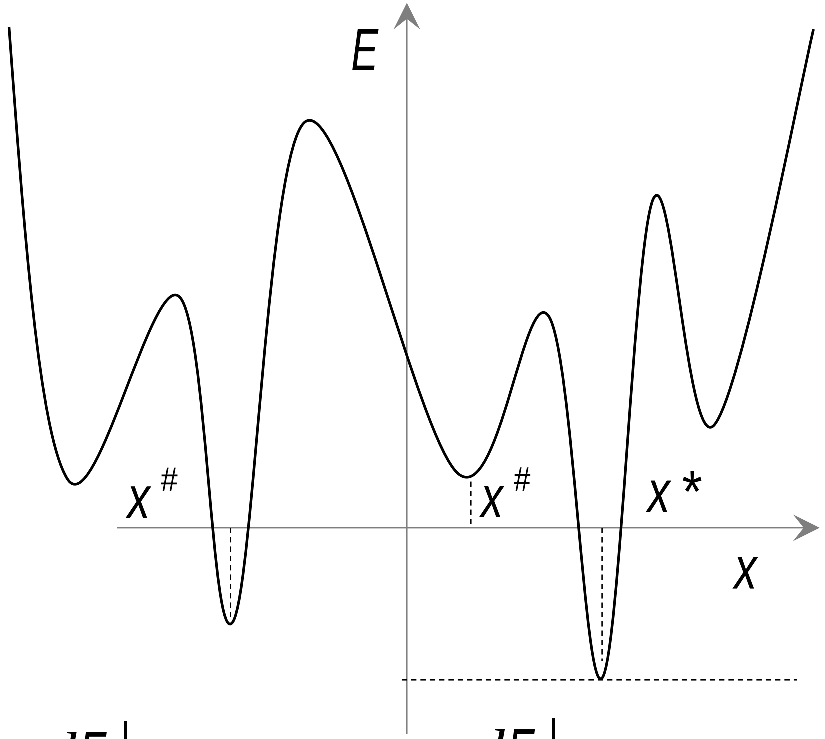


$$\left.\frac{dE}{dx}\right|_{x=x^*} = 0$$

$$f(\mathrm{x}) = \mathrm{b}$$

$$E = \|f(\mathrm{x}) - \mathrm{b}\|^2$$

Nonlinear System

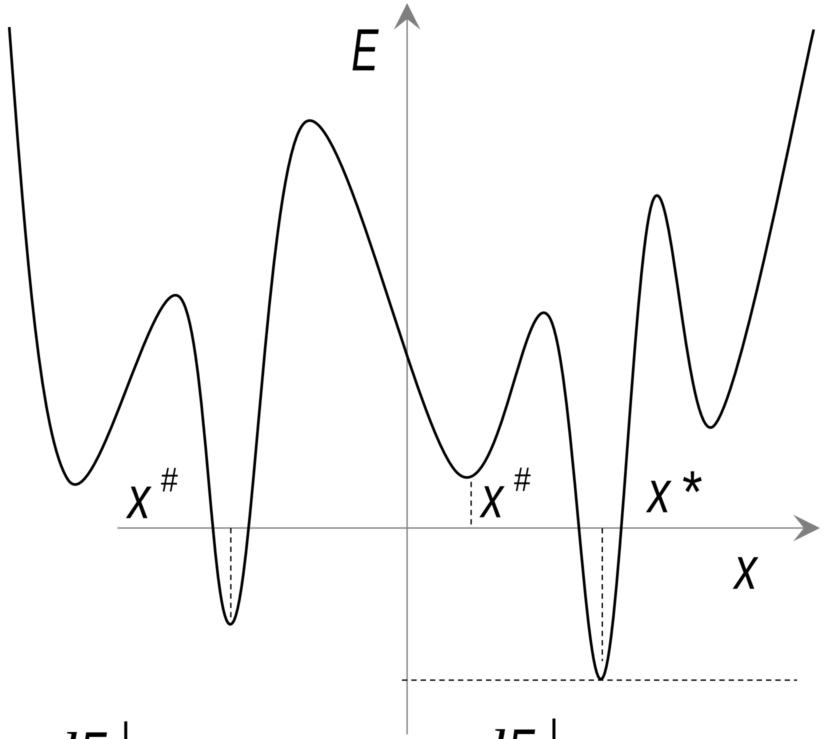


$$\left.\frac{dE}{dx}\right|_{x=x^{\#}}=0 \qquad \left.\frac{dE}{dx}\right|_{x=x^{*}}=0$$

$$f(\mathrm{x})=\mathrm{b}$$

$$E=\left\|f(\mathrm{x})-\mathrm{b}\right\|^{2}$$

Nonlinear System



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$$E = \left(f(\mathrm{x}) - \mathrm{b}\right)^{\mathsf{T}} \left(f(\mathrm{x}) - \mathrm{b}\right)$$

$$= f(\mathrm{x})^{\mathsf{T}} f(\mathrm{x}) - 2f(\mathrm{x})^{\mathsf{T}}\mathrm{b} + \mathrm{b}^{\mathsf{T}}\mathrm{b}$$

Nonlinear System

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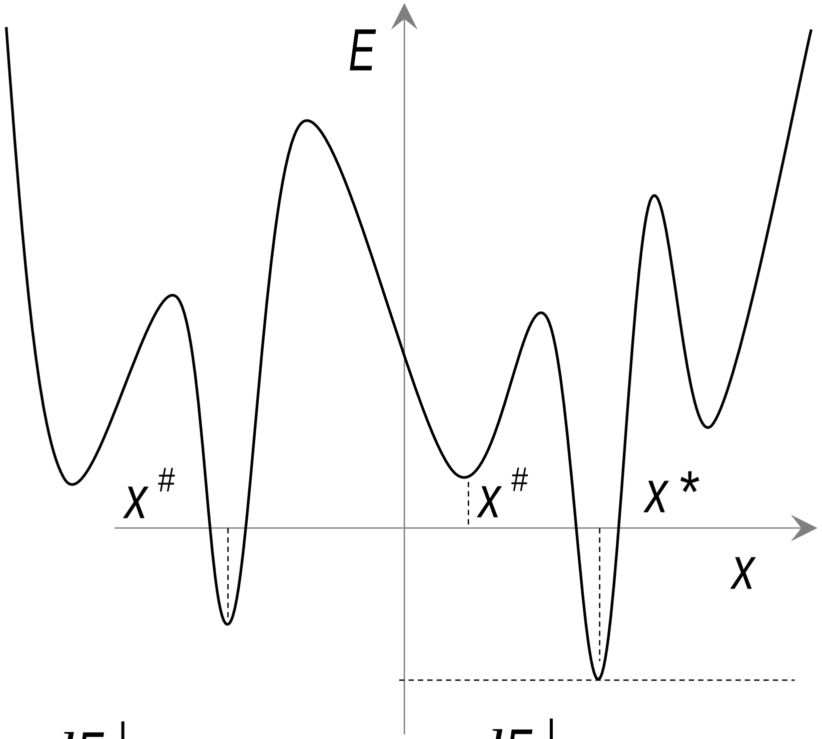

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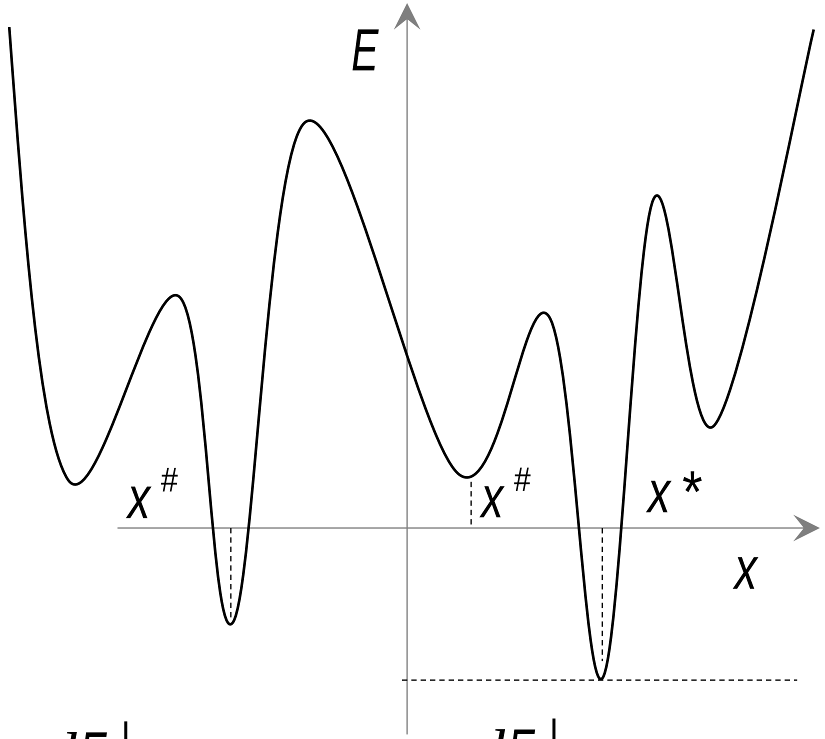
Nonlinear System



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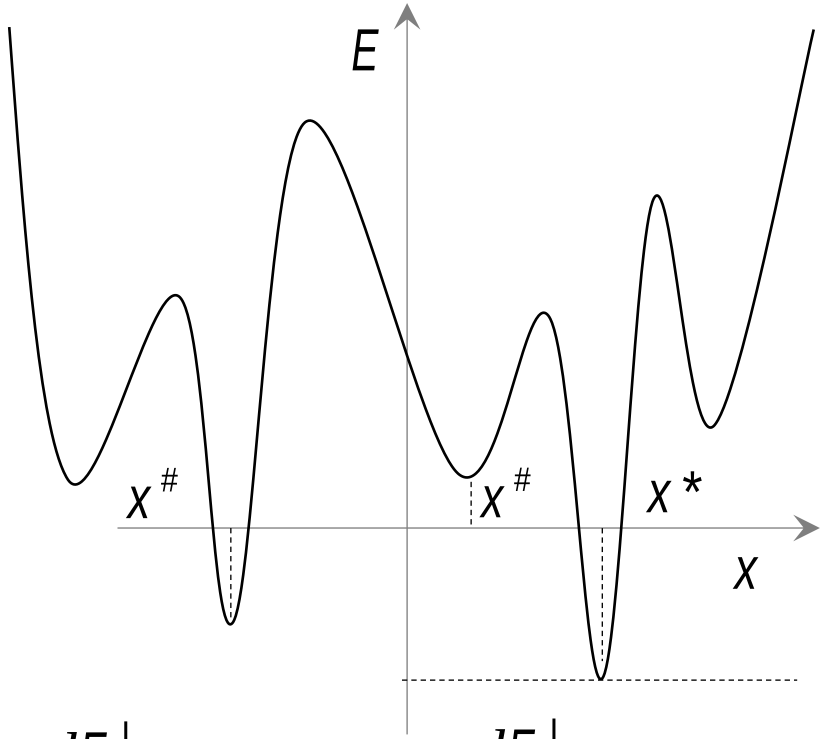
where $$\frac{\partial f(\mathrm{x})}{\partial \mathrm{x}}=\begin{bmatrix}\frac{\partial f_1}{\partial \mathrm{x}_1} & \cdots & \frac{\partial f_1}{\partial \mathrm{x}_n}\\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial \mathrm{x}_n} & \cdots & \frac{\partial f_m}{\partial \mathrm{x}_n}\end{bmatrix}$$: Jacobian

Nonlinear System



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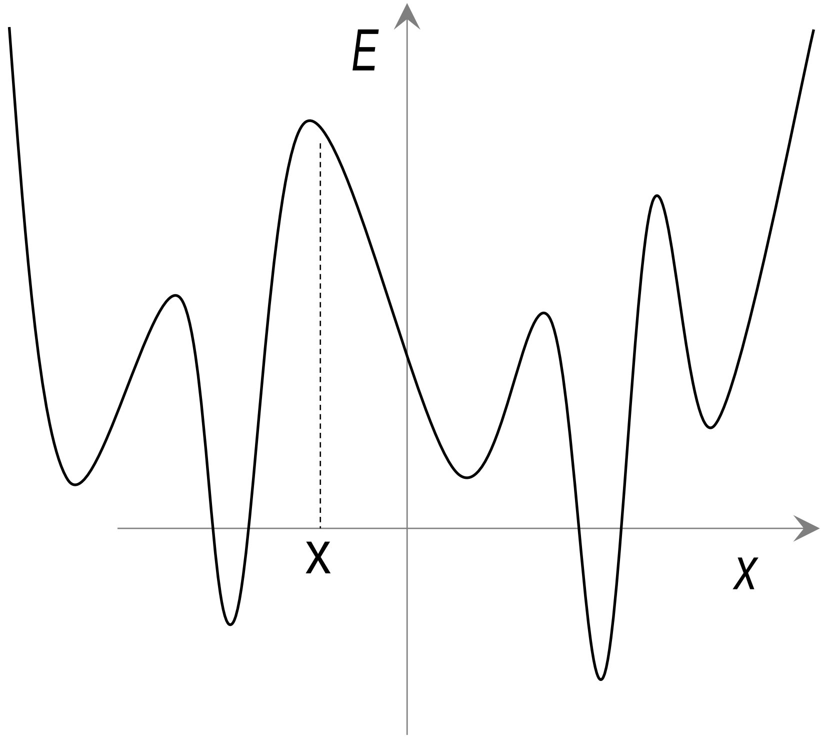
Find x such that the following equation is satisfied:

$$\frac{\partial f(\mathrm{x})^{\mathsf{T}}}{\partial \mathrm{x}}f(\mathrm{x})=\frac{\partial f(\mathrm{x})^{\mathsf{T}}}{\partial \mathrm{x}}\mathrm{b}$$

How?

Nonlinear System



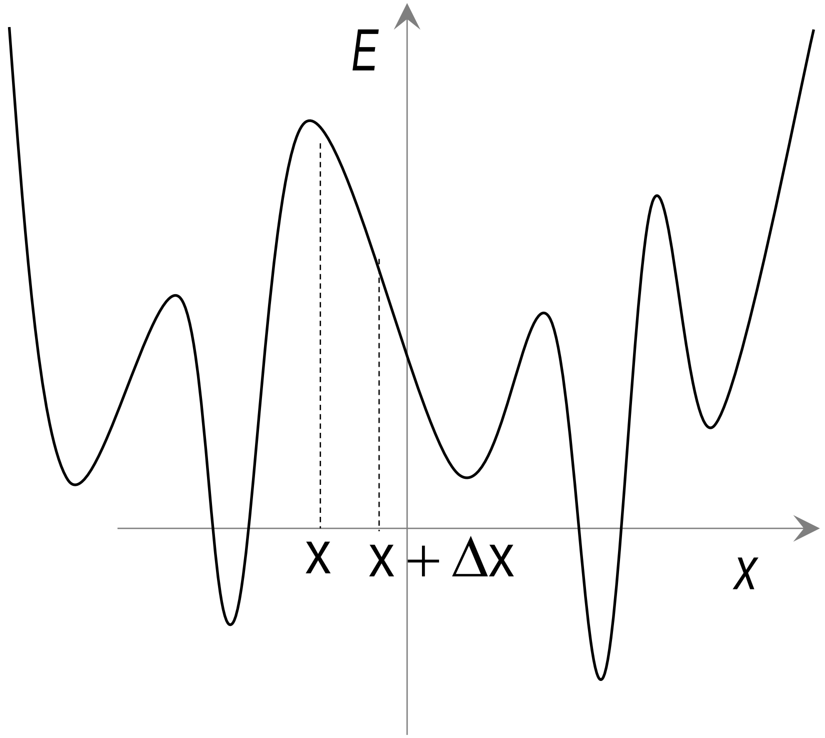
Find x such that the following equation is satisfied:

$$\frac{\partial f(x)}{\partial x}^{T} f(x) = \frac{\partial f(x)}{\partial x}^{T} b$$

How?

Strategy: Given x,

Nonlinear System



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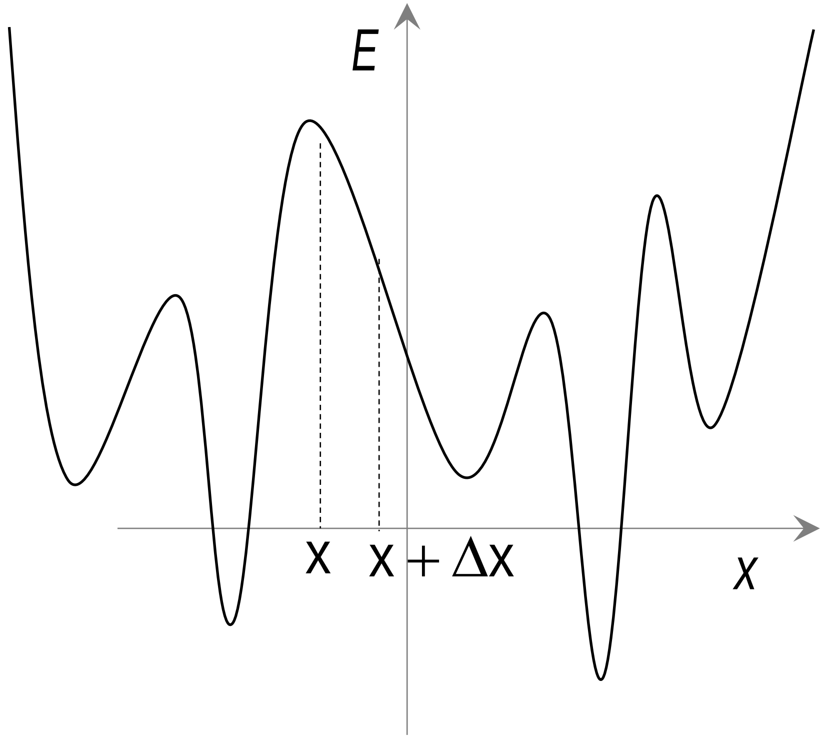
$$\frac{\partial f(x)}{\partial x}^{T} f(x) = \frac{\partial f(x)}{\partial x}^{T} b$$

How?

Strategy: Given x, move Δx such that $E(x + \Delta x) \leq E(x)$

Nonlinear System



Find x such that the following equation is satisfied:

$$\frac{\partial f(\mathrm{x})}{\partial \mathrm{x}}^{T} f(\mathrm{x}) = \frac{\partial f(\mathrm{x})}{\partial \mathrm{x}}^{T} \mathrm{b}$$

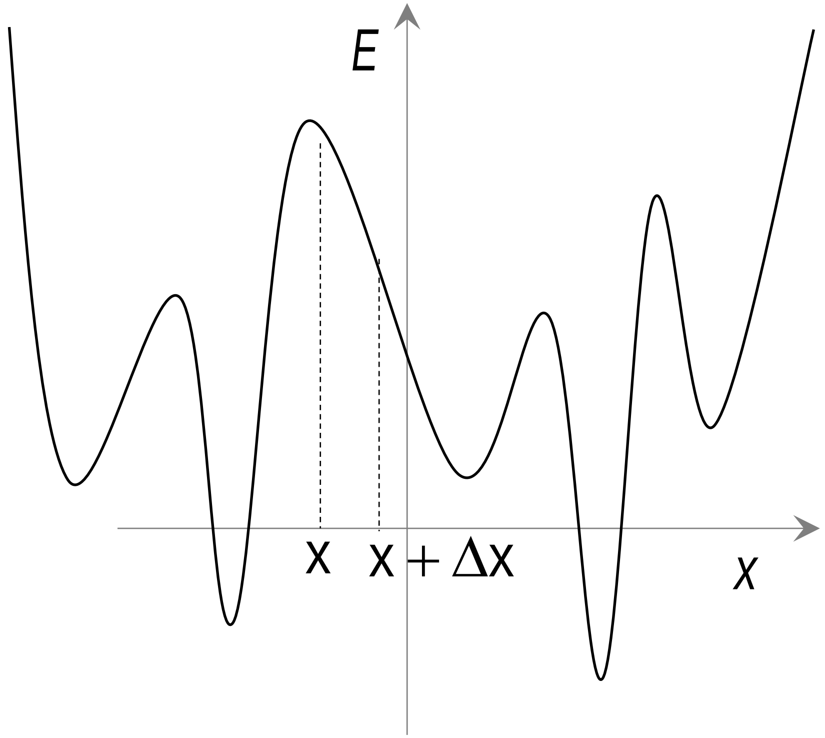
How?

Strategy: Given x, move $\Delta \mathrm{x}$ such that $E(\mathrm{x}+\Delta \mathrm{x}) \leq E(\mathrm{x})$

Taylor expansion:

$$f(\mathrm{x}+\Delta \mathrm{x}) =$$

Nonlinear System



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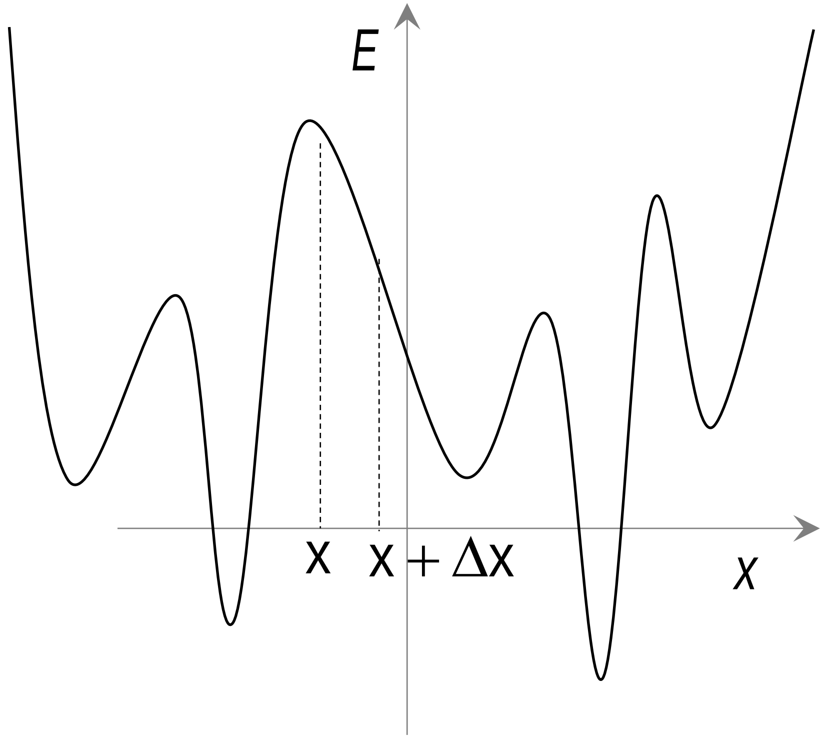
Strategy: Given x, move Δx such that $E(x+\Delta x) \le E(x)$

Taylor expansion:

$$f(x+\Delta x) = f(x+\Delta x) + \frac{\partial f(x)}{\partial x}\Delta x + \text{H.O.T.}$$

Nonlinear System



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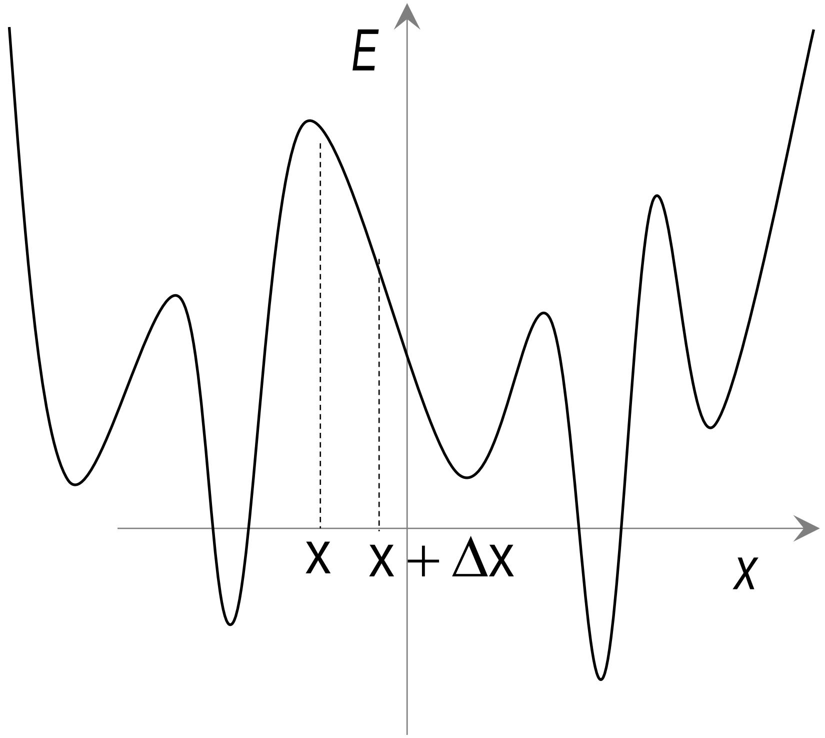
Nonlinear System



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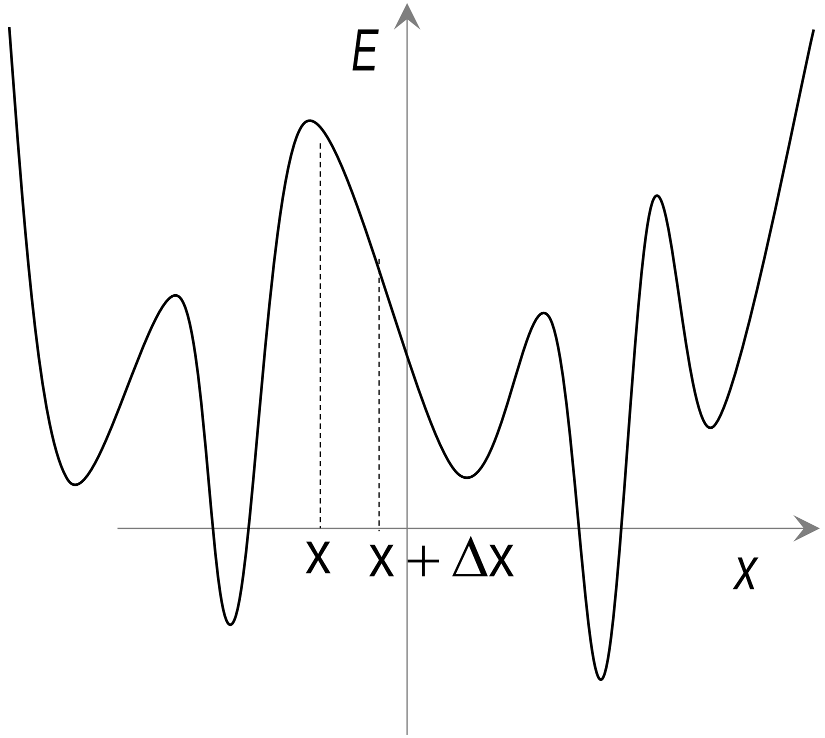
Nonlinear System



Cf.) $x = \left[A^T A \right]^{-1} A^T b$

Find x such that the following equation is satisfied:

$$\frac{\partial f(x)}{\partial x}^T f(x) = \frac{\partial f(x)}{\partial x}^T b \qquad \text{How?}$$

Strategy: Given x, move Δx such that $E(x+\Delta x) \leq E(x)$

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$$f(x+\Delta x) = f(x+\Delta x) + \frac{\partial f(x)}{\partial x}\Delta x + \text{H.O.T.}$$

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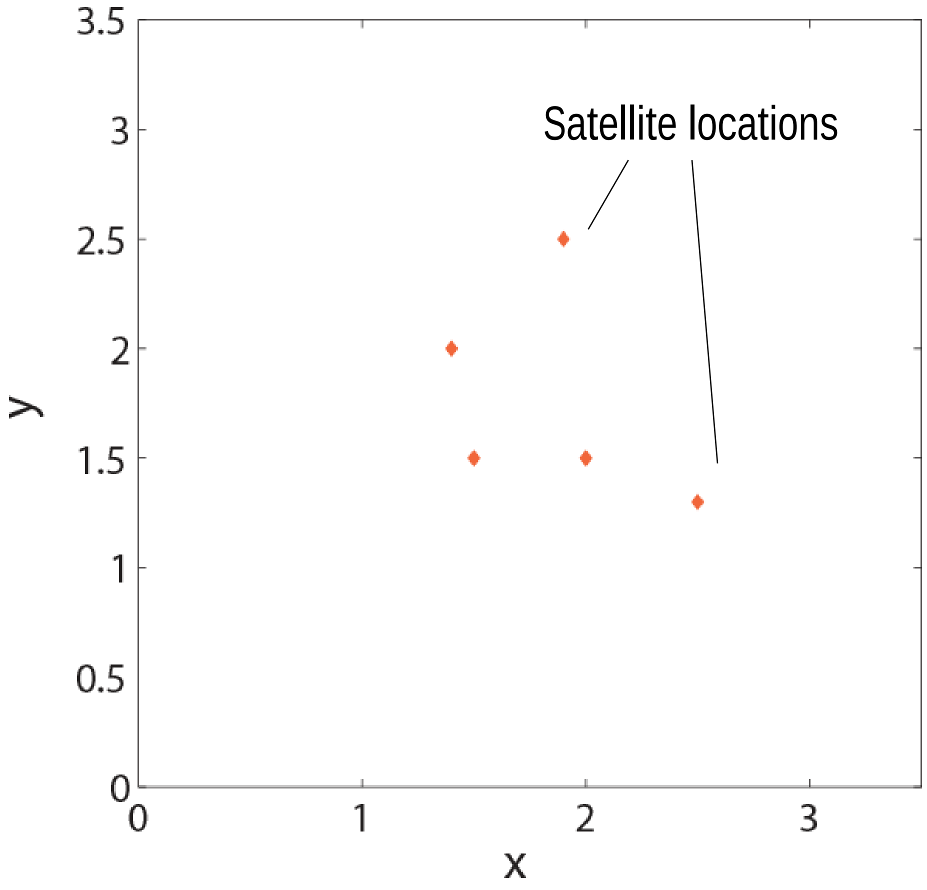
$$\rightarrow \frac{\partial f(x)}{\partial x}^T \frac{\partial f(x)}{\partial x}\Delta x = \frac{\partial f(x)}{\partial x}^T \left(b - f(x) \right)$$

Where am I?: GPS Localization

Example:
Localization using range data from beacons



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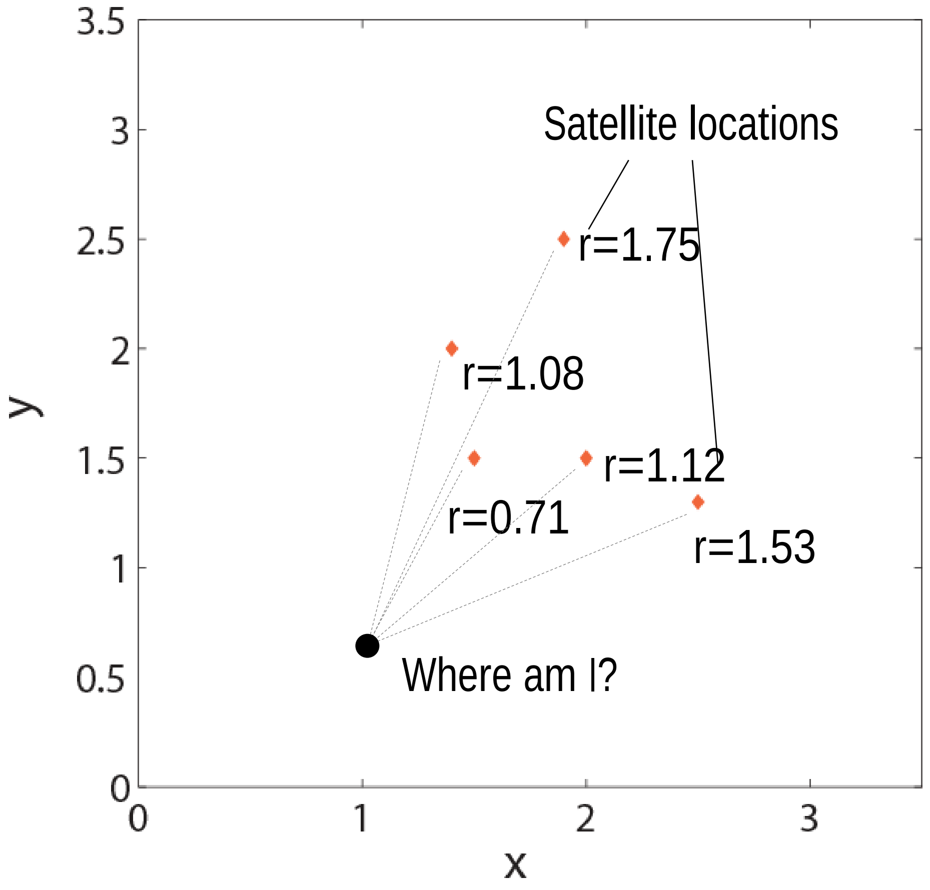

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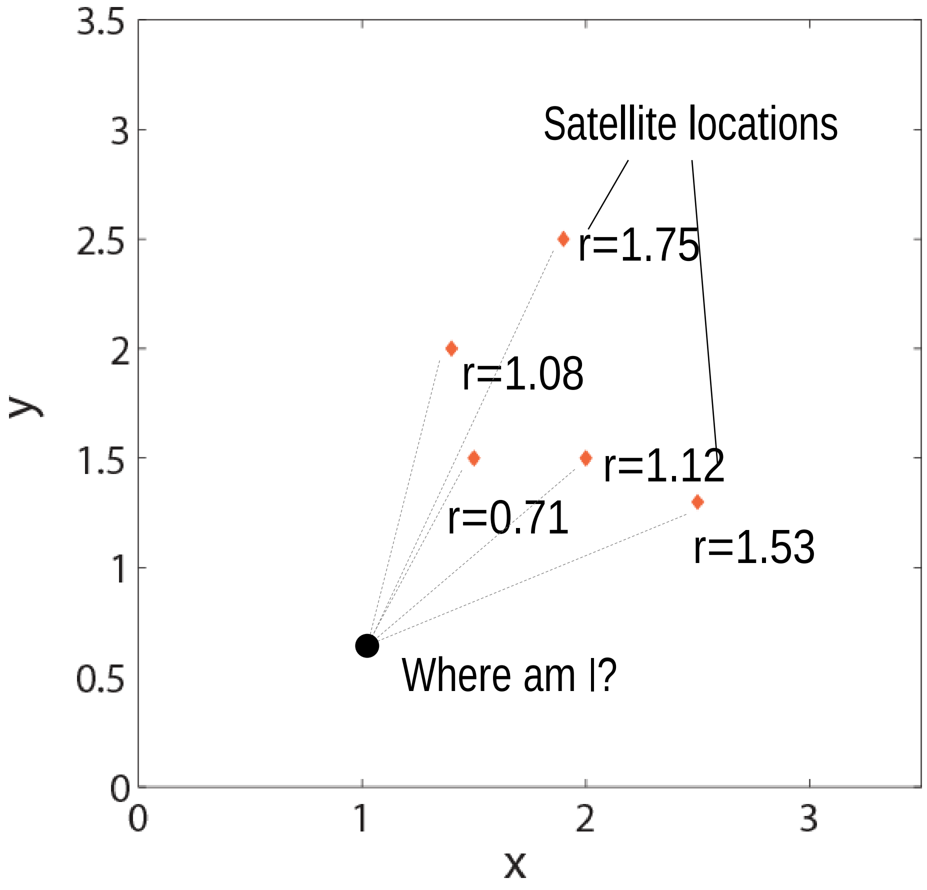

$E =$

Where am I?: GPS Localization

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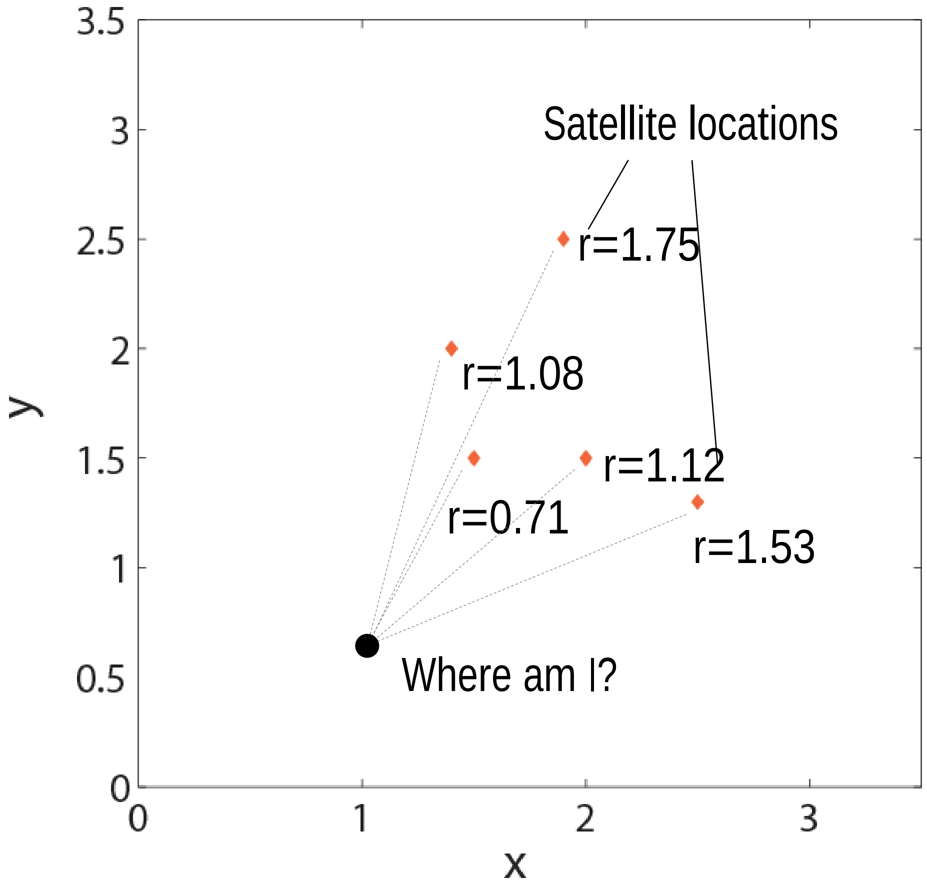
Localization using range data from beacons



$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

Where am I?: GPS Localization

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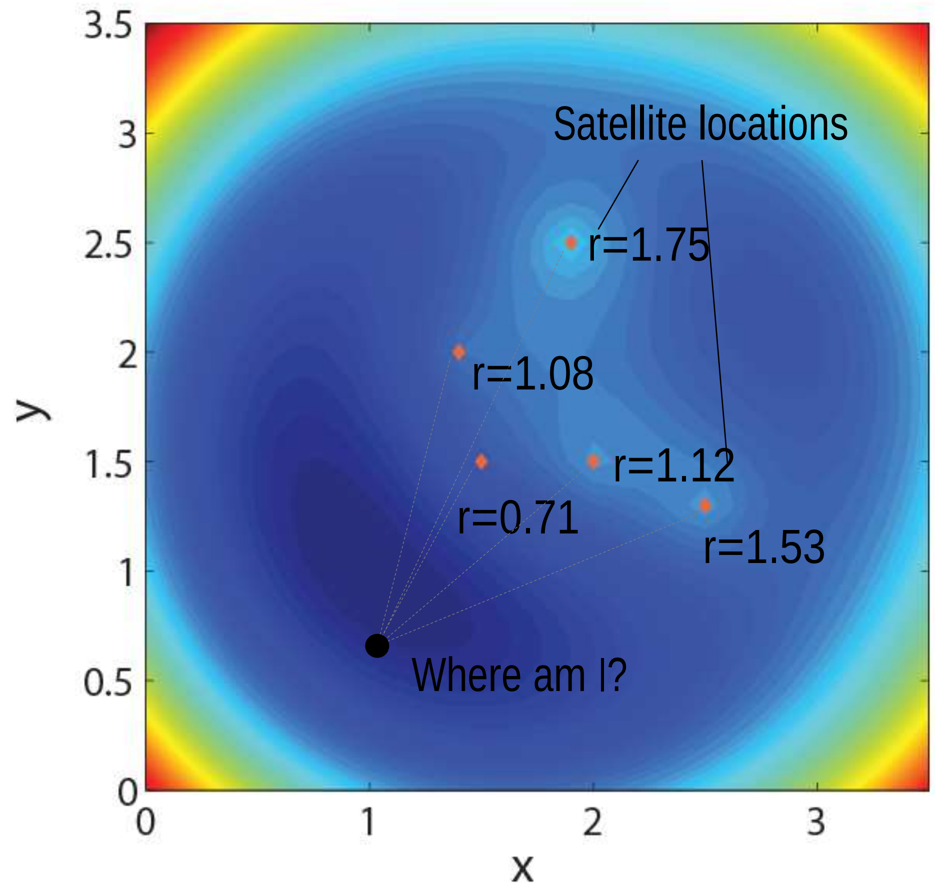
Localization using range data from beacons



$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(x)} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{b} \right\|^2$$

Where am I?: GPS Localization

Example:
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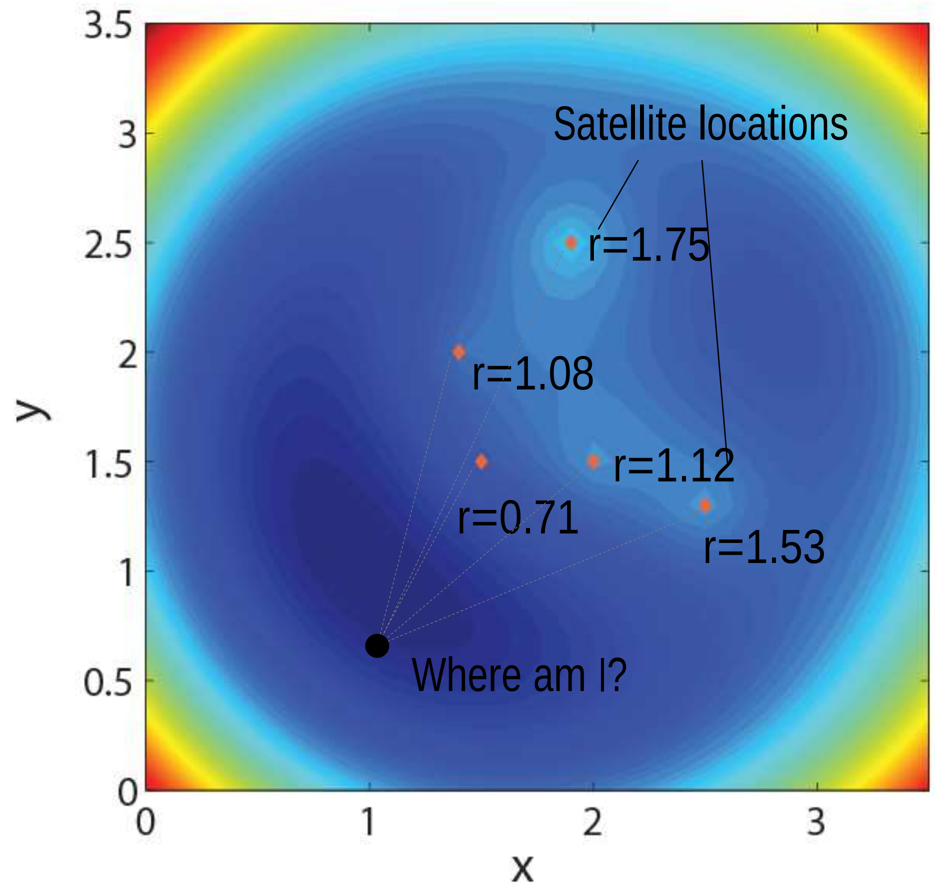

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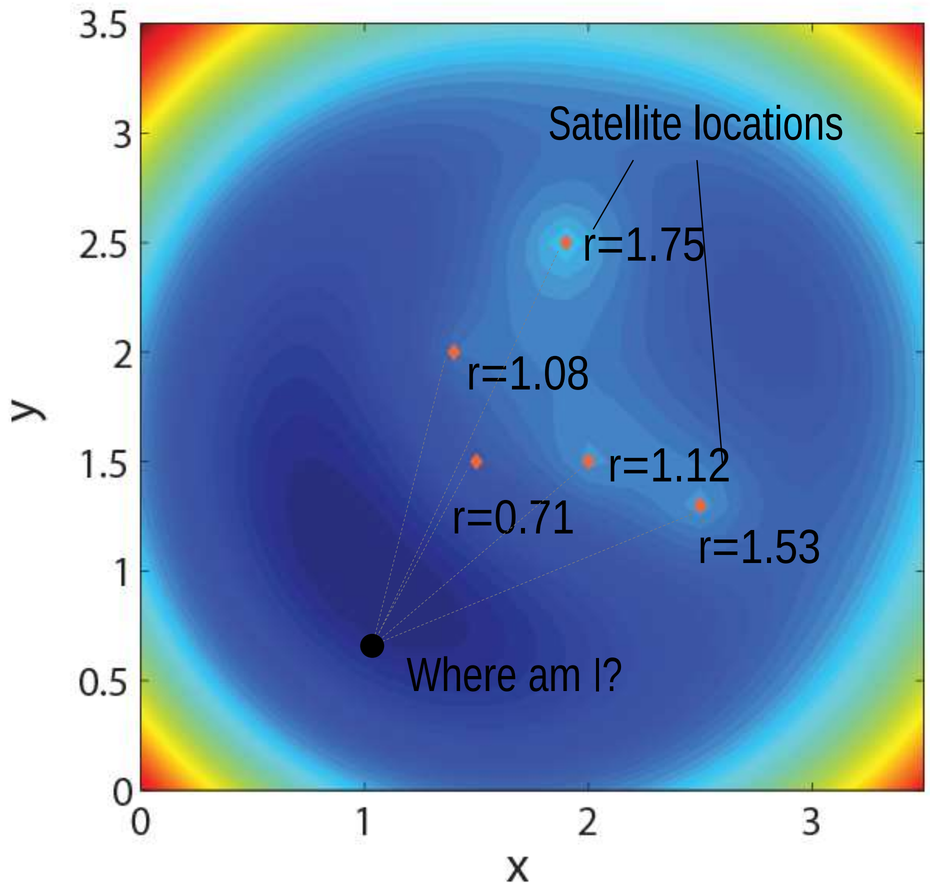
$$\frac{\partial f(x)}{\partial x}^{T} \frac{\partial f(x)}{\partial x} \Delta x = \frac{\partial f(x)}{\partial x}^{T} \left(b - f(x) \right)$$

where $$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$: Jacobian

Where am I?: GPS Localization

Example:
Localization using range data from beacons



$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathrm{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathrm{b}} \right\|^2$$

$$\frac{\partial f(\mathrm{x})}{\partial \mathrm{x}}^{\mathsf{T}} \frac{\partial f(\mathrm{x})}{\partial \mathrm{x}} \Delta \mathrm{x} = \frac{\partial f(\mathrm{x})}{\partial \mathrm{x}}^{\mathsf{T}} \left(\mathrm{b} - f(\mathrm{x})\right)$$

$$\frac{\partial \mathrm{f}(\mathrm{x})}{\partial \mathrm{x}} = \begin{bmatrix} -\frac{u_1 - x}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} & -\frac{v_1 - y}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} \\ \vdots & \vdots \\ -\frac{u_5 - x}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} & -\frac{v_5 - y}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} \end{bmatrix}$$

Where am I?: GPS Localization

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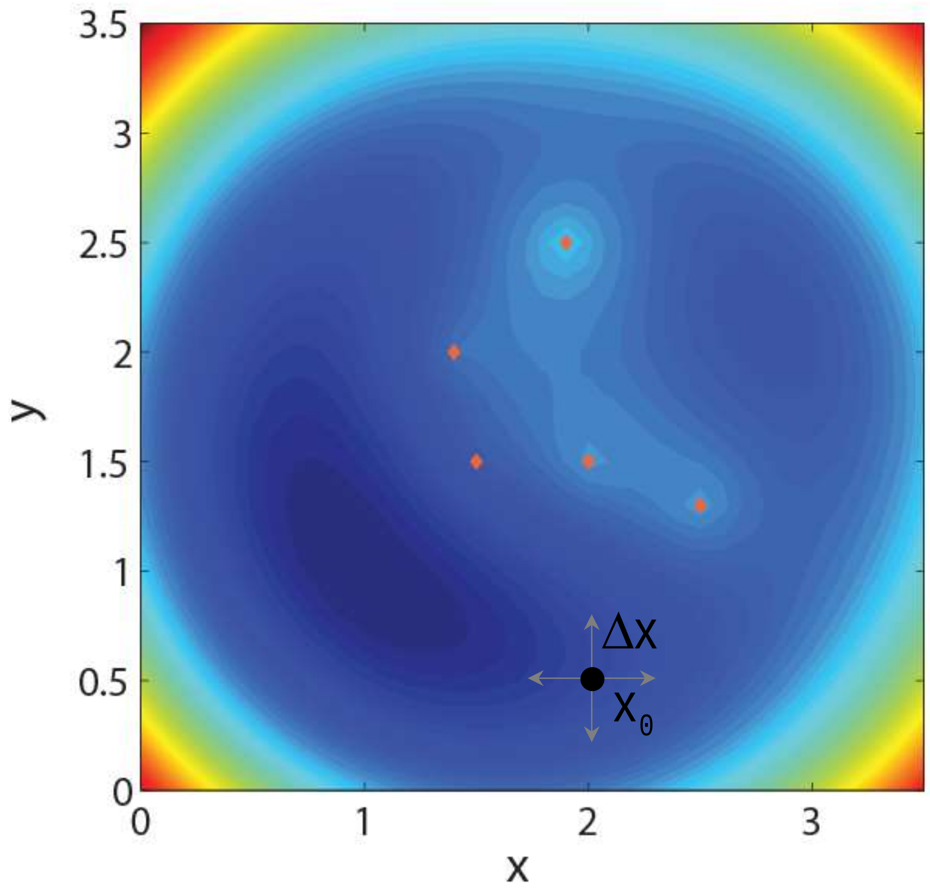

$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathrm{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathrm{b}} \right\|^2$$

$$\frac{\partial f(\mathrm{x})^T}{\partial \mathrm{x}} \frac{\partial f(\mathrm{x})}{\partial \mathrm{x}} \Delta\mathrm{x} = \frac{\partial f(\mathrm{x})^T}{\partial \mathrm{x}} \left(\mathrm{b} - f(\mathrm{x})\right)$$

$$\Delta\mathrm{x} = \left(\frac{\partial f(\mathrm{x})^T}{\partial \mathrm{x}} \frac{\partial f(\mathrm{x})}{\partial \mathrm{x}} \right)^{-1} \frac{\partial f(\mathrm{x})^T}{\partial \mathrm{x}} \left(\mathrm{b} - f(\mathrm{x})\right)$$

Where am I?: GPS Localization

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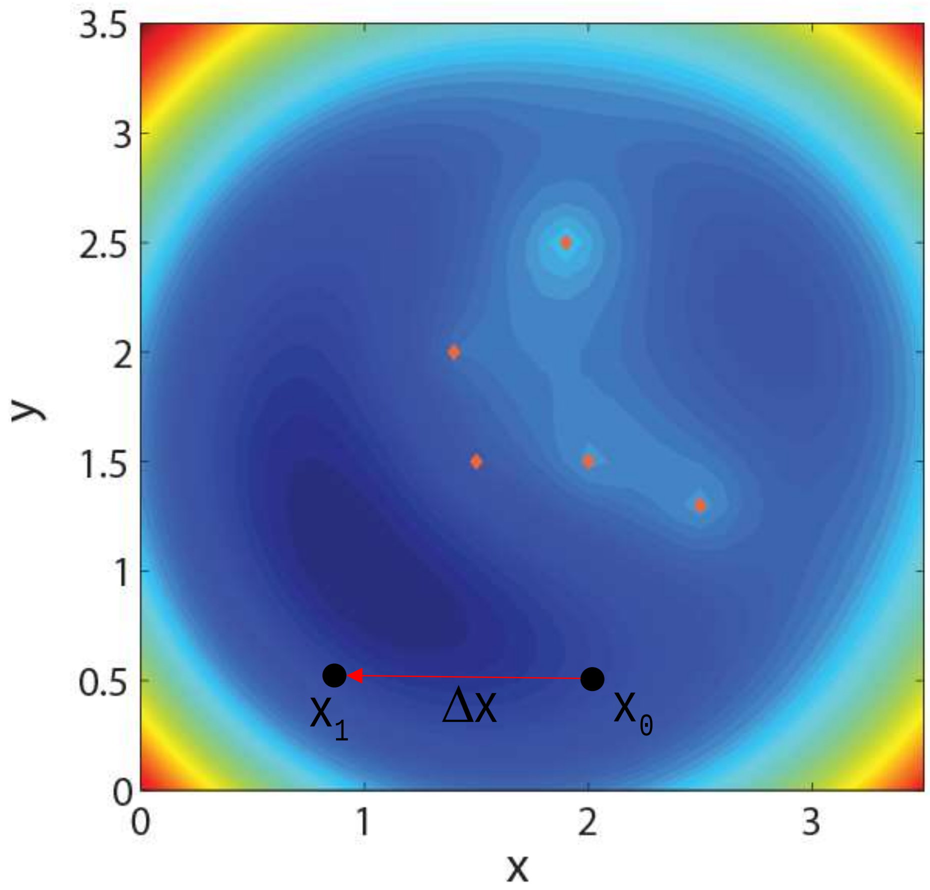
Localization using range data from beacons



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Where am I?: GPS Localization

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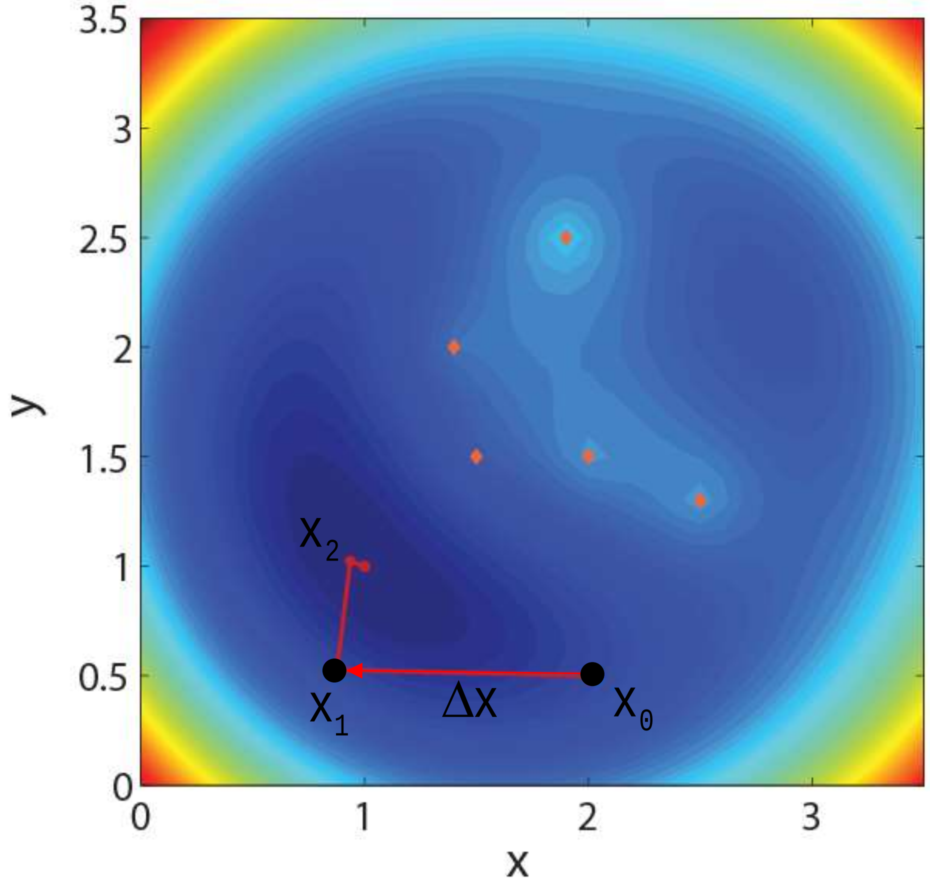

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$$\frac{\partial f(\mathrm{x})^{\mathrm{T}}}{\partial \mathrm{x}} \frac{\partial f(\mathrm{x})}{\partial \mathrm{x}} \Delta \mathrm{x} = \frac{\partial f(\mathrm{x})^{\mathrm{T}}}{\partial \mathrm{x}} \left(\mathrm{b} - f(\mathrm{x})\right)$$

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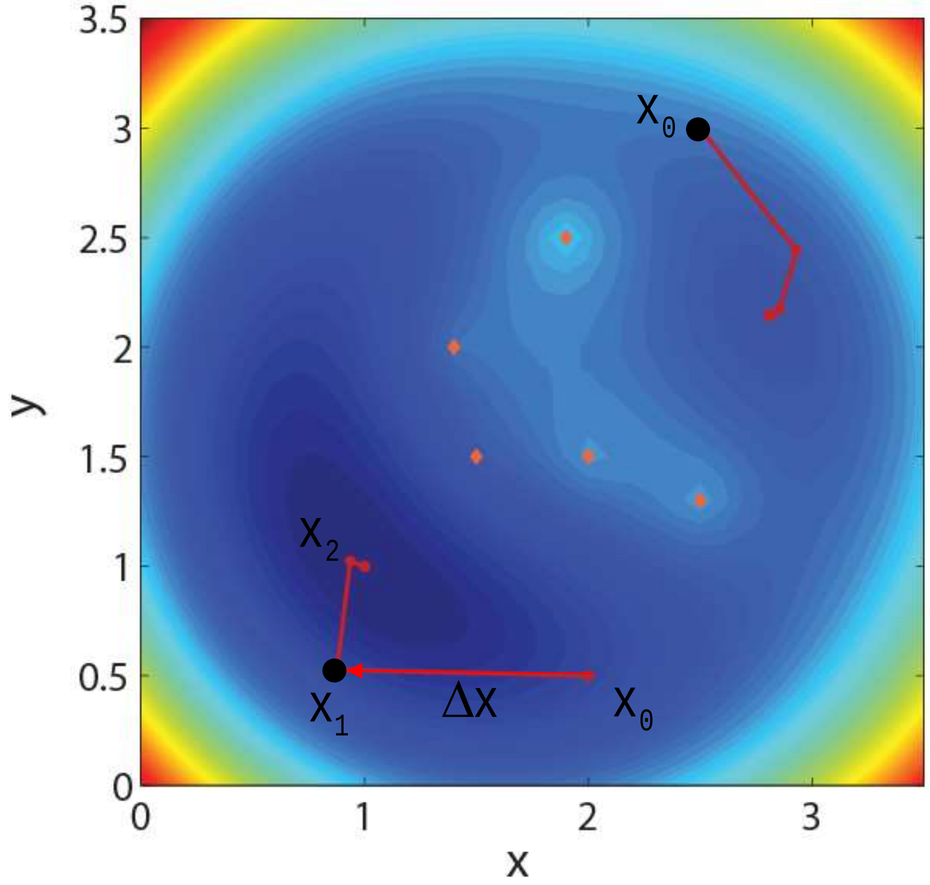
Where am I?: GPS Localization

Example:
Localization using range data from beacons



$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathrm{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathrm{b}} \right\|^2$$

$$\frac{\partial f(\mathrm{x})}{\partial \mathrm{x}}^{\mathsf{T}} \frac{\partial f(\mathrm{x})}{\partial \mathrm{x}} \Delta \mathrm{x} = \frac{\partial f(\mathrm{x})}{\partial \mathrm{x}}^{\mathsf{T}} \left(\mathrm{b} - f(\mathrm{x}) \right)$$

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Where am I?: GPS Localization

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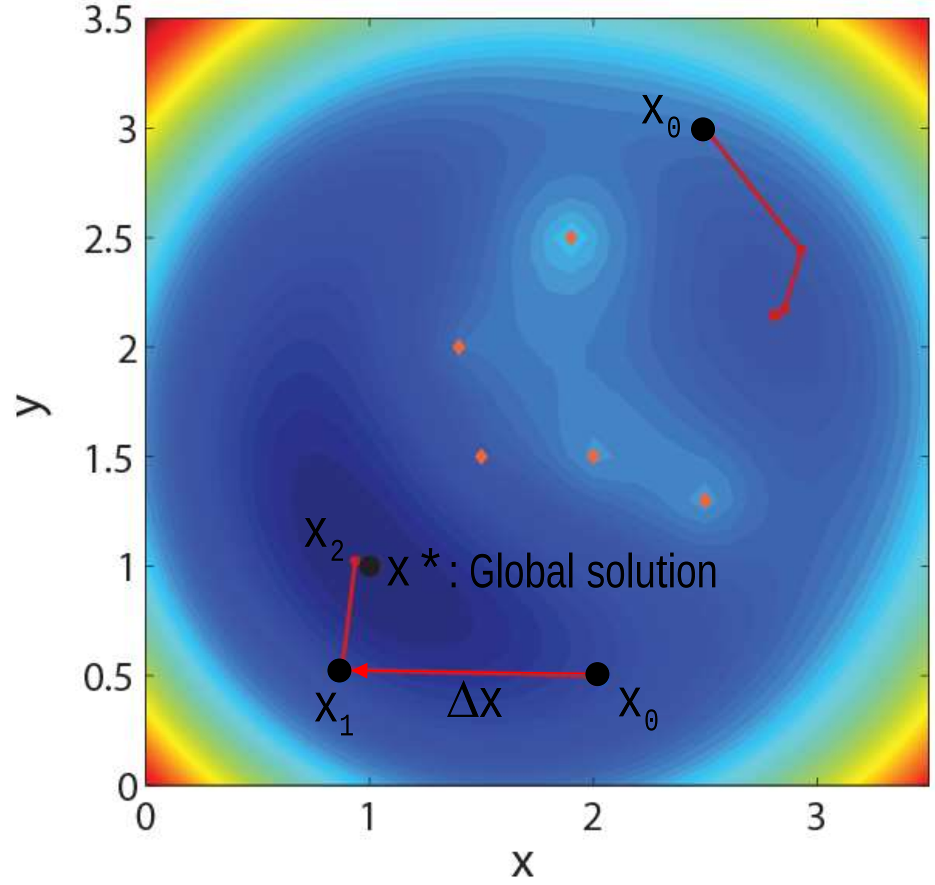

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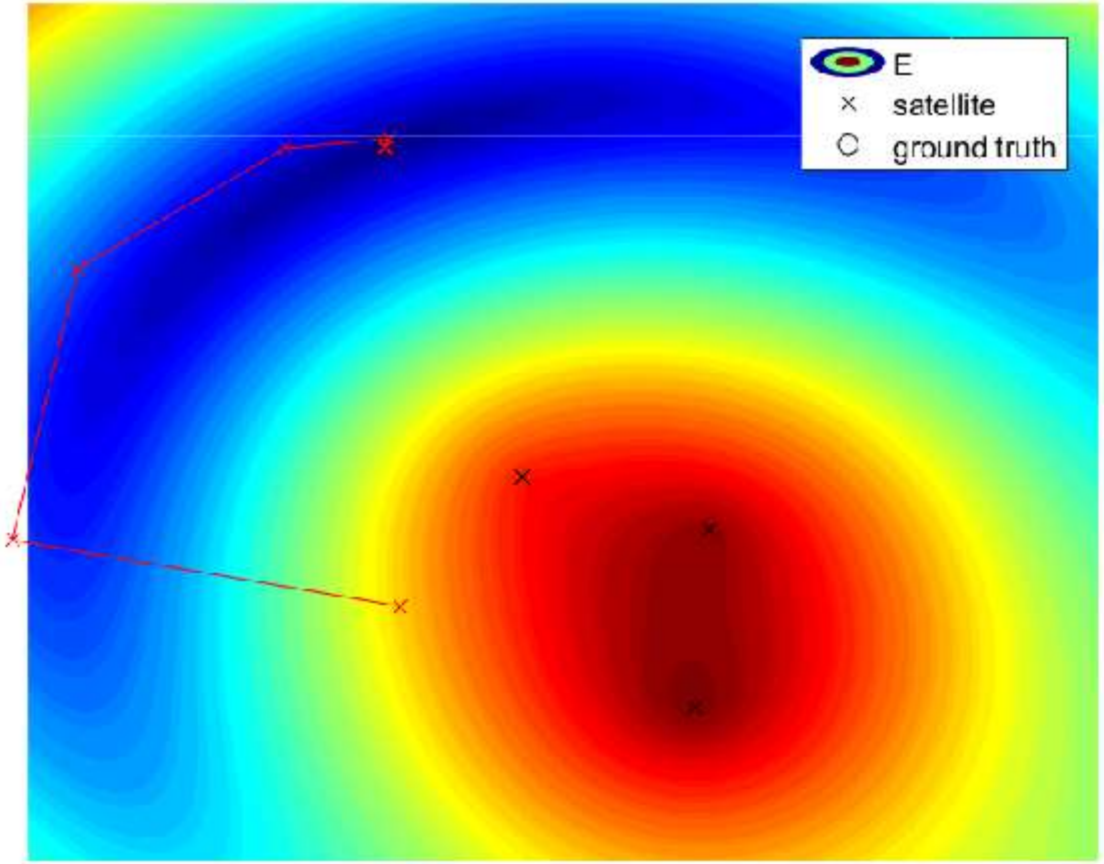

Beacon.m

```
u = 2*(rand(3,2)-0.5);
x = [-0.5 1];

for i = 1 : size(u,1)
   d(i,1) = norm(x-u(i,:));
end

[x_grid, y_grid] = meshgrid(-1.5:0.01:1.5, -1.5:0.01:1.5);

E = zeros(size(x_grid));
for i = 1 : size(u,1)
   E = E + (sqrt((x_grid-u(i,1)).^2 +(y_grid-u(i,2)).^2)-d(i)).^2 ;
end
E = sqrt(E);

x0 = 2*(rand(2,1)-0.5);
for j = 1 : 10
   J = [];
   fx = [];
   for i = 1 : size(u,1)
      denom = norm(u(i,:)-x0');
      J = [J; (u(i,:)-x0')/denom];
      fx = [fx; norm(u(i,:)-x0')];
   end
   b = d;
   delta_x = -inv(J'*J)*J'*(b-fx);
   x0 = x0 + delta_x;
end
```