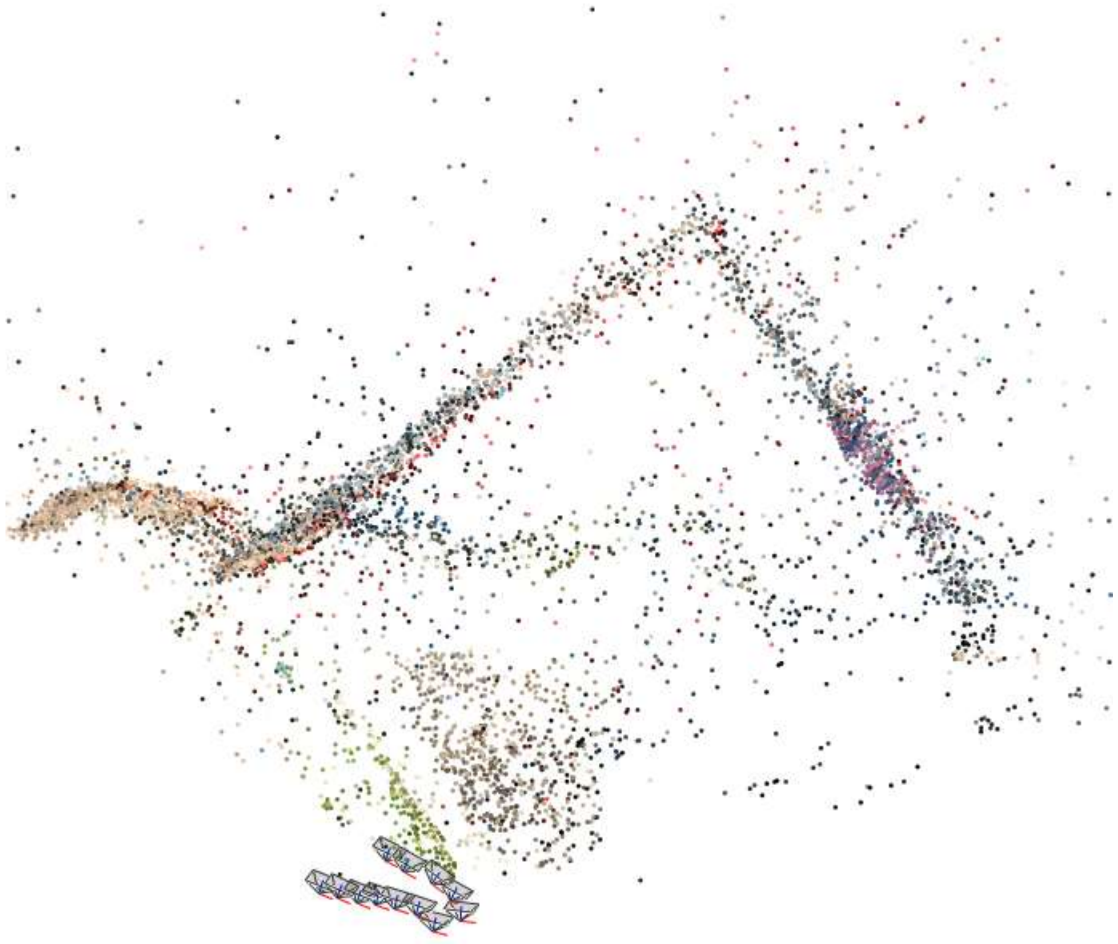
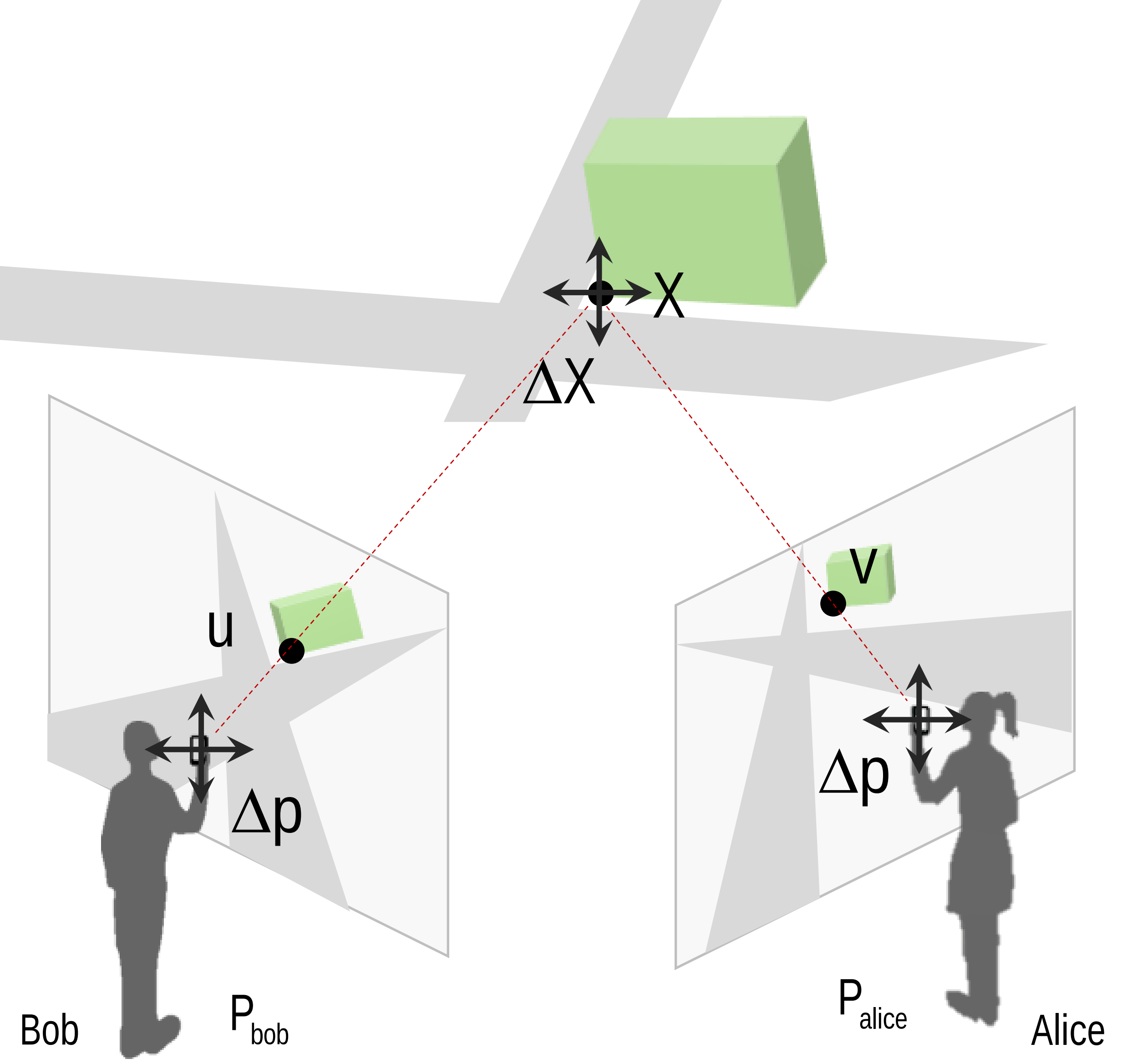


Bundle Adjustment

Black: given variables
Red: unknowns

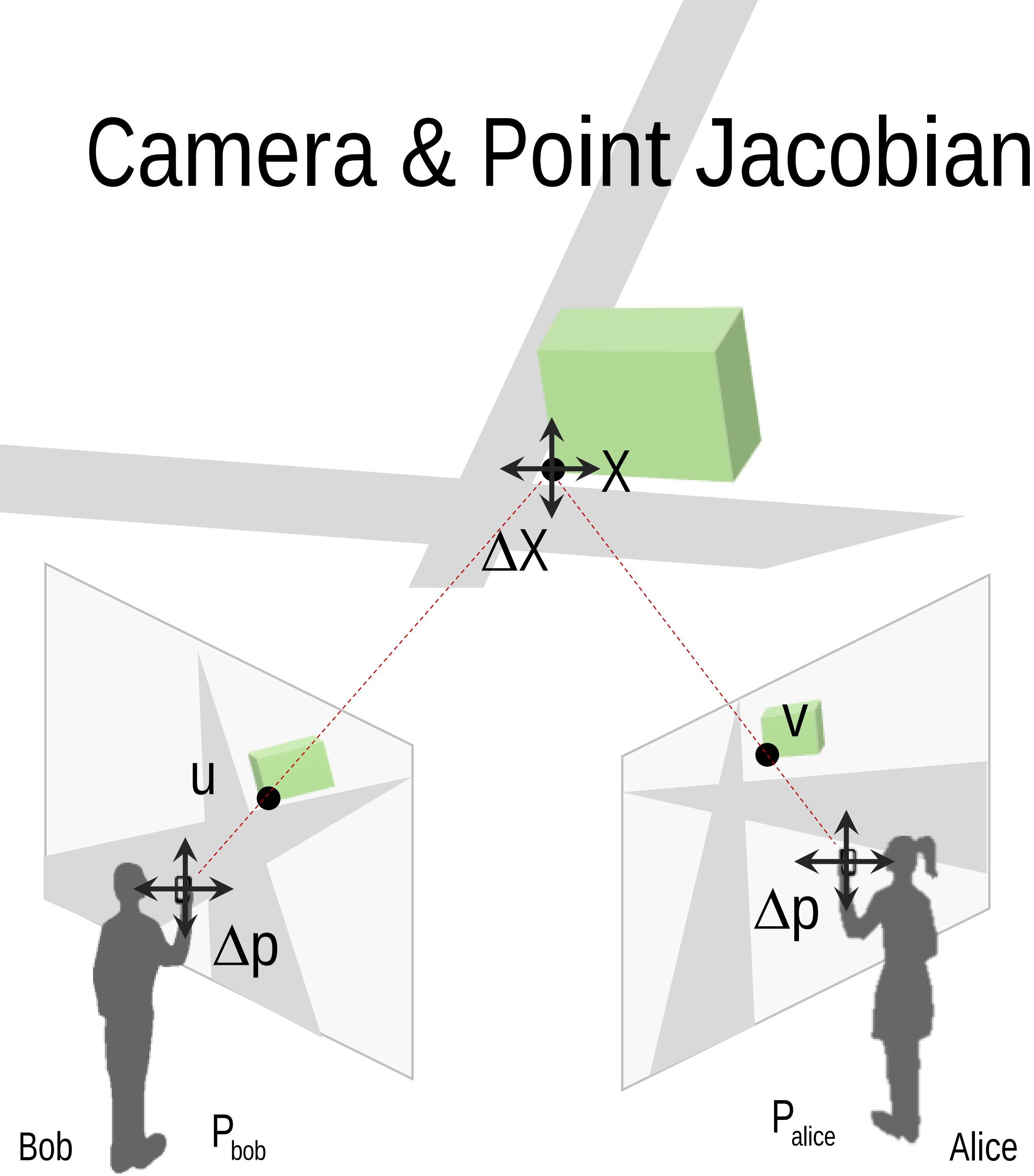
Camera & Point Jacobian

$$E_{geom} = \left\| \hat{u} - u \right\|^2 = \left(\frac{P_1 X}{P_3 X} - x \right)^2 + \left(\frac{P_2 X}{P_3 X} - y \right)^2$$



Black: given variables
Red: unknowns

Camera & Point Jacobian



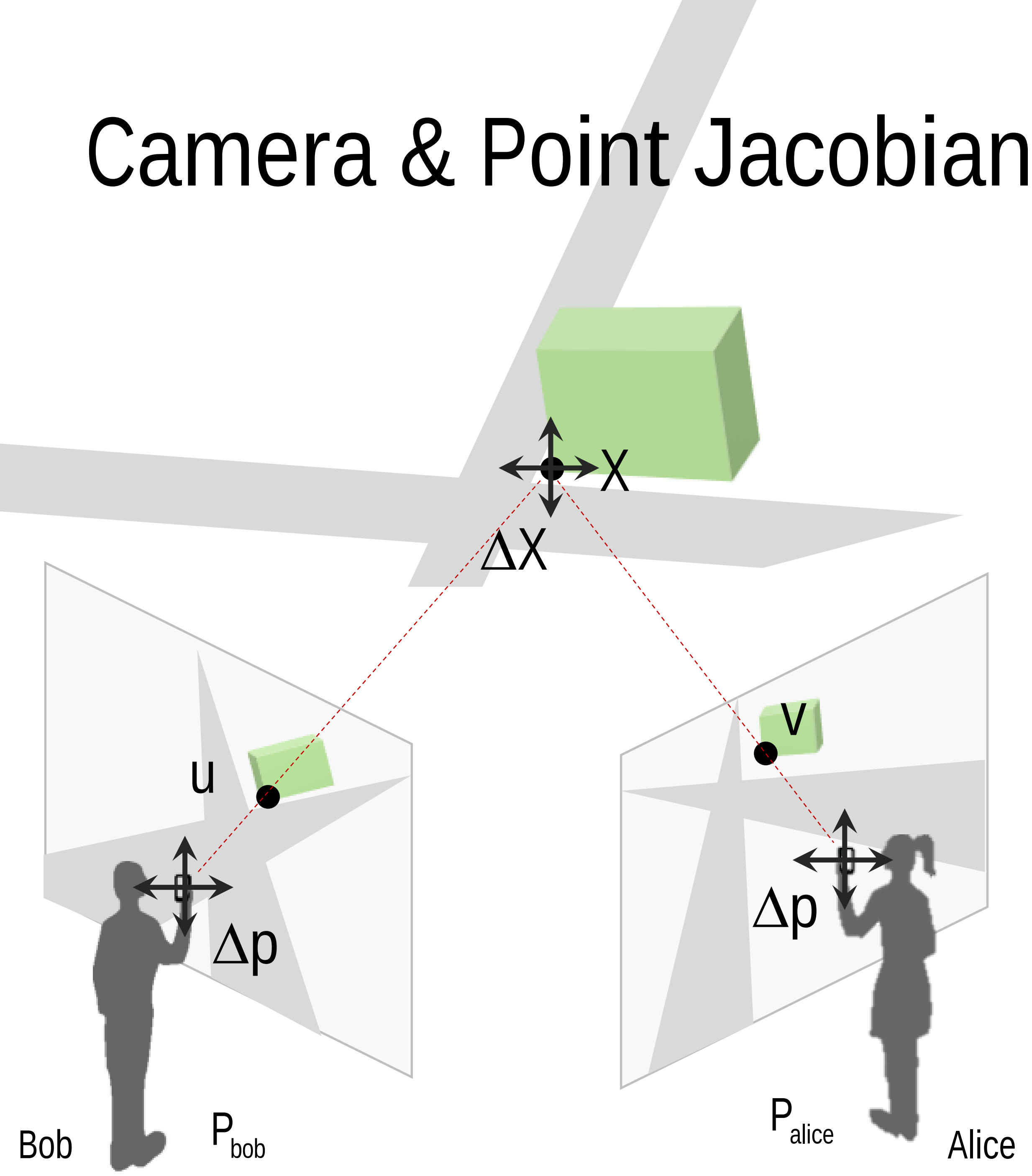
$$E_{geom} = \left\| \hat{u} - u \right\|^2$$

$$= \left(\frac{P_1 X}{P_3 X} - x \right)^2 + \left(\frac{P_2 X}{P_3 X} - y \right)^2$$

$$f(p, X) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix}$$

Camera & Point Jacobian

Black: given variables
Red: unknowns



$$E_{geom} = \left\| \hat{u} - u \right\|^2$$

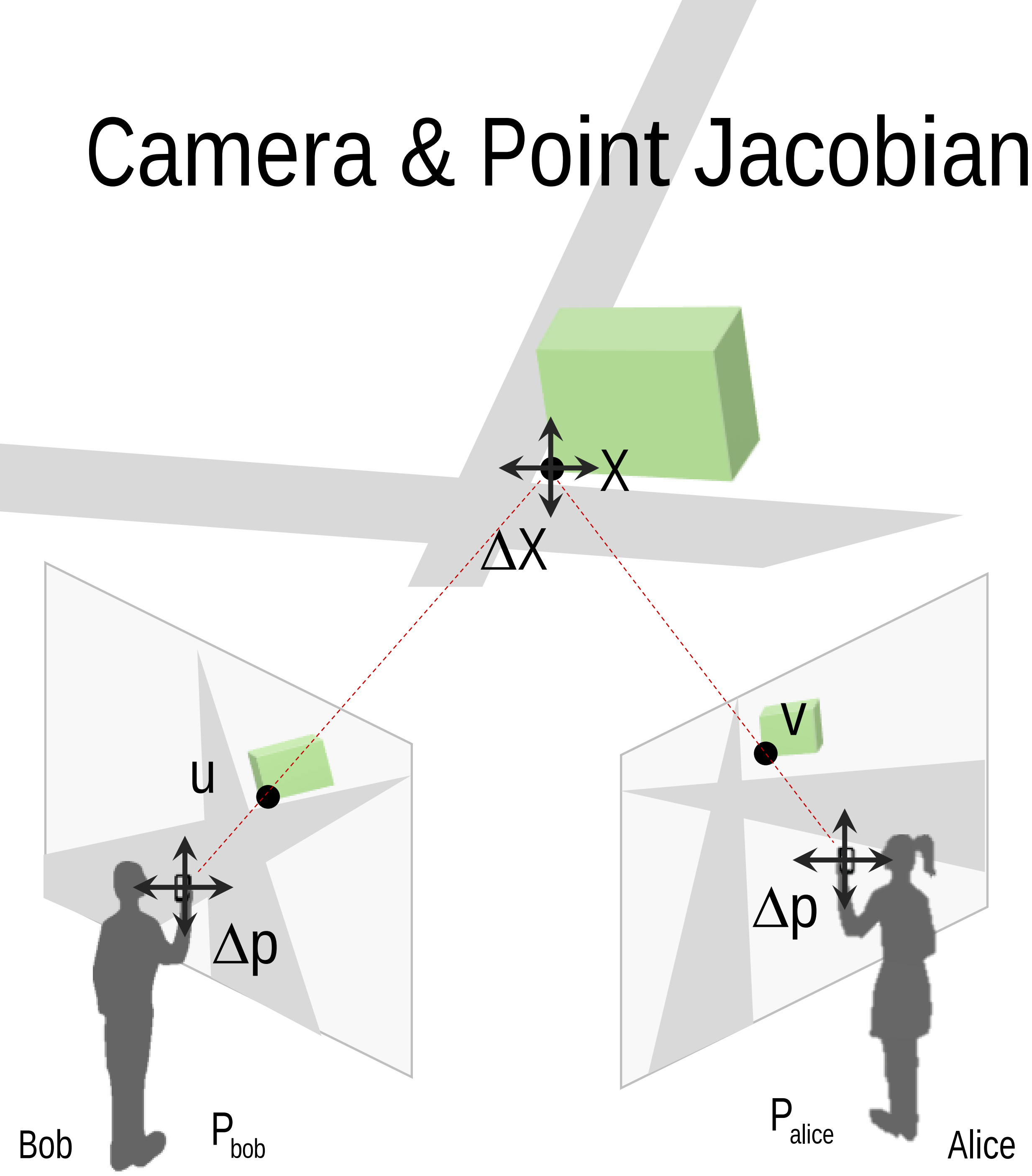
$$= \left(\frac{P_1 X}{P_3 X} - x \right)^2 + \left(\frac{P_2 X}{P_3 X} - y \right)^2$$

$$f(p, X) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} \rightarrow \frac{\partial f(p,X)}{\partial p} = \frac{\partial}{\partial p} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \dfrac{w \frac{\partial u}{\partial p} - u \frac{\partial w}{\partial p}}{w^2} \\ \dfrac{v \frac{\partial u}{\partial p} - v \frac{\partial w}{\partial p}}{w^2} \end{bmatrix}$$

$$\rightarrow \frac{\partial f(p,X)}{\partial X} = \frac{\partial}{\partial X} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \dfrac{w \frac{\partial u}{\partial X} - u \frac{\partial w}{\partial X}}{w^2} \\ \dfrac{v \frac{\partial u}{\partial X} - v \frac{\partial w}{\partial X}}{w^2} \end{bmatrix}$$

Black: given variables
Red: unknowns

Camera & Point Jacobian

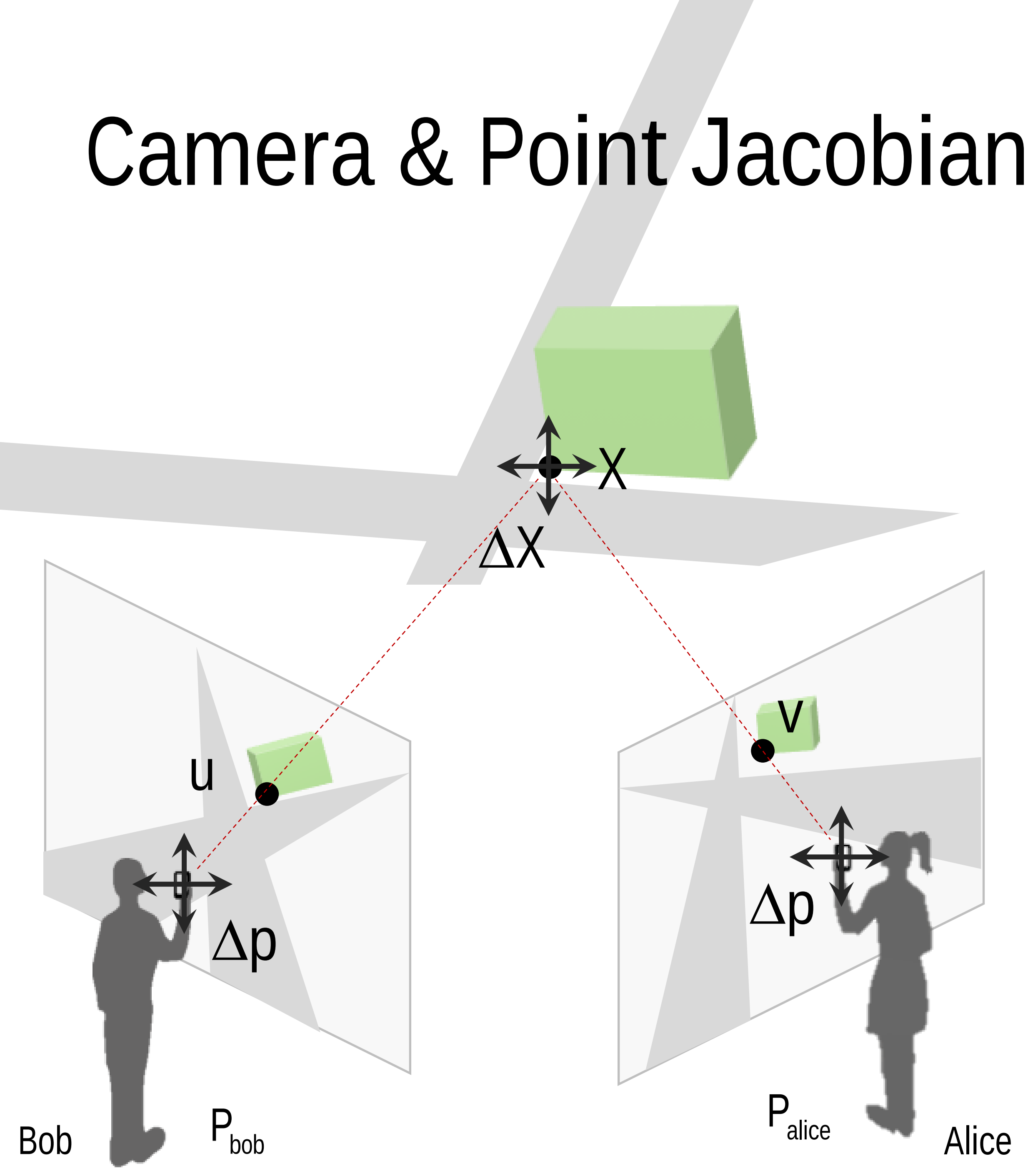


$$E_{\text{geom}} = \left\| \hat{u} - u \right\|^2 = \left(\frac{P_1 X}{P_3 X} - x \right)^2 + \left(\frac{P_2 X}{P_3 X} - y \right)^2$$

$$f(p, X) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} \longrightarrow \frac{\partial f(p, X)}{\partial p} = \frac{\partial}{\partial p} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \dfrac{w \frac{\partial u}{\partial p} - u \frac{\partial w}{\partial p}}{w^2} \\ \dfrac{v \frac{\partial u}{\partial p} - v \frac{\partial w}{\partial p}}{w^2} \end{bmatrix}$$

Black: given variables
Red: unknowns

Camera & Point Jacobian



$$E_{geom} = \left\| \hat{u} - u \right\|^2$$

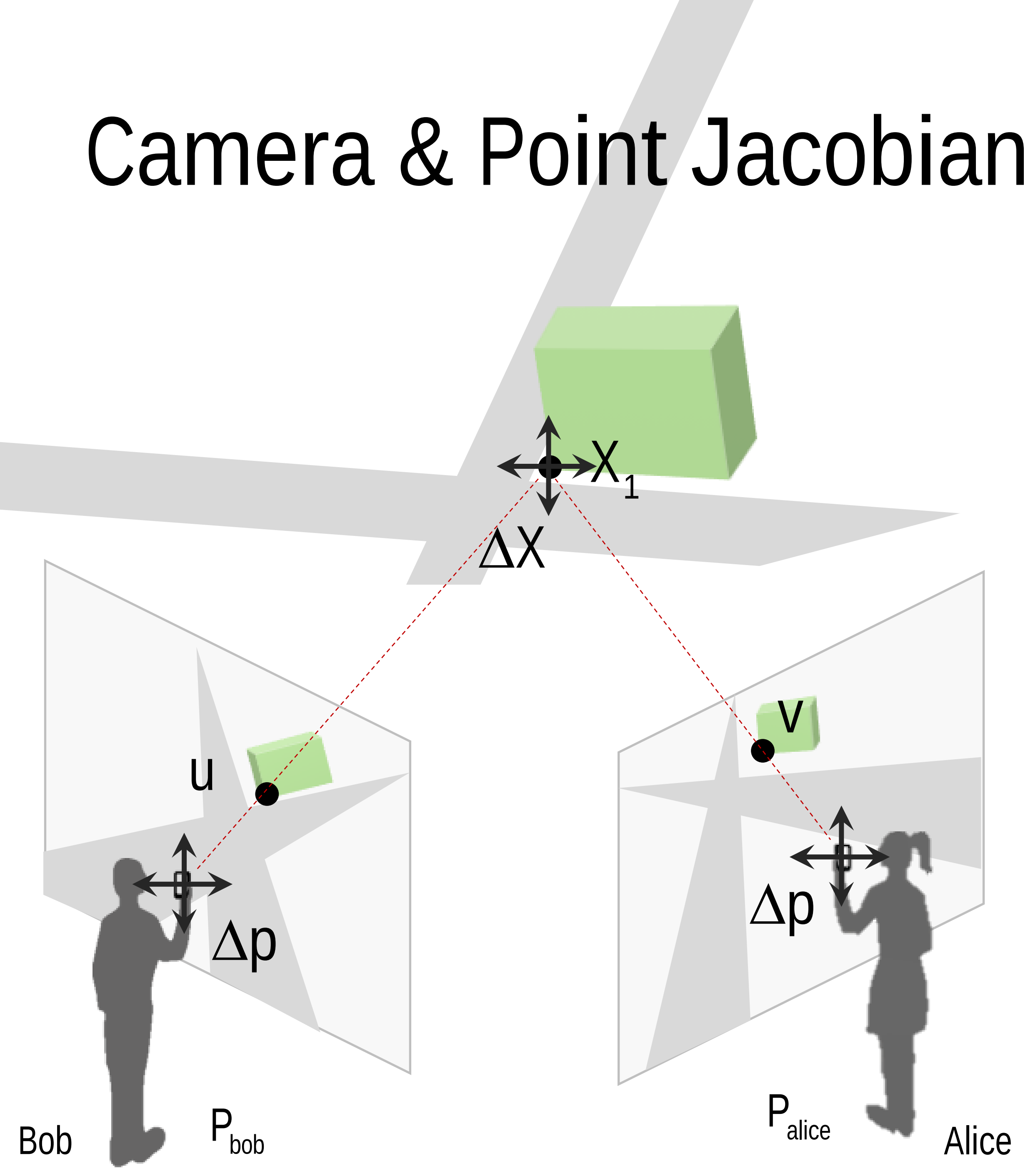
$$= \left(\frac{P_1 X}{P_3 X} - x \right)^2 + \left(\frac{P_2 X}{P_3 X} - y \right)^2$$

$$f(p, X) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} \longrightarrow \frac{\partial f(p,X)}{\partial p} = \frac{\partial}{\partial p} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \dfrac{w\frac{\partial u}{\partial p} - u\frac{\partial w}{\partial p}}{w^2} \\ \dfrac{v\frac{\partial u}{\partial p} - v\frac{\partial w}{\partial p}}{w^2} \end{bmatrix}$$

$$\longrightarrow \frac{\partial f(p,X)}{\partial X} = \frac{\partial}{\partial X} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \dfrac{w\frac{\partial u}{\partial X} - u\frac{\partial w}{\partial X}}{w^2} \\ \dfrac{v\frac{\partial u}{\partial X} - v\frac{\partial w}{\partial X}}{w^2} \end{bmatrix}$$

Black: given variables
Red: unknowns

Camera & Point Jacobian

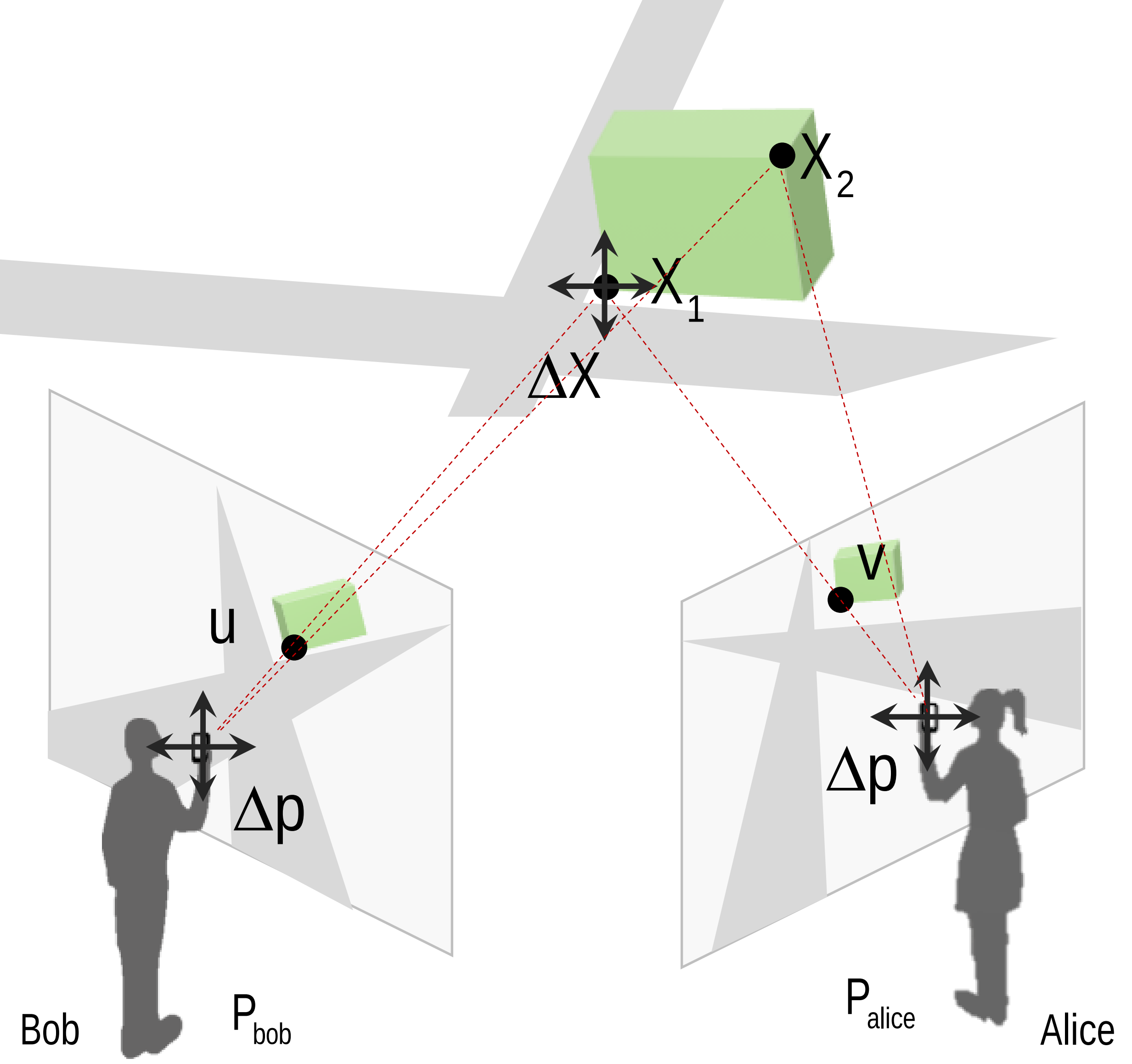


$$J_{ij} = \begin{bmatrix} \frac{\partial f(p_j, X_i)}{\partial p_j} & \frac{\partial f(p_j, X_i)}{\partial X_i} \end{bmatrix} = \begin{bmatrix} J_{pij} & J_{Xij} \end{bmatrix}$$

$$J = \begin{bmatrix} J_{p1,bob} & 0_{2\times7} & J_{X1,bob} \\ 0_{2\times7} & J_{p1,alice} & J_{X1,alice} \end{bmatrix}$$

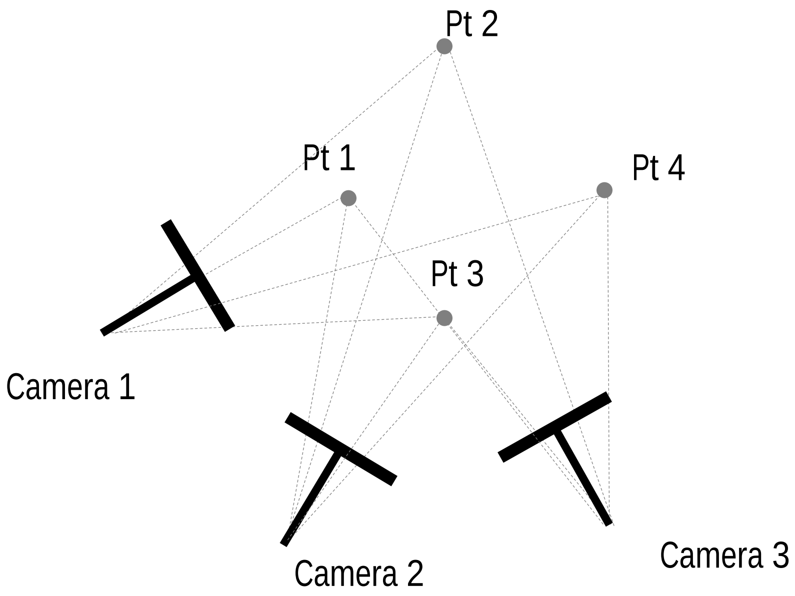
Black: given variables
Red: unknowns

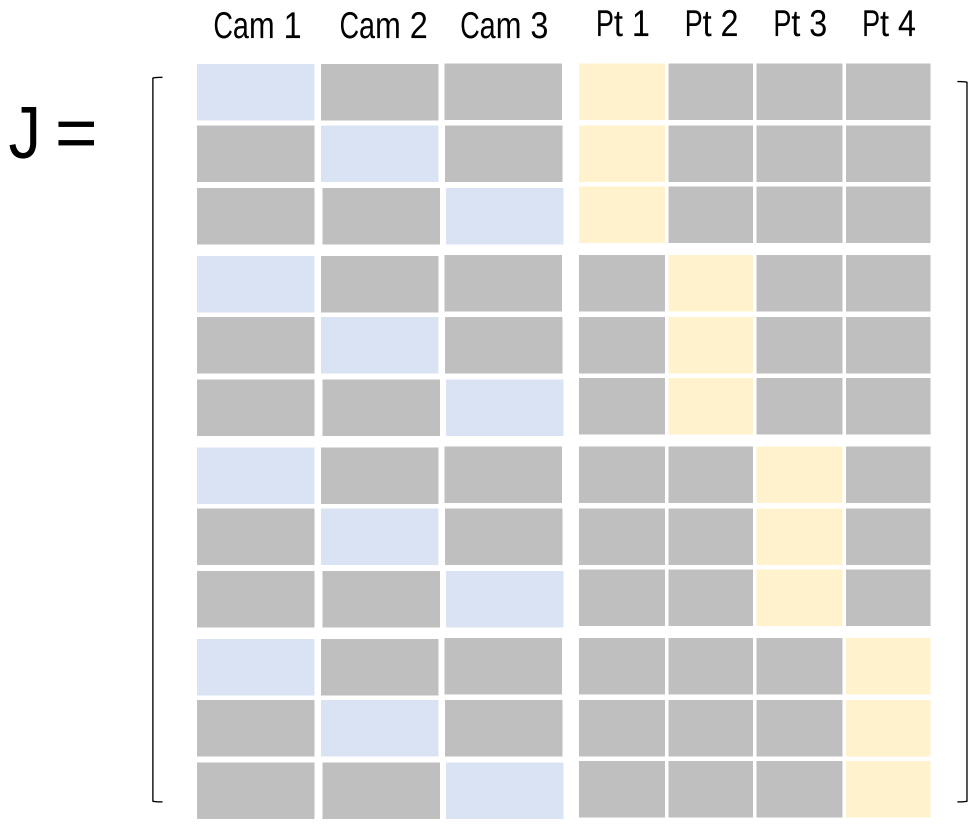
Camera & Point Jacobian



$$J_{ij} = \begin{bmatrix} \dfrac{\partial f(p_j, X_i)}{\partial p_j} & \dfrac{\partial f(p_j, X_i)}{\partial X_i} \end{bmatrix} = \begin{bmatrix} J_{pij} & J_{Xij} \end{bmatrix}$$

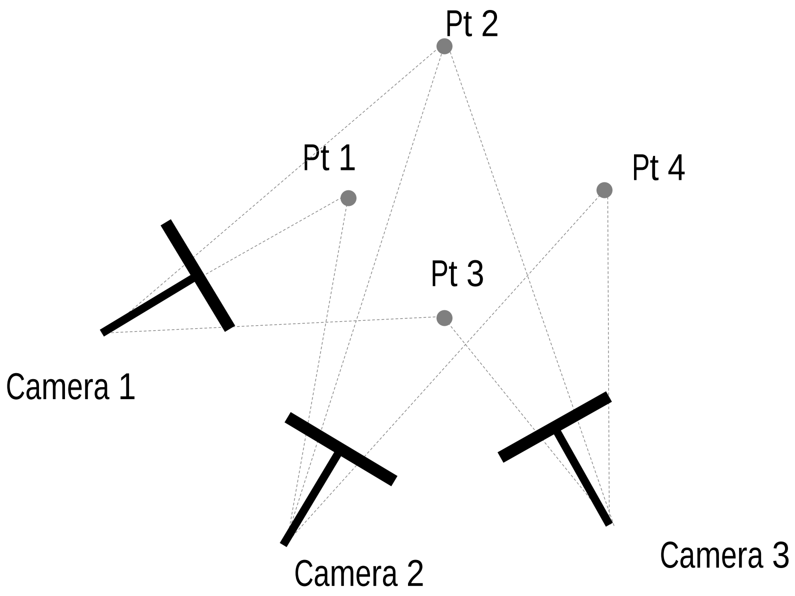
$$J = \begin{matrix} J_{p1,bob} & 0_{2\times7} & J_{X1,bob} & 0_{2\times3} \\ 0_{2\times7} & J_{p1,alice} & J_{X1,alice} & 0_{2\times3} \\ J_{p2,bob} & 0_{2\times7} & 0_{2\times3} & J_{X2,bob} \\ 0_{2\times7} & J_{p2,alice} & 0_{2\times3} & J_{X2,alice} \end{matrix}$$

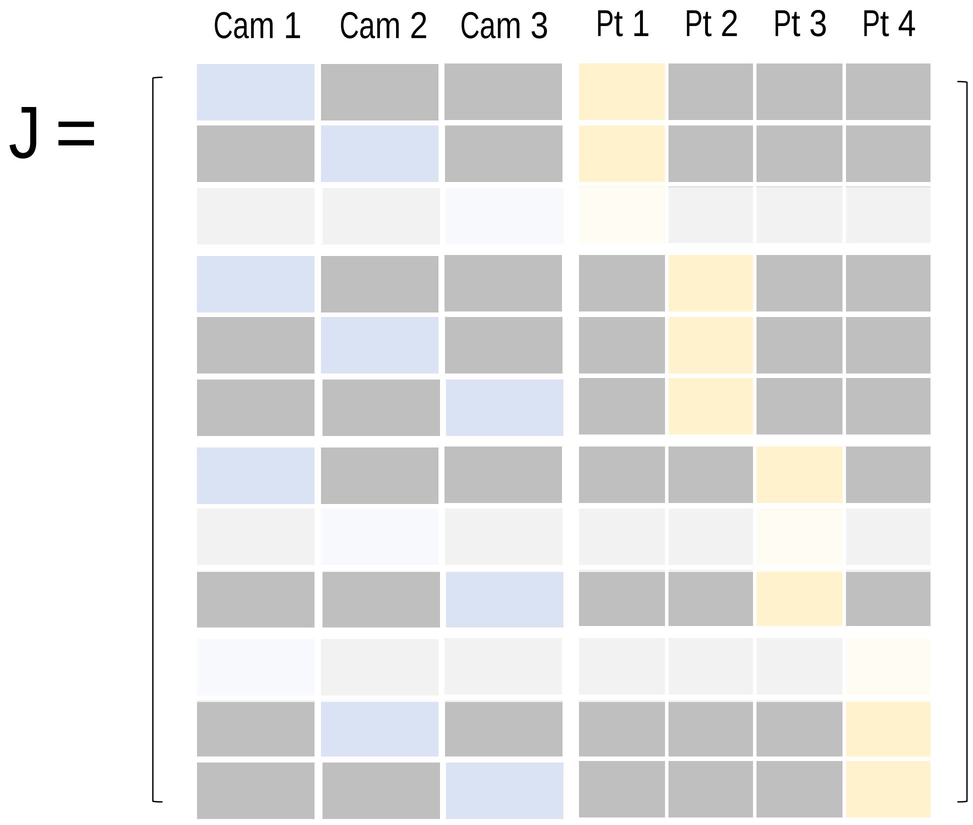




# of unknowns: 3x7+4x3

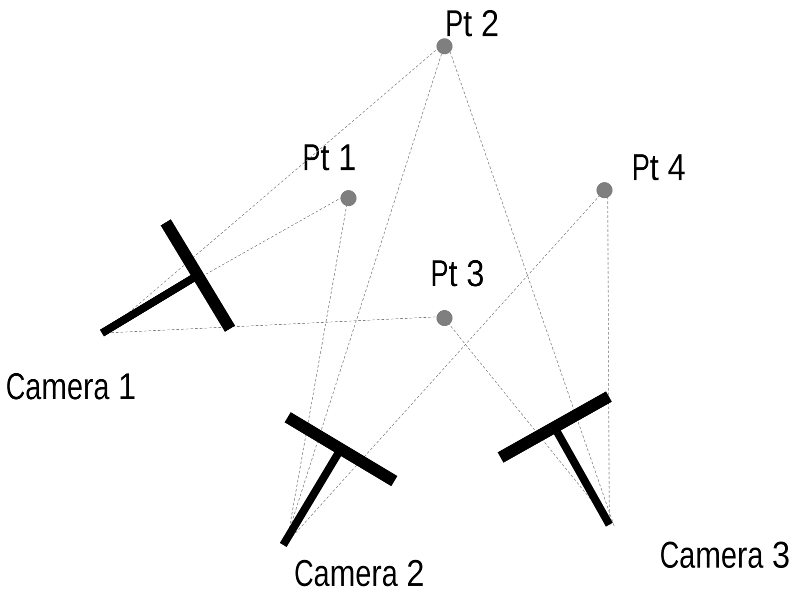
# of projections: 3x4

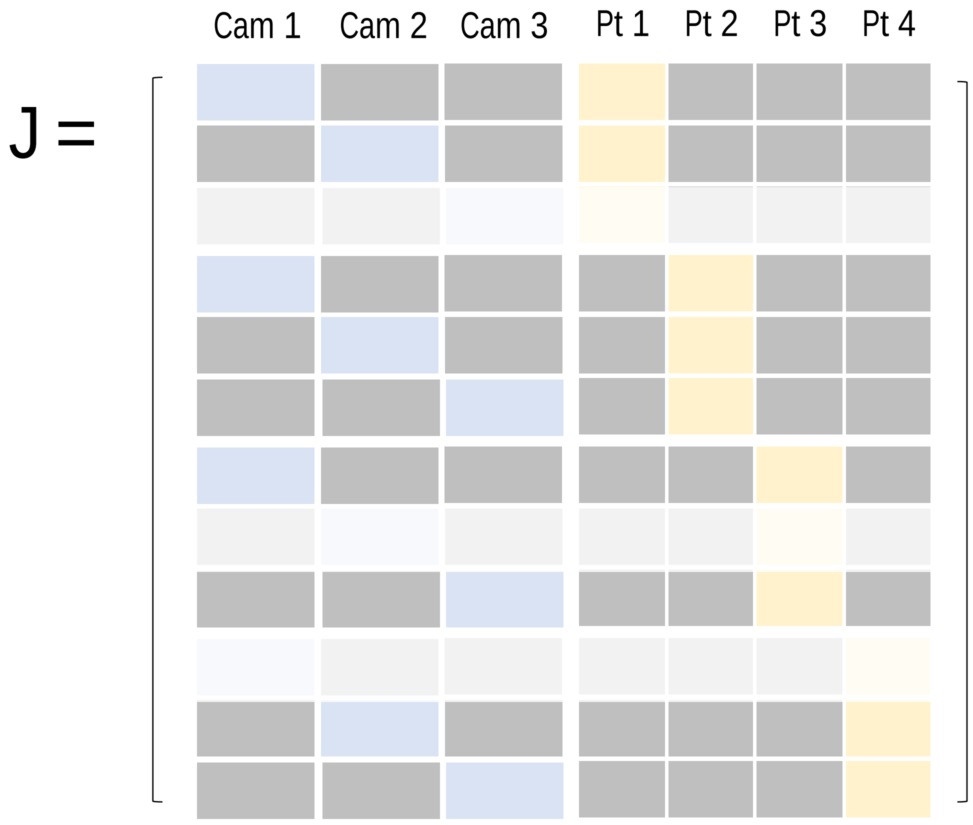




# of unknowns: 3x7+4x3

# of projections: 9 (not all points are visible from cameras)



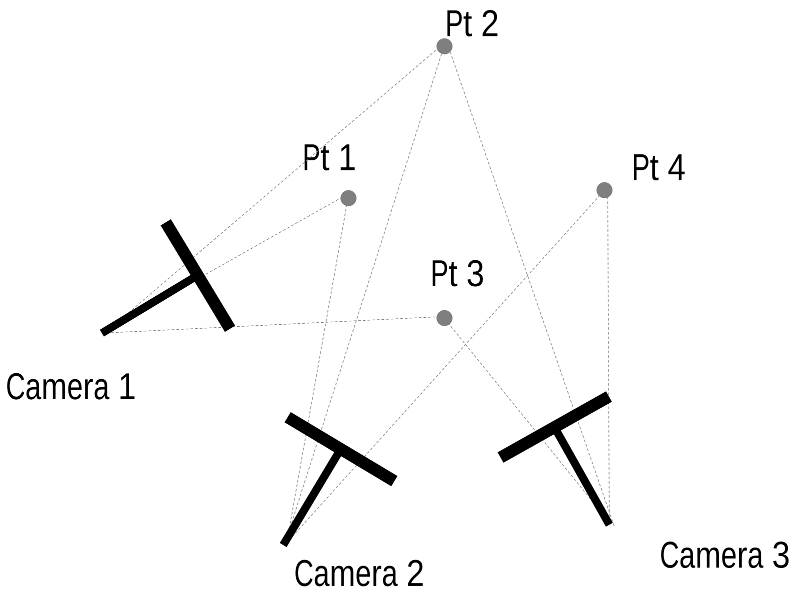


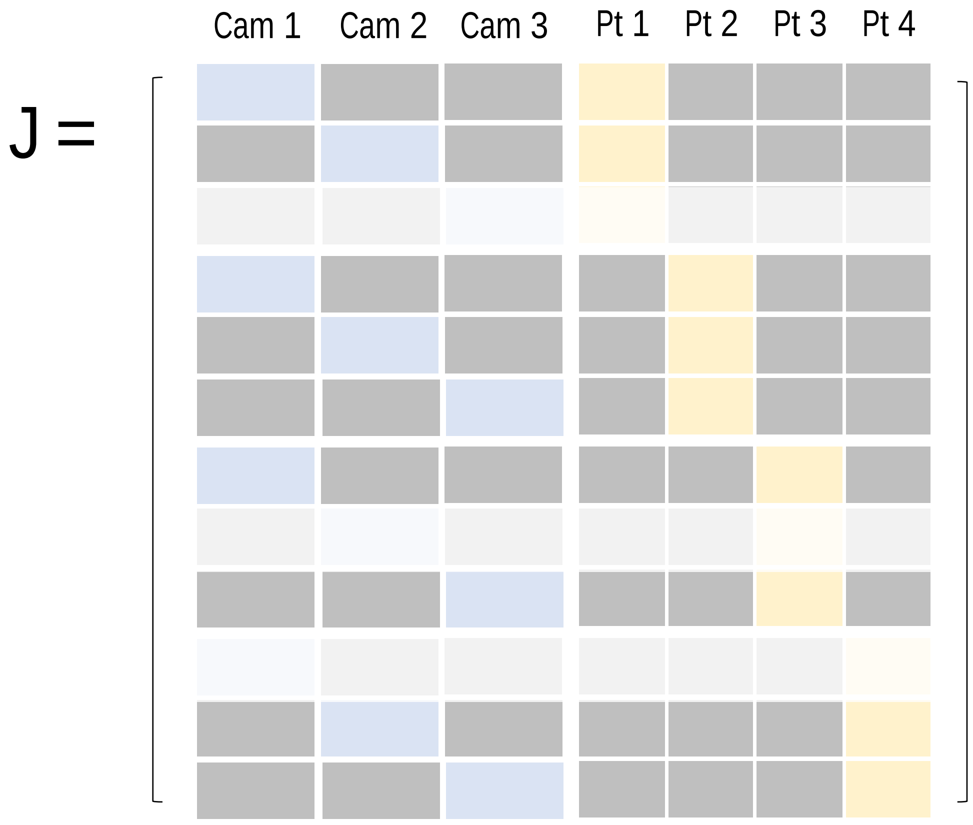
# of unknowns: 3x7+4x3

# of projections: 9 (not all points are visible from cameras)

$$\Delta x = \left(J^T J\right)^{-1} J^T \left(b - f(x)\right)$$

size of $J^T J$: 24x24





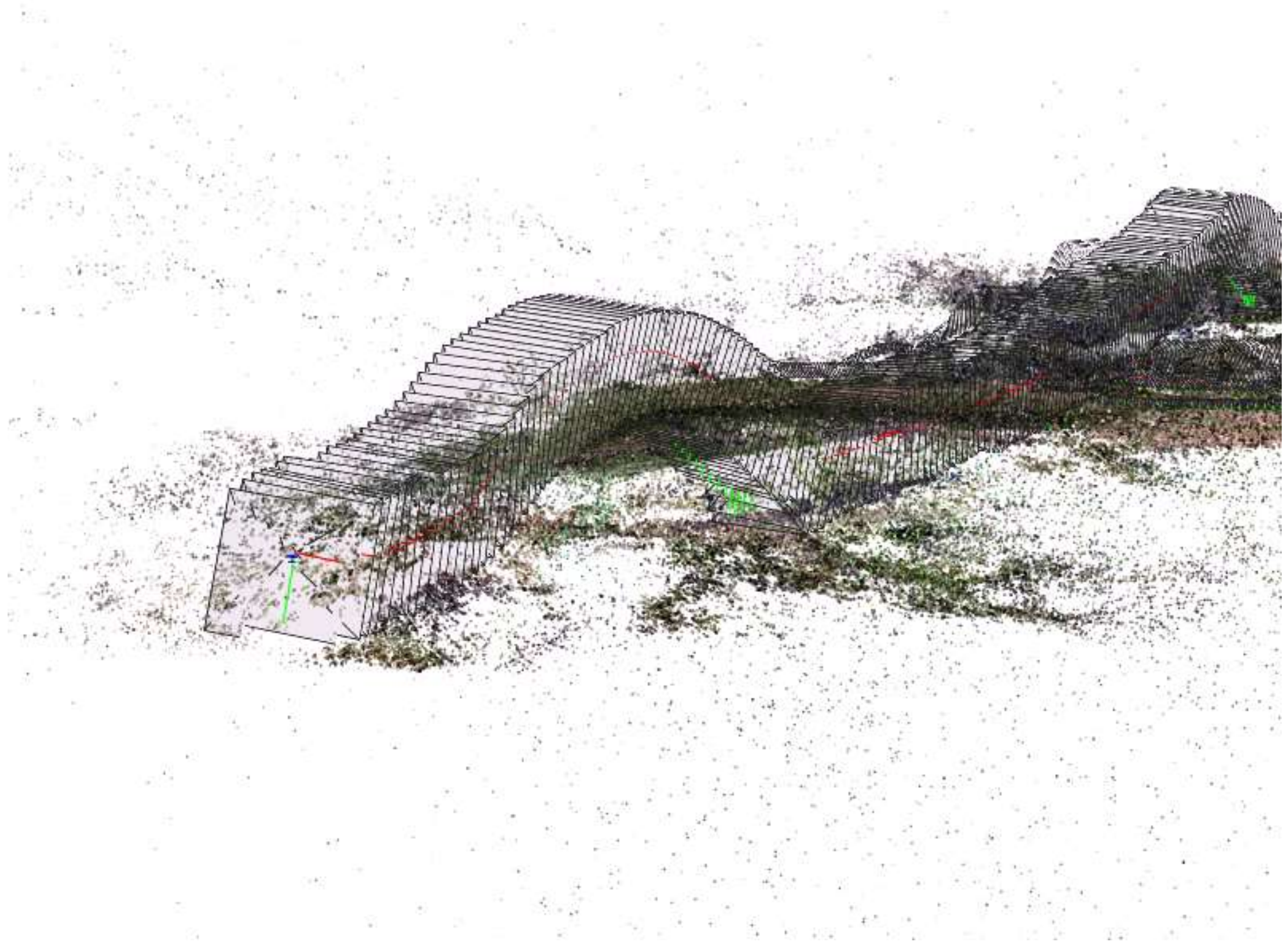
# of unknowns: 3x7+4x3
# of projections: 9 (not all points are visible from cameras)

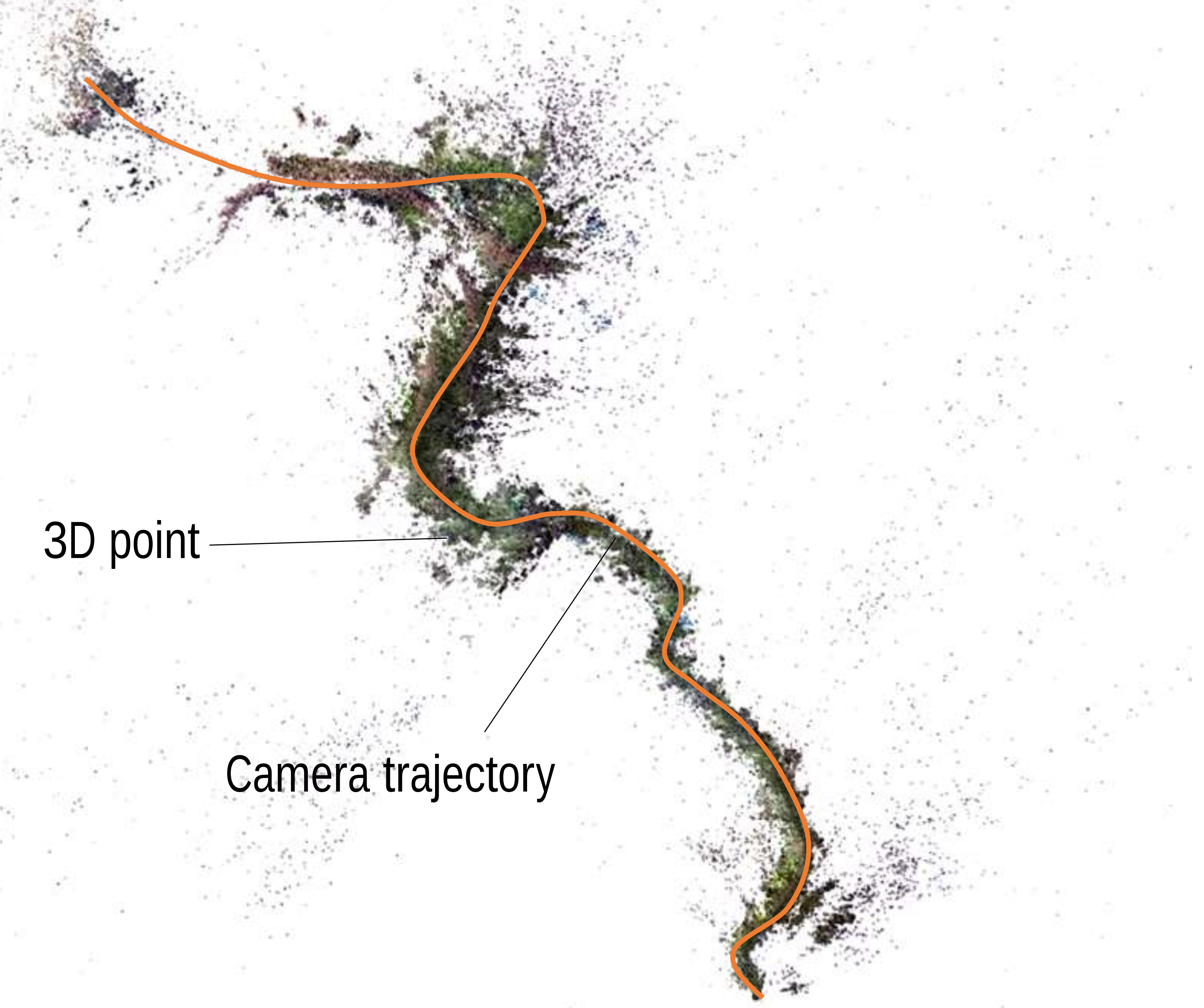
$$\Delta x = \underline{\left(J^T J\right)^{-1}} J^T \left(b - f(x)\right)$$

Main computational bottle neck

size of $J^T J$: 24x24

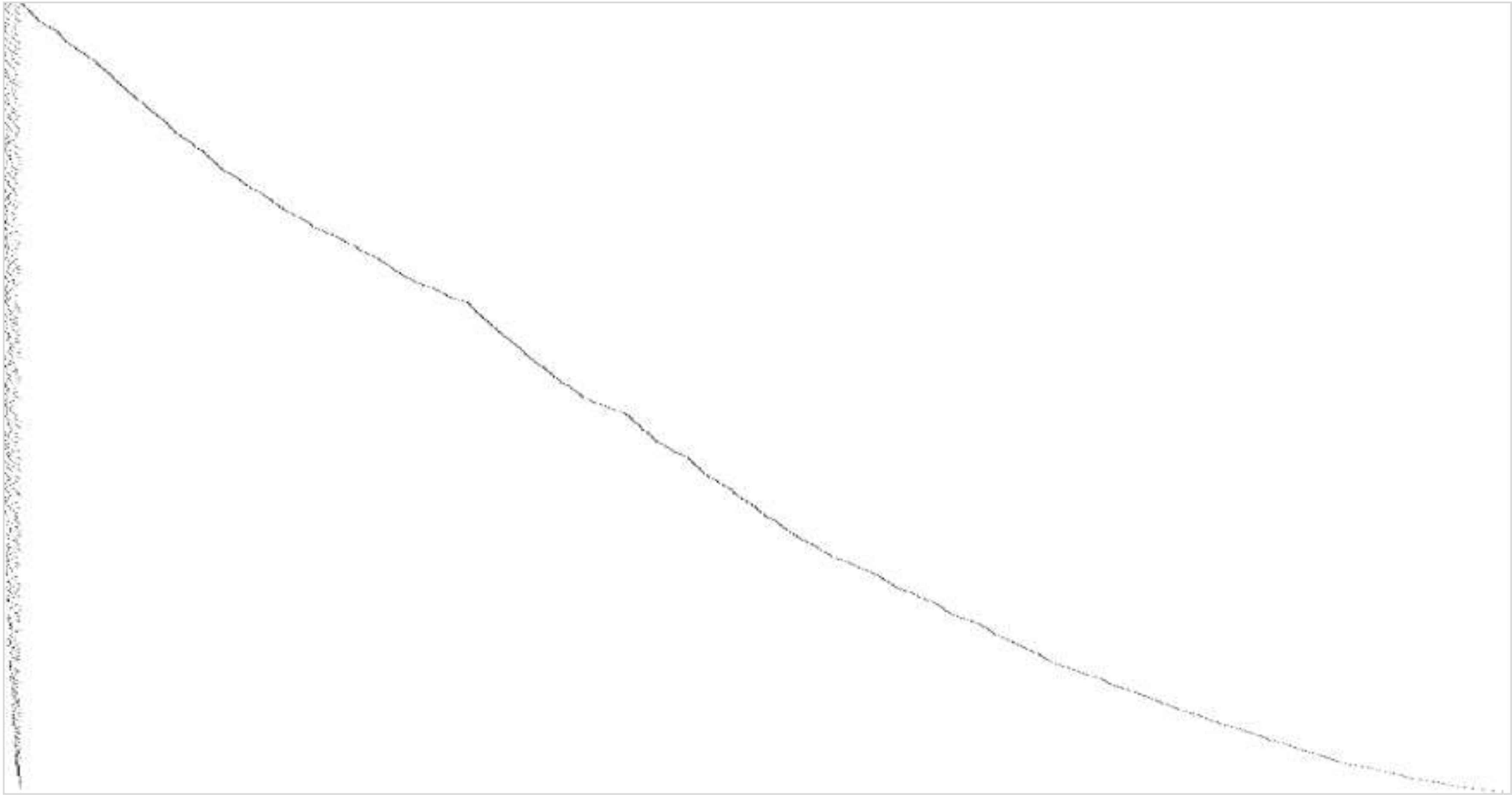

Input: first person video
https://www.youtube.com/watch?v=bKU0Z4bj7Mw



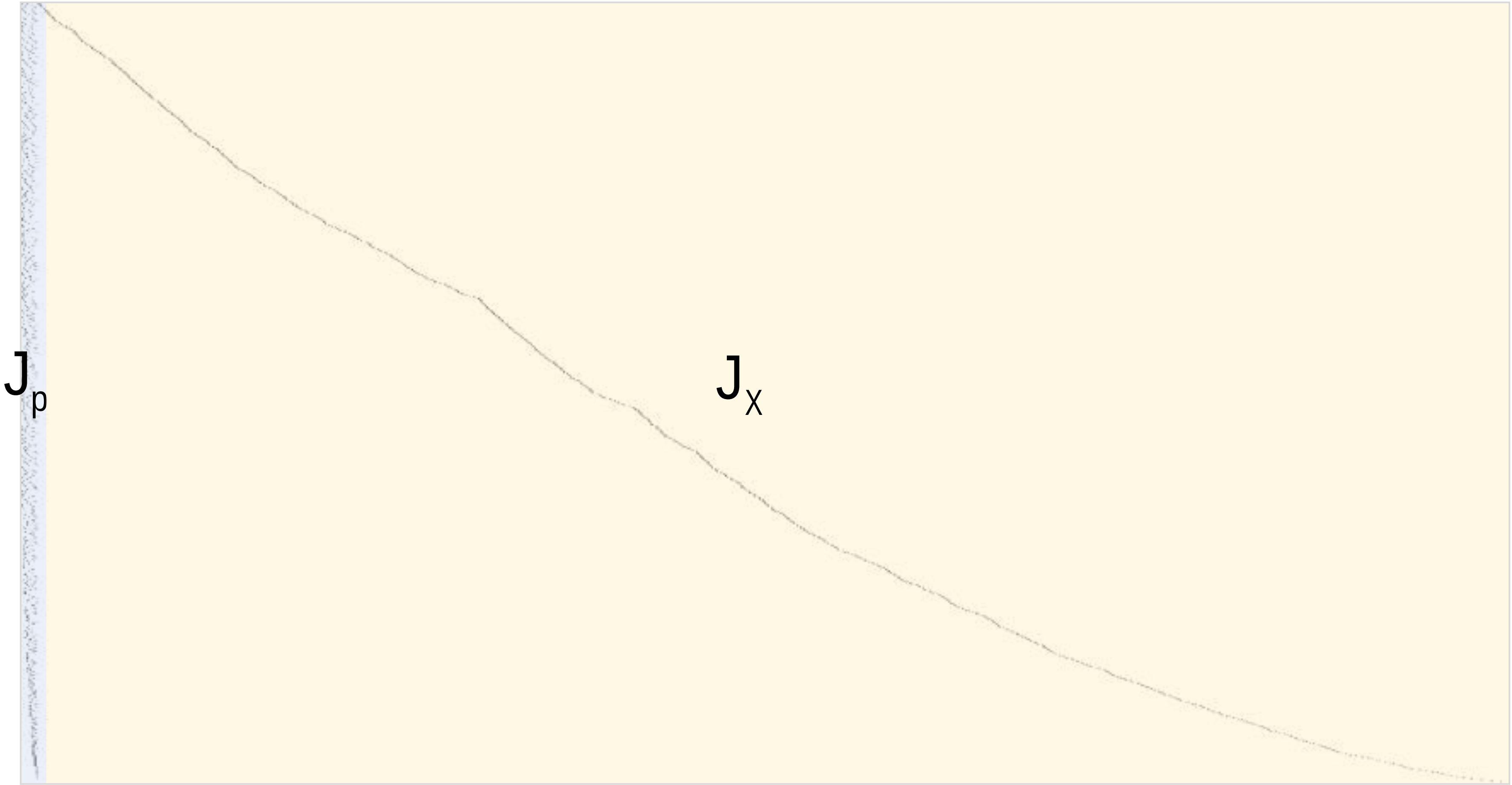


# of cameras: 3009
# of 3D points: 1,298,317

# of unknowns: 7x3009+3x1,298,317
size of J^TJ: 6,167,690x6,167,690

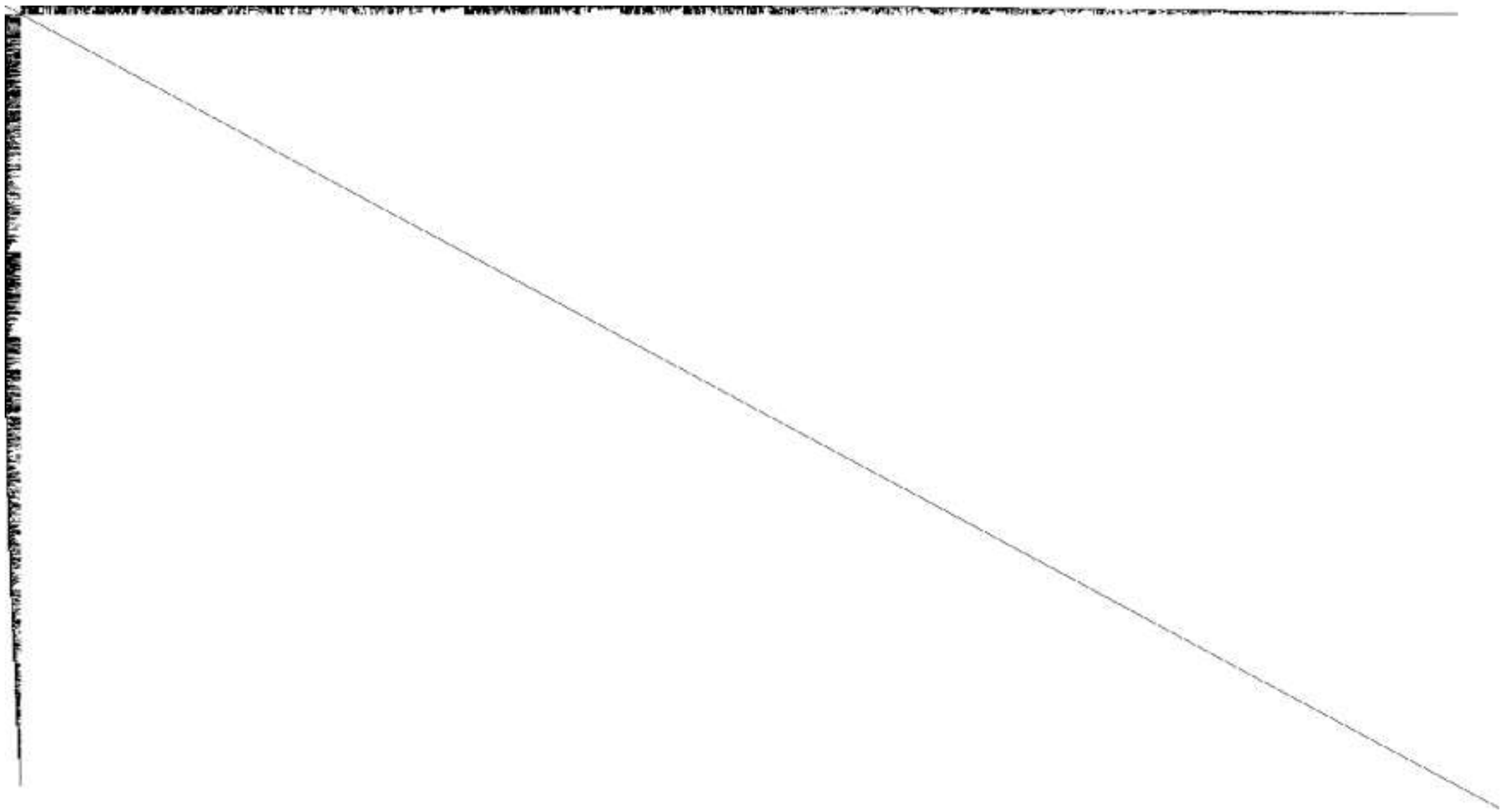
$J =$ 

$J =$



$$J = \begin{bmatrix} J_p & J_X \end{bmatrix}$$

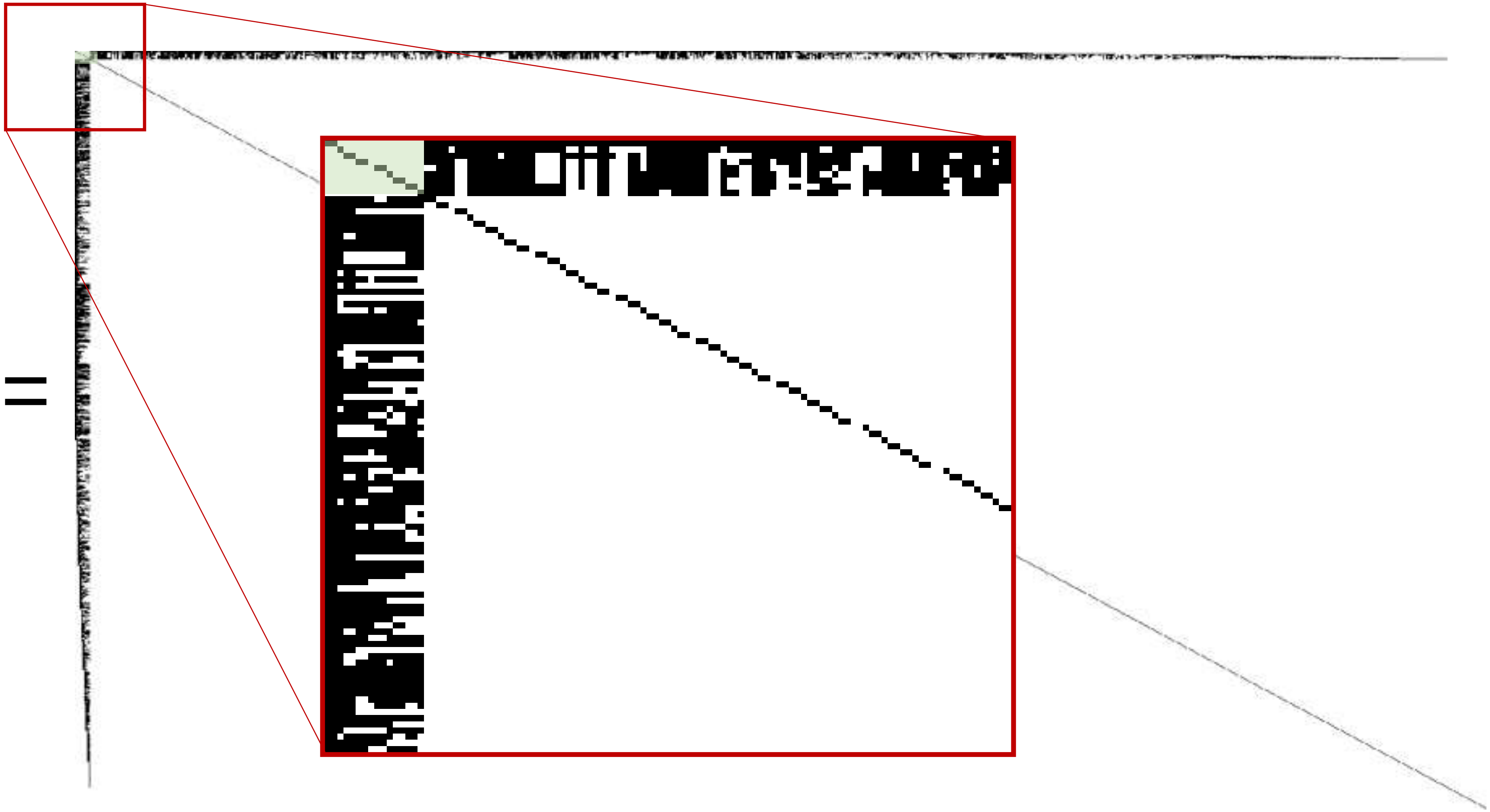
$$J^T J =$$



$$J = \begin{bmatrix} J_p & J_X \end{bmatrix}$$

$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix}$$

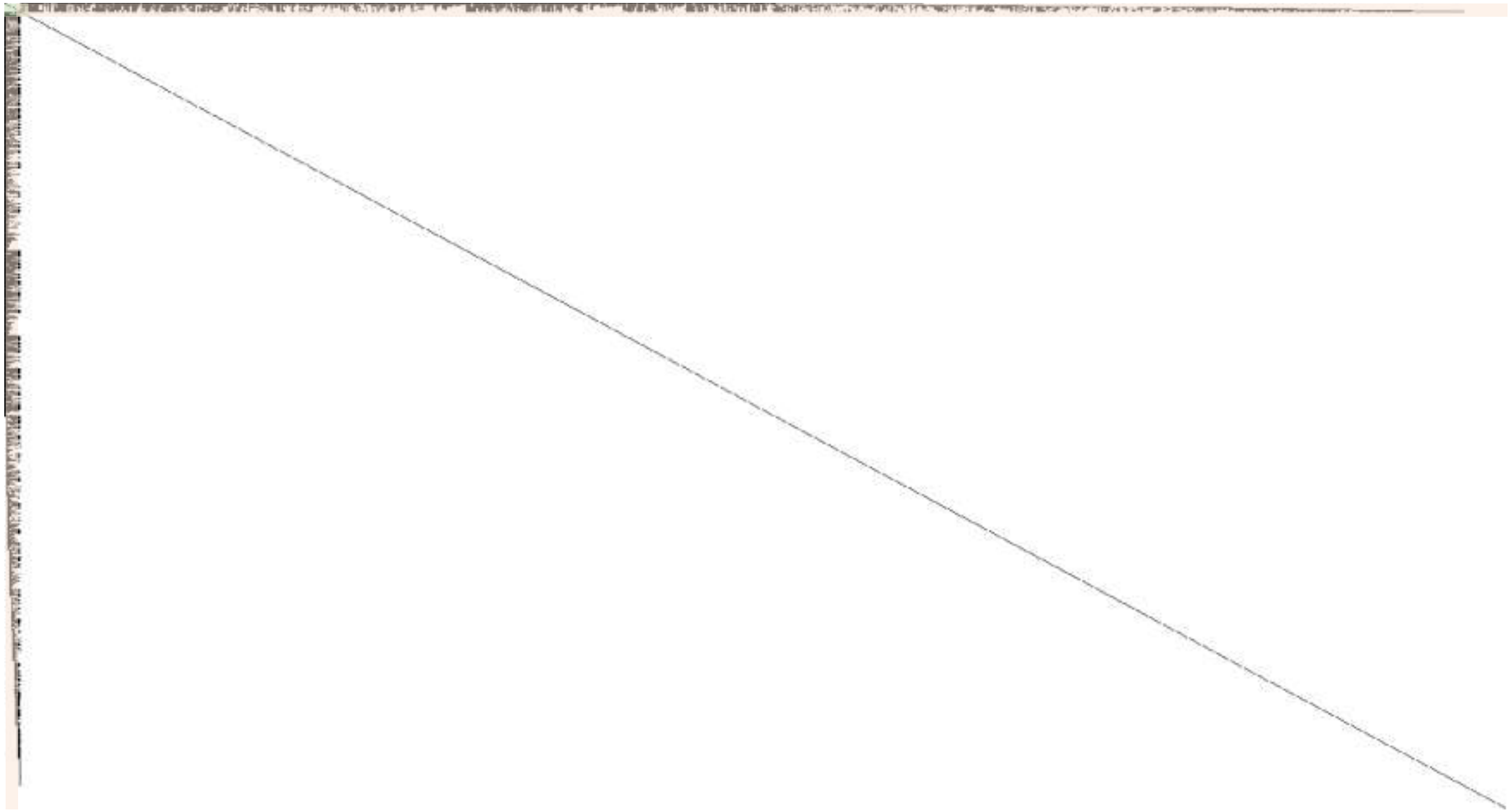
$J^T J =$



$$J = \begin{bmatrix} J_p & J_X \end{bmatrix}$$

$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

$J^T J =$

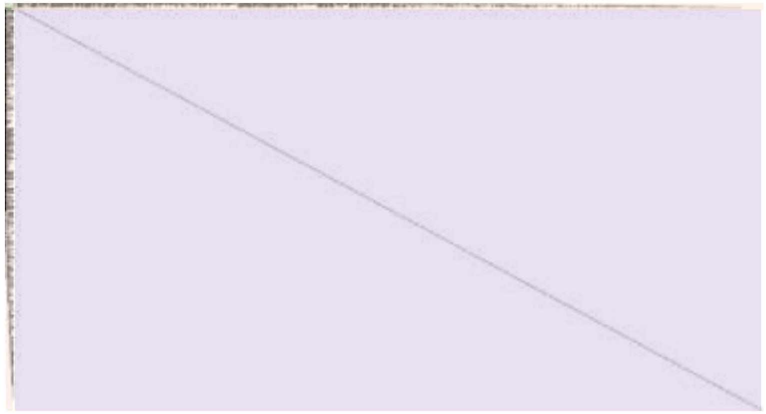


$$J = \begin{bmatrix} J_p & J_X \end{bmatrix}$$

$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

Normal equation:

$$J^T J \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = J^T \left(b - f(X) \right)$$

$J^T J =$ 

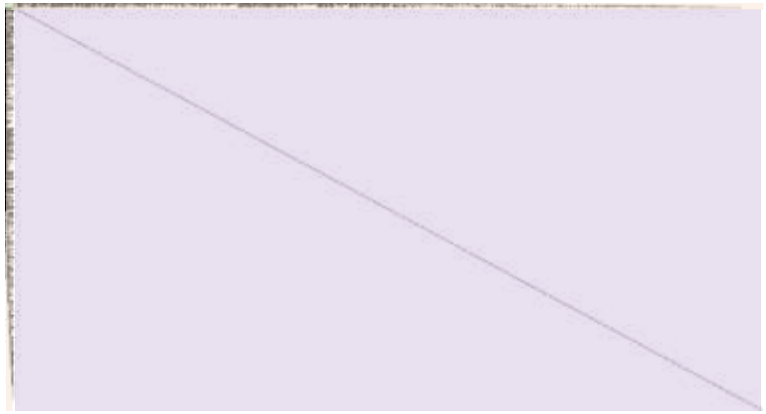
$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

Normal equation:

$$J^T J \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = J^T \left(b - f(X) \right)$$

$$\rightarrow \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} J_p^T \\ J_X^T \end{bmatrix} \left(b - f(X) \right) = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$J^T J =$



$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

Normal equation:

$$J^T J \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = J^T \left(b - f(X)\right)$$

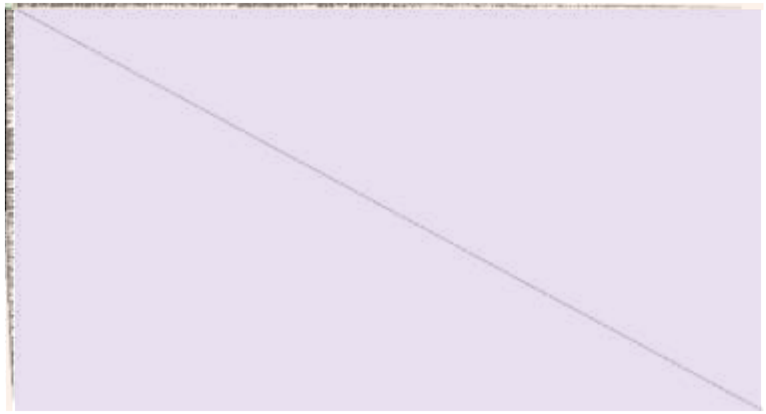
$$\rightarrow \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} J_p^T \\ J_X^T \end{bmatrix} \left(b - f(X)\right) = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

or

$$\begin{bmatrix} J_p^T J_p + \mu I & J_p^T J_X \\ J_X^T J_p & J_X^T J_X + \mu I \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

or

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$J^T J =$ 

$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

Normal equation:

$$J^T J \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = J^T \left(b - f(X) \right)$$

$$\longrightarrow \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} J_p^T \\ J_X^T \end{bmatrix} \left(b - f(X) \right) = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

or $$\begin{bmatrix} J_p^T J_p + \mu I & J_p^T J_X \\ J_X^T J_p & J_X^T J_X + \mu I \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

or $$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$$\begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$$J^T J = \quad D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_M \end{bmatrix} \longrightarrow D^{-1} = \begin{bmatrix} d_1^{-1} & & \\ & \ddots & \\ & & d_M^{-1} \end{bmatrix}$$

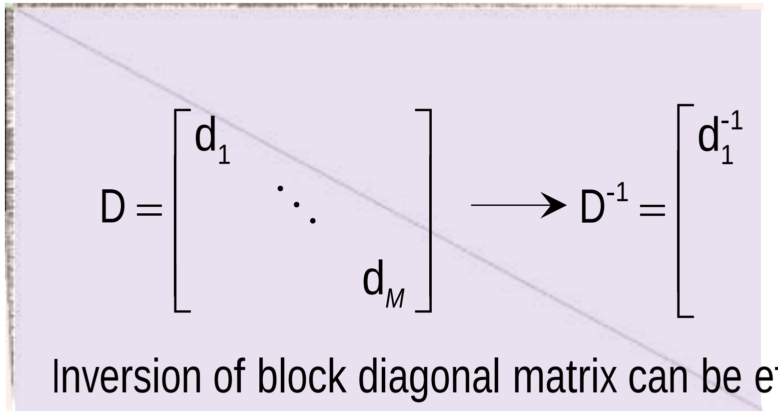
Inversion of block diagonal matrix can be efficiently computed.

$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

Normal equation:

$$J^T J \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = J^T \left(b - f(X) \right)$$

$$\longrightarrow \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$J^T J =$ 

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_M \end{bmatrix} \longrightarrow D^{-1} = \begin{bmatrix} d_1^{-1} & & \\ & \ddots & \\ & & d_M^{-1} \end{bmatrix}$$

Inversion of block diagonal matrix can be efficiently computed.

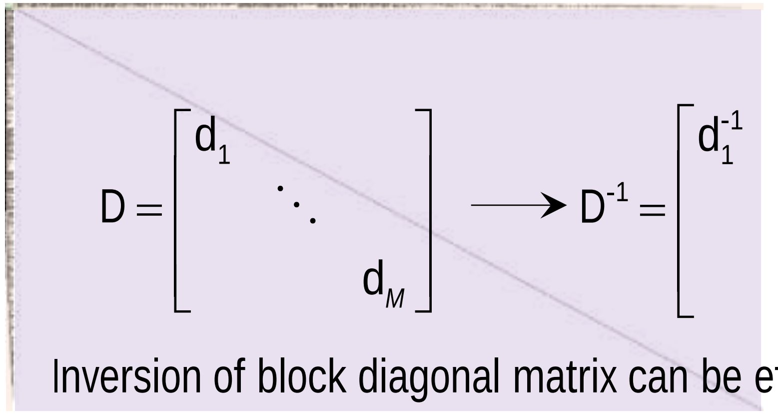
$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

Normal equation:

$$J^T J \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = J^T \left(b - f(X) \right)$$

$$\longrightarrow \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$J^T J =$ 

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_M \end{bmatrix} \longrightarrow D^{-1} = \begin{bmatrix} d_1^{-1} & & \\ & \ddots & \\ & & d_M^{-1} \end{bmatrix}$$

Inversion of block diagonal matrix can be efficiently computed.

$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

Normal equation:

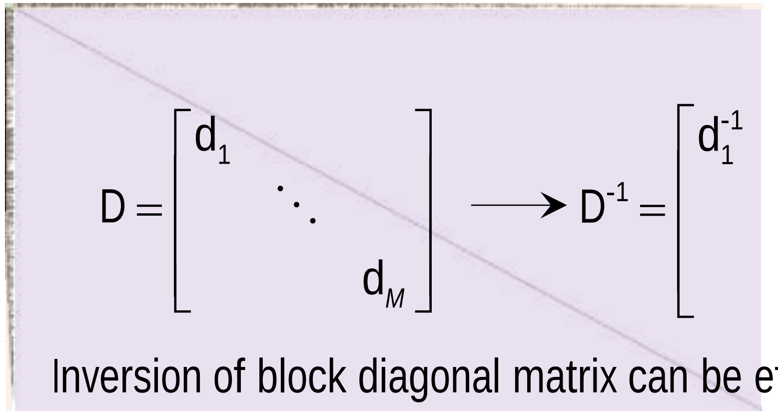
$$J^T J \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = J^T \left(b - f(X) \right)$$

$$\longrightarrow \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} A - BD^{-1}B^T & 0 \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p - BD^{-1}e_X \\ e_X \end{bmatrix}$$

$J^T J =$



$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_M \end{bmatrix} \longrightarrow D^{-1} = \begin{bmatrix} d_1^{-1} & & \\ & \ddots & \\ & & d_M^{-1} \end{bmatrix}$$

Inversion of block diagonal matrix can be efficiently computed.

$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

$$J^T J =$$

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_M \end{bmatrix} \longrightarrow D^{-1} = \begin{bmatrix} d_1^{-1} & & \\ & \ddots & \\ & & d_M^{-1} \end{bmatrix}$$

Inversion of block diagonal matrix can be efficiently computed.

$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

Normal equation:

$$J^T J \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = J^T \left(b - f(X) \right)$$

$$\longrightarrow \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

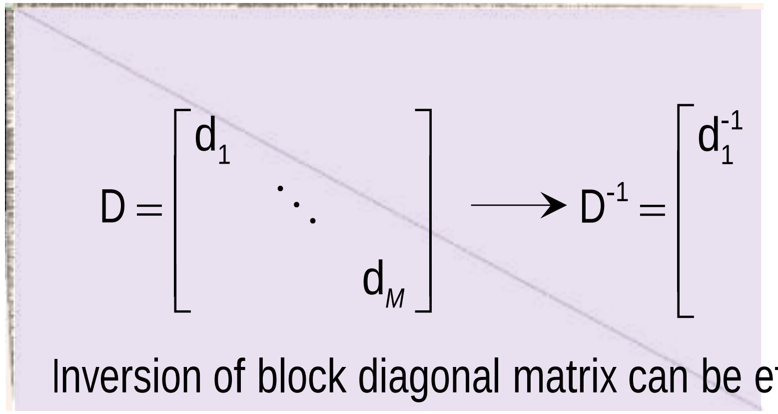
$$\longrightarrow \begin{bmatrix} A - BD^{-1}B^T & 0 \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p - BD^{-1}e_X \\ e_X \end{bmatrix}$$

$$\longrightarrow \Delta p = \left(A - BD^{-1}B^T \right)^{-1} \left(e_p - BD^{-1}e_X \right)$$

$$\Delta X = D^{-1} \left(e_X - B^T \Delta p \right)$$

Note: $A - BD^{-1}B^T$ is Schur complement of D

$$J^T J = $$



$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_M \end{bmatrix} \longrightarrow D^{-1} = \begin{bmatrix} d_1^{-1} & & \\ & \ddots & \\ & & d_M^{-1} \end{bmatrix}$$

Inversion of block diagonal matrix can be efficiently computed.

$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

Normal equation:

$$J^T J \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = J^T \left(b - f(X)\right)$$

$$\longrightarrow \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

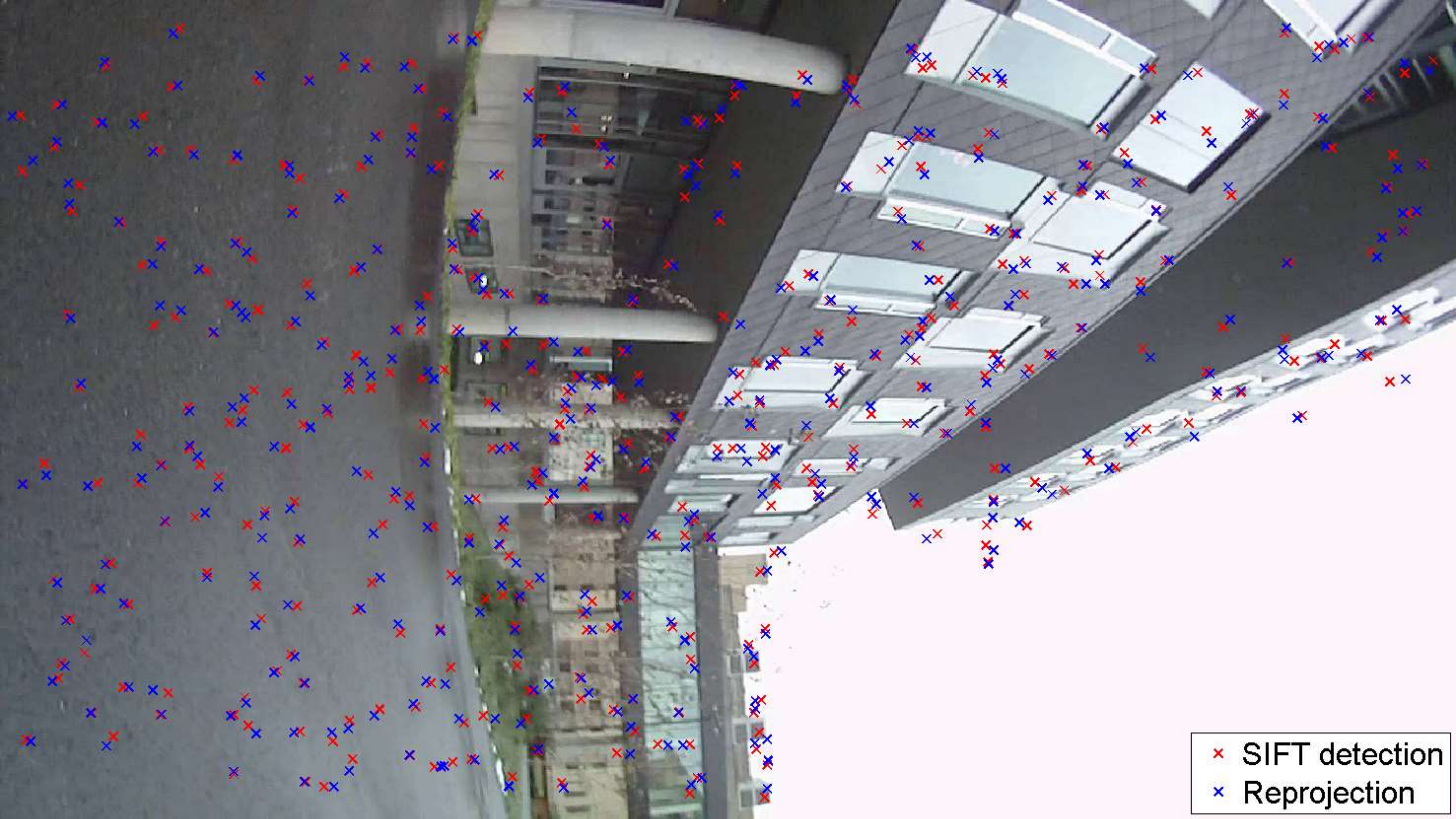
$$\longrightarrow \begin{bmatrix} A - BD^{-1}B^T & 0 \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p - BD^{-1}e_X \\ e_X \end{bmatrix}$$

Small size matrix

$$\longrightarrow \Delta p = \left(A - BD^{-1}B^T\right)^{-1} \left(e_p - BD^{-1}e_X\right)$$

$$\Delta X = D^{-1}\left(e_X - B^T \Delta p\right)$$

Note: $A - BD^{-1}B^T$ is Schur complement of D


SIFT detection
Reprojection


SIFT detection
Reprojection


Measurement
Linear estimate (reproj: 0.199104)
Nonlinear estimate (reproj: 0.119272)


Measurement
Linear estimate (reproj: 0.199104)
Nonlinear estimate (reproj: 0.119272)