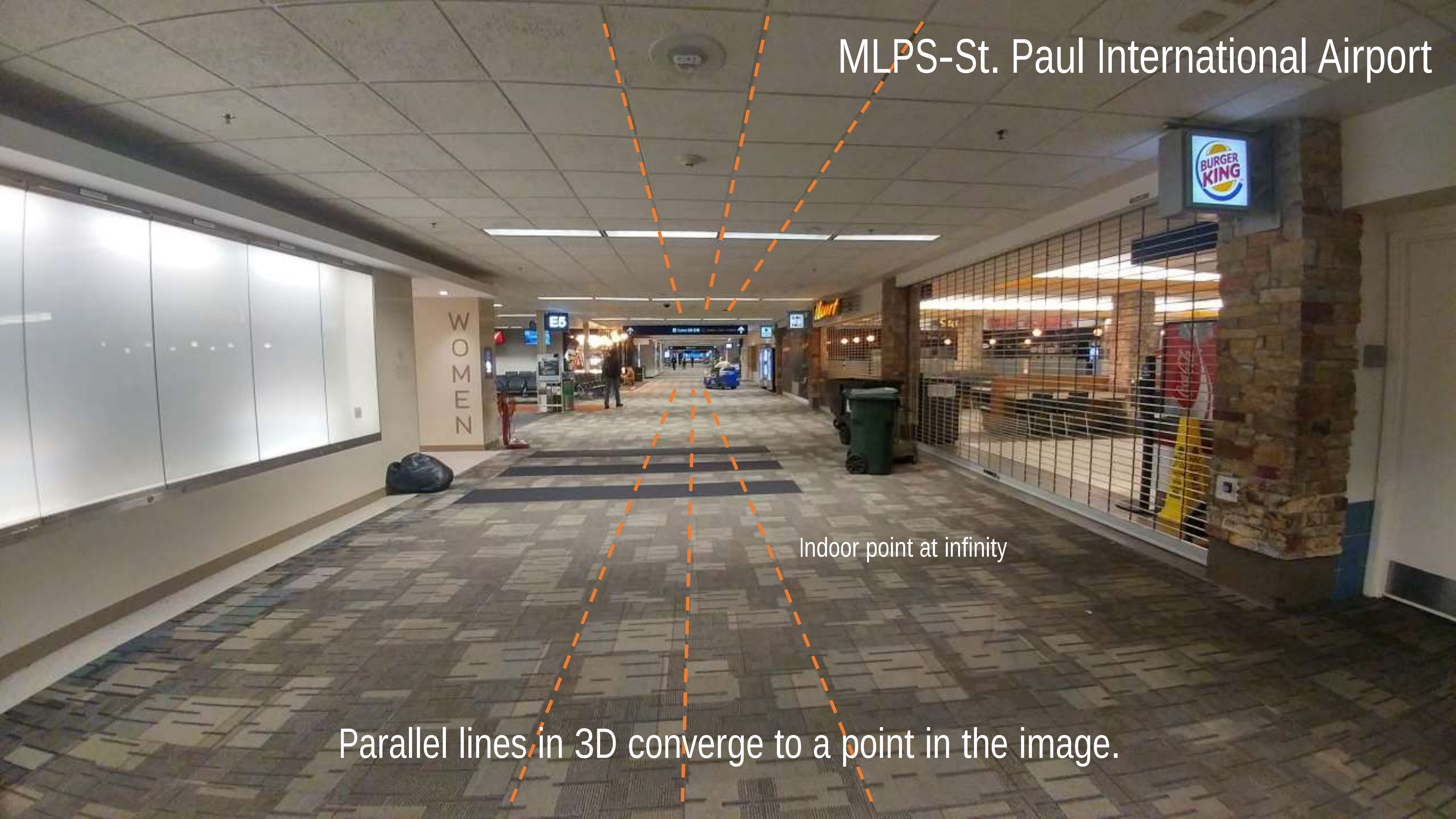
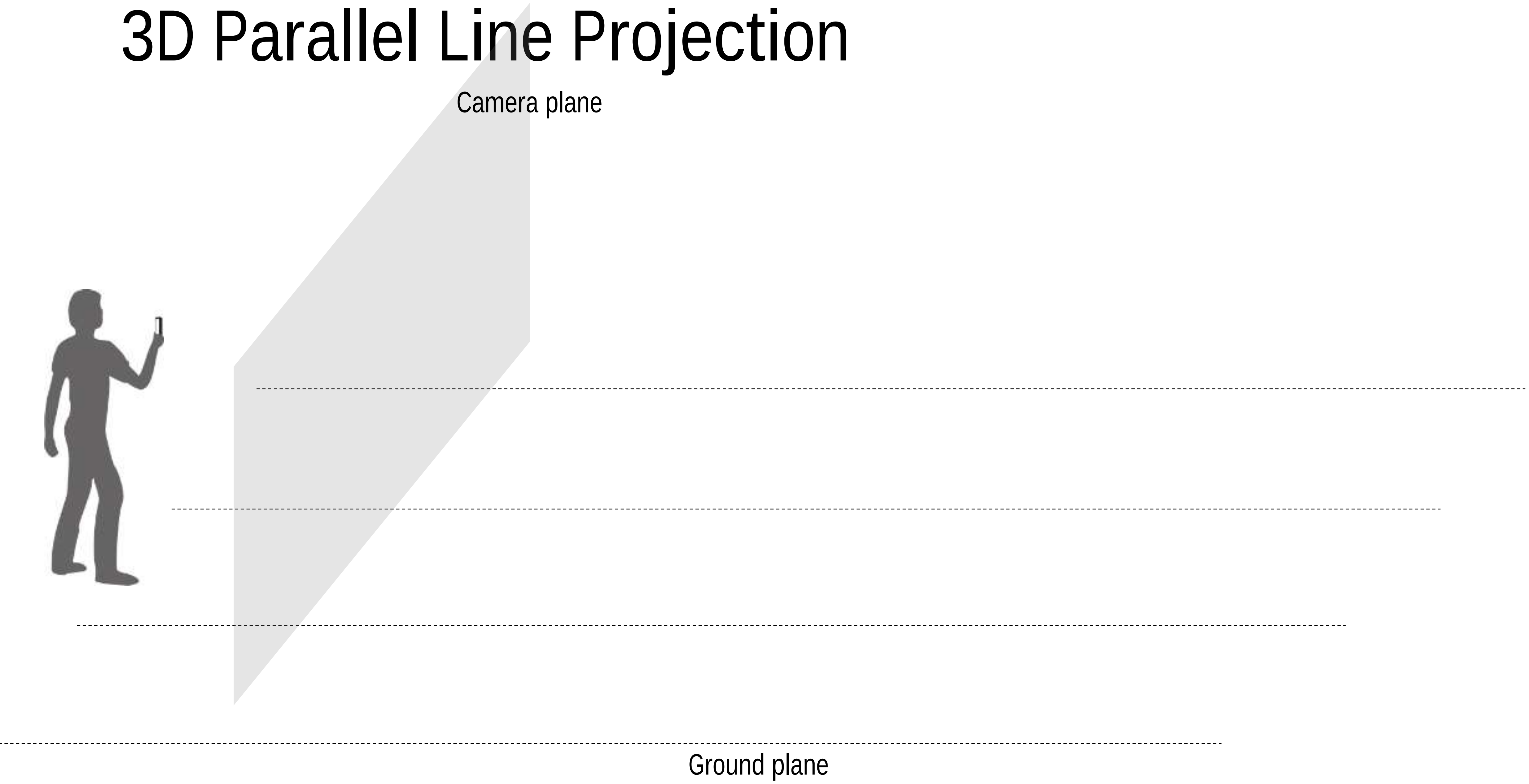

Projective Line

THE MAKING OF

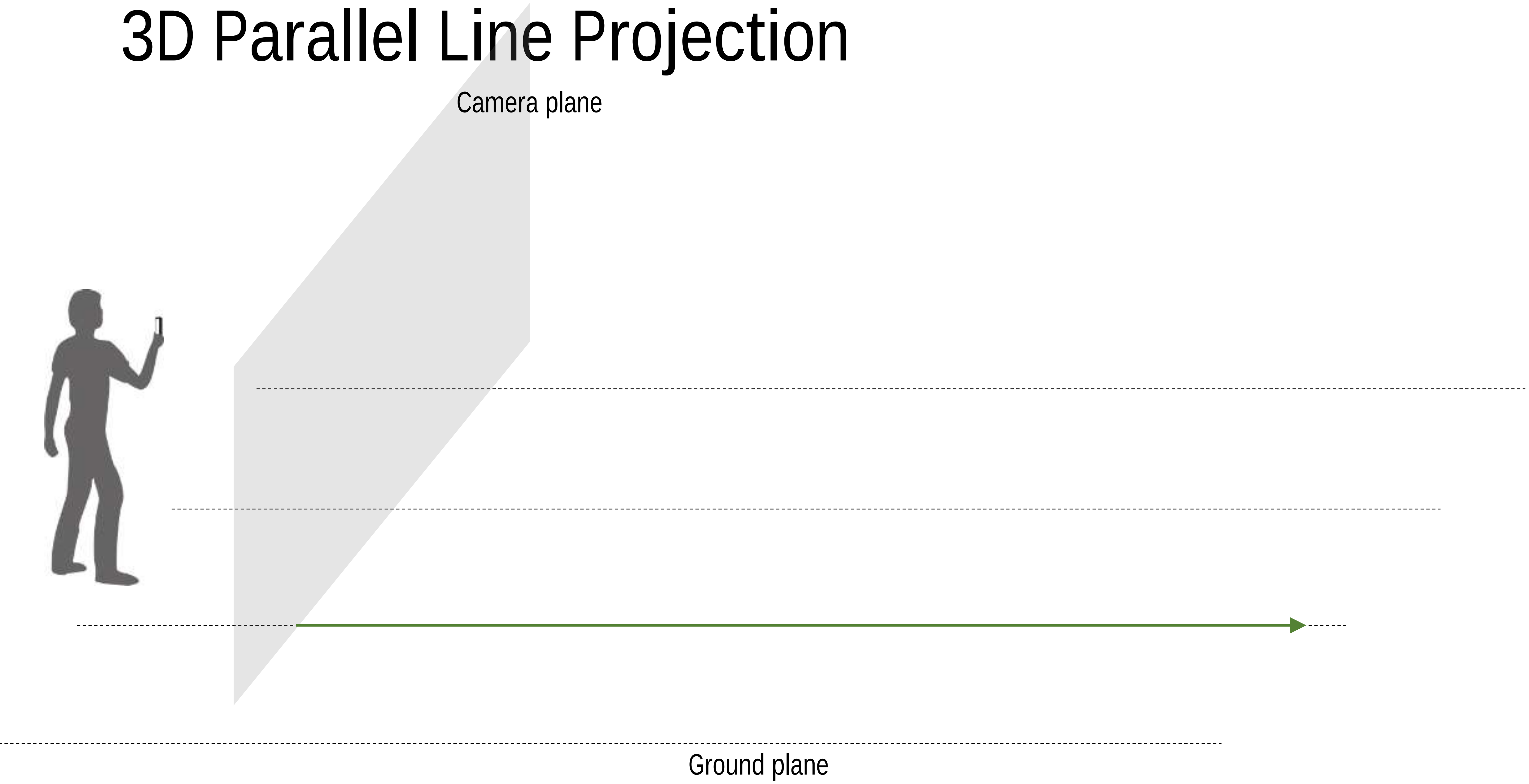
CHEMICAL BROTHERS 'WIDE OPEN'


MLPS-St. Paul International Airport
WOMEN
E5
BURGER KING
Indoor point at infinity
Parallel lines in 3D converge to a point in the image.

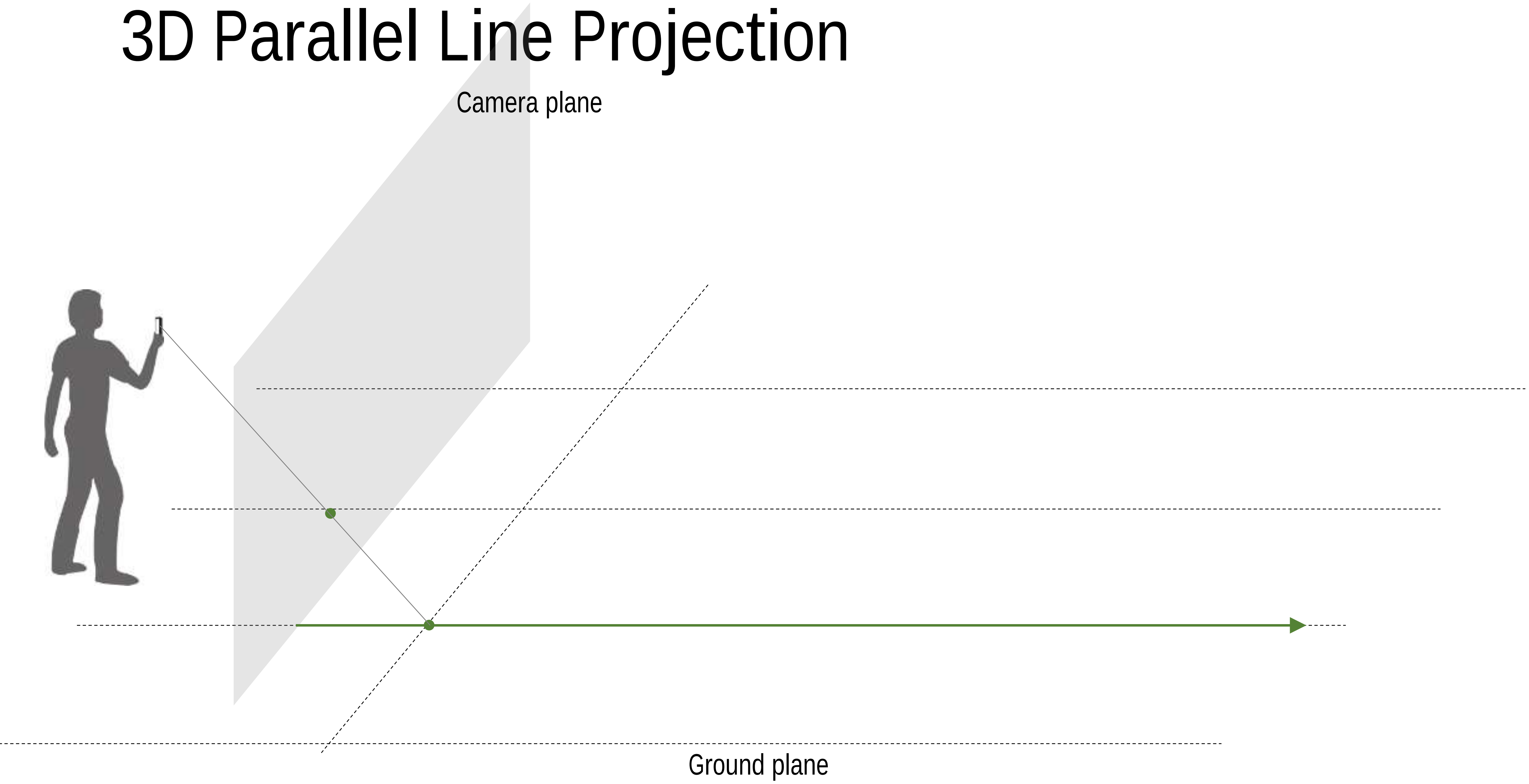
3D Parallel Line Projection



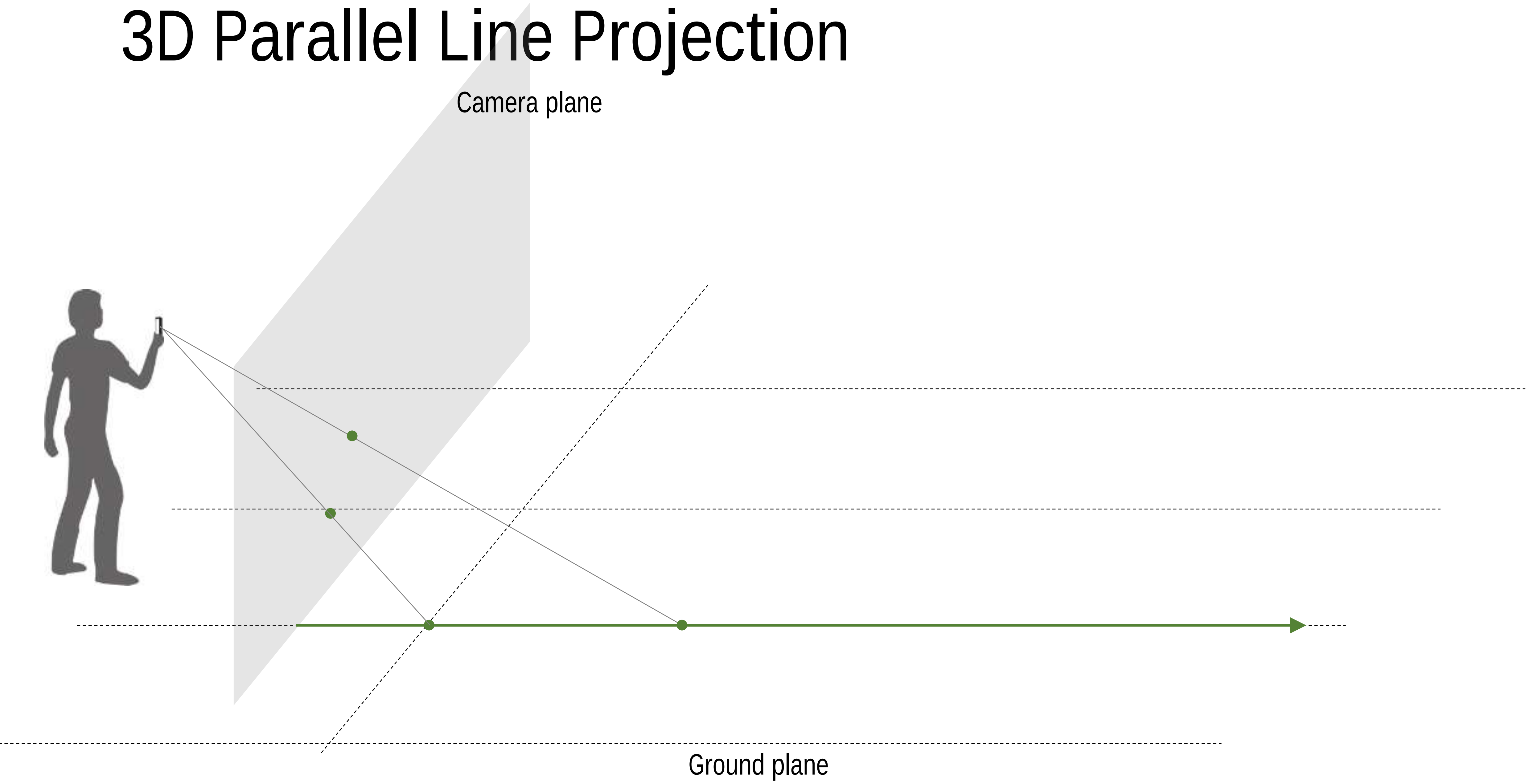
3D Parallel Line Projection



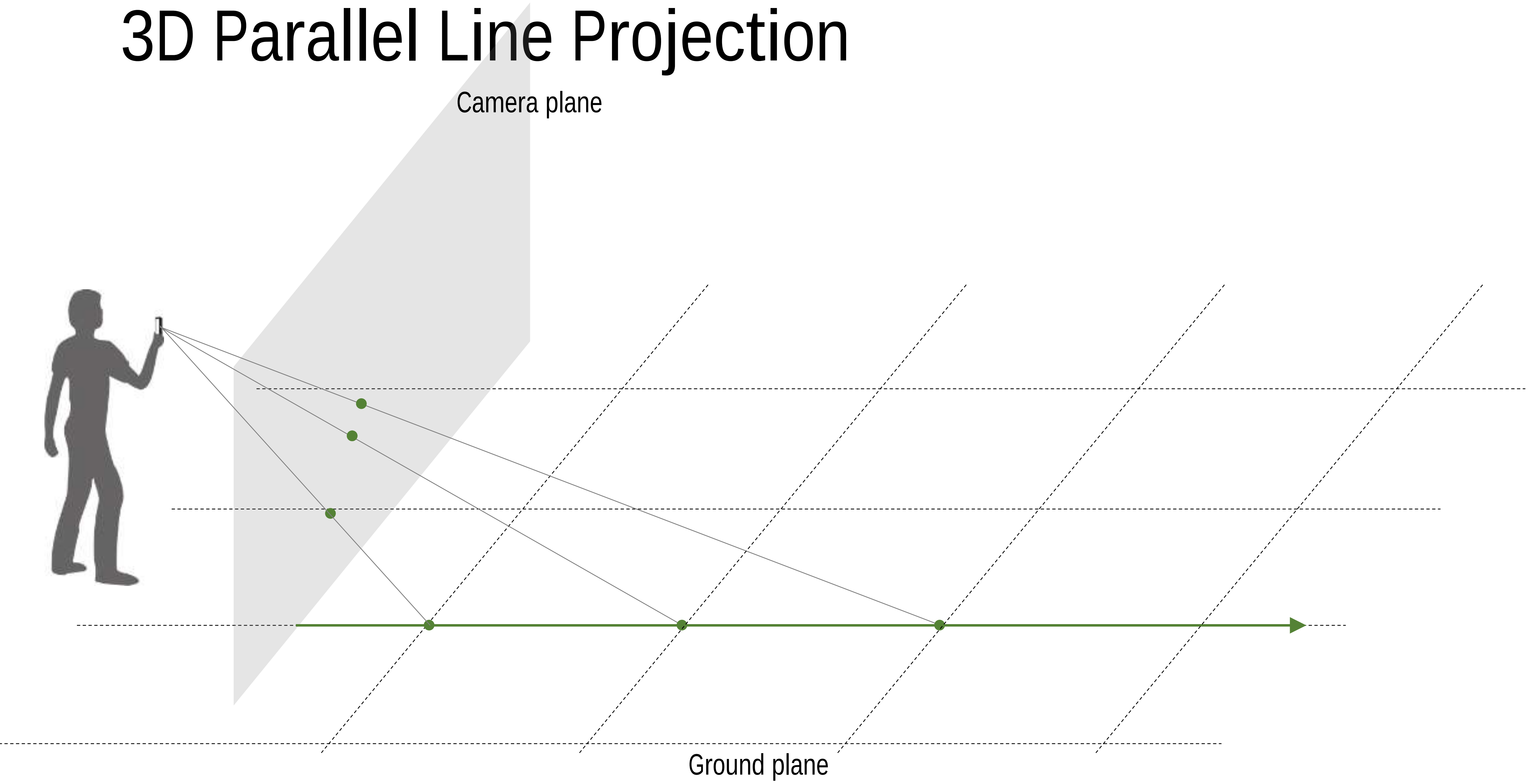
3D Parallel Line Projection



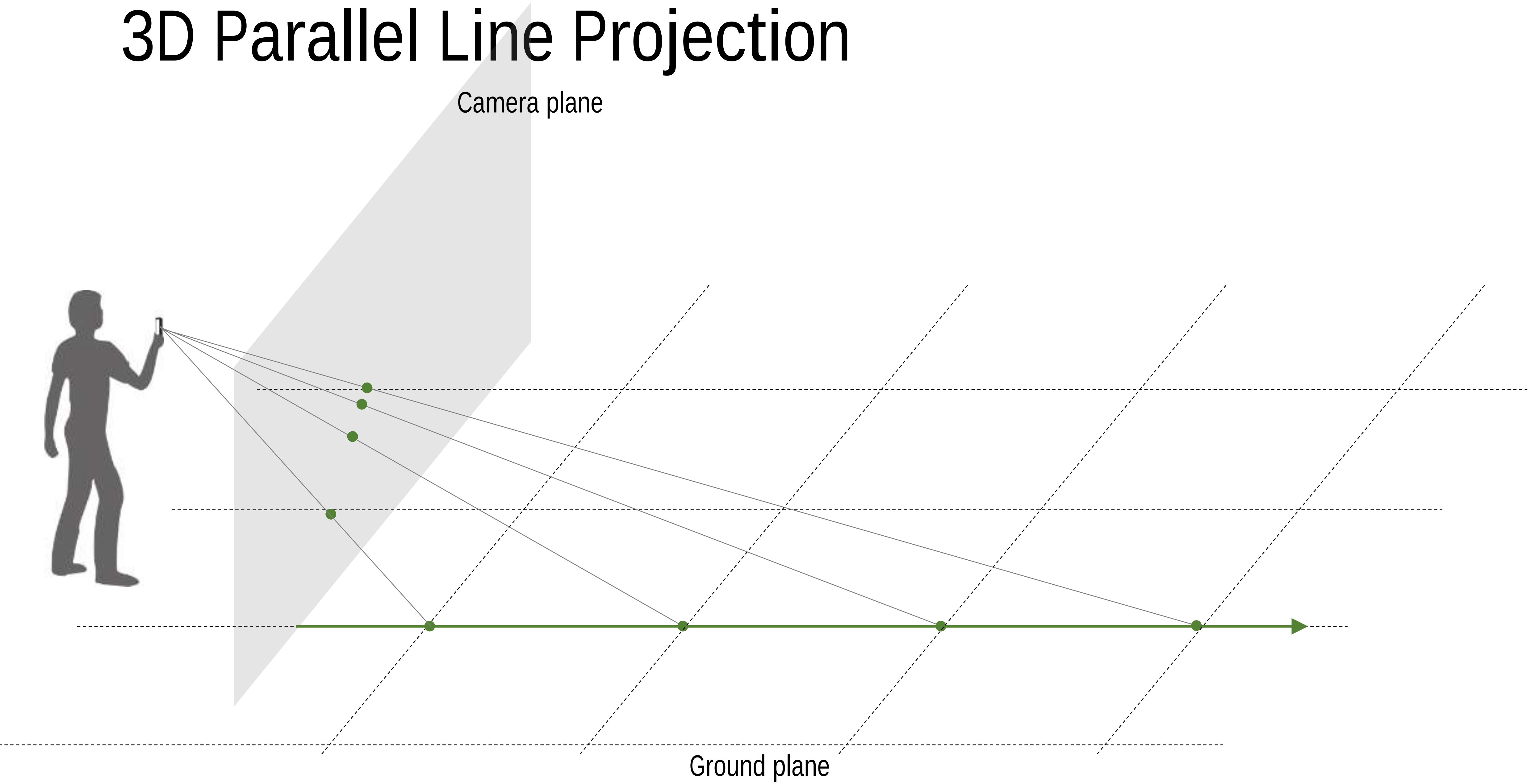
3D Parallel Line Projection



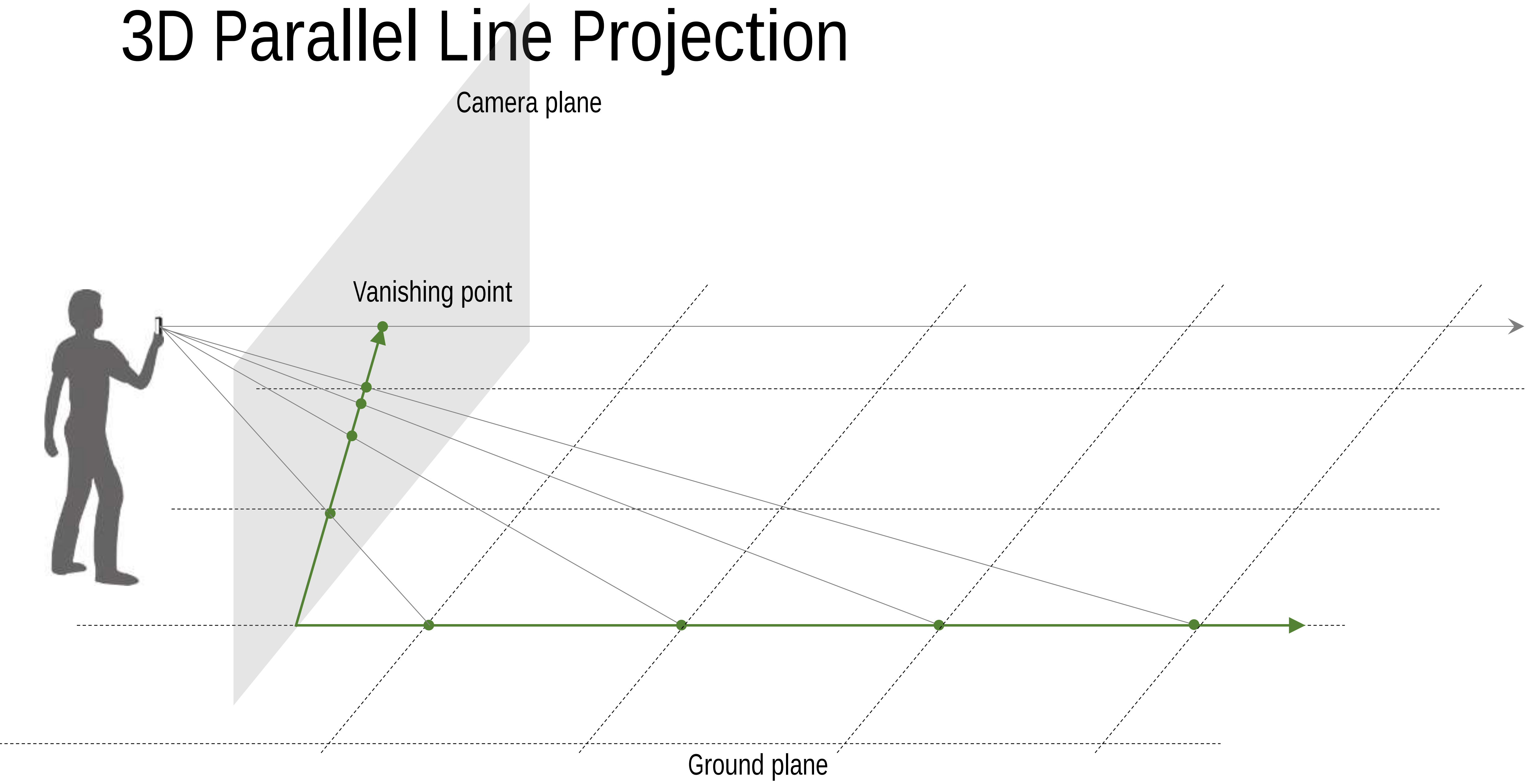
3D Parallel Line Projection



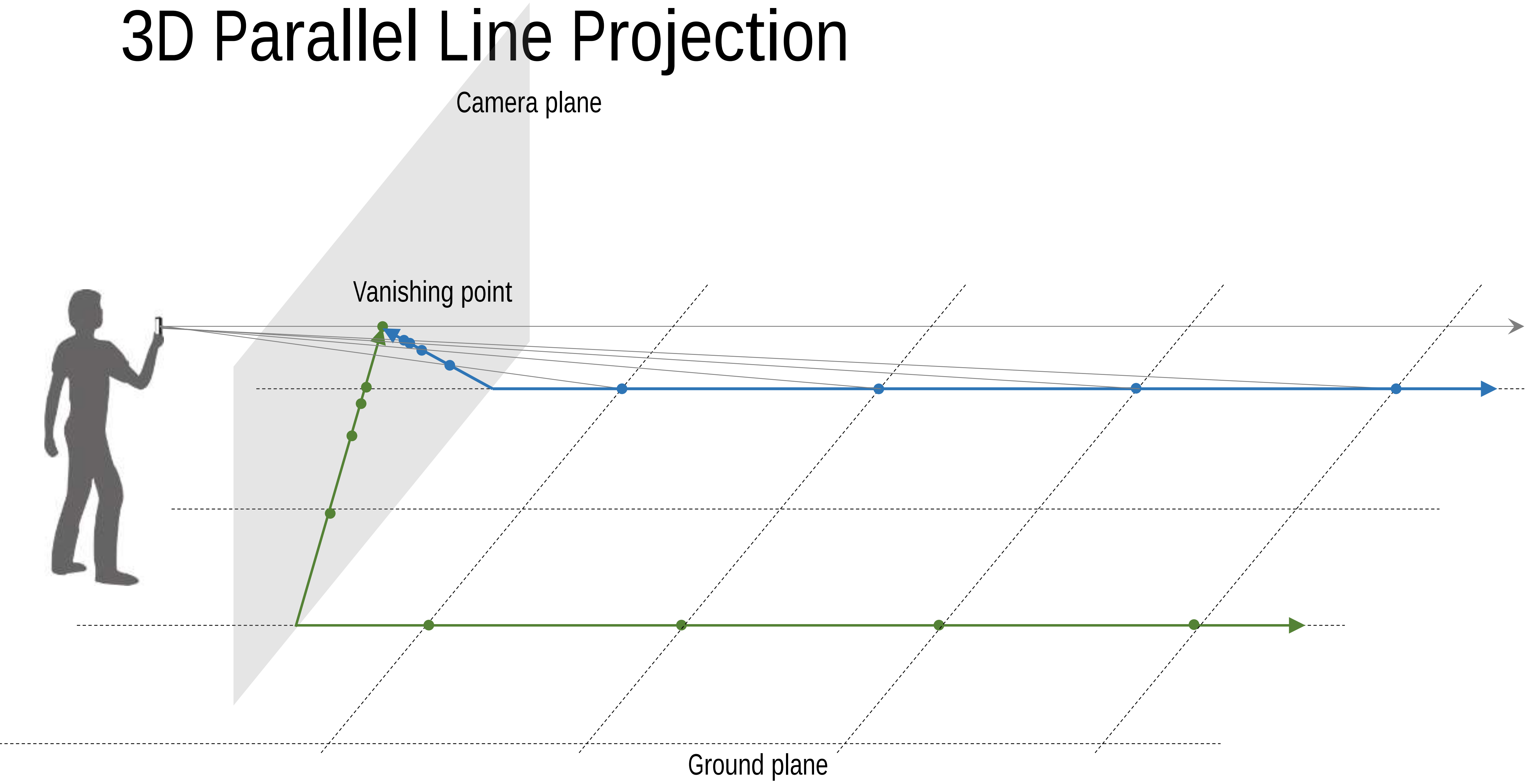
3D Parallel Line Projection



3D Parallel Line Projection

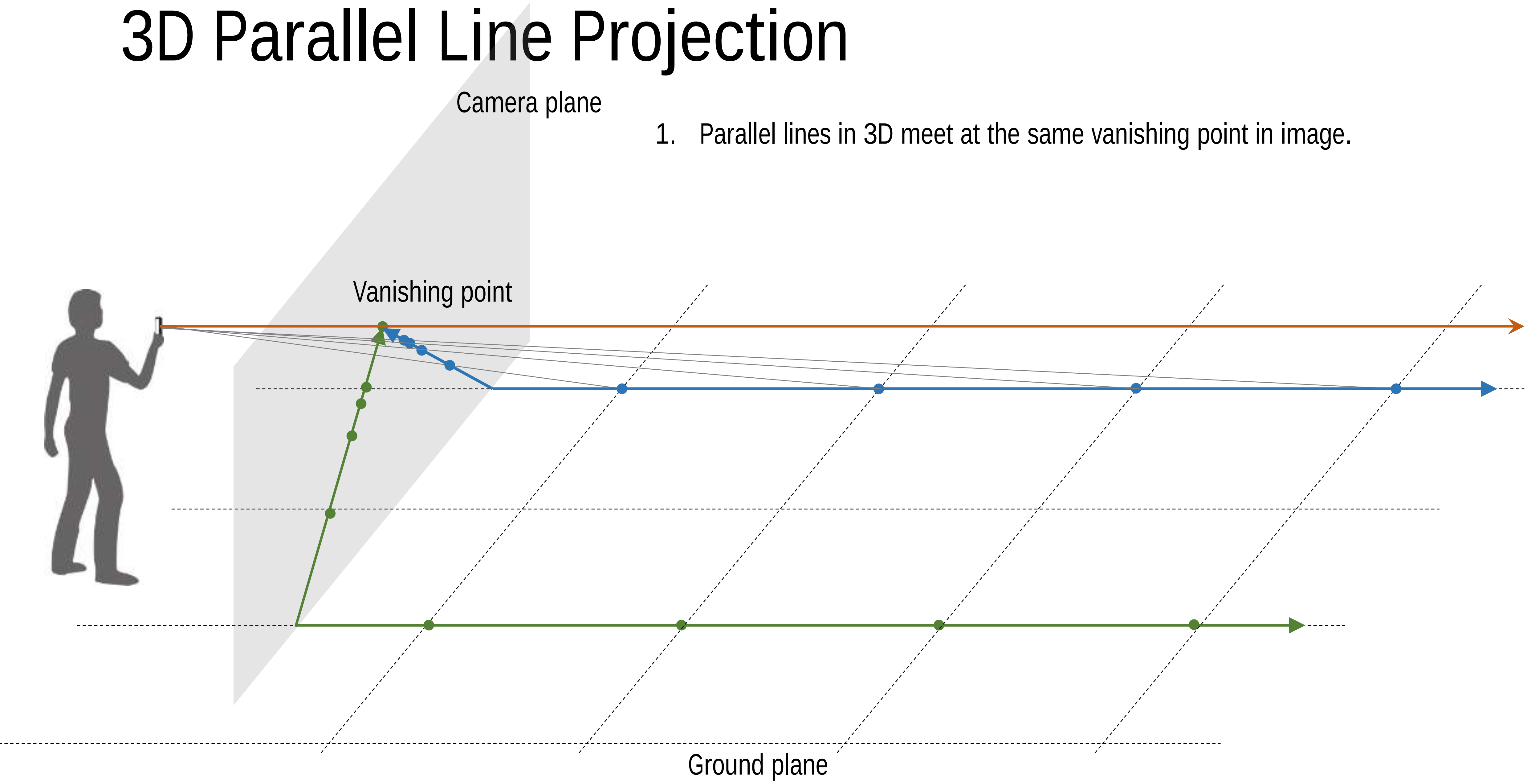


3D Parallel Line Projection



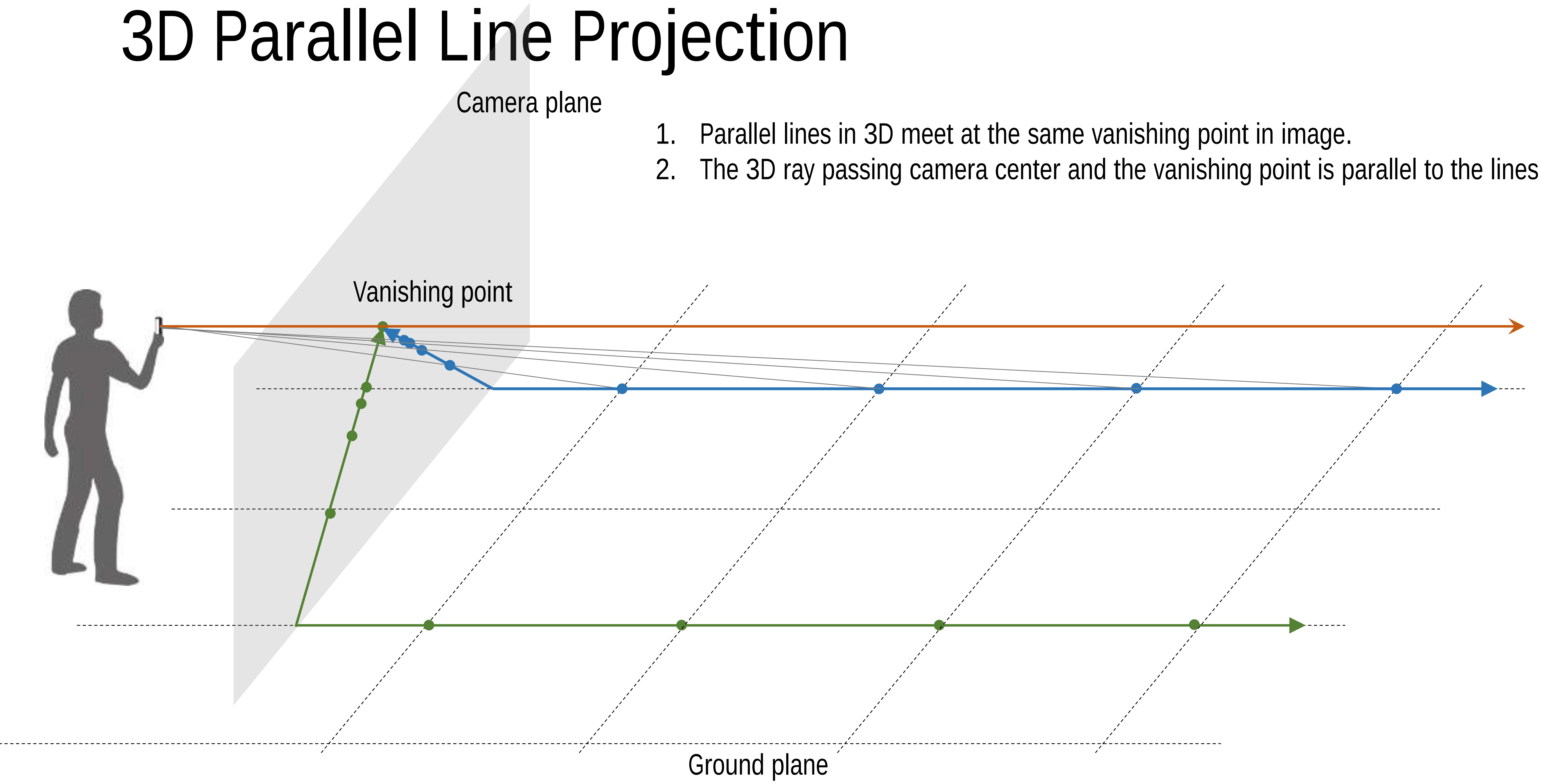
3D Parallel Line Projection

1. Parallel lines in 3D meet at the same vanishing point in image.



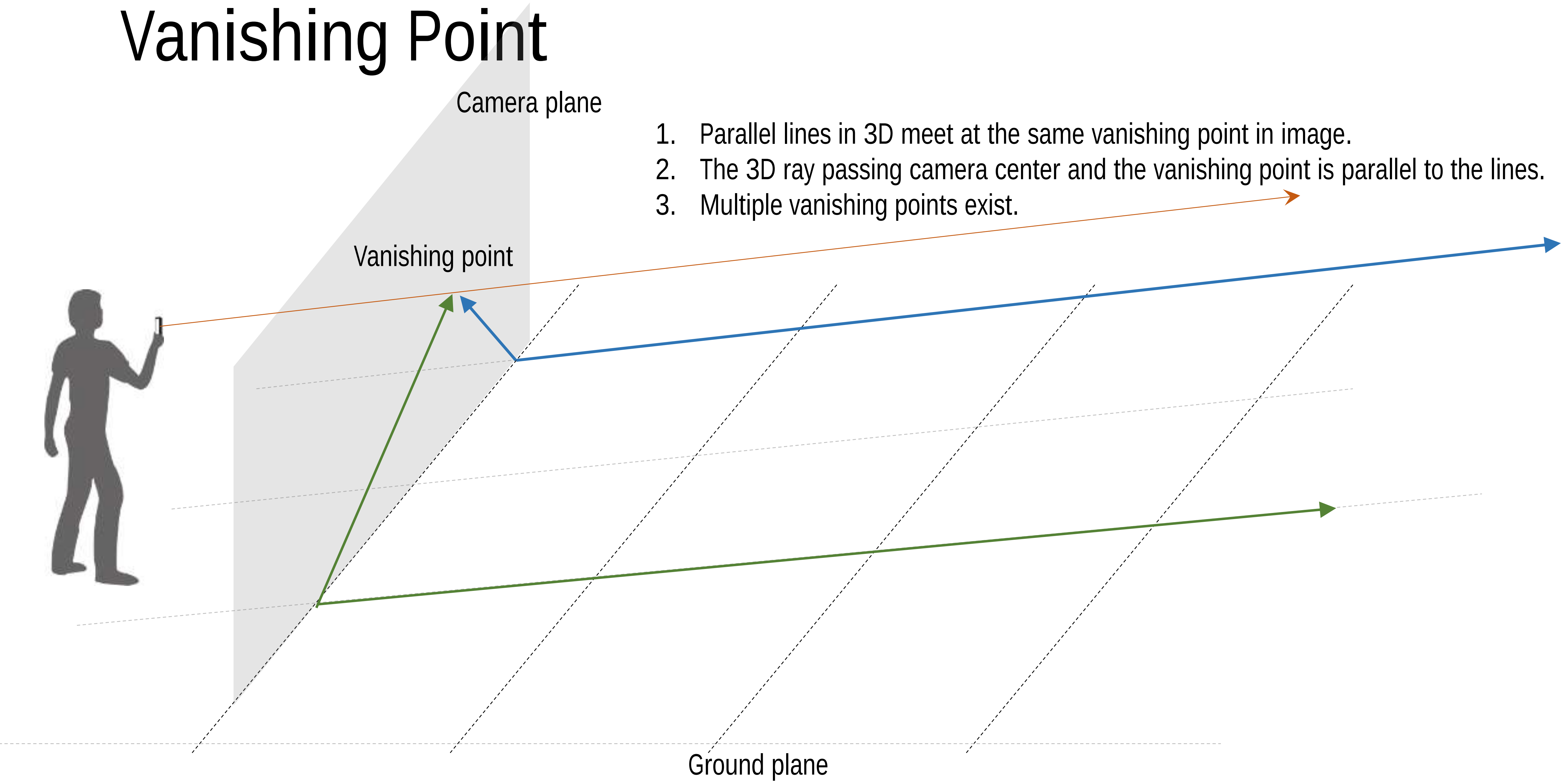
3D Parallel Line Projection

1. Parallel lines in 3D meet at the same vanishing point in image.
2. The 3D ray passing camera center and the vanishing point is parallel to the lines.



Vanishing Point

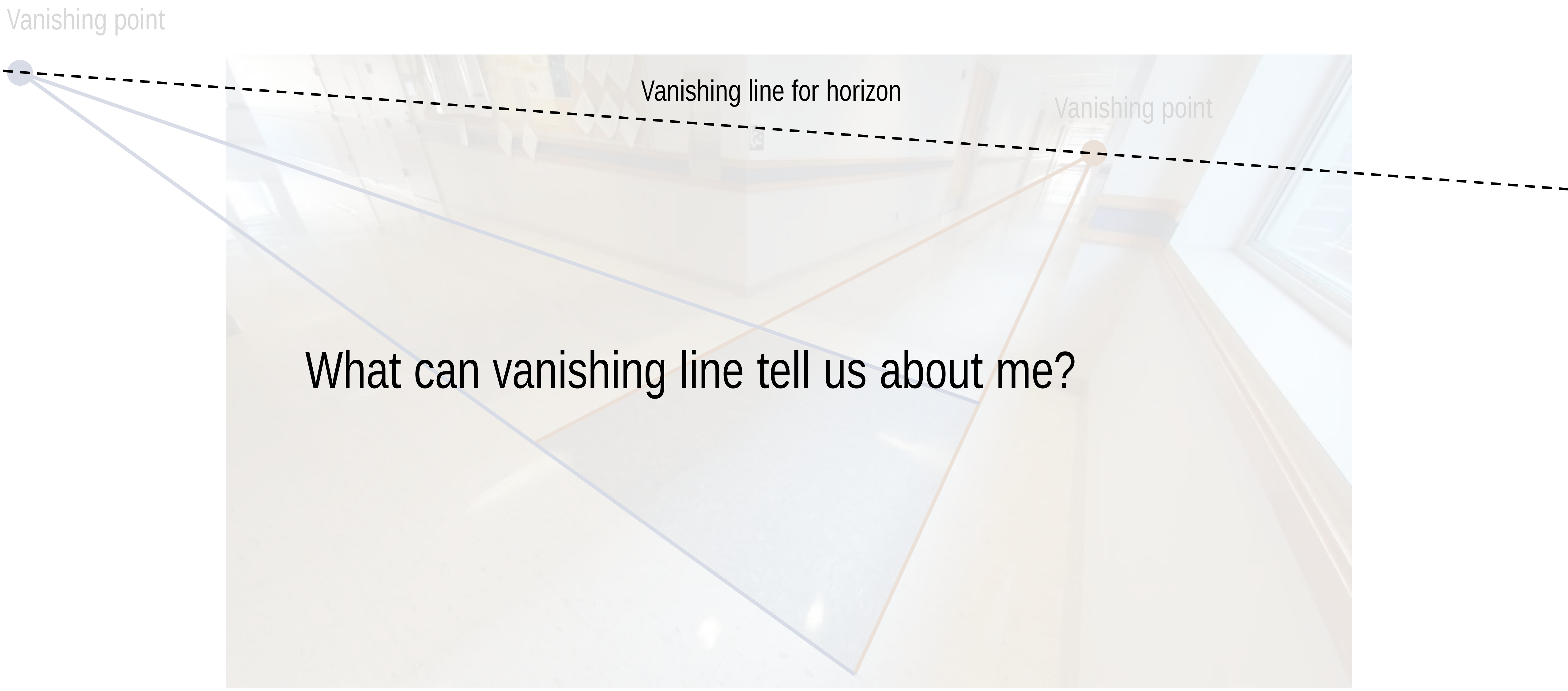
1. Parallel lines in 3D meet at the same vanishing point in image.
2. The 3D ray passing camera center and the vanishing point is parallel to the lines.
3. Multiple vanishing points exist.



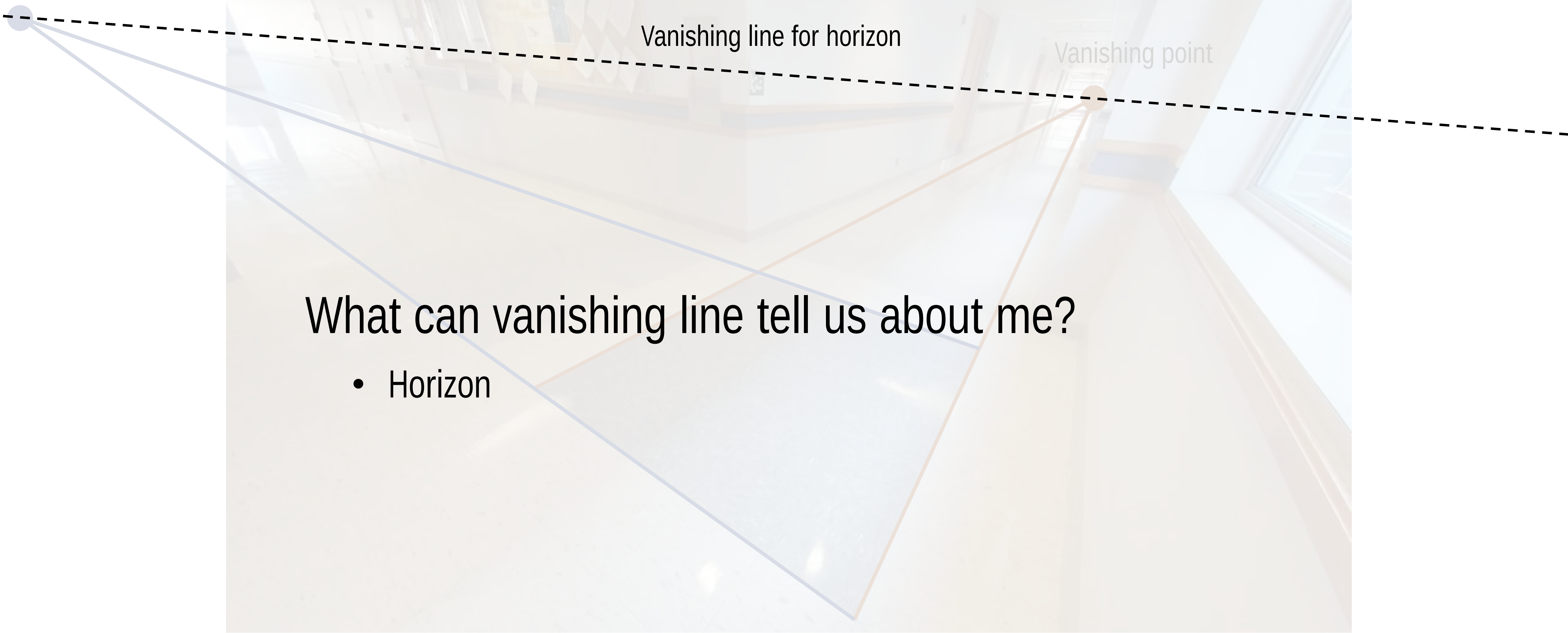

Vanishing point

Multiple vanishing point

Vanishing line: Horizon

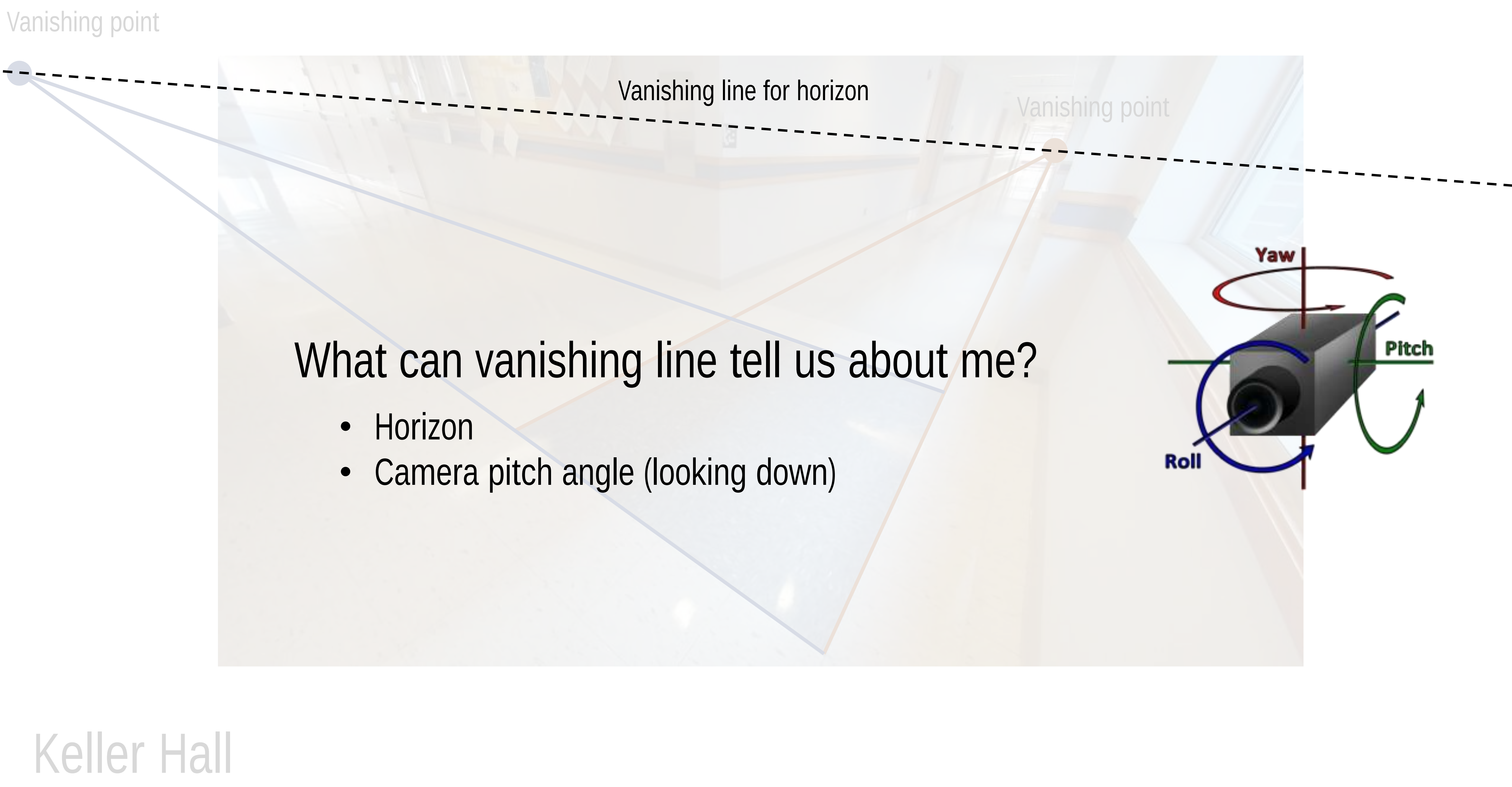


What can vanishing line tell us about me?



What can vanishing line tell us about me?

- Horizon



What can vanishing line tell us about me?

- Horizon
- Camera pitch angle (looking down)

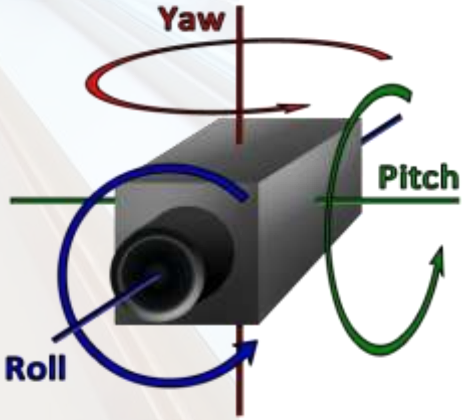
Vanishing point

Vanishing line for horizon

Vanishing point

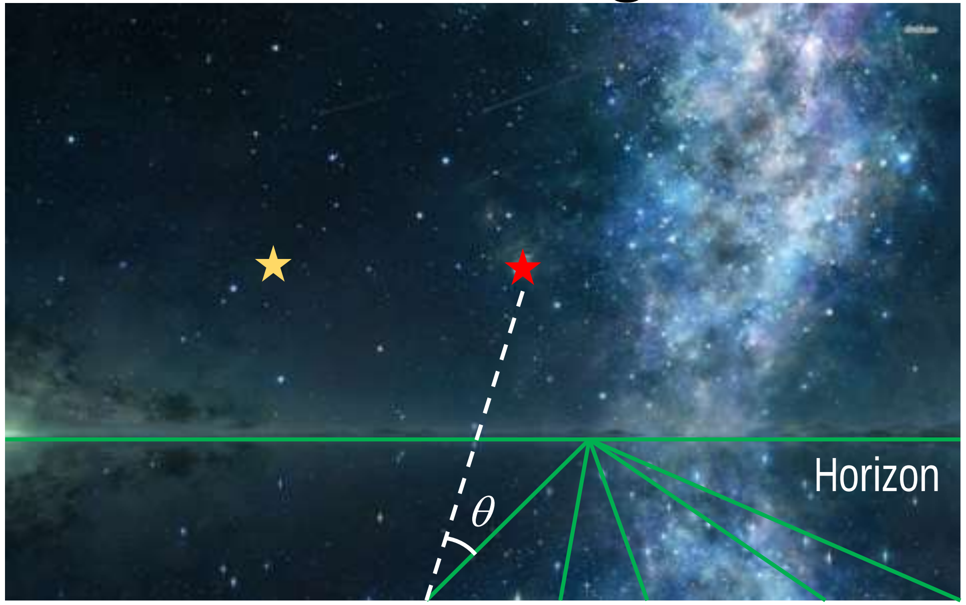
What can vanishing line tell us about me?

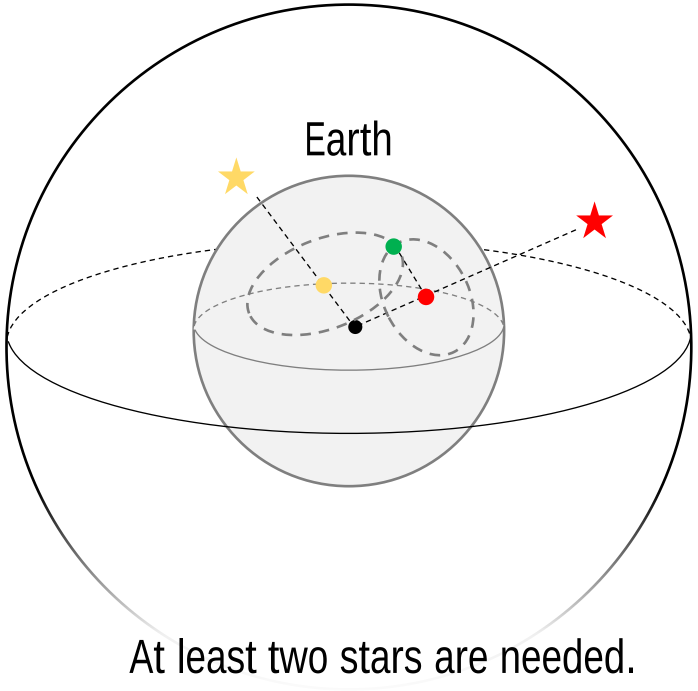
- Horizon
- Camera pitch angle (looking down)
- Camera roll angle (tilted toward right)



Keller Hall

Celestial Navigation



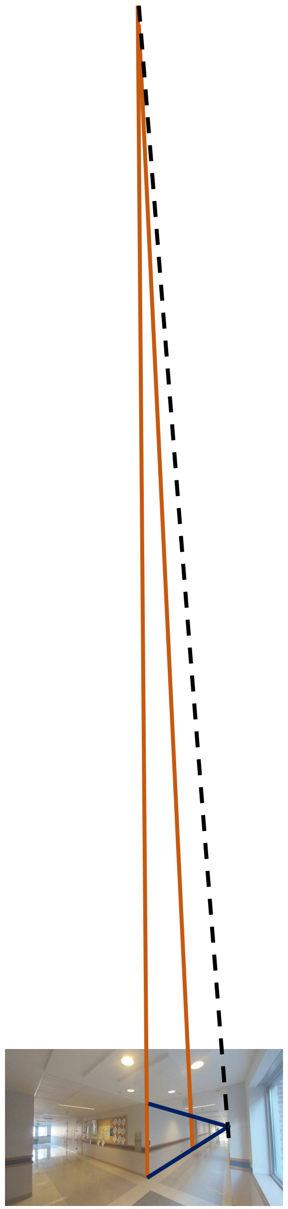


At least two stars are needed.

Two points at infinity (vanishing points) tells us about where I am.

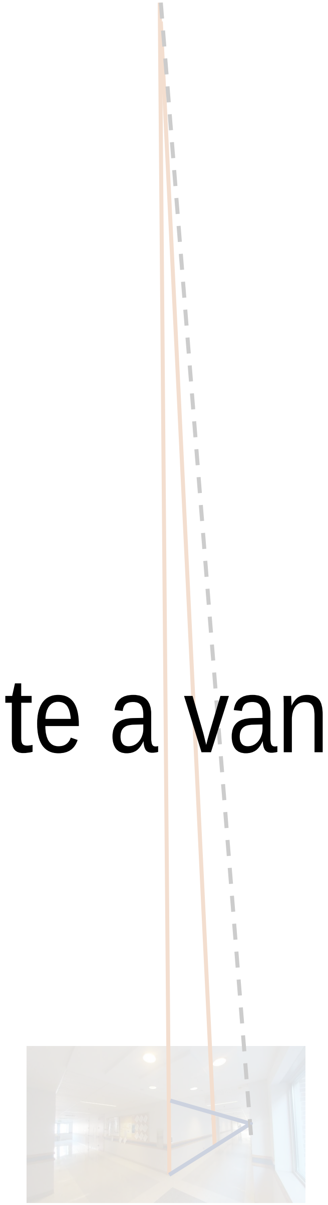
Parallel 3D planes share the vanishing line.

Different plane produces different vanishing line.



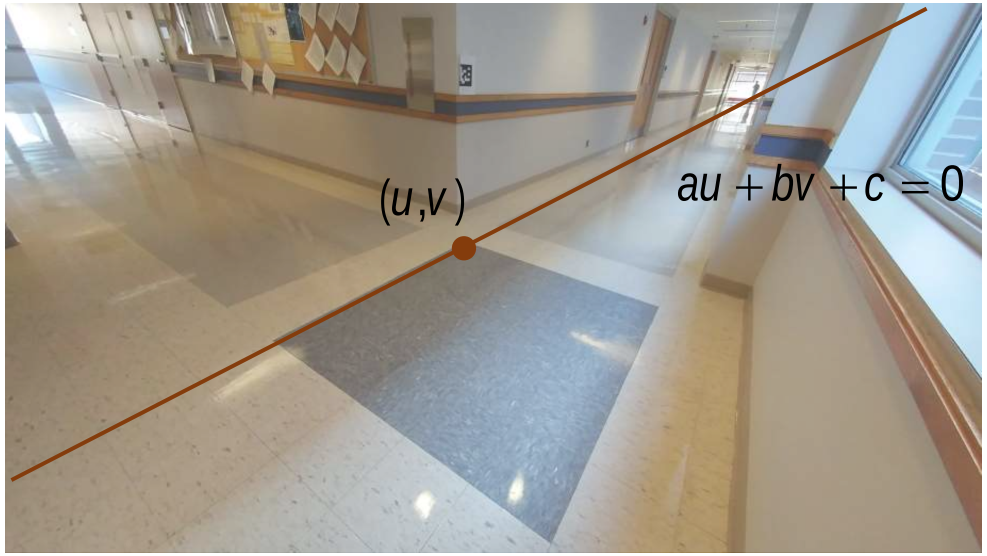
Different plane produces different vanishing line.

How to compute a vanishing point?



Different plane produces different vanishing line.

Point-Line in Image

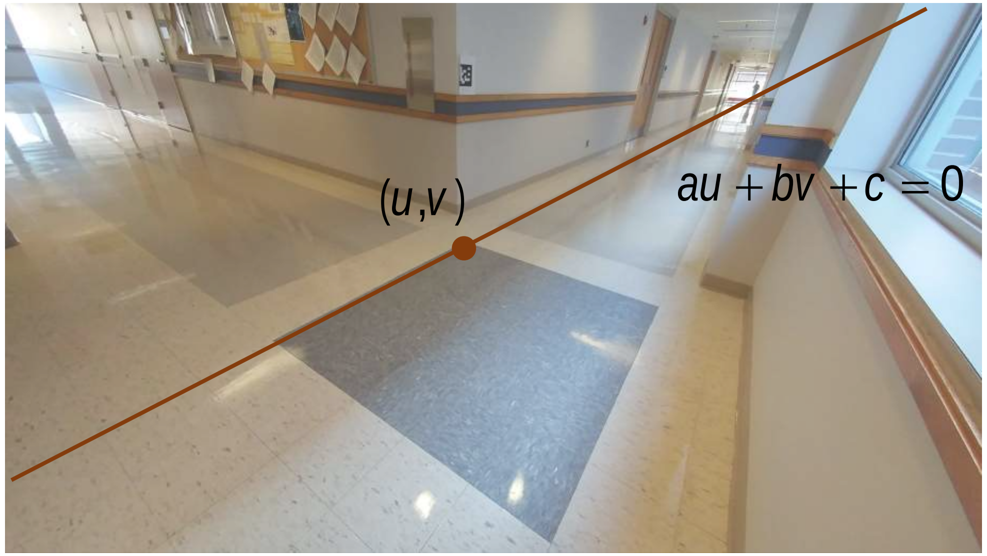


A 2D line passing through 2D point (u,v):

$au + bv + c = 0$

Line parameter: (a,b,c)

Point-Line in Image



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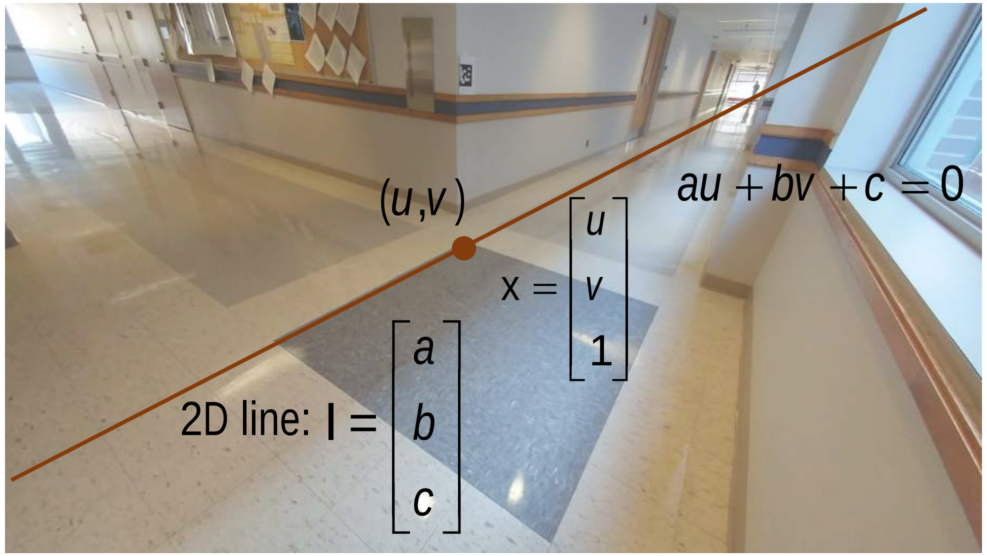
Line parameter: (a,b,c)

$$au + bv + c = 0 \longrightarrow \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathrm{l}^{\mathsf{T}}\mathrm{x} = 0$$

Point-Line in Image



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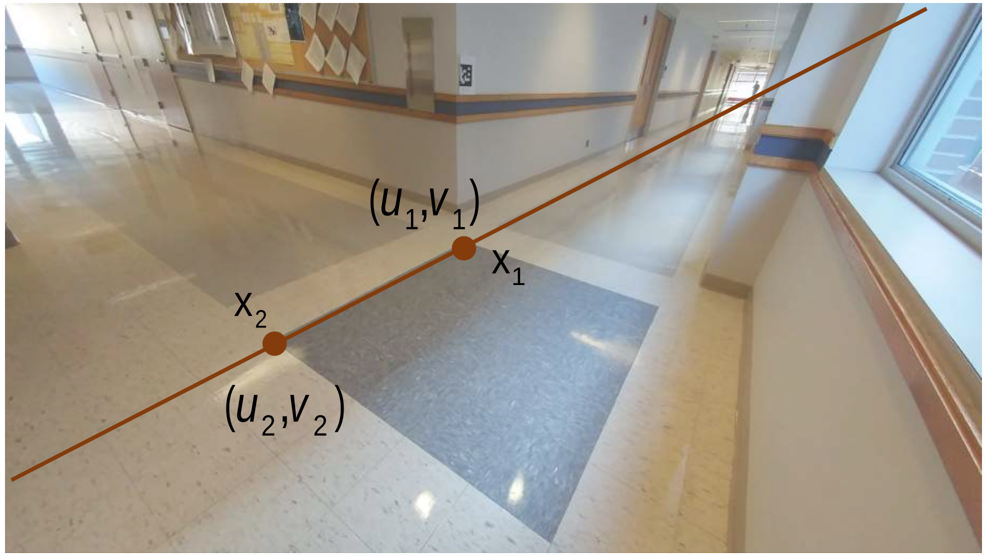
where $\mathbf{x} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$ and $\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

2D point   Line parameter

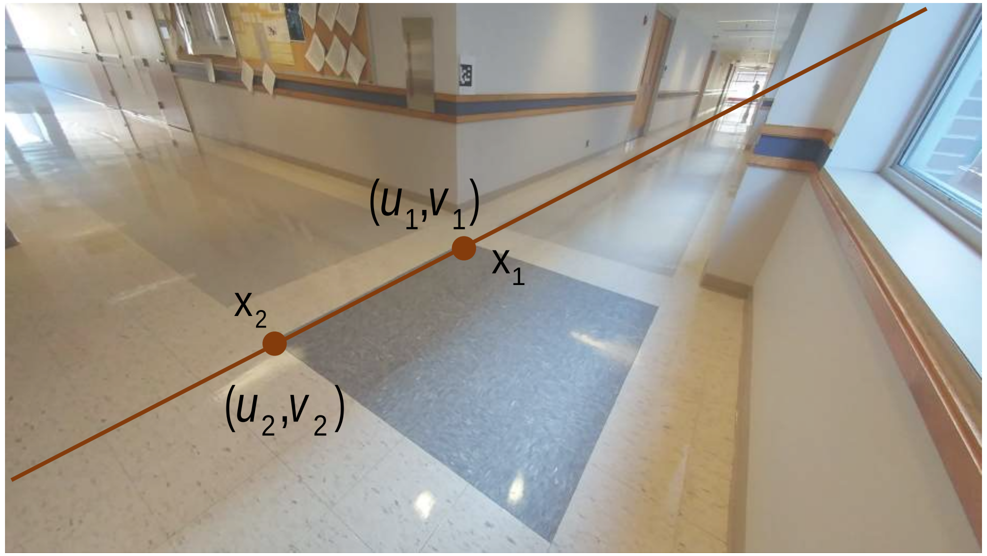
Point-Point in Image



A 2D line passing through two 2D points:

$$au_1 + bv_1 + c = 0 \qquad au_2 + bv_2 + c = 0$$

Point-Point in Image



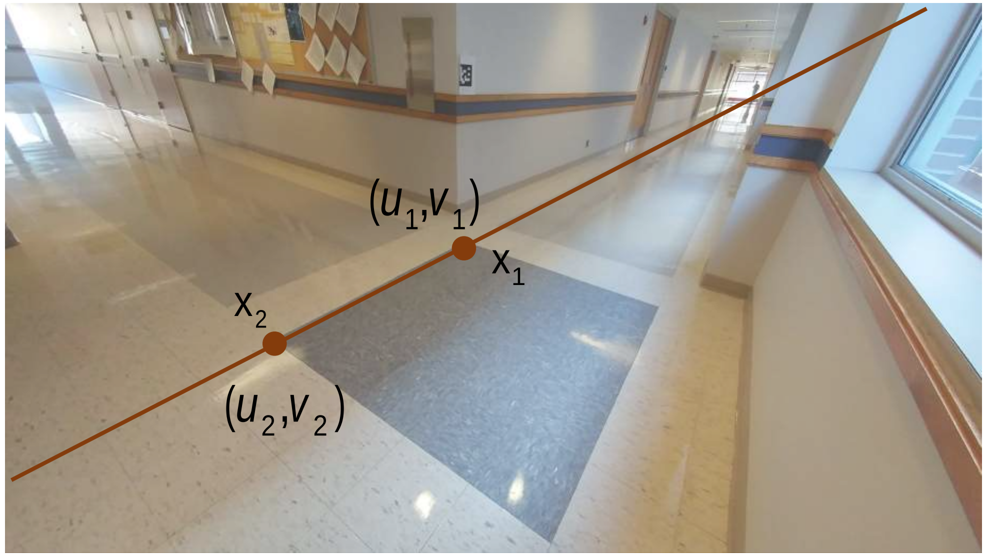
A 2D line passing through two 2D points:

$$au_1 + bv_1 + c = 0 \qquad au_2 + bv_2 + c = 0$$

$$\mathrm{x}_1^{\top}\mathbf{l} = 0 \qquad \mathrm{x}_2^{\top}\mathbf{l} = 0$$

where $\mathrm{x}_1 = \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix} \quad \mathrm{x}_2 = \begin{bmatrix} u_2 \\ v_2 \\ 1 \end{bmatrix} \quad \mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Point-Point in Image



A 2D line passing through two 2D points:

$$au_1 + bv_1 + c = 0 \qquad au_2 + bv_2 + c = 0$$

$$x_1^T l = 0 \qquad x_2^T l = 0$$

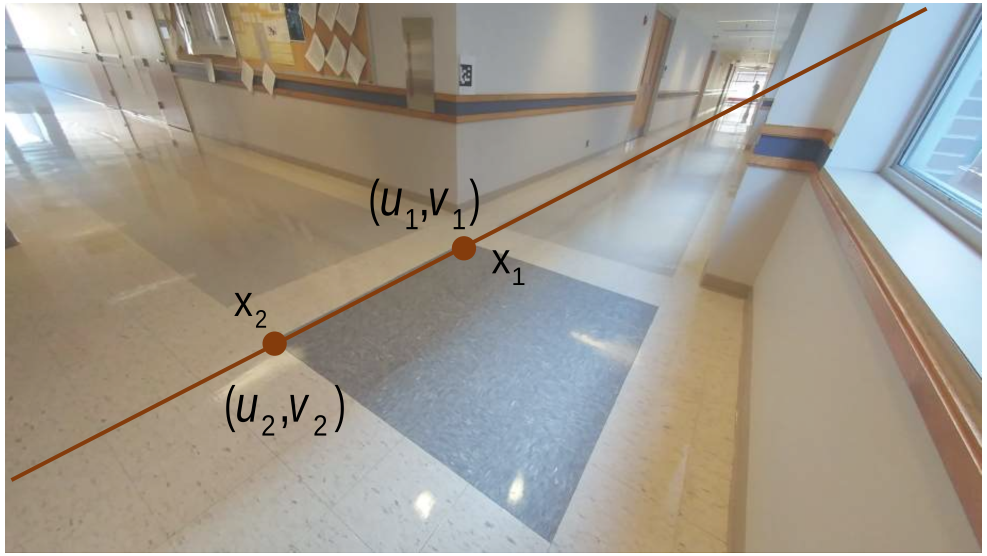
where $x_1 = \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} u_2 \\ v_2 \\ 1 \end{bmatrix}$ $l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\rightarrow \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} l = 0$$

$$A \; l = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2x3

Point-Point in Image



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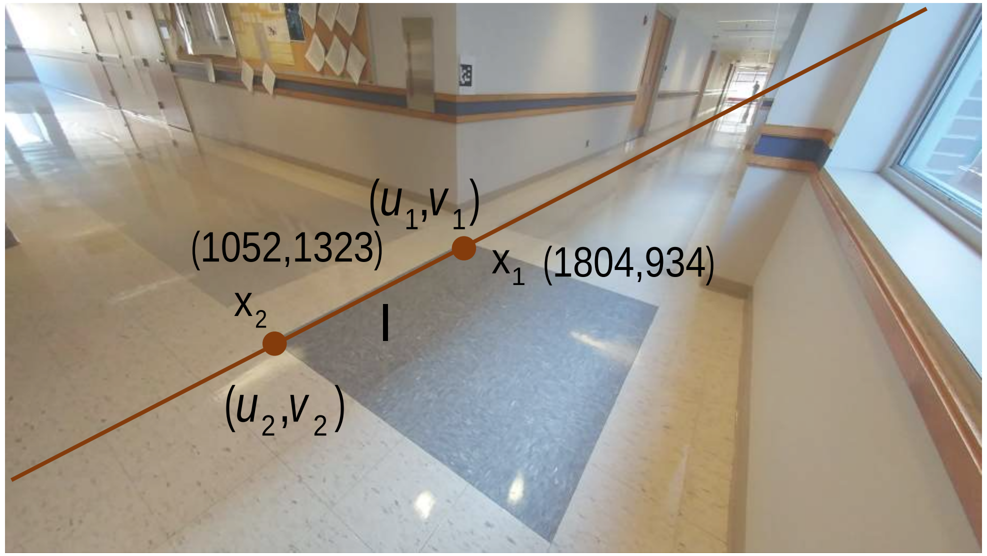
$$\rightarrow \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} l = 0$$

$$A \, l = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad l = \text{null}(A) \quad \text{or} \quad l = x_1 \times x_2$$

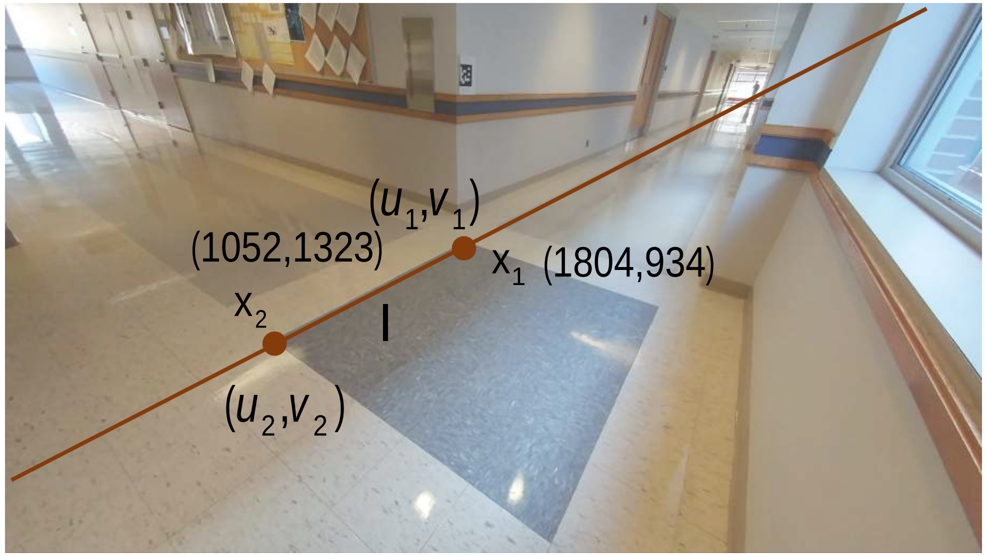
2x3

Point-Point in Image

x1 = [1804;934;1];
x2 = [1052;1323;1];



Point-Point in Image



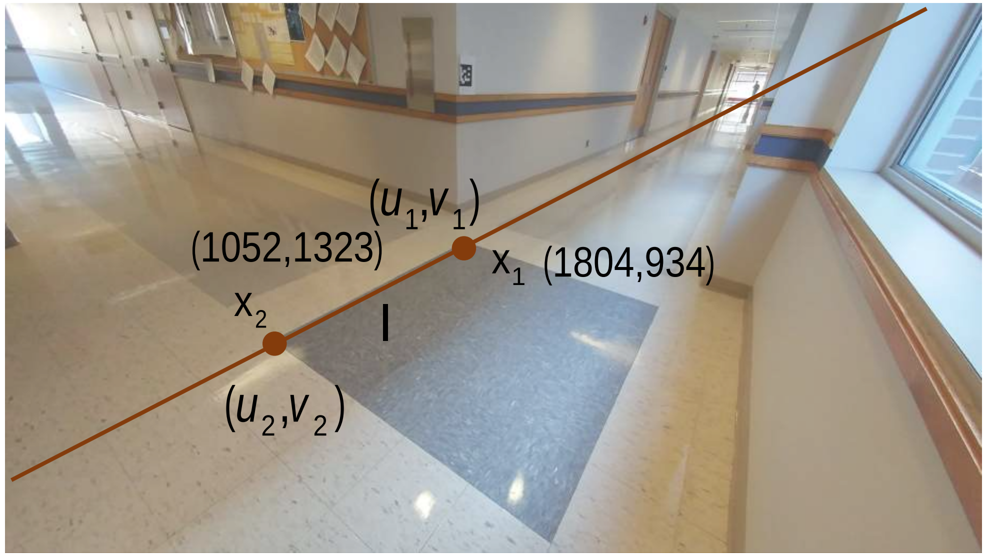
GetLineFromTwoPoints.m

```
x1 = [1804;934;1];
x2 = [1052;1323;1];

l = Vec2Skew(x1)*x2;
```

Cross product

Point-Point in Image



GetLineFromTwoPoints.m

```
x1 = [1804;934;1];
x2 = [1052;1323;1];

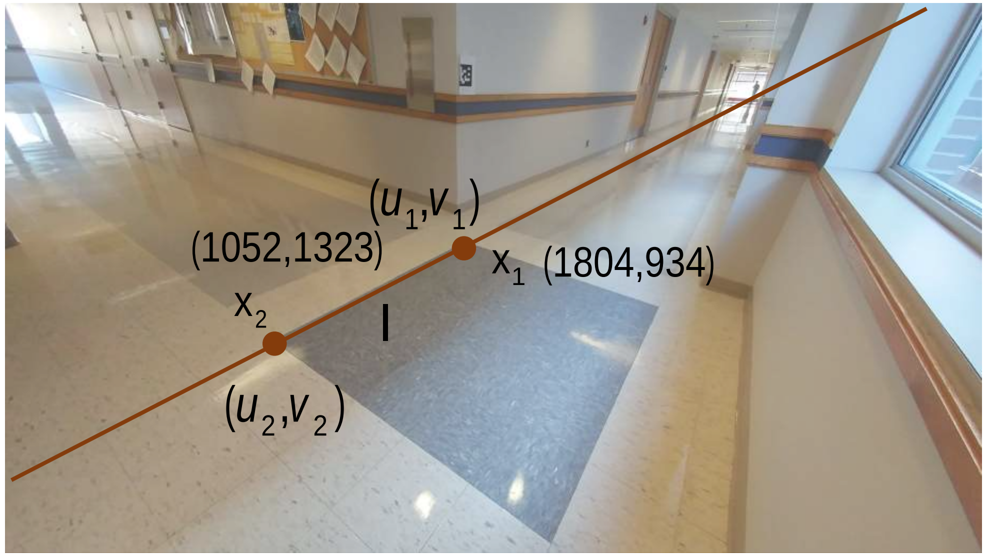
l = Vec2Skew(x1)*x2;
```

Cross product

```
l =

    -389
    -752
 1404124
```

Point-Point in Image



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```

Cross product

```
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   -389
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1404124
```

Cross product with skew-symmetric matrix representation:

$$\mathbf{a}\times\mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [\mathbf{a}]_\times \mathbf{b}$$

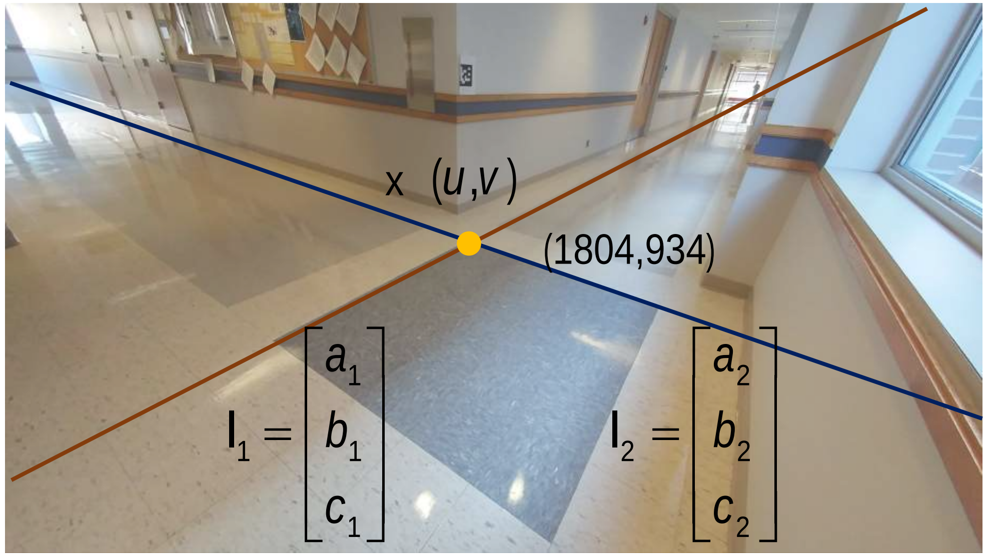
Vec2Skew.m

```
function skew = Vec2Skew(v)
skew = [0 -v(3) v(2);
      v(3) 0 -v(1);
      -v(2) v(1) 0];
```

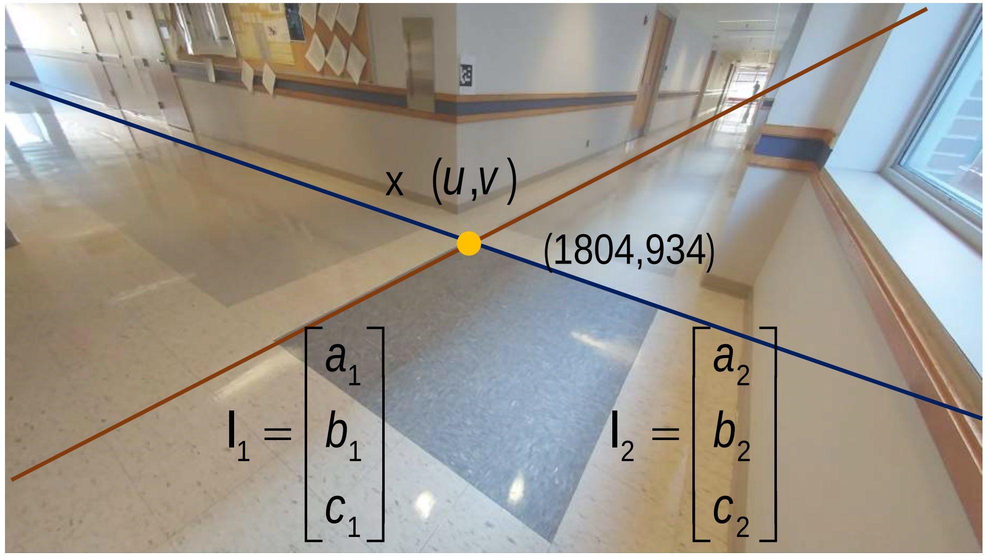
Line-Line in Image

Two 2D lines in an image intersect at a 2D point:

$$a_1 u + b_1 v + c_1 = 0 \qquad a_2 u + b_2 v + c_2 = 0$$



Line-Line in Image



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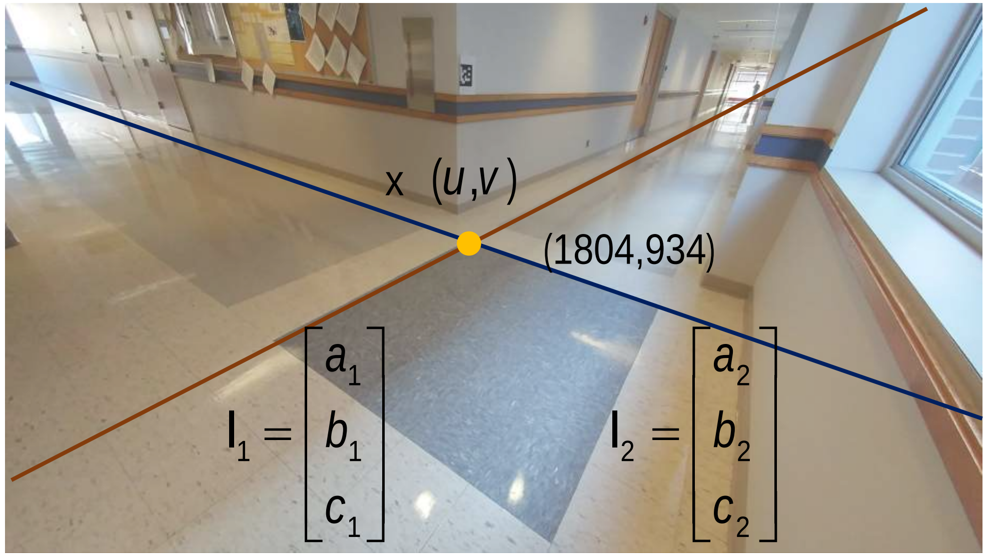
$$\mathbf{l}_1^T \mathbf{x} = 0 \qquad \mathbf{l}_2^T \mathbf{x} = 0$$

where $\mathbf{x} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad \mathbf{l}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \quad \mathbf{l}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$

Line-Line in Image



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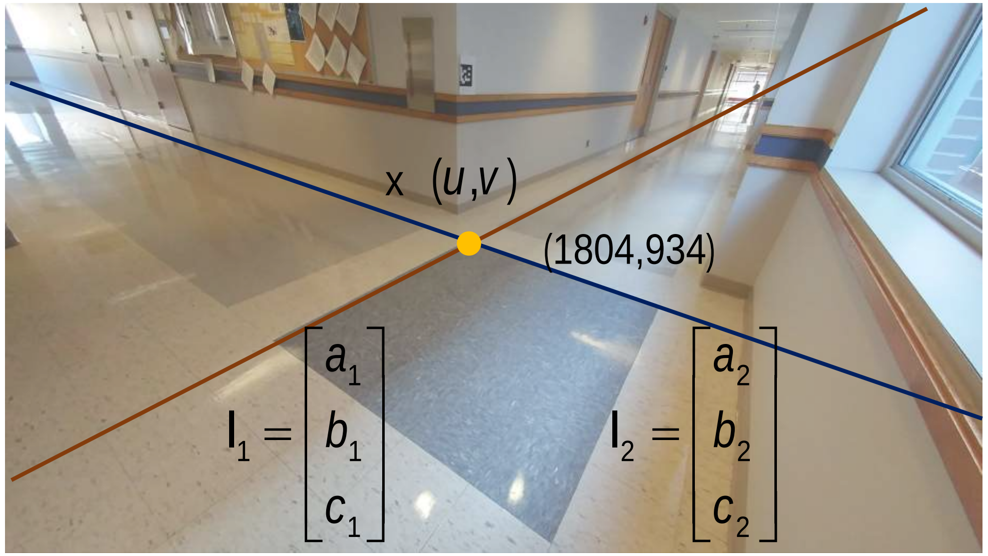
$$\rightarrow \begin{bmatrix} \mathbf{l}_1^T \\ \mathbf{l}_2^T \end{bmatrix} \mathbf{x} = 0$$

A $=$ $\begin{matrix} 0 \\ 0 \end{matrix}$ $\rightarrow$ $=$ null(A)

2x3

or $\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$

Line-Line in Image



GetPointFromTwoLines.m

```
l1 = [-398;-752;1404124];
l2 = [310;-924;303790];
x = Vec2Skew(l1)*l2;
x = x/x(3)


x =

    1779.0
    925.6
    1
```

similar to (1804,934)

2D Point and Line Duality

The 2D line joining two points:

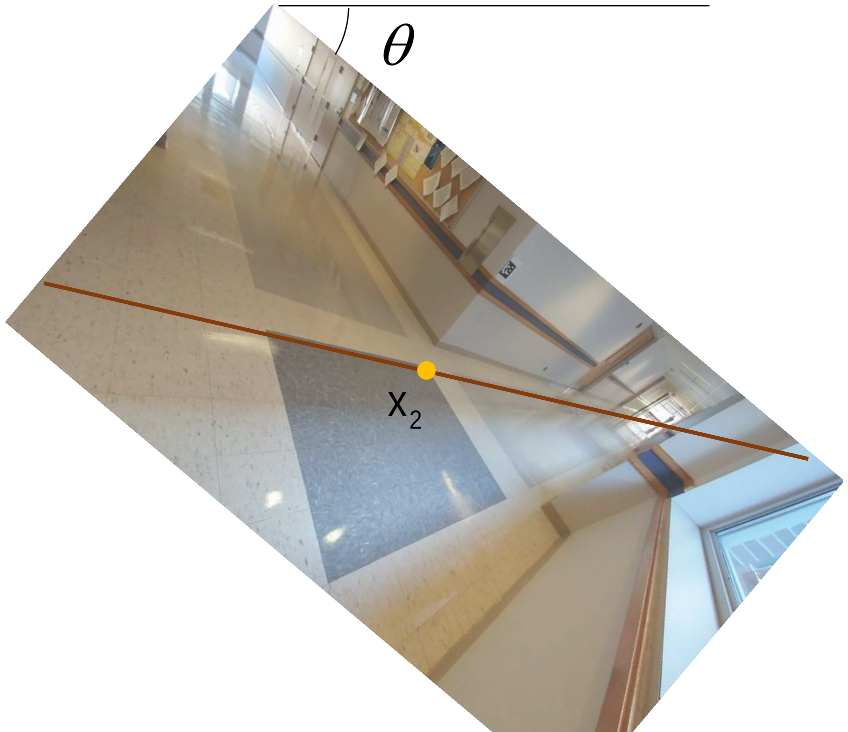
$$l = x_1 \times x_2$$

The intersection between two lines:

$$x = l_1 \times l_2$$

Given any formula, we can switch the meaning of point and line to get another formula.

2D Point and Line Duality



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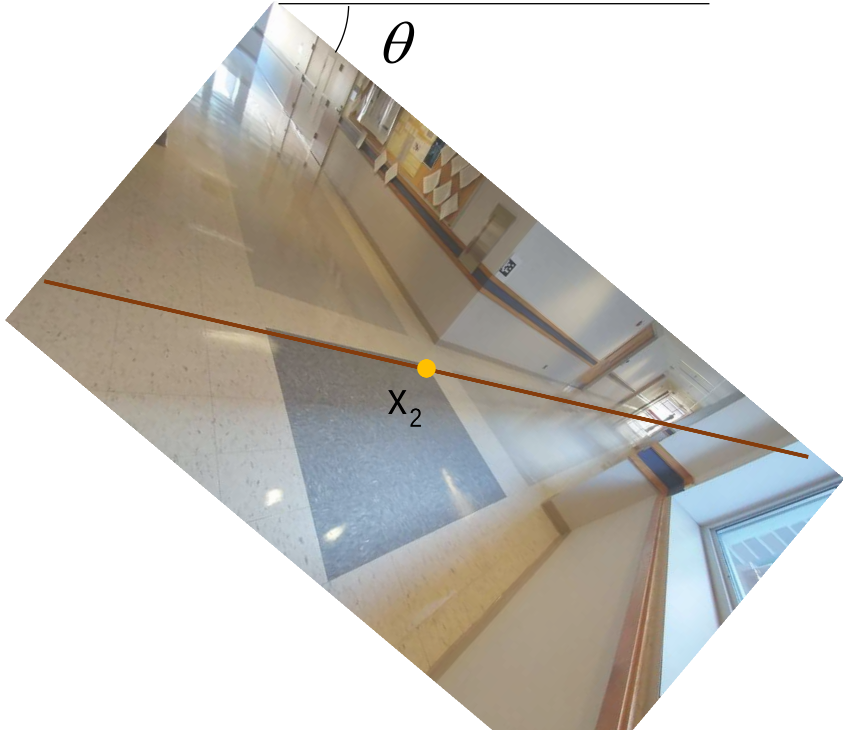
$$x_2 = Tx_1 \leftrightarrow l_2 = T^{-T} l_1$$

T: Transformation

2D Point and Line Duality



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$$\because l_1^T x_1 = 0$$

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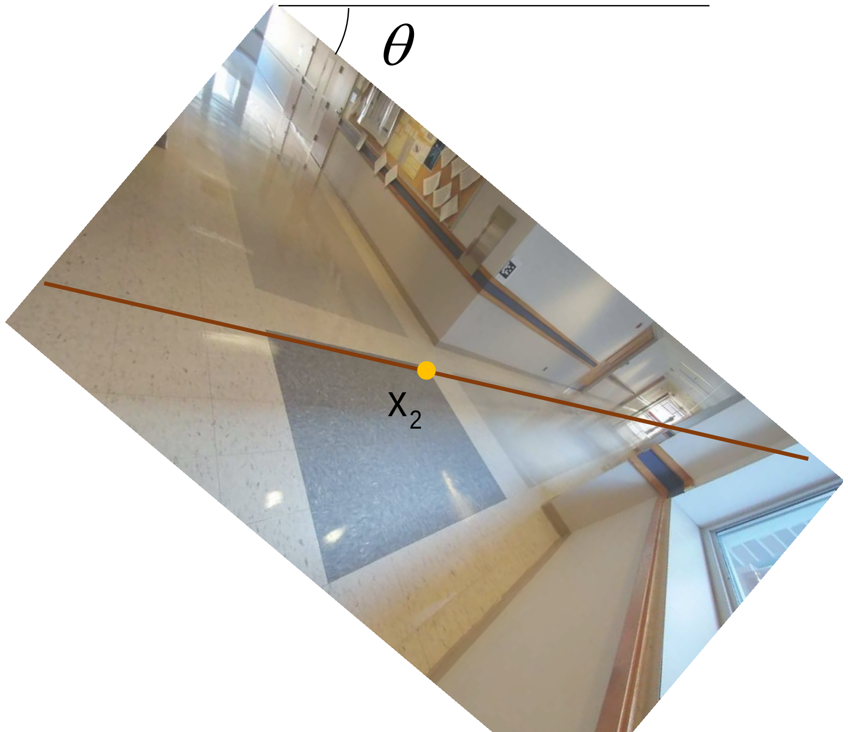

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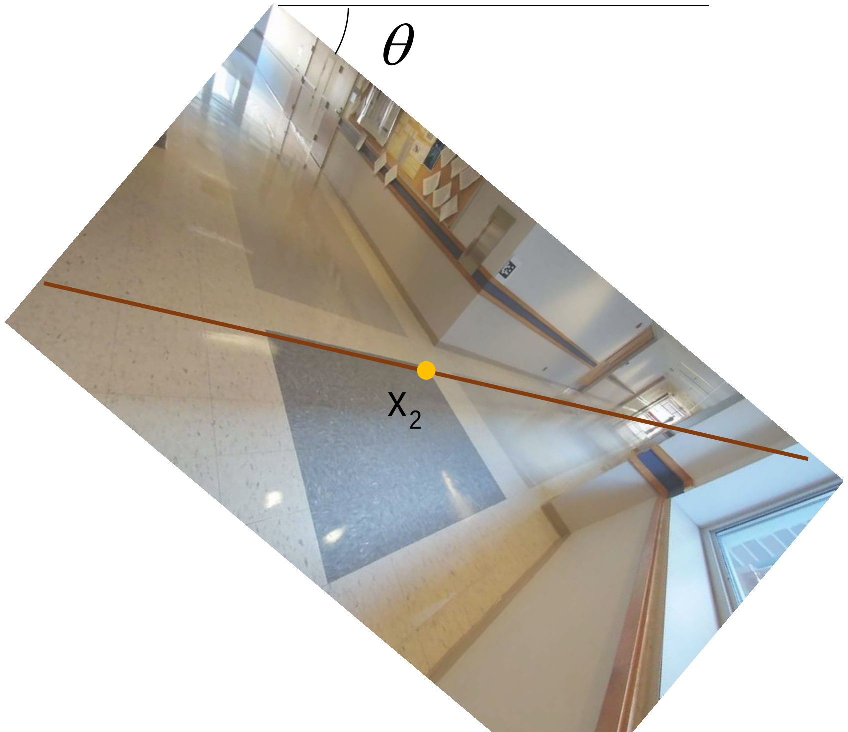
$$x_2 = Tx_1 \leftrightarrow l_2 = T^{-T} l_1$$ T: Transformation

$$\because l_1^T x_1 = \left(l_1^T T^{-1}\right)\left(Tx_1\right) = 0$$

2D Point and Line Duality



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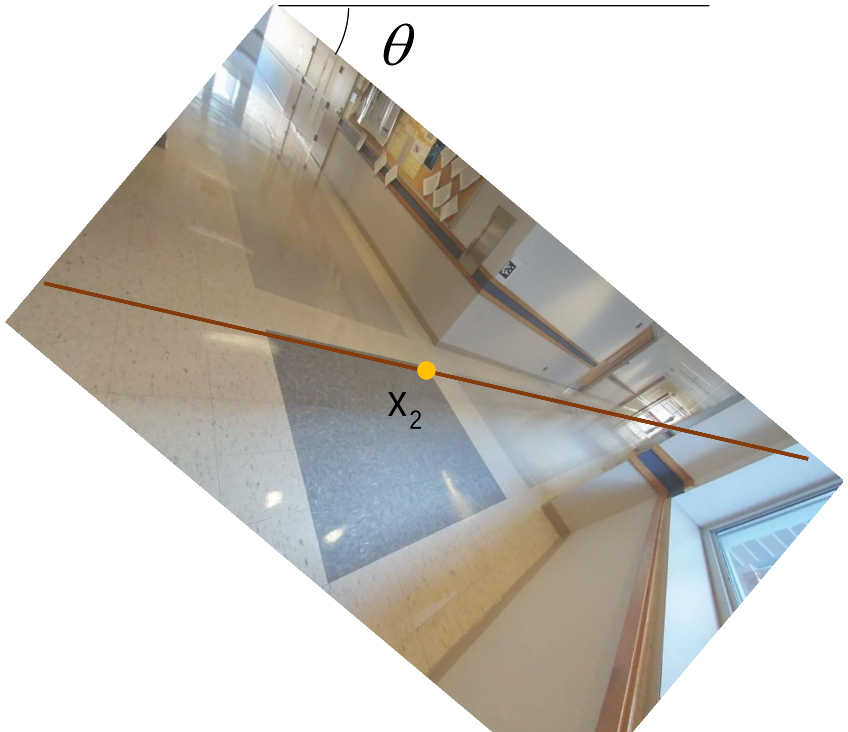
$$\because l_1^T x_1 = \left(l_1^T T^{-1}\right)\left(Tx_1\right) = \left(T^{-T} l_1\right)^T \left(Tx_1\right) = l_2^T x_2$$

2D Point and Line Duality



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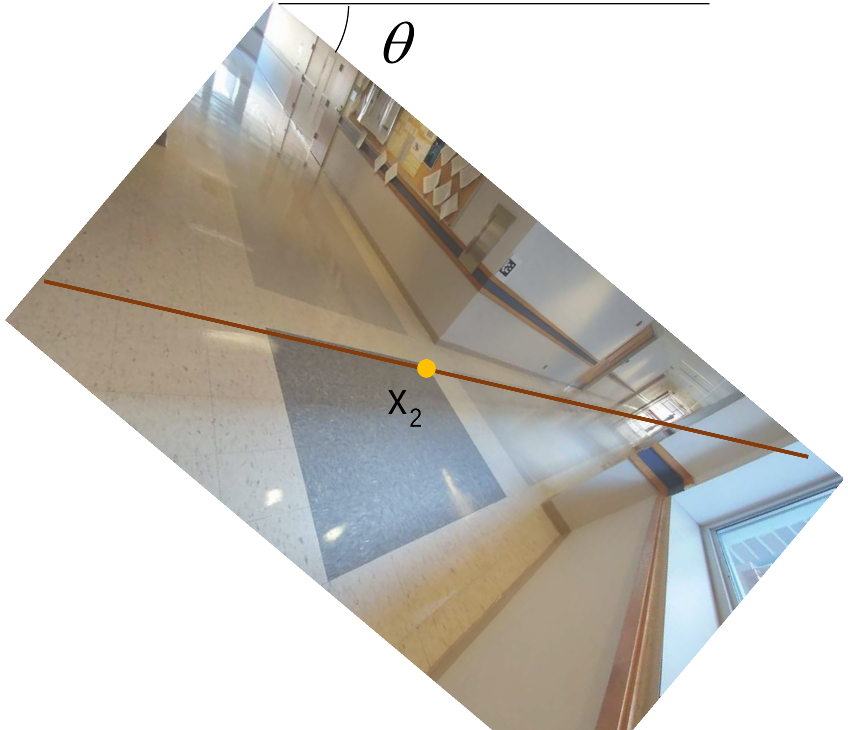
Given any formula, we can switch the meaning of point and line to get another formula.

$$\mathrm{x}_2 = \mathrm{T}\mathrm{x}_1 \leftrightarrow \mathrm{l}_2 = \mathrm{T}^{-\mathrm{T}}\mathrm{l}_1 \qquad \text{T: Transformation}$$

$$\because \mathrm{l}_1^{\mathrm{T}}\mathrm{x}_1 = \left(\mathrm{l}_1^{\mathrm{T}}\mathrm{T}^{-1}\right)\left(\mathrm{T}\mathrm{x}_1\right) = \left(\mathrm{T}^{-\mathrm{T}}\mathrm{l}_1\right)^{\mathrm{T}}\left(\mathrm{T}\mathrm{x}_1\right) = \mathrm{l}_2^{\mathrm{T}}\mathrm{x}_2$$

$$\mathrm{x}_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{x}_1 \longleftrightarrow \qquad ?$$

2D Point and Line Duality



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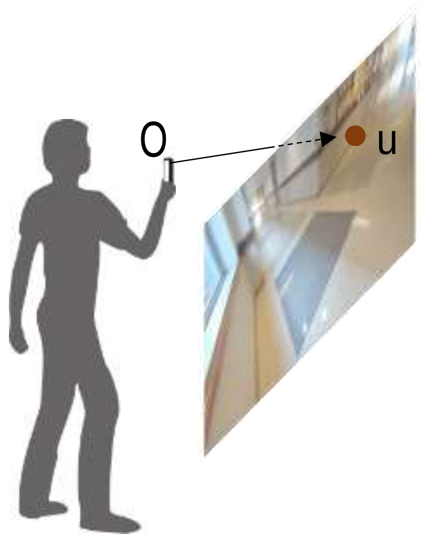
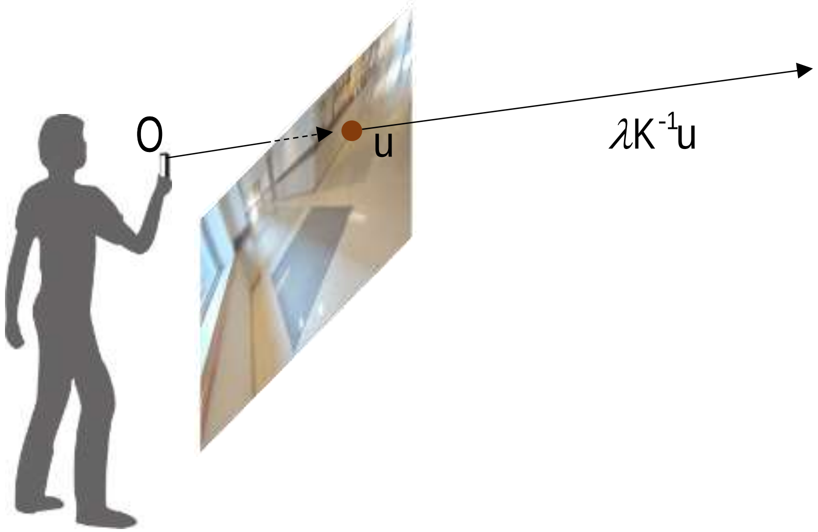
$$\because l_1^T x_1 = \left(l_1^T T^{-1}\right)\left(Tx_1\right) = \left(T^{-T} l_1\right)^T \left(Tx_1\right) = l_2^T x_2$$

$$x_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} x_1 \longleftrightarrow l_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-T} l_1 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} l_1$$

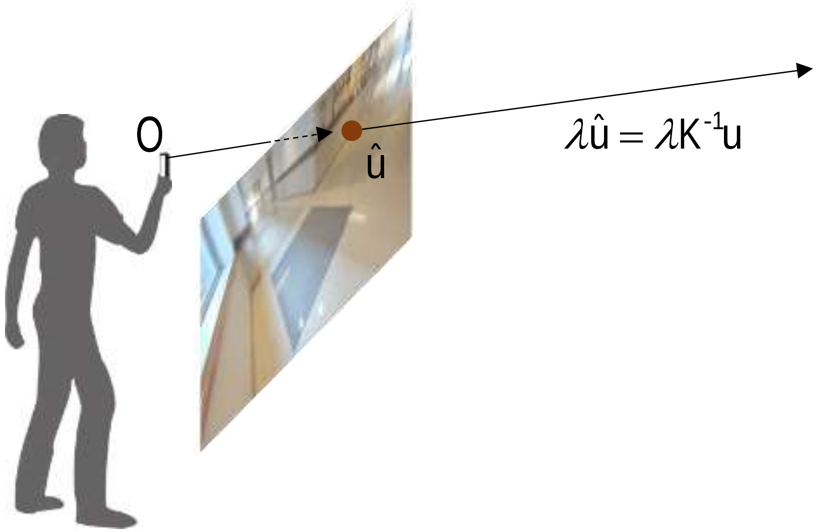
Geometric Interpretation (Point)



Geometric Interpretation (Point)



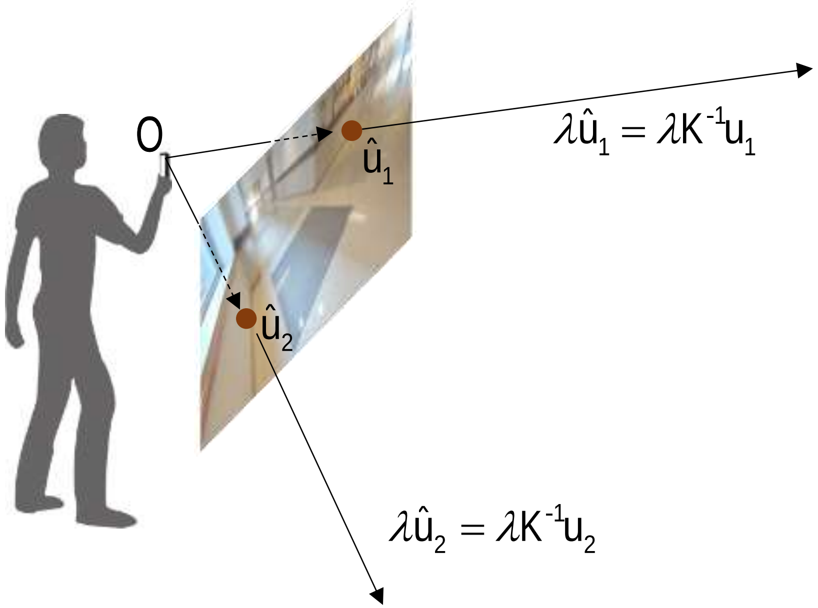
Geometric Interpretation (Point)

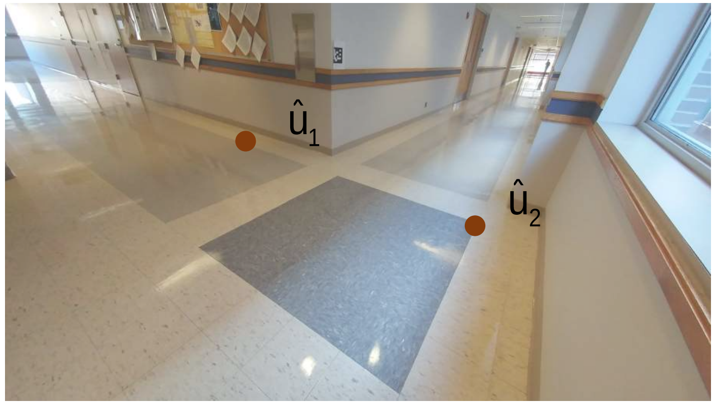


Normalized coordinate:

$$\hat{u} = K^{-1} u$$

Geometric Interpretation (Point)

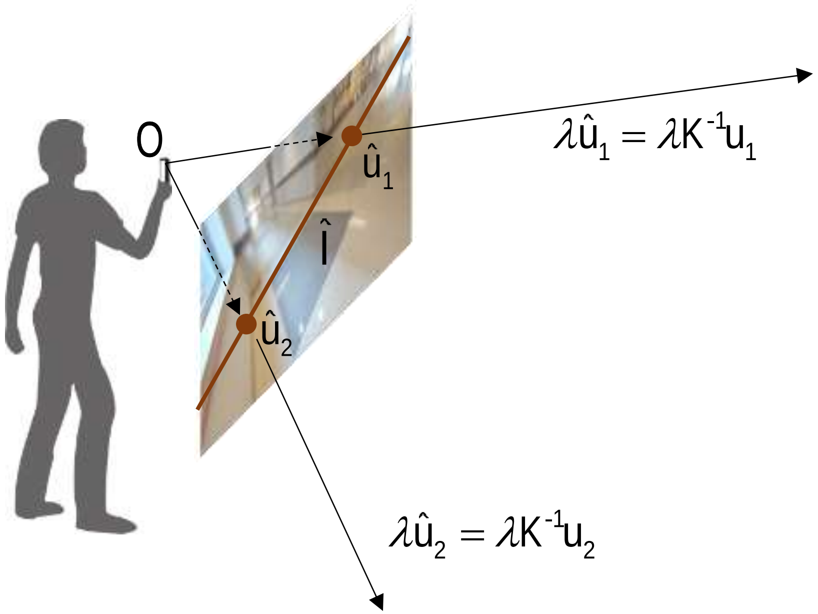


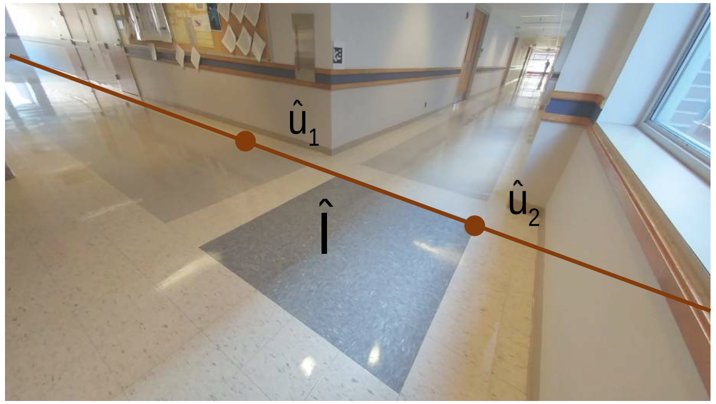


Normalized coordinate:

$$\hat{u}_1 = K^{-1}u_1 \qquad \hat{u}_2 = K^{-1}u_2$$

Geometric Interpretation (Line)





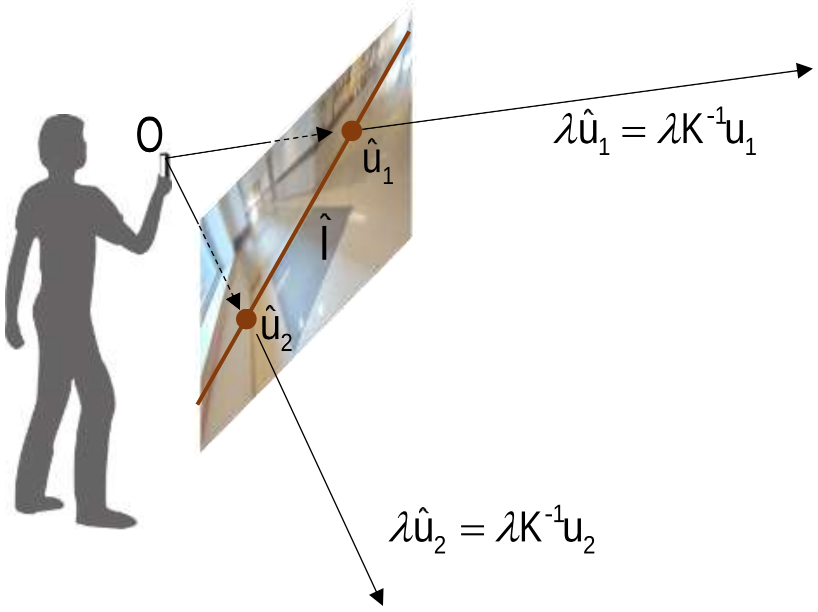
Normalized coordinate:

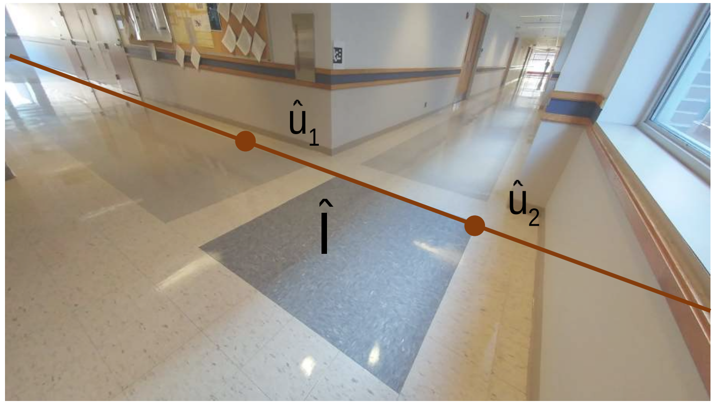
$$\hat{u}_1 = K^{-1}u_1 \qquad \hat{u}_2 = K^{-1}u_2$$

$$\longrightarrow \quad \hat{l} = \hat{u}_1 \times \hat{u}_2$$

where $\hat{l} = \ ?$

Geometric Interpretation (Line)





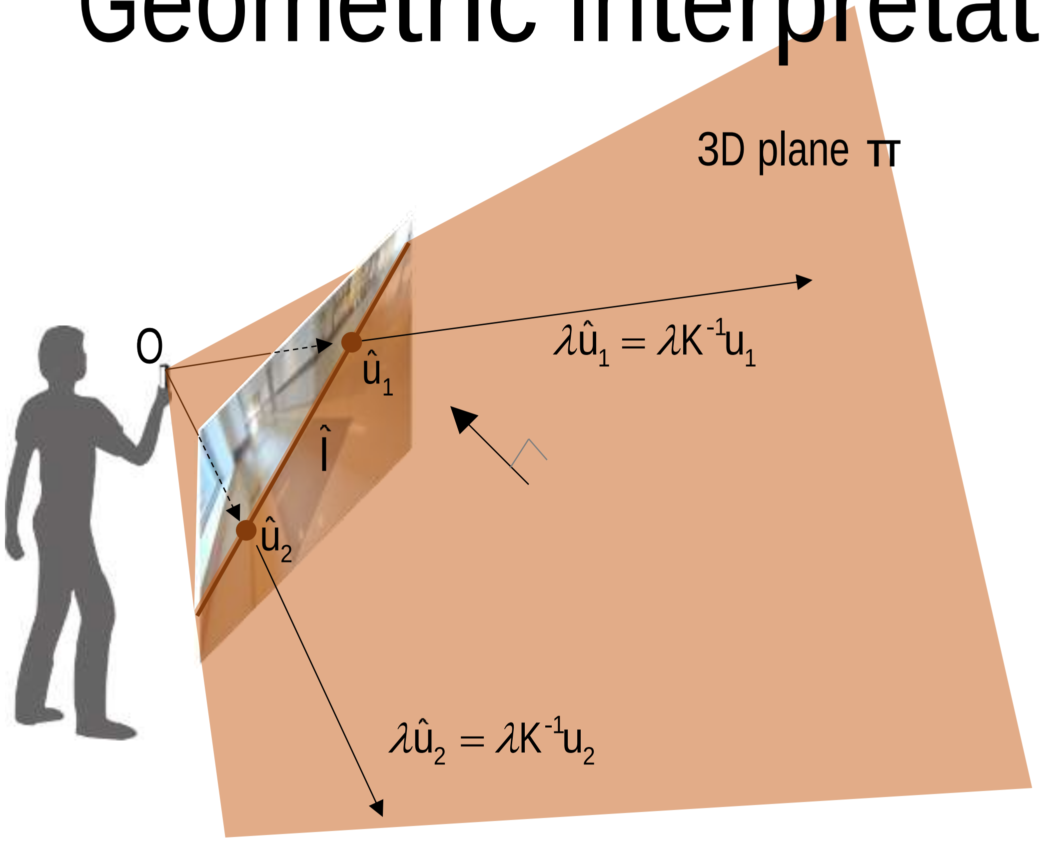
Normalized coordinate:

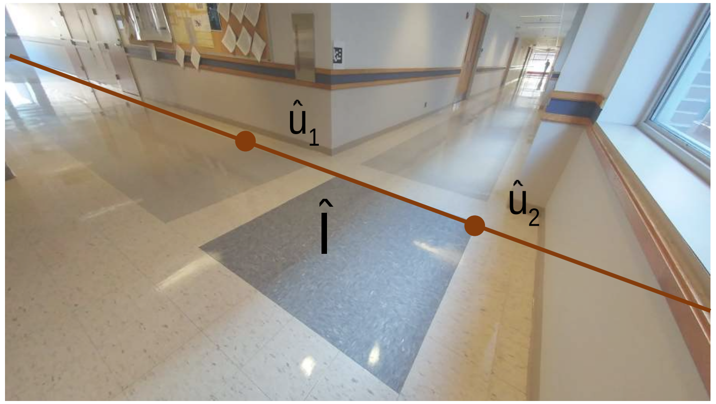
$$\hat{u}_1 = K^{-1}u_1 \qquad \hat{u}_2 = K^{-1}u_2$$

$$\longrightarrow \quad \hat{l} = \hat{u}_1 \times \hat{u}_2$$

where $\hat{l} = \left(K^{-1}\right)^{-T} l = K^{T} l$ due to duality

Geometric Interpretation (Line)





Normalized coordinate:

$$\hat{u}_1 = K^{-1}u_1 \qquad \hat{u}_2 = K^{-1}u_2$$

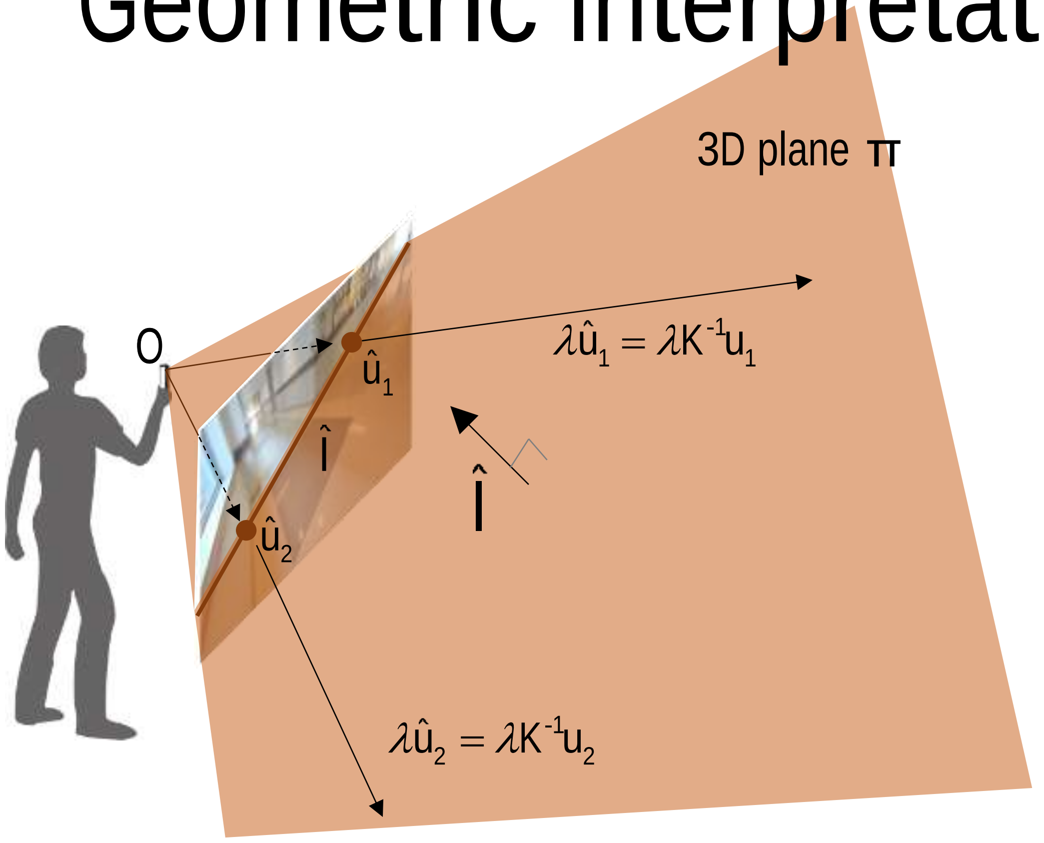
$$\longrightarrow \hat{l} = \hat{u}_1 \times \hat{u}_2$$

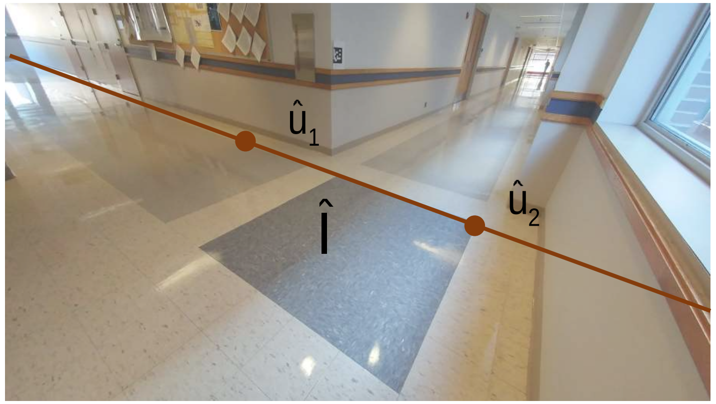
where $\hat{l} = \left(K^{-1}\right)^{-T} l = K^{T} l$ due to duality

A 2D line in an image defines to a 3D plane passing the camera center:

$$\hat{l} \rightarrow \pi$$

Geometric Interpretation (Line)





Normalized coordinate:

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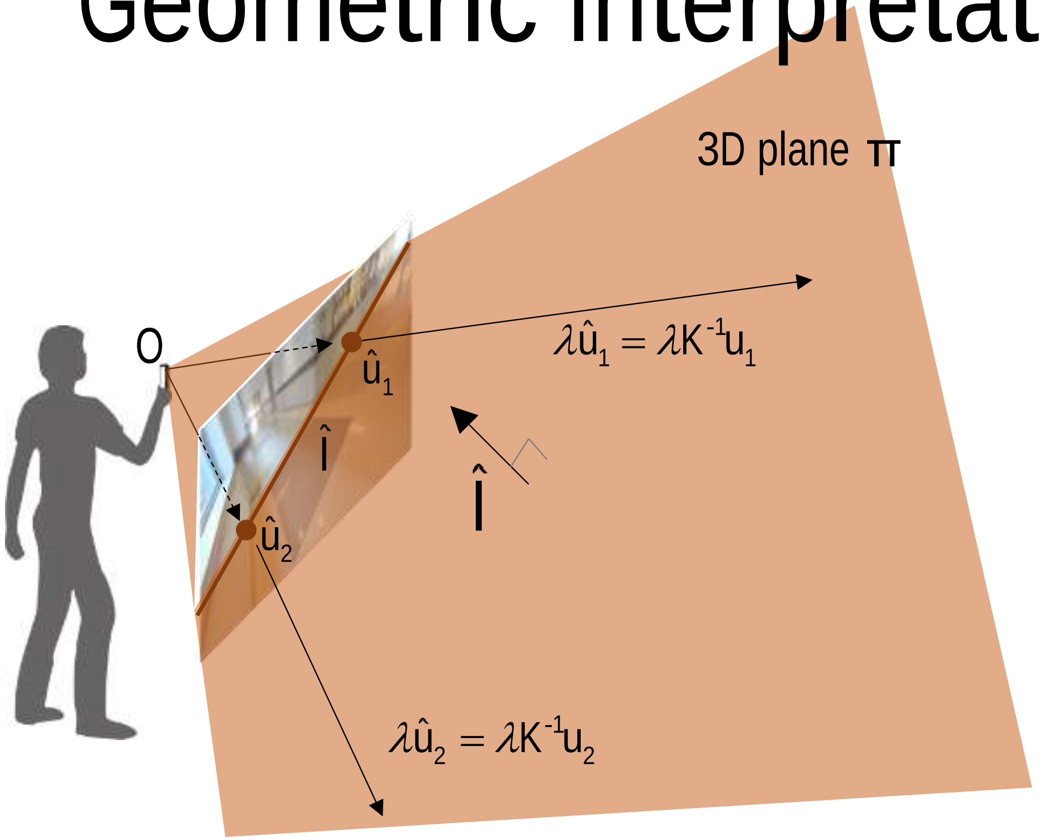
where $\hat{l} = \left(K^{-1}\right)^{-T} l = K^{T} l$ due to duality

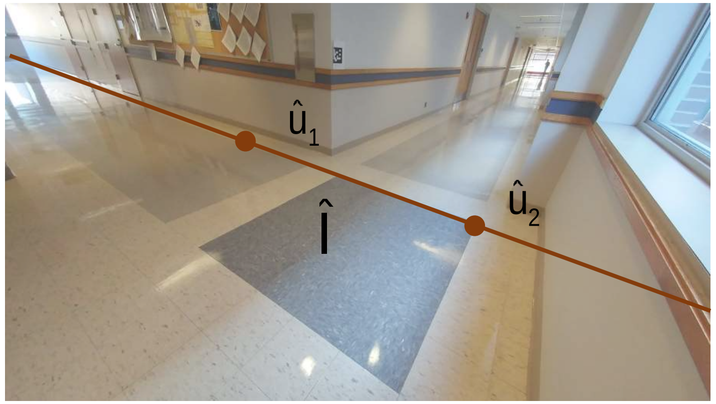
A 2D line in an image defines to a 3D plane passing the camera center:

$$\hat{l} \rightarrow \pi$$

Plane normal: ? $= \lambda\hat{l}$

Geometric Interpretation (Line)





Normalized coordinate:

$$\hat{u}_1 = K^{-1}u_1 \qquad \hat{u}_2 = K^{-1}u_2$$

$$\longrightarrow \quad \hat{l} = \hat{u}_1 \times \hat{u}_2$$

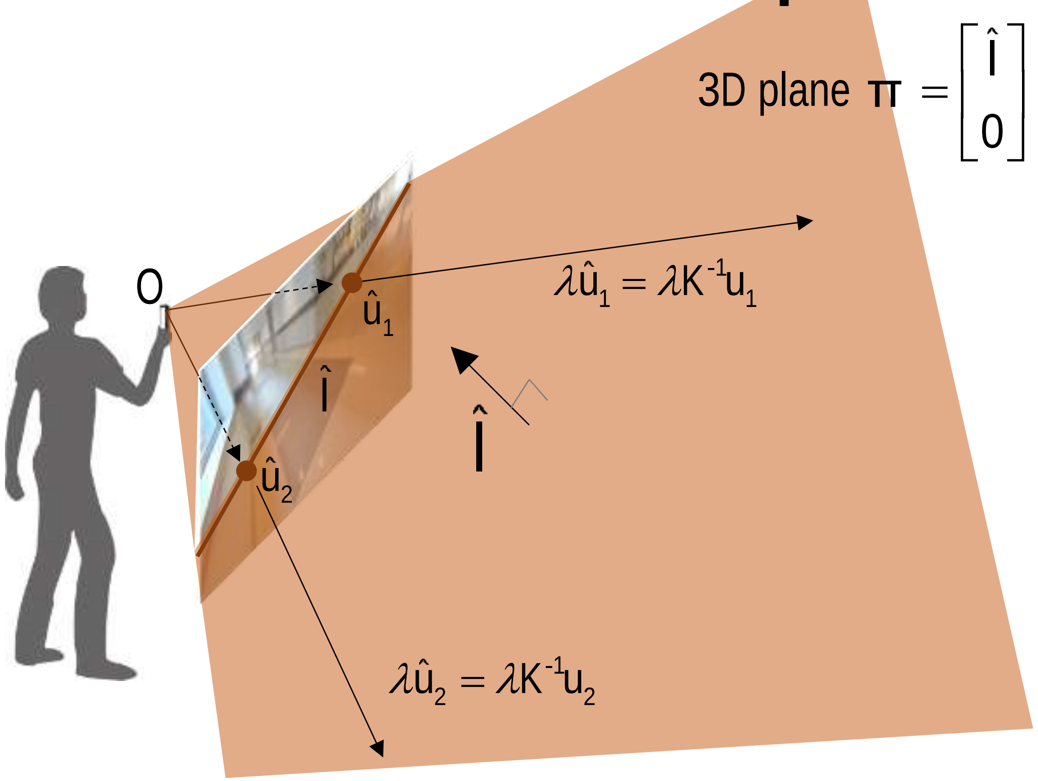
where $\hat{l} = \left(K^{-1}\right)^{-T} l = K^{T} l$ due to duality

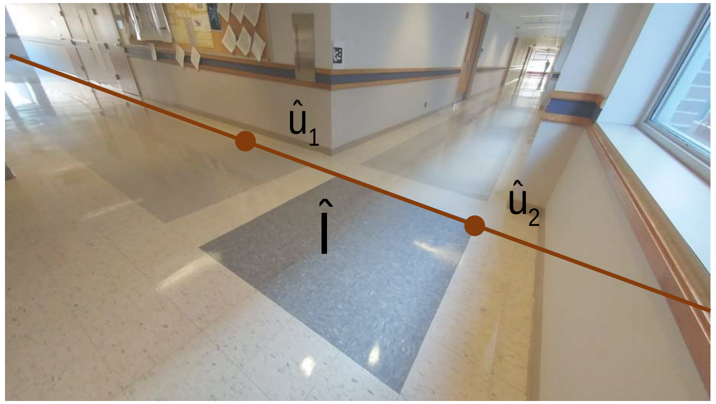
A 2D line in an image defines to a 3D plane passing the camera center:

$$\hat{l} \rightarrow \pi$$

Plane normal: $\left(\lambda_1\hat{u}_1\right) \times \left(\lambda_2\hat{u}_2\right) = \lambda\hat{l}$

Geometric Interpretation (Line)





Normalized coordinate:

$$\hat{u}_1 = K^{-1}u_1 \qquad \hat{u}_2 = K^{-1}u_2$$

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where $\hat{l} = \left(K^{-1}\right)^{-T} l = K^{T} l$ due to duality

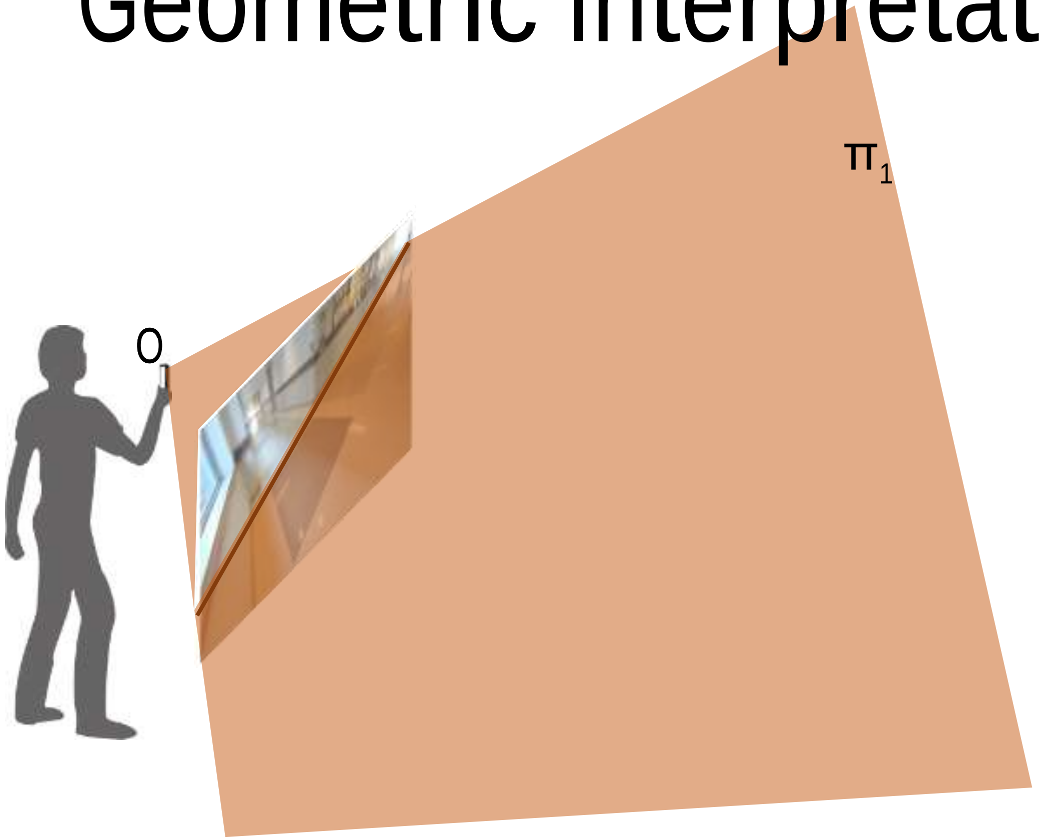
A 2D line in an image defines to a 3D plane passing the camera center:

$$\hat{l} \rightarrow \pi$$

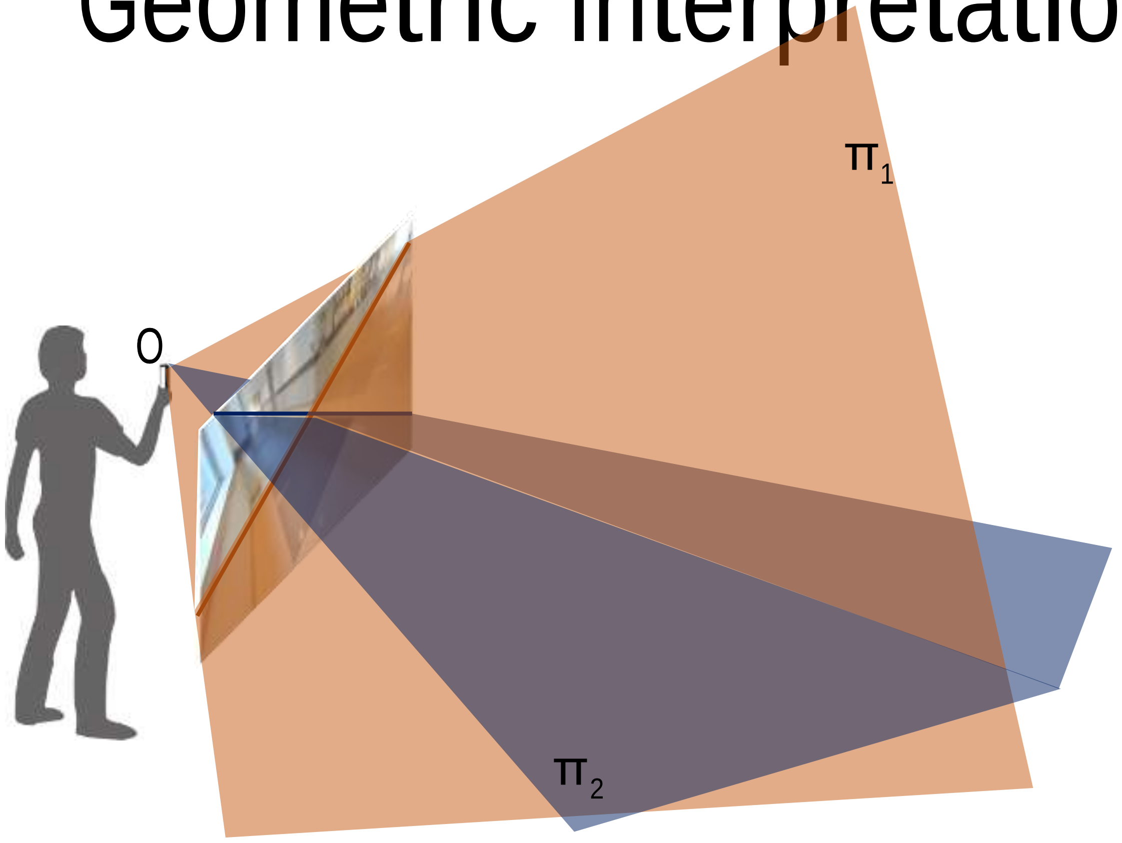
Plane normal: $\left(\lambda_1\hat{u}_1\right) \times \left(\lambda_2\hat{u}_2\right) = \lambda\hat{l}$

$$\therefore \pi = \begin{bmatrix} \hat{l} \\ 0 \end{bmatrix}$$

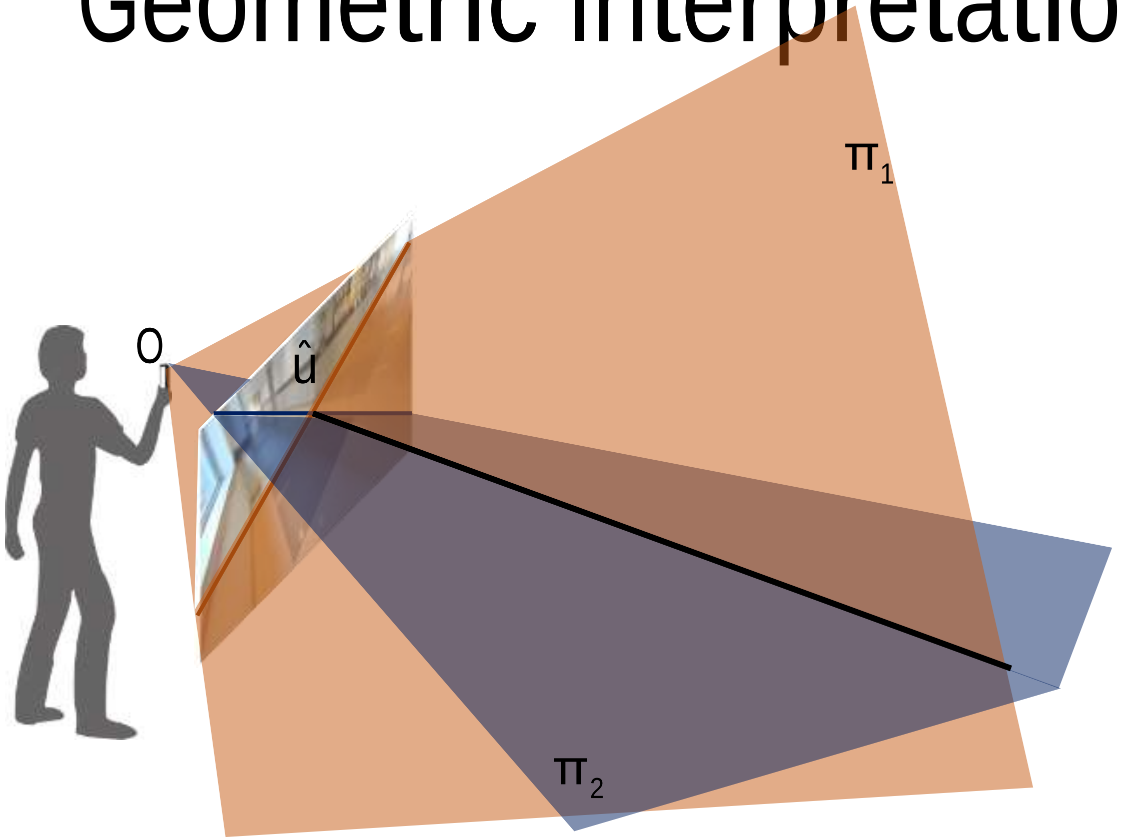
Geometric Interpretation (Line-Line)

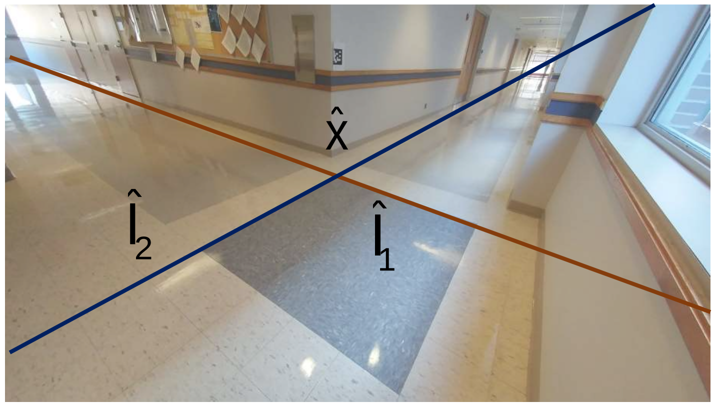


Geometric Interpretation (Line-Line)



Geometric Interpretation (Line-Line)

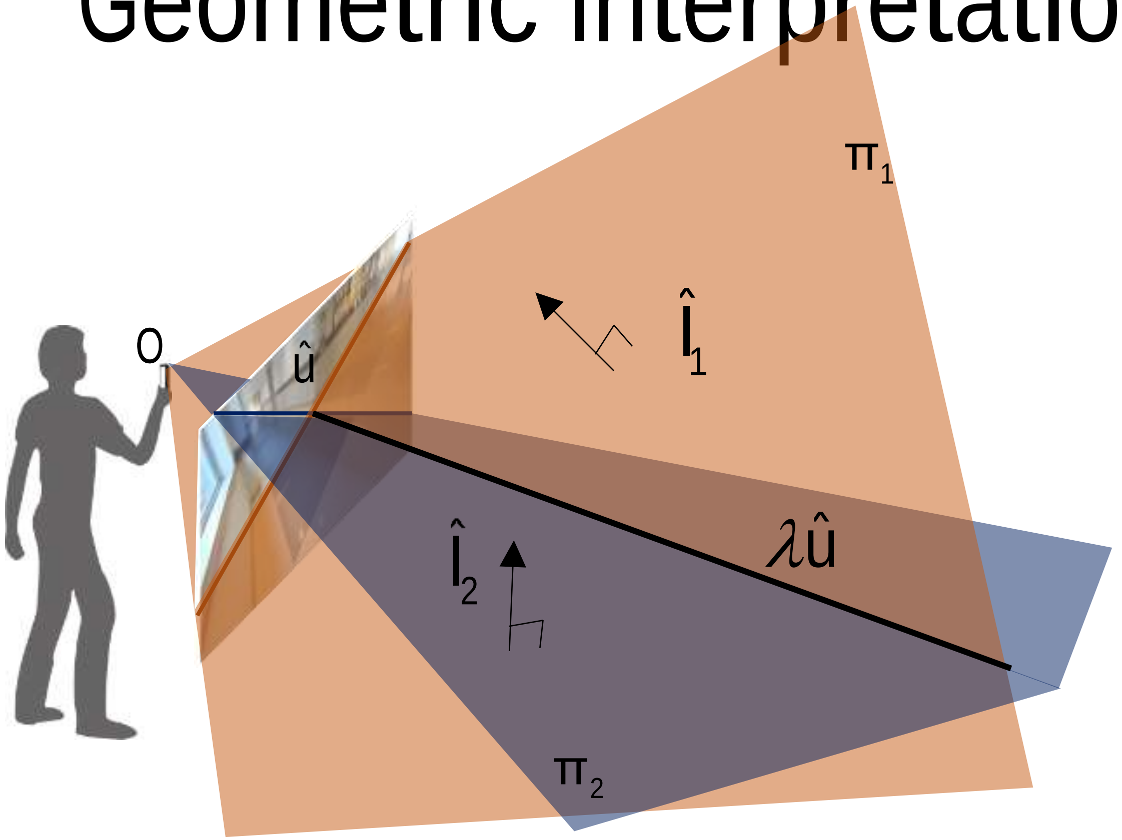


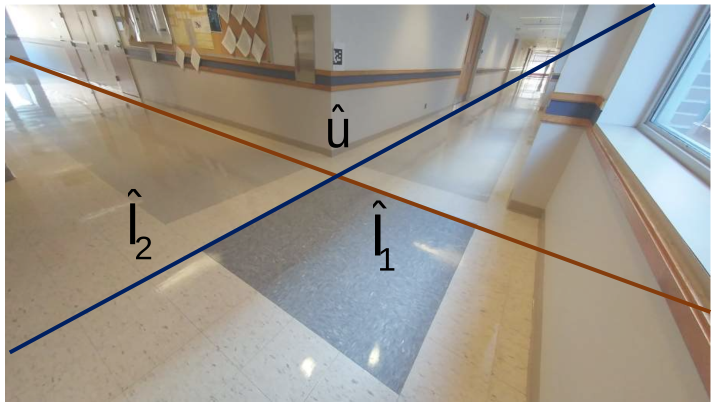


2D lines in an image intersect a 2D point corresponding to a 3D ray:

$$\hat{u} = \hat{l}_1 \times \hat{l}_2$$

Geometric Interpretation (Line-Line)



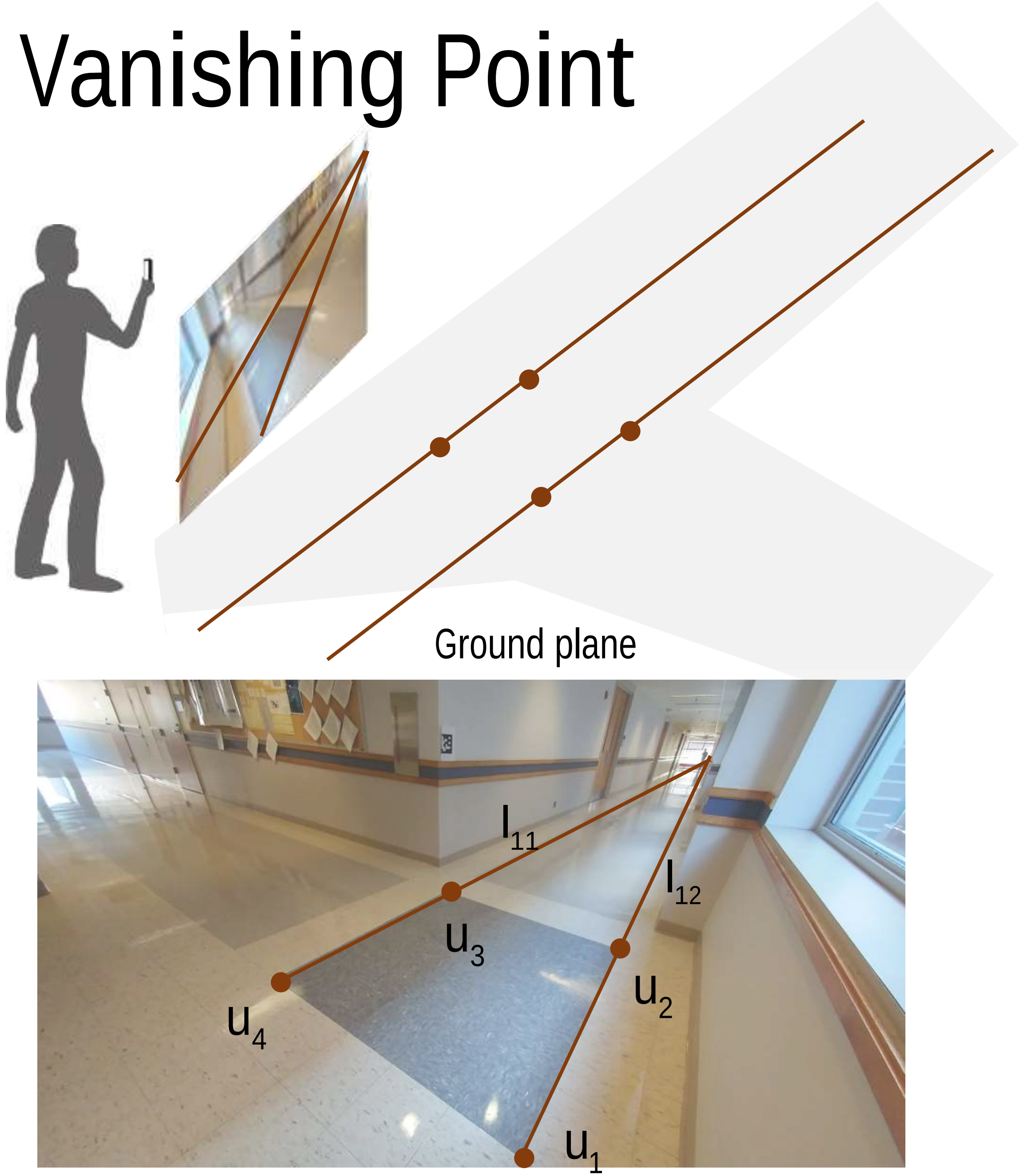


2D lines in an image intersect a 2D point corresponding to a 3D ray:

$$\hat{u} = \hat{l}_1 \times \hat{l}_2$$

: the 3D ray is perpendicular to two plane normals.

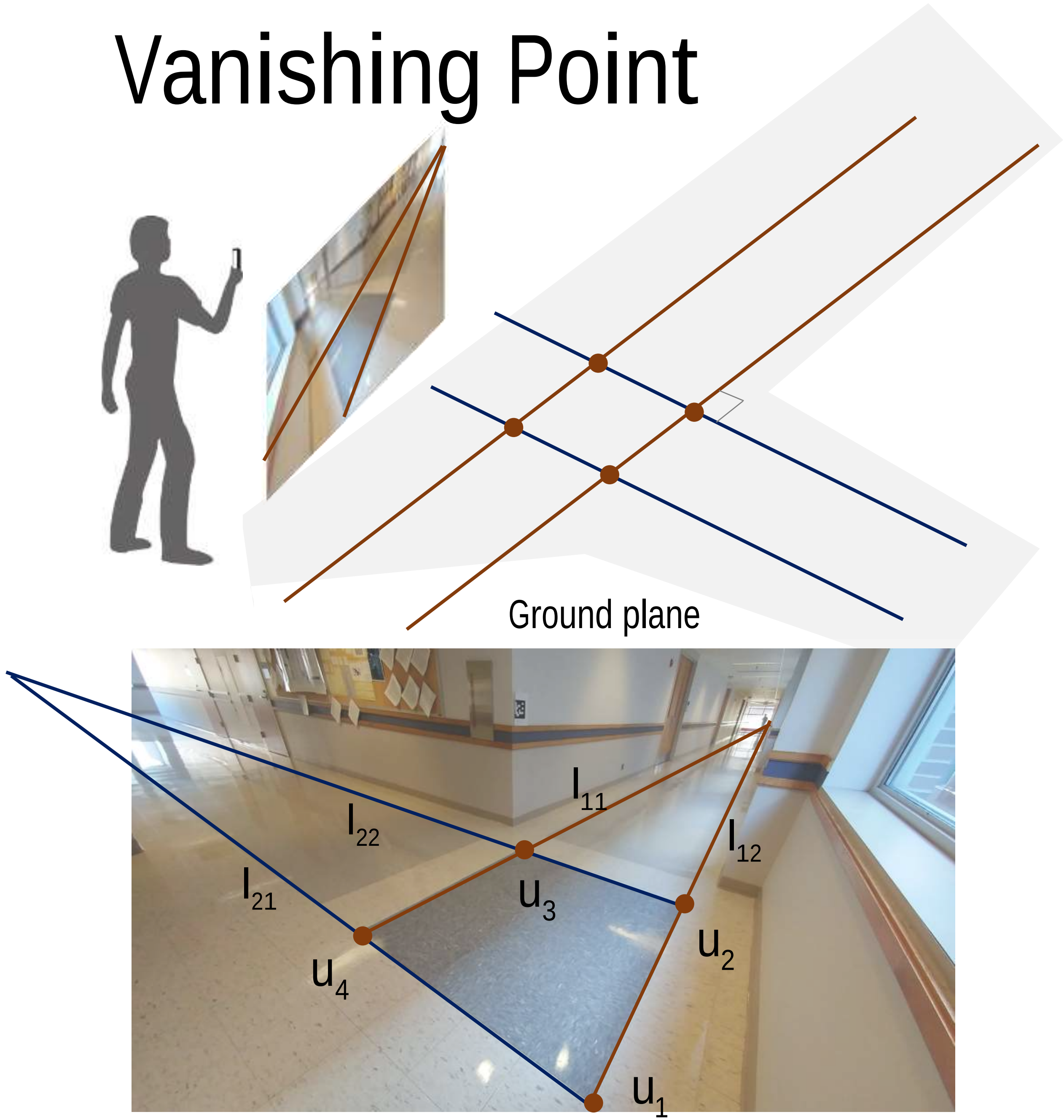
Vanishing Point



Parallel lines:

$$l_{11} = u_4 \times u_3 \qquad l_{12} = u_1 \times u_2$$

Vanishing Point

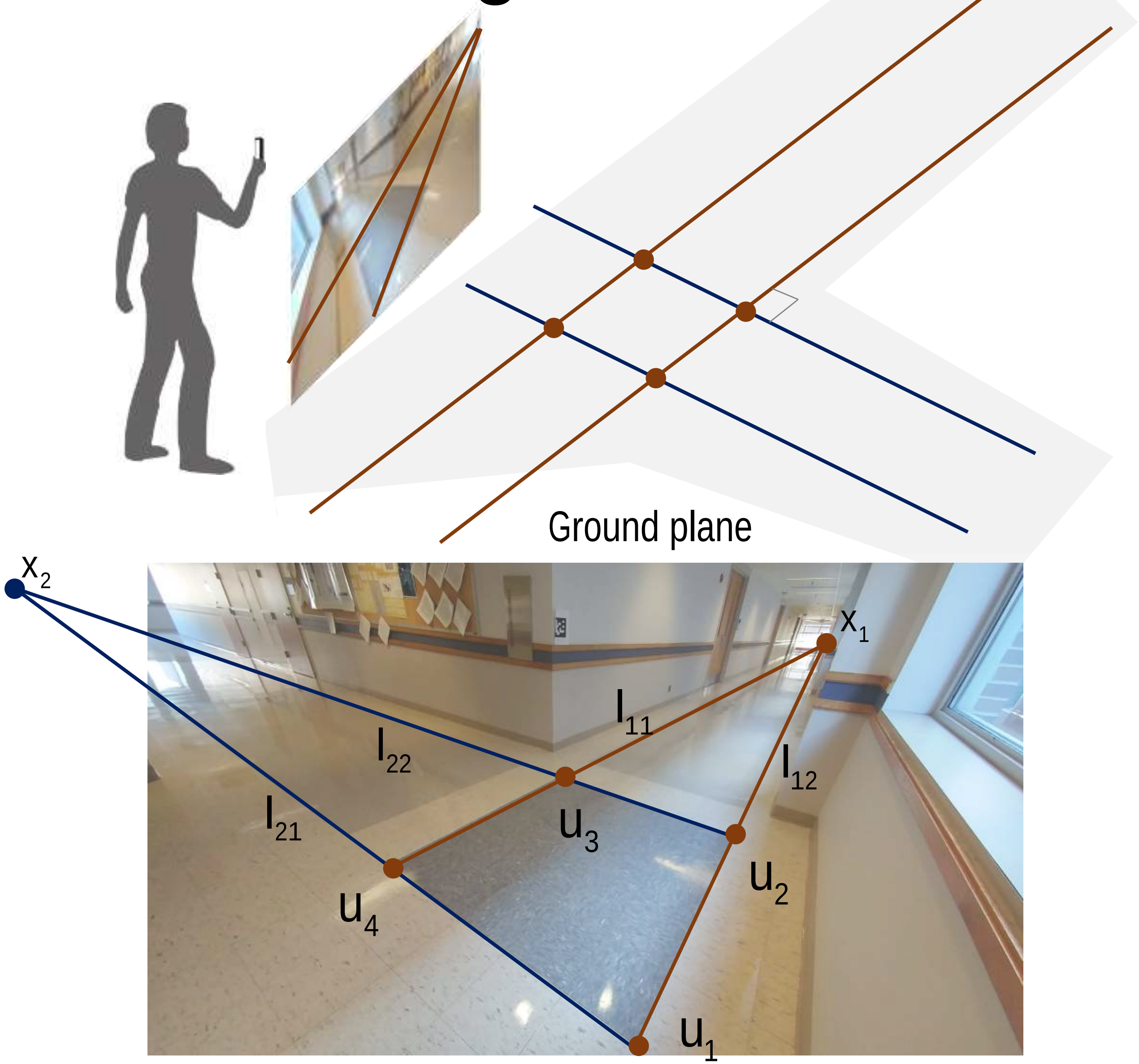


Parallel lines:

$$l_{11} = u_4 \times u_3 \qquad l_{12} = u_1 \times u_2$$

$$l_{21} = u_4 \times u_1 \qquad l_{22} = u_3 \times u_4$$

Vanishing Point



Parallel lines:

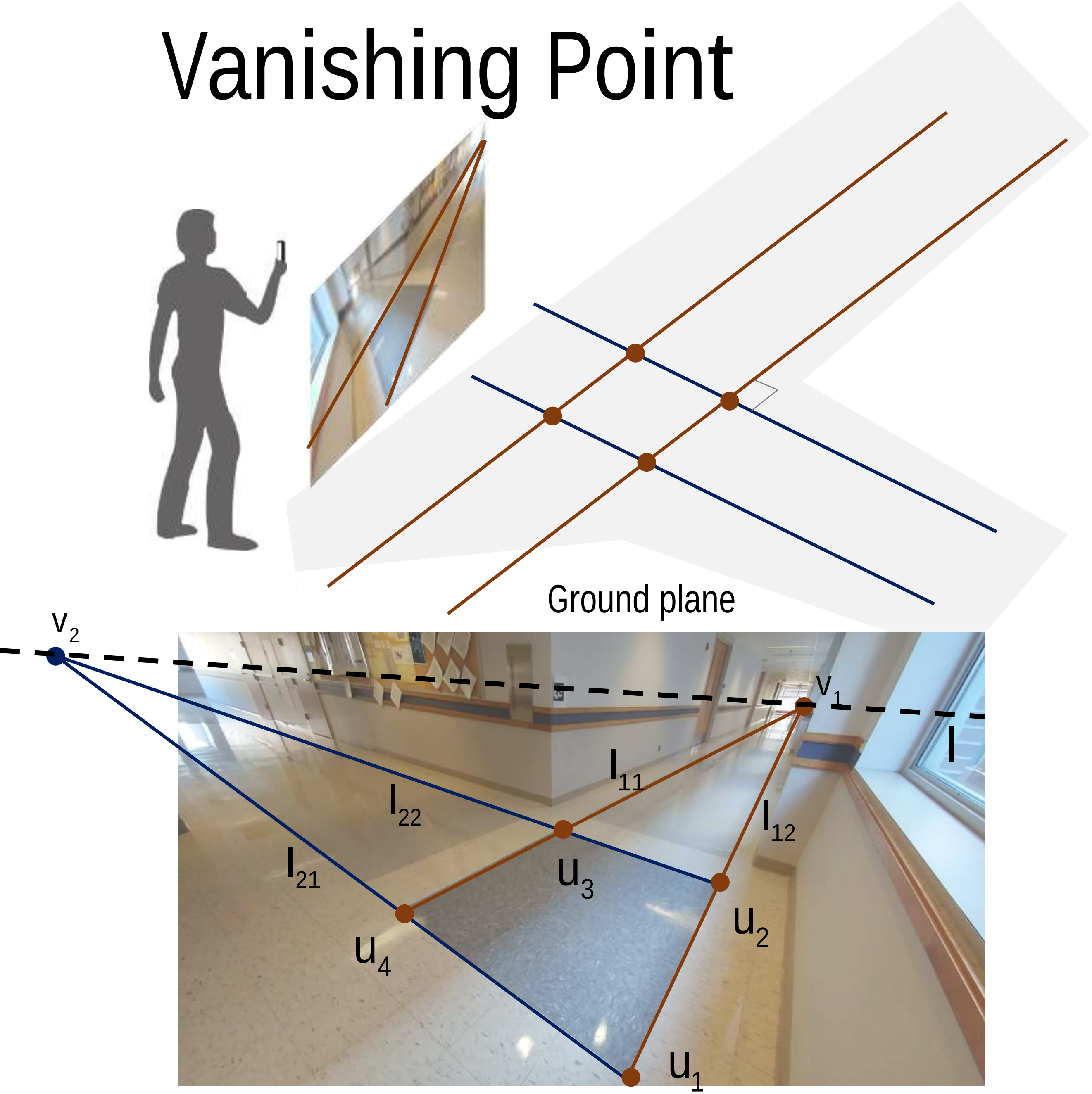
$$l_{11} = u_4 \times u_3 \qquad l_{12} = u_1 \times u_2$$

$$l_{21} = u_4 \times u_1 \qquad l_{22} = u_3 \times u_4$$

Vanishing points:

$$x_1 = l_{11} \times l_{12} \qquad x_2 = l_{21} \times l_{22}$$

Vanishing Point



Parallel lines:

$$l_{11} = u_4 \times u_3 \qquad l_{12} = u_1 \times u_2$$

$$l_{21} = u_4 \times u_1 \qquad l_{22} = u_3 \times u_4$$

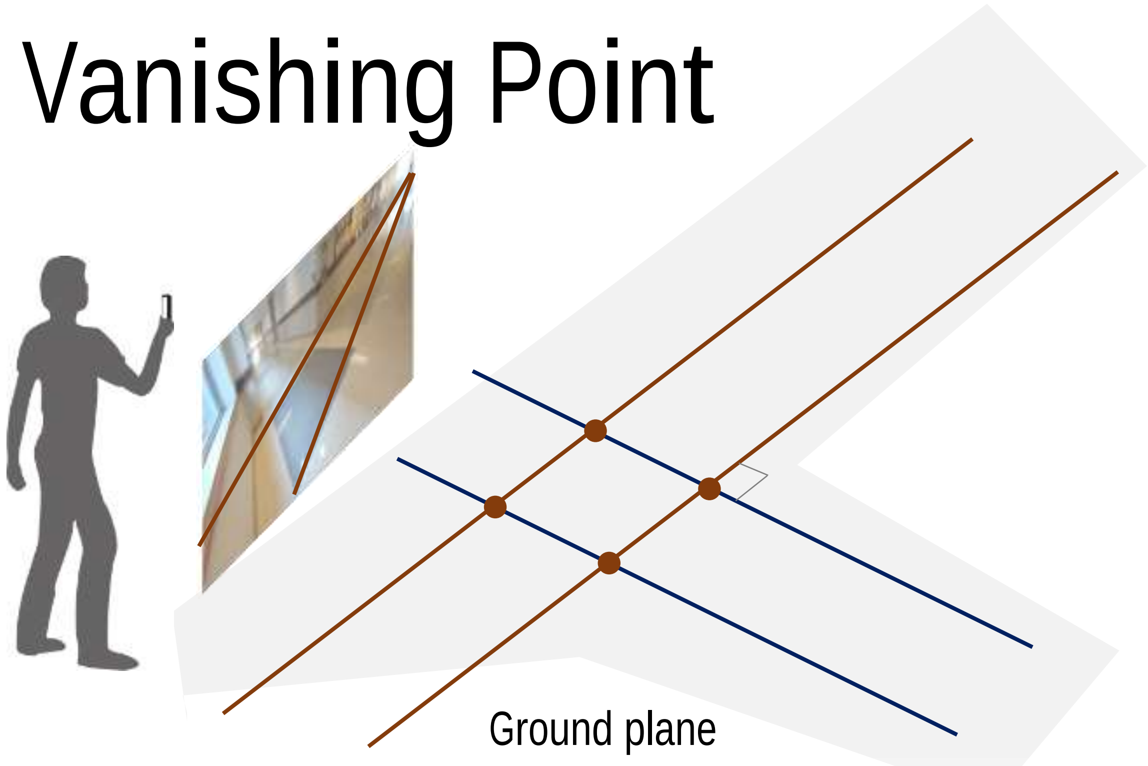
Vanishing points:

$$v_1 = l_{11} \times l_{12} \qquad v_2 = l_{21} \times l_{22}$$

Vanishing line:

$$l = v_1 \times v_2$$

Vanishing Point



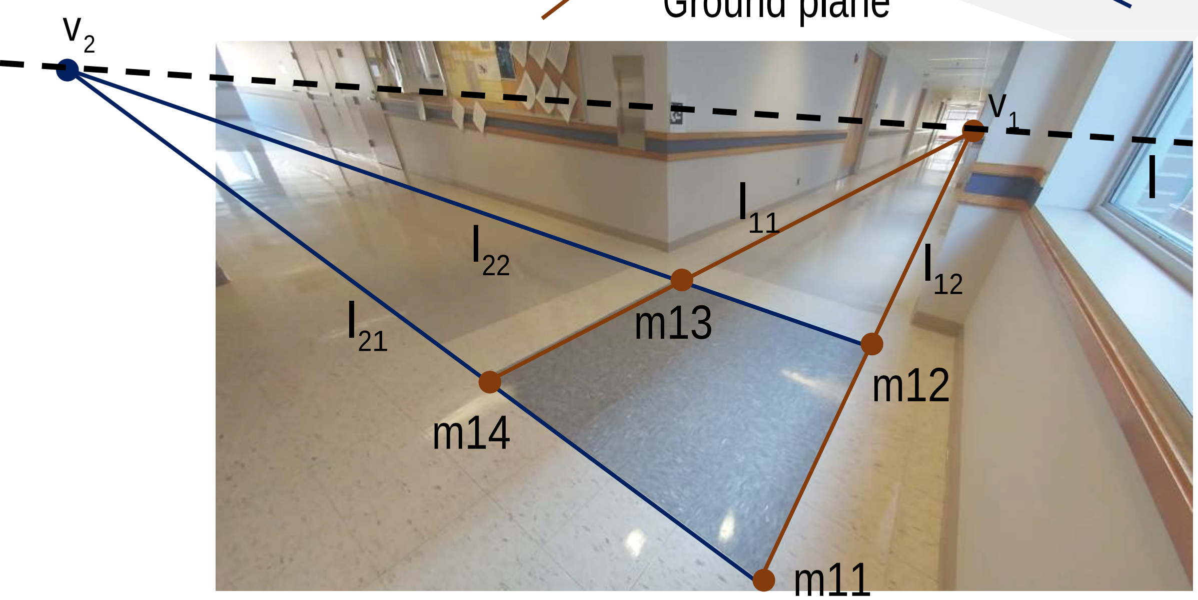

```
l11 = GetLineFromTwoPoints(m11,m12);
l12 = GetLineFromTwoPoints(m13,m14);

l21 = GetLineFromTwoPoints(m21,m22);
l22 = GetLineFromTwoPoints(m23,m24);

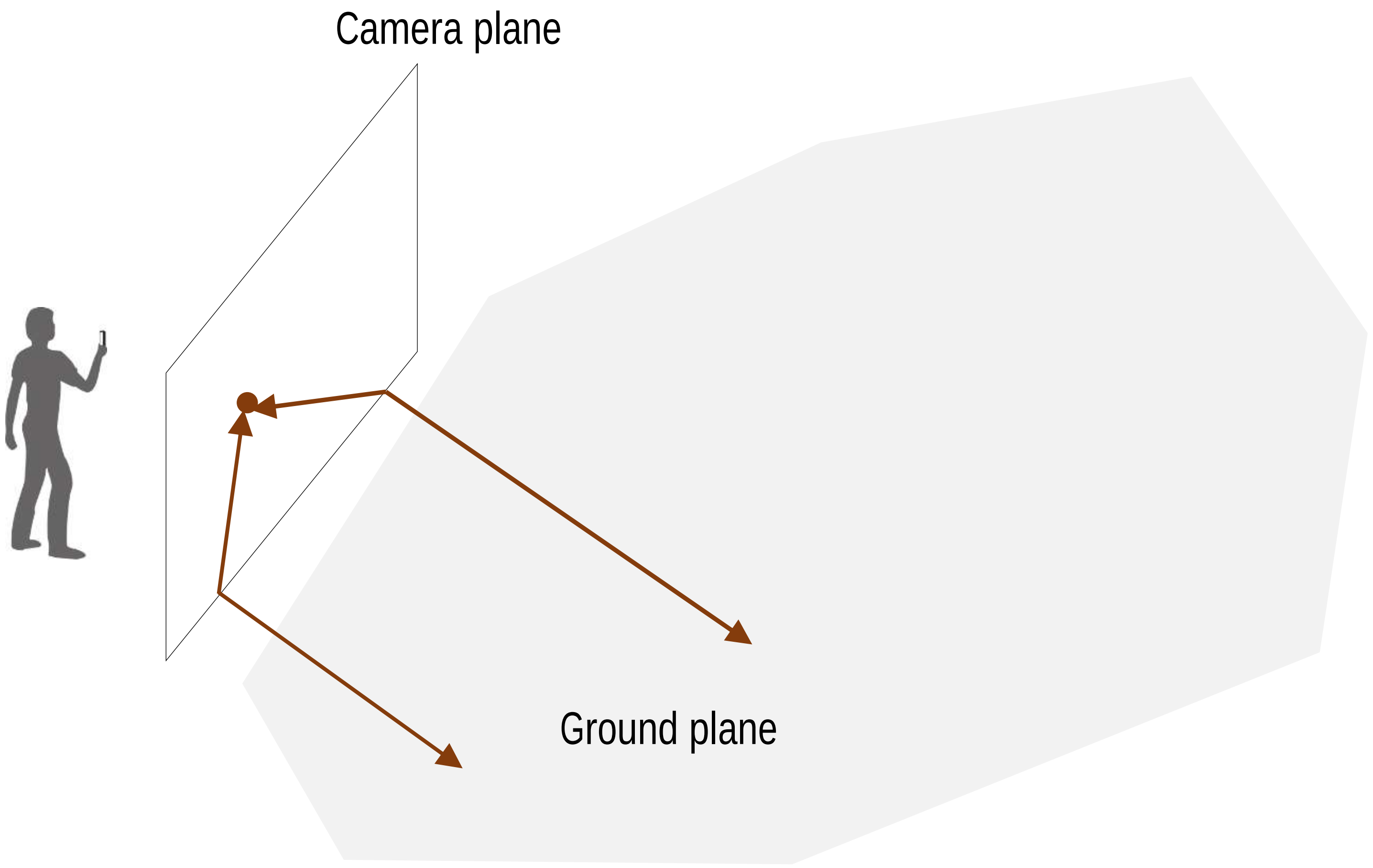
v1 = GetPointFromTwoLines(l11,l12);
v2 = GetPointFromTwoLines(l21,l22);

vanishing_line = GetLineFromTwoPoints(x1, x2);
```

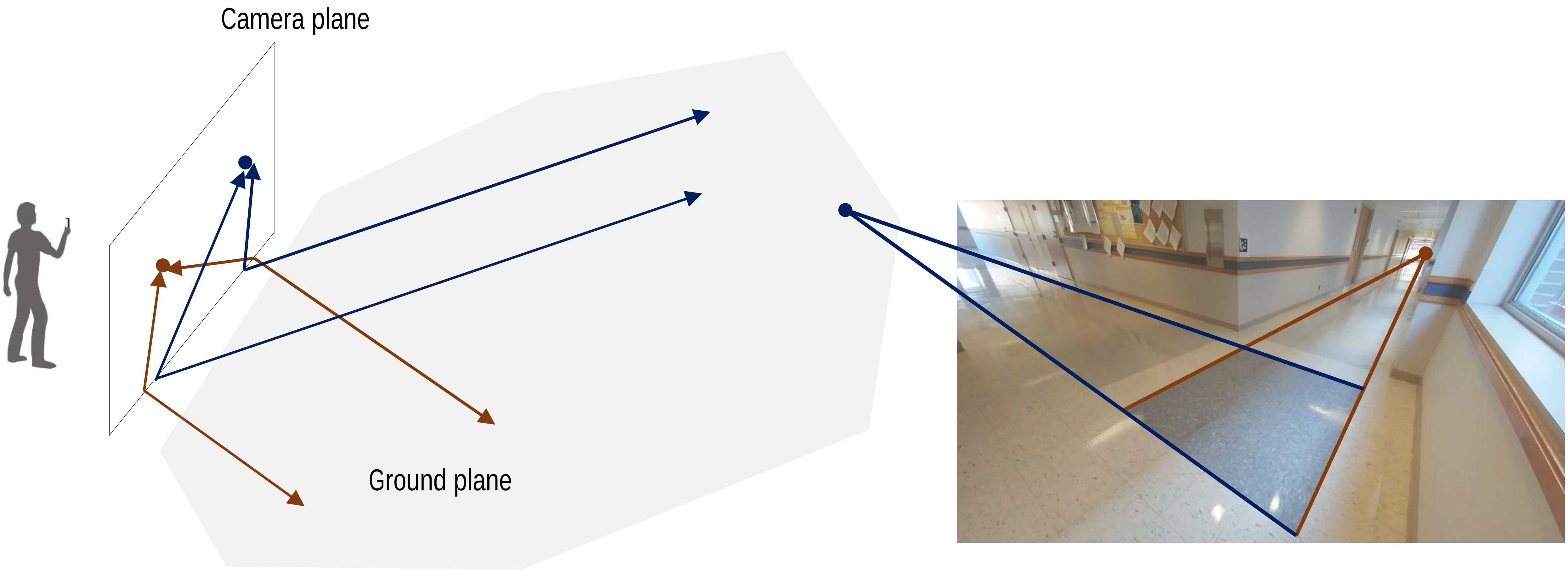
ComputeVanishingLine.m



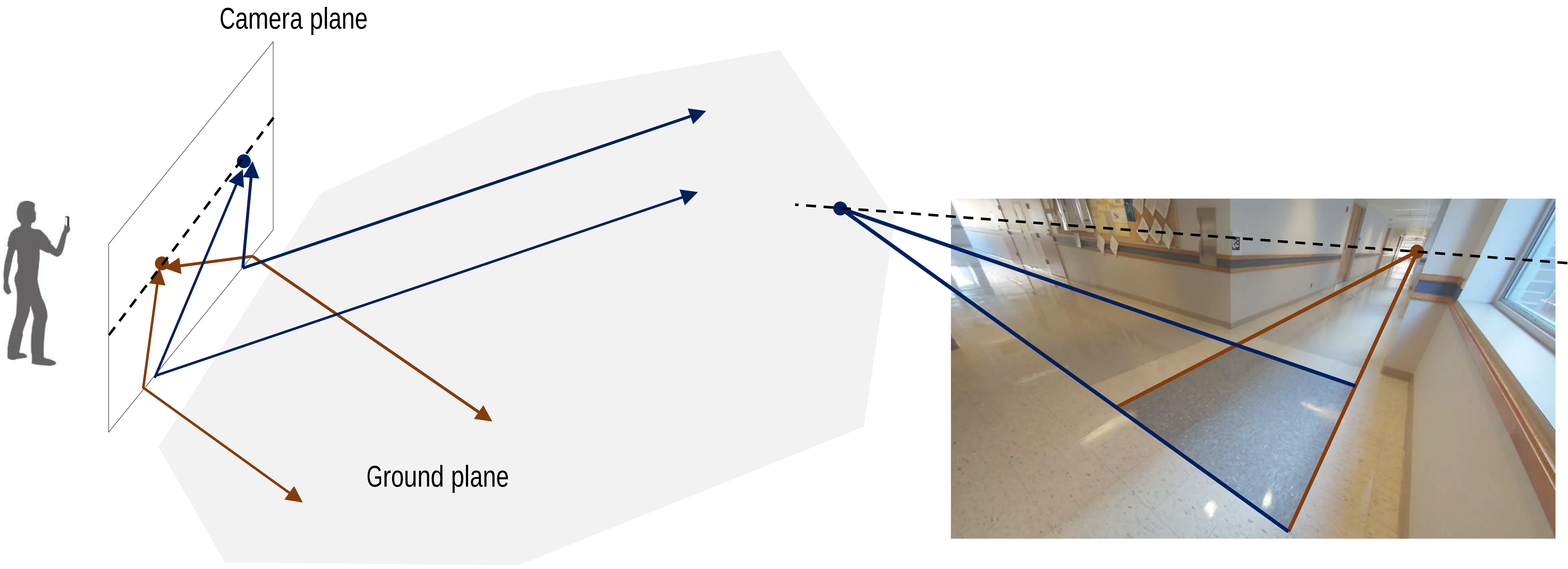
Geometric Interpretation of Vanishing Line



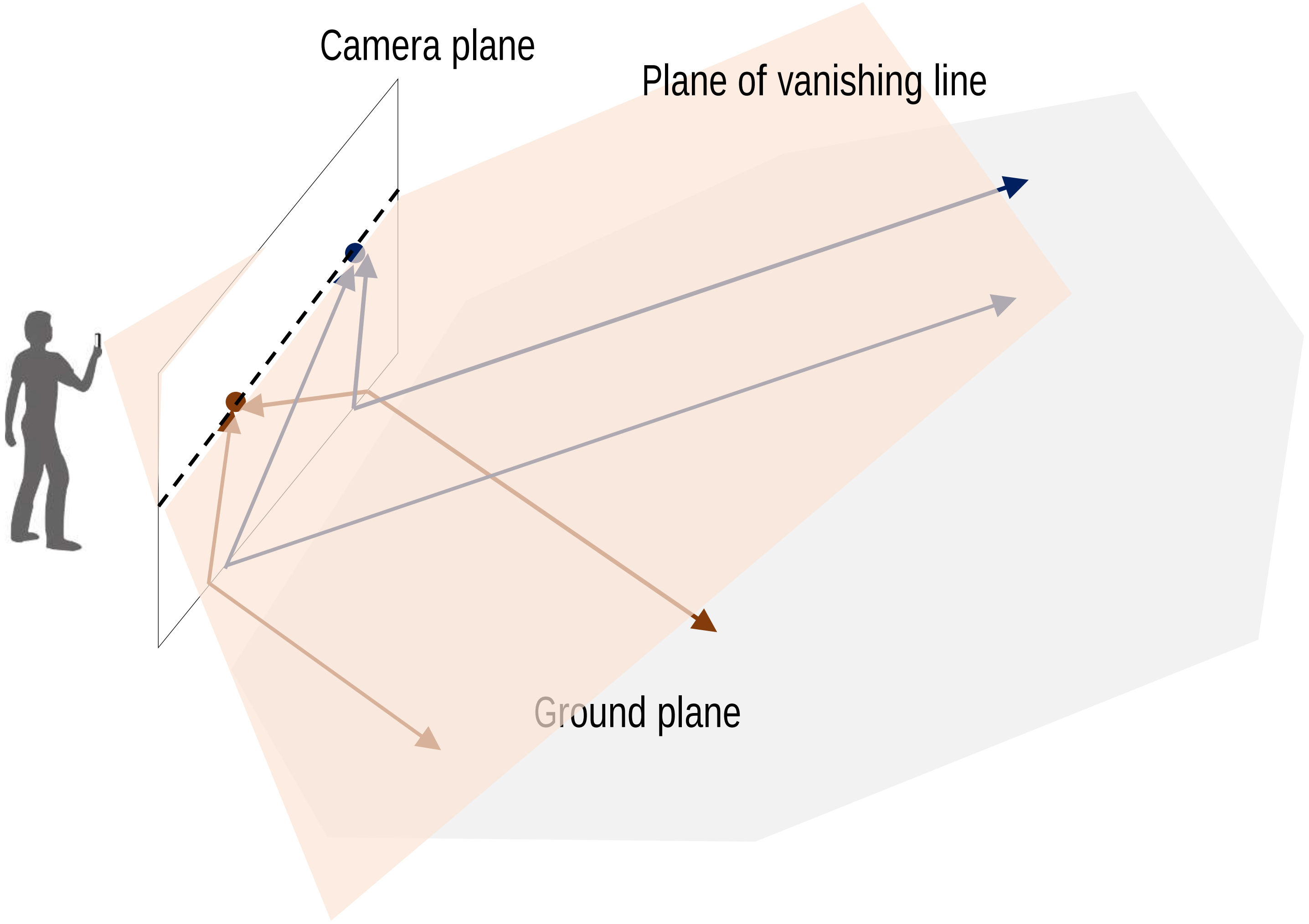
Geometric Interpretation of Vanishing Line

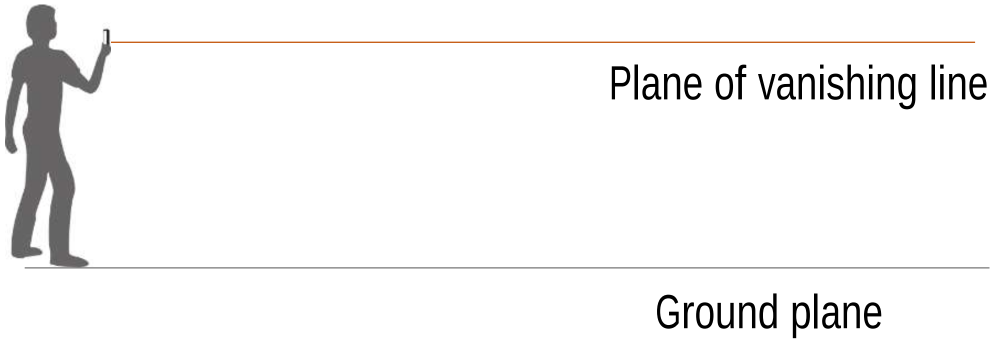


Geometric Interpretation of Vanishing Line



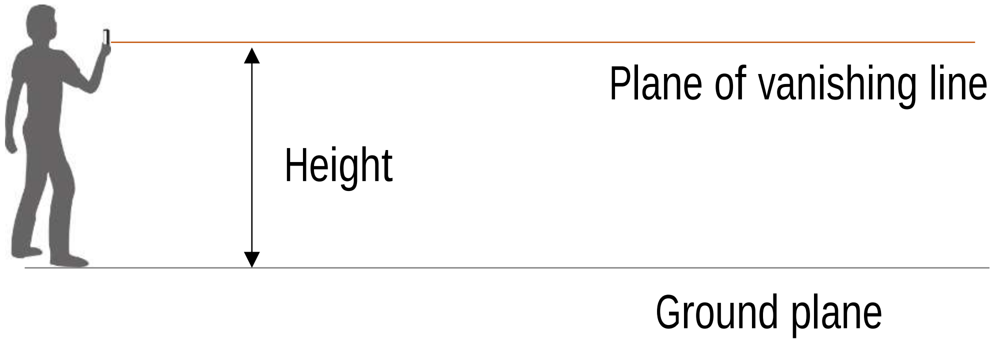
Geometric Interpretation of Vanishing Line





Side view

Geometric Interpretation of Vanishing Line



Side view

Where was I (how high)?


Where was I (how high)?

Taken from my hotel room (6th floor)

Taken from beach

Where was I (how high)?

Taken from my hotel room (6th floor)

Taken from beach

First person video

Cylindrical projection