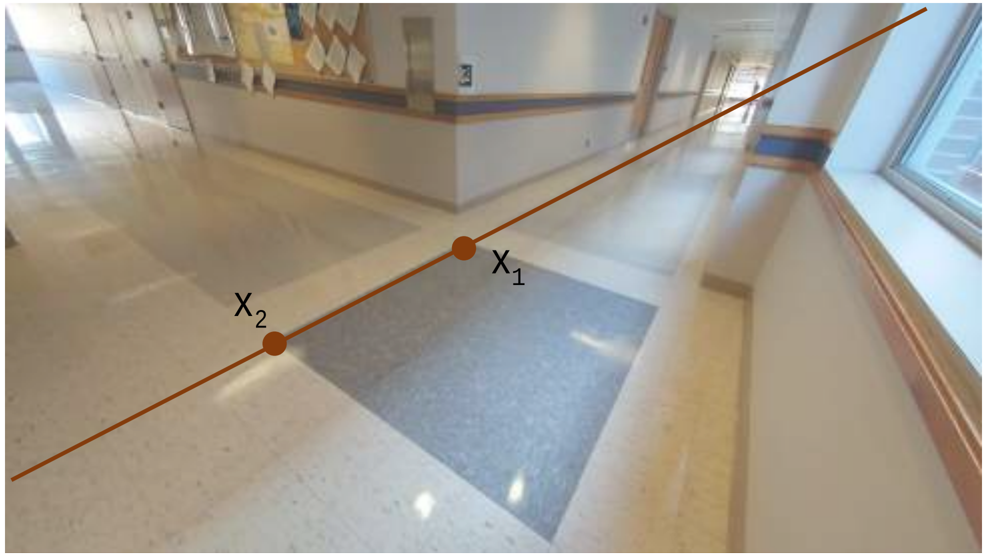
Linear Parameter Estimation

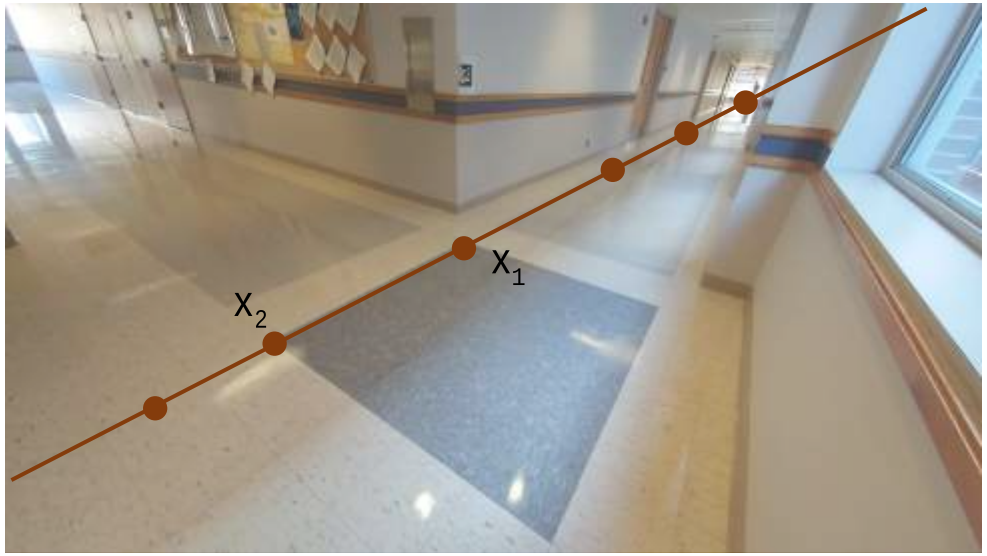
Point-Point in Image

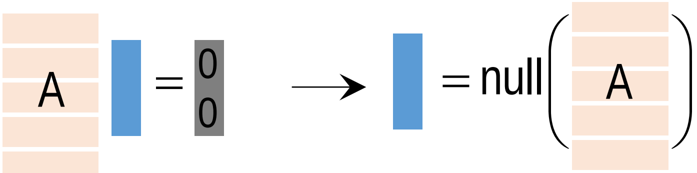


$$\begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} l = 0$$

$$\underset{3x2}{A} \; l = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \longrightarrow \quad l = \text{null}\left(A \right) \quad \text{or} \quad l = x_1 \times x_2$$

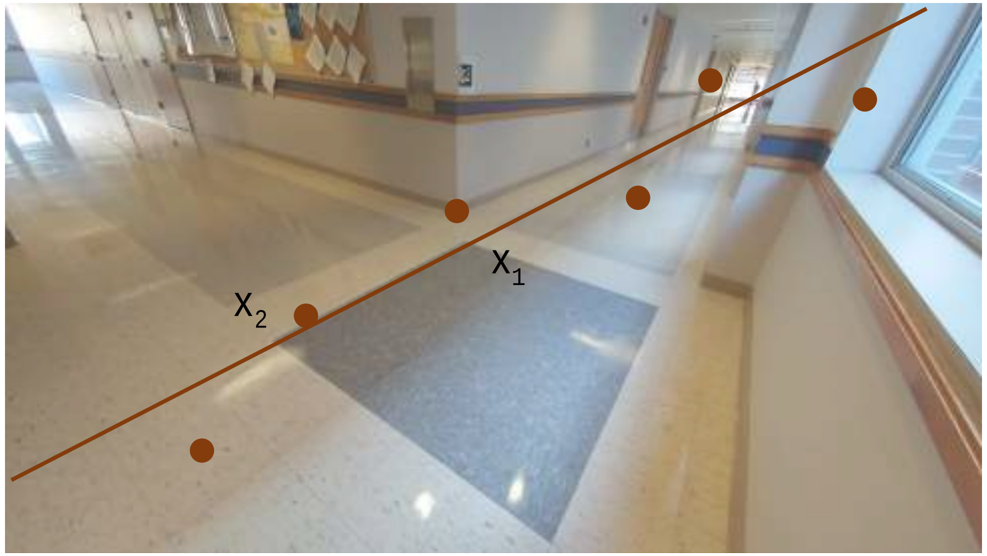
Point-Point in Image

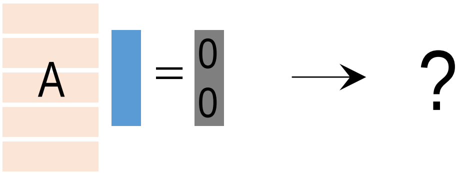




or $l = x_1 \times x_2$

Line Fitting

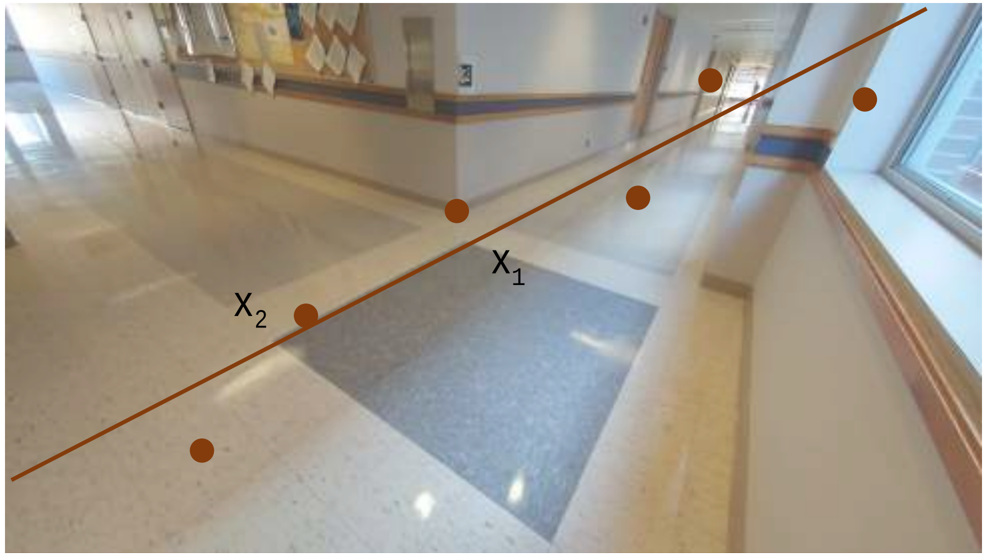




Line Fitting

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



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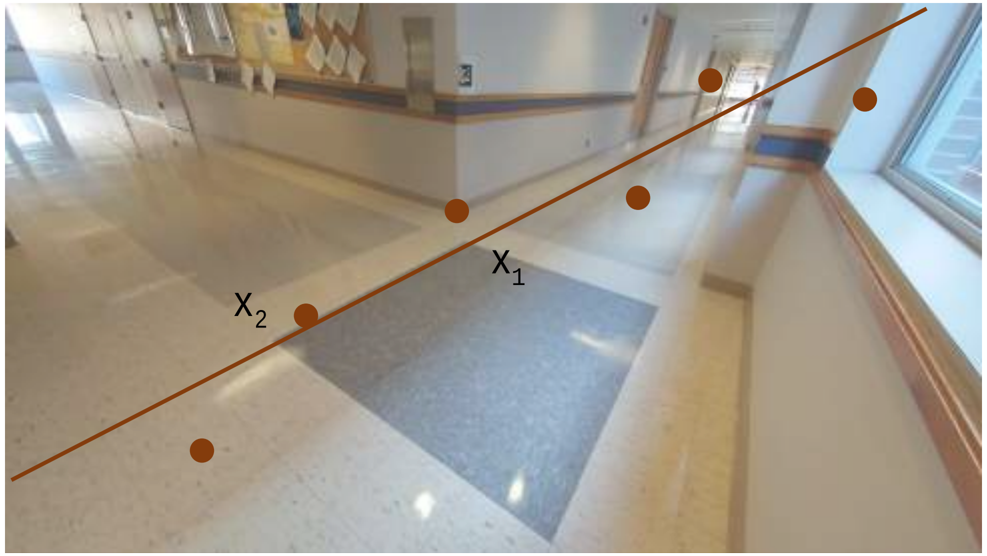


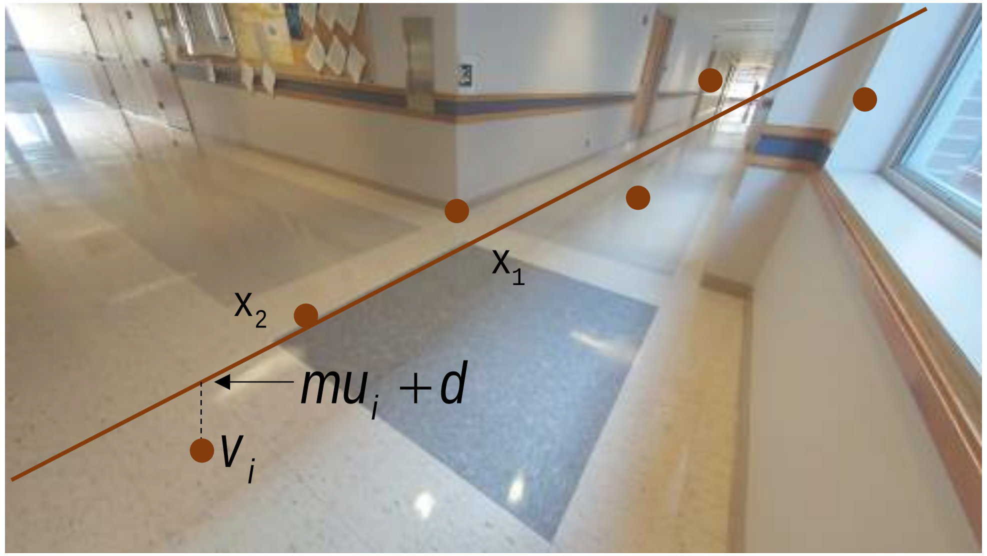
$$\longrightarrow au_1 + bv_1 + c \approx 0$$

$$au_2 + bv_2 + c \approx 0$$

$$\vdots$$

$$au_n + bv_n + c \approx 0$$

Line Fitting



Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: $(a, b, c) \longrightarrow (\underline{m}, \underline{d})$

slope   y-intercept

$$\longrightarrow \quad v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

$$\vdots$$

$$v_n \approx mu_n + d$$

Line Fitting

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m},\underline{d})$

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$$\longrightarrow \quad v_1 \approx mu_1 + d$$
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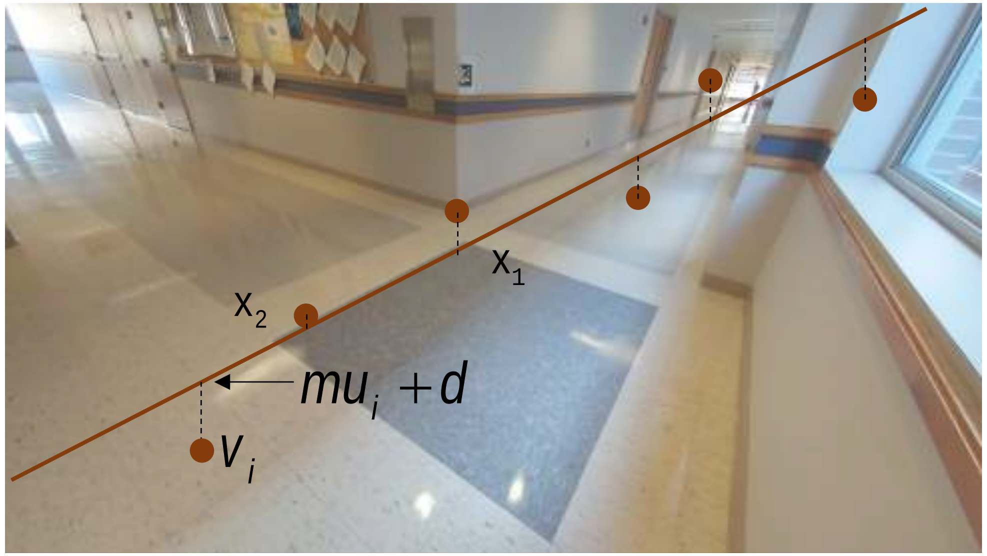


Error: $e_i = v_i - (mu_i + d)$

$$e_1 = v_1 - (mu_1 + d)$$
$$e_2 = v_2 - (mu_2 + d)$$

$$e_n = v_n - (mu_n + d)$$

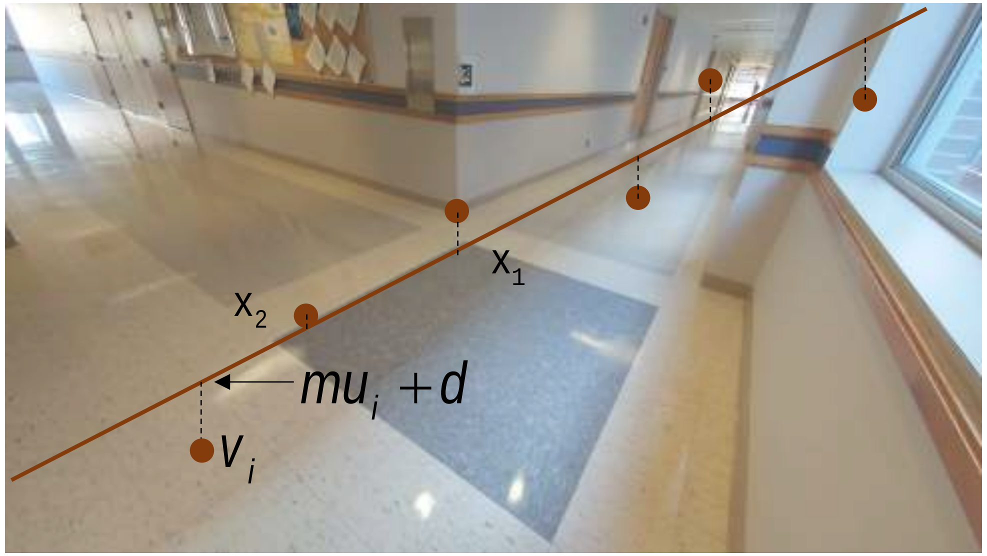
Line Fitting

Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: $(a, b, c) \longrightarrow (\underline{m}, \underline{d})$

slope y-intercept

$$\longrightarrow \quad \begin{aligned} v_1 &\approx m u_1 + d \\ v_2 &\approx m u_2 + d \\ &\vdots \\ v_n &\approx m u_n + d \end{aligned}$$



Total error: $E = \sum_{i=1}^{n} \left(v_i - (m u_i + d)\right)^2$

$$e_1 = v_1 - (m u_1 + d)$$
$$e_2 = v_2 - (m u_2 + d)$$

$$e_n = v_n - (m u_n + d)$$

Unknowns:

Number of eq.:

Line Fitting

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m},\underline{d})$

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$$\longrightarrow \quad v_1 \approx mu_1 + d$$
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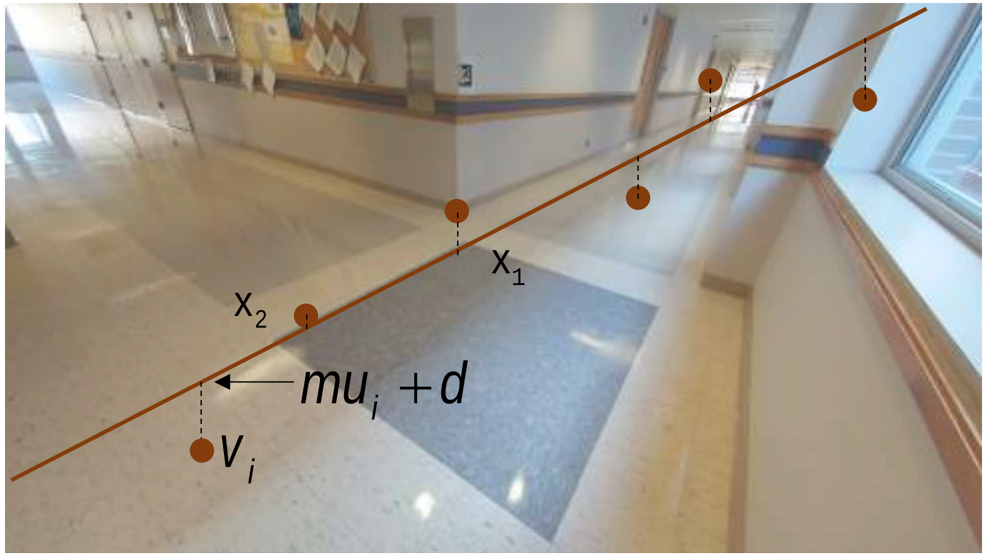

$$e_1 = v_1 - (mu_1 + d)$$
$$e_2 = v_2 - (mu_2 + d)$$

$$e_n = v_n - (mu_n + d)$$

Total error: $E = \sum_{i=1}^{n}\left(v_i - (mu_i + d)\right)^2$

Unknowns: m,d

Number of eq.:

Line Fitting

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m},\underline{d})$

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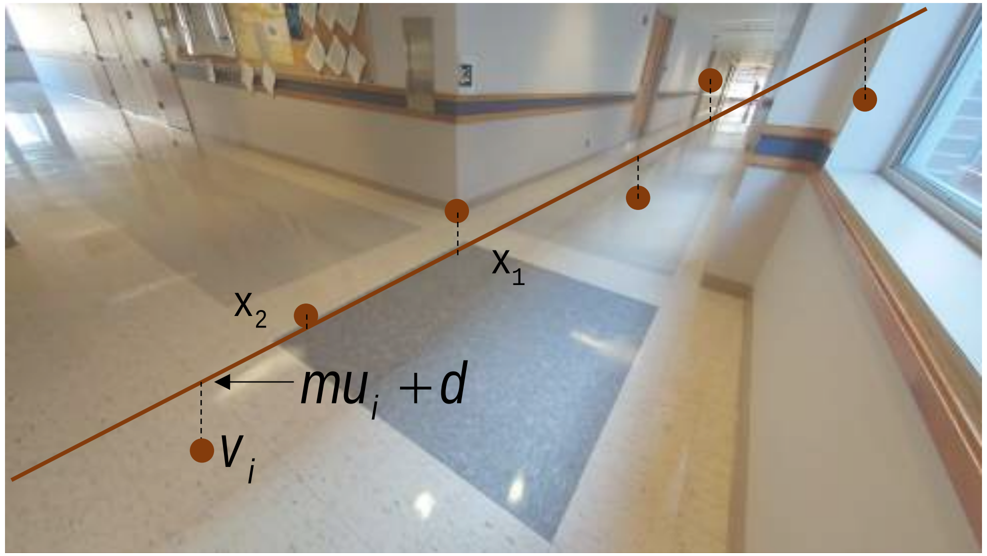


$$e_1 = v_1 - (mu_1 + d)$$
$$e_2 = v_2 - (mu_2 + d)$$

$$e_n = v_n - (mu_n + d)$$

Total error: $E = \sum_{i=1}^{n}\left(v_i - (mu_i + d)\right)^2$

Unknowns: m,d

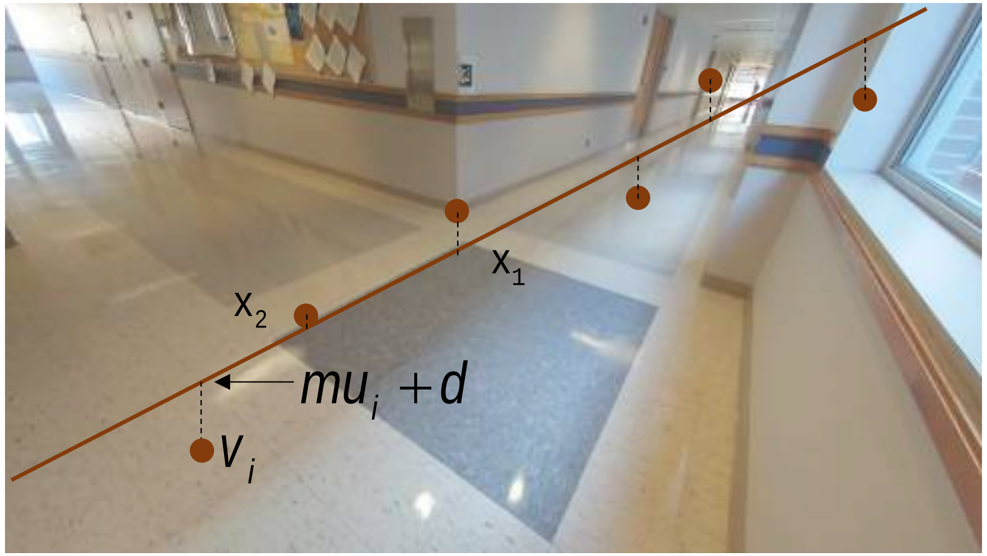
Number of eq.: n

Line Fitting

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m},\underline{d})$

slope y-intercept



$$\longrightarrow \quad v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$\vdots$$
$$v_n \approx mu_n + d$$

Total error: $E = \sum_{i=1}^{n} (v_i - (mu_i + d))^2$

$$e_1 = v_1 - (mu_1 + d)$$
$$e_2 = v_2 - (mu_2 + d)$$
$$e_n = v_n - (mu_n + d)$$

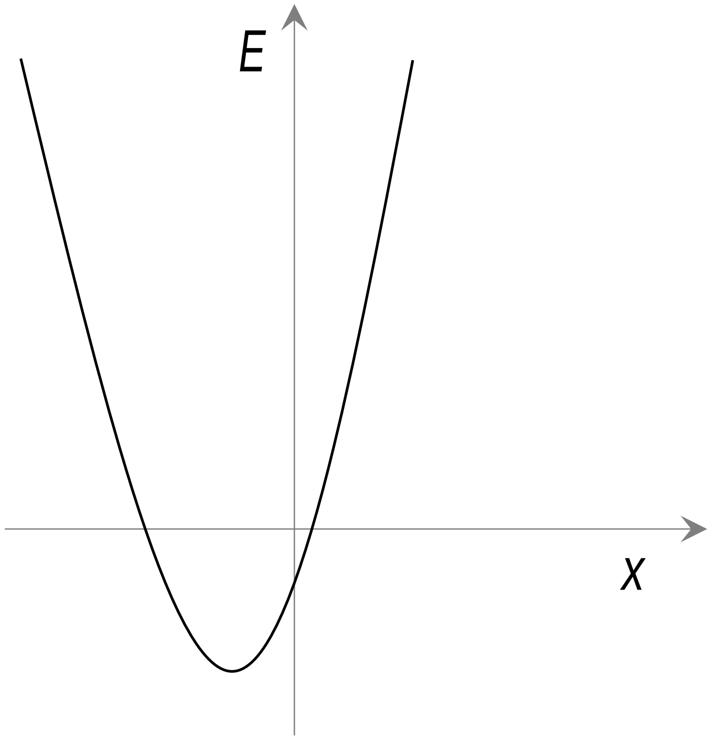
Unknowns: m,d

Number of eq.: n

$$\underset{m,d}{\text{minimize}} \sum_{i=1}^{n} (v_i - (mu_i + d))^2$$

How to minimize?

Line Fitting



Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: $(a, b, c) \longrightarrow (\underline{m}, \underline{d})$

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$$\longrightarrow \quad v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
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$$v_n \approx mu_n + d$$

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

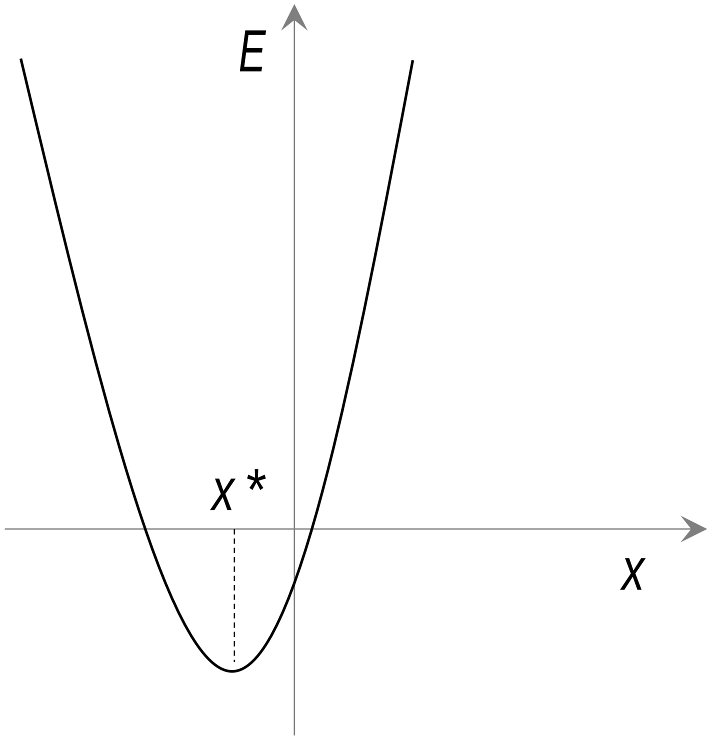
Unknowns: m, d

Number of eq.: n

$$\underset{m,d}{\text{minimize}} \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$$

How to minimize?

Line Fitting



Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: $(a, b, c) \longrightarrow (\underline{m}, \underline{d})$

slope y-intercept

$$\longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

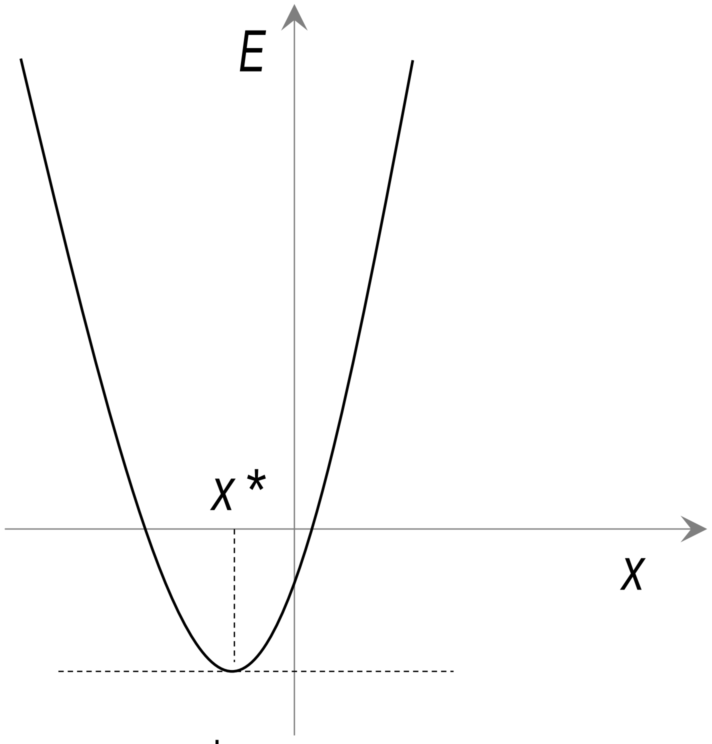
Unknowns: m, d

Number of eq.: n

$$\underset{m,d}{\text{minimize}} \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$$

How to minimize?

Line Fitting



$$\left.\frac{dE}{dx}\right|_{x=x^*}=0$$

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m},\underline{d})$

slope y-intercept

$$\longrightarrow \quad v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$\vdots$$
$$v_n \approx mu_n + d$$

Total error: $E = \sum_{i=1}^{n}\left(v_i - (mu_i + d)\right)^2$

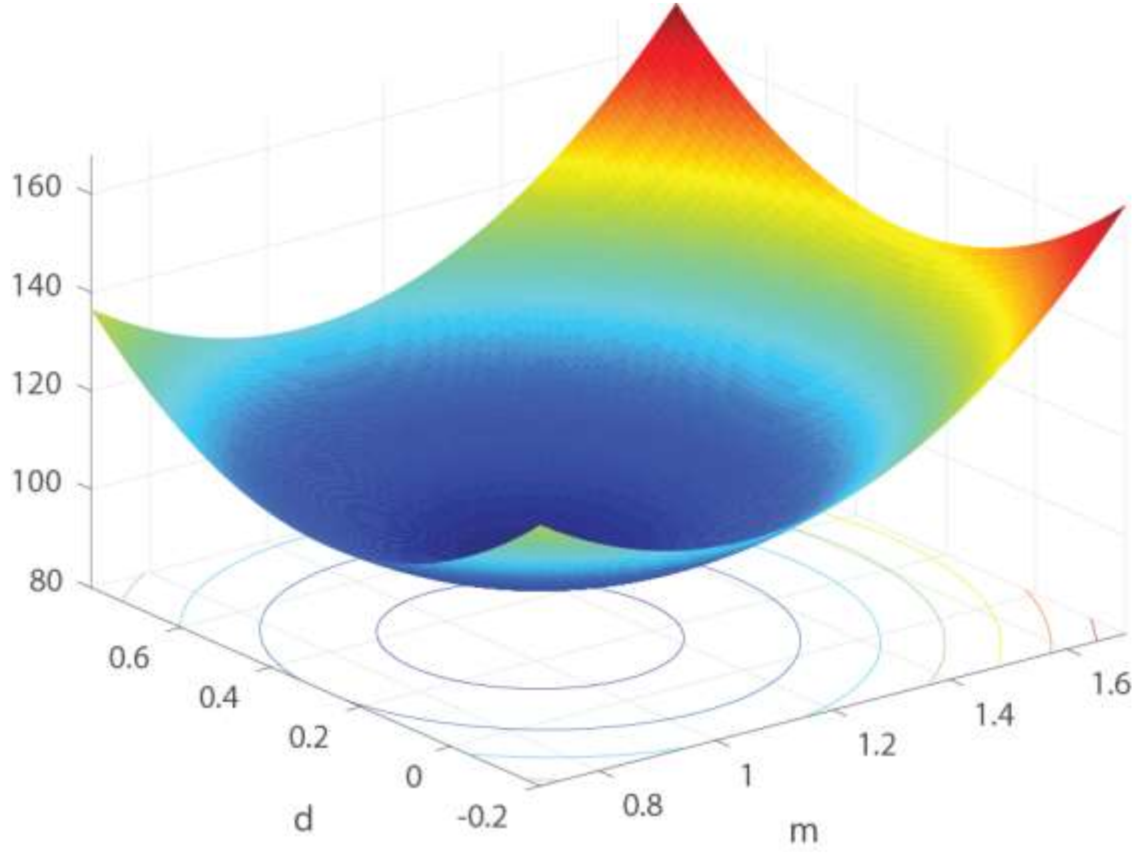
Unknowns: m,d

Number of eq.: n

$$\underset{m,d}{\text{minimize}} \sum_{i=1}^{n}\left(v_i - (mu_i + d)\right)^2$$

How to minimize?

Line Fitting



Total error: $$E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$$

Optimal point: $$\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$$

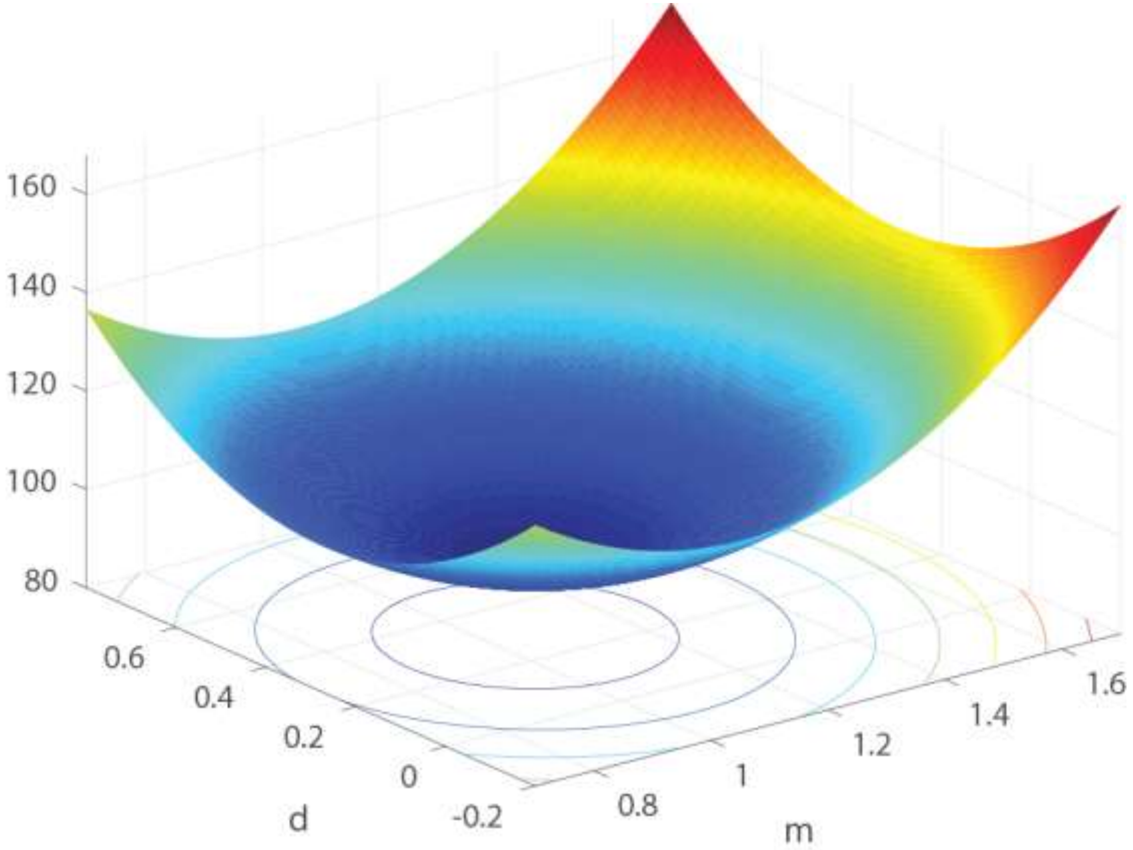
$$\frac{\partial E}{\partial m} =$$

$$\frac{\partial E}{\partial d} =$$

Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

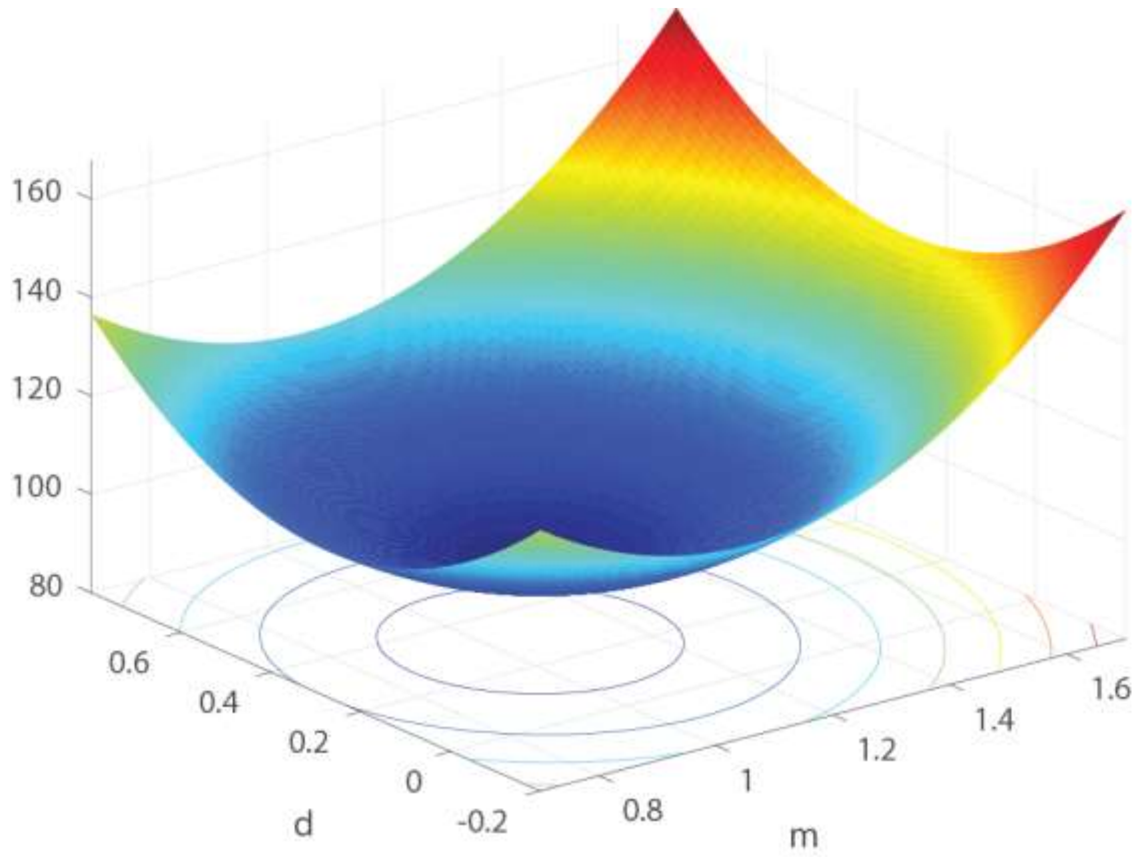
Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$



$$\frac{\partial E}{\partial m} = -\sum_{i=1}^{n} 2u_i \left(v_i - (mu_i + d)\right) = 0$$

$$\frac{\partial E}{\partial d} = -\sum_{i=1}^{n} 2\left(v_i - (mu_i + d)\right) = 0$$

Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$\frac{\partial E}{\partial m} = -\sum_{i=1}^{n} 2u_i \left(v_i - (mu_i + d)\right) = 0$$

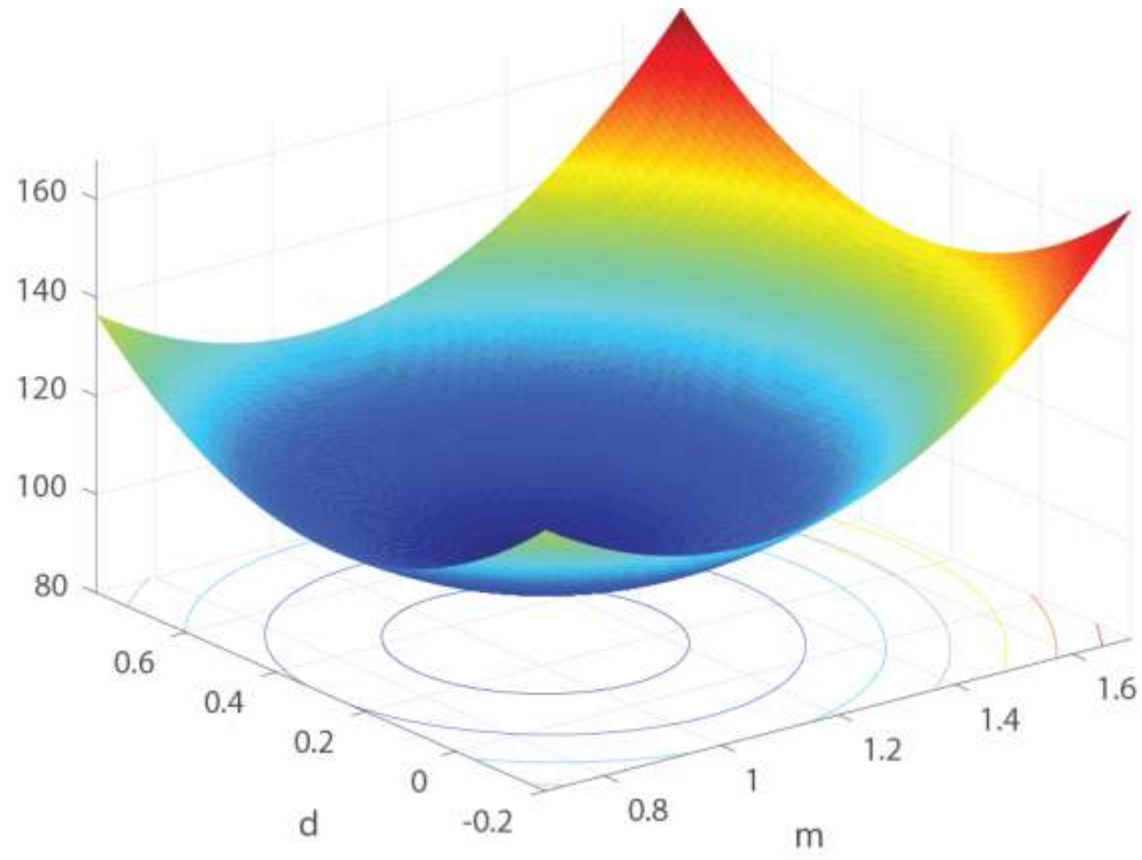
$$\frac{\partial E}{\partial d} = -\sum_{i=1}^{n} 2\left(v_i - (mu_i + d)\right) = 0$$

$\longrightarrow$

$$m\sum_{i=1}^{n} u_i^2 + d\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} u_i v_i$$

$$m\sum_{i=1}^{n} u_i + nd = \sum_{i=1}^{n} v_i$$

Line Fitting



Total error: $E = \sum_{i=1}^{n} (v_i - (mu_i + d))^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$\frac{\partial E}{\partial m} = -\sum_{i=1}^{n} 2u_i (v_i - (mu_i + d)) = 0$$

$$\frac{\partial E}{\partial d} = -\sum_{i=1}^{n} 2(v_i - (mu_i + d)) = 0$$

$\longrightarrow$

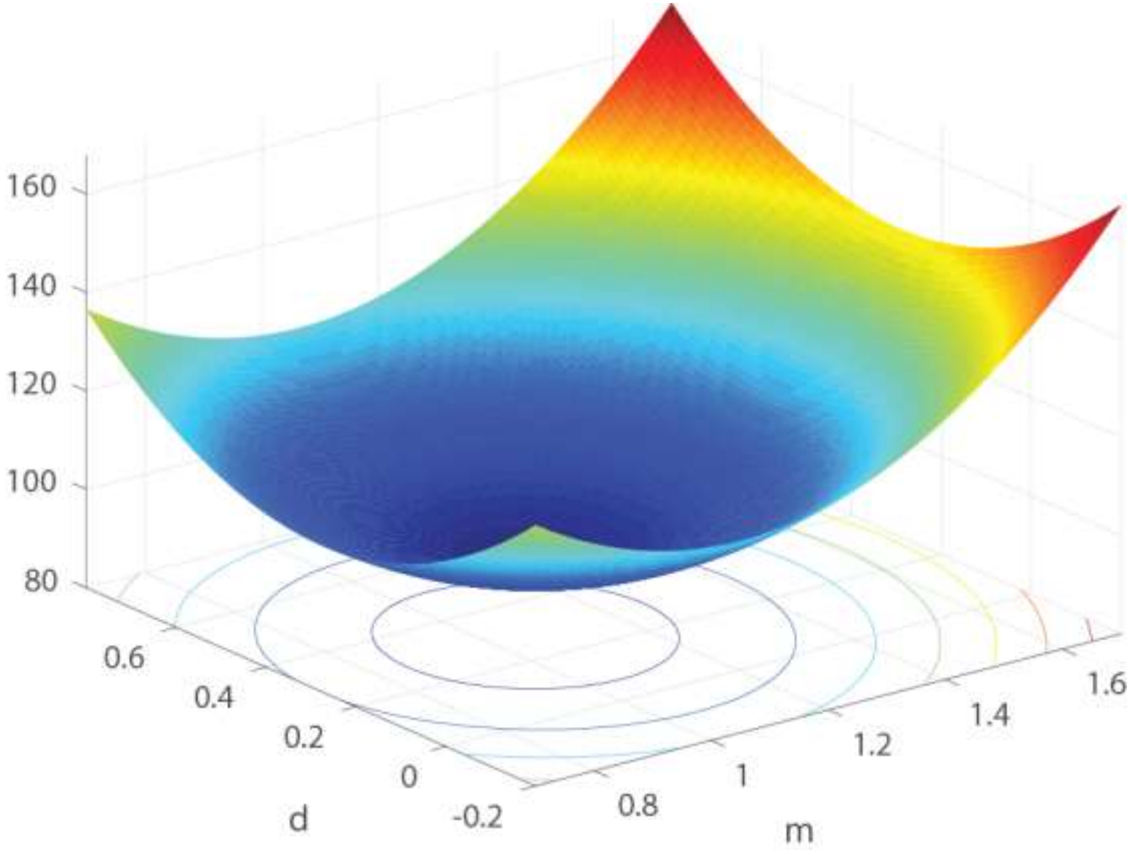
$$m\sum_{i=1}^{n} u_i^2 + d\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} u_i v_i$$

$$m\sum_{i=1}^{n} u_i + nd = \sum_{i=1}^{n} v_i$$

$$\begin{bmatrix} \sum_{i=1}^{n} u_i^2 & \sum_{i=1}^{n} u_i \\ \sum_{i=1}^{n} u_i & n \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} u_i v_i \\ \sum_{i=1}^{n} v_i \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} m \\ d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} u_i^2 & \sum_{i=1}^{n} u_i \\ \sum_{i=1}^{n} u_i & n \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} u_i v_i \\ \sum_{i=1}^{n} v_i \end{bmatrix}$$

Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

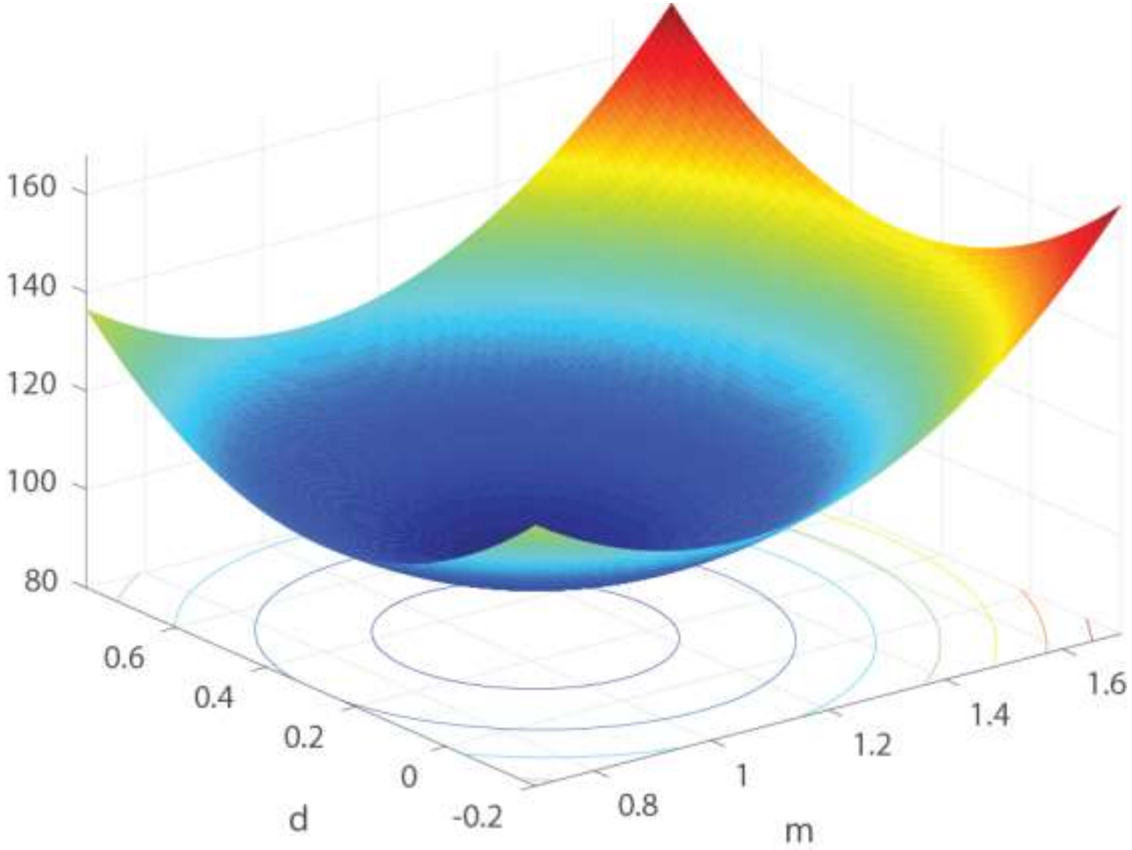
Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$v_1 \approx mu_1 + d$

$v_2 \approx mu_2 + d$

$v_n \approx mu_n + d$

Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d \qquad \begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$

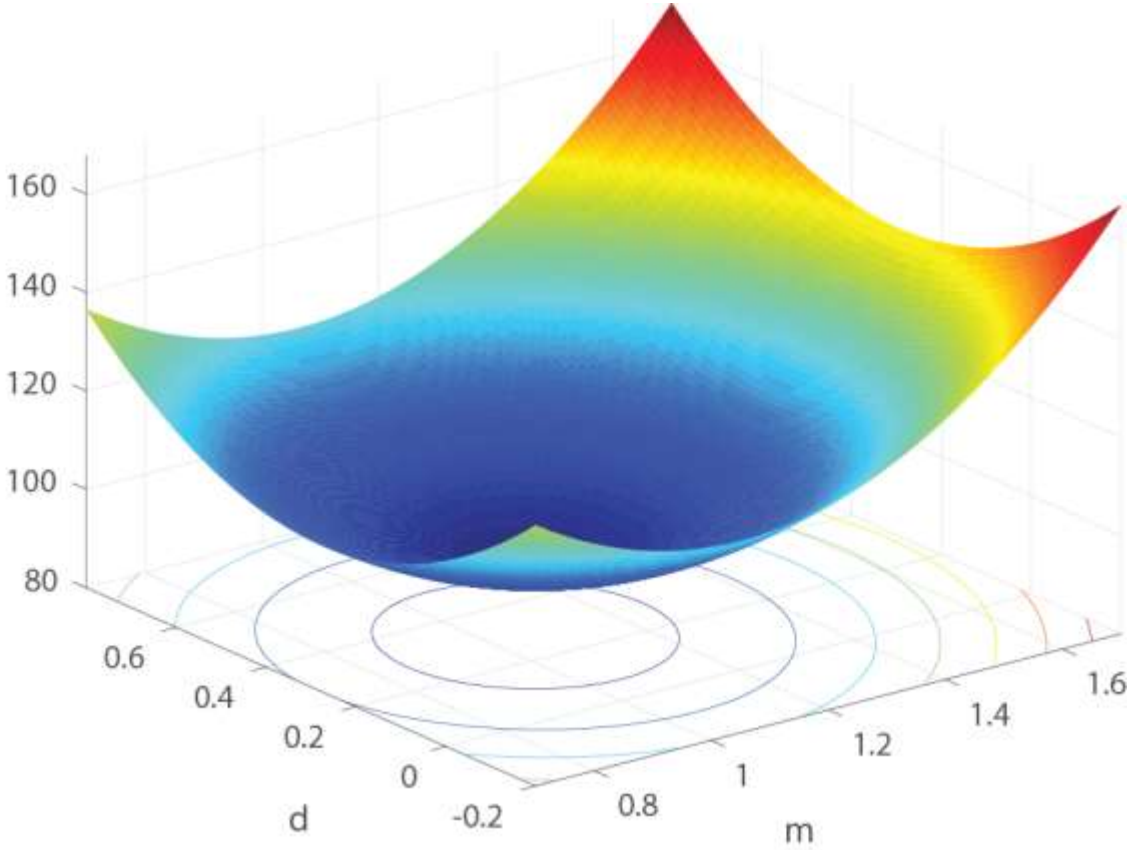
$$v_2 \approx mu_2 + d \quad \longrightarrow \quad \begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$

$$v_n \approx mu_n + d \qquad \begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} (v_i - (mu_i + d))^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$



$v_1 \approx mu_1 + d$

$v_2 \approx mu_2 + d$

$v_n \approx mu_n + d$

$\rightarrow$

$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$

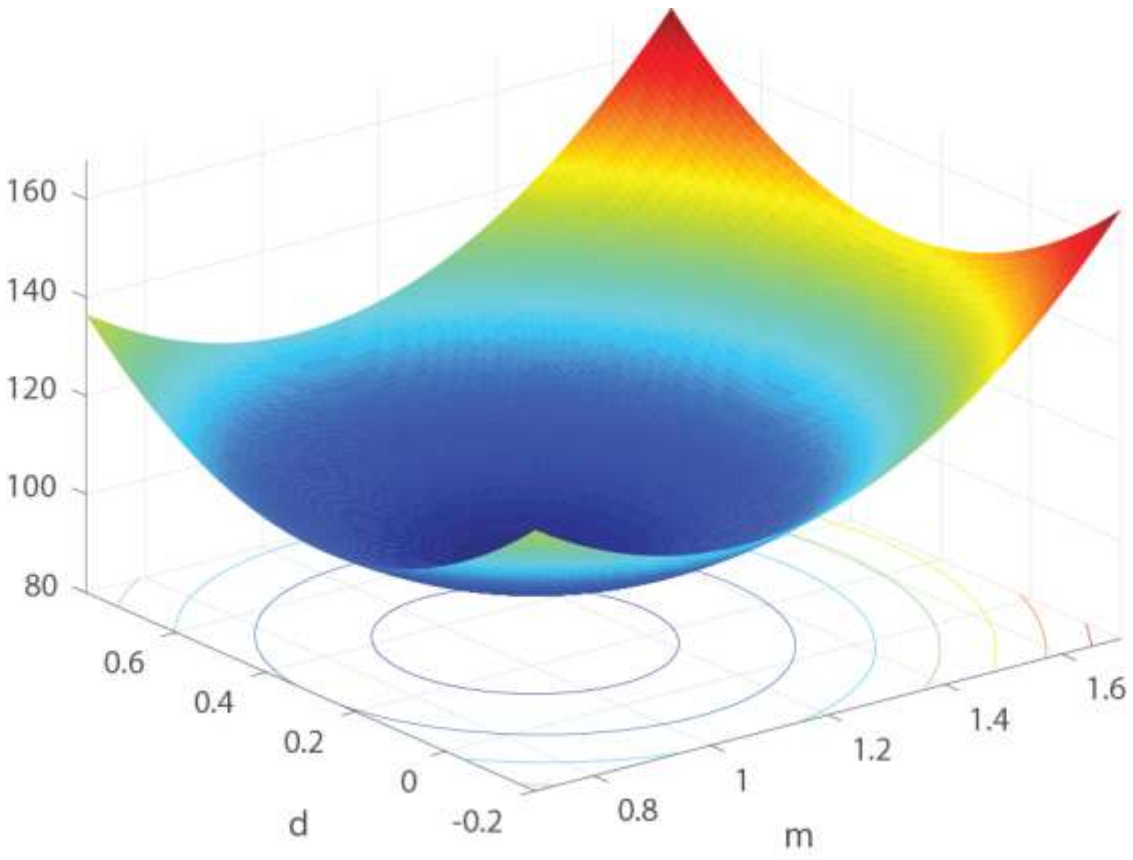
$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$

$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$

$\rightarrow$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$v_n \approx mu_n + d$$

→

$$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$
$$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$
$$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

→

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$$

We can't invert A.

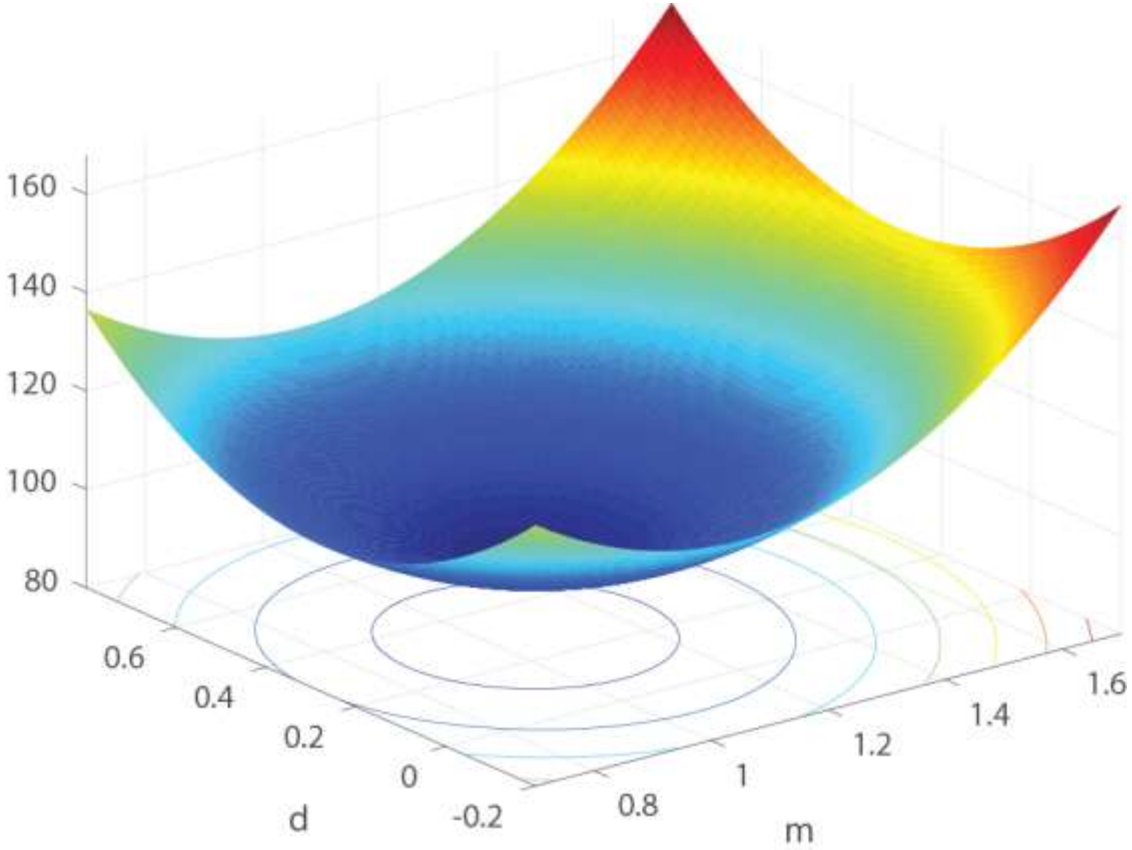
Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} (v_i - (mu_i + d))^2$

$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$

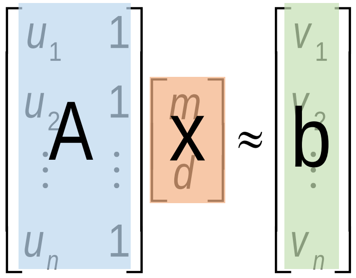
$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

A x ≈ b



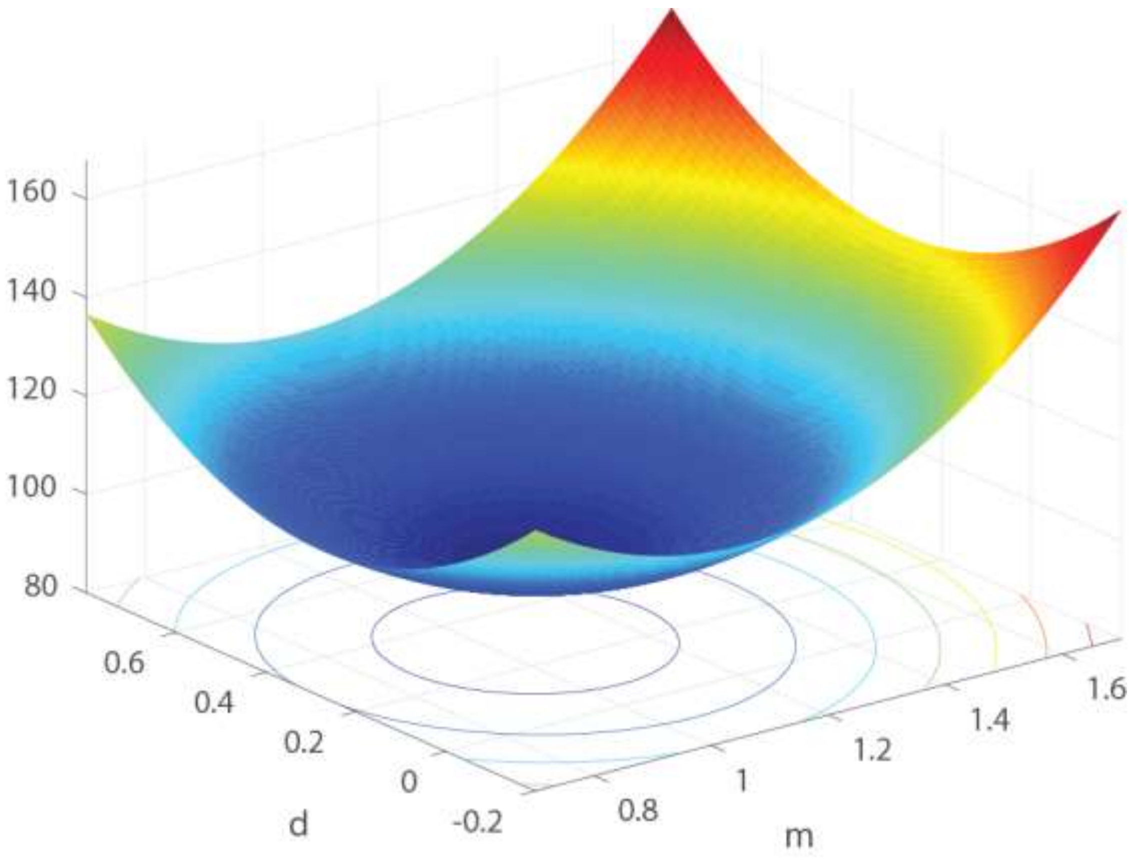
Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$



$$E = (\mathrm{Ax}-\mathrm{b})^{\mathrm{T}}(\mathrm{Ax}-\mathrm{b}) = \|\mathrm{Ax}-\mathrm{b}\|^2$$

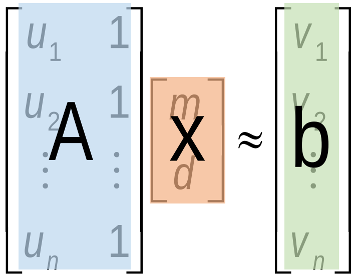
$$= \mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{b} + \mathrm{b}^{\mathrm{T}}\mathrm{b}$$



$$\frac{\partial E}{\partial \mathrm{x}} = \qquad ?$$

Line Fitting (Ax=b)

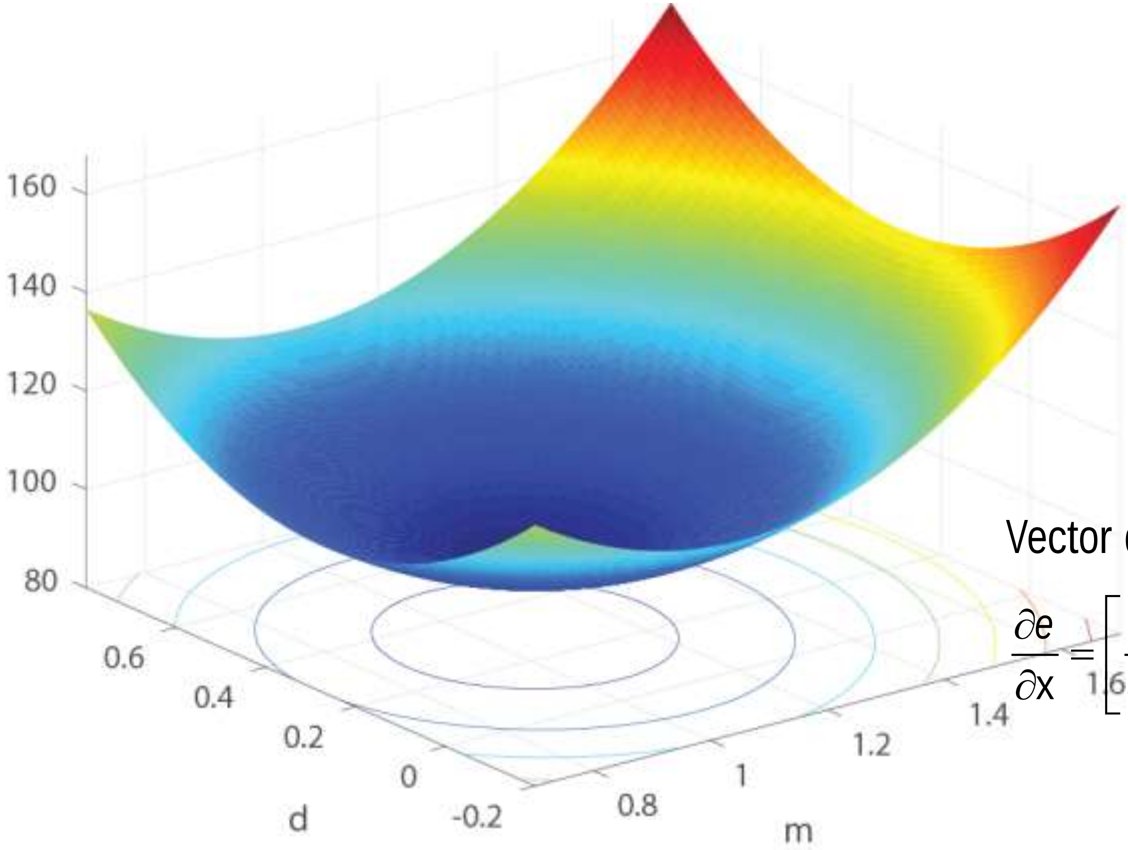
Total error: $E = \sum_{i=1}^{n} (v_i - (mu_i + d))^2$



$$E = (\mathrm{Ax} - \mathrm{b})^T (\mathrm{Ax} - \mathrm{b}) = \|\mathrm{Ax} - \mathrm{b}\|^2$$

$$= \mathrm{x}^T\mathrm{A}^T\mathrm{Ax} - 2\mathrm{x}^T\mathrm{A}^T\mathrm{b} + \mathrm{b}^T\mathrm{b}$$

$$\frac{\partial E}{\partial \mathrm{x}} = \qquad ?$$



Vector derivative:

$$\frac{\partial e}{\partial \mathrm{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

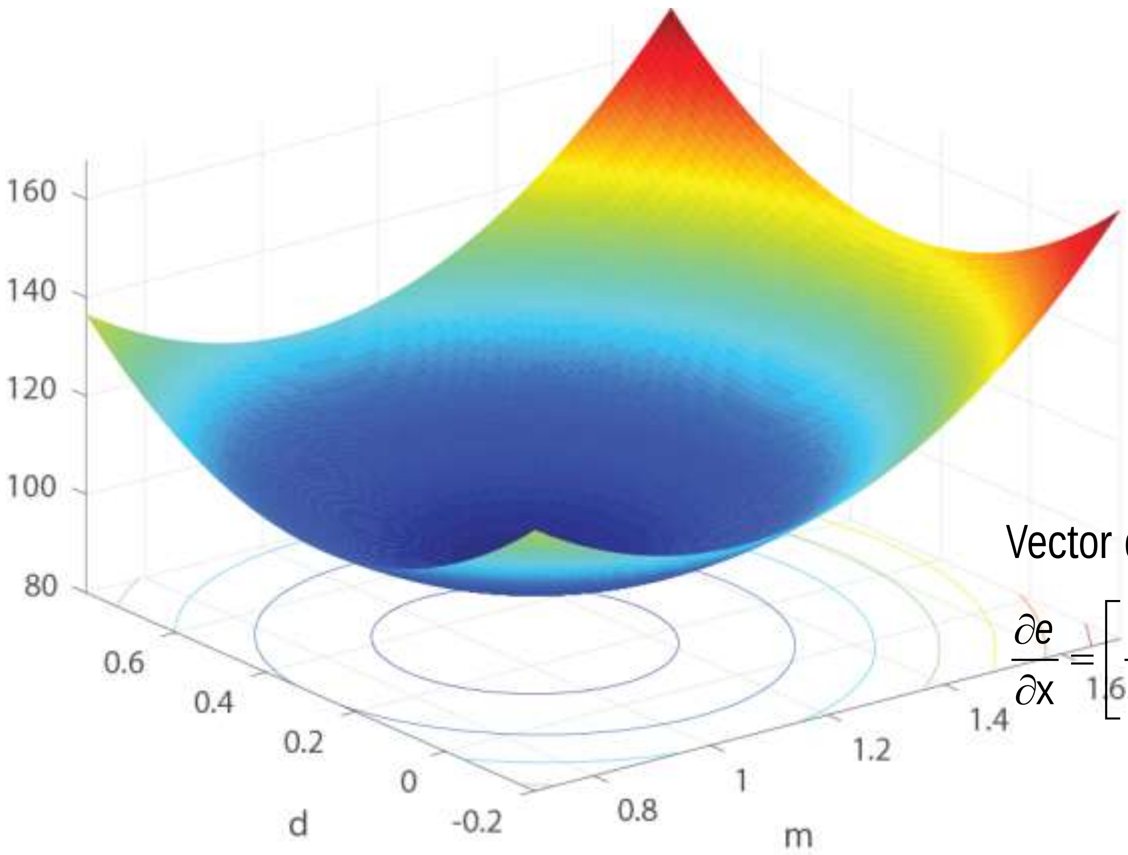
Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathrm{A}\,\mathrm{x} \approx \mathrm{b}$$

$$E = (\mathrm{Ax}-\mathrm{b})^{\mathrm{T}}(\mathrm{Ax}-\mathrm{b}) = \|\mathrm{Ax}-\mathrm{b}\|^2$$
$$= \mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{b} + \mathrm{b}^{\mathrm{T}}\mathrm{b}$$

$$\frac{\partial E}{\partial \mathrm{x}} = \qquad ?$$



Vector derivative:

$$\frac{\partial e}{\partial \mathrm{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathrm{c}^{\mathrm{T}}\mathrm{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathrm{c}^{\mathrm{T}}\mathrm{x}}{\partial \mathrm{x}} =$$

Line Fitting (Ax=b)

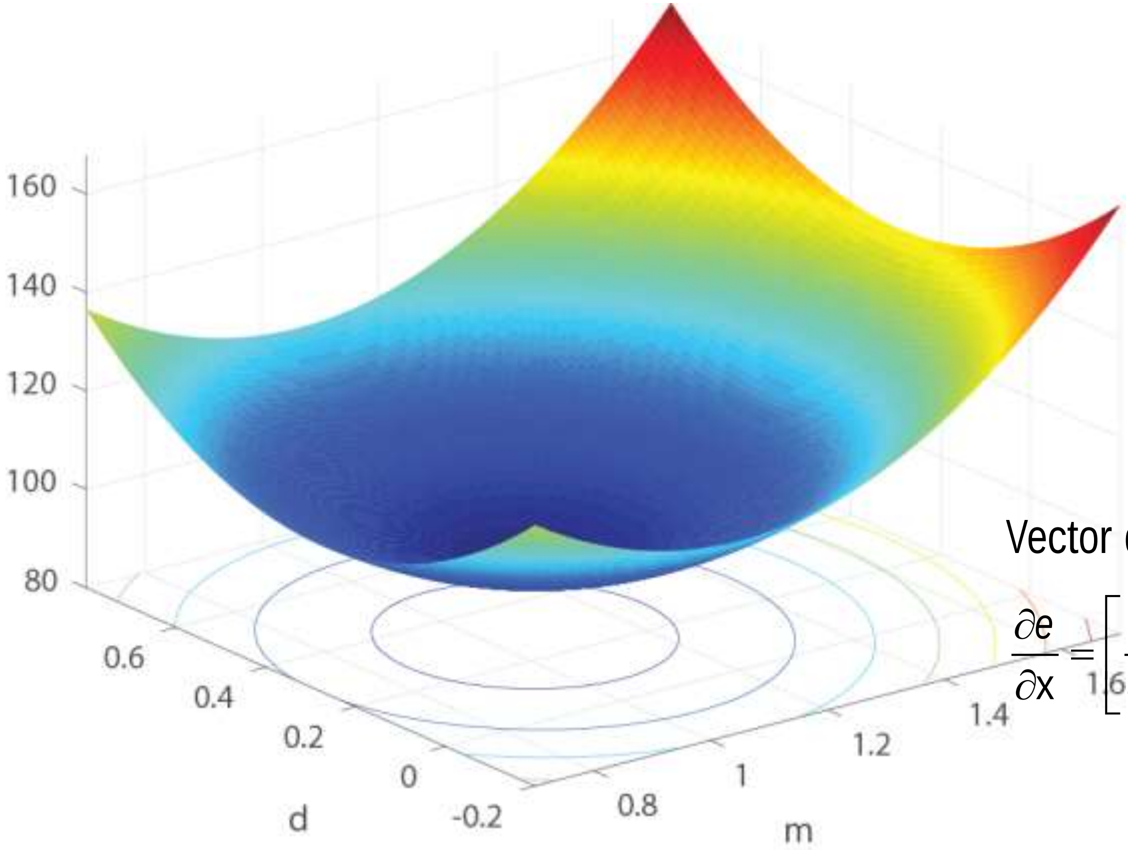
Total error: $E = \sum_{i=1}^{n} (v_i - (mu_i + d))^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

A x ≈ b

$$E = (\mathrm{Ax-b})^{\mathrm{T}} (\mathrm{Ax-b}) = \|\mathrm{Ax-b}\|^2$$

$$= \mathrm{x^T A^T A x} - 2\mathrm{x^T A^T b} + \mathrm{b^T b}$$



$$\frac{\partial E}{\partial \mathrm{x}} = \quad ?$$

Vector derivative:

$$\frac{\partial e}{\partial \mathrm{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathrm{c^T x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathrm{c^T x}}{\partial \mathrm{x}} = \frac{\partial}{\partial \mathrm{x}} (c_1 x_1 + \cdots + c_n x_n)$$

Line Fitting (Ax=b)

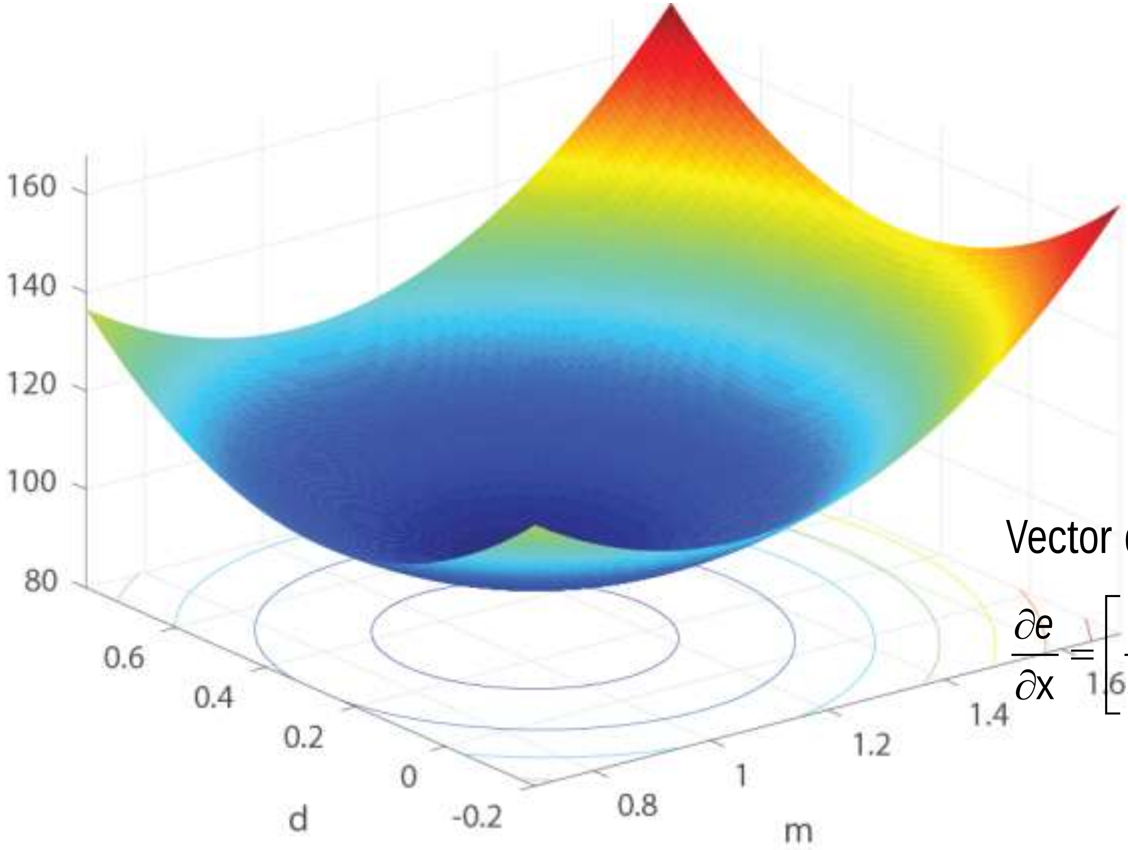
Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathrm{A}\,\mathrm{x} \approx \mathrm{b}$$

$$E = (\mathrm{Ax}-\mathrm{b})^{\mathrm{T}}(\mathrm{Ax}-\mathrm{b}) = \|\mathrm{Ax}-\mathrm{b}\|^2$$

$$= \mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{b} + \mathrm{b}^{\mathrm{T}}\mathrm{b}$$

$$\frac{\partial E}{\partial \mathrm{x}} = \qquad ?$$



Vector derivative:

$$\frac{\partial e}{\partial \mathrm{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

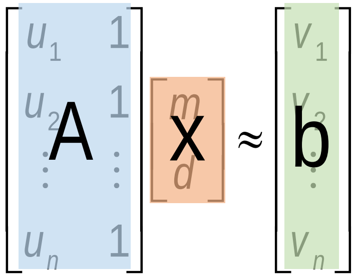
Ex) $e = \mathrm{c}^{\mathrm{T}}\mathrm{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathrm{c}^{\mathrm{T}}\mathrm{x}}{\partial \mathrm{x}} = \frac{\partial}{\partial \mathrm{x}}(c_1x_1 + \cdots + c_nx_n)$$

$$= \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$$

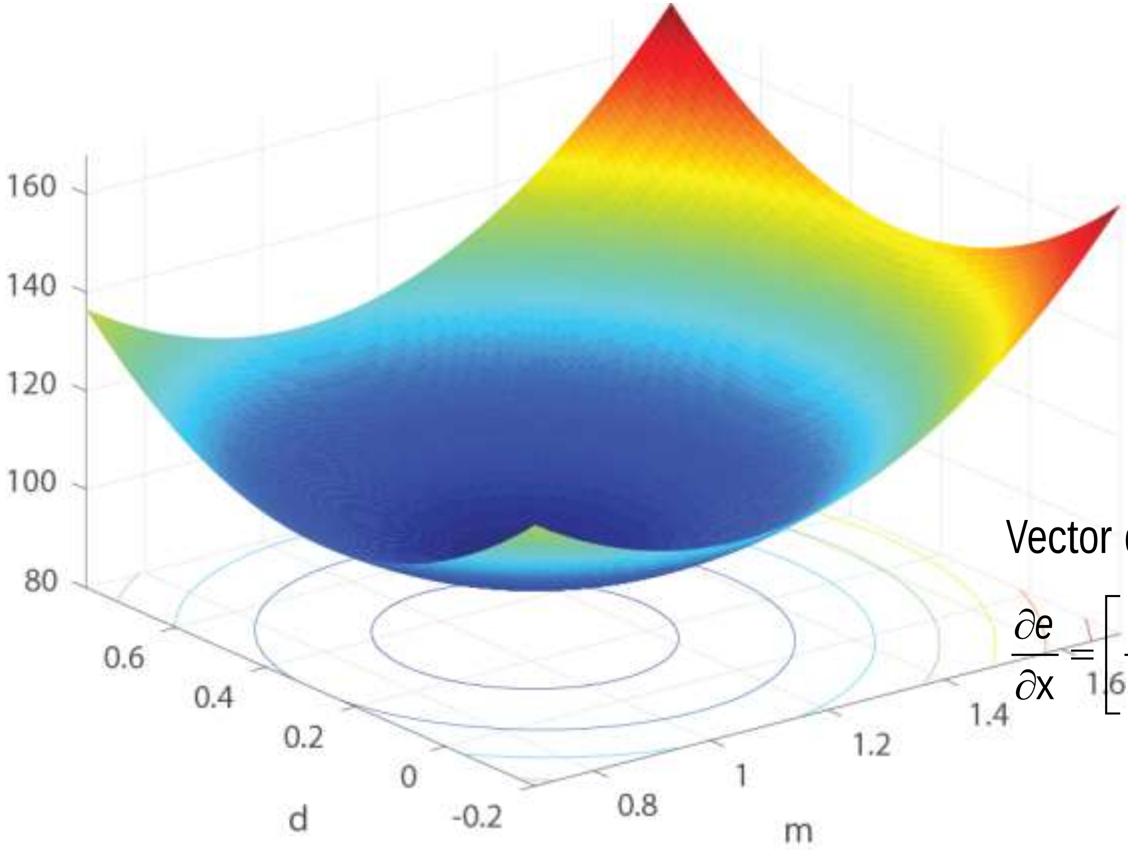
Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$



$$E = (\mathrm{Ax} - \mathrm{b})^T(\mathrm{Ax} - \mathrm{b}) = \|\mathrm{Ax} - \mathrm{b}\|^2$$
$$= \mathrm{x}^T\mathrm{A}^T\mathrm{Ax} - 2\mathrm{x}^T\mathrm{A}^T\mathrm{b} + \mathrm{b}^T\mathrm{b}$$

$$\frac{\partial E}{\partial \mathrm{x}} = 2\mathrm{A}^T\mathrm{Ax} - 2\mathrm{A}^T\mathrm{b} = 0$$



Vector derivative:

$$\frac{\partial e}{\partial \mathrm{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

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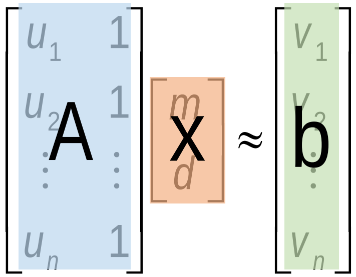
$$= \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$$

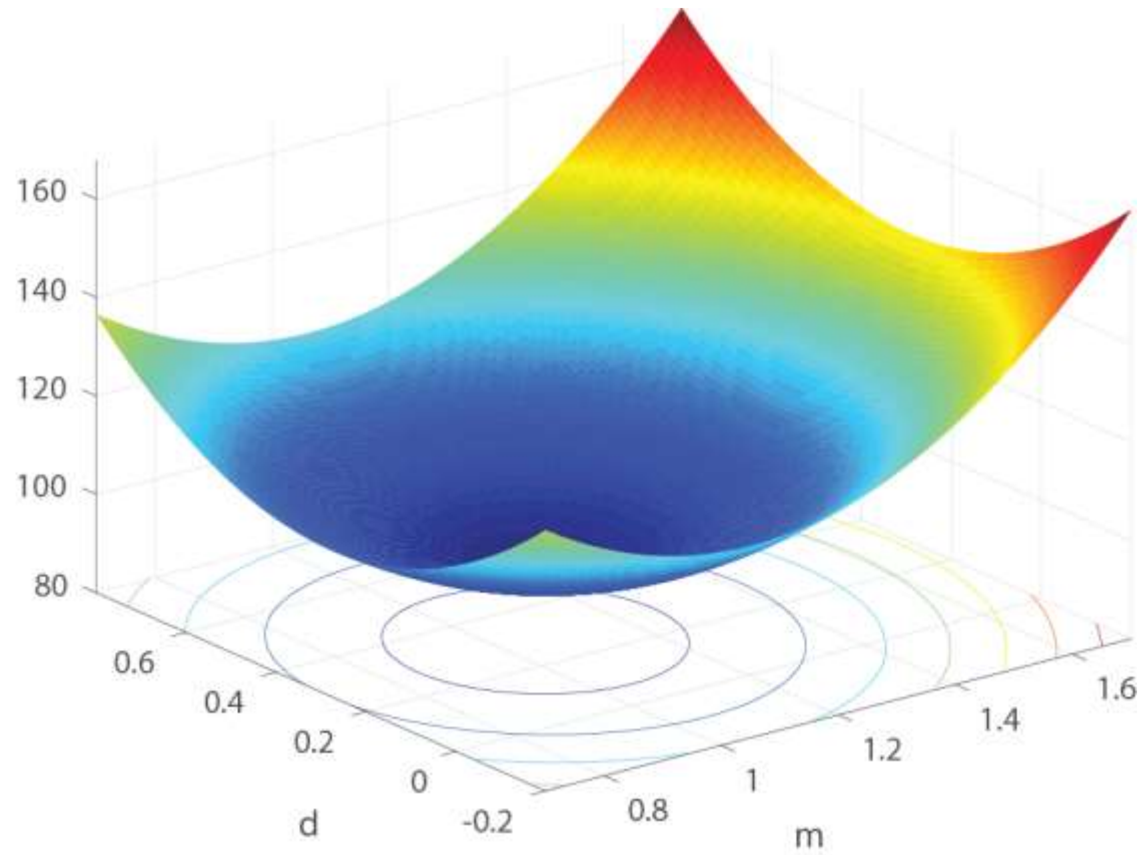
Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$



$$E = (\mathrm{Ax}-\mathrm{b})^T(\mathrm{Ax}-\mathrm{b}) = \|\mathrm{Ax}-\mathrm{b}\|^2$$

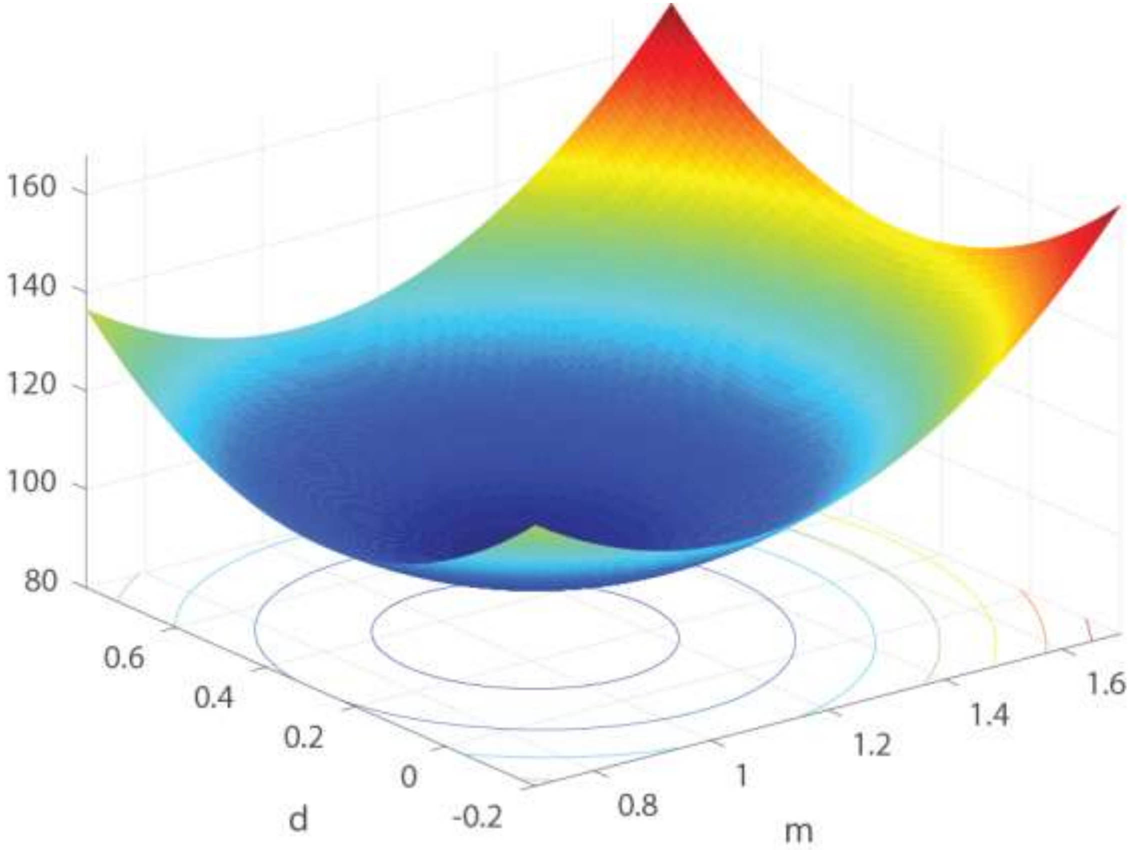
$$= \mathrm{x}^T\mathrm{A}^T\mathrm{Ax} - 2\mathrm{x}^T\mathrm{A}^T\mathrm{b} + \mathrm{b}^T\mathrm{b}$$



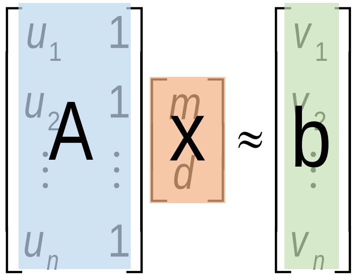
$$\frac{\partial E}{\partial \mathrm{x}} = 2\mathrm{A}^T\mathrm{Ax} - 2\mathrm{A}^T\mathrm{b} = 0$$

$$\longrightarrow \mathrm{A}^T\mathrm{Ax} = \mathrm{A}^T\mathrm{b}$$

Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} (v_i - (mu_i + d))^2$



$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$$

$$= x^T A^T A x - 2x^T A^T b + b^T b$$

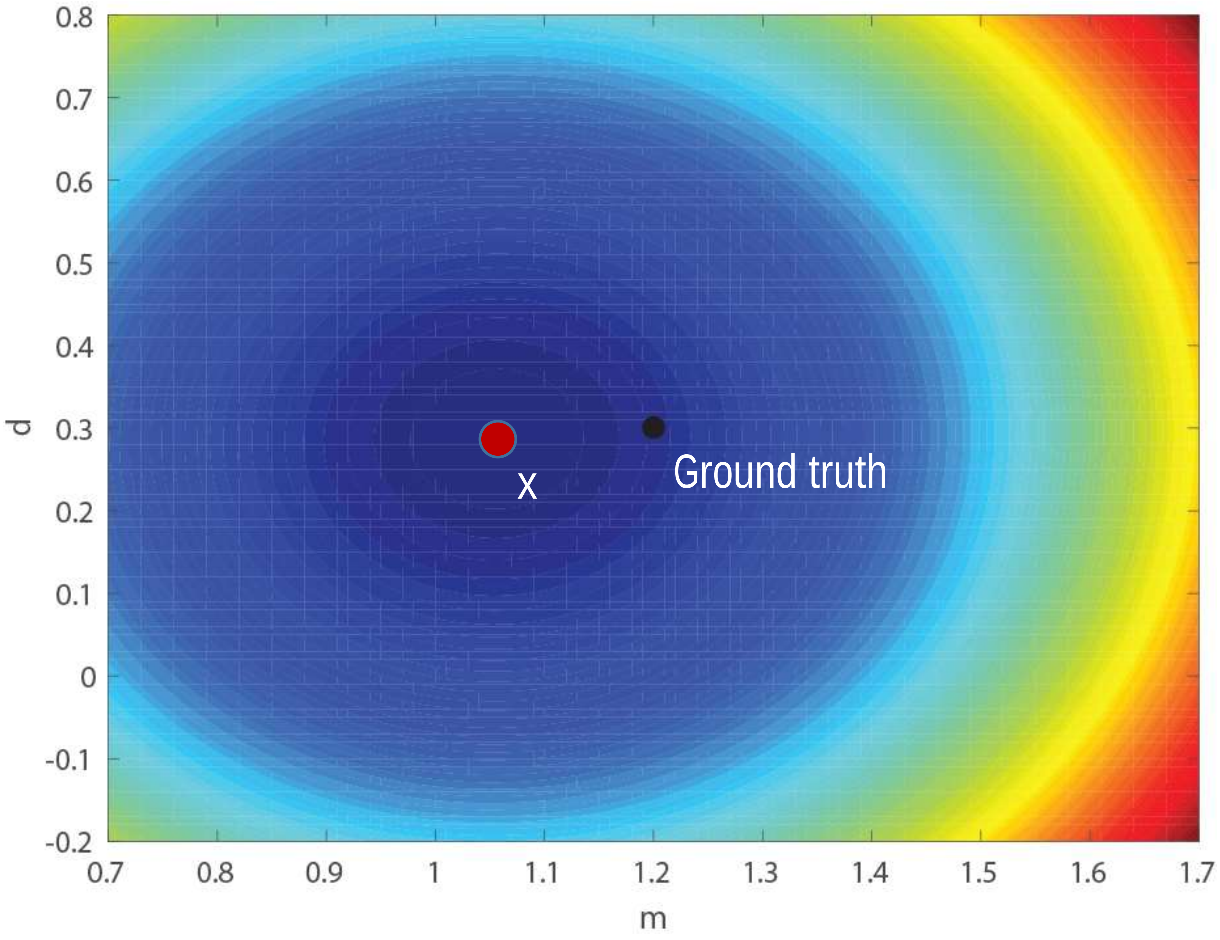
$$\frac{\partial E}{\partial x} = 2A^T A x - 2A^T b = 0$$

$$\longrightarrow A^T A x = A^T b$$

$$A^T \; A \; x = A^T \; b$$

Normal equation

Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} (v_i - (mu_i + d))^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A} \, \mathbf{x} \approx \mathbf{b}$$

$$E = (Ax - b)^T (Ax - b) = \|Ax - b\|^2$$

$$= x^T A^T A x - 2 x^T A^T b + b^T b$$

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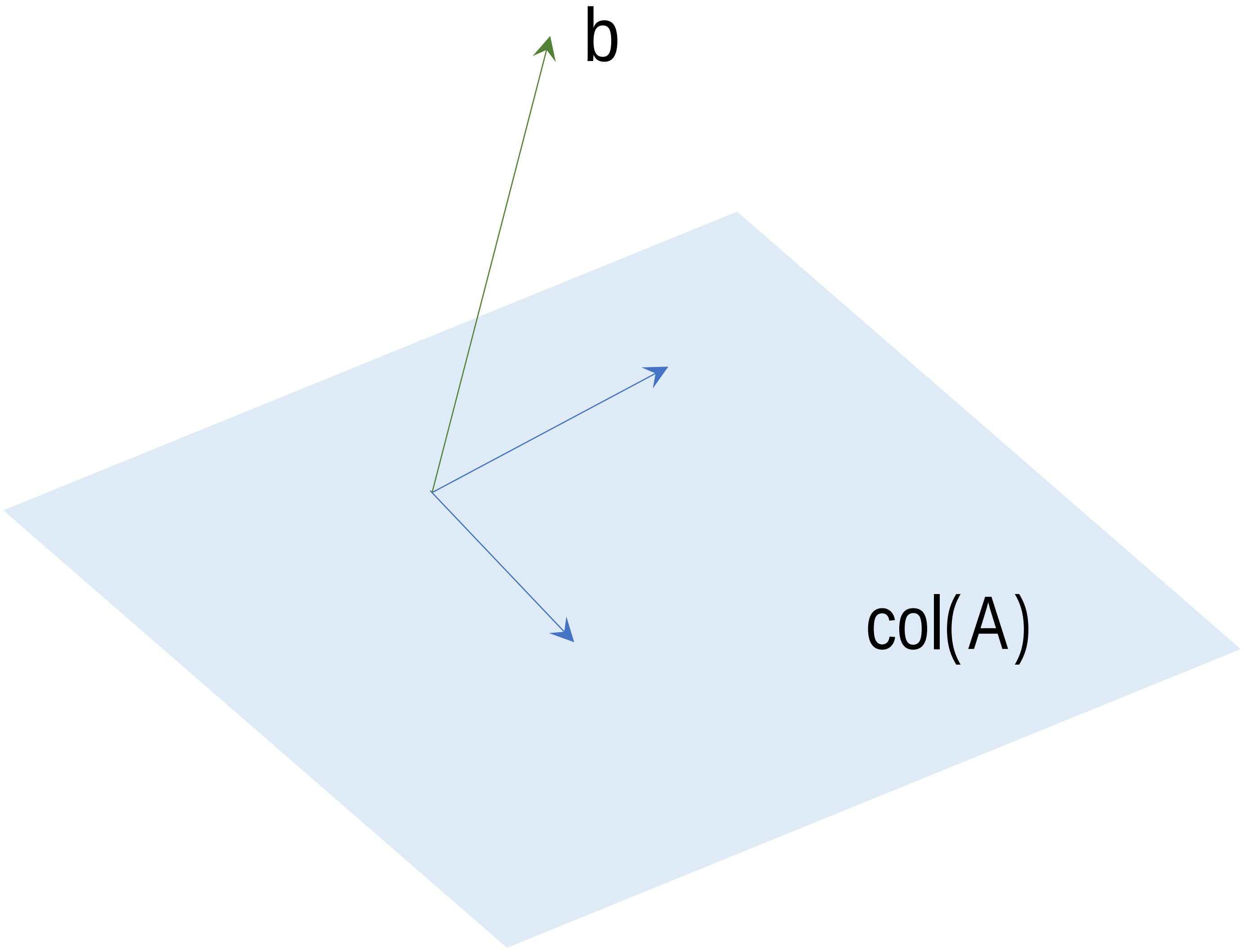
$$\longrightarrow A^T A x = A^T b$$

$$A^T \; A \; x = A^T \; b$$

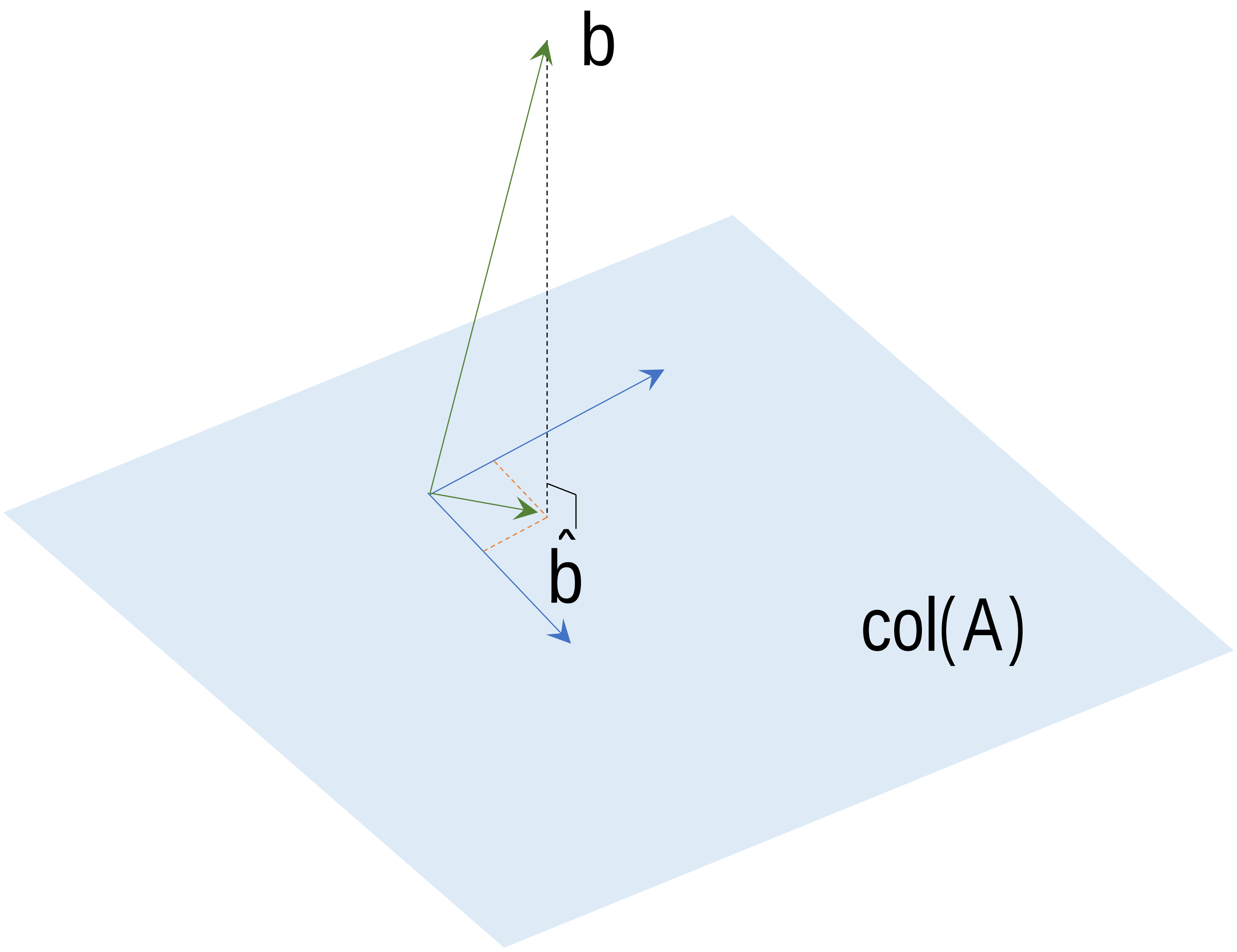
$$x = \left[A^T \; A \right]^{-1} A^T \; b$$

Geometric Interpretation

A x ≈ b



Geometric Interpretation

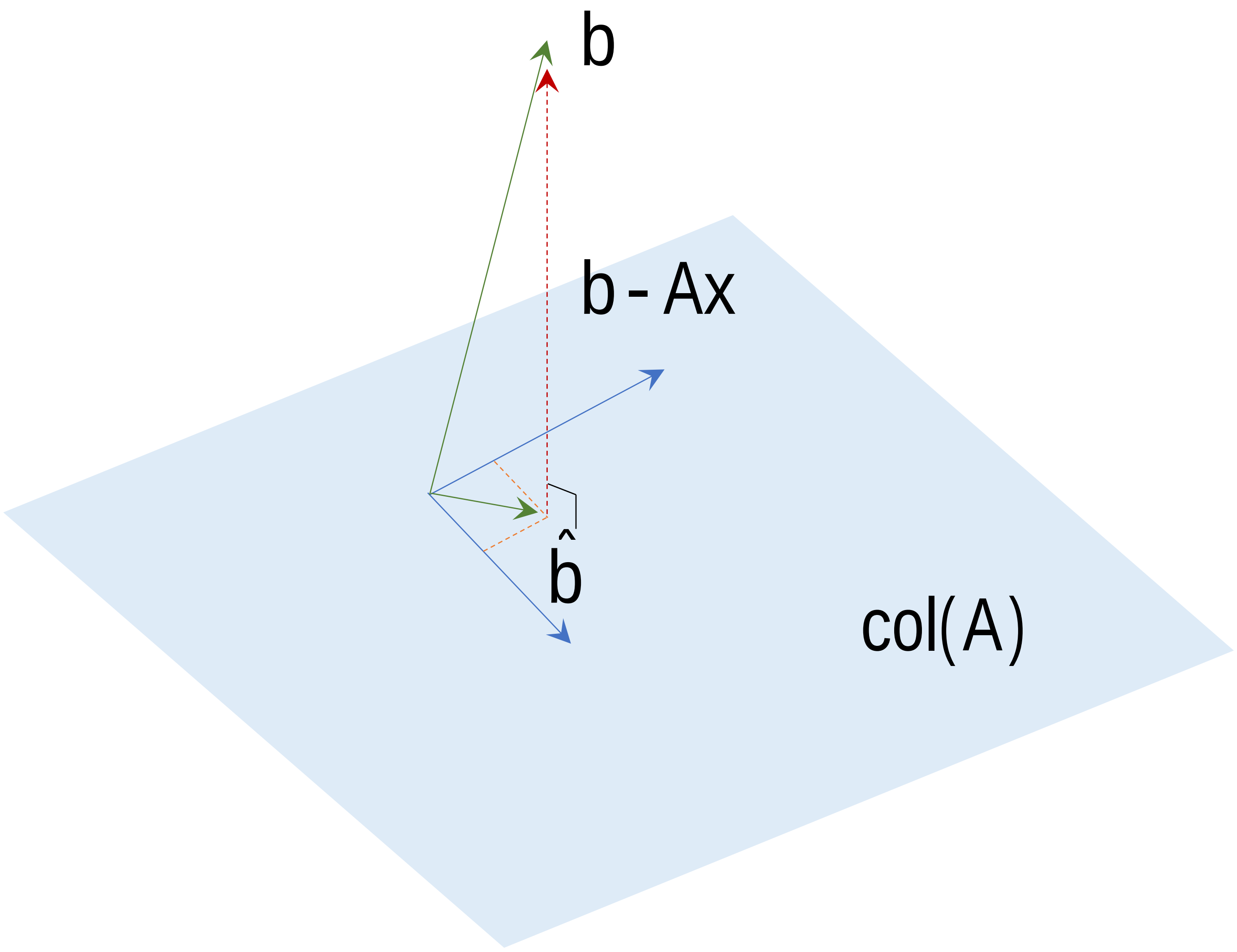


$$A\,x \approx b$$

$$A\,x = \hat{b}$$

$\hat{b}$: Best approximation of b spanned by col(A)

Geometric Interpretation



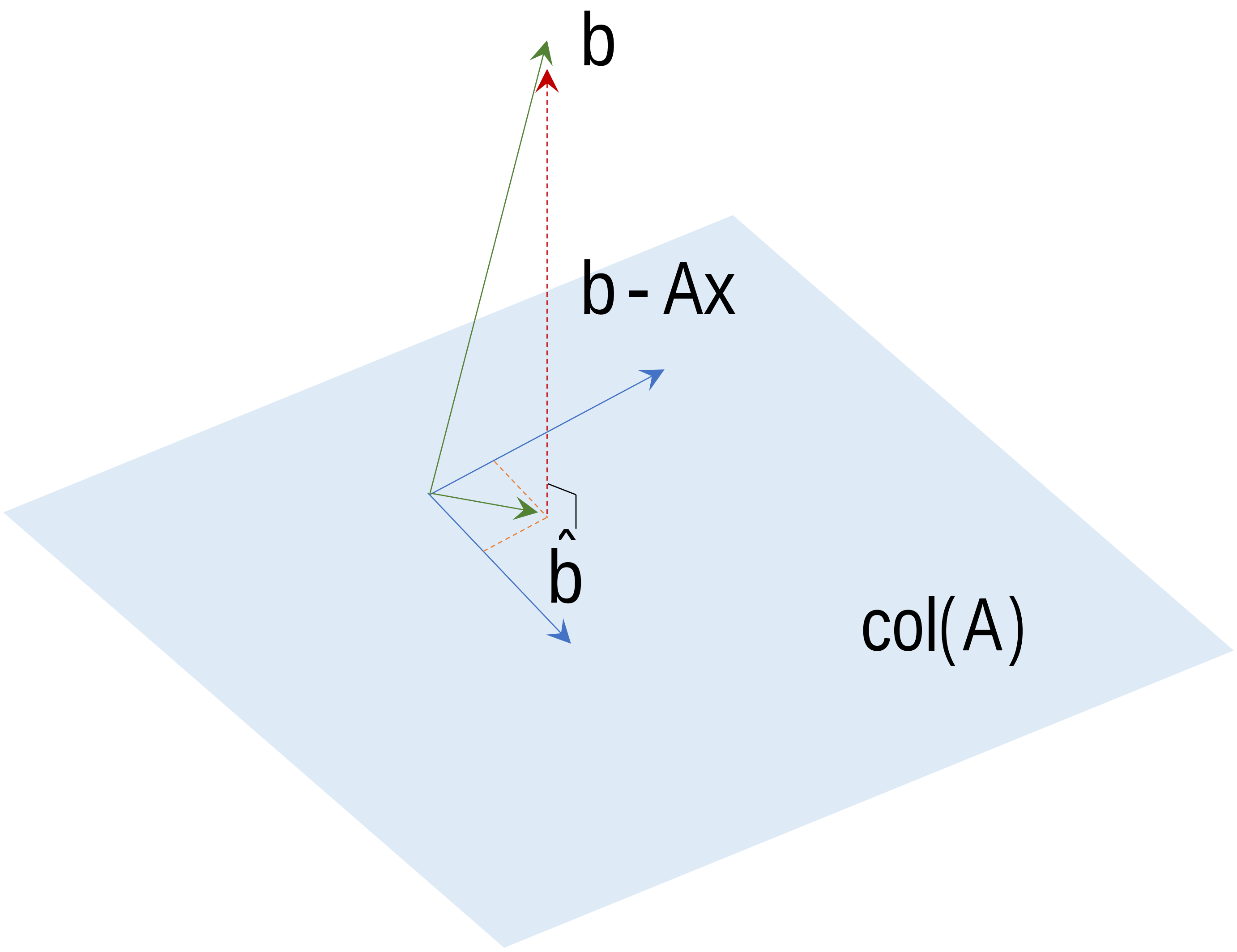
$$A\,x \approx b$$

$$A\,x = \hat{b}$$

$\hat{b}$: Best approximation of b spanned by col(A)

$$(b - Ax) \perp col(A)$$

Geometric Interpretation



$$A\,x \approx b$$

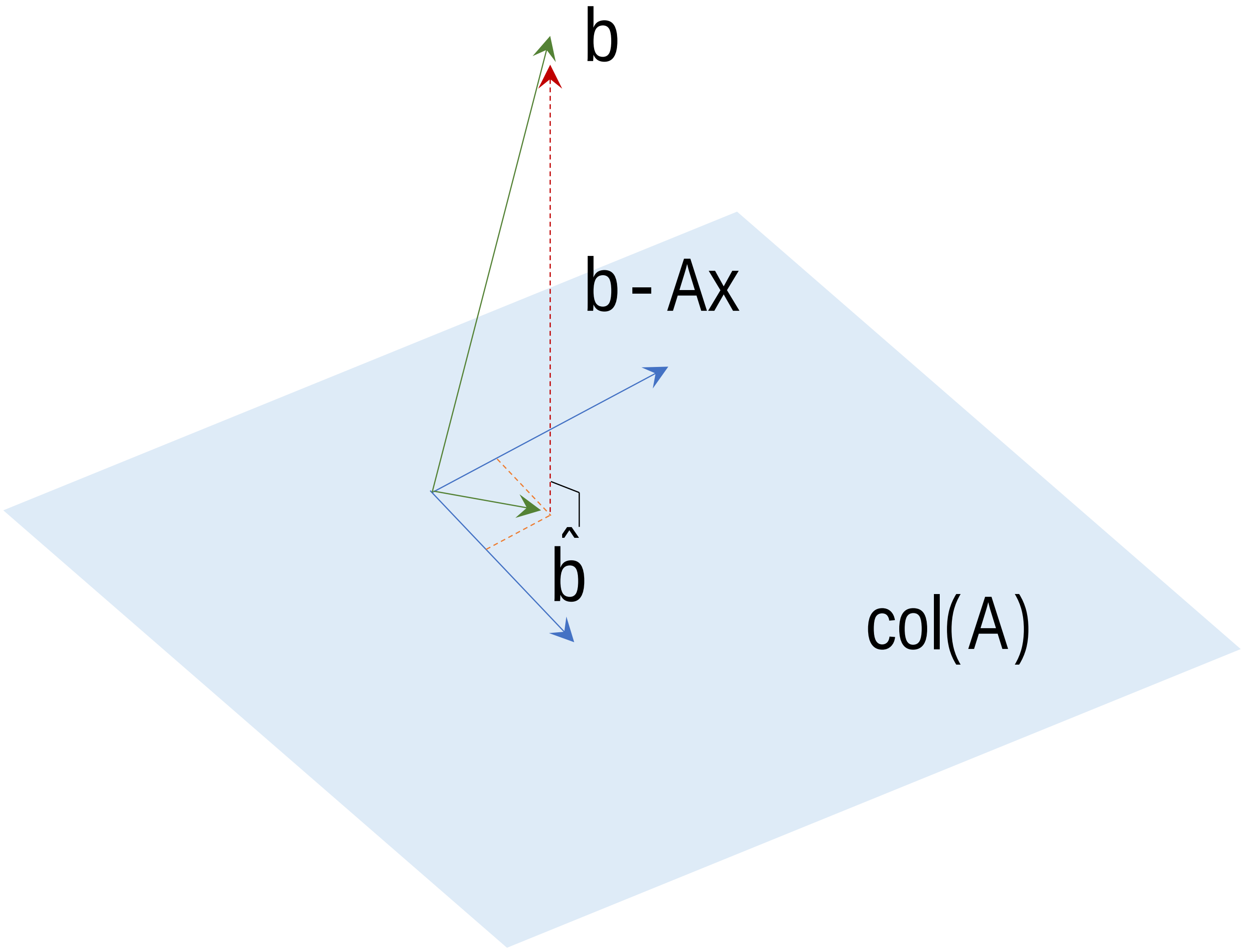
$$A\,x = \hat{b}$$

$\hat{b}$: Best approximation of b spanned by col(A)

$$(b - Ax) \perp col(A)$$

$$A^T(b - Ax) = 0$$

Geometric Interpretation



$$A \; x \approx b$$

$$A \; x = \hat{b}$$

$\hat{b}$: Best approximation of b spanned by col(A)

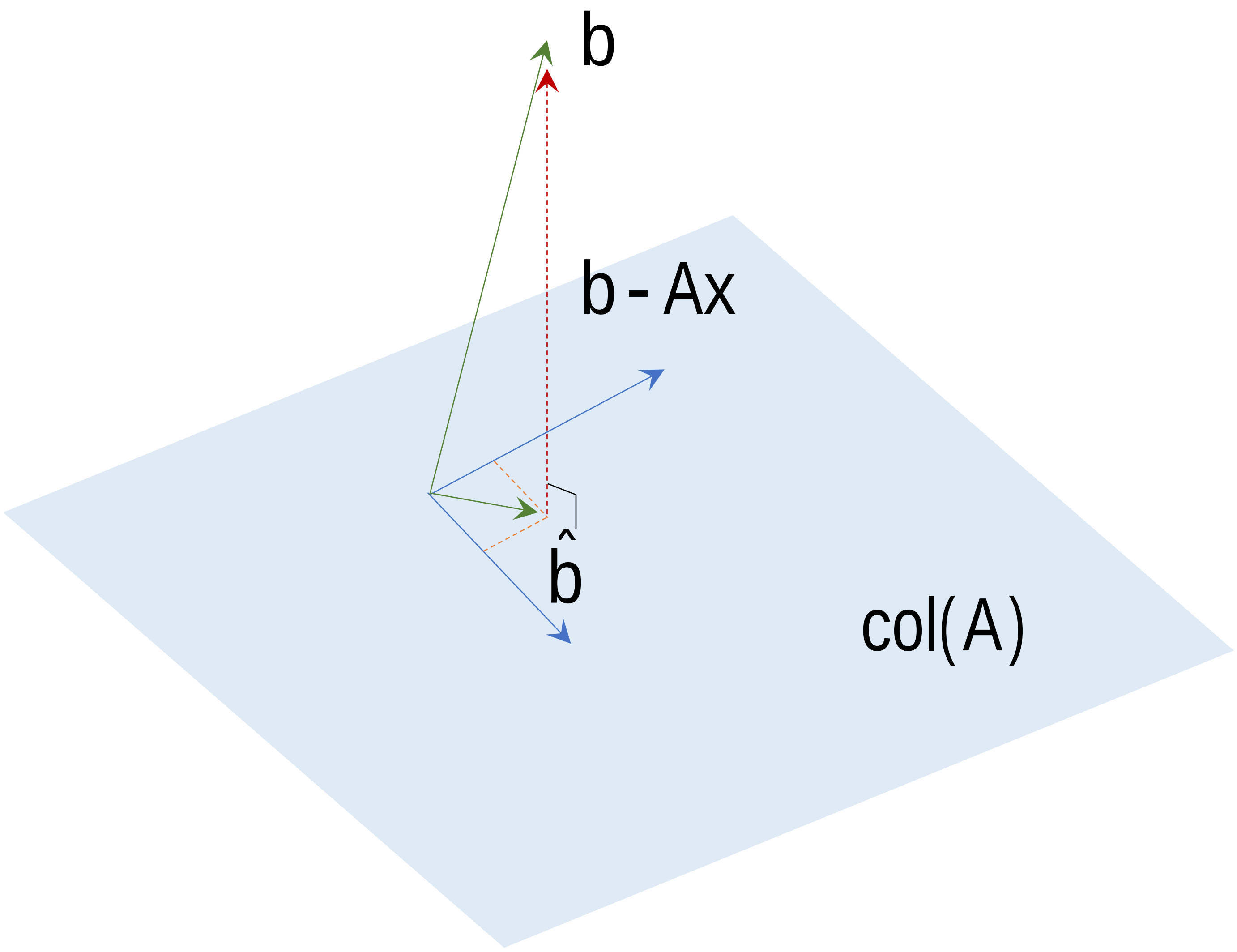
$$(b - Ax) \perp col(A)$$

$$A^T(b - Ax) = 0$$

$$A^T \; A \; x = A^T \; b$$

: Normal equation

Geometric Interpretation



$$Ax \approx b$$

$$Ax = \hat{b}$$

$\hat{b}$: Best approximation of b spanned by col(A)

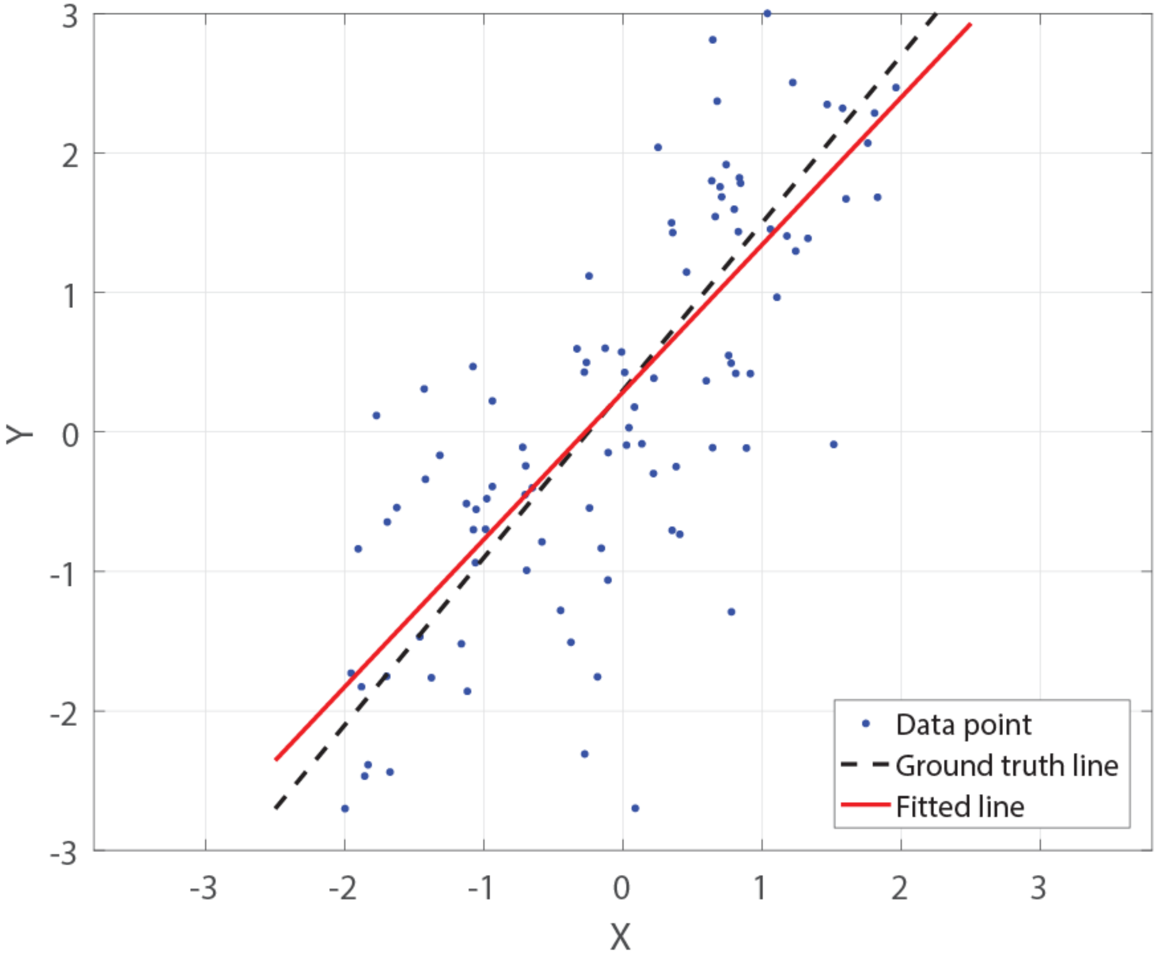
$$(b - Ax) \perp col(A)$$

$$A^T(b - Ax) = 0$$

$$A^T A x = A^T b$$

$$x = \left[A^T A\right]^{-1} A^T b$$

Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathrm{A} \; \mathrm{x} \approx \mathrm{b}$$

$$E = (\mathrm{Ax}-\mathrm{b})^{\mathrm{T}}(\mathrm{Ax}-\mathrm{b}) = \|\mathrm{Ax}-\mathrm{b}\|^2$$

$$= \mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{b} + \mathrm{b}^{\mathrm{T}}\mathrm{b}$$

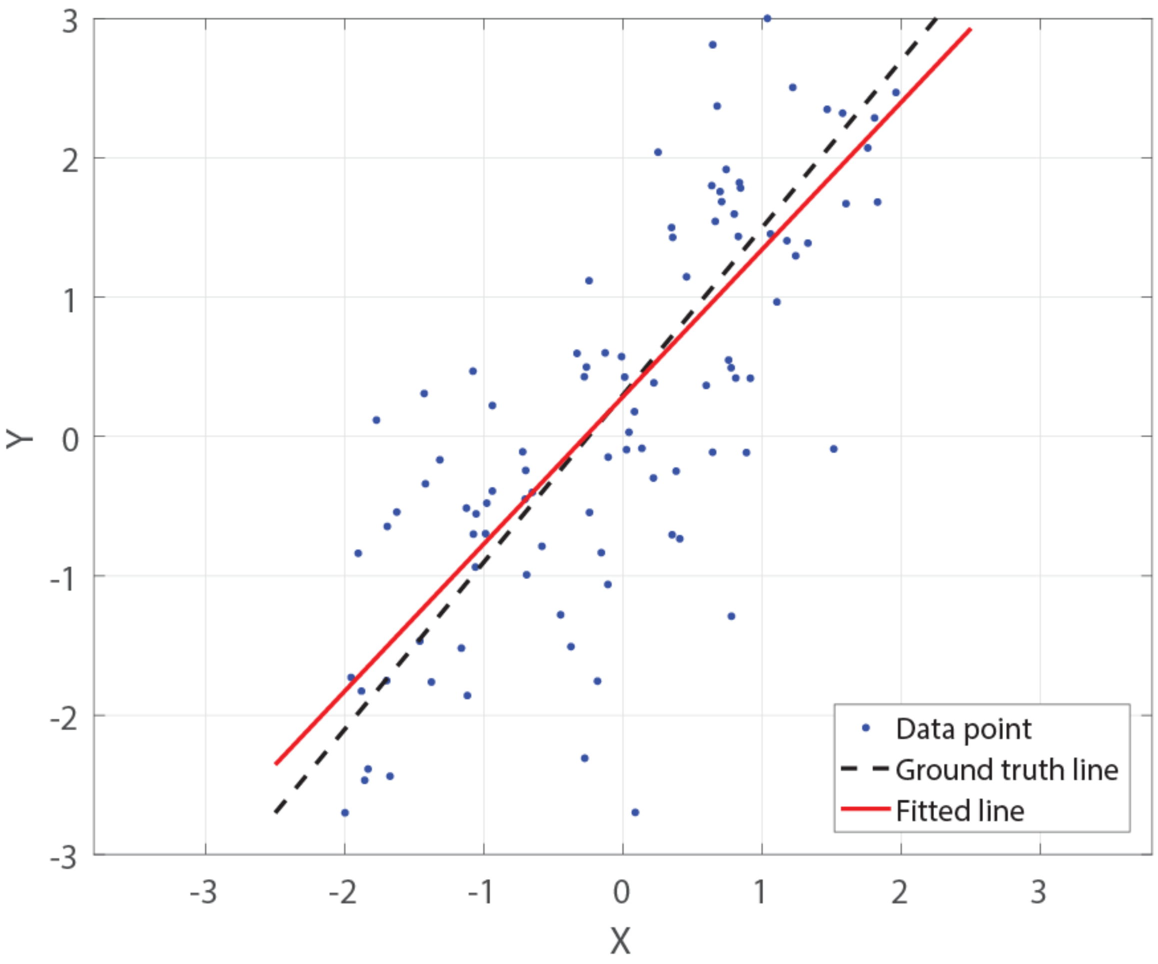
$$\frac{\partial E}{\partial \mathrm{x}} = 2\mathrm{A}^{\mathrm{T}}\mathrm{Ax} - 2\mathrm{A}^{\mathrm{T}}\mathrm{b} = 0$$

$$\longrightarrow \mathrm{A}^{\mathrm{T}}\mathrm{Ax} = \mathrm{A}^{\mathrm{T}}\mathrm{b}$$

$$\mathrm{A}^{\mathrm{T}} \; \mathrm{A} \; \mathrm{x} = \mathrm{A}^{\mathrm{T}} \; \mathrm{b}$$

$$\mathrm{x} = \left[\mathrm{A}^{\mathrm{T}} \; \mathrm{A} \right]^{-1} \mathrm{A}^{\mathrm{T}} \; \mathrm{b}$$

Line Fitting (Ax=b)



LineFitting.m

```
m = 1.2;
d = 0.3;

x = 4*(rand(100,1)-.5);
y = m*x + d + randn(size(x));

A = [x ones(size(x))];
b = y;

u = A\b;
```

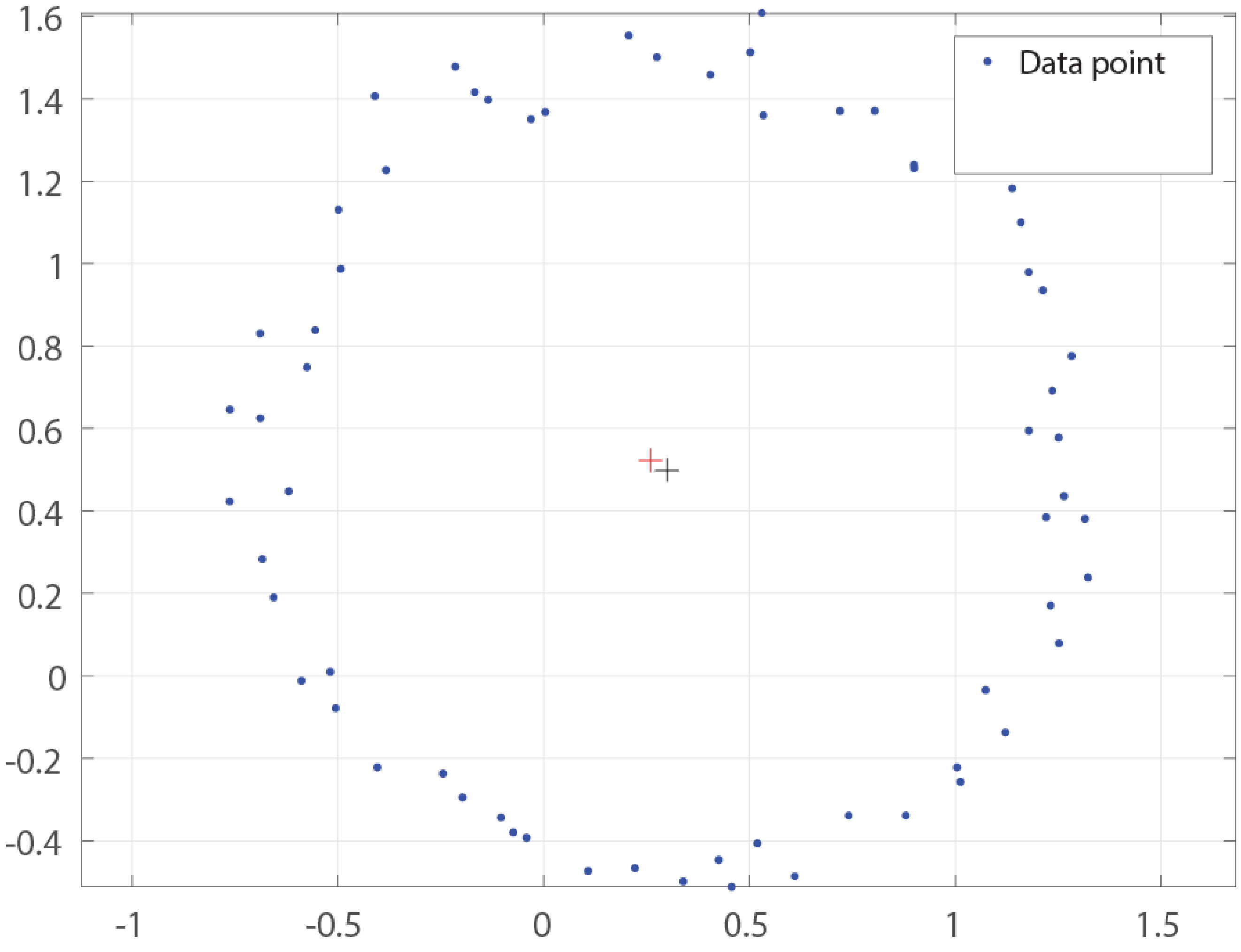
Ground truth

Random data point w/ Gaussian noise

$$x = \left(A^T A\right)^{-1} A^T b$$

\ backslush: solve linear least squares

Circle Fitting (Ax=b)



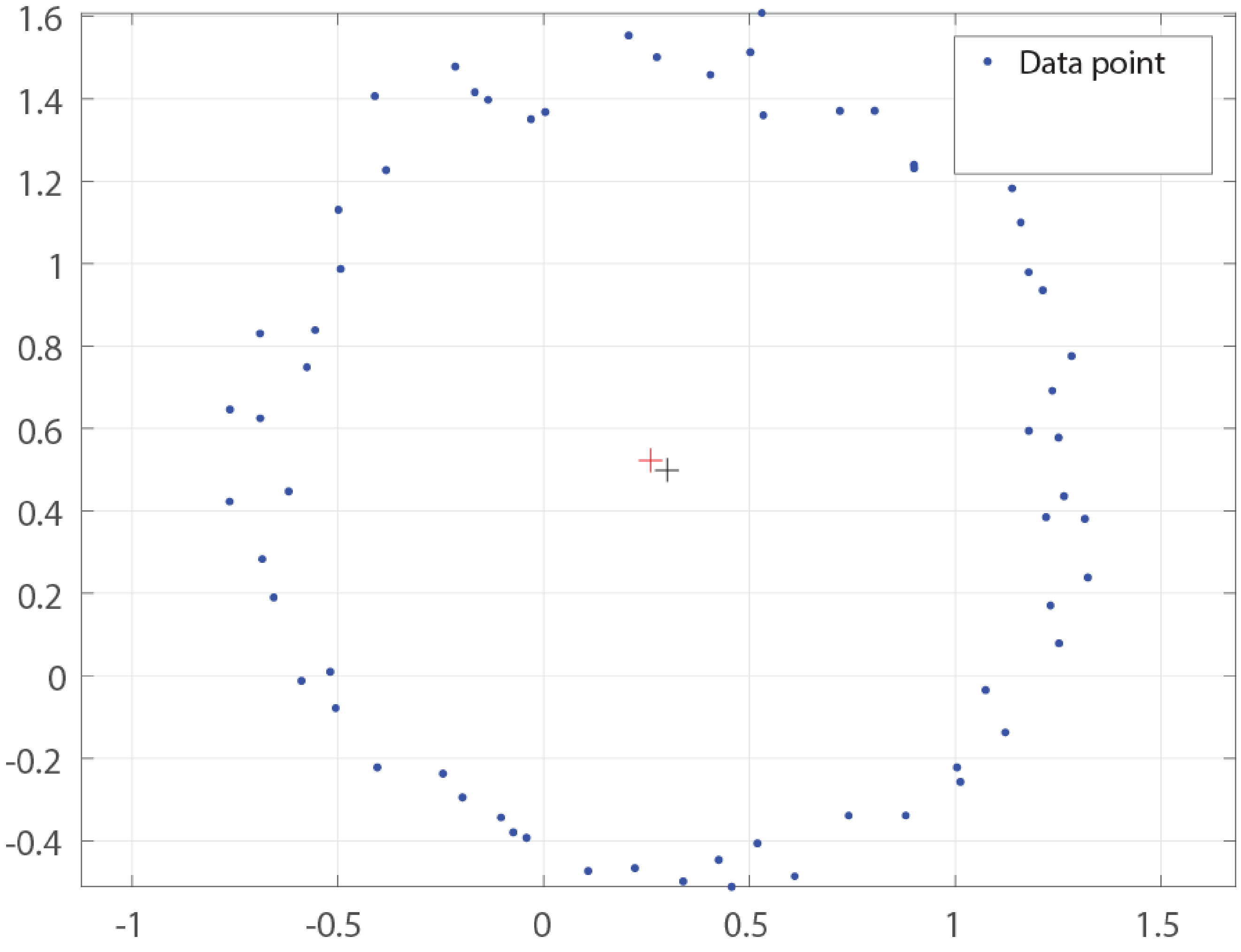
$$(x_1 - c_x)^2 + (y_1 - c_y)^2 = r^2$$

$$\vdots$$

$$(x_n - c_x)^2 + (y_n - c_y)^2 = r^2$$

Unkowns: c_x, c_y, r

Circle Fitting (Ax=b)

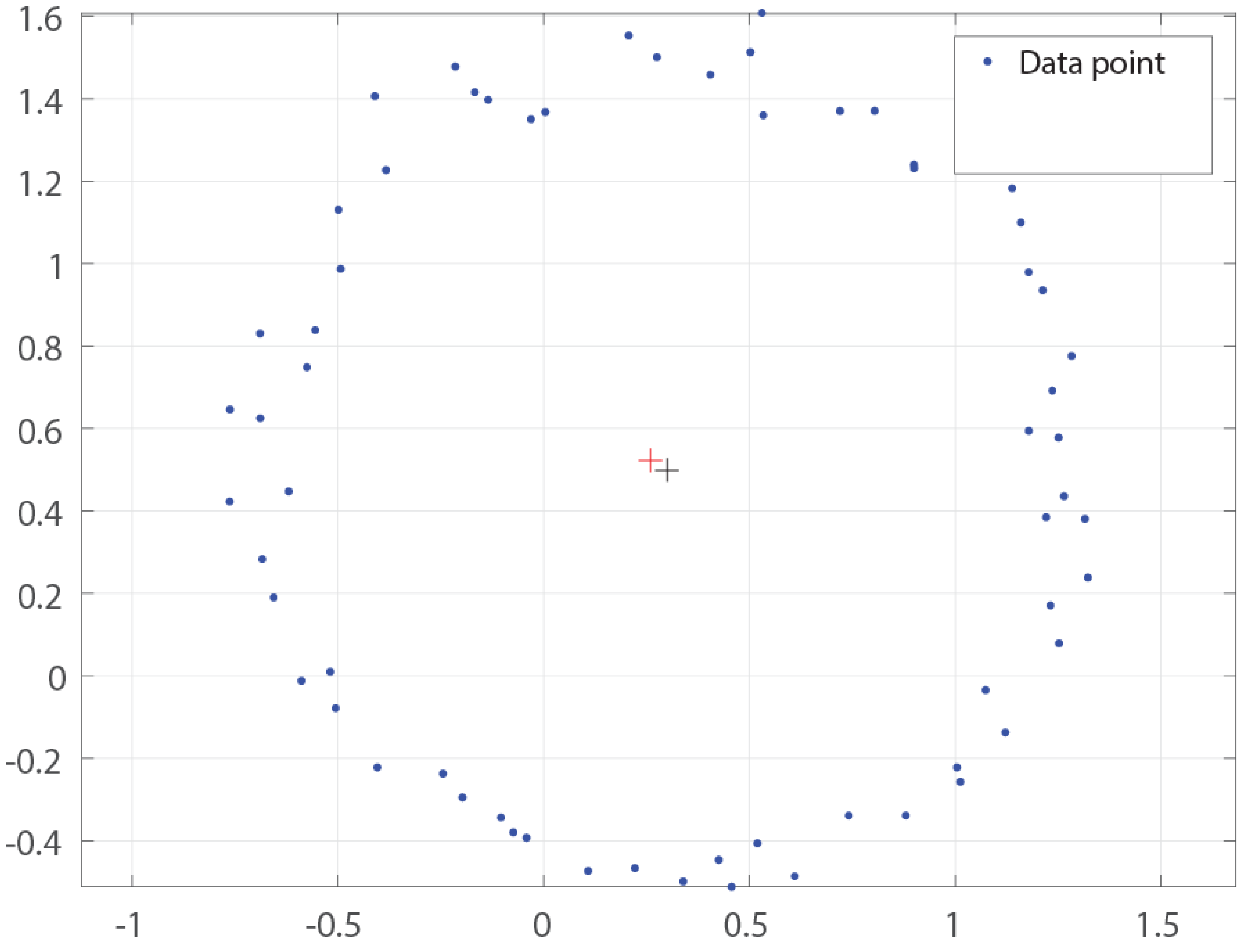


$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

Circle Fitting (Ax=b)



$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

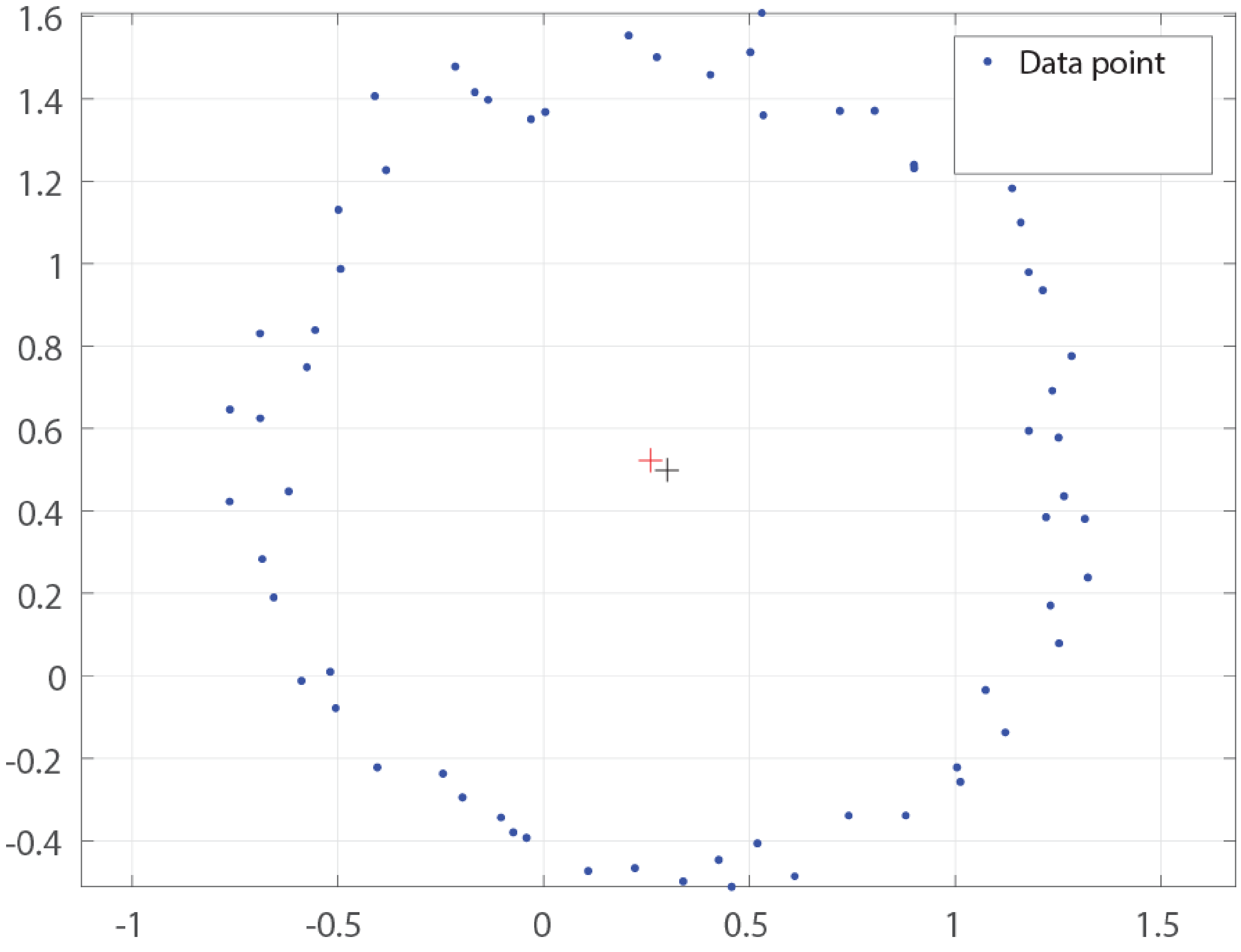
$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

$$\downarrow$$

$$x_i^2 - x_1^2 - 2c_x\left(x_i - x_1\right) + y_i^2 - y_1^2 - 2\left(y_i - y_1\right)c_y = 0$$

Circle Fitting (Ax=b)



$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

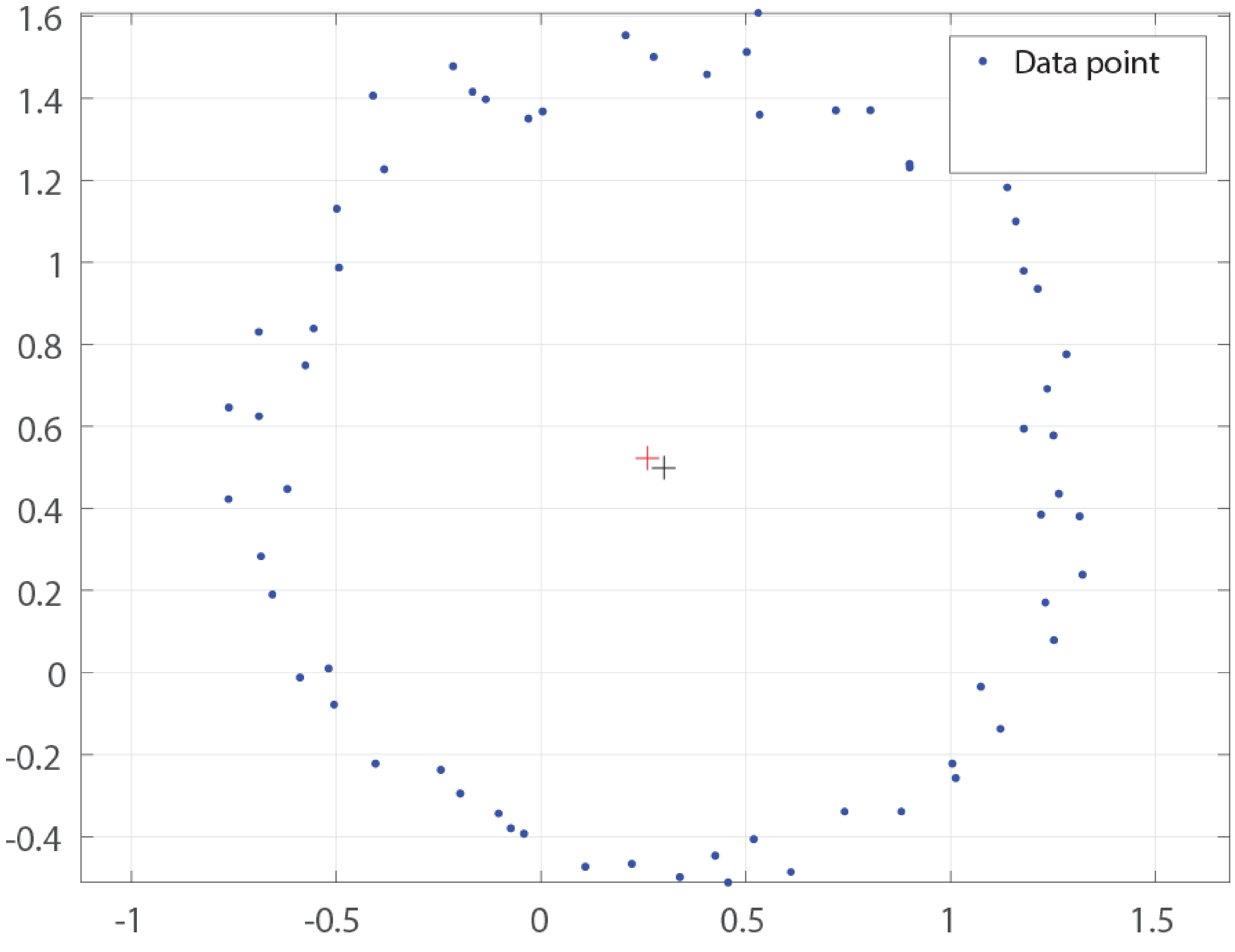
$$\downarrow$$

$$x_i^2 - x_1^2 - 2c_x\left(x_i - x_1\right) + y_i^2 - y_1^2 - 2\left(y_i - y_1\right)c_y = 0$$

$$\downarrow$$

$$\begin{bmatrix} 2\left(x_2 - x_1\right) & 2\left(y_2 - y_1\right) \\ \vdots & \vdots \\ 2\left(x_n - x_1\right) & 2\left(y_n - y_1\right) \end{bmatrix} \begin{bmatrix} c_x \\ c_y \end{bmatrix} = \begin{bmatrix} x_2^2 - x_1^2 + y_2^2 - y_1^2 \\ \\ x_n^2 - x_1^2 + y_n^2 - y_1^2 \end{bmatrix}$$

Circle Fitting (Ax=b)



$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

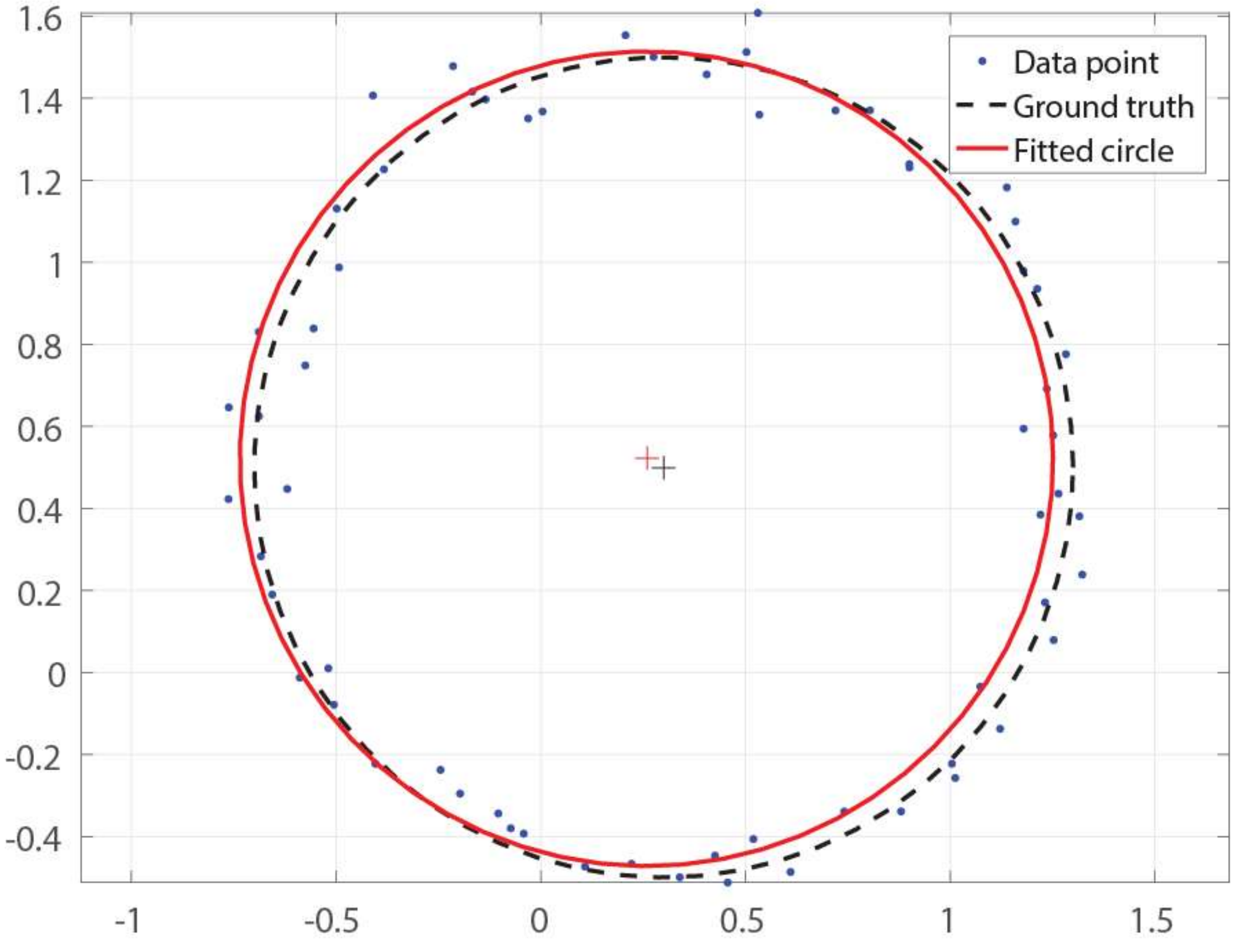
$$\downarrow$$

$$x_i^2 - x_1^2 - 2c_x(x_i - x_1) + y_i^2 - y_1^2 - 2(y_i - y_1)c_y = 0$$

$$\downarrow$$

$$\underbrace{\begin{bmatrix} 2(x_2 - x_1) & 2(y_2 - y_1) \\ \vdots & \vdots \\ 2(x_n - x_1) & 2(y_n - y_1) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_x \\ c_y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} x_2^2 - x_1^2 + y_2^2 - y_1^2 \\ \vdots \\ x_n^2 - x_1^2 + y_n^2 - y_1^2 \end{bmatrix}}_{b}$$

Circle Fitting (Ax=b)



$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

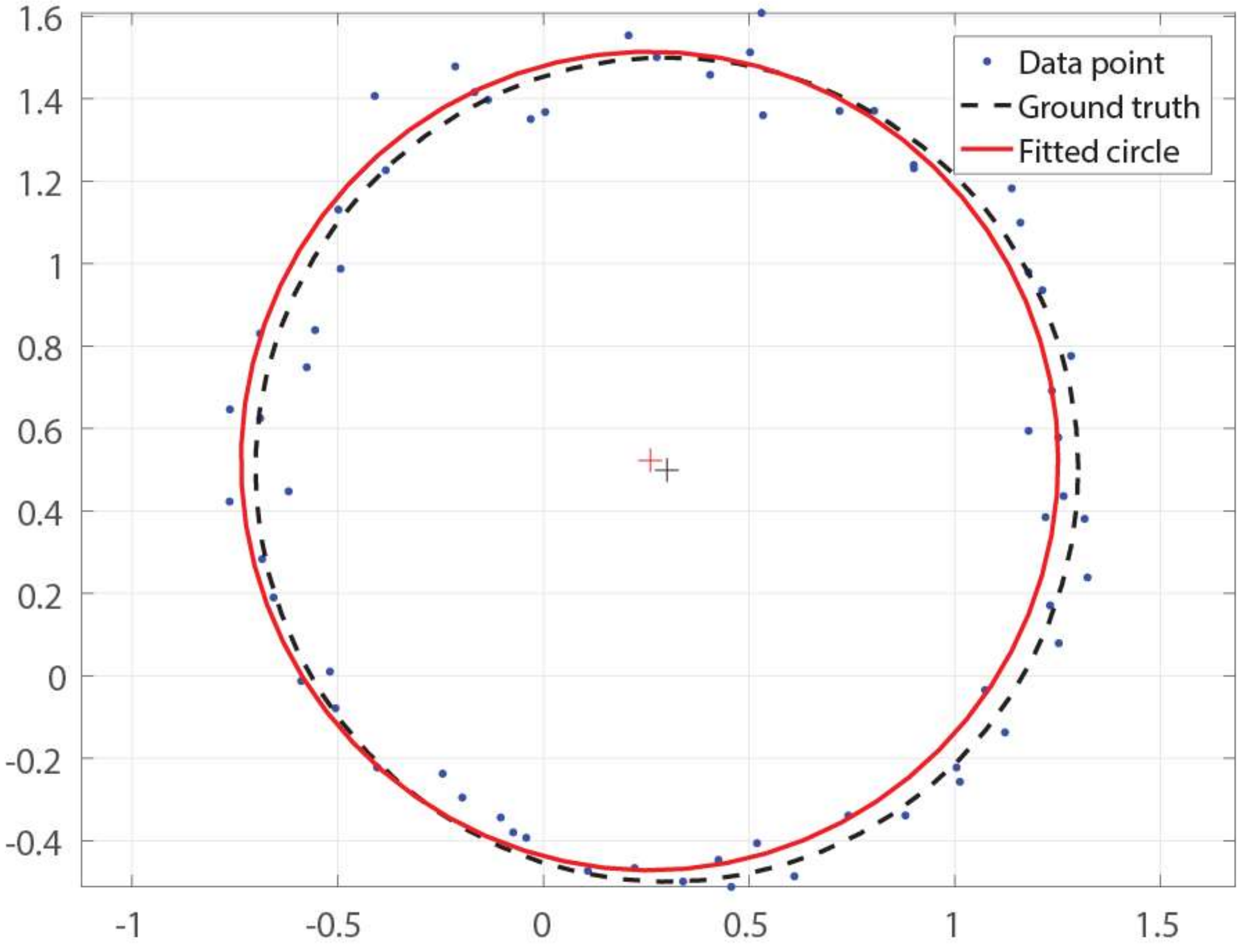
$$\downarrow$$

$$x_i^2 - x_1^2 - 2c_x (x_i - x_1) + y_i^2 - y_1^2 - 2(y_i - y_1) c_y = 0$$

$$\downarrow$$

$$\underbrace{\begin{bmatrix} 2(x_2 - x_1) & 2(y_2 - y_1) \\ \vdots & \vdots \\ 2(x_n - x_1) & 2(y_n - y_1) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_x \\ c_y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} x_2^2 - x_1^2 + y_2^2 - y_1^2 \\ \vdots \\ x_n^2 - x_1^2 + y_n^2 - y_1^2 \end{bmatrix}}_{b}$$

Circle Fitting (Ax=b)



CircleFitting.m

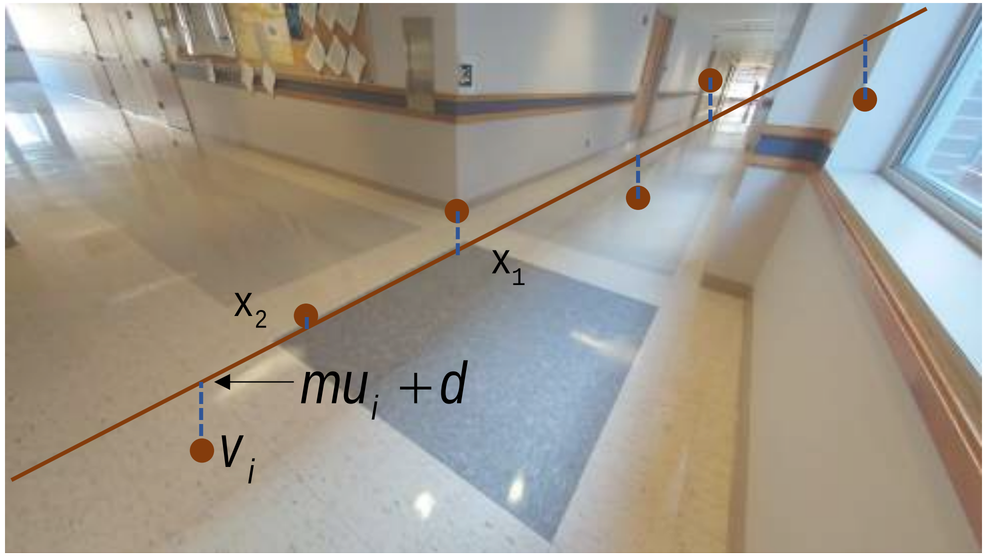
```
cx = 0.3;
cy = 0.5;
r = 1;

theta = 0:0.1:2*pi+0.1;
theta = theta';
x = cos(theta) + cx + 0.05*randn(size(theta));
y = sin(theta) + cy + 0.05*randn(size(theta));

A = [2*(x(2:end)-x(1)) 2*(y(2:end)-y(1))];
b = x(2:end).^2-x(1)^2 + y(2:end).^2-y(1)^2;
u = A\b;
dist = [x y] - ones(size(x)) * [u(1) u(2)];
dist = dist';
dist = mean(sqrt(sum(dist.^2)));
```

Line Fitting (Ax=b)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m},\underline{d})$

slope y-intercept



$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \longrightarrow \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

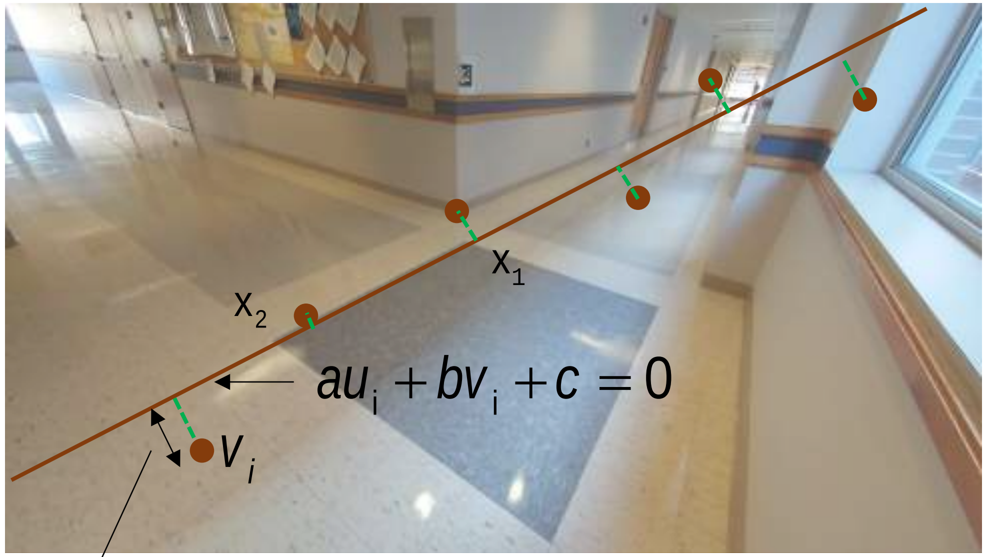
$$\mathrm{Ax} = \mathrm{b}$$

What is different?

Line Fitting (Ax=0)

Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: (a, b, c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

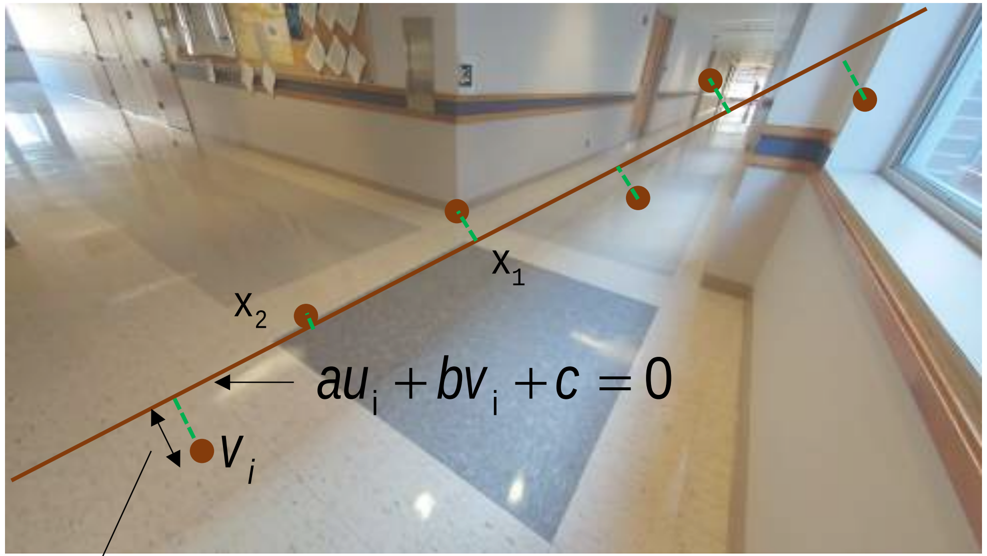
$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$\mathrm{Ax} = 0$ $\qquad$ $\mathrm{Ax} = \mathrm{b}$

Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$\mathrm{Ax} = 0$

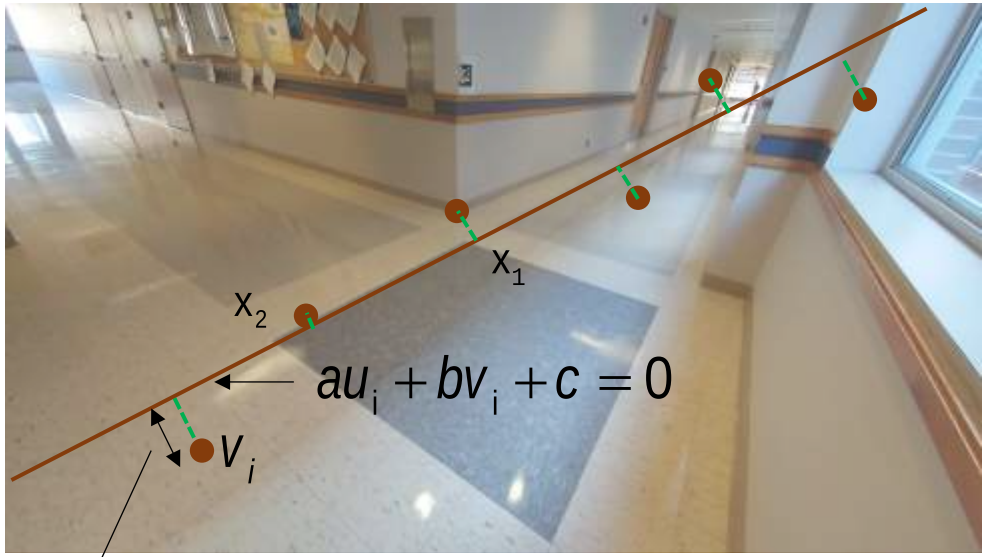
$\mathrm{Ax} = \mathrm{b}$

Trivial solution: $\mathrm{x} = 0$

Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$\mathrm{Ax} = 0$

$\mathrm{Ax} = \mathrm{b}$

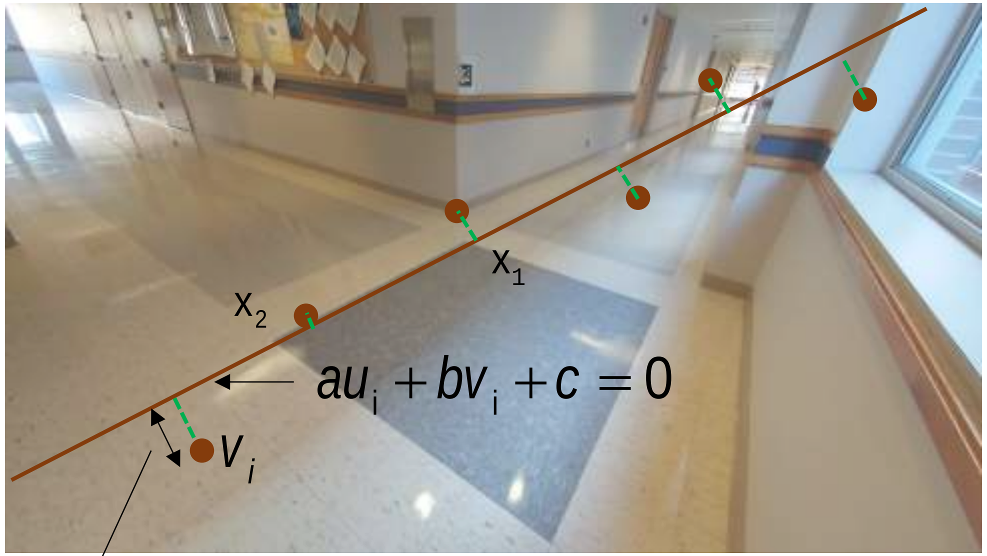
Trivial solution: $\mathrm{x} = 0$

$$\underset{\mathrm{x}}{\mathrm{minimize}} \|\mathrm{Ax}\|^2$$

Line Fitting (Ax=0)

Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: (a, b, c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$\mathrm{Ax} = 0$ $\qquad\qquad$ $\mathrm{Ax} = \mathrm{b}$

Trivial solution: $\mathrm{x} = 0$

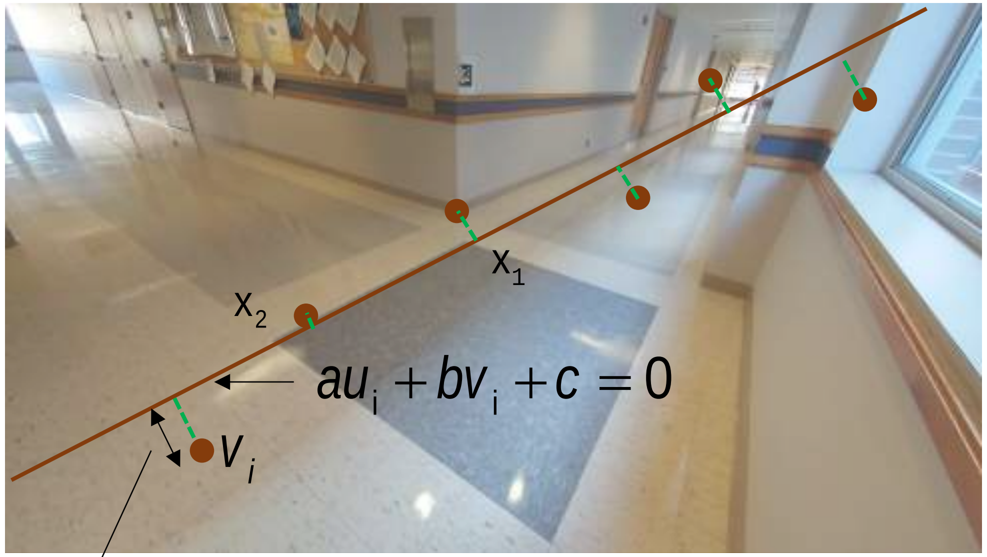
$\underset{\mathrm{x}}{\text{minimize}} \|\mathrm{Ax}\|^2$ $\quad$ subject to $\|\mathrm{x}\| = 1$

Condition to avoid the trivial solution

Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



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$\mathrm{Ax} = 0$ $\qquad$ $\mathrm{Ax} = \mathrm{b}$

Trivial solution: $\mathrm{x} = 0$

$\underset{\mathrm{x}}{\text{minimize}} \|\mathrm{Ax}\|^2$ subject to $\|\mathrm{x}\| = 1$

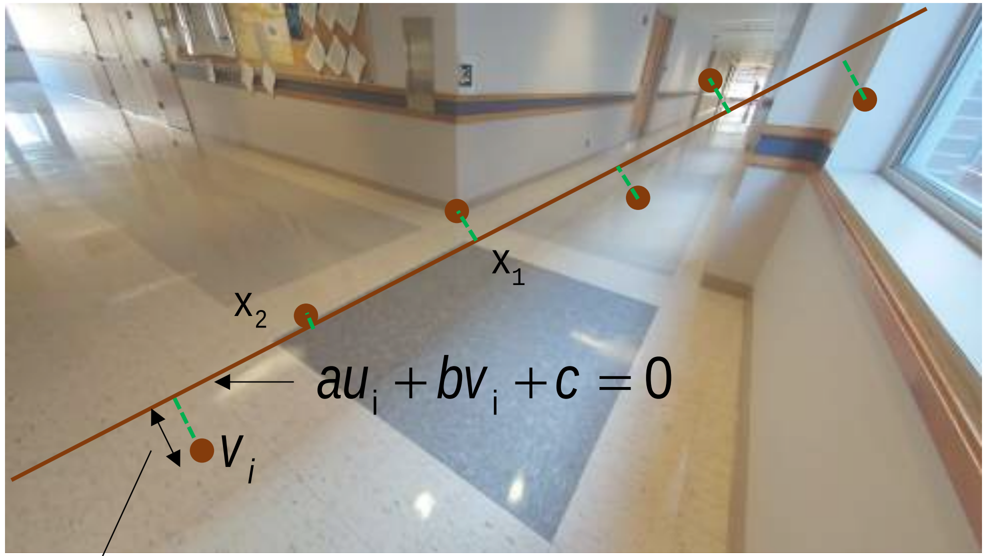
Condition to avoid the trivial solution

How to solve?

Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$\mathrm{Ax} = 0$ $\qquad$ $\mathrm{Ax} = \mathrm{b}$

Trivial solution: $\mathrm{x} = 0$

$$\underset{\mathrm{x}}{\text{minimize}} \|\mathrm{Ax}\|^2 \qquad \text{subject to} \quad \|\mathrm{x}\| = 1$$

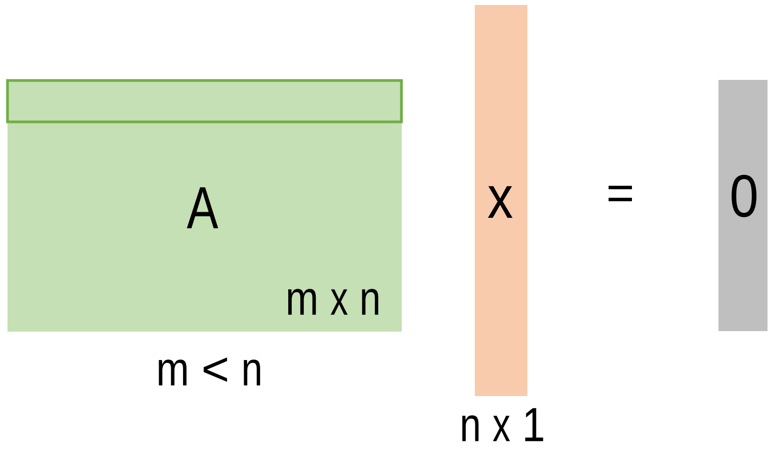
Condition to avoid the trivial solution

How to solve? approximated null space $\mathrm{x} = \mathrm{null}(\mathrm{A})$

Nullspace

eqs < # unknowns

There exist at least n-m vectors that makes Ax=0.

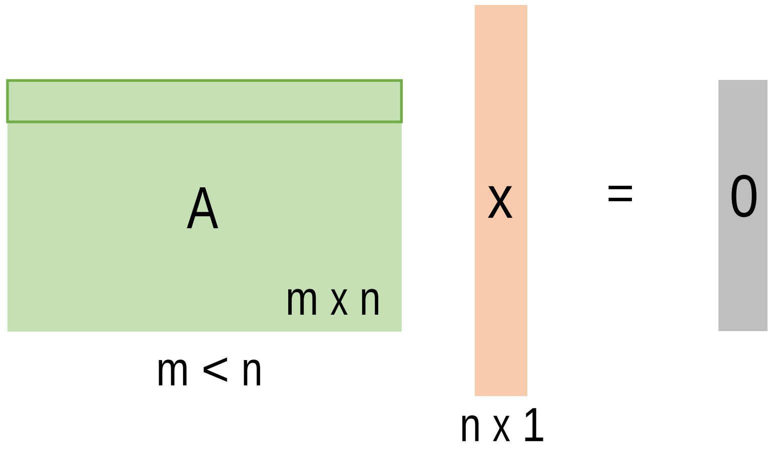


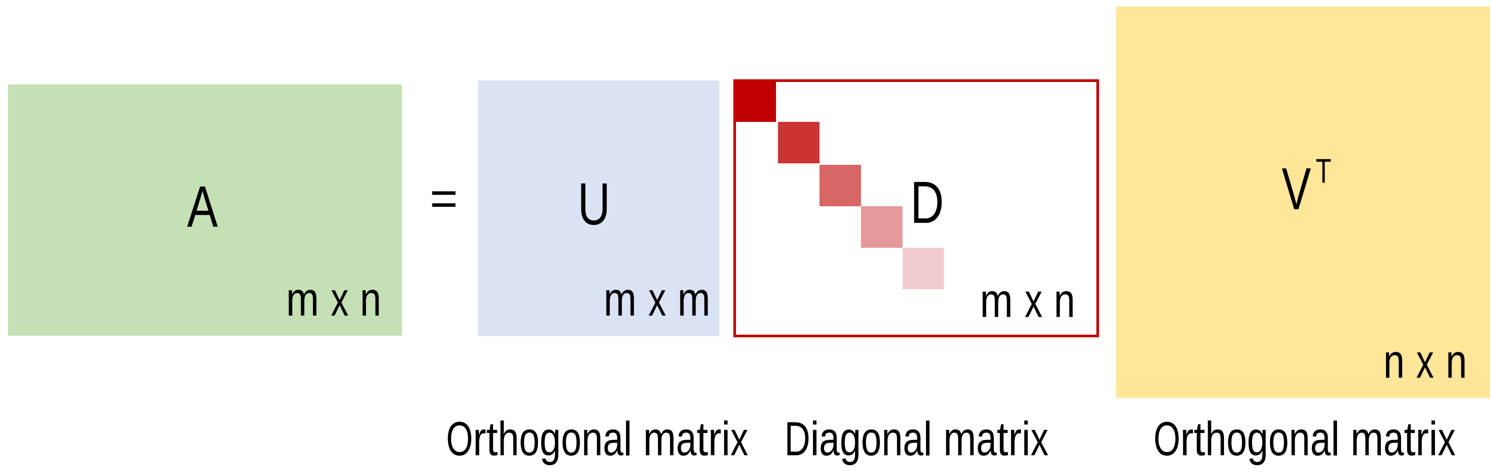
$\longrightarrow \quad x = \mathrm{null}(A)$

Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least n-m vectors that makes Ax=0.



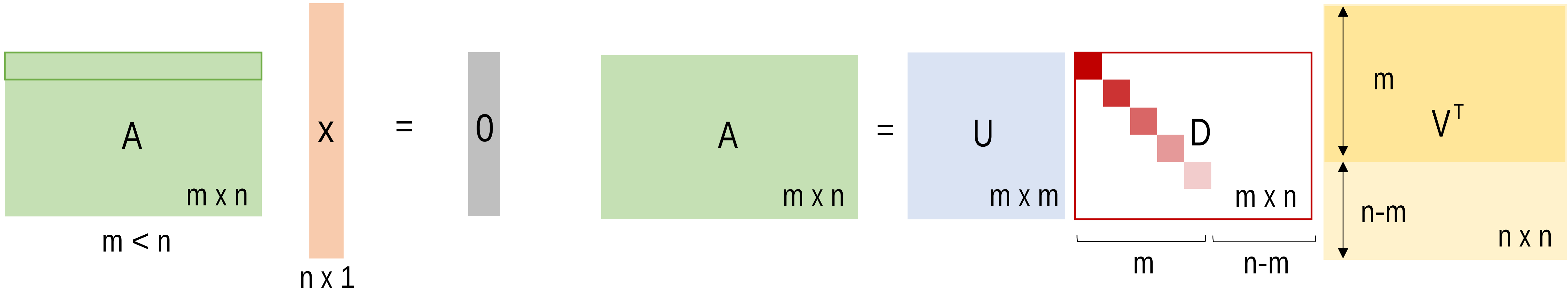


$\longrightarrow x = \mathrm{null}(A)$

Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least n-m vectors that makes Ax=0.

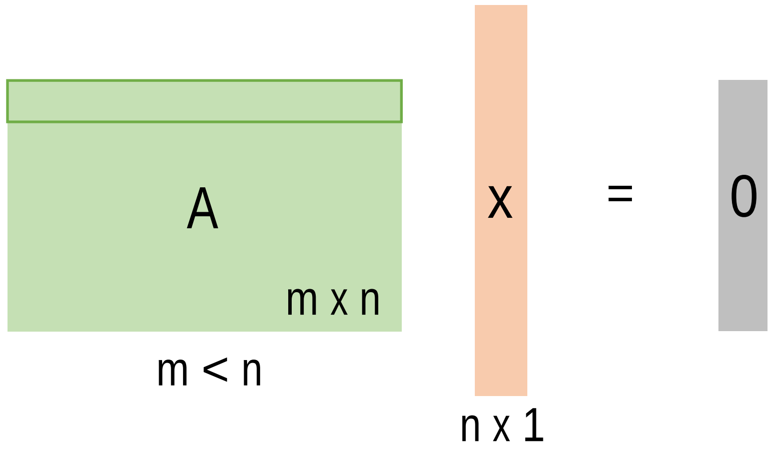


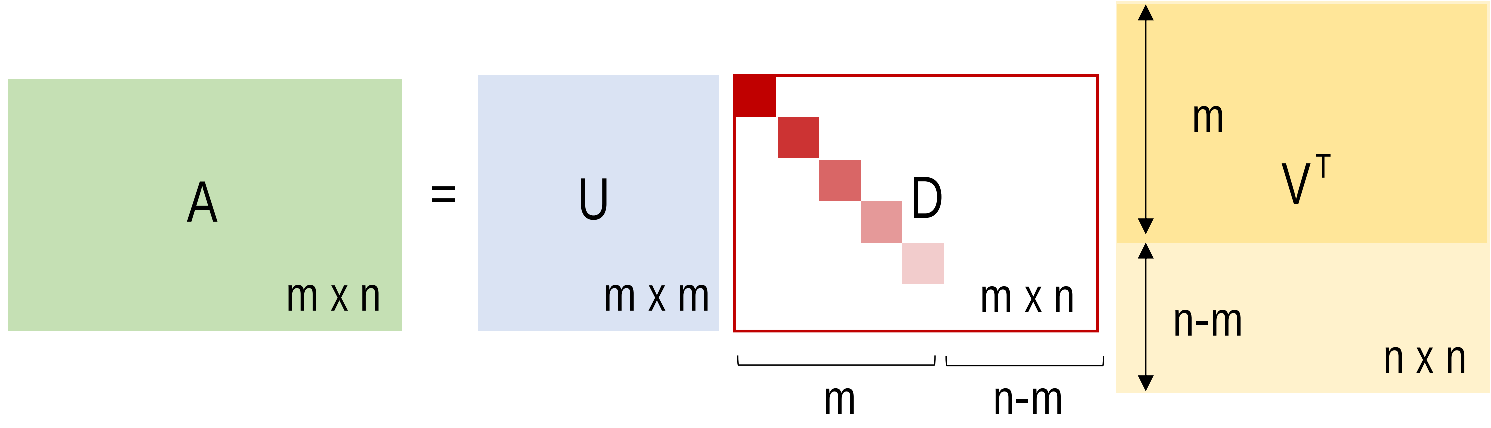
$\longrightarrow x = \text{null}(A)$

Singular Value Decomposition (SVD)

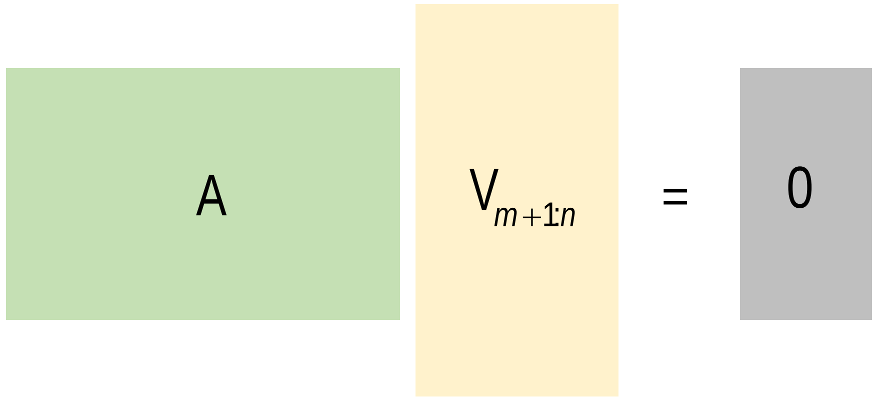
\# eqs < # unknowns

There exist at least n-m vectors that makes Ax=0.





$\longrightarrow x = \text{null}(A)$

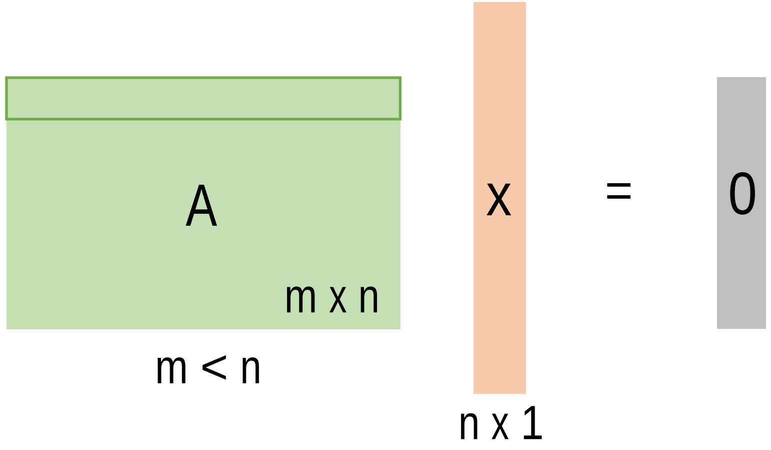


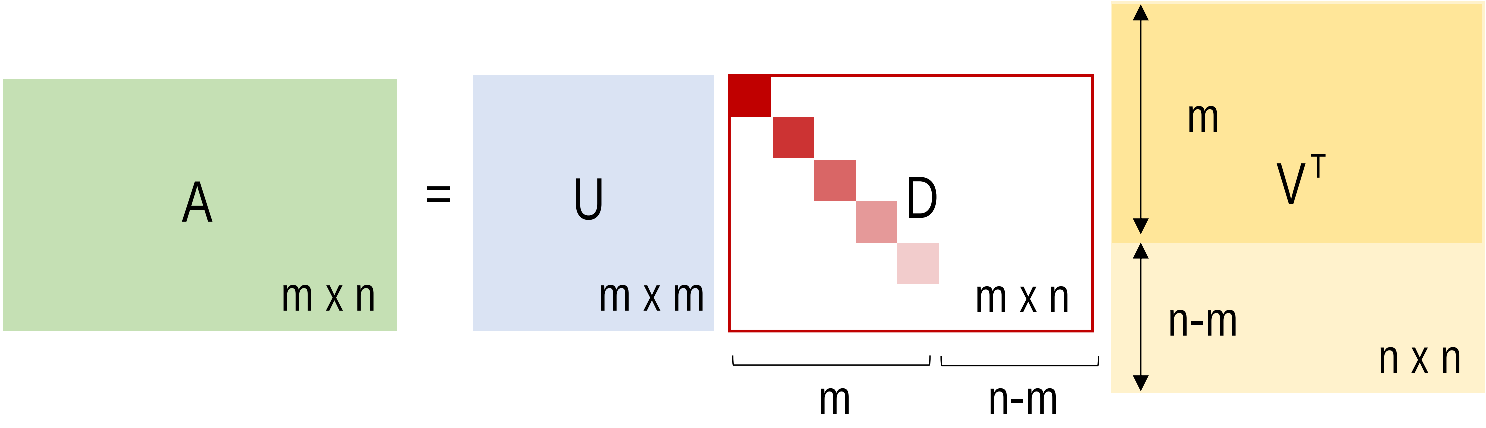
$$V_{m+1:n} = \text{null}(A)$$

Singular Value Decomposition (SVD)

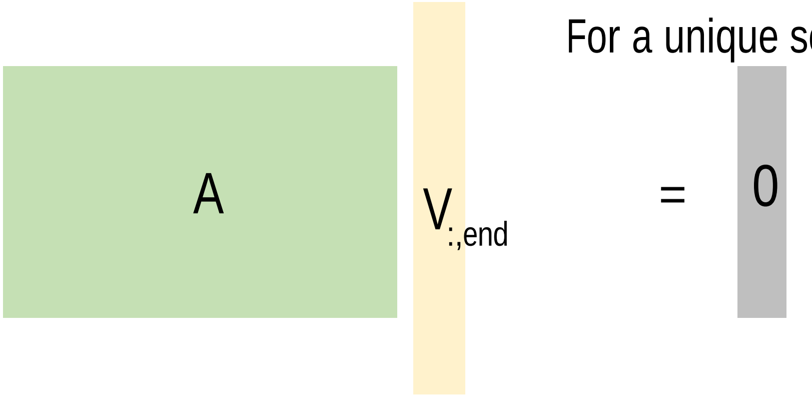
eqs < # unknowns

There exist at least n-m vectors that makes Ax=0.





$\longrightarrow x = \text{null}(A)$



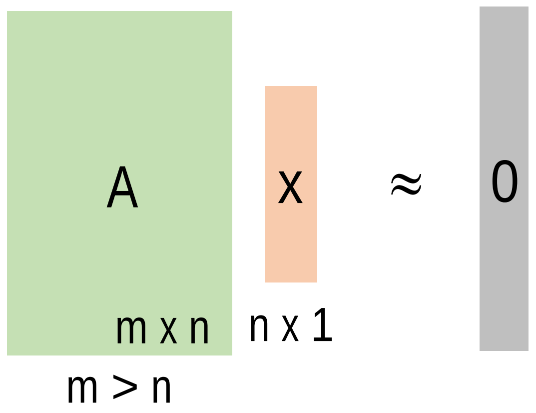
For a unique solution, A should be m x (m+1)

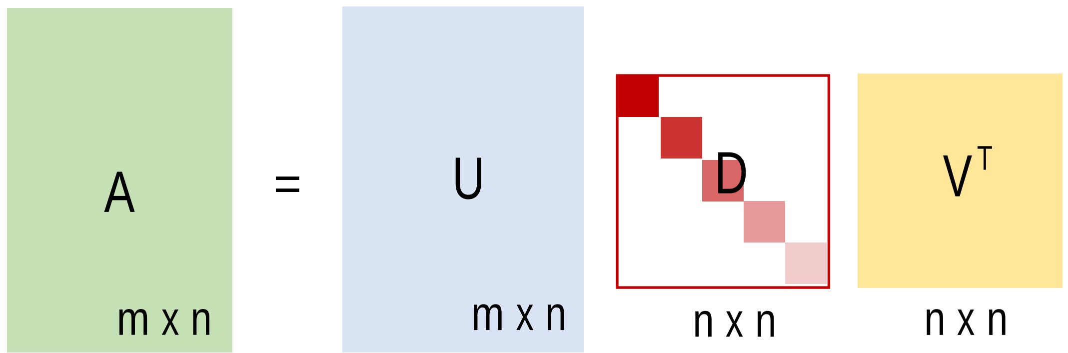
$V_{:end} = \text{null}(A)$

Singular Value Decomposition (SVD)

eqs > # unknowns

There exist no nullspace of A.



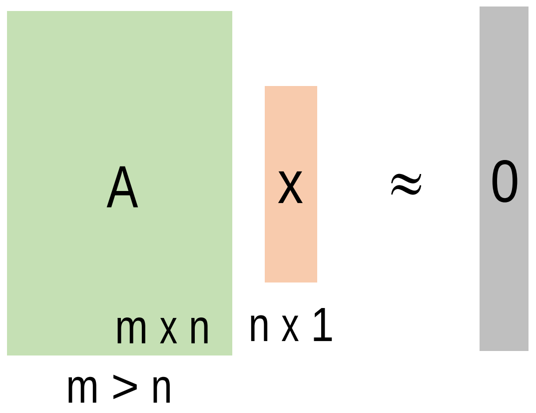


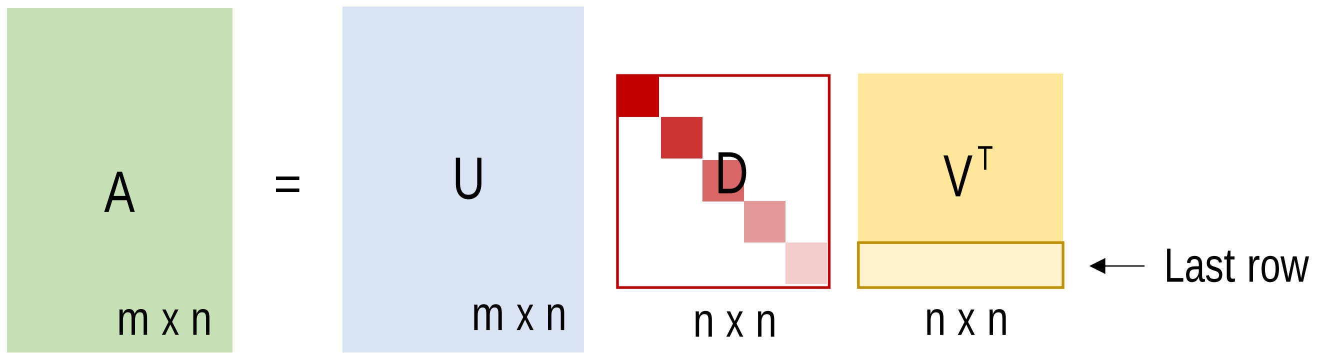
$$\underset{x}{\text{minimize}} \|Ax\|^2 \text{ subject to } \|x\| = 1$$

Singular Value Decomposition (SVD)

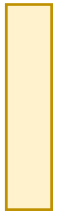
eqs > # unknowns

There exist no nullspace of A.

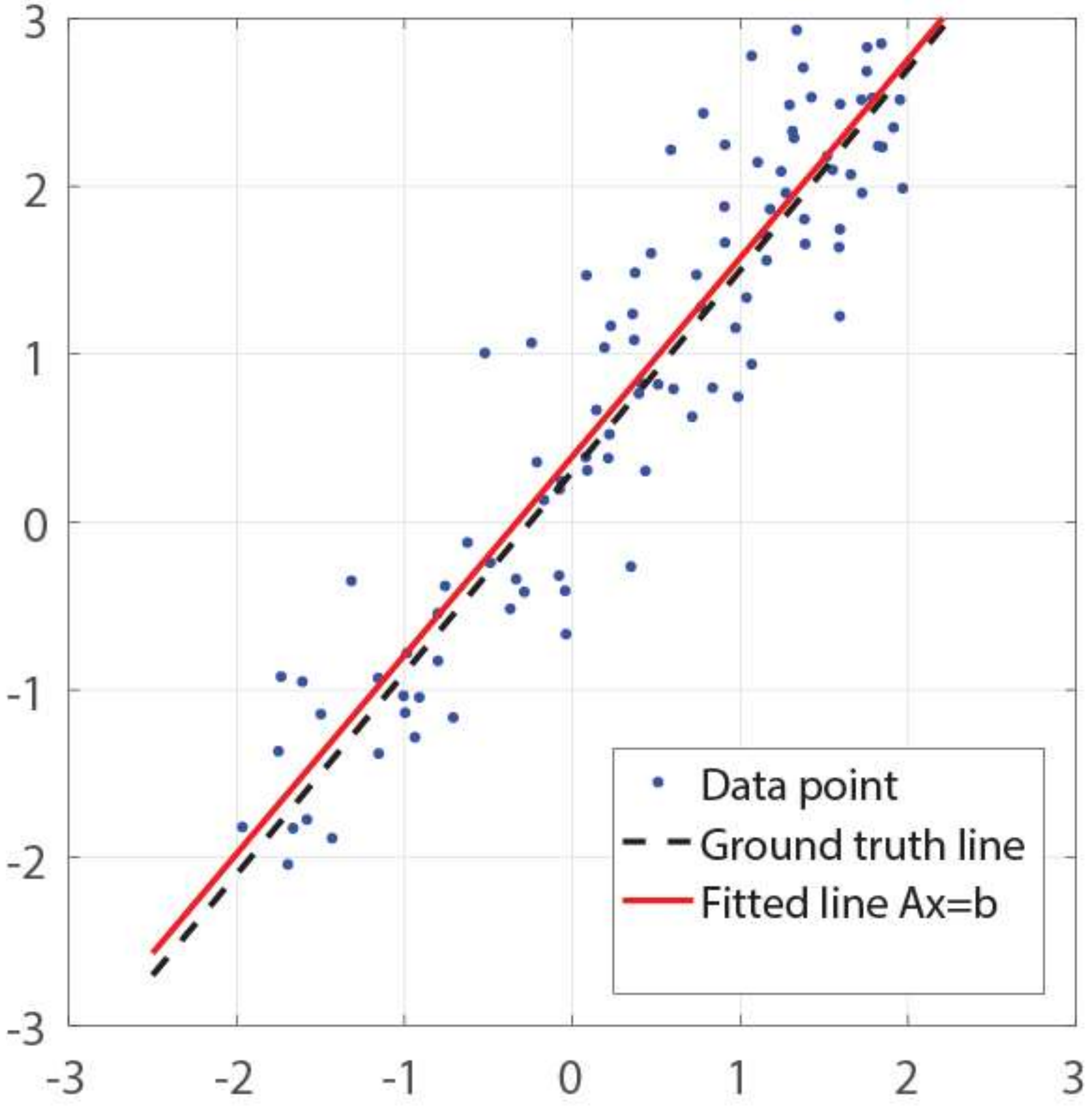




$$\underset{x}{\text{minimize}} \|Ax\|^2 \text{ subject to } \|x\| = 1$$

Approximated nullspace of A:  $V_{:,end}$

Line Fitting (Ax=b)



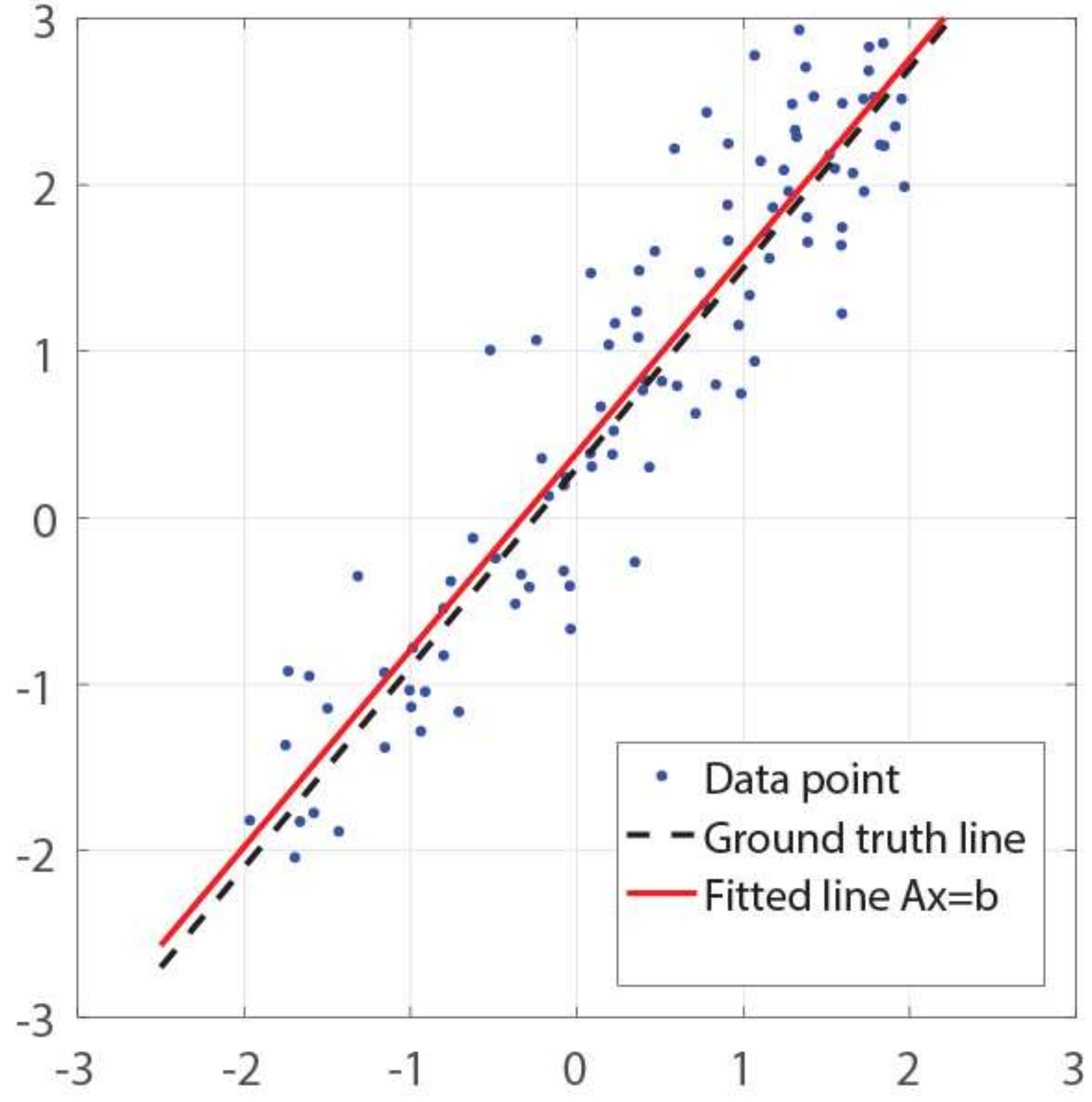
Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (m,d)

$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$\vdots$$
$$v_n \approx mu_n + d$$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$$

Line Fitting (Ax=b)

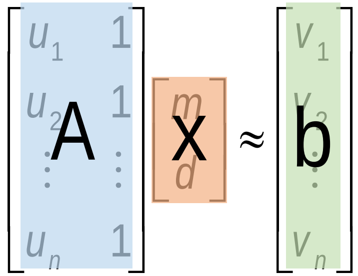


Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

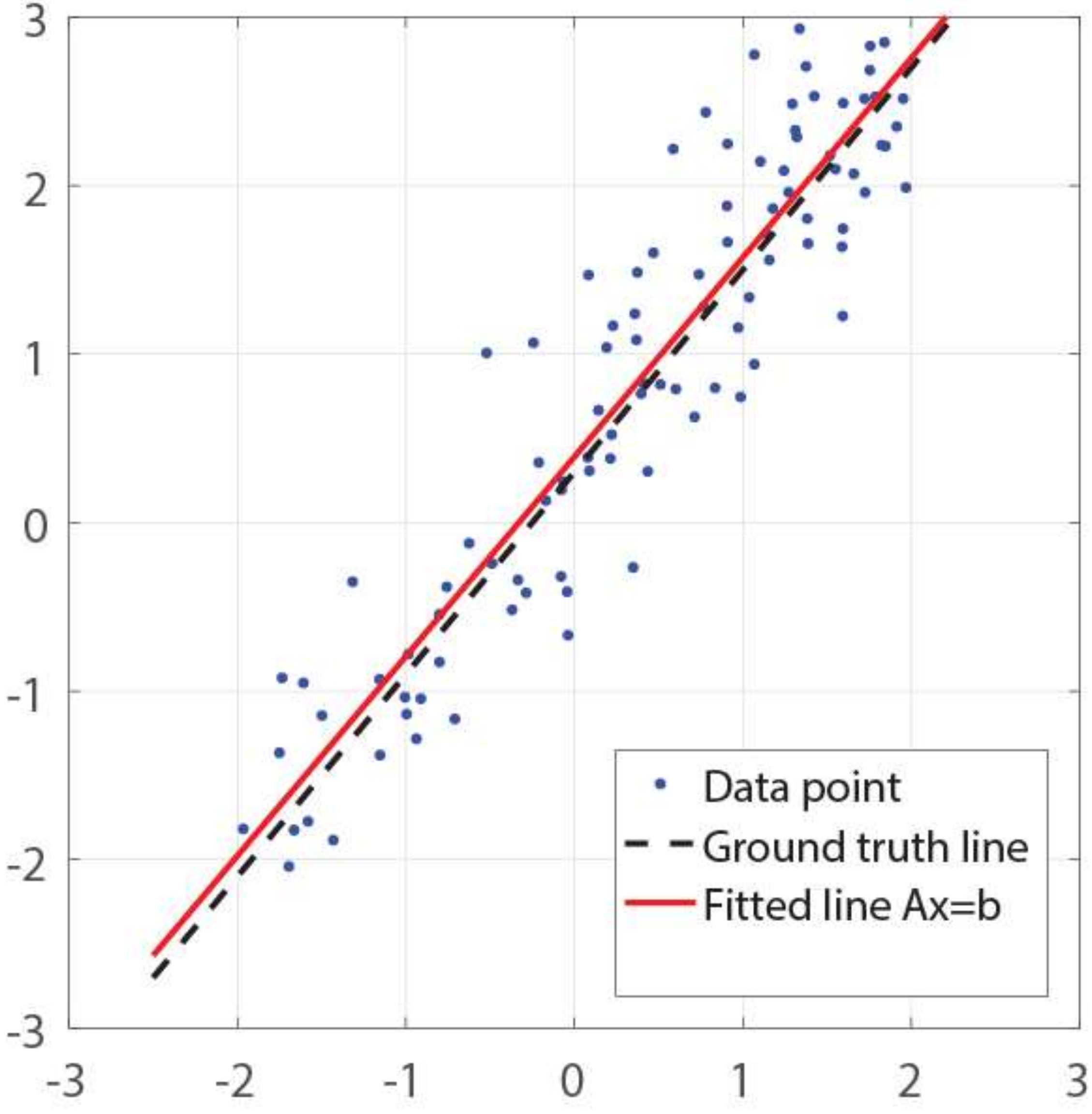
Find the best line: (a,b,c) (m,d)

$$au_1 + bv_1 + c \approx 0$$
$$au_2 + bv_2 + c \approx 0$$
$$\vdots$$
$$au_n + bv_n + c \approx 0$$

$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$\vdots$$
$$v_n \approx mu_n + d$$



Line Fitting (Ax=b)



Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: (a, b, c) (m, d)

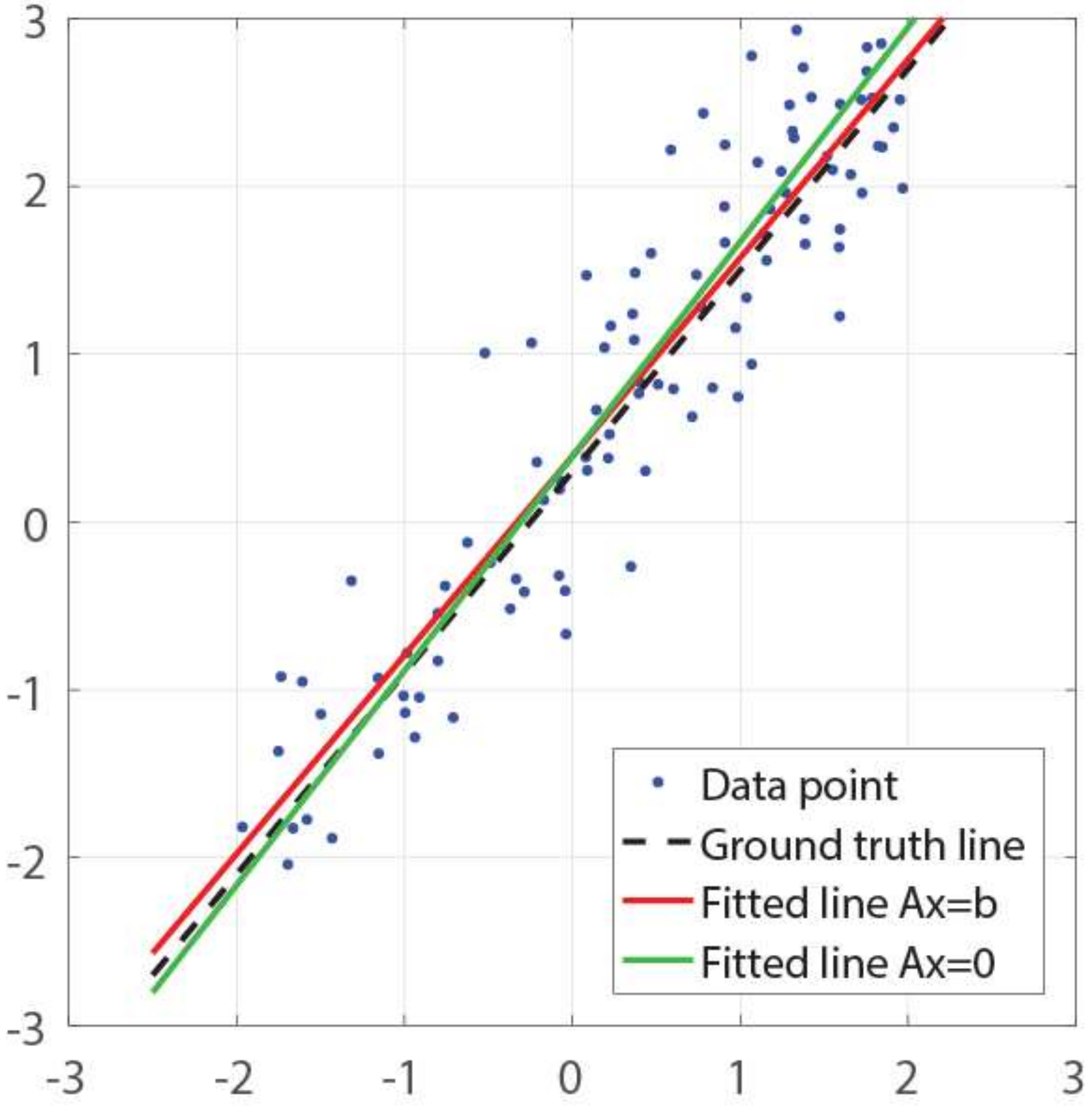
$$au_1 + bv_1 + c \approx 0$$
$$au_2 + bv_2 + c \approx 0$$
$$\vdots$$
$$au_n + bv_n + c \approx 0$$

$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$\vdots$$
$$v_n \approx mu_n + d$$

$$\begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ \vdots & \vdots & 1 \\ u_n & v_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$$

Line Fitting (Ax=b)



Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: (a, b, c)

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$$\underbrace{\begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ \vdots & \vdots & 1 \\ u_n & v_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{0} \qquad \underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$$

How to compute homography?

Homography Computation

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Homography Computation

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$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\longrightarrow \begin{aligned} h_{11}u_x + h_{12}u_y + h_{13} + h_{31}u_x v_x + h_{32}u_y v_x + h_{33}v_x &= 0 \\ h_{21}u_x + h_{22}u_y + h_{23} + h_{31}u_x v_y + h_{32}u_y v_y + h_{33}v_y &= 0 \end{aligned}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Unknowns: $h_{11}, \cdots, h_{33}$

Equations: 2 per correspondence

Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\longrightarrow \begin{aligned} h_{11}u_x + h_{12}u_y + h_{13} - h_{31}u_x v_x - h_{32}u_y v_x - h_{33}v_x &= 0 \\ h_{21}u_x + h_{22}u_y + h_{23} - h_{31}u_x v_y - h_{32}u_y v_y - h_{33}v_y &= 0 \end{aligned}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\rightarrow \begin{array}{l} h_{11}u_x + h_{12}u_y + h_{13} - h_{31}u_x v_x - h_{32}u_y v_x - h_{33}v_x = 0 \\ h_{21}u_x + h_{22}u_y + h_{23} - h_{31}u_x v_y - h_{32}u_y v_y - h_{33}v_y = 0 \end{array}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

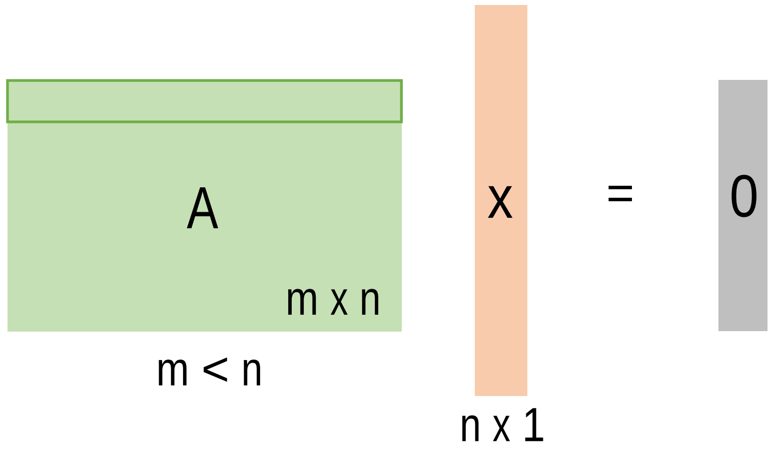
$$\mathbf{A}\,\mathbf{x} = \mathbf{0}$$

2x9

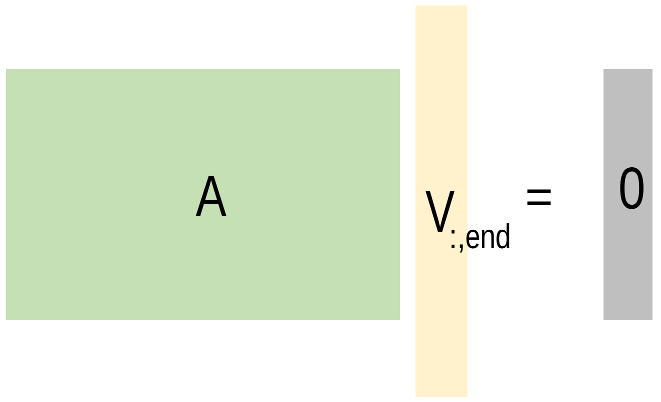
Recall: Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least n-m vectors that makes Ax=0.



$$\longrightarrow \; x = \text{null}(A)$$



$$V_{:end} = \text{null}(A)$$

For a unique solution, A should be m x (m+1)

Homography Computation

How many correspondences are needed?

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

2x9

Homography Computation

How many correspondences are needed? 4

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

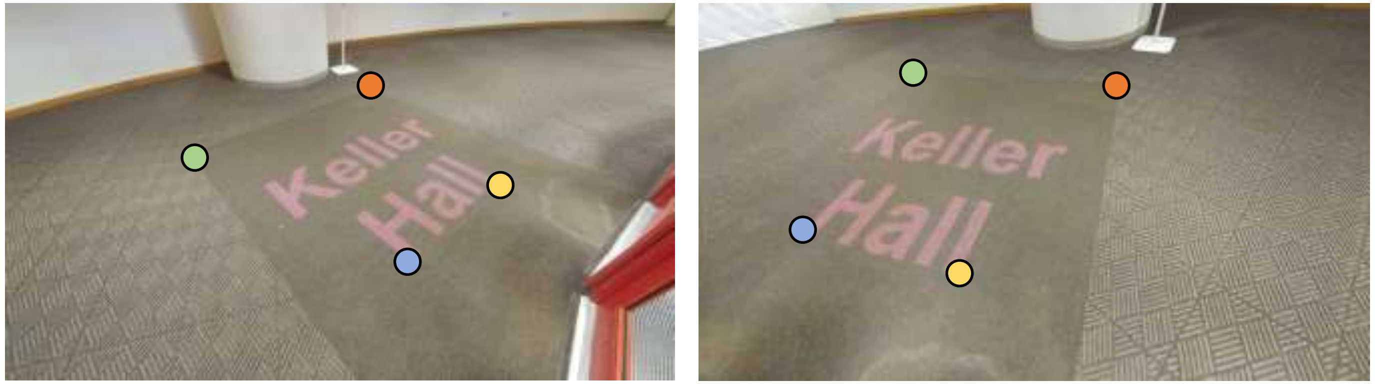
$\mathrm{A}\,\mathrm{x} = 0$

8x9

$$\mathrm{x} = \mathrm{V}_{:end} = \mathrm{null}(\mathrm{A})$$

Homography Computation

How many correspondences are needed? 4



$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

ComputeHomography.m

```
function H = ComputeHomography(u, X)

A = [];
for i = 1 : size(u,1)
   A = [A; X(i,:) zeros(1,3) -u(i,1)*X(i,:)];
   A = [A; zeros(1,3) X(i,:) -u(i,2)*X(i,:)];
end
```

Constructing A

Homography Computation

How many correspondences are needed? 4

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

ComputeHomography.m

```
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   A = [A; X(i,:) zeros(1,3) -u(i,1)*X(i,:)];
   A = [A; zeros(1,3) X(i,:) -u(i,2)*X(i,:)];
end

[u, d, v] = svd(A);
h = v(:,end);
H = [h(1:3)'; h(4:6)'; h(7:9)'];
H = H/norm(H);
```

Constructing A

Solving Ax=0

KellerEntranceHomography.m

```
im1 = imread('keller_left.png');
im2 = imread('keller_right.png');

im_warped = zeros(2000,4000,3);

u1 = [2806 1004;     2456 753;
      1677 1234;     2325 1474];

u2 = [1483 1541;     1948 997;
      860 843;     587 1316];

u1 = [u1 ones(4,1)];
u2 = [u2 ones(4,1)];

H1 = ComputeHomography(u2, u1);
H2 = ComputeHomography(u1, u2);

im_warped1 = ImageWarping(im1, H1);
im_warped2 = ImageWarping(im2, H2);

im_1 = 0.5*im_warped1 + 0.5*im2;
im_2 = 0.5*im_warped2 + 0.5*im1;
```