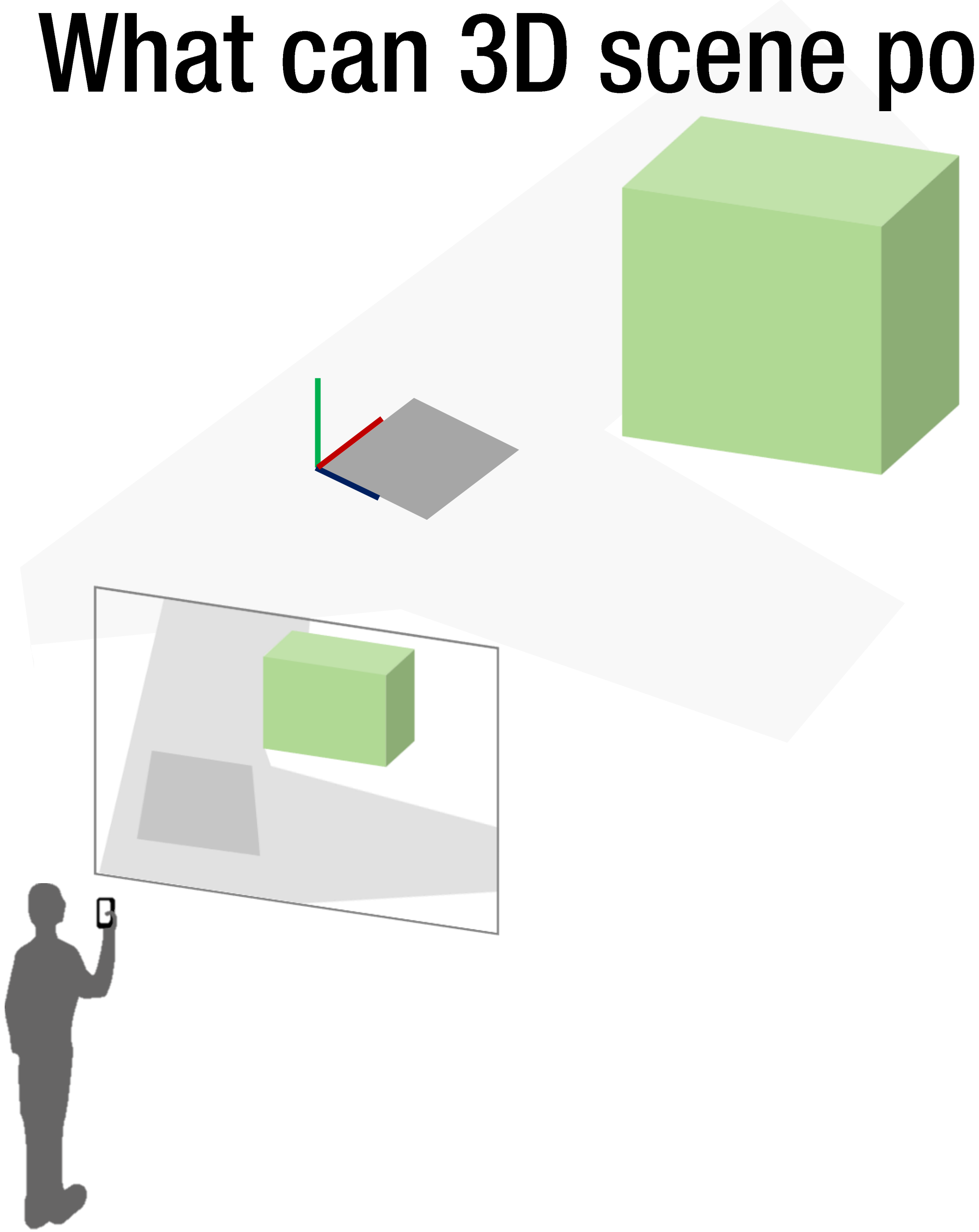
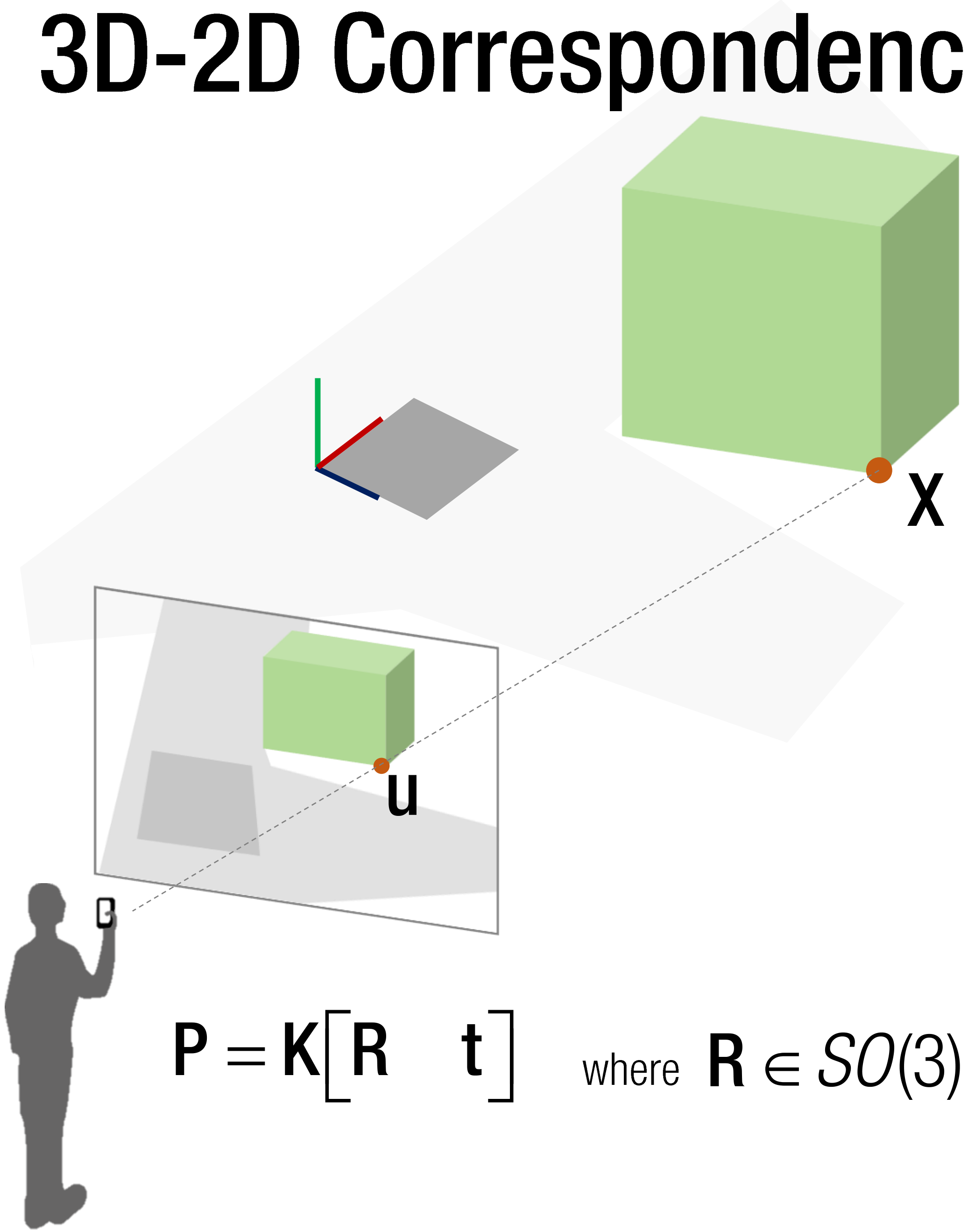
# Where am I?
# Perspective-n-Point

# What can 3D scene points tell us about?



http://www.wired.com/gadgetlab/2010/07/camera-software-lets-you-see-into-the-past/

# 3D-2D Correspondence




$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

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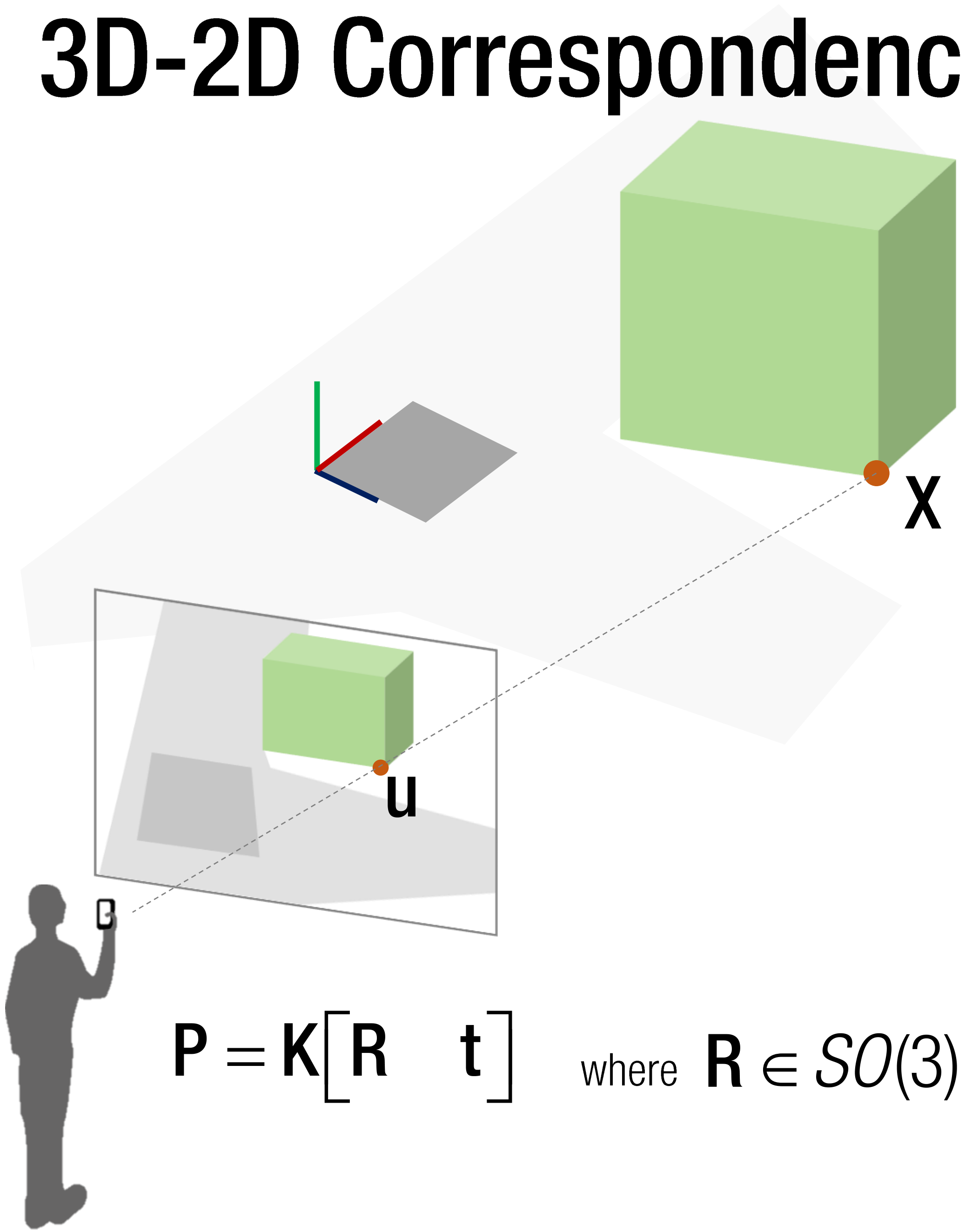


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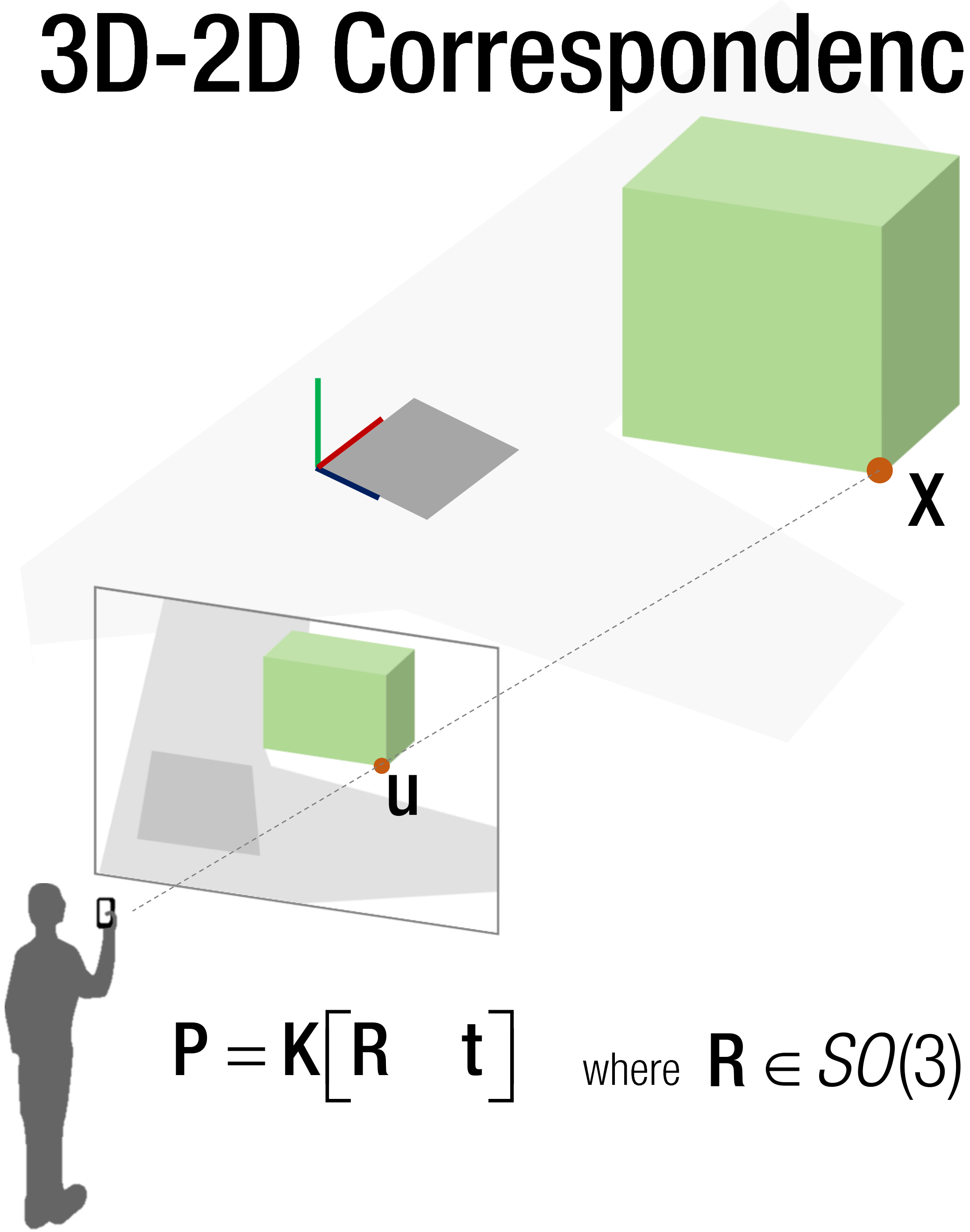


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Linear in camera matrix

$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

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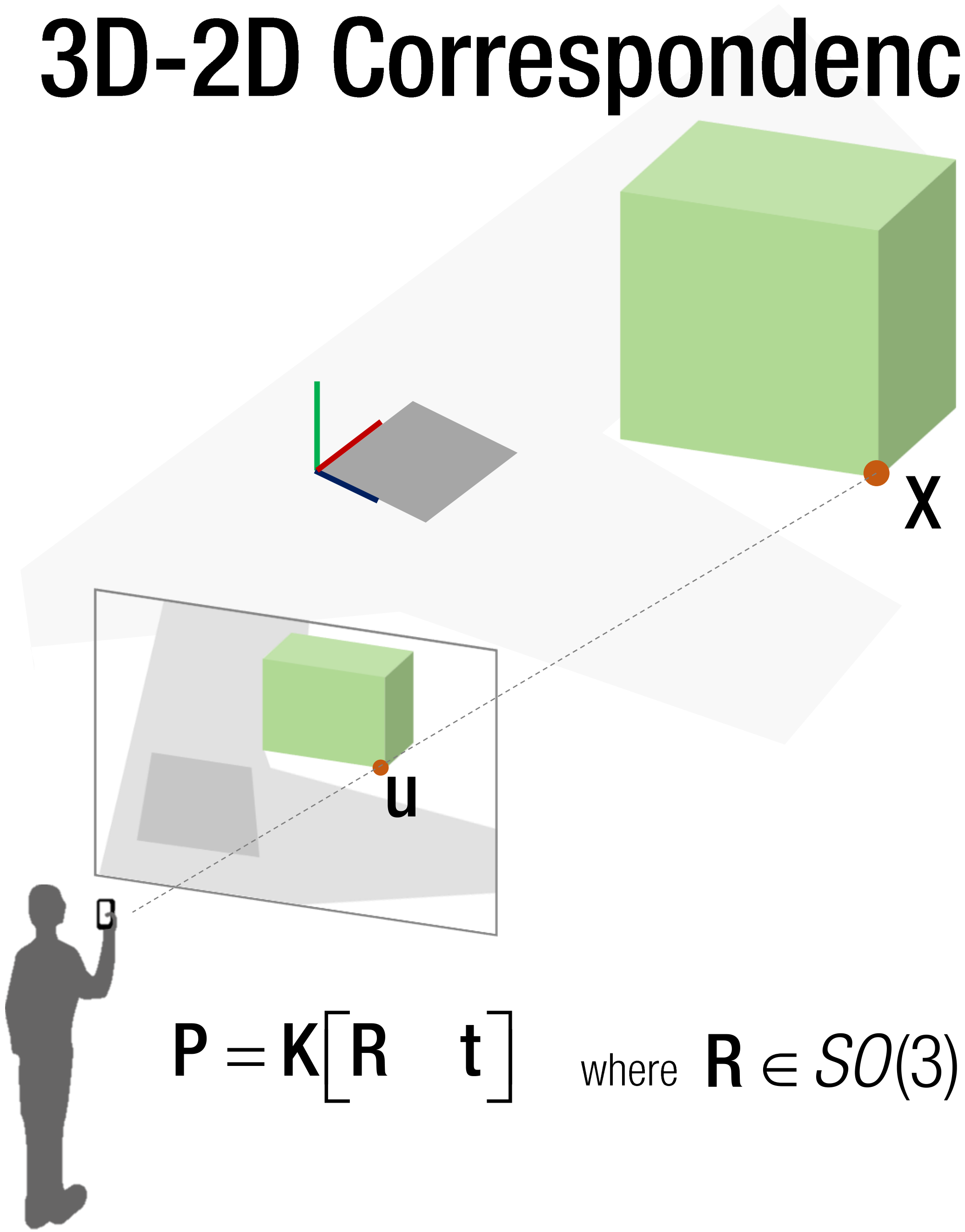


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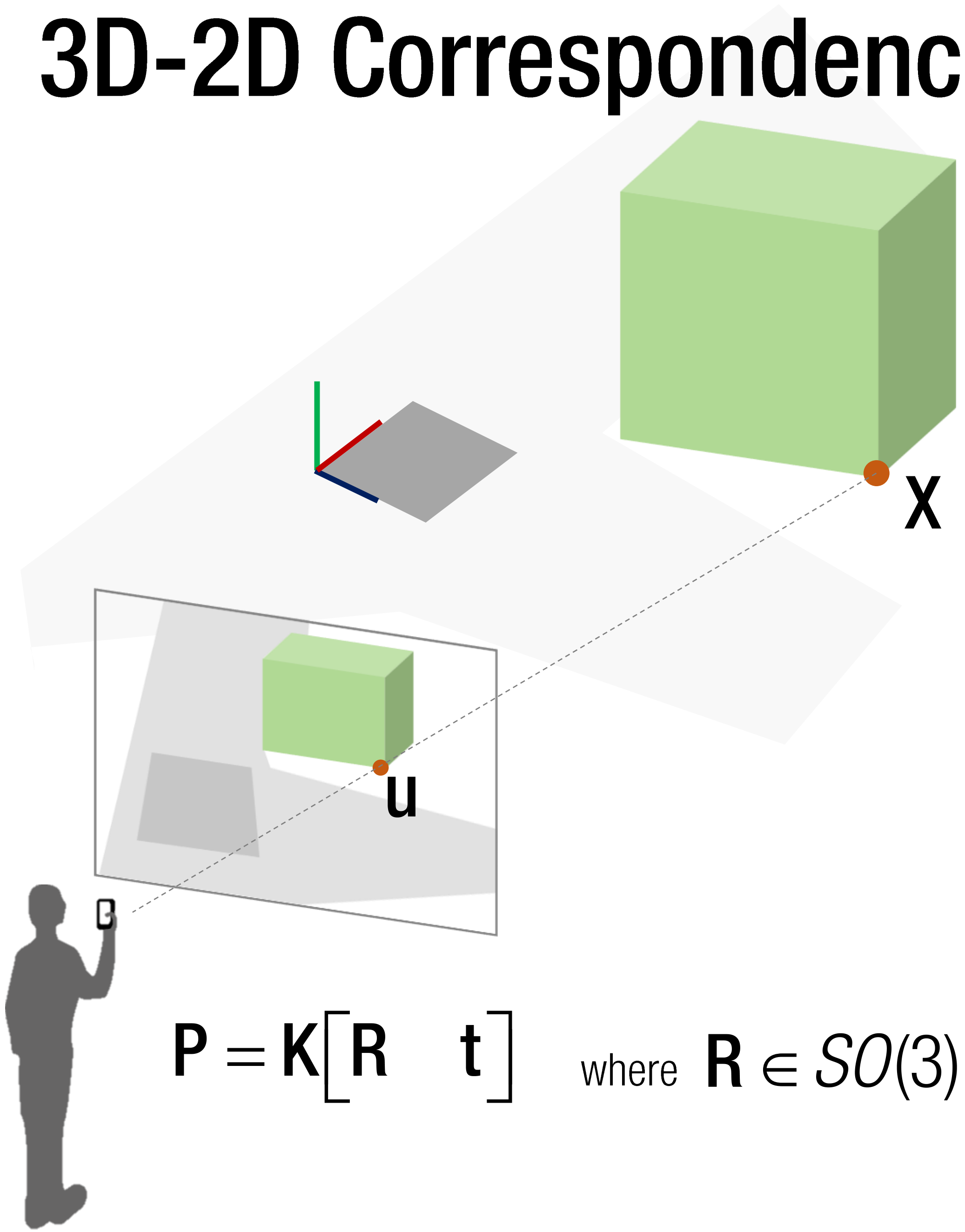
# of unknowns: 11 = 12 (3x4 matrix) – 1 (scale)
# of equations per correspondence: 2

# 3D-2D Correspondence

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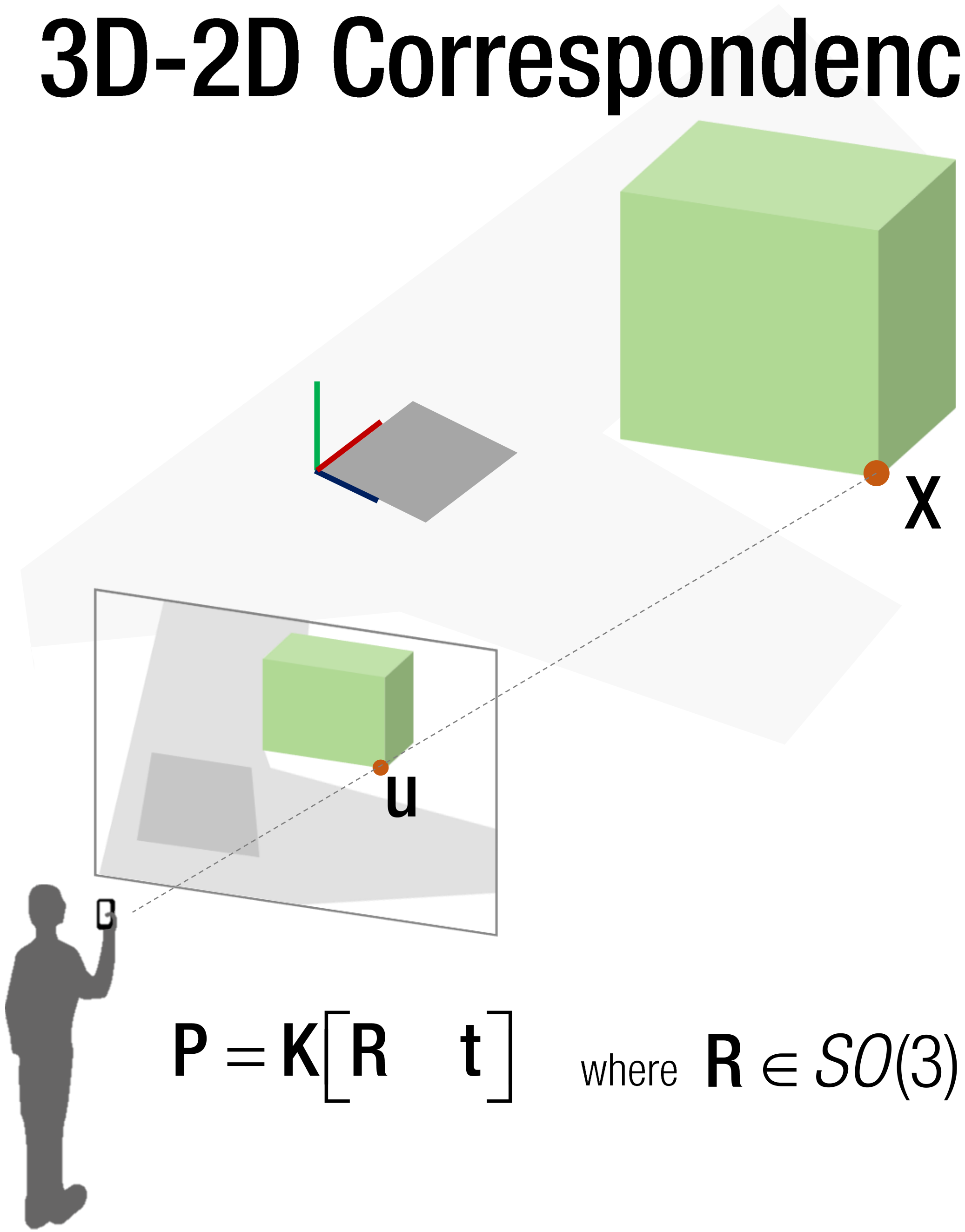


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$$\underbrace{\begin{bmatrix} X & Y & Z & 1 & & & & & -u^xX & -u^xY & -u^xZ & -u^x \\ & & & & X & Y & Z & 1 & -u^yX & -u^yY & -u^yZ & -u^y \end{bmatrix}}_{2\text{x}12} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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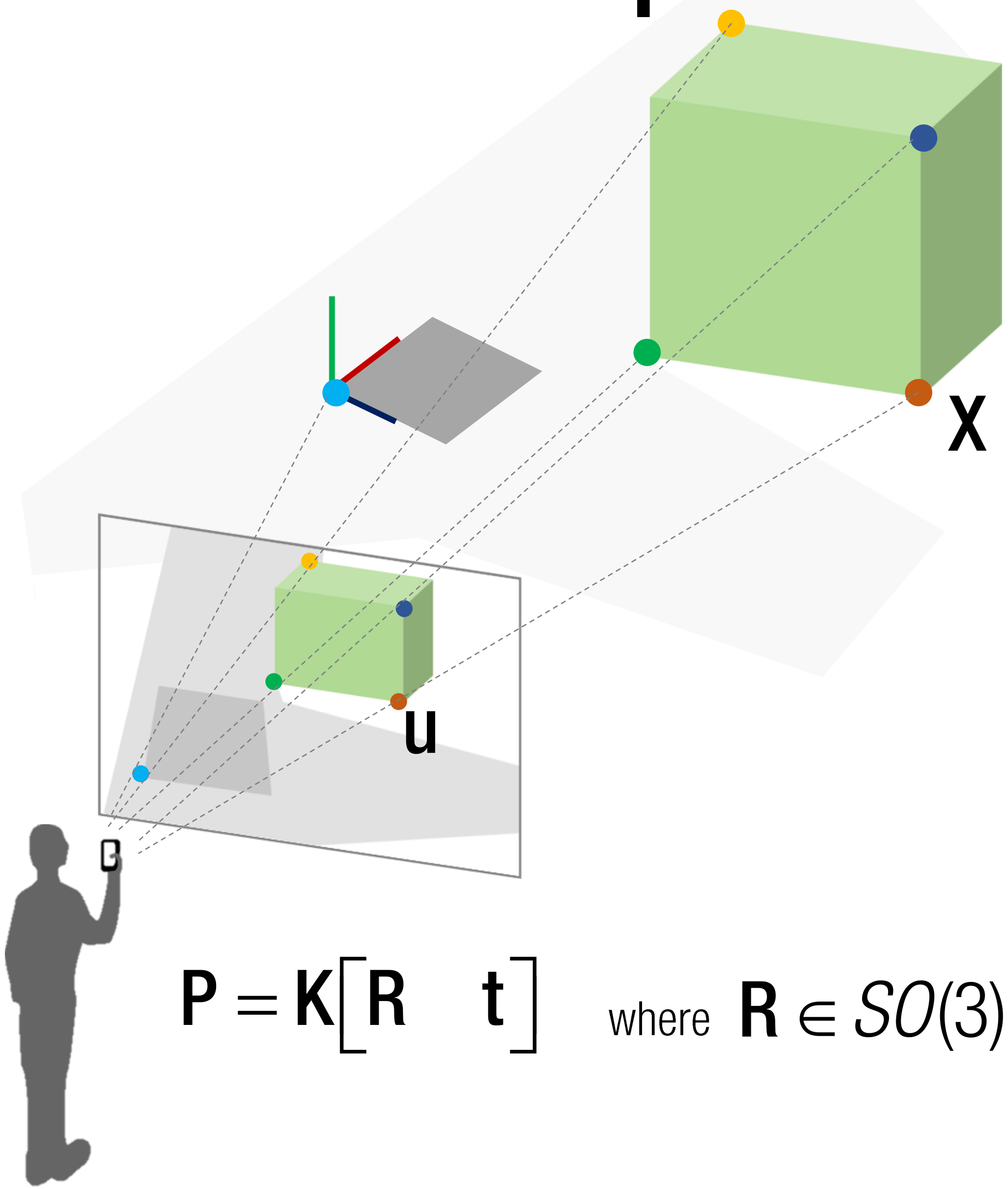


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$2m$x12

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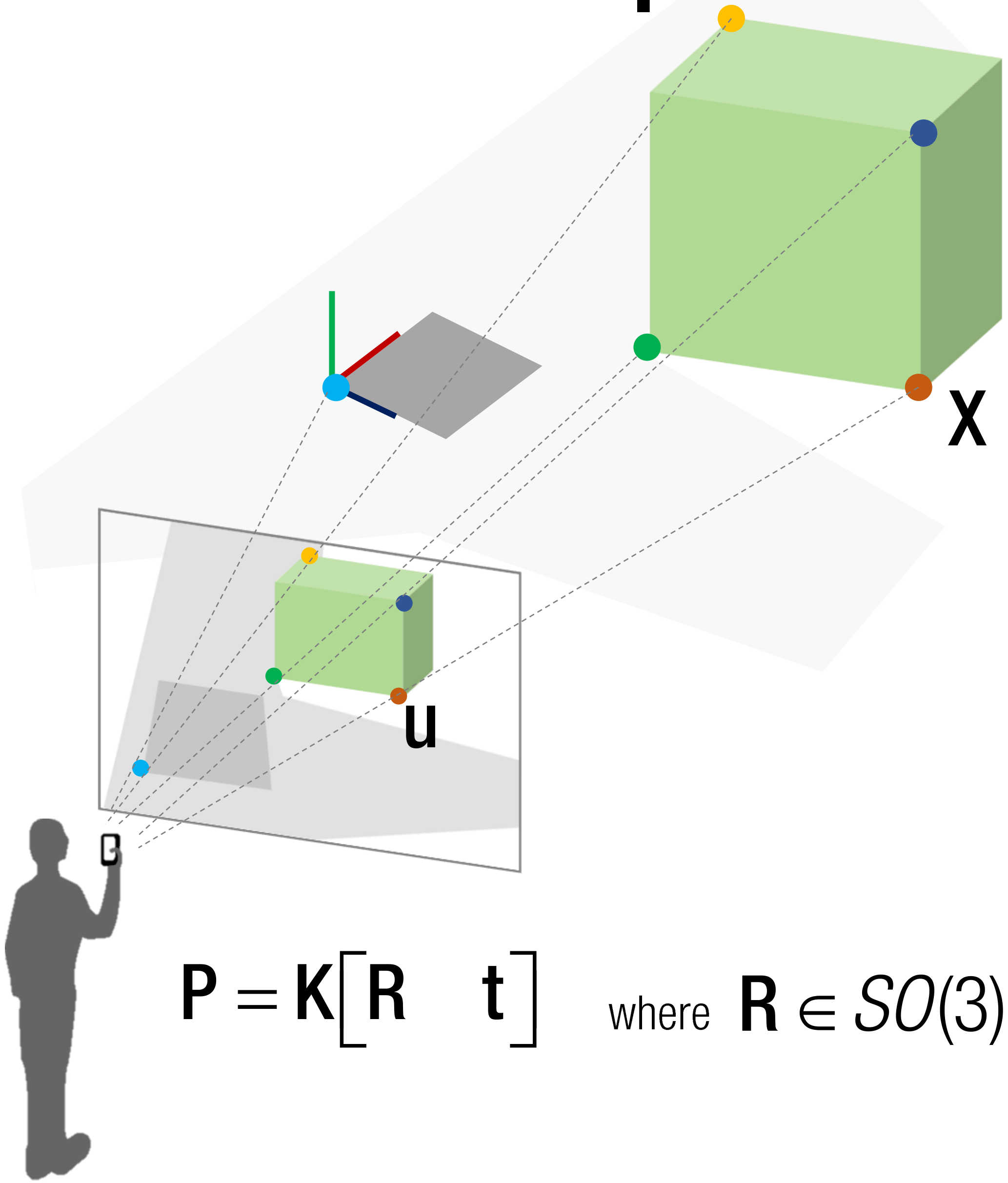


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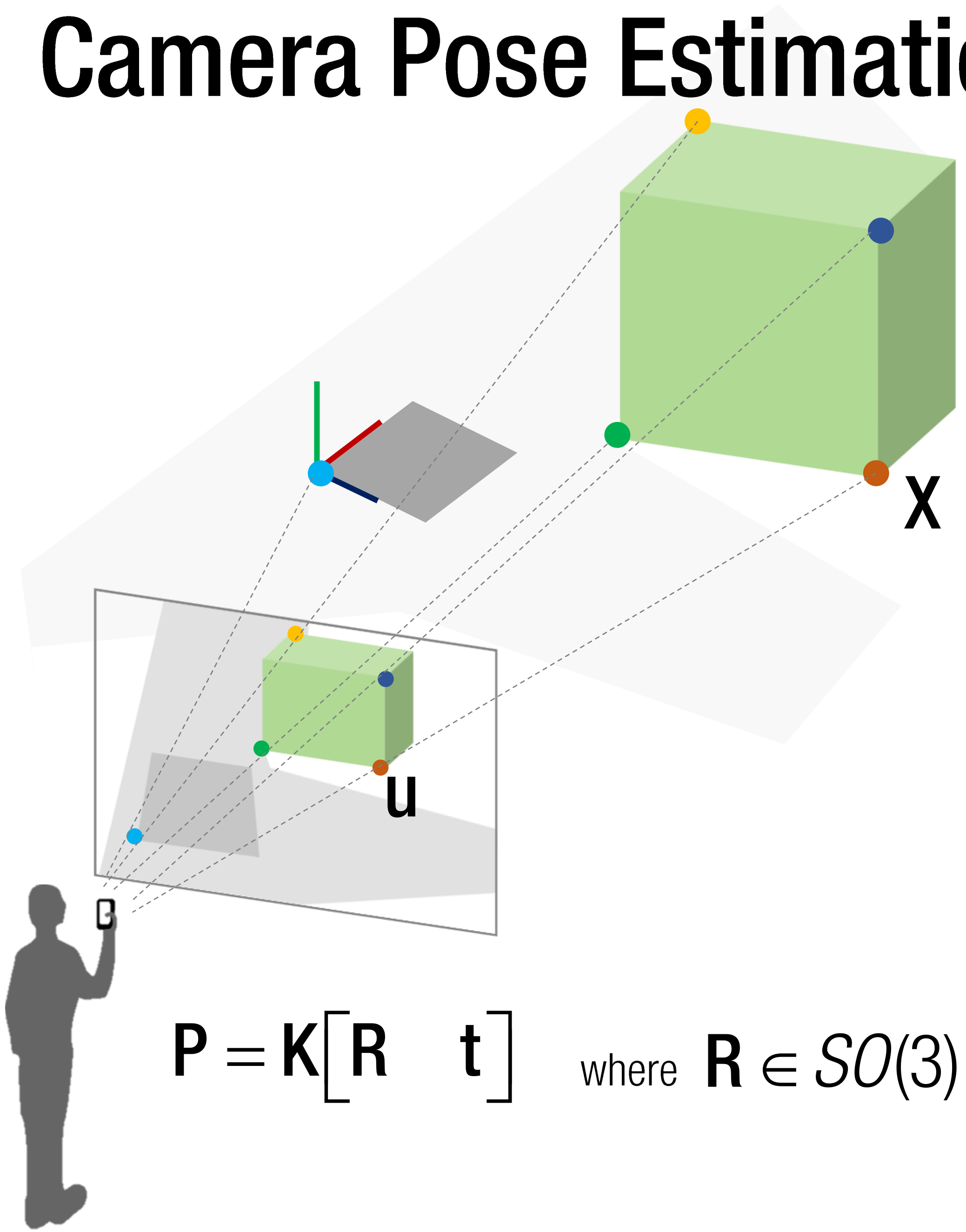
$2m$x12

# Camera Pose Estimation




If **K** is given,

$$\mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \gamma\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4\end{bmatrix}$$

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

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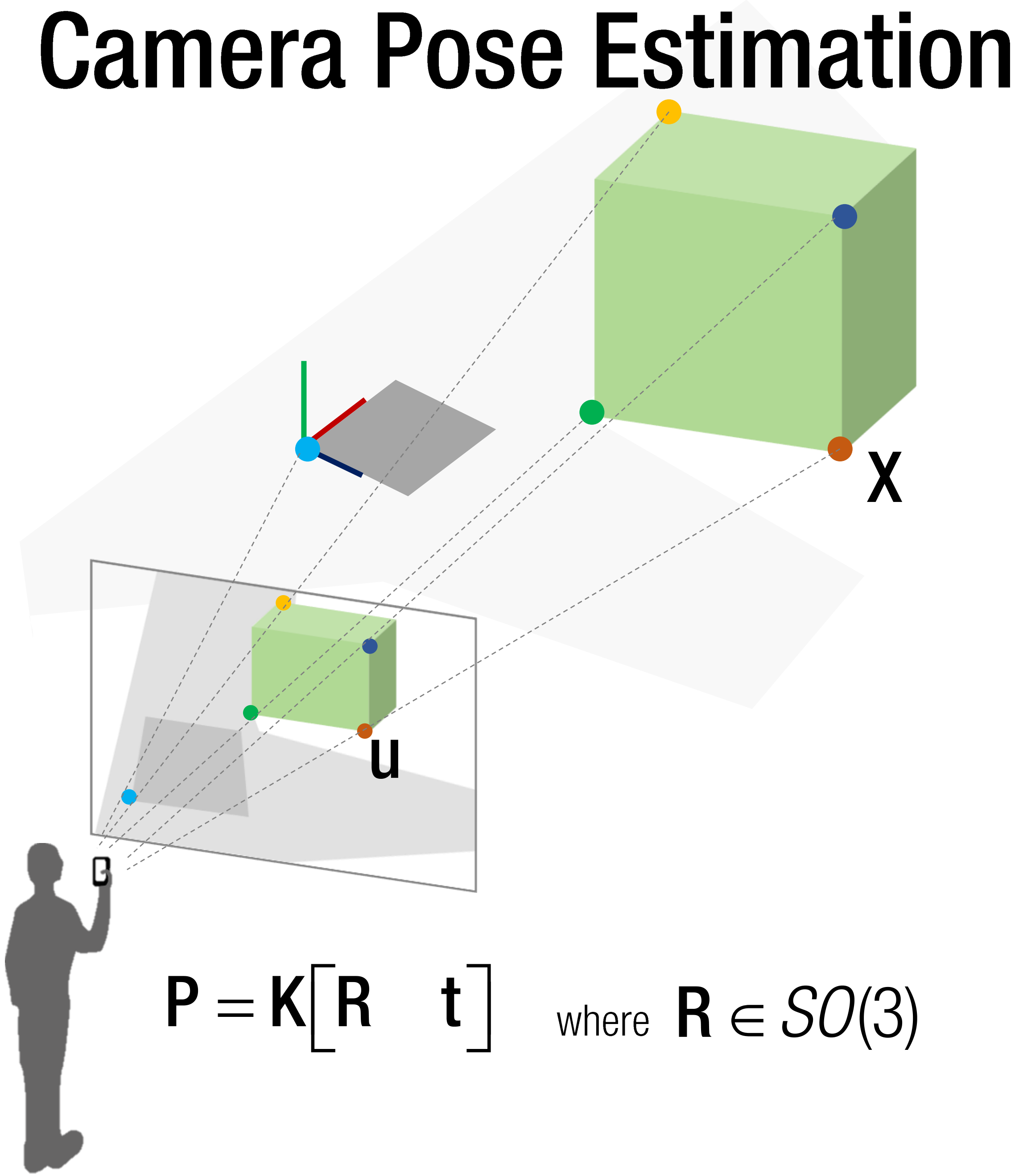


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$$\longrightarrow \gamma\mathbf{R} = \mathbf{K}^{-1}\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix}$$

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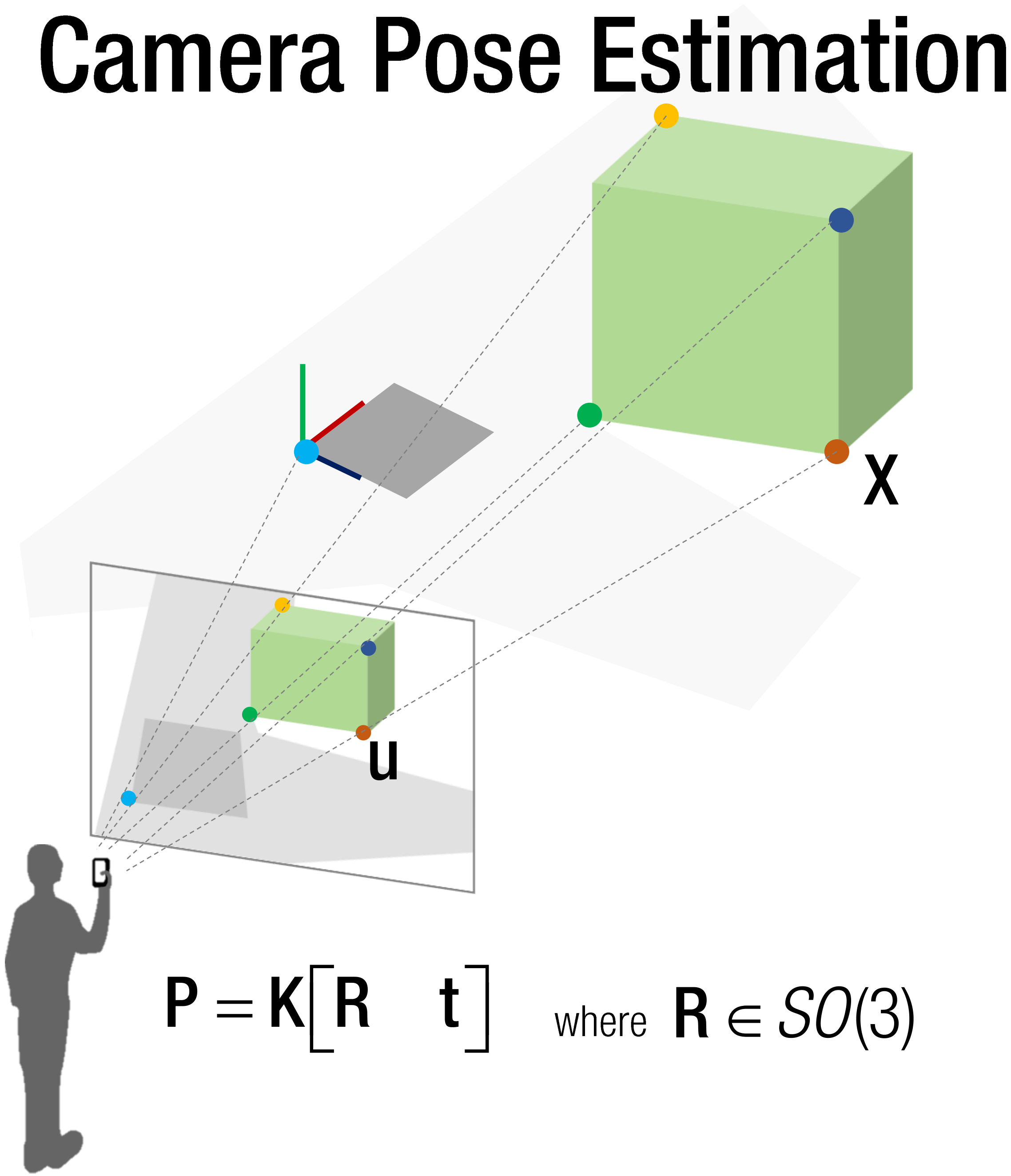


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$$\mathbf{K}^{-1}\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix} = \mathbf{U}\begin{bmatrix}d_{11} & & \\ & d_{22} & \\ & & d_{33}\end{bmatrix}\mathbf{V}^{\mathsf{T}}$$

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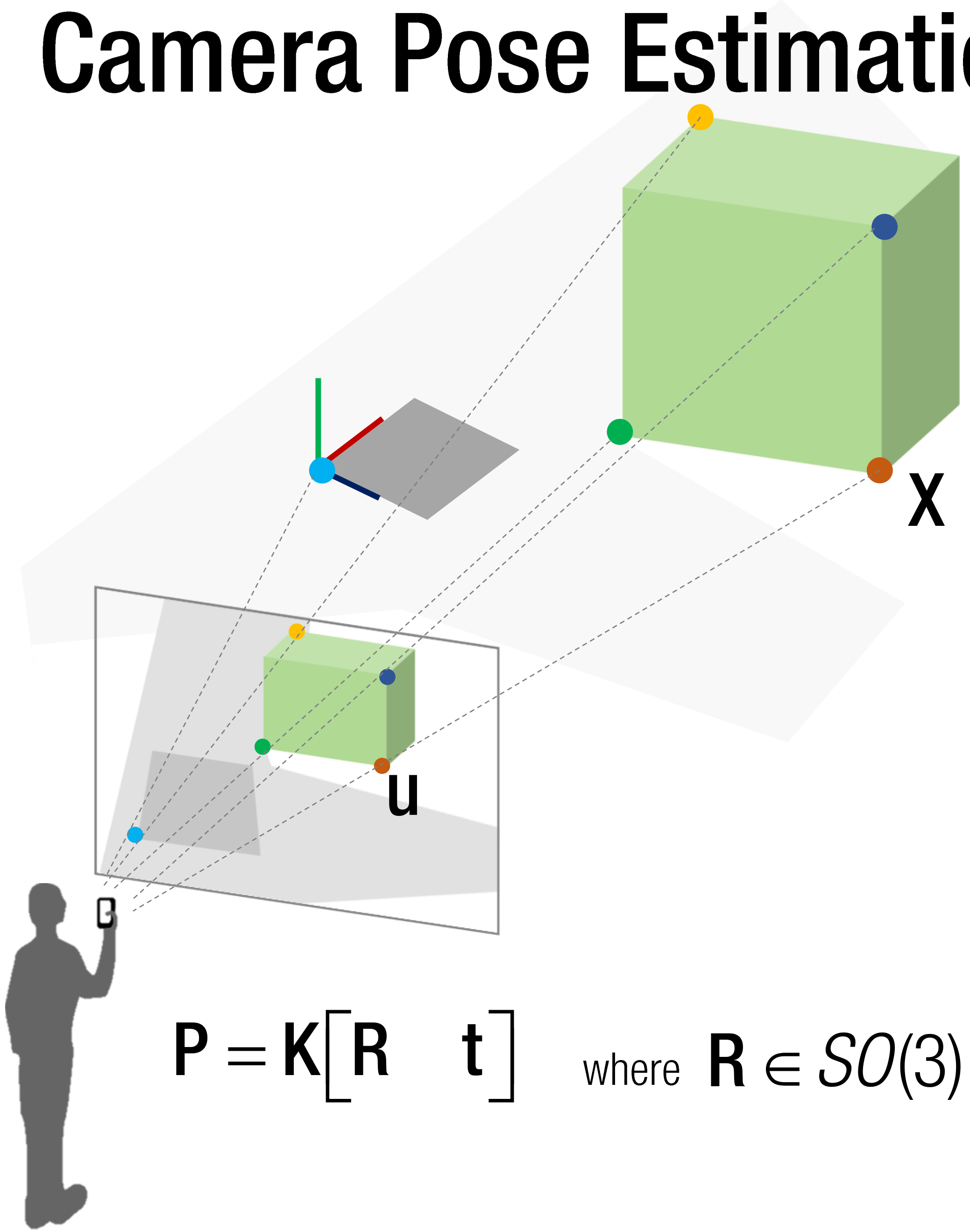


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$$\longrightarrow \gamma \approx d_{11}$$

$\mathbf{R} = \mathbf{U}\mathbf{V}^{\mathsf{T}}$ : SVD cleanup

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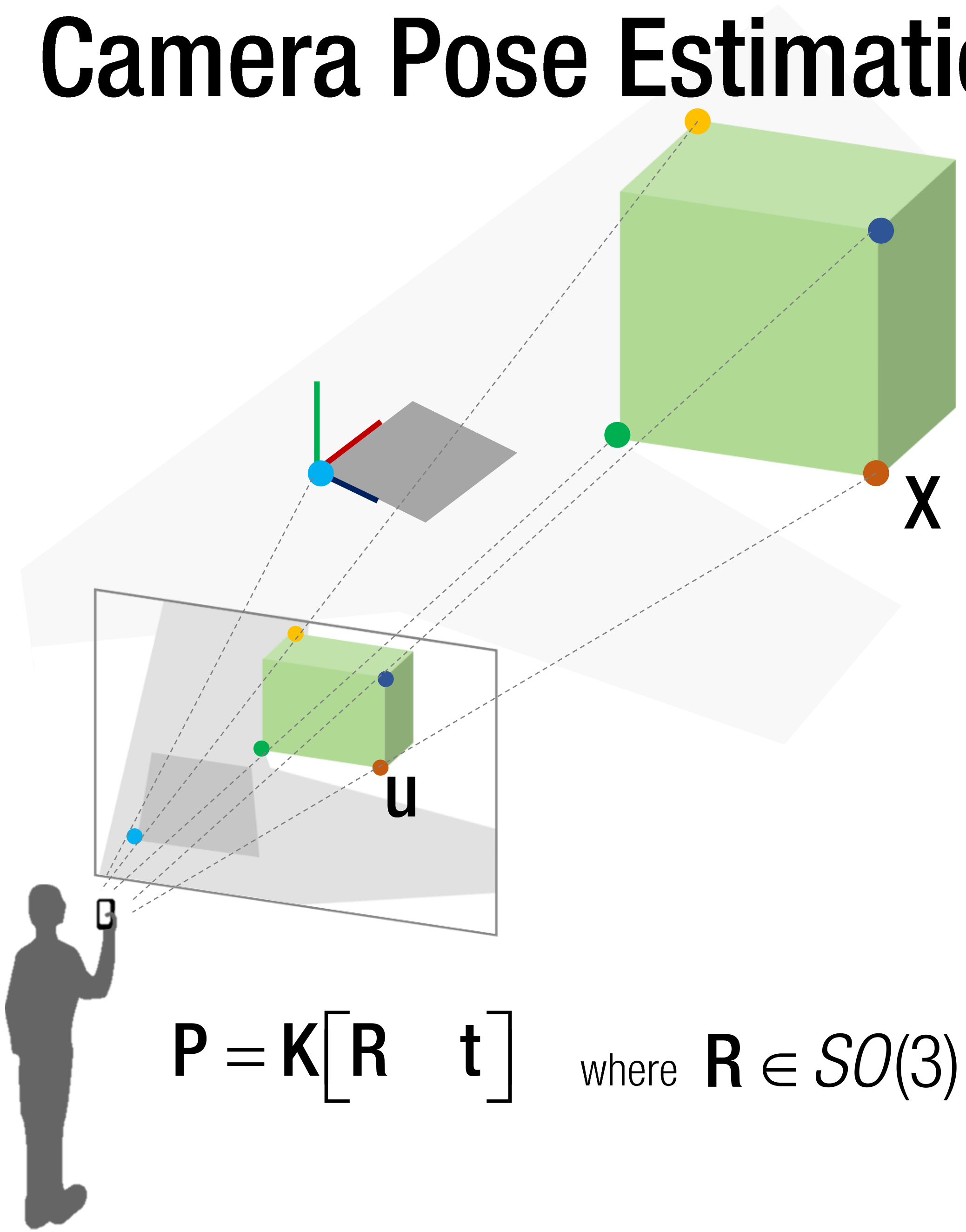


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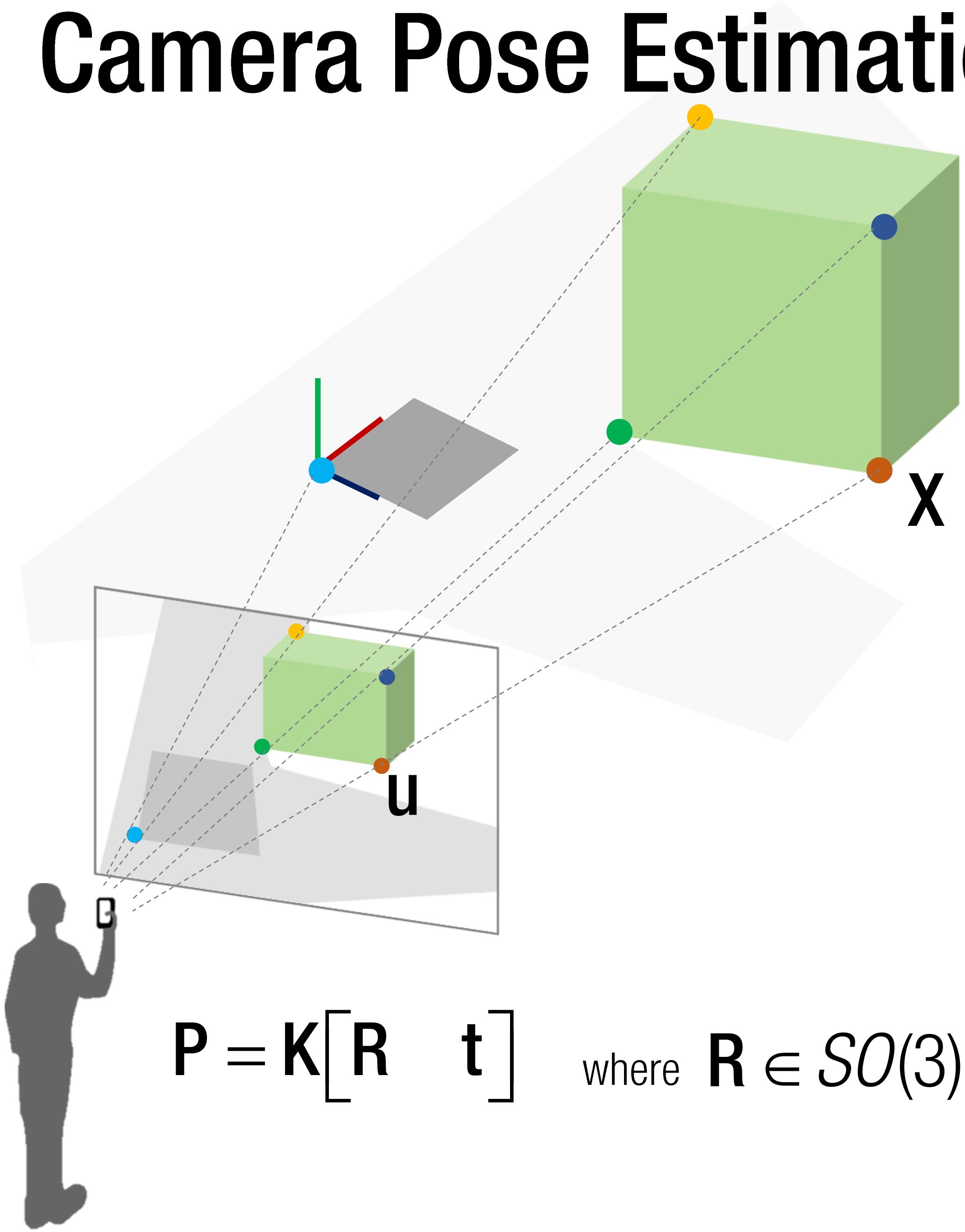
$$\mathbf{R} = \mathbf{U}\mathbf{V}^{\mathsf{T}}$$ : SVD cleanup

$$\longrightarrow \mathbf{t} = \frac{\mathbf{K}^{-1}\mathbf{p}_4}{d_{11}}$$ : Translation and scale recovery

# Camera Pose Estimation




$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

```
function [R t] = LinearPnP(X, u, K)
A = [];
for i = 1 : size(X,1)
   %% Build A matrix here
End

[u d v] = svd(A);
P = v(:,end);
P = [P(1:4)'; P(5:8)'; P(9:12)'];
R = inv(K)*P(:,1:3);
t = inv(K)*P(:,4);

% SVD clean up
[u d v] = svd(R);
R = u * v';
t = t/d(1,1);

if det(R) < 0
   R = -R;
   t = -t;
end
```

$$\mathbf{R} = \mathbf{U}\mathbf{V}^{\mathsf{T}}$$

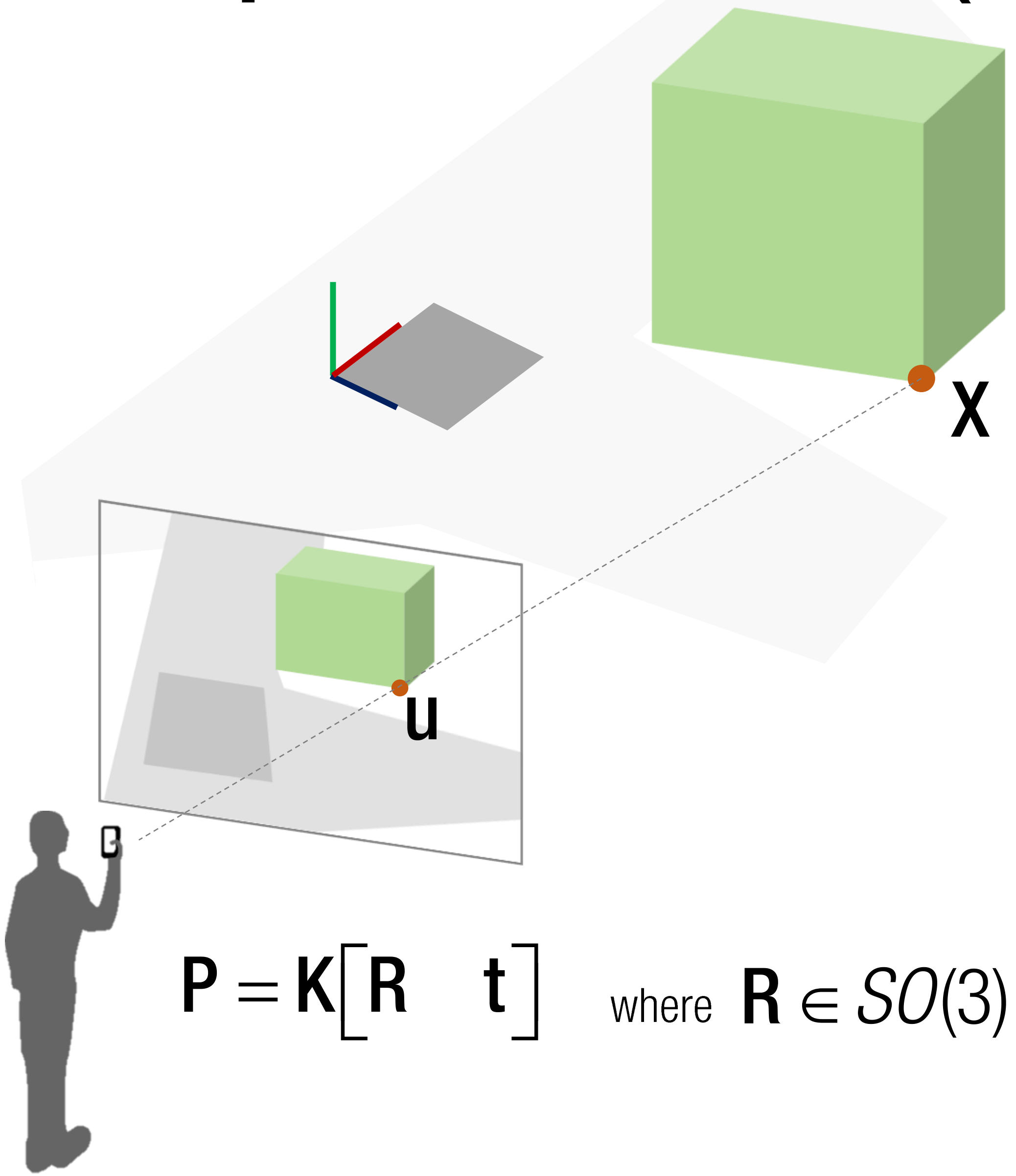
$$\mathbf{t} = \frac{\mathbf{K}^{-1}\mathbf{p}_4}{d_{11}}$$

# RANSAC Camera Pose Estimation

**Algorithm 1** PnP RANSAC

$nInliers \leftarrow 0$
**for** $i = 1 : M$ **do**
    Choose 6 correspondences, $\mathbf{X}_r$ and $\mathbf{w}_r$, randomly from $\mathbf{X}$ and $\mathbf{w}$.
    $[\mathbf{R}_r,\ \mathbf{t}_r]$ = `LinearPnP`$(\mathbf{X}_r,\ \mathbf{w}_r,\ \mathbf{K})$
    Compute the number of inliers, $n_r$, with respect to $\mathbf{R}_r$, $\mathbf{t}_r$.
    **if** $n_r > nInliers$ **then**
        $nInliers \leftarrow n_r$
        $\mathbf{R} = \mathbf{R}_r$ and $\mathbf{t} = \mathbf{t}_r$
    **end if**
**end for**

# Perspective-3-Point (P3P)

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Linear in camera matrix

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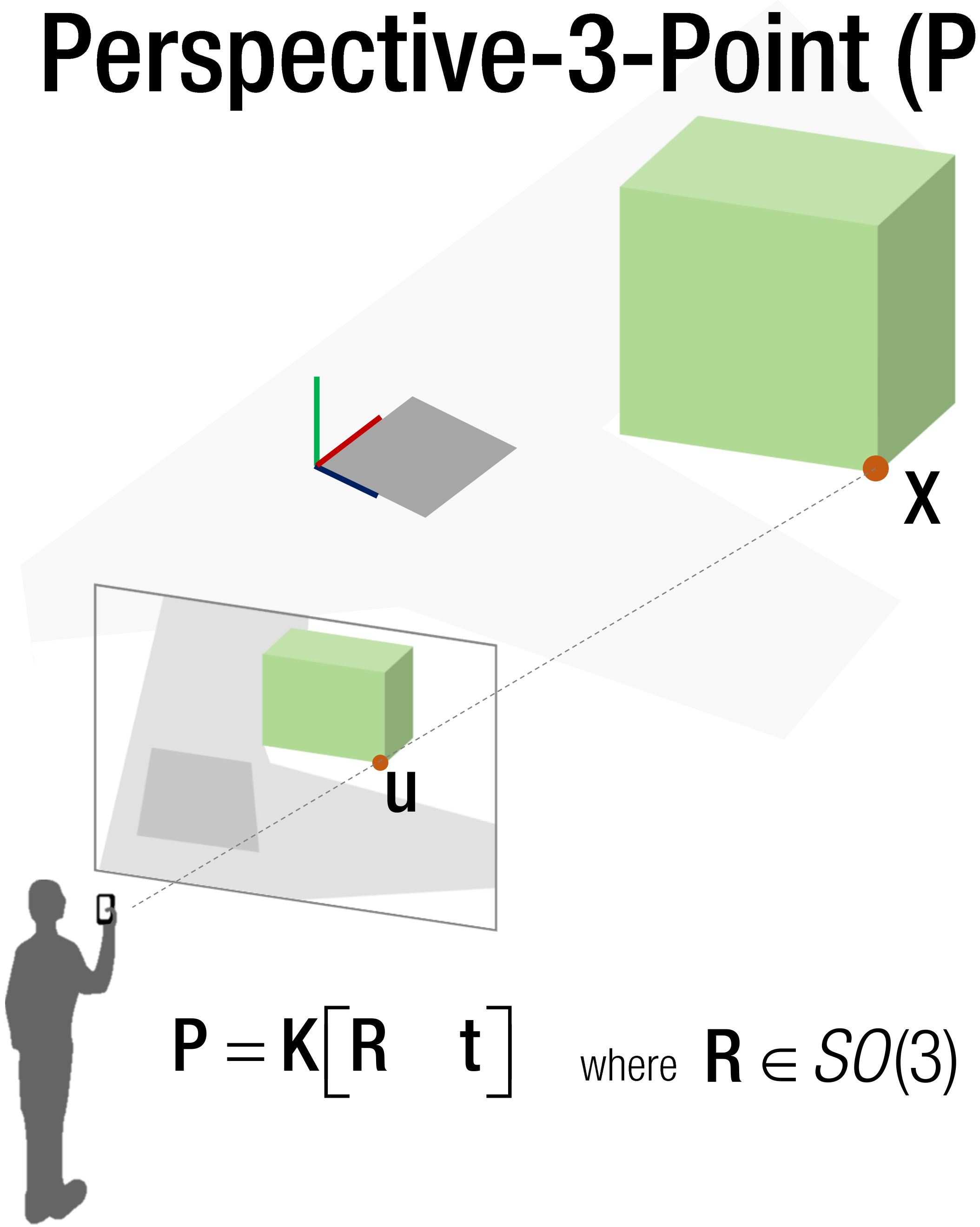


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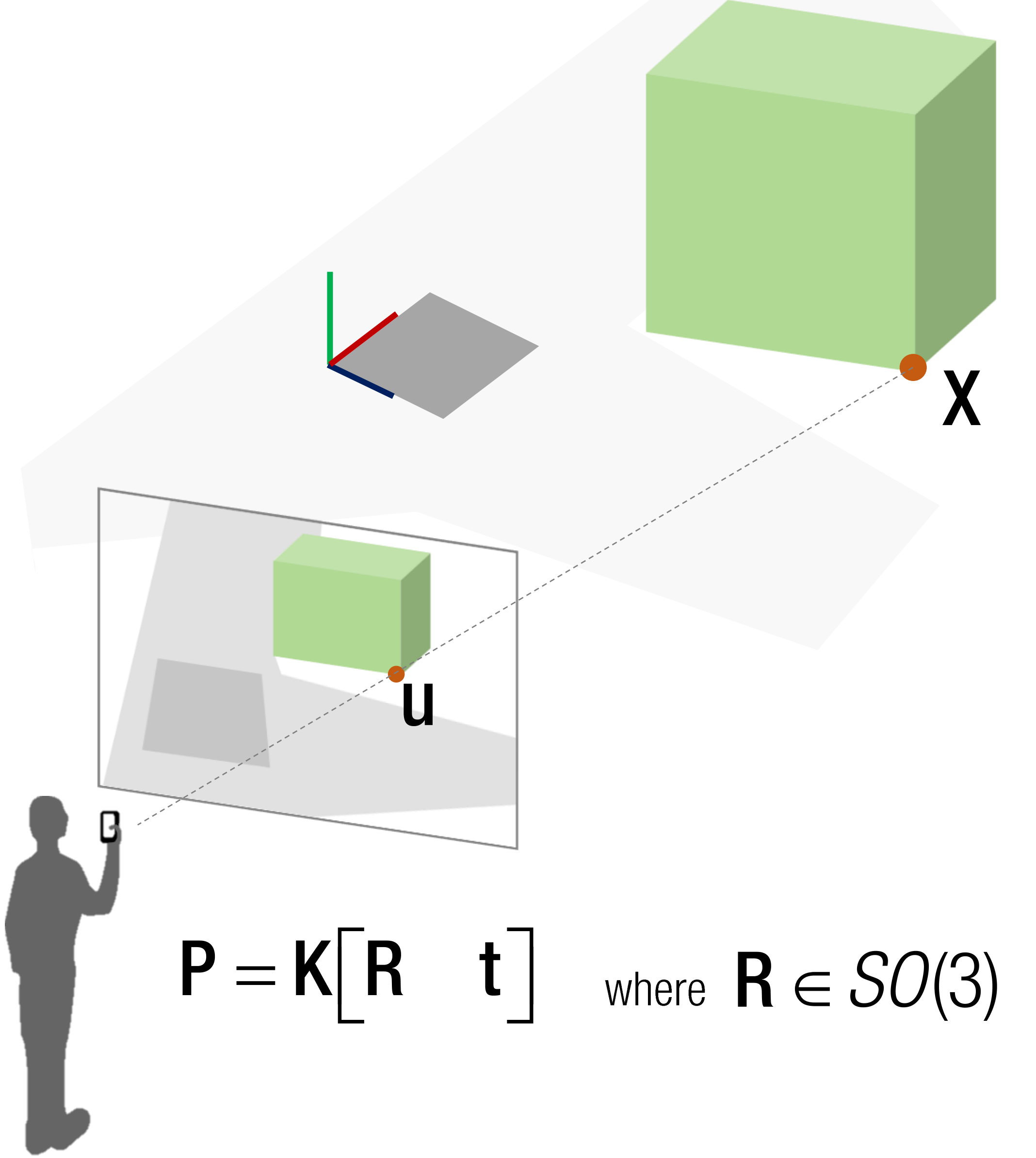


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6 dof when **K** is known.

# of equations per correspondence: 2

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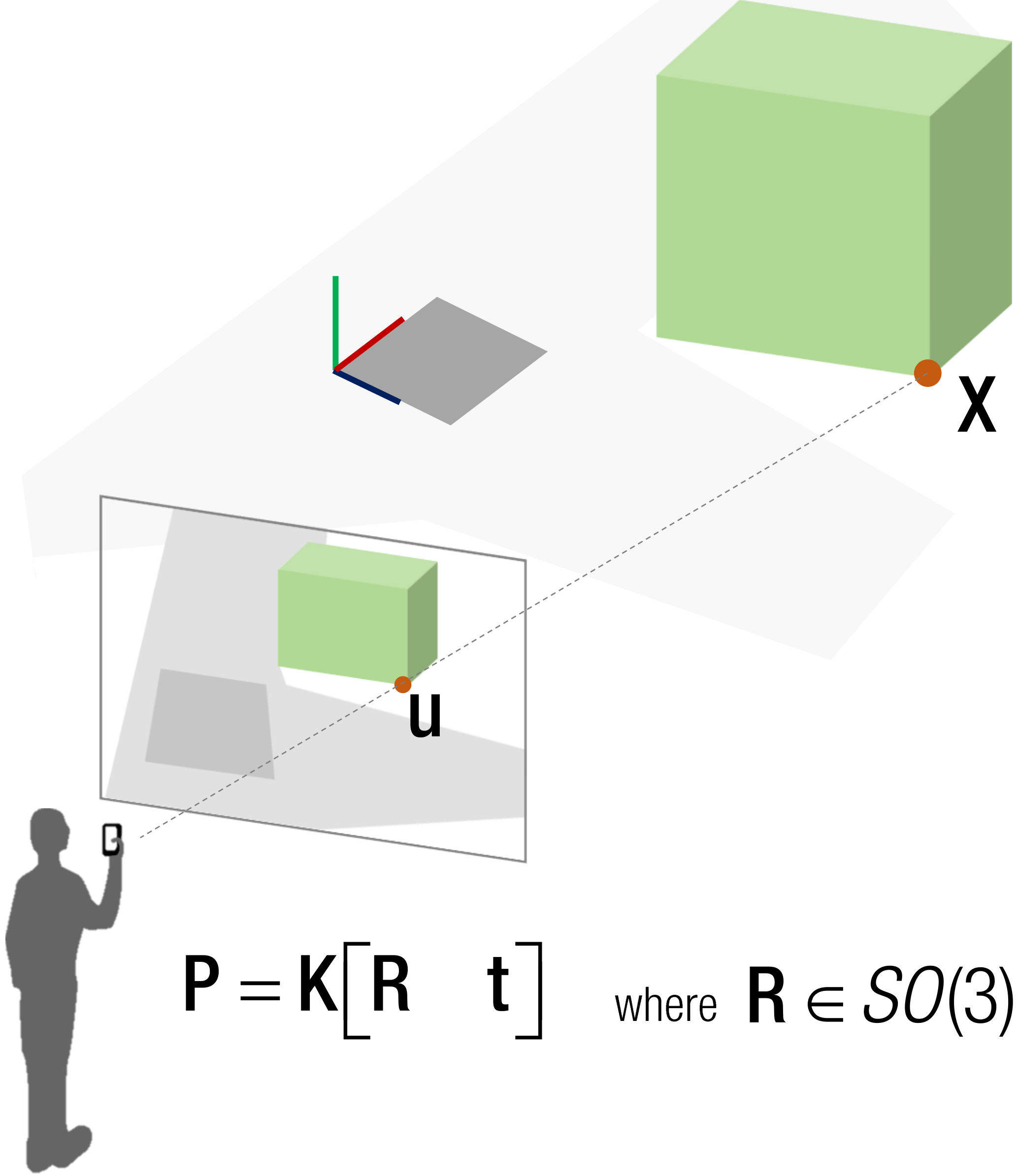


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3 correspondences should be enough.

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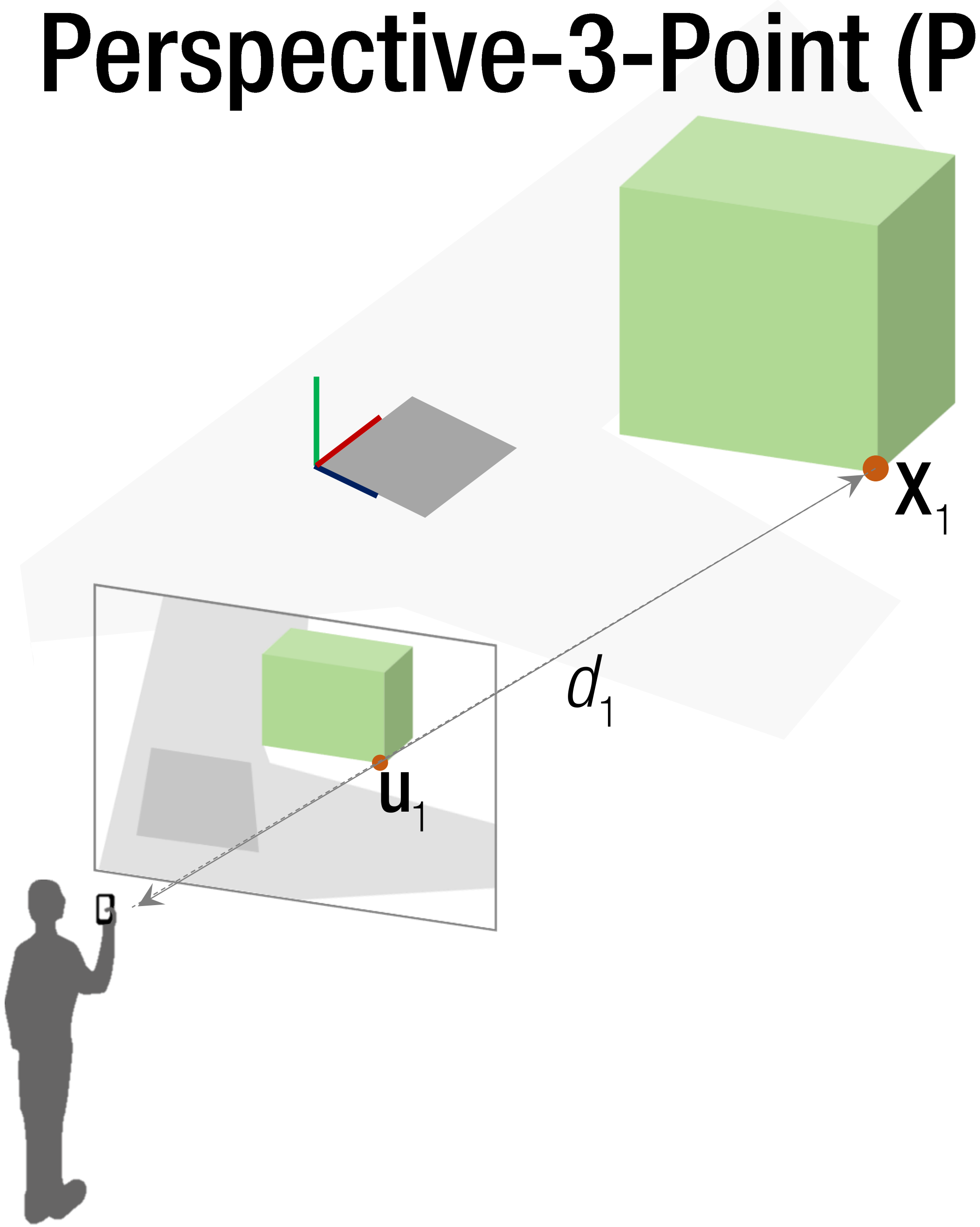
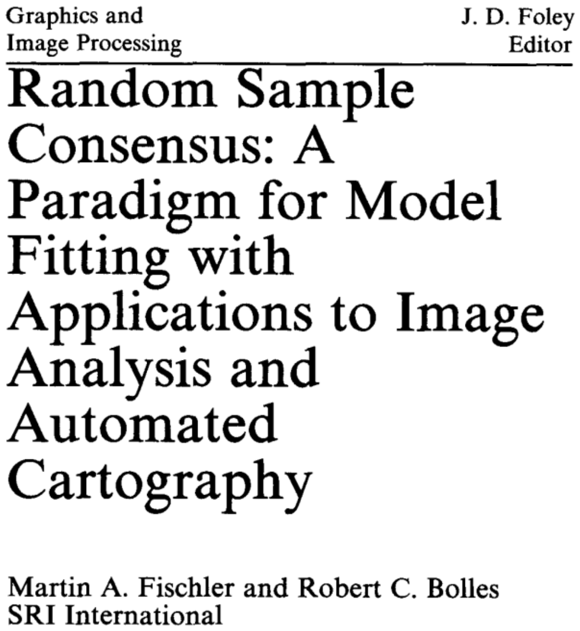
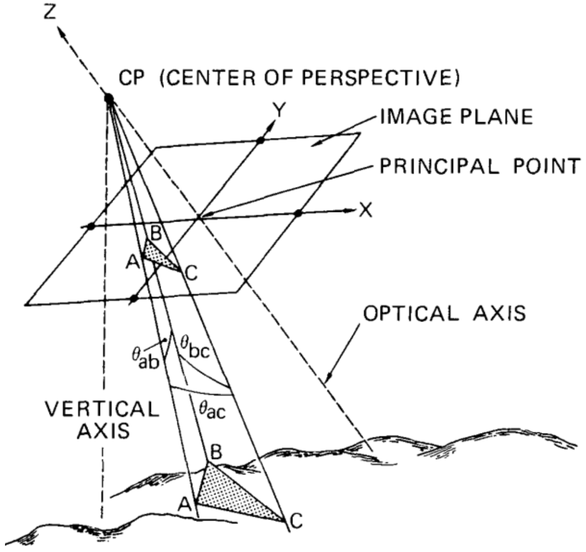



RANSAC with PnP

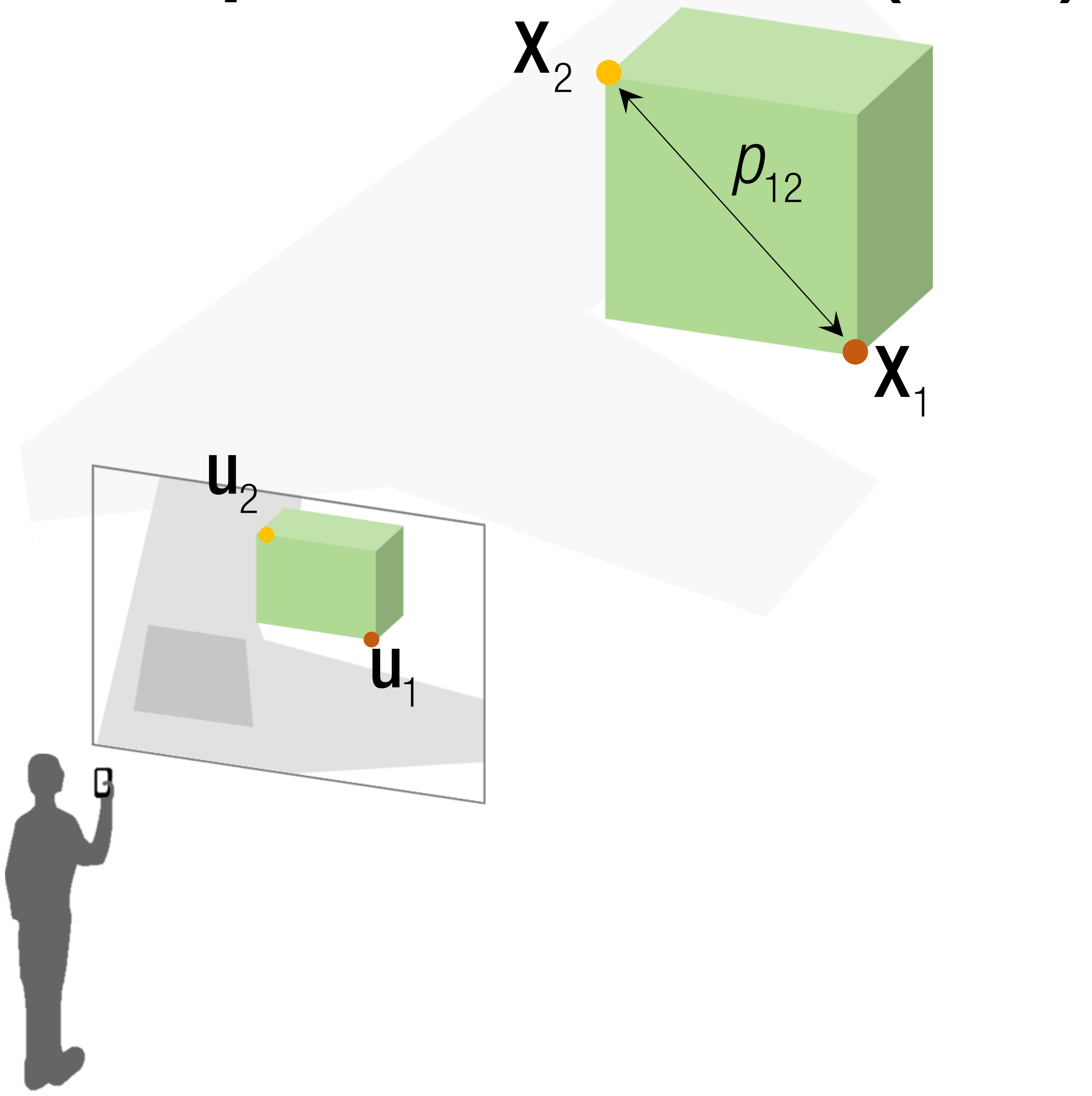

Graphics and Image Processing
J. D. Foley Editor

Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography

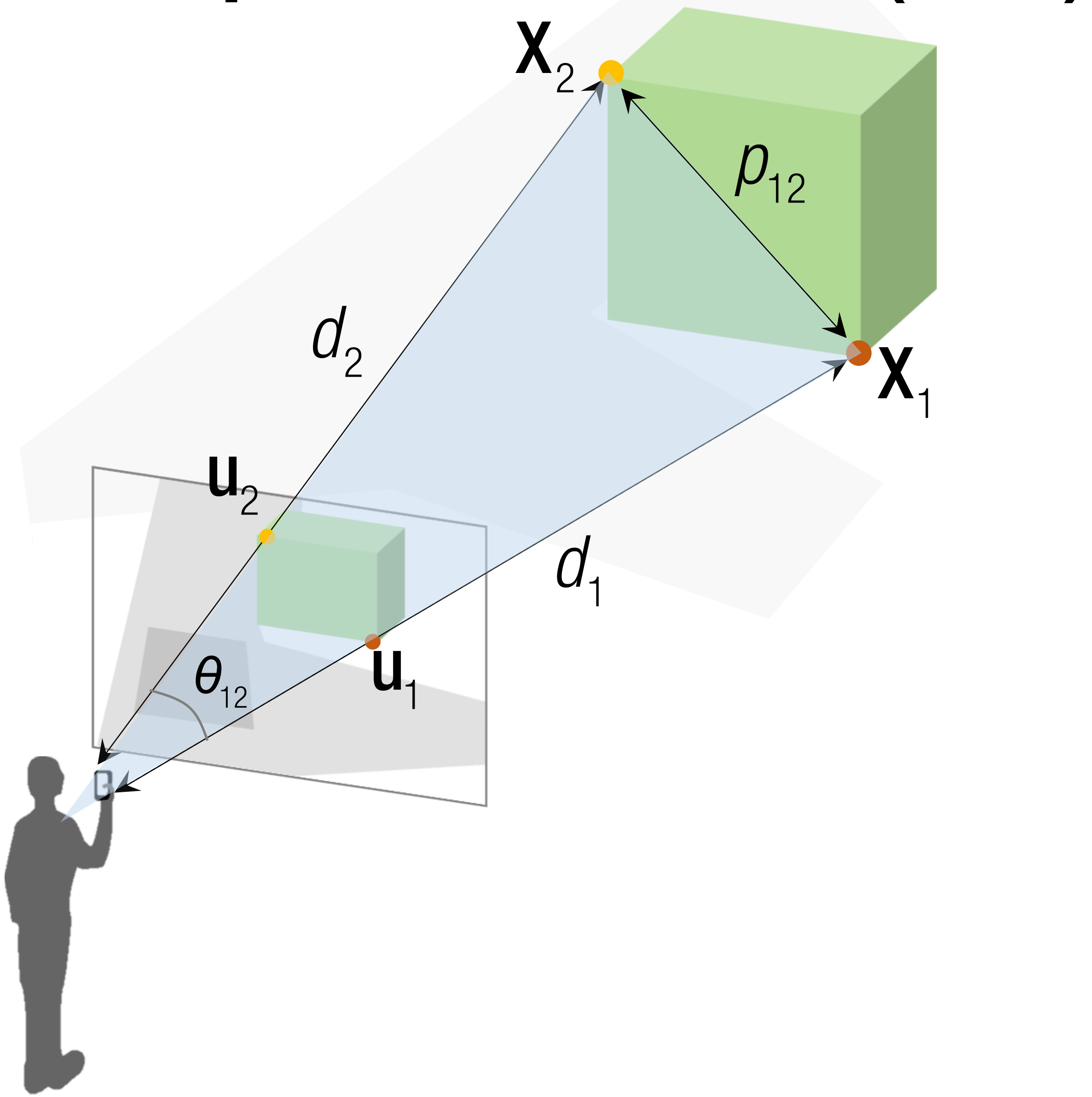
Martin A. Fischler and Robert C. Bolles
SRI International



# Perspective-3-Point (P3P)



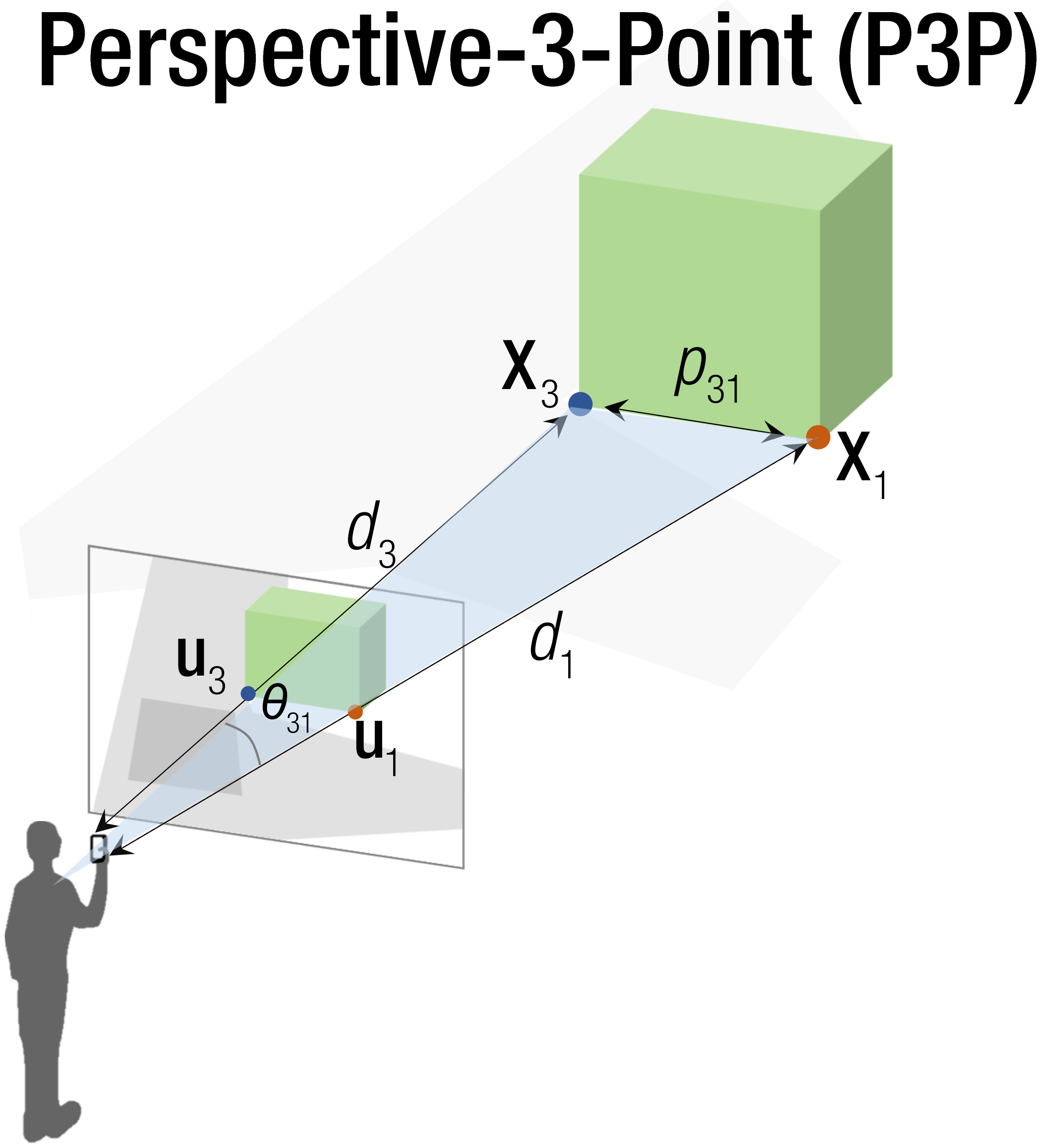
# Perspective-3-Point (P3P)




2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1 d_2 \cos\theta_{12} = p_{12}^2$$

# Perspective-3-Point (P3P)

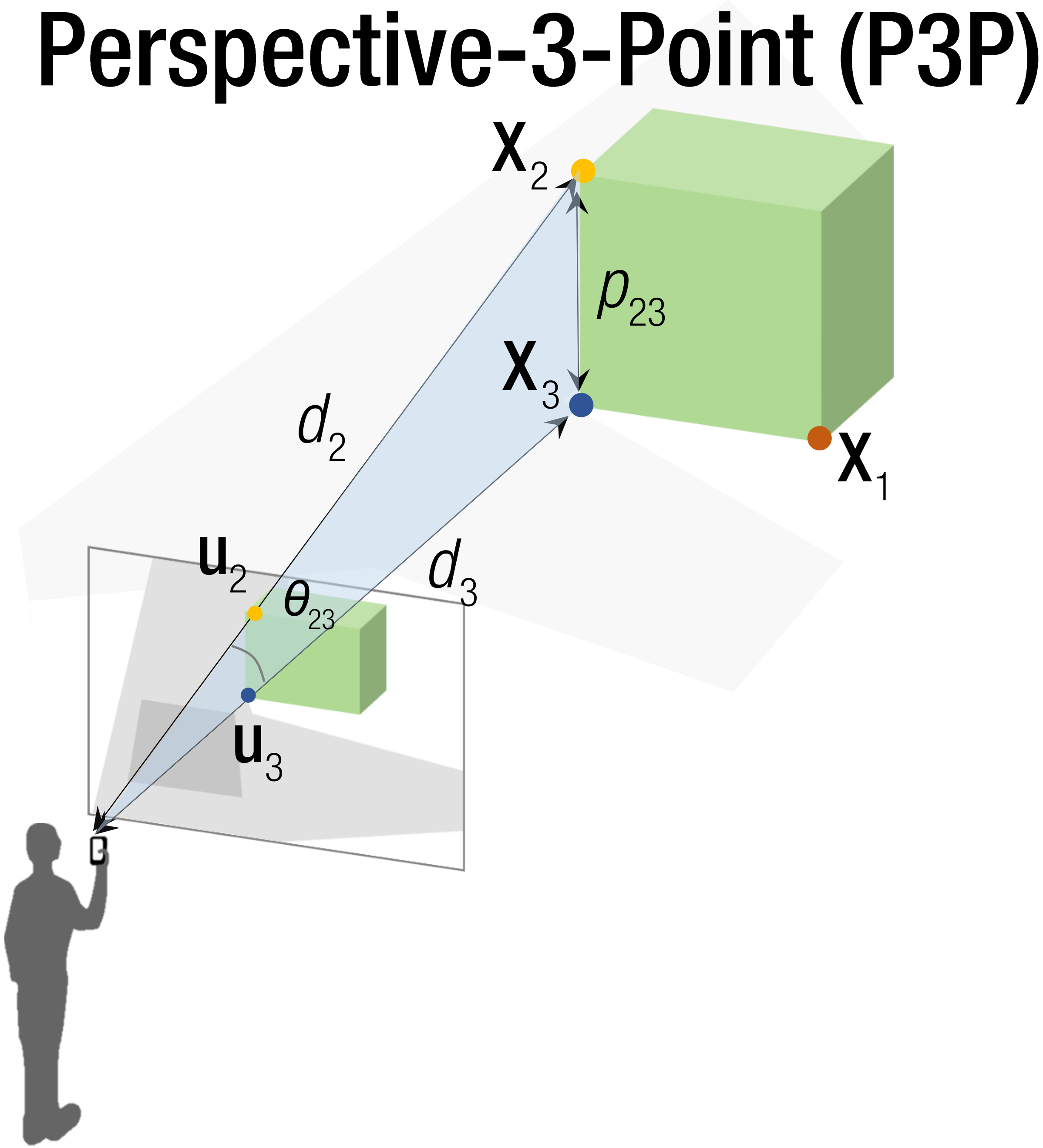



2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\boldsymbol{\theta}_{12} = p_{12}^2$$

$$d_3^2 + d_1^2 - 2d_3d_1\cos\boldsymbol{\theta}_{31} = p_{31}^2$$

# Perspective-3-Point (P3P)



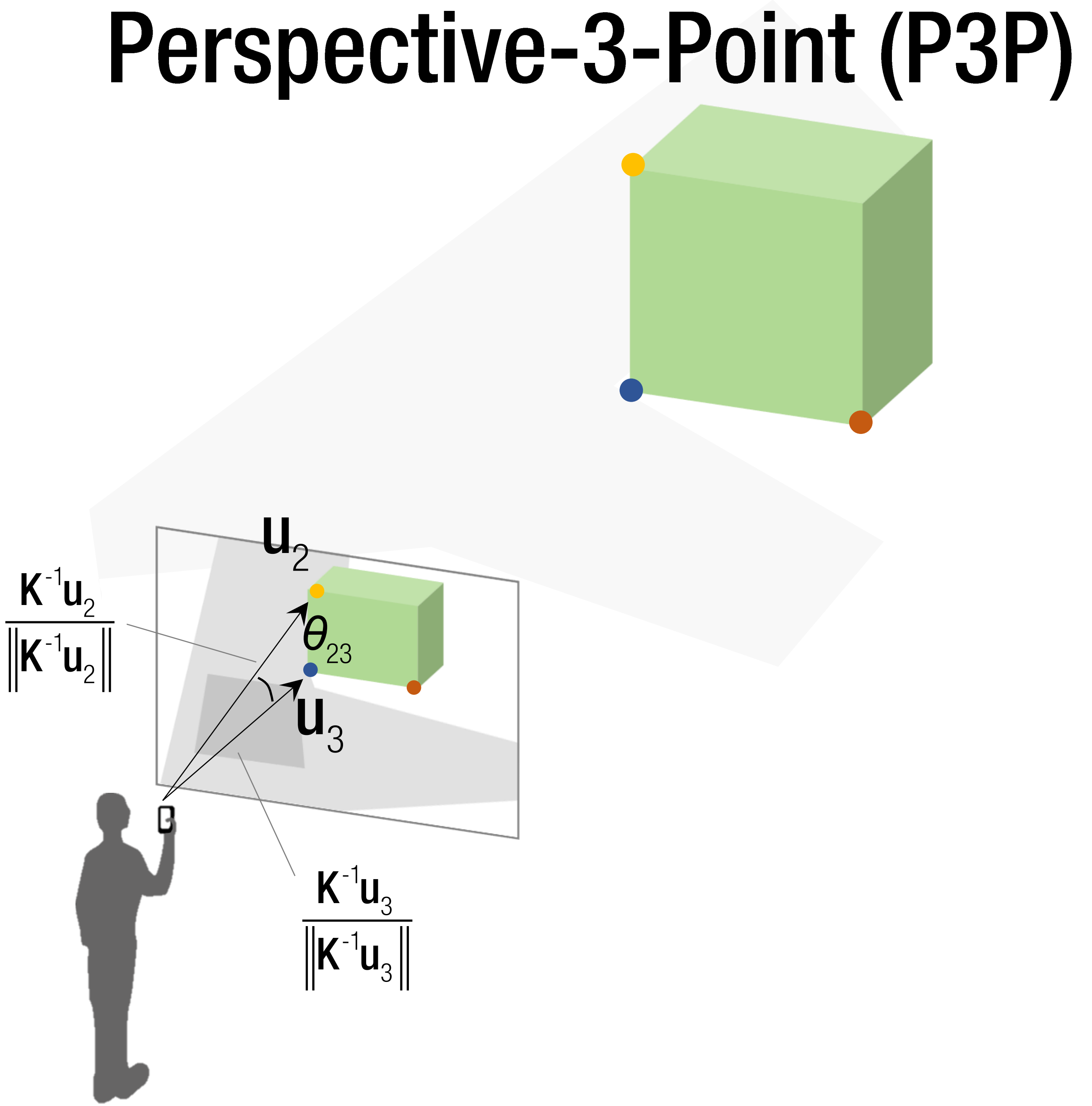

2[nd] Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2$$
$$d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} = p_{31}^2$$
$$d_2^2 + d_3^2 - 2d_2d_3\cos\theta_{23} = p_{23}^2$$

3 equations

Unknowns: $d_1, d_2, d_3$

# Perspective-3-Point (P3P)




2nd Cosine law:

$$\left.\begin{aligned} d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} &= p_{12}^2 \\ d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} &= p_{31}^2 \\ d_2^2 + d_3^2 - 2d_2d_3\cos\theta_{23} &= p_{23}^2 \end{aligned}\right| \quad \text{3 equations}$$

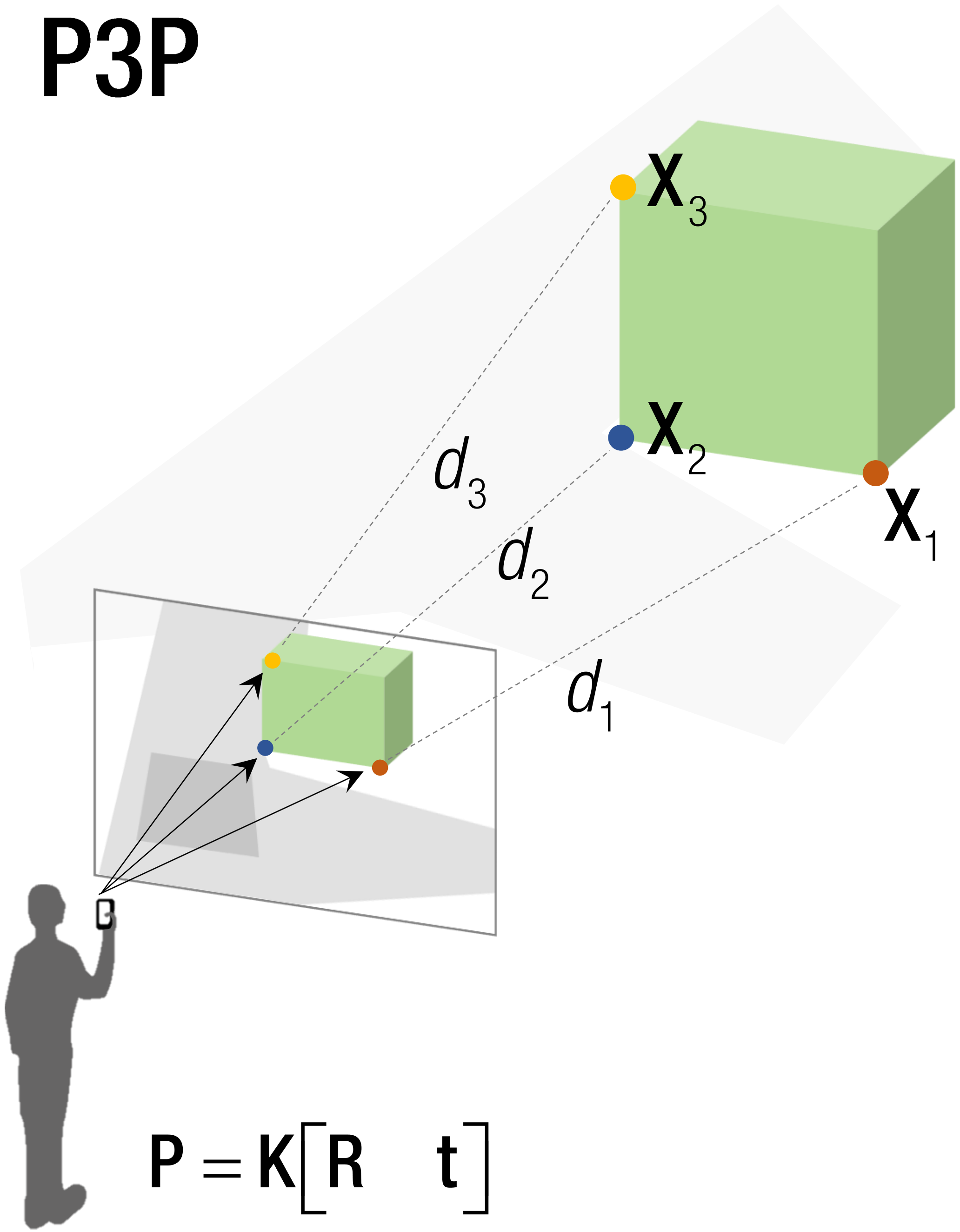
Unknowns: $d_1, d_2, d_3$

Note:

$$\cos\theta_{12} = \frac{\left(\mathbf{K}^{-1}\mathbf{u}_1\right)^{\mathsf{T}}\left(\mathbf{K}^{-1}\mathbf{u}_2\right)}{\|\mathbf{K}^{-1}\mathbf{u}_1\|\|\mathbf{K}^{-1}\mathbf{u}_2\|} \qquad \cos\theta_{23} = \frac{\left(\mathbf{K}^{-1}\mathbf{u}_2\right)^{\mathsf{T}}\left(\mathbf{K}^{-1}\mathbf{u}_3\right)}{\|\mathbf{K}^{-1}\mathbf{u}_2\|\|\mathbf{K}^{-1}\mathbf{u}_3\|}$$

$$\cos\theta_{31} = \frac{\left(\mathbf{K}^{-1}\mathbf{u}_1\right)^{\mathsf{T}}\left(\mathbf{K}^{-1}\mathbf{u}_3\right)}{\|\mathbf{K}^{-1}\mathbf{u}_1\|\|\mathbf{K}^{-1}\mathbf{u}_3\|}$$

# P3P




$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2$$
$$d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} = p_{31}^2$$
$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{23} = p_{23}^2$$

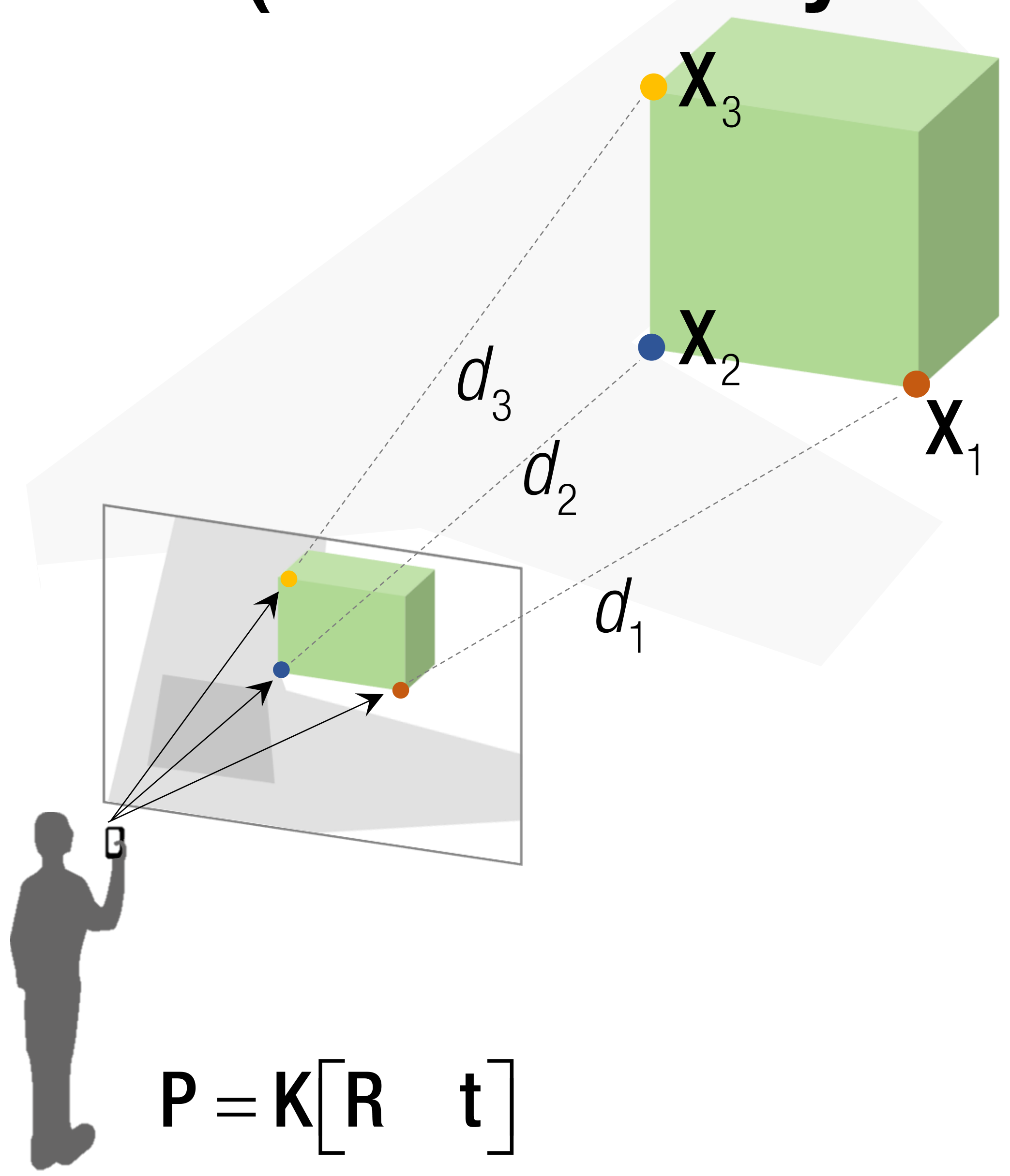
3 equations

The number of possible solutions: 8 = 2x2x2

$d_1 > 0 \quad d_2 > 0 \quad d_3 > 0$ : 4 = 2x2x2/2

→ requires additional fourth point to verify the solution.

# P3P (4th order Polynomial)




$$P = K\begin{bmatrix} R & t \end{bmatrix}$$

2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2$$
$$d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} = p_{31}^2$$
$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{23} = p_{23}^2$$

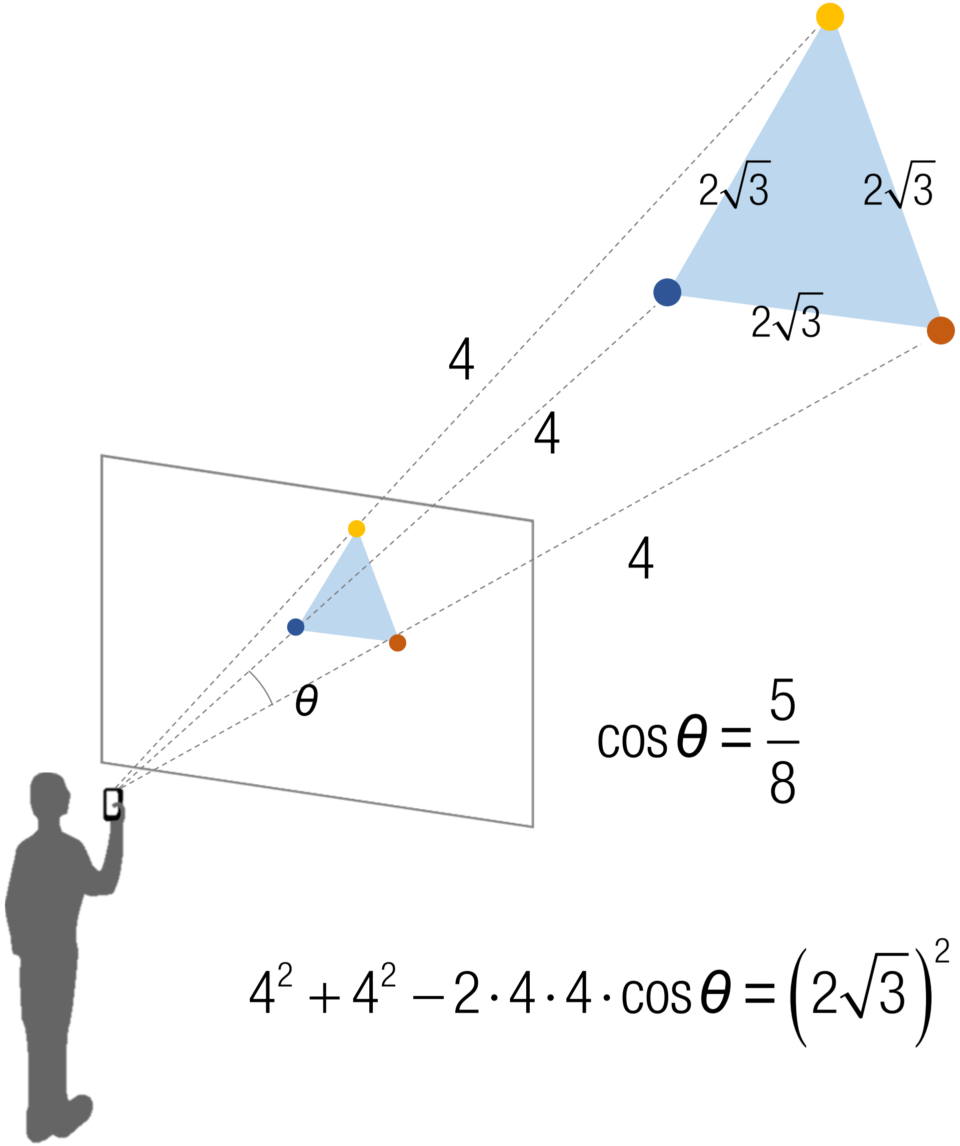
3 equations

4th order polynomial:

$$\rightarrow a_4y^4 + a_3y^3 + a_2y^2 + a_1y + a_0 = 0$$

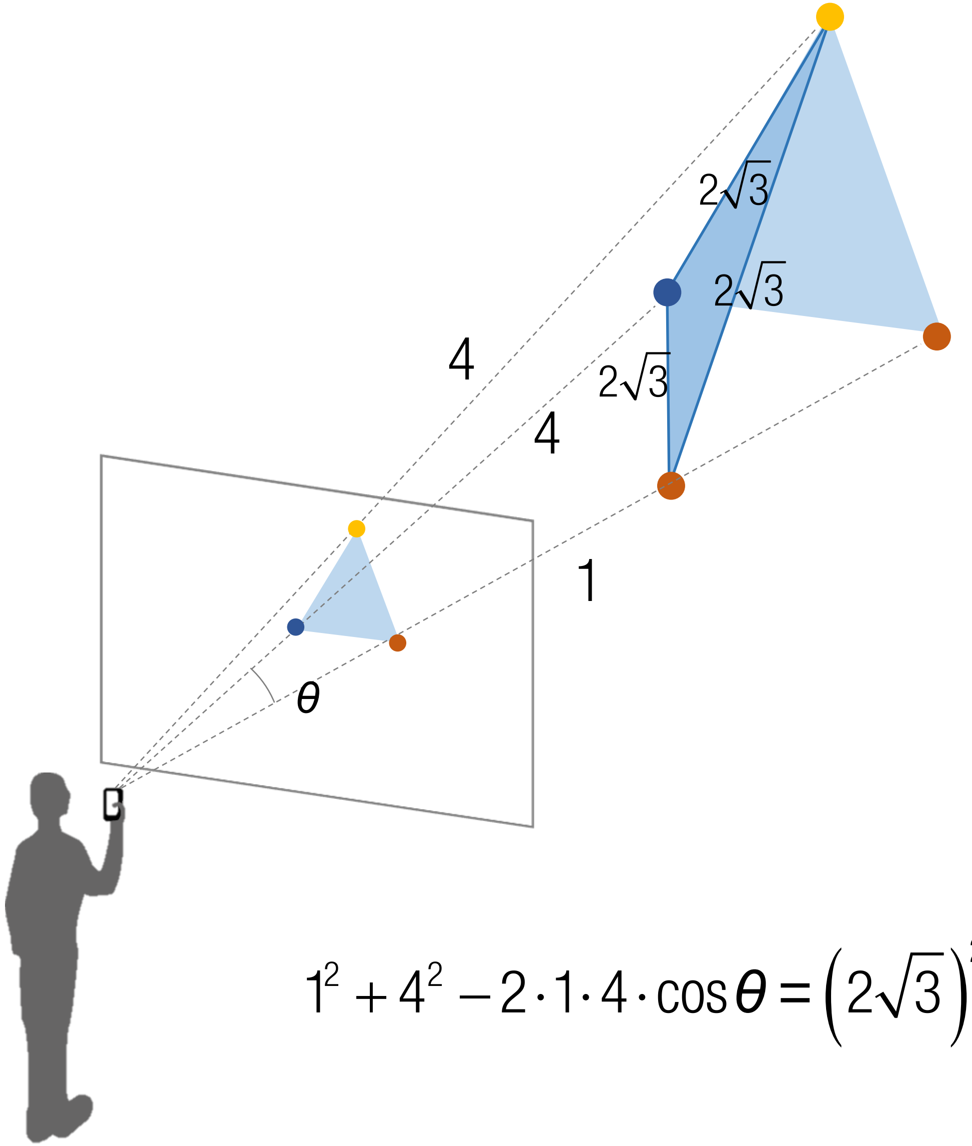
Closed form solutions exist.

# Four Solution Example



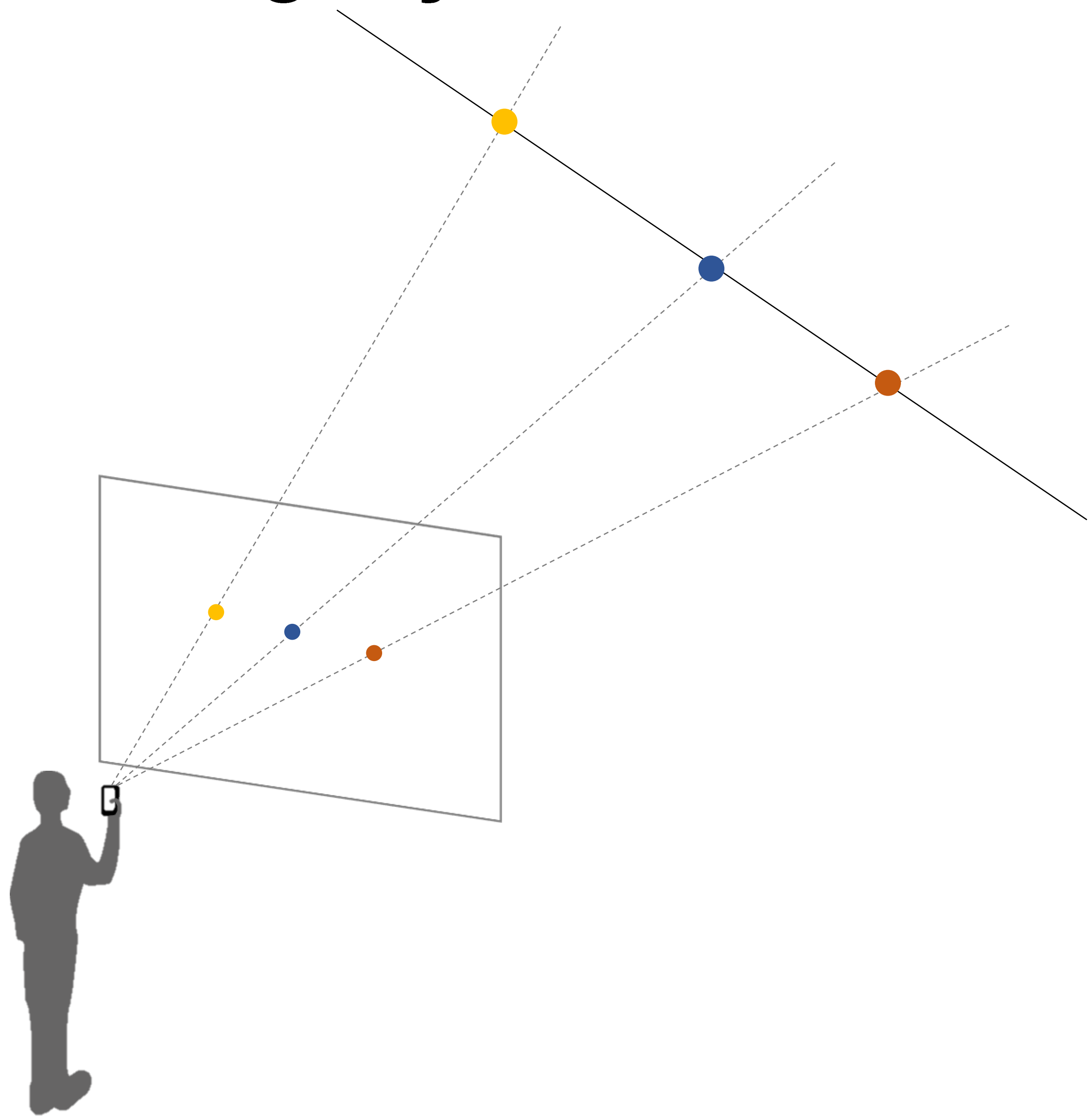

$$\cos\theta = \frac{5}{8}$$

$$4^2 + 4^2 - 2 \cdot 4 \cdot 4 \cdot \cos\theta = \left(2\sqrt{3}\right)^2$$

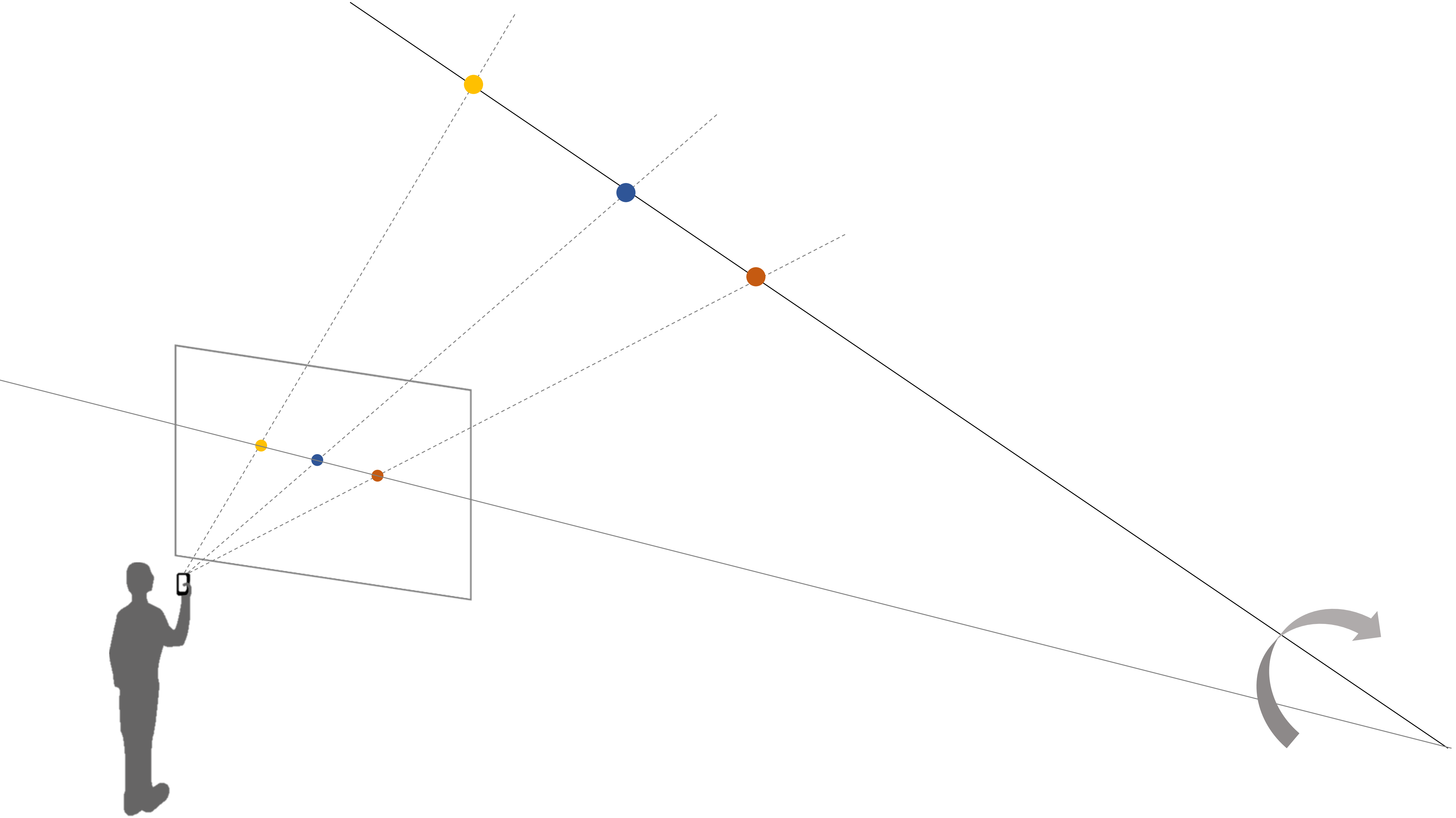



$$1^2 + 4^2 - 2 \cdot 1 \cdot 4 \cdot \cos\theta = \left(2\sqrt{3}\right)^2$$

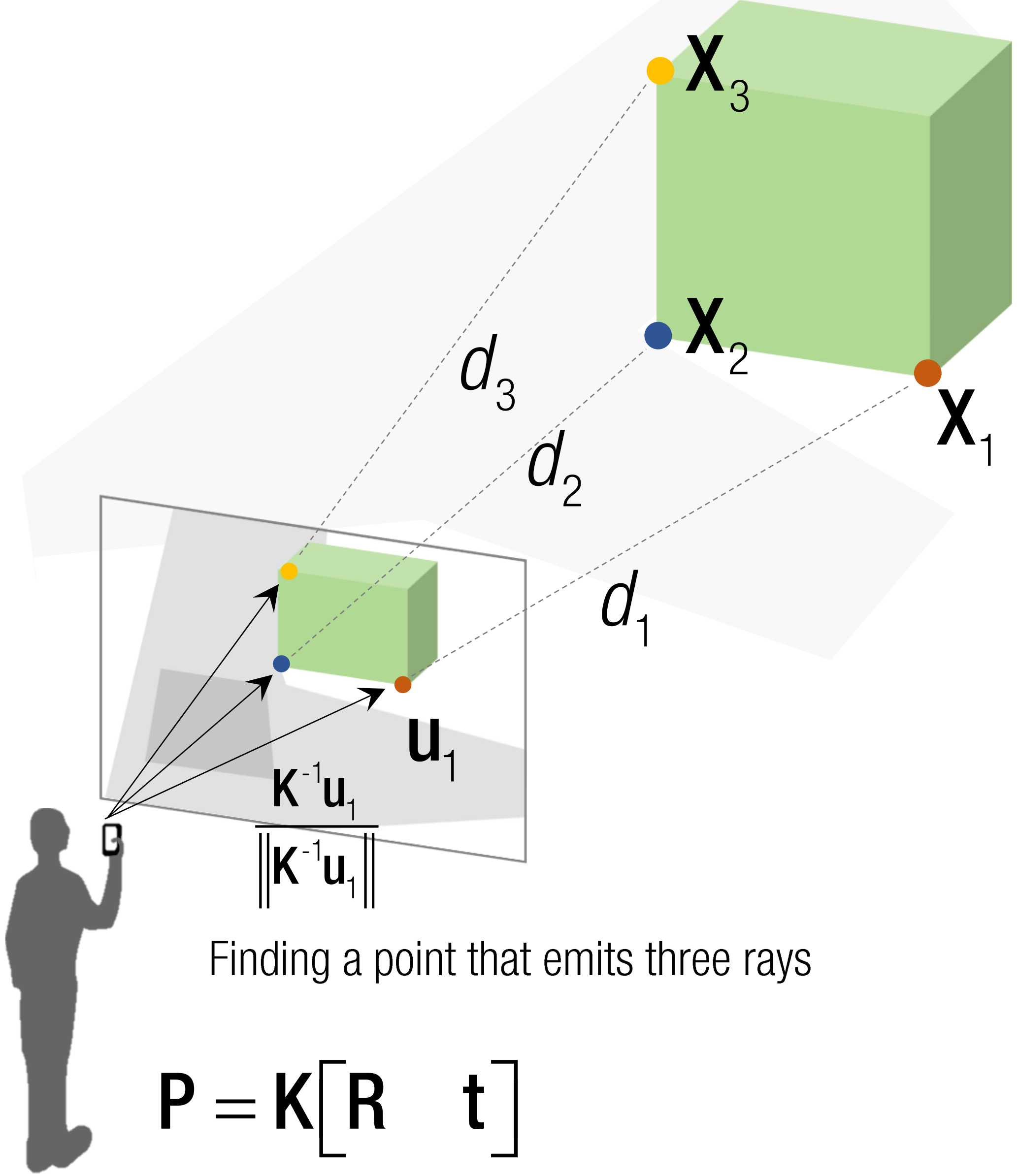
# Ambiguity



# Ambiguity (Colinear points)



# P3P (4th order Polynomial)




Finding a point that emits three rays

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2$$
$$d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} = p_{31}^2$$
$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{23} = p_{23}^2$$

3 equations

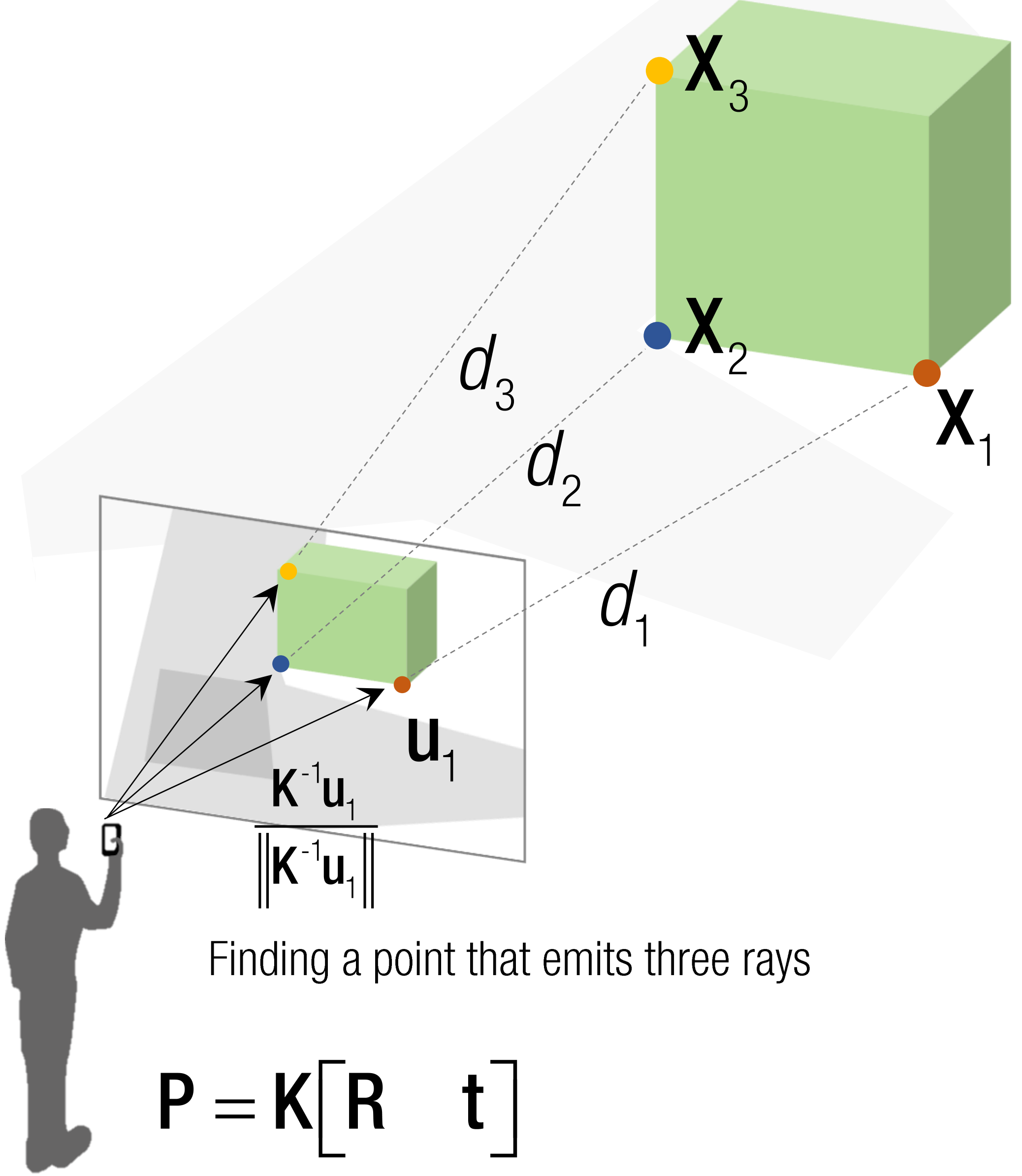
4th order polynomial:

$\rightarrow$ $a_4y^4 + a_3y^3 + a_2y^2 + a_1y + a_0 = 0$

Closed form solutions exist.

$\rightarrow$ Compute **t** using $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $d_1$, $d_2$, and $d_3$.

# P3P ($4^{th}$ order Polynomial)




Finding a point that emits three rays

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

$2^{nd}$ Cosine law:

$$\left.\begin{aligned} d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} &= p_{12}^2 \\ d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} &= p_{31}^2 \\ d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{23} &= p_{23}^2 \end{aligned}\right| \text{ 3 equations}$$

$4^{th}$ order polynomial:

$$\rightarrow \underline{a_4y^4 + a_3y^3 + a_2y^2 + a_1y + a_0 = 0}$$

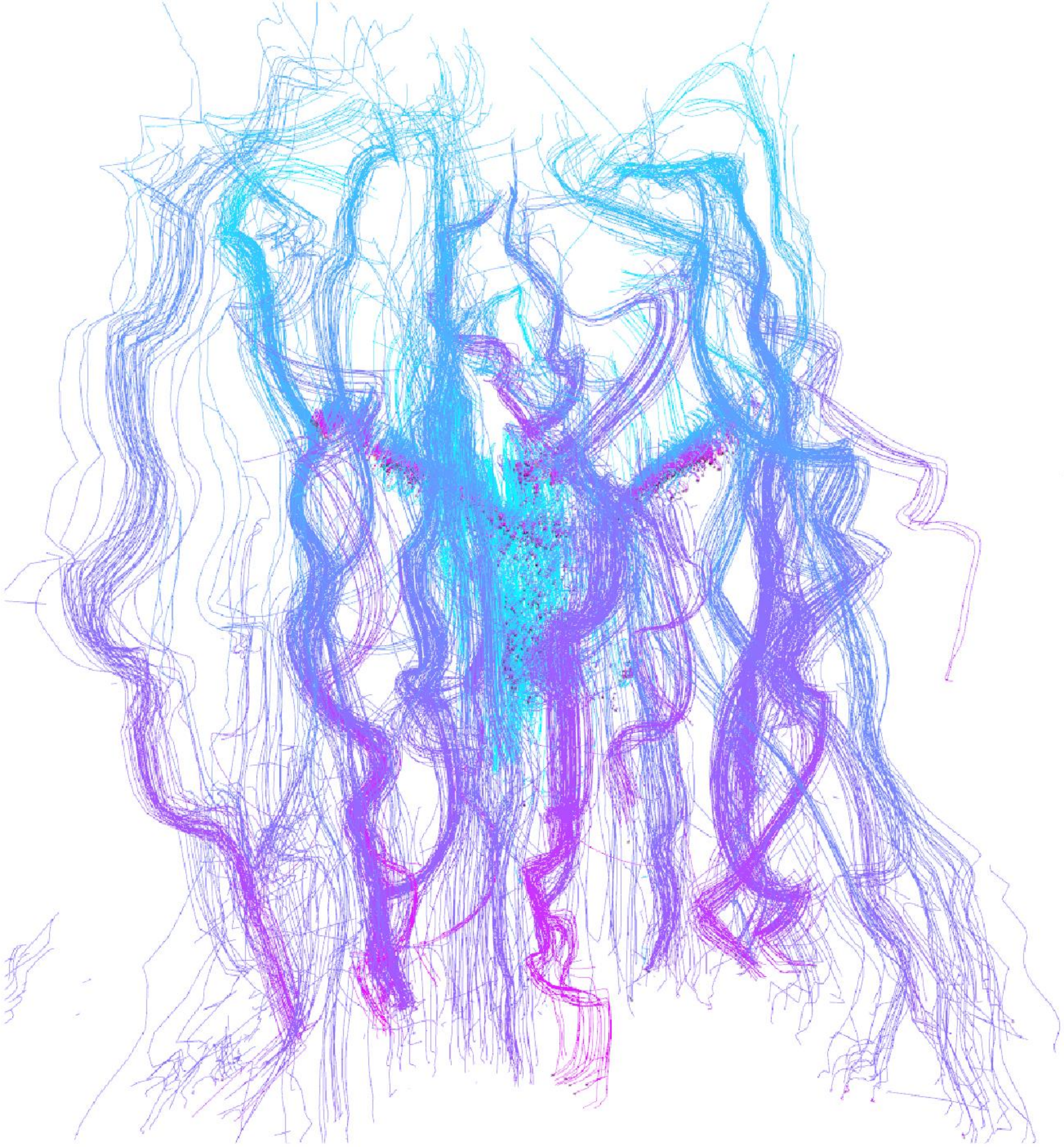
Closed form solutions exist.

$\rightarrow$ Compute $\mathbf{t}$ using $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $d_1$, $d_2$, and $d_3$.

$$\rightarrow \underline{\begin{bmatrix}\tilde{\mathbf{X}}_1 & \tilde{\mathbf{X}}_2 & \tilde{\mathbf{X}}_3\end{bmatrix} = \mathbf{R}\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3\end{bmatrix}}$$

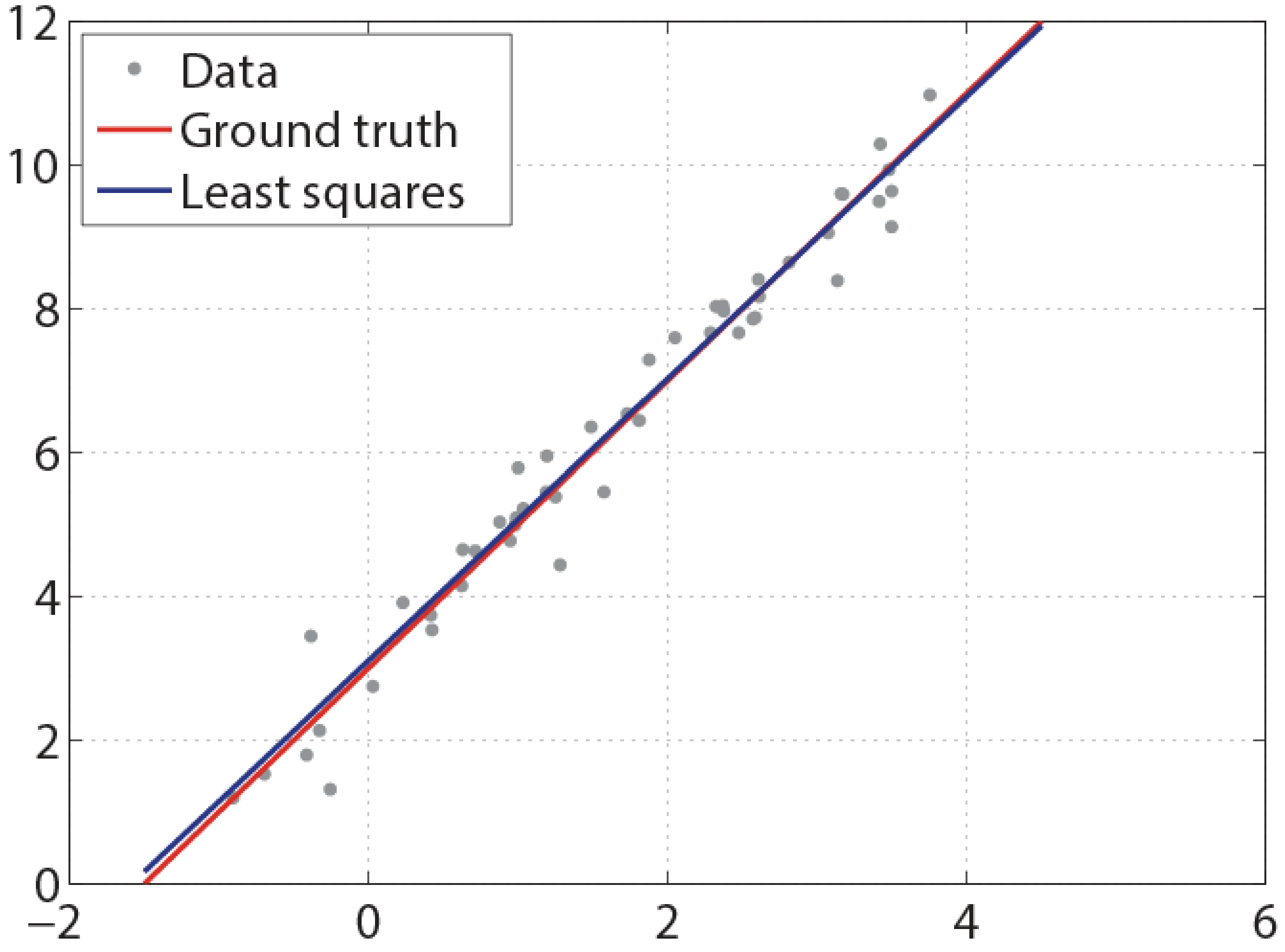
Rotation matrix computation

where $\tilde{\mathbf{X}}_1 = d_1\frac{\mathbf{K}^{-1}\mathbf{u}_1}{\left\|\mathbf{K}^{-1}\mathbf{u}_1\right\|}$ $\tilde{\mathbf{X}}_2 = d_2\frac{\mathbf{K}^{-1}\mathbf{u}_2}{\left\|\mathbf{K}^{-1}\mathbf{u}_2\right\|}$ $\tilde{\mathbf{X}}_3 = d_3\frac{\mathbf{K}^{-1}\mathbf{u}_3}{\left\|\mathbf{K}^{-1}\mathbf{u}_3\right\|}$



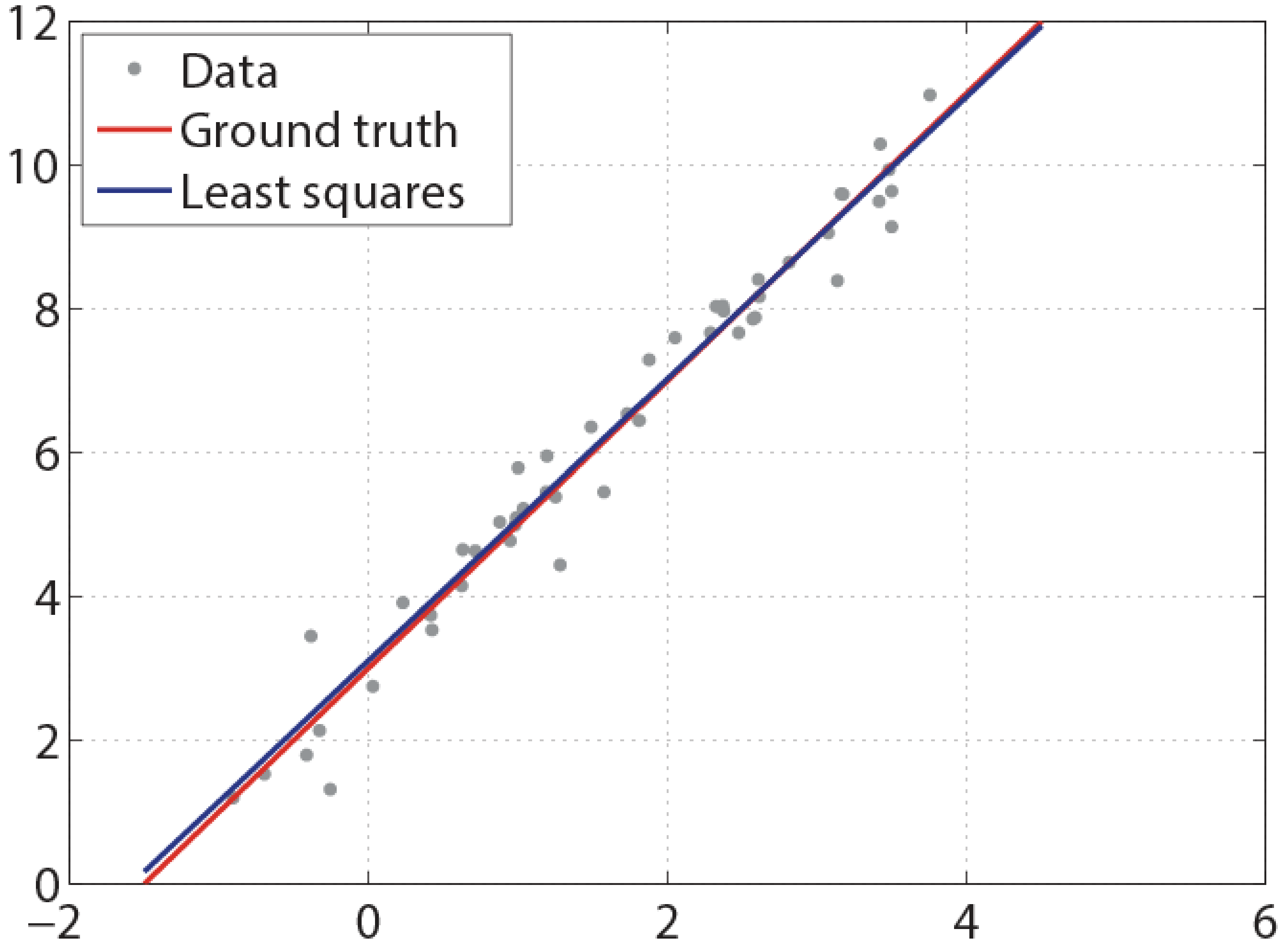
# Nonlinear Estimation

# Recall: Line Fitting (Ax=b)




$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

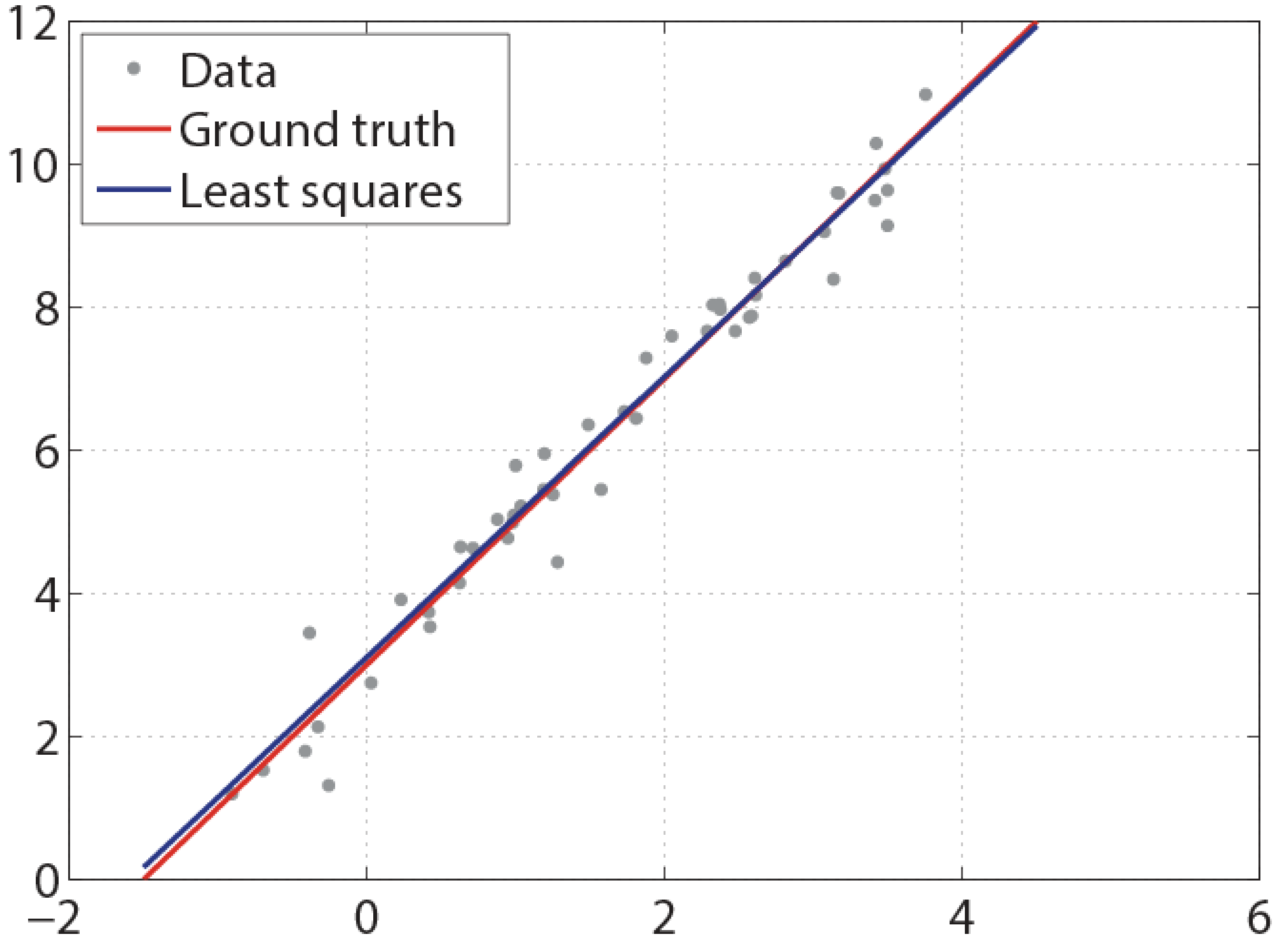
# Recall: Line Fitting (Ax=b)




$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$\mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{A}^T \mathbf{b}$$

# Recall: Line Fitting (Ax=b)



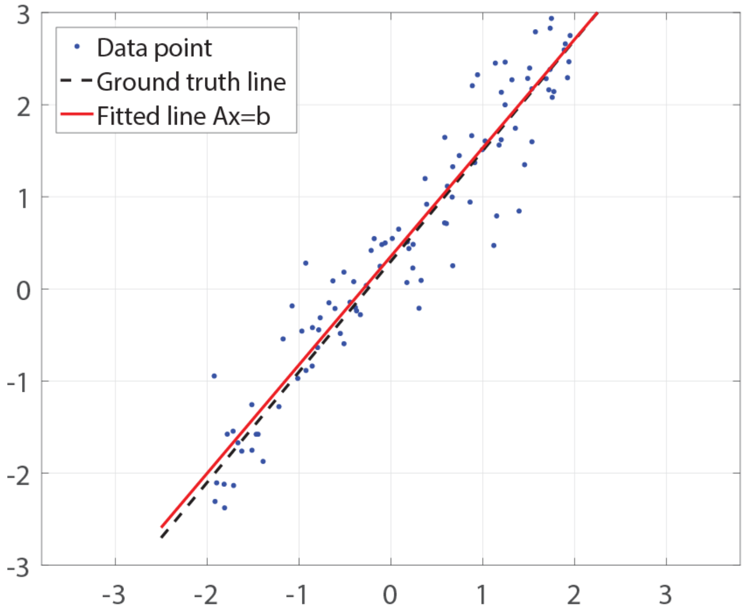

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$\mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{A}^T \mathbf{b}$$

$$\mathbf{x} = \left[ \mathbf{A}^T \mathbf{A} \right]^{-1} \mathbf{A}^T \mathbf{b}$$

# Recall: Line Fitting (Ax=b)

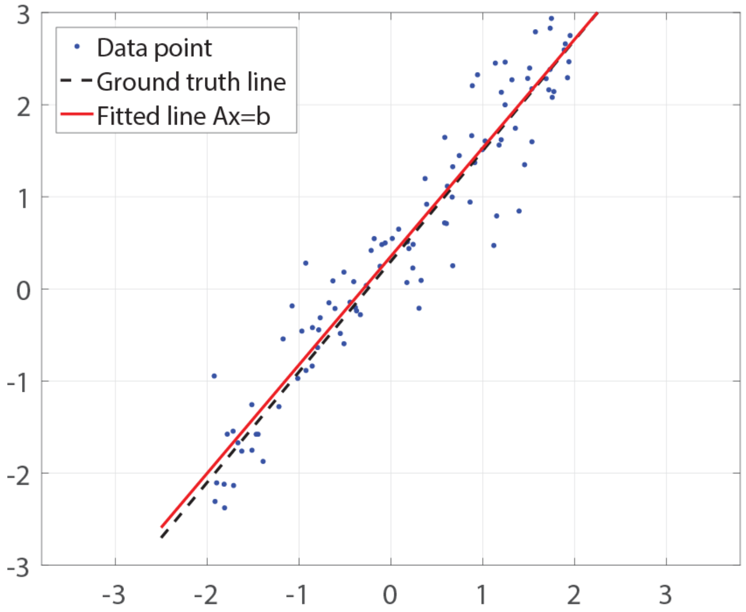
$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

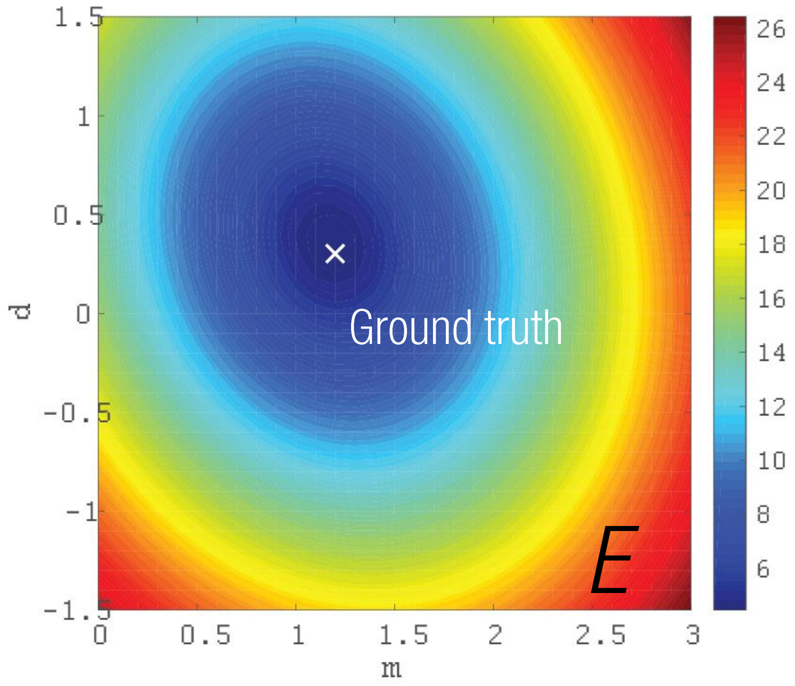



Error:

# Recall: Line Fitting (Ax=b)

$$\mathbf{A}\,\mathbf{x} \approx \mathbf{b}: \quad \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$



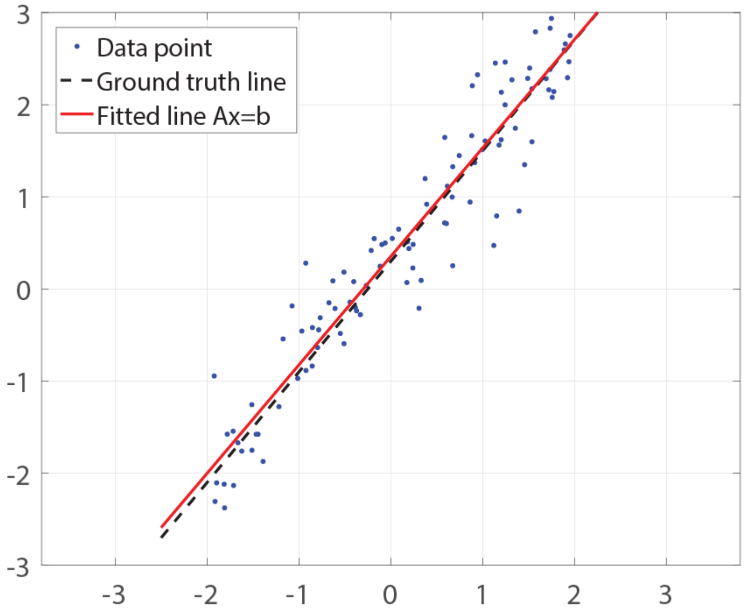




Error:

$$E = \left\| \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|^2$$

# Recall: Line Fitting

Total error: $$E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$$



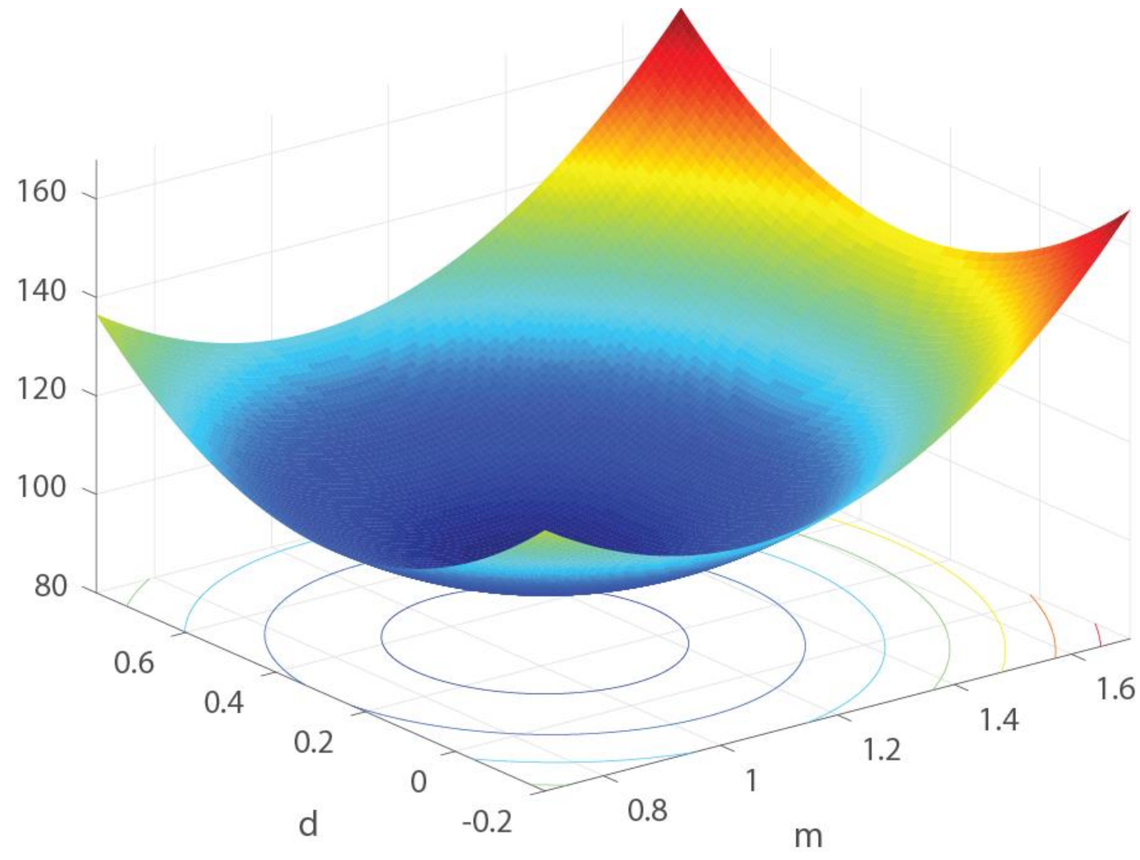

Error:

$$E = \left\| \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|^2$$

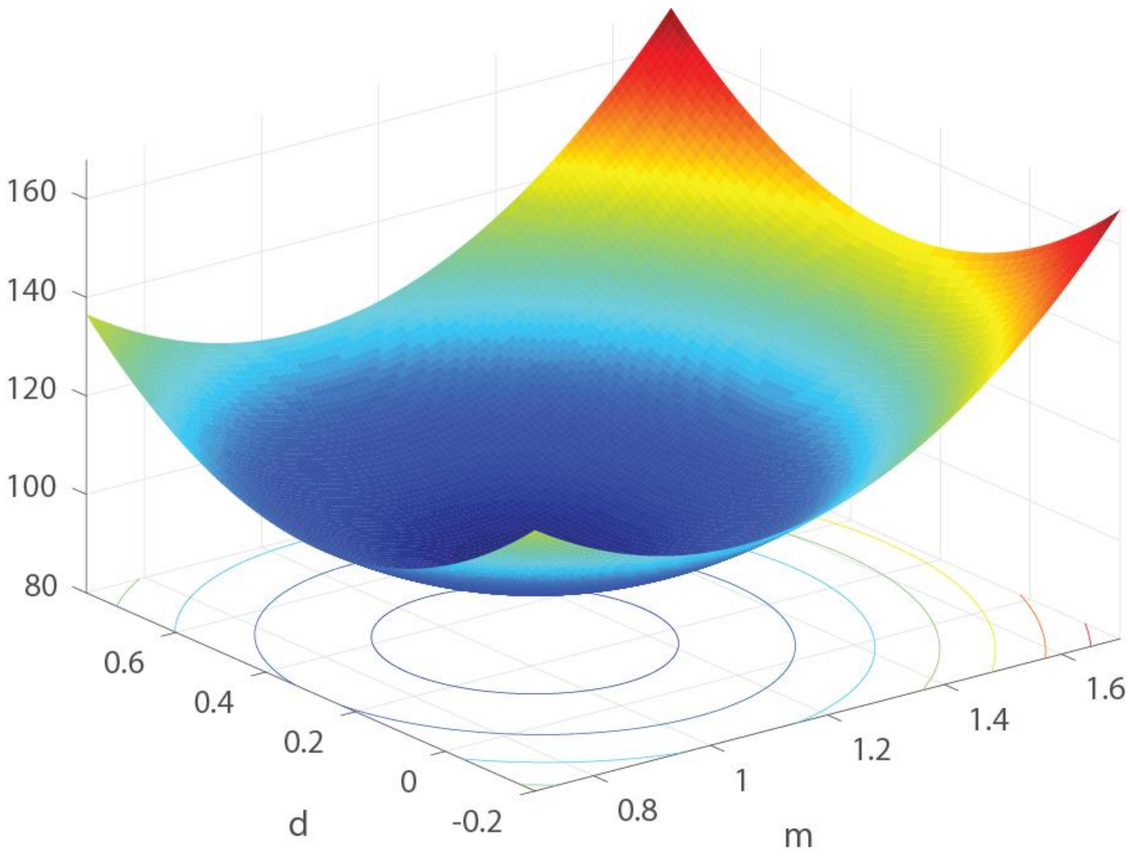
# Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$



# Recall: Line Fitting




Total error: $E=\sum_{i=1}^{n}\left(v_i-(mu_i+d)\right)^2$

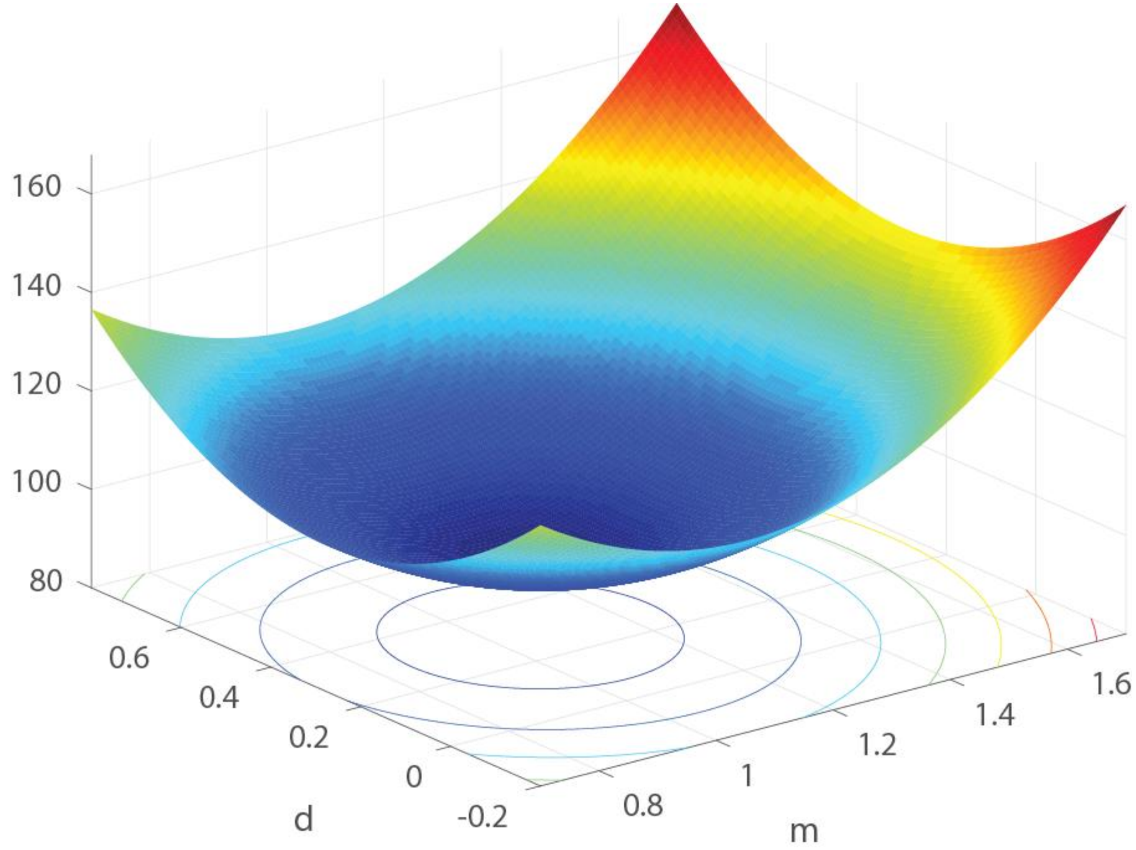
Optimal point: $\frac{\partial E}{\partial m}=0,\quad \frac{\partial E}{\partial d}=0$

$$v_1 \approx mu_1+d \qquad \begin{bmatrix} u_1 & 1 \end{bmatrix}\begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$

$$v_2 \approx mu_2+d \longrightarrow \begin{bmatrix} u_2 & 1 \end{bmatrix}\begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$

$$v_n \approx mu_n+d \qquad \begin{bmatrix} u_n & 1 \end{bmatrix}\begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

# Recall: Line Fitting




Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$v_n \approx mu_n + d$$

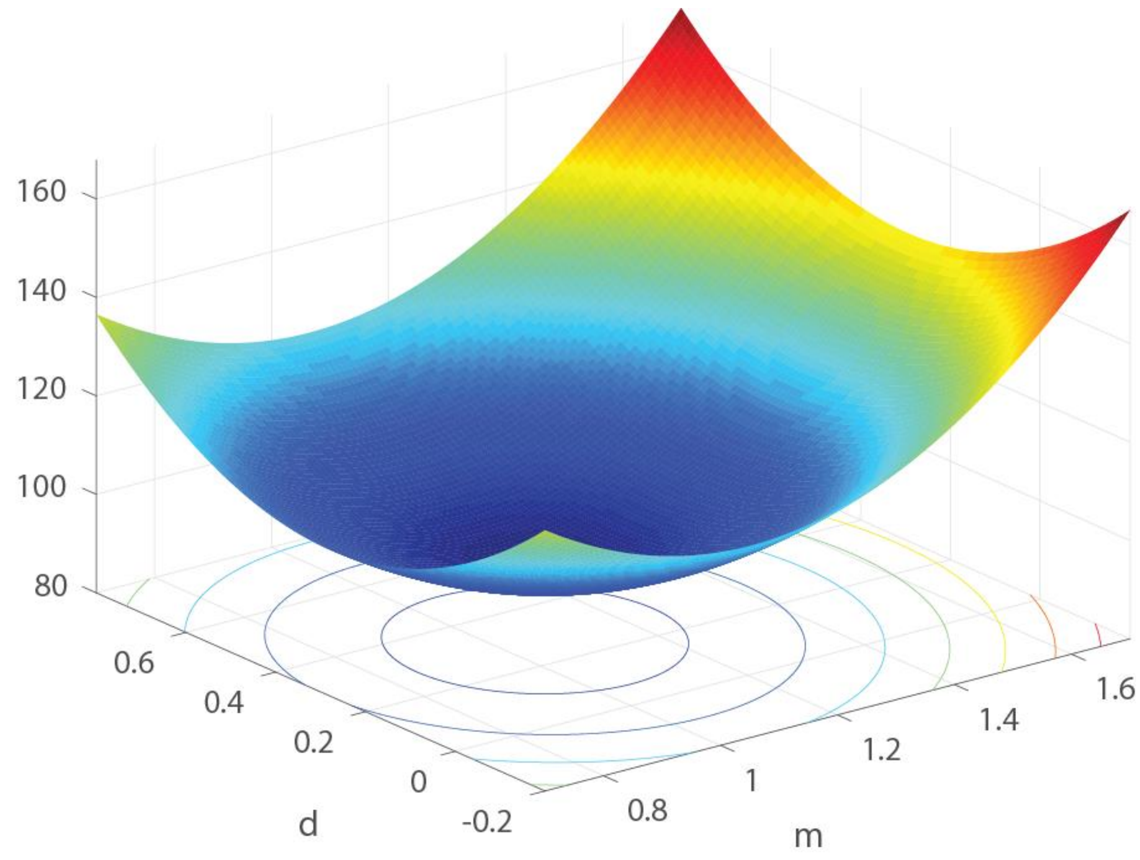
$\longrightarrow$

$$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$
$$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$
$$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

$\longrightarrow$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

# Recall: Line Fitting




Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\dfrac{\partial E}{\partial m} = 0, \quad \dfrac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d \qquad \begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$

$$v_2 \approx mu_2 + d \longrightarrow \begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2 \longrightarrow \underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$v_n \approx mu_n + d \qquad \begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

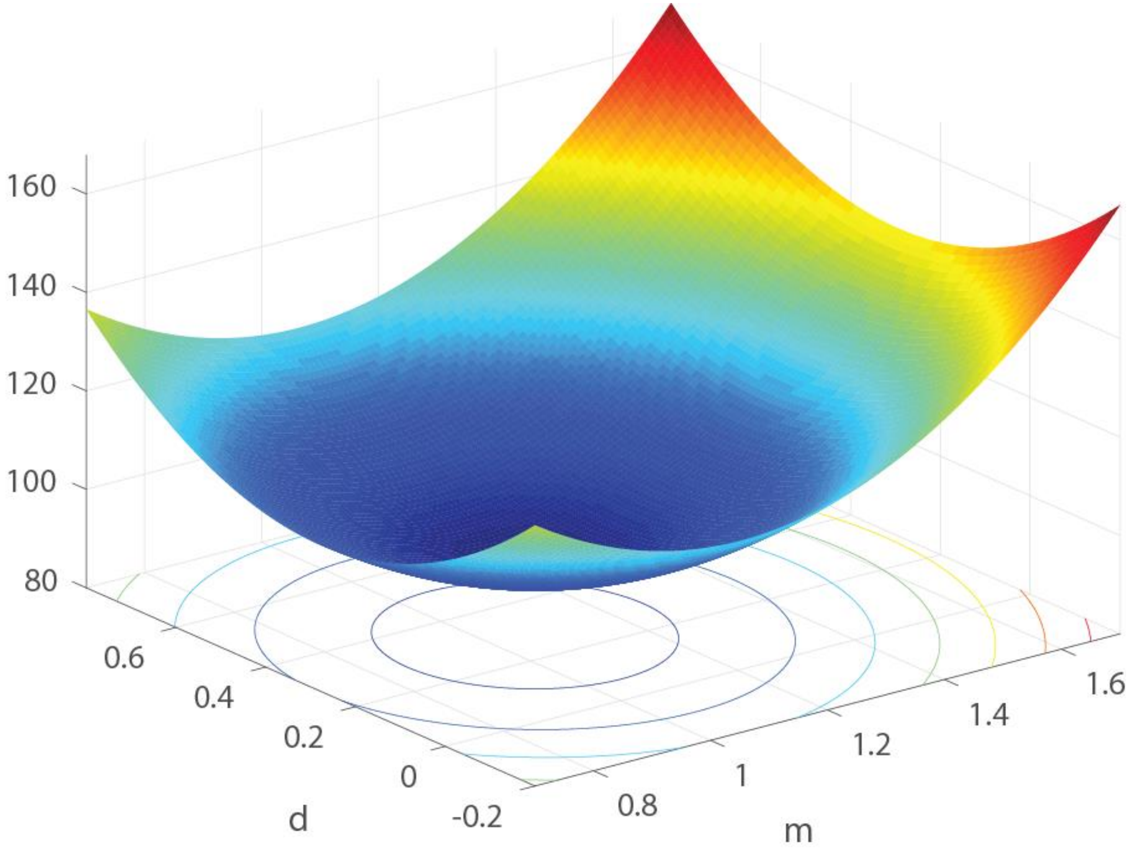
We can't invert **A**.

# Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left( v_i - (mu_i + d) \right)^2$

$$E = \left(\mathbf{Ax} - \mathbf{b}\right)^{\mathrm{T}} \left(\mathbf{Ax} - \mathbf{b}\right) = \left\| \mathbf{Ax} - \mathbf{b} \right\|^2$$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

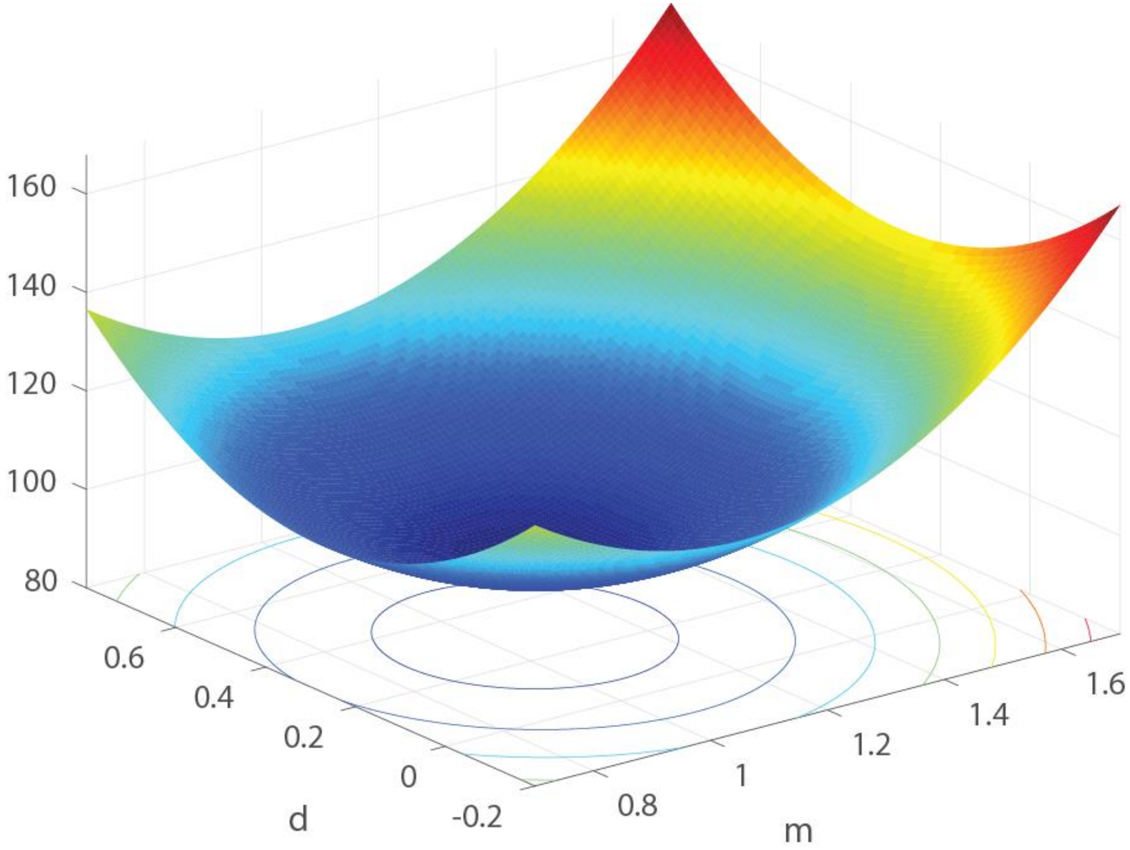


# Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A} \quad \mathbf{x} \approx \mathbf{b}$$




$$E = (\mathbf{Ax} - \mathbf{b})^{\mathrm{T}} (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{b} + \mathbf{b}^{\mathrm{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \quad ?$$

# Recall: Line Fitting

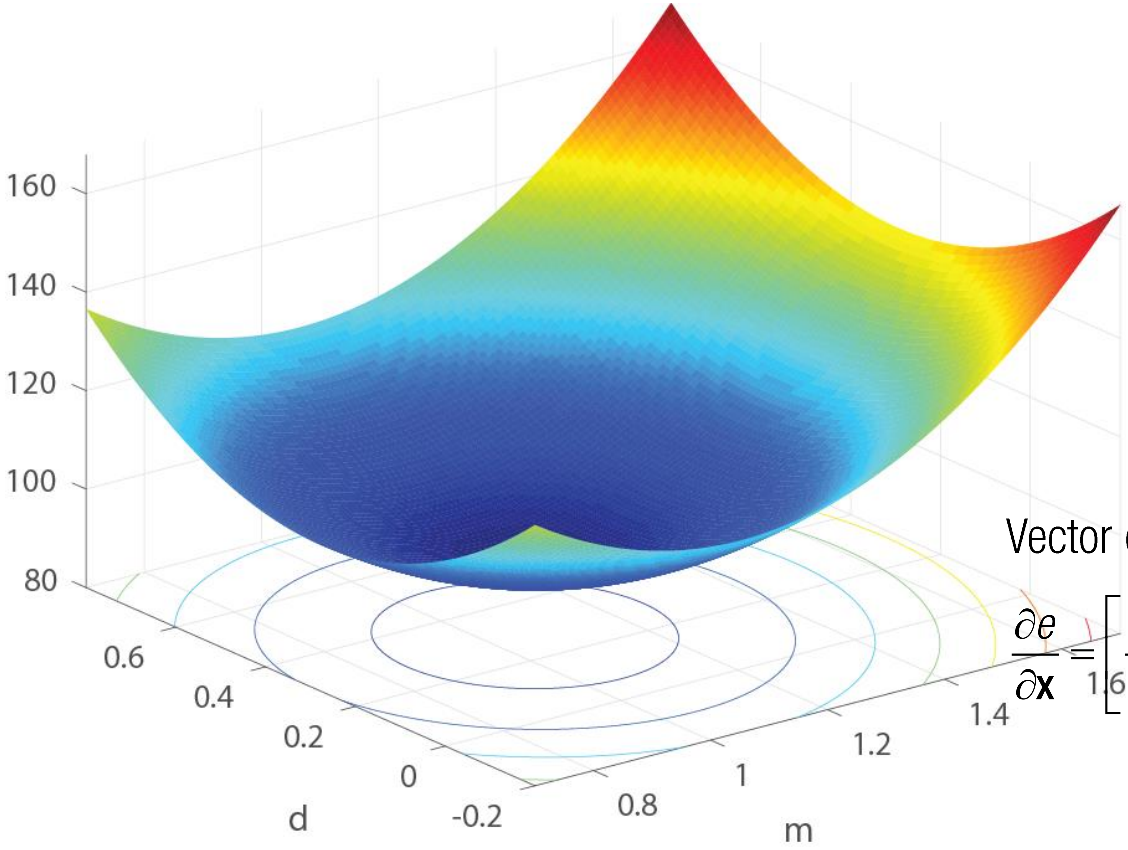
Total error: $E=\sum_{i=1}^{n}\left(v_i-(mu_i+d)\right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$E=\left(\mathbf{Ax}-\mathbf{b}\right)^{\mathrm{T}}\left(\mathbf{Ax}-\mathbf{b}\right)=\left\|\mathbf{Ax}-\mathbf{b}\right\|^2$$

$$=\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax}-2\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{b}+\mathbf{b}^{\mathrm{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}}= \quad ?$$




Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}}=\begin{bmatrix}\frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n}\end{bmatrix}$$

# Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

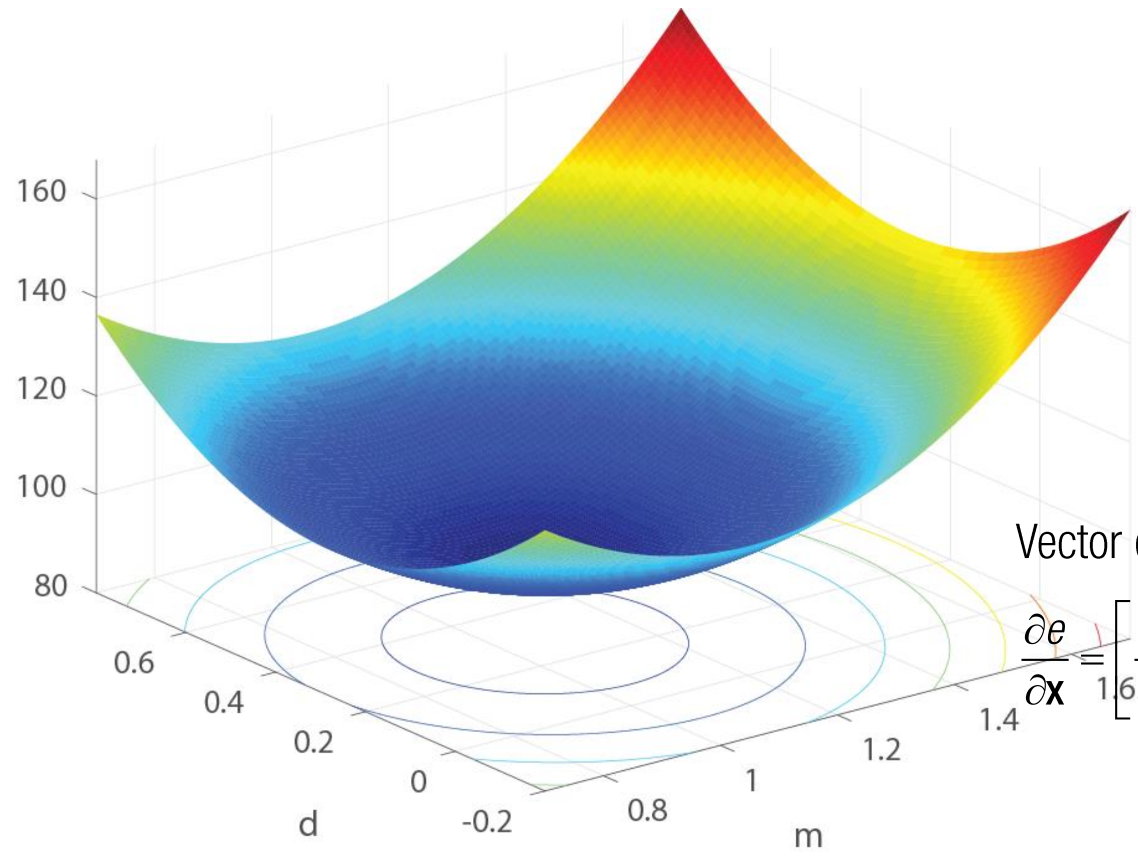
$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A} \quad \mathbf{x} \approx \mathbf{b}$$

$$E = \left(\mathbf{Ax} - \mathbf{b}\right)^{\mathsf{T}} \left(\mathbf{Ax} - \mathbf{b}\right) = \left\|\mathbf{Ax} - \mathbf{b}\right\|^2$$

$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \quad ?$$




Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

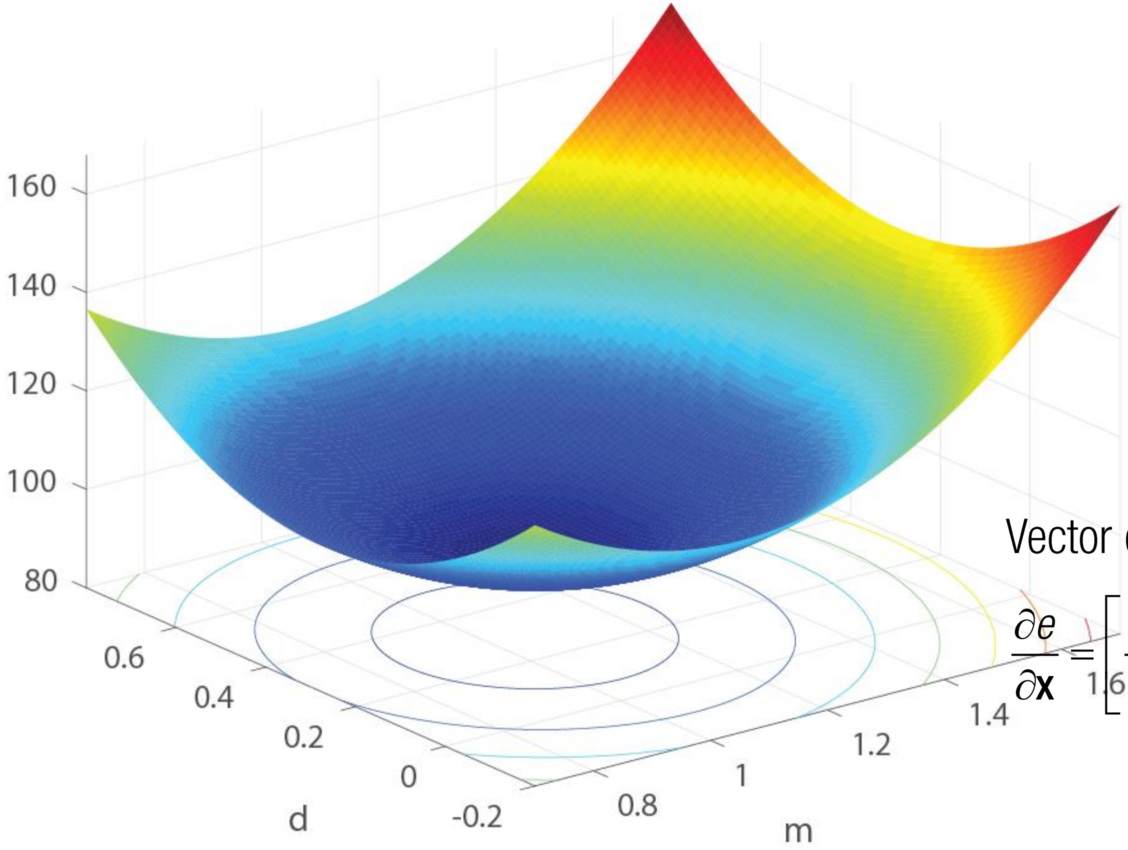
Ex) $e = \mathbf{c}^{\mathsf{T}}\mathbf{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^{\mathsf{T}}\mathbf{x}}{\partial \mathbf{x}} =$$

# Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left( v_i - (mu_i + d) \right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$




$$E = (\mathbf{Ax} - \mathbf{b})^{\mathsf{T}} (\mathbf{Ax} - \mathbf{b}) = \left\| \mathbf{Ax} - \mathbf{b} \right\|^2$$

$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \qquad ?$$

Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial e}{\partial x_1} & \cdots & \dfrac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathbf{c}^{\mathsf{T}}\mathbf{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^{\mathsf{T}}\mathbf{x}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}} \left( c_1 x_1 + \cdots + c_n x_n \right)$$

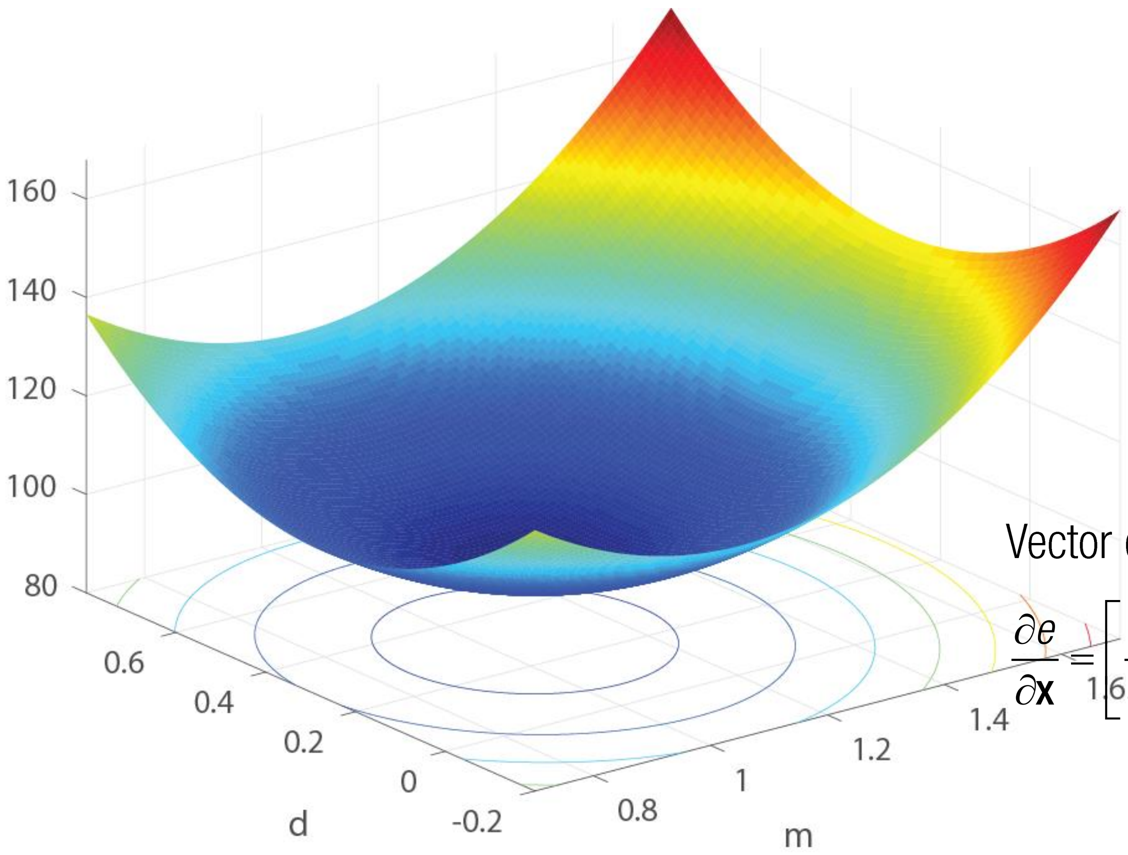
# Recall: Line Fitting

Total error: $E=\sum_{i=1}^{n}\left(v_i-(mu_i+d)\right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$E=\left(\mathbf{Ax}-\mathbf{b}\right)^{\mathsf{T}}\left(\mathbf{Ax}-\mathbf{b}\right)=\left\|\mathbf{Ax}-\mathbf{b}\right\|^2$$
$$=\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax}-2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b}+\mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}}= \quad ?$$




Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}}=\begin{bmatrix}\frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n}\end{bmatrix}$$

Ex) $e=\mathbf{c}^{\mathsf{T}}\mathbf{x}=\begin{bmatrix}c_1 & \cdots & c_n\end{bmatrix}\begin{bmatrix}x_1 \\ \vdots \\ x_n\end{bmatrix}$

$$\frac{\partial \mathbf{c}^{\mathsf{T}}\mathbf{x}}{\partial \mathbf{x}}=\frac{\partial}{\partial \mathbf{x}}\left(c_1x_1+\cdots+c_nx_n\right)$$
$$=\begin{bmatrix}c_1 & \cdots & c_n\end{bmatrix}$$

# Recall: Line Fitting

Total error: $E=\sum_{i=1}^{n}\left(v_i-(mu_i+d)\right)^2$

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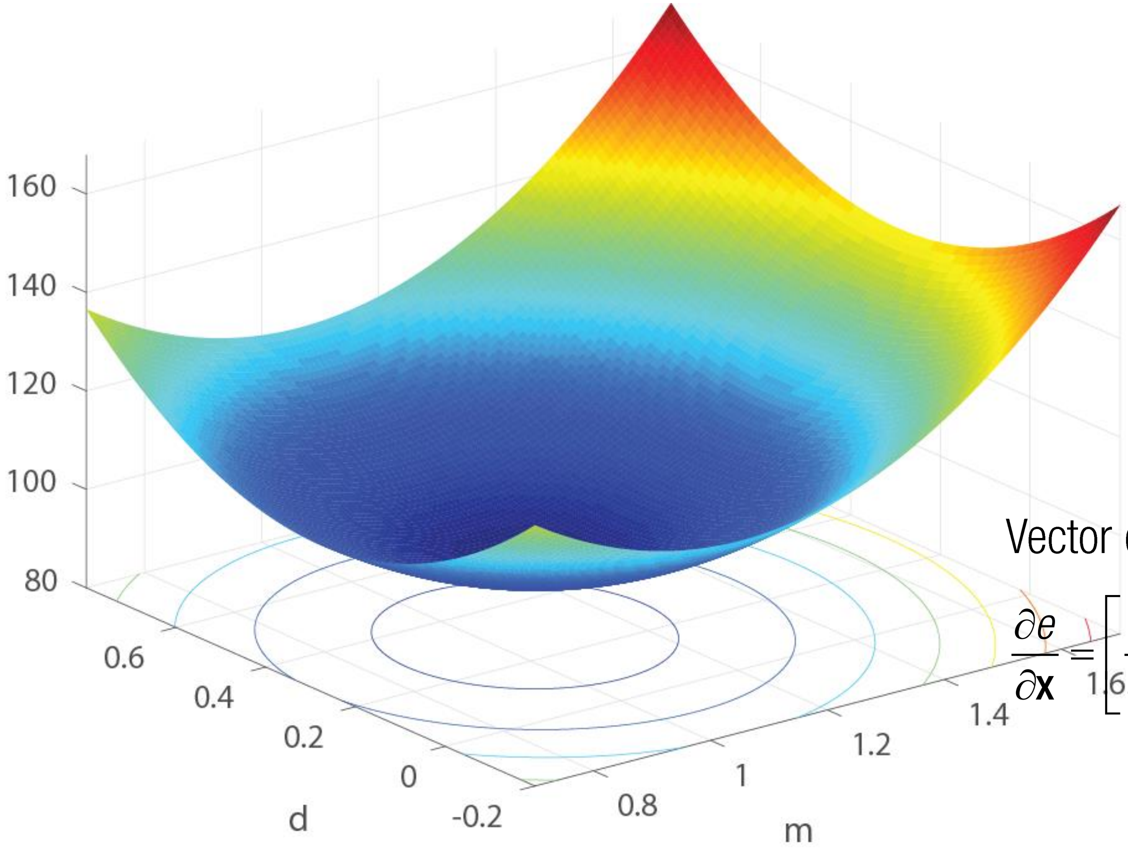
$$E=(\mathbf{Ax}-\mathbf{b})^{\mathsf{T}}(\mathbf{Ax}-\mathbf{b})=\|\mathbf{Ax}-\mathbf{b}\|^2$$
$$=\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax}-2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b}+\mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}}=2\mathbf{A}^{\mathsf{T}}\mathbf{Ax}-2\mathbf{A}^{\mathsf{T}}\mathbf{b}=\mathbf{0}$$




Vector derivative:

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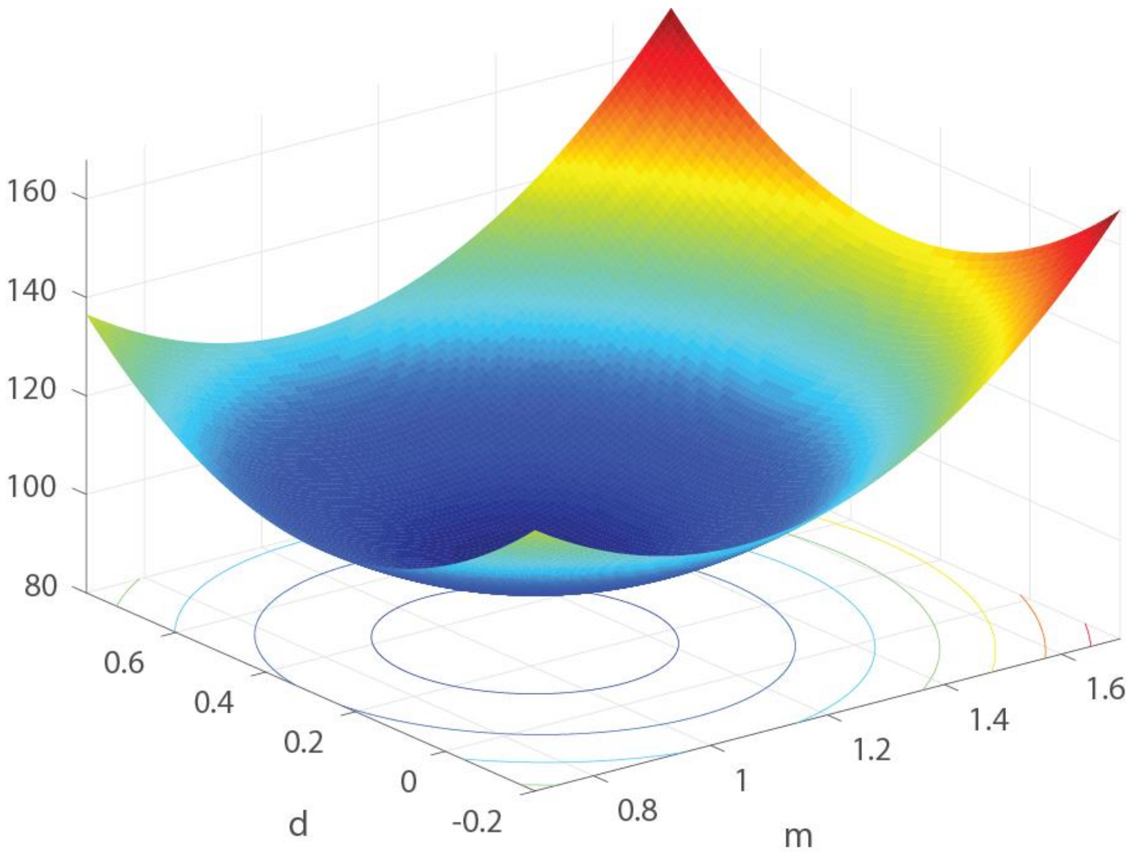
$$=\begin{bmatrix}c_1 & \cdots & c_n\end{bmatrix}$$

# Recall: Line Fitting




Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

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$$\mathbf{A} \quad \mathbf{x} \approx \mathbf{b}$$

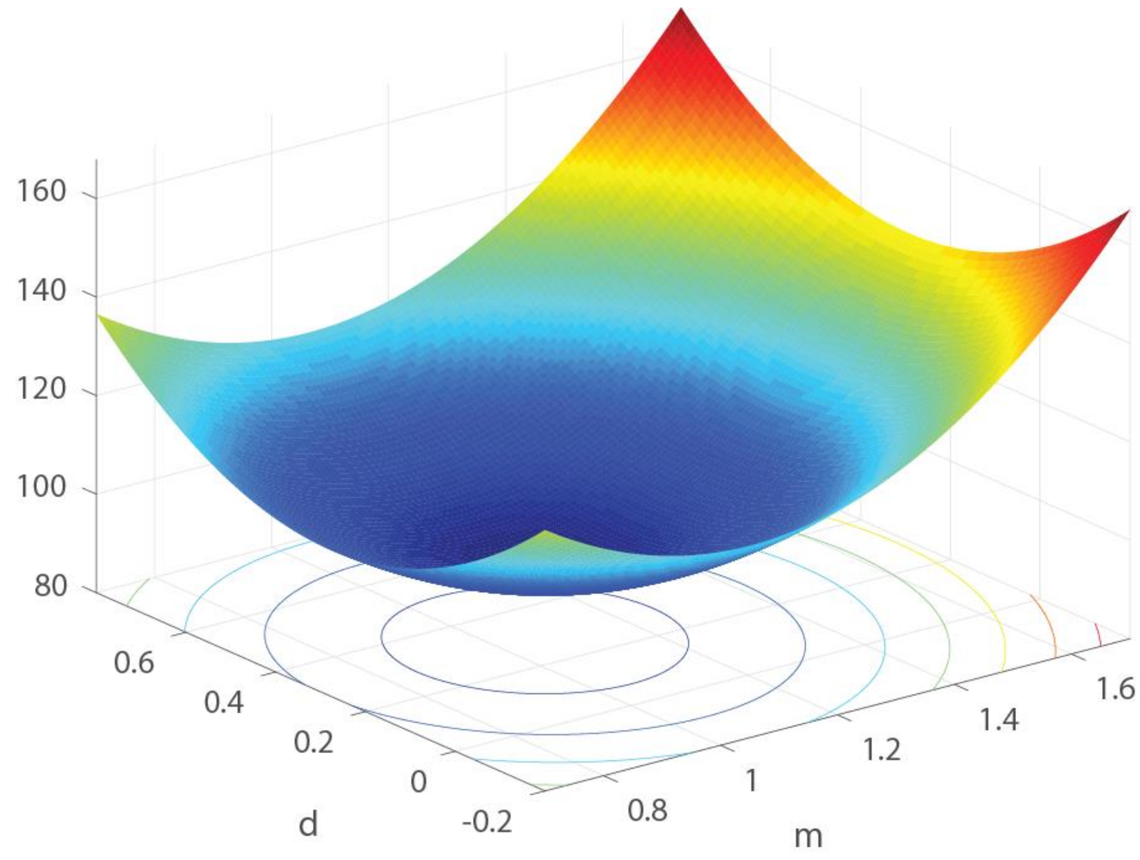
$$E = (\mathbf{Ax} - \mathbf{b})^{\mathsf{T}} (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^{\mathsf{T}} \mathbf{A}^{\mathsf{T}} \mathbf{A} \mathbf{x} - 2\mathbf{x}^{\mathsf{T}} \mathbf{A}^{\mathsf{T}} \mathbf{b} + \mathbf{b}^{\mathsf{T}} \mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^{\mathsf{T}} \mathbf{A} \mathbf{x} - 2\mathbf{A}^{\mathsf{T}} \mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^{\mathsf{T}} \mathbf{A} \mathbf{x} = \mathbf{A}^{\mathsf{T}} \mathbf{b}$$

# Recall: Line Fitting

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$$E = (\mathbf{Ax} - \mathbf{b})^{\mathrm{T}} (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{b} + \mathbf{b}^{\mathrm{T}}\mathbf{b}$$

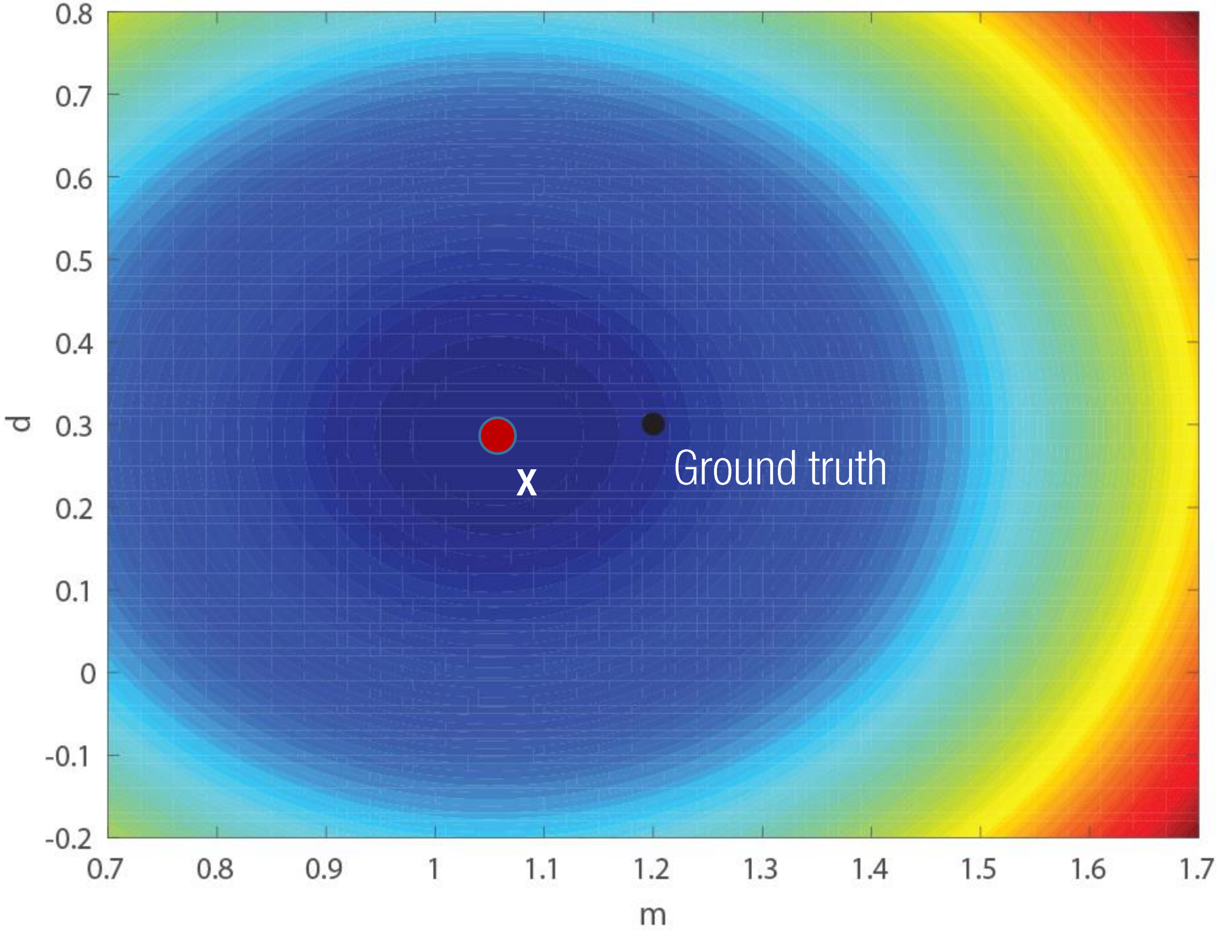
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{A}^{\mathrm{T}}\mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

$$\mathbf{A}^{\mathrm{T}} \; \mathbf{A} \; \mathbf{x} = \mathbf{A}^{\mathrm{T}} \; \mathbf{b}$$

Normal equation

# Recall: Line Fitting




Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

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$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

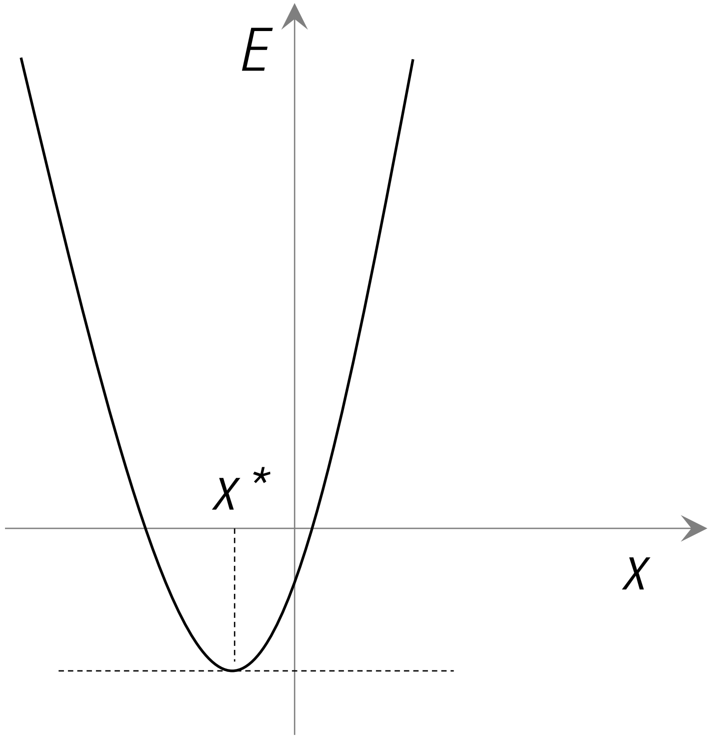
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{A}^{\mathsf{T}}\mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^{\mathsf{T}}\mathbf{Ax} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$$

$$\mathbf{A}^{\mathsf{T}} \; \mathbf{A} \; \mathbf{x} = \mathbf{A}^{\mathsf{T}} \; \mathbf{b}$$

$$\mathbf{x} = \left[\mathbf{A}^{\mathsf{T}} \; \mathbf{A}\right]^{-1} \mathbf{A}^{\mathsf{T}} \; \mathbf{b}$$

# Linear System Recap




$$\left.\frac{dE}{dx}\right|_{x=x^*} = 0$$

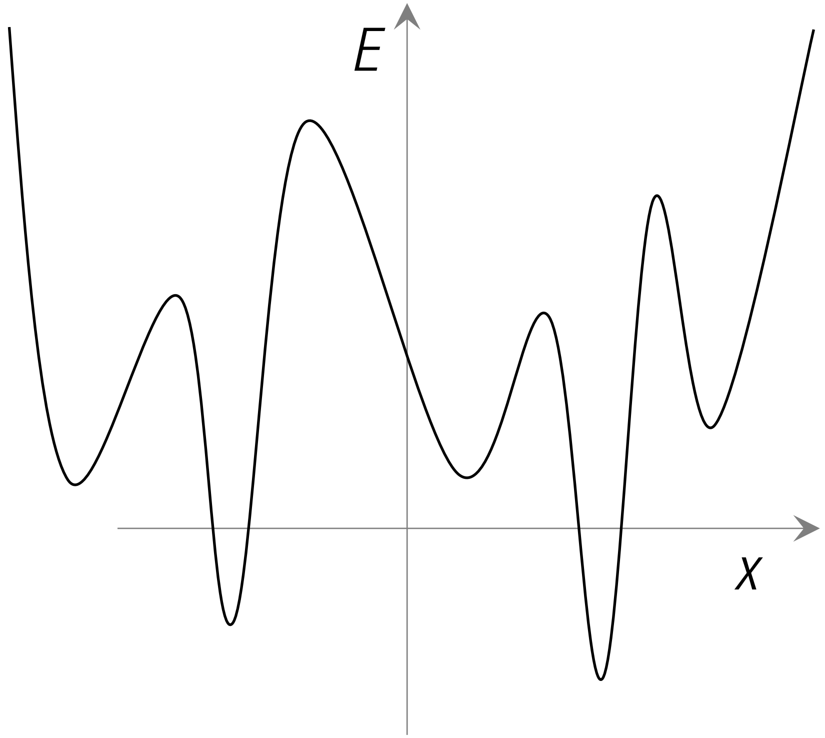
$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$\mathbf{x} = \left[\mathbf{A}^T \mathbf{A}\right]^{-1} \mathbf{A}^T \mathbf{b}$$

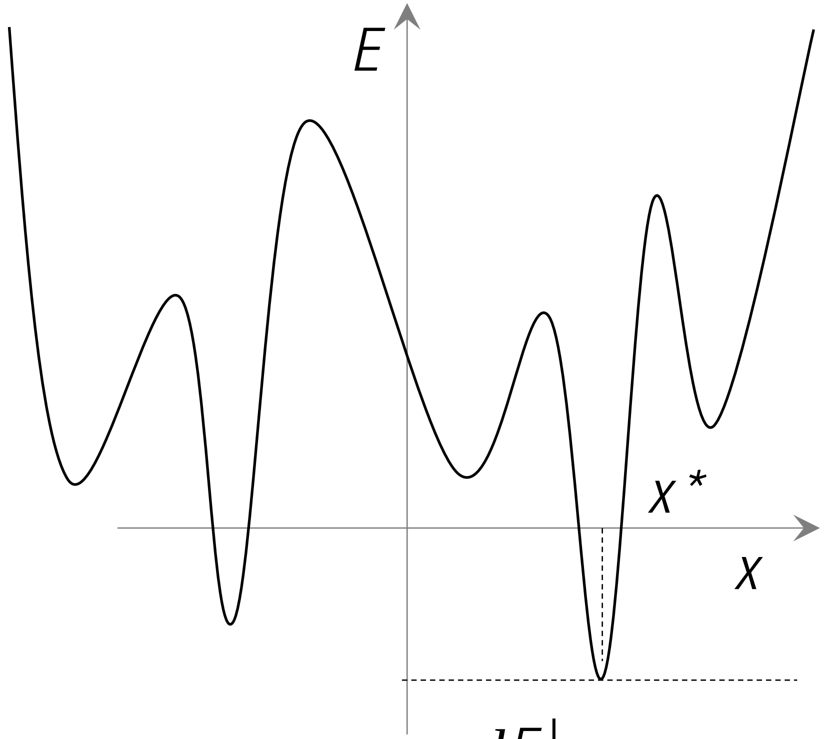
- Has the global solution
- Has the closed form solution (non-iterative solve)
- Is solved efficiently ($O(n^2)$)
- Does not require an initialization

# Nonlinear System

$$f(\mathbf{x}) = \mathbf{b}$$



# Nonlinear System




$$\left.\frac{dE}{dx}\right|_{x=x^*} = 0$$

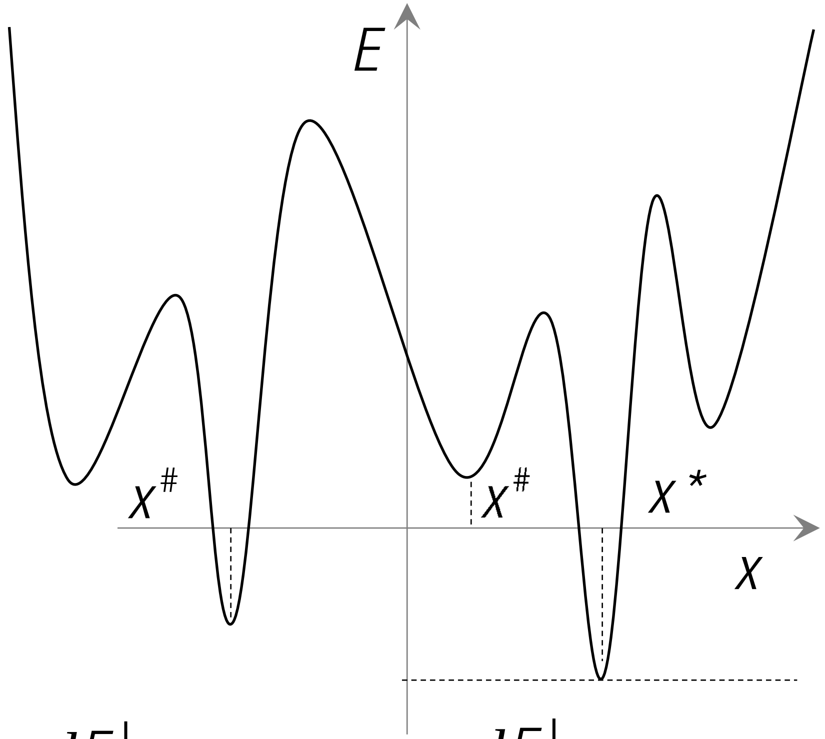
$$f(\mathbf{x}) = \mathbf{b}$$

$$E = \left\| f(\mathbf{x}) - \mathbf{b} \right\|^2$$

# Nonlinear System

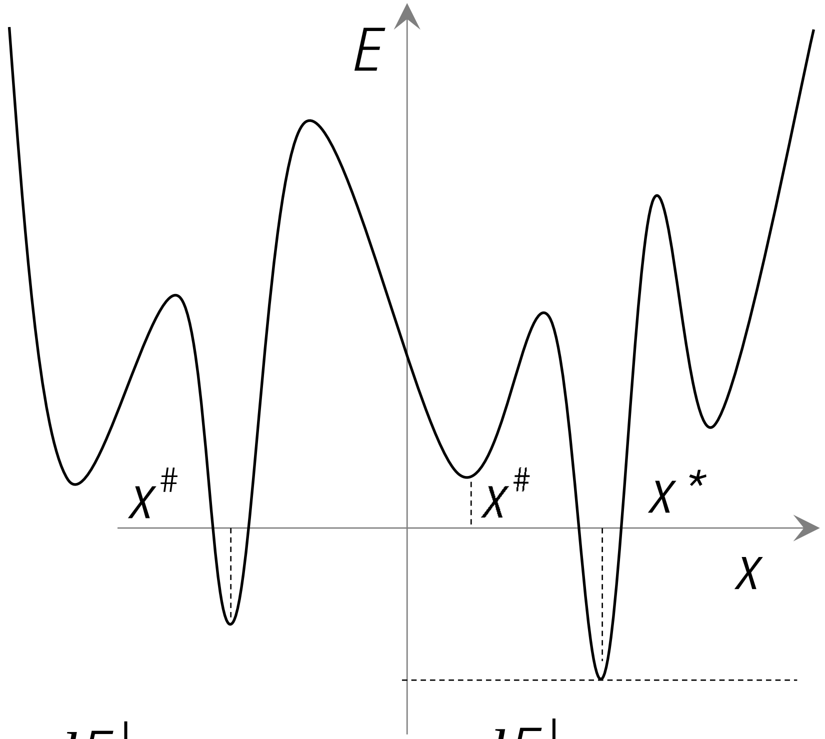
$$f(\mathbf{x}) = \mathbf{b}$$

$$E = \left\| f(\mathbf{x}) - \mathbf{b} \right\|^2$$




$$\left.\frac{dE}{dx}\right|_{x=x^{\#}} = 0 \qquad \left.\frac{dE}{dx}\right|_{x=x^{*}} = 0$$

# Nonlinear System



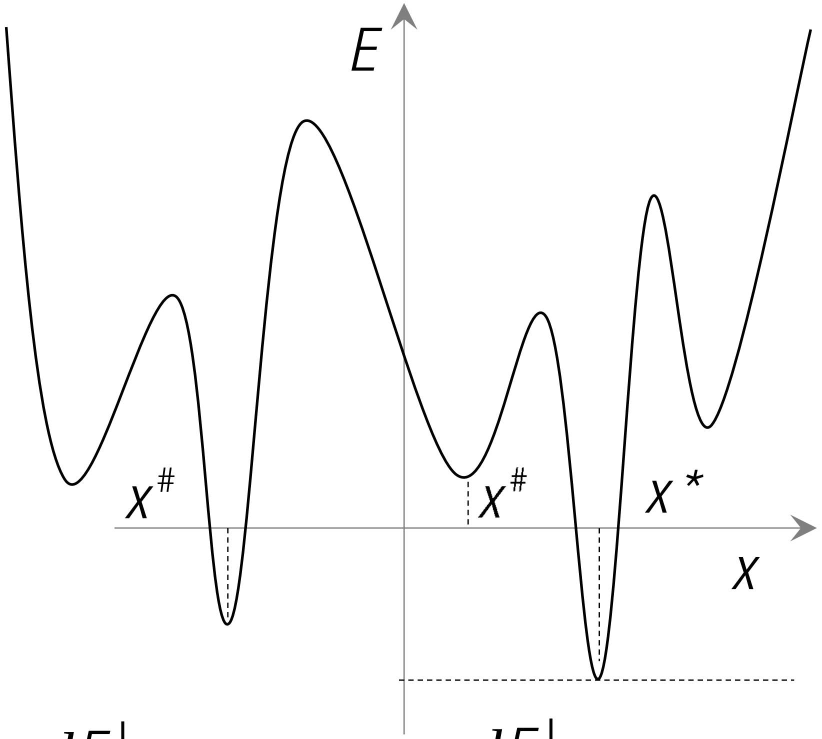

$$\left.\frac{dE}{dx}\right|_{x=x^{\#}}=0 \qquad \left.\frac{dE}{dx}\right|_{x=x^{*}}=0$$

$$f(\mathbf{x})=\mathbf{b}$$

$$E=\left\|f(\mathbf{x})-\mathbf{b}\right\|^{2}$$

$$\begin{aligned}E&=\left(f(\mathbf{x})-\mathbf{b}\right)^{\mathrm{T}}\left(f(\mathbf{x})-\mathbf{b}\right)\\&=f(\mathbf{x})^{\mathrm{T}}f(\mathbf{x})-2f(\mathbf{x})^{\mathrm{T}}\mathbf{b}+\mathbf{b}^{\mathrm{T}}\mathbf{b}\end{aligned}$$

# Nonlinear System

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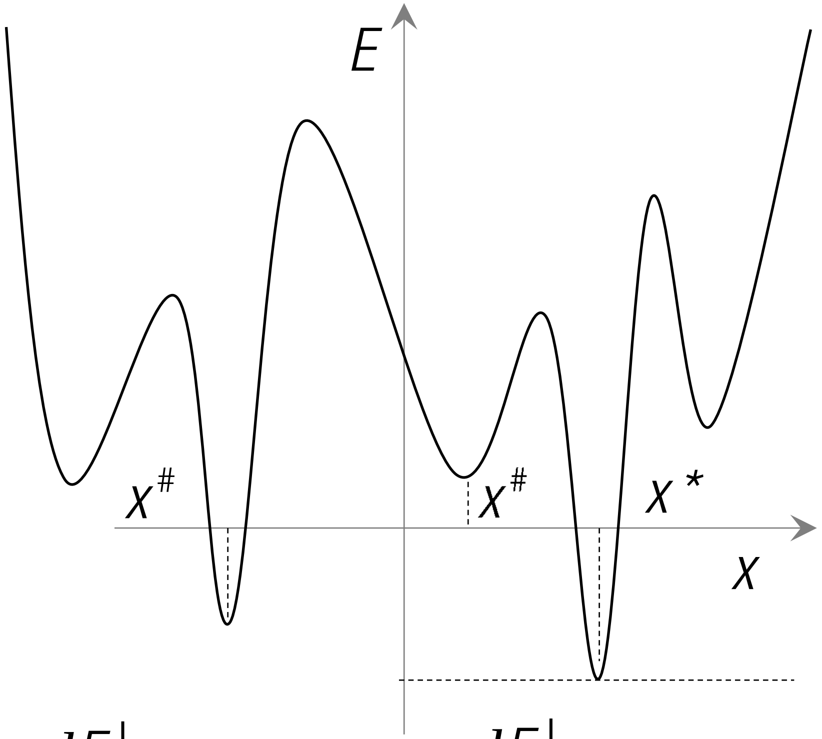
$$\frac{\partial E}{\partial \mathbf{x}}=2\frac{\partial f(\mathbf{x})^{\mathsf{T}}}{\partial \mathbf{x}}f(\mathbf{x})-2\frac{\partial f(\mathbf{x})^{\mathsf{T}}}{\partial \mathbf{x}}\mathbf{b}=\mathbf{0}$$

# Nonlinear System

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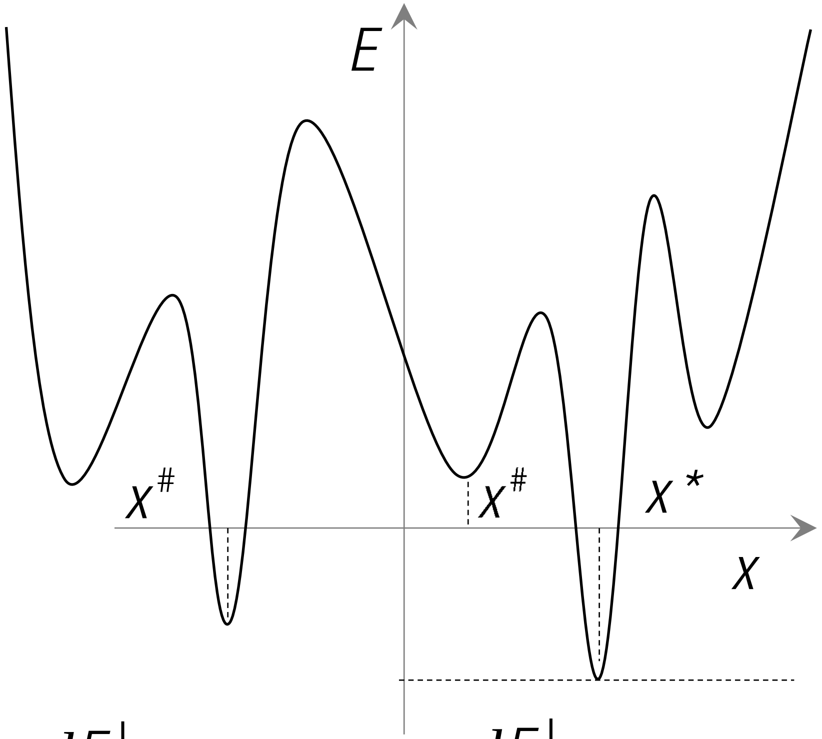
$$\frac{\partial E}{\partial \mathbf{x}}=2\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}}f(\mathbf{x})-2\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}}\mathbf{b}=\mathbf{0}$$

where $\dfrac{\partial f(\mathbf{x})}{\partial \mathbf{x}}=\begin{bmatrix}\dfrac{\partial f_1}{\partial \mathbf{x}_1} & \cdots & \dfrac{\partial f_1}{\partial \mathbf{x}_n}\\ \vdots & \ddots & \vdots\\ \dfrac{\partial f_1}{\partial \mathbf{x}_n} & \cdots & \dfrac{\partial f_m}{\partial \mathbf{x}_n}\end{bmatrix}$ : Jacobian

# Nonlinear System




$$\left.\frac{dE}{dx}\right|_{x=x^{\#}} = 0 \qquad \left.\frac{dE}{dx}\right|_{x=x^{*}} = 0$$

$$f(\mathbf{x}) = \mathbf{b}$$

$$E = \left\| f(\mathbf{x}) - \mathbf{b} \right\|^2$$

$$E = \left(f(\mathbf{x}) - \mathbf{b}\right)^{\mathrm{T}} \left(f(\mathbf{x}) - \mathbf{b}\right)$$
$$= f(\mathbf{x})^{\mathrm{T}} f(\mathbf{x}) - 2f(\mathbf{x})^{\mathrm{T}}\mathbf{b} + \mathbf{b}^{\mathrm{T}}\mathbf{b}$$

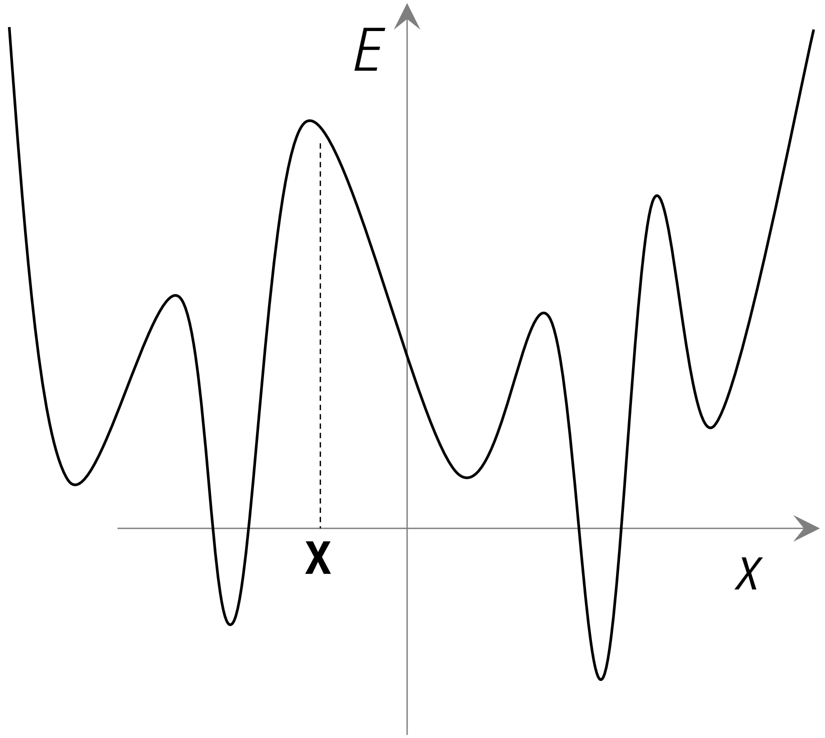
$$\frac{\partial E}{\partial \mathbf{x}} = 2\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}} f(\mathbf{x}) - 2\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}}\mathbf{b} = \mathbf{0}$$

Find **x** such that the following equation is satisfied:

$$\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}} f(\mathbf{x}) = \frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}}\mathbf{b}$$

How?

# Nonlinear System



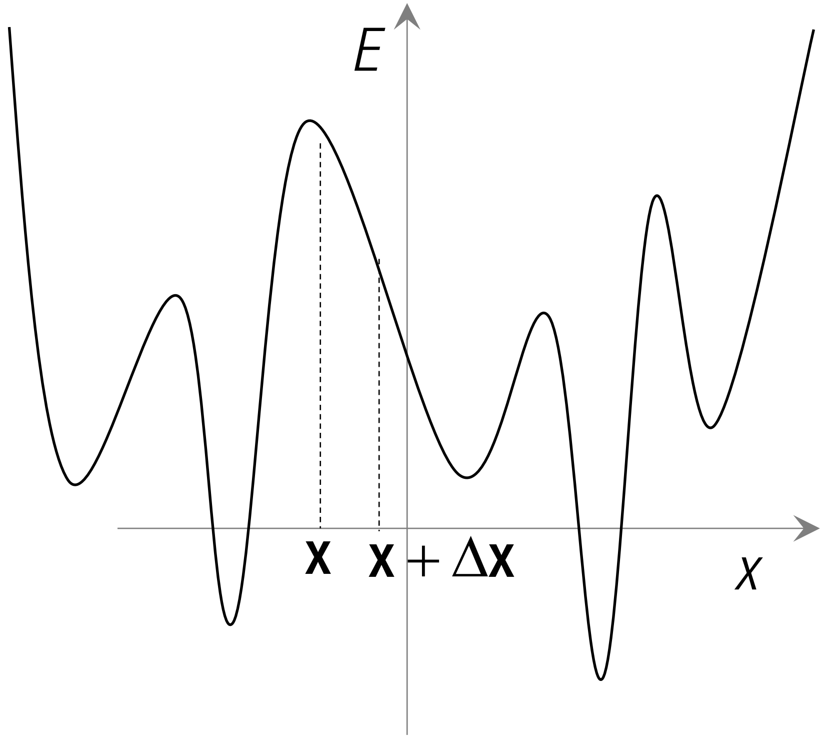

Find **x** such that the following equation is satisfied:

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} f(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{b}$$

How?

Strategy: Given **x**,

# Nonlinear System



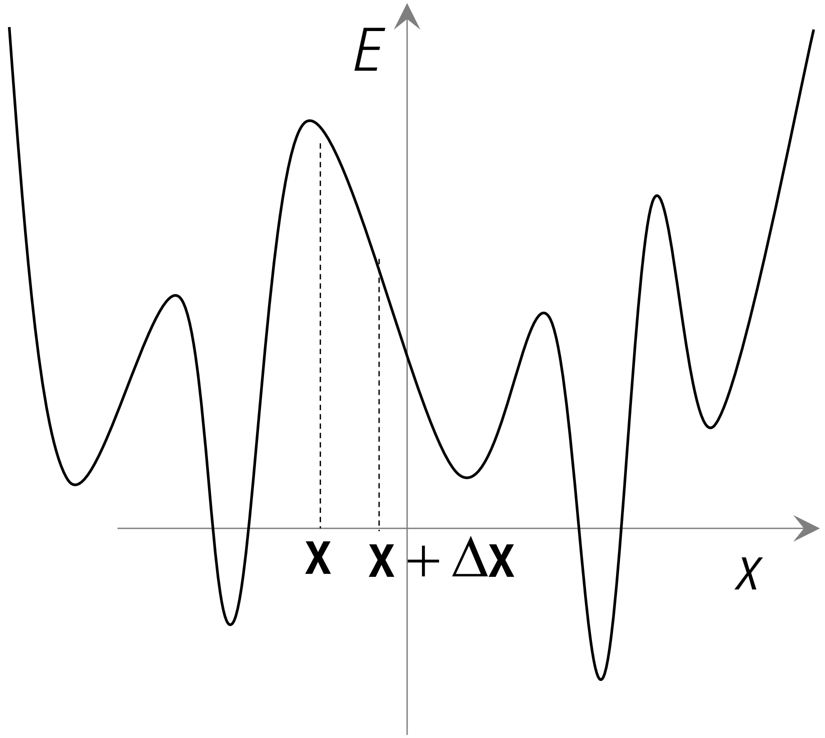

Find $\mathbf{x}$ such that the following equation is satisfied:

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} f(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{b}$$

How?

Strategy: Given $\mathbf{x}$, move $\Delta\mathbf{x}$ such that $E(\mathbf{x}+\Delta\mathbf{x}) \leq E(\mathbf{x})$

# Nonlinear System

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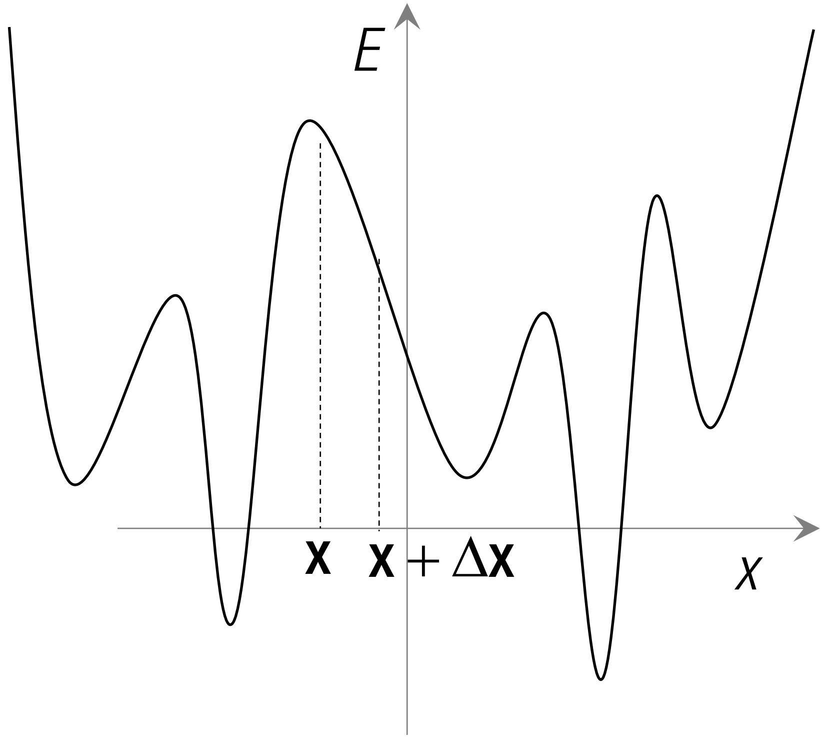
How?

Strategy: Given **x,** move $\Delta\mathbf{x}$ such that $E(\mathbf{x}+\Delta\mathbf{x}) \leq E(\mathbf{x})$

Taylor expansion:

$$f(\mathbf{x}+\Delta\mathbf{x}) =$$

# Nonlinear System

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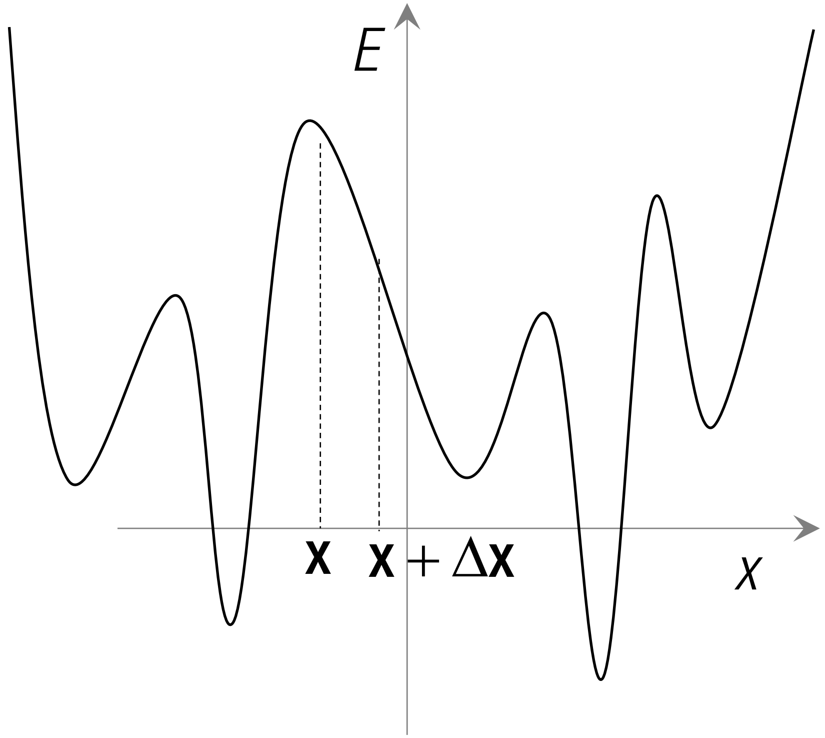
Taylor expansion:

$$f(\mathbf{x}+\Delta\mathbf{x}) = f(\mathbf{x}+\Delta\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} + \mathrm{H.O.T.}$$

# Nonlinear System

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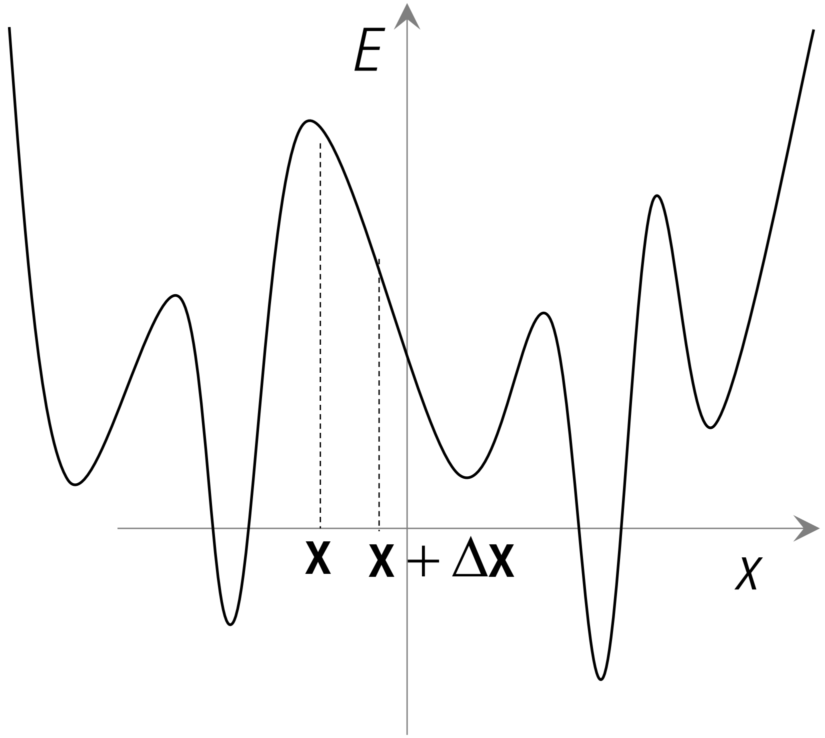
$$\rightarrow \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left( f(\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} \right) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \mathbf{b}$$

# Nonlinear System

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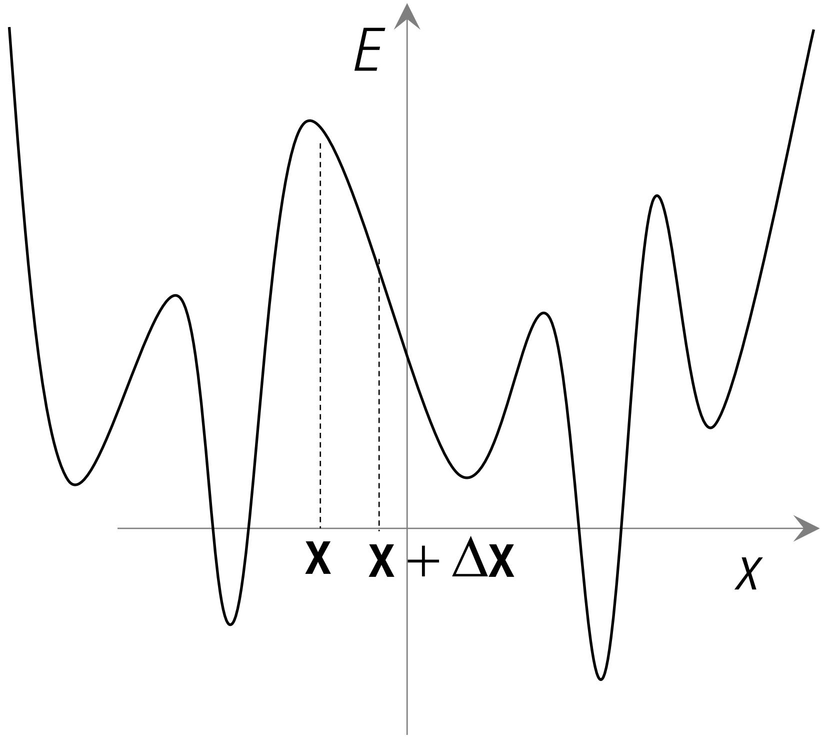
$$\rightarrow \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

# Nonlinear System




Find **x** such that the following equation is satisfied:

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} f(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \mathbf{b}$$

How?

Strategy: Given **x**, move $\Delta\mathbf{x}$ such that $E(\mathbf{x}+\Delta\mathbf{x}) \le E(\mathbf{x})$

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$$f(\mathbf{x}+\Delta\mathbf{x}) = f(\mathbf{x}+\Delta\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x} + \mathrm{H.O.T.}$$

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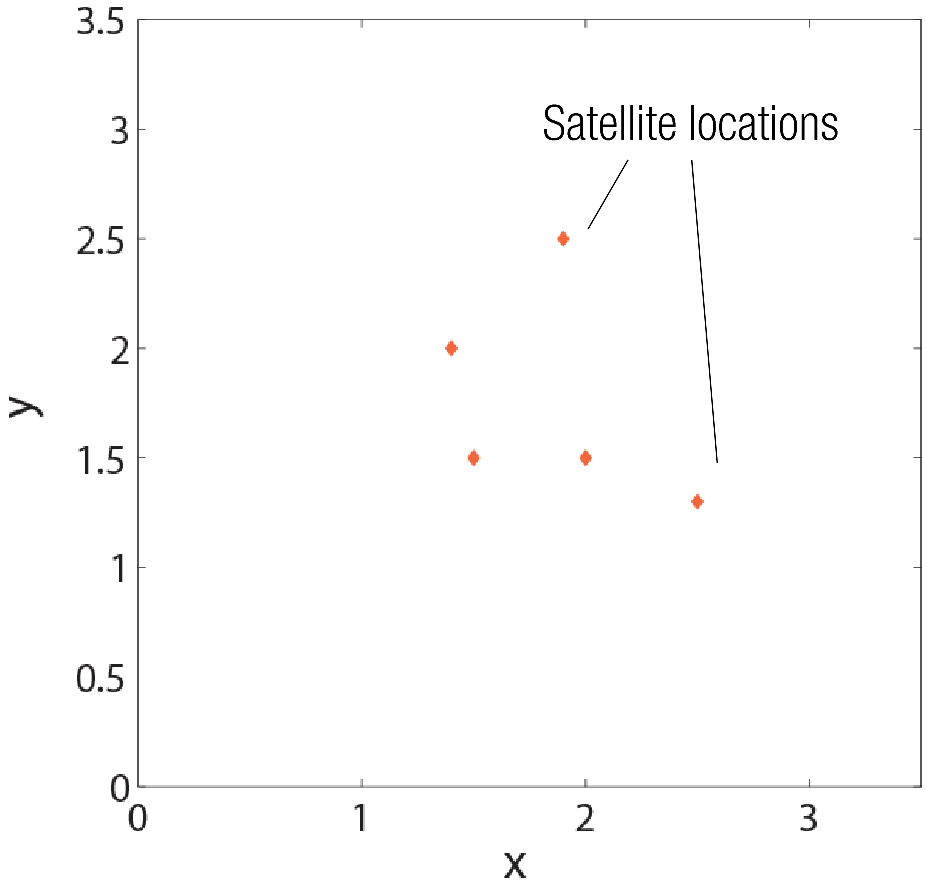
$$\rightarrow \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} (\mathbf{b} - f(\mathbf{x}))$$

Cf.) $\mathbf{x} = \left[\mathbf{A}^{\mathrm{T}} \mathbf{A}\right]^{-1} \mathbf{A}^{\mathrm{T}} \mathbf{b}$

# Where am I?: GPS Localization

Example:
Localization using range data from beacons



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Example:
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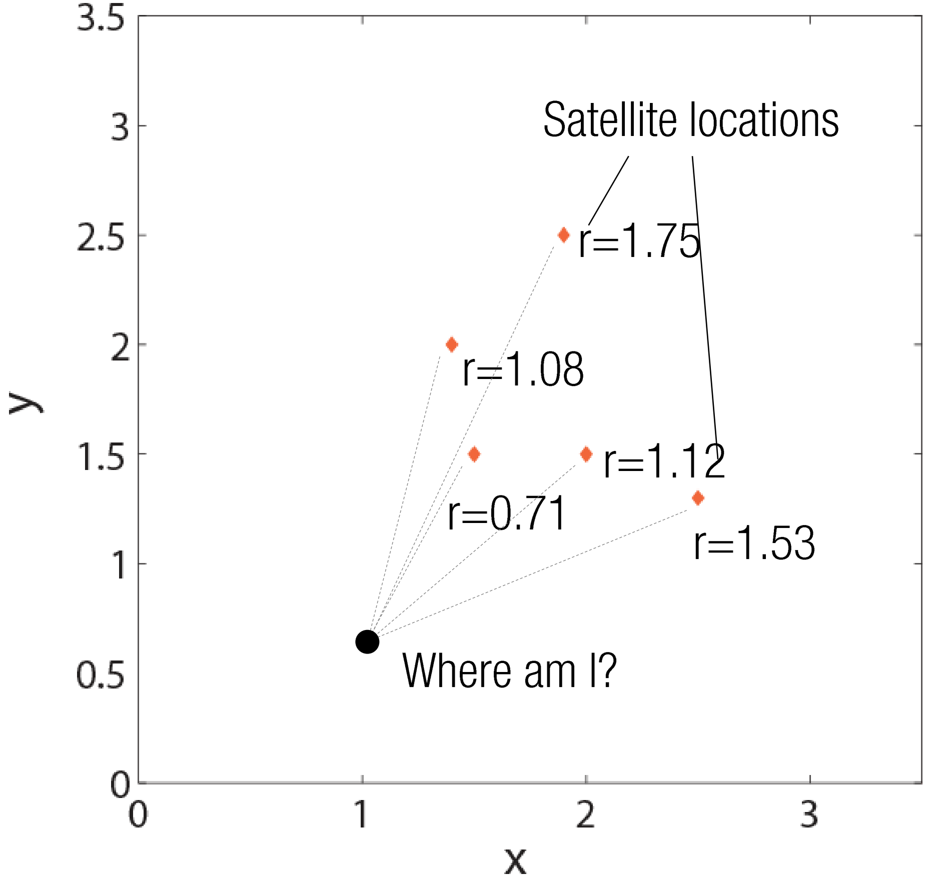

# Where am I?: GPS Localization

Example:
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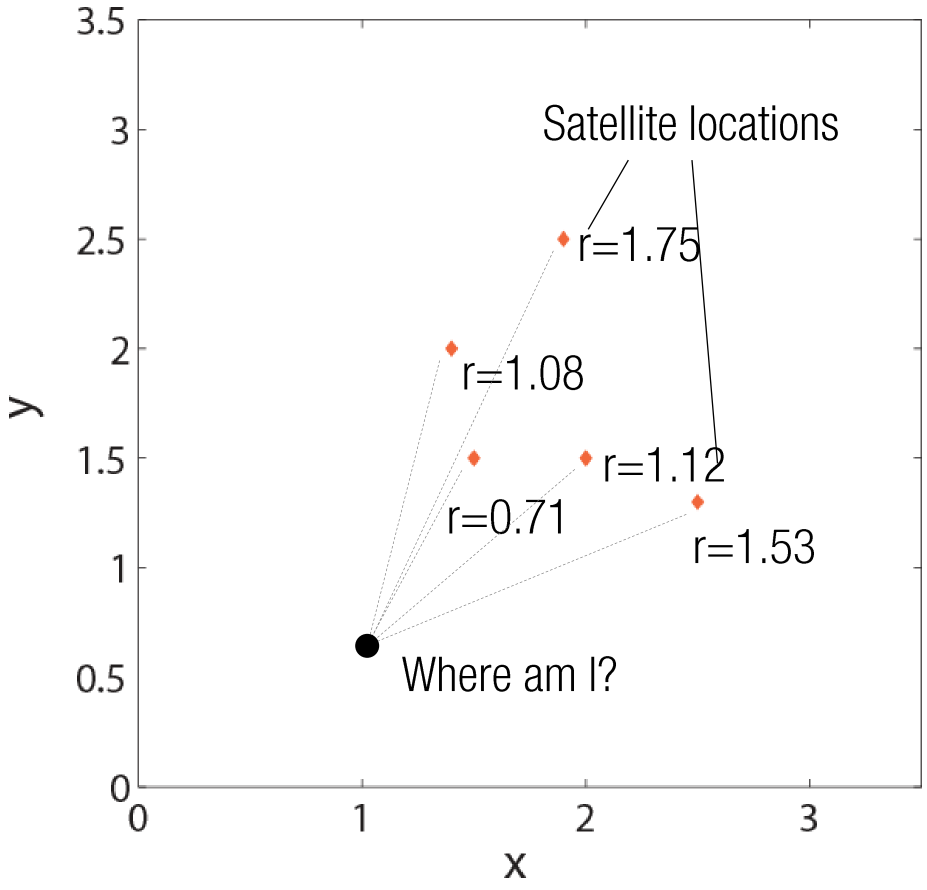
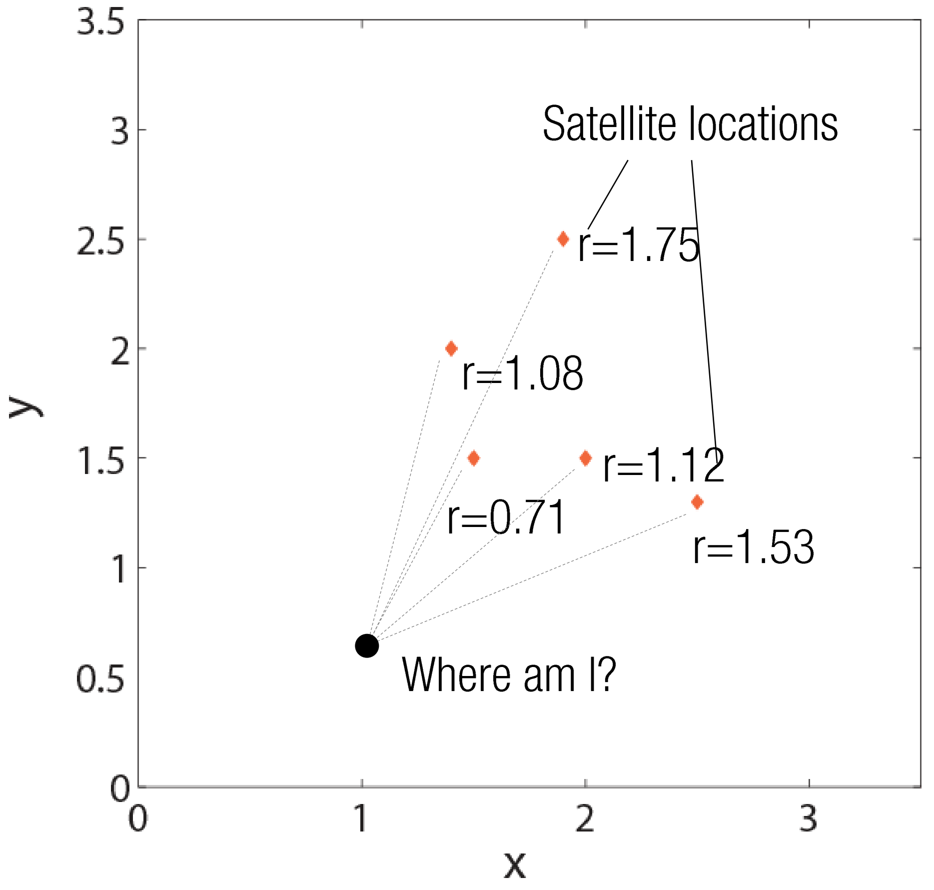



$E =$

# Where am I?: GPS Localization

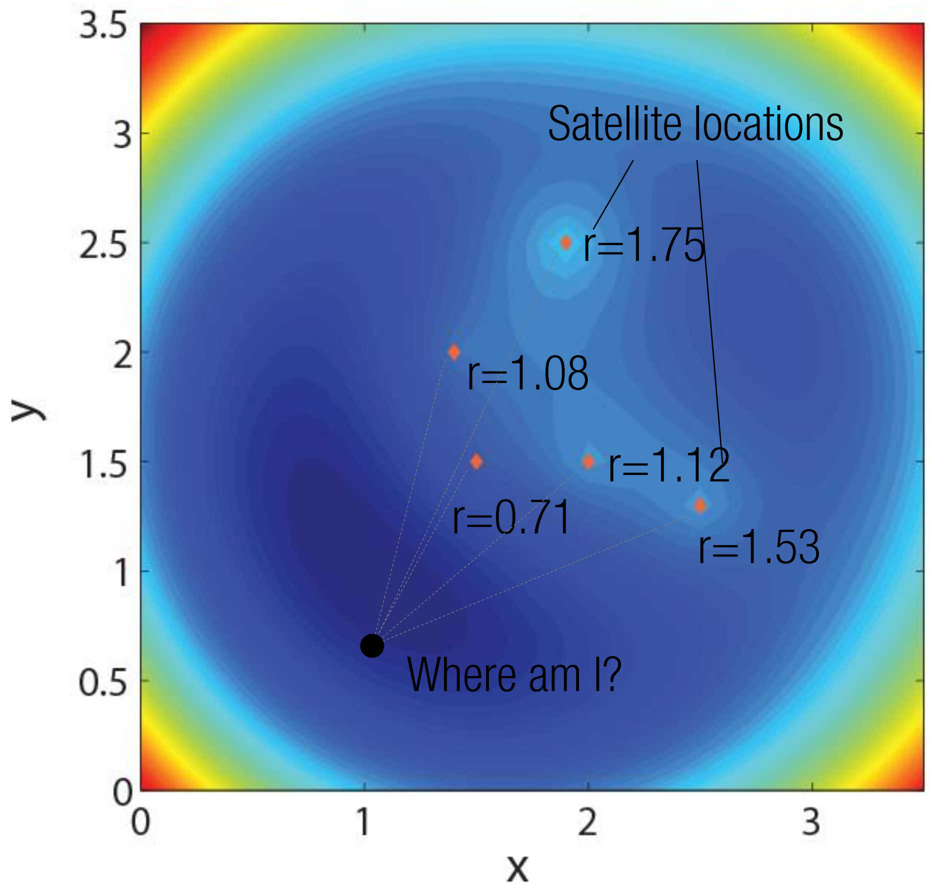
Example:
Localization using range data from beacons




$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

# Where am I?: GPS Localization

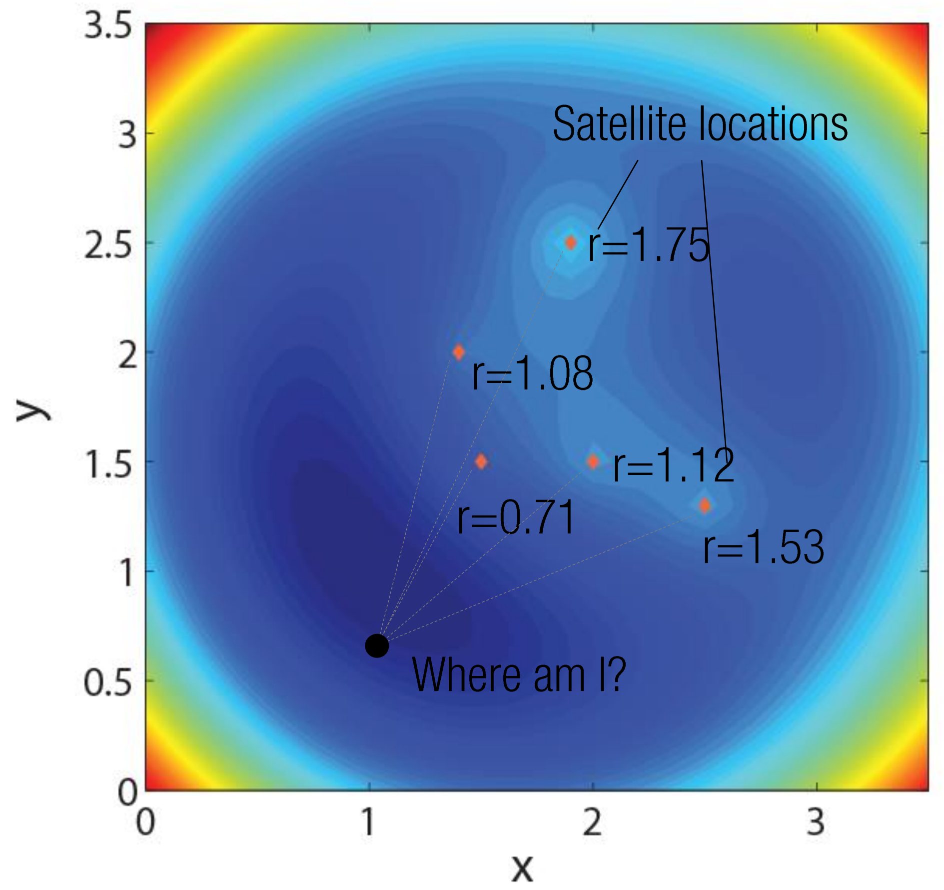
Example:
Localization using range data from beacons




$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

# Where am I?: GPS Localization

Example:
Localization using range data from beacons



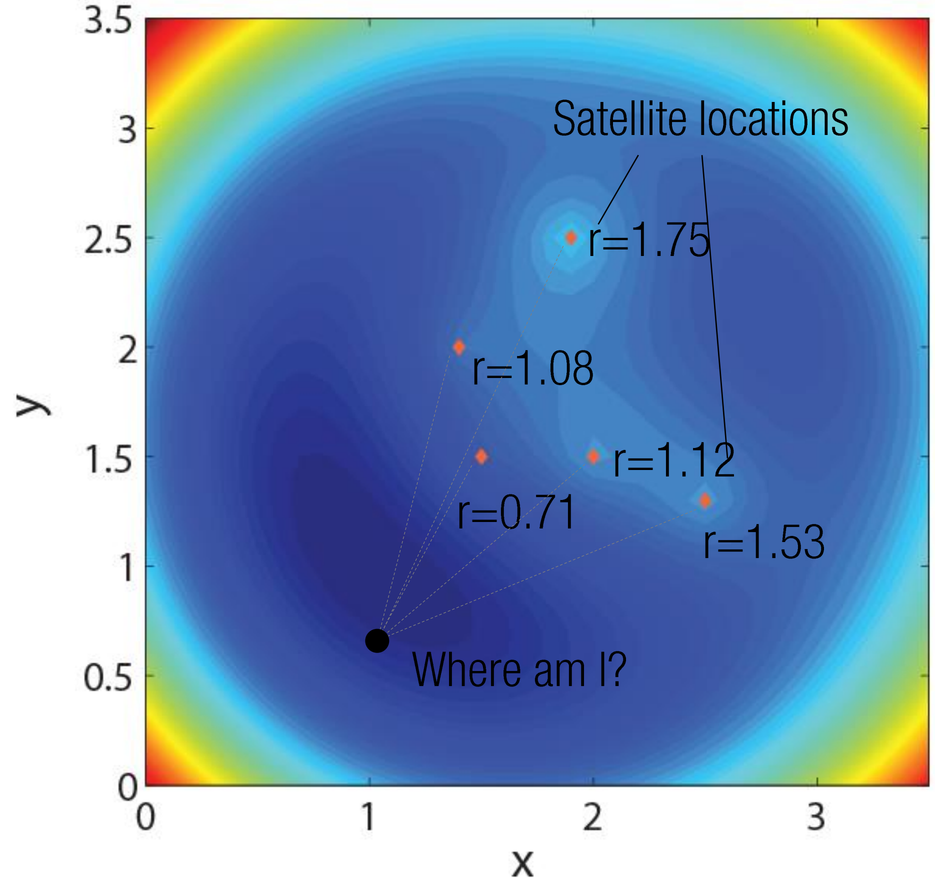

$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

where $\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial \mathbf{x}_1} & \cdots & \frac{\partial f_1}{\partial \mathbf{x}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial \mathbf{x}_n} & \cdots & \frac{\partial f_m}{\partial \mathbf{x}_n} \end{bmatrix}$ : Jacobian

# Where am I?: GPS Localization

Example:
Localization using range data from beacons



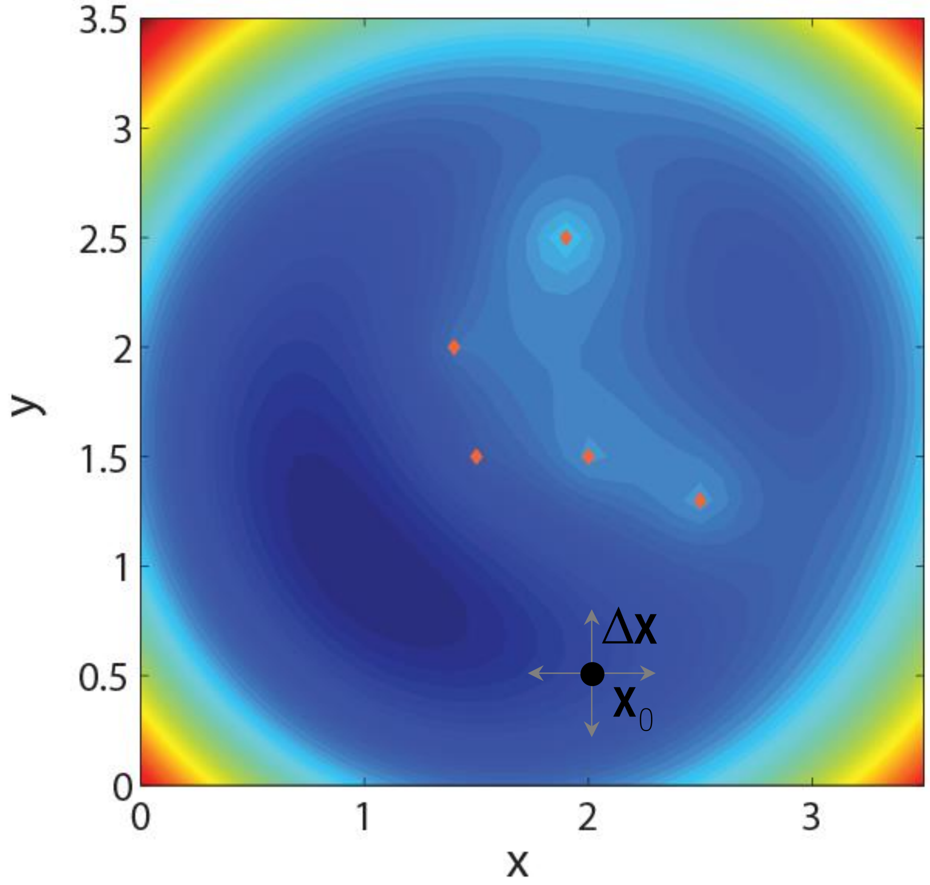

$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} -\dfrac{u_1 - x}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} & -\dfrac{v_1 - y}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} \\ \vdots & \vdots \\ -\dfrac{u_5 - x}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} & -\dfrac{v_5 - y}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} \end{bmatrix}$$

# Where am I?: GPS Localization

Example:
Localization using range data from beacons



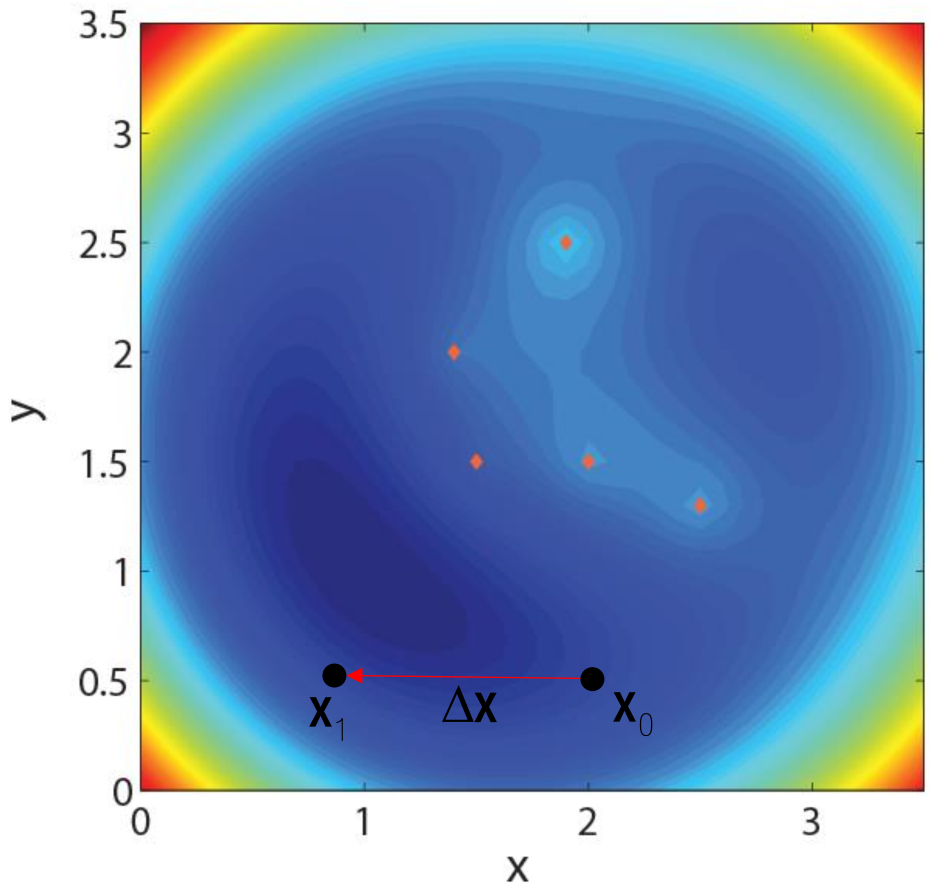

$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

$$\frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} = \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

$$\Delta \mathbf{x} = \left( \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})^T}{\partial \mathbf{x}} \left( \mathbf{b} - f(\mathbf{x}) \right)$$

# Where am I?: GPS Localization

Example:
Localization using range data from beacons



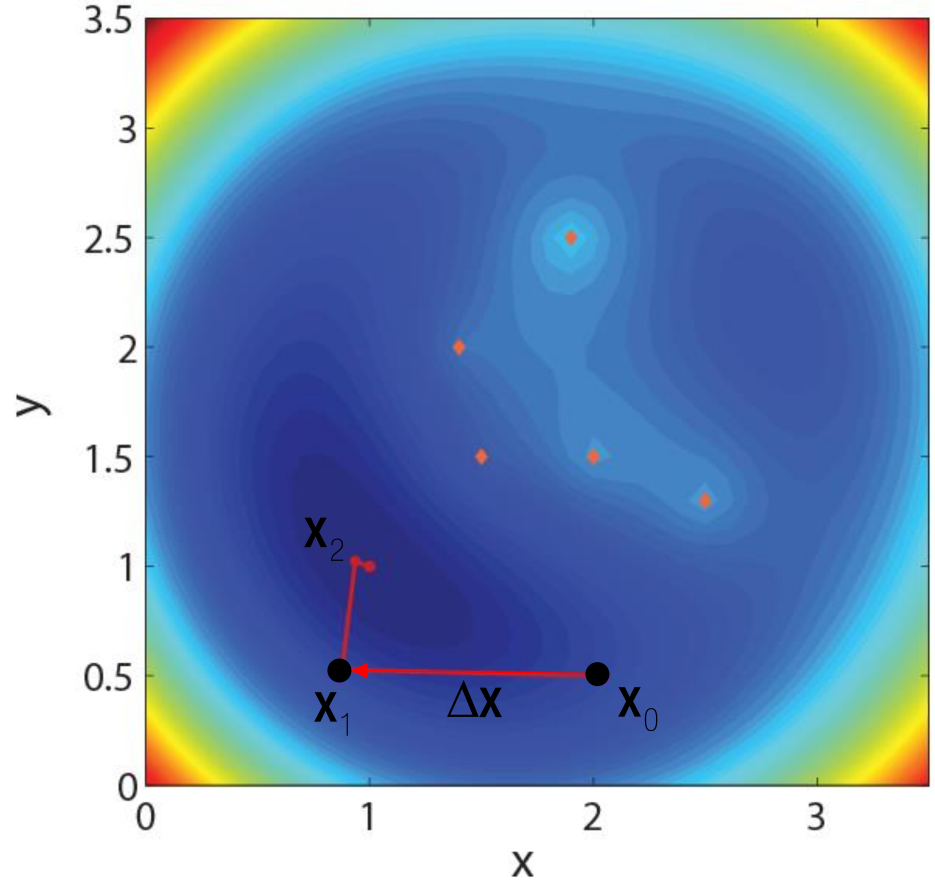

$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

$$\Delta\mathbf{x} = \left( \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

# Where am I?: GPS Localization

Example:
Localization using range data from beacons



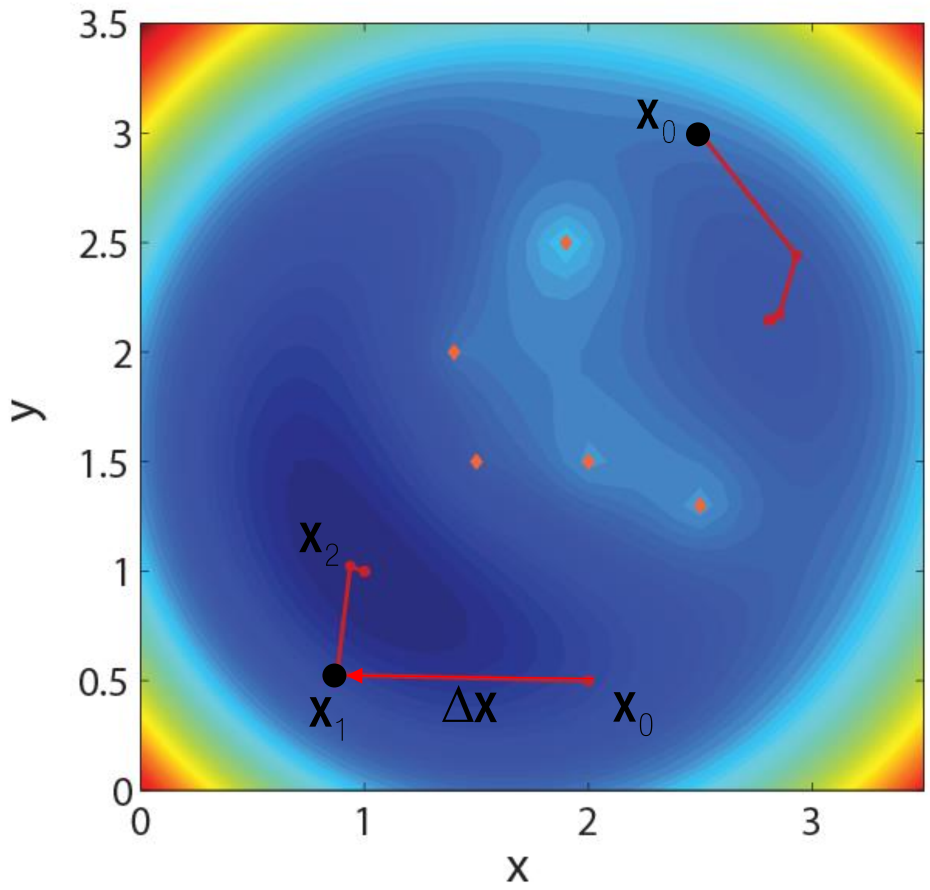

$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

$$\Delta\mathbf{x} = \left( \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

# Where am I?: GPS Localization

Example:
Localization using range data from beacons



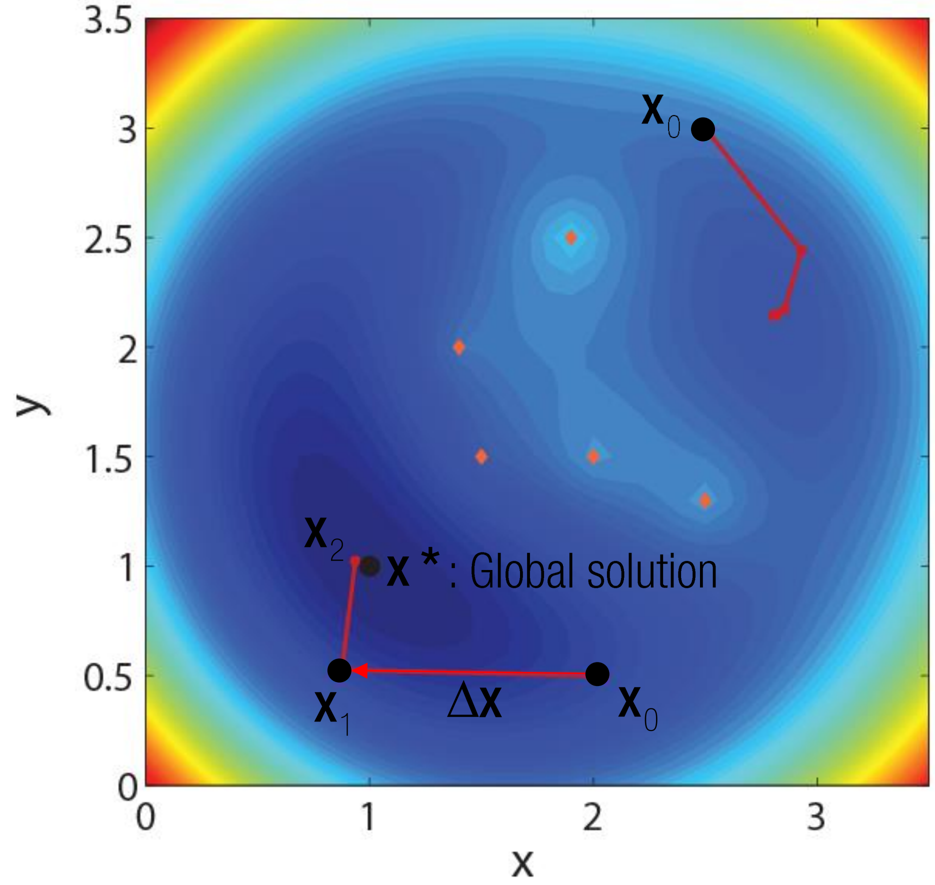

$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

$$\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} = \frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

$$\Delta\mathbf{x} = \left( \frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

# Where am I?: GPS Localization

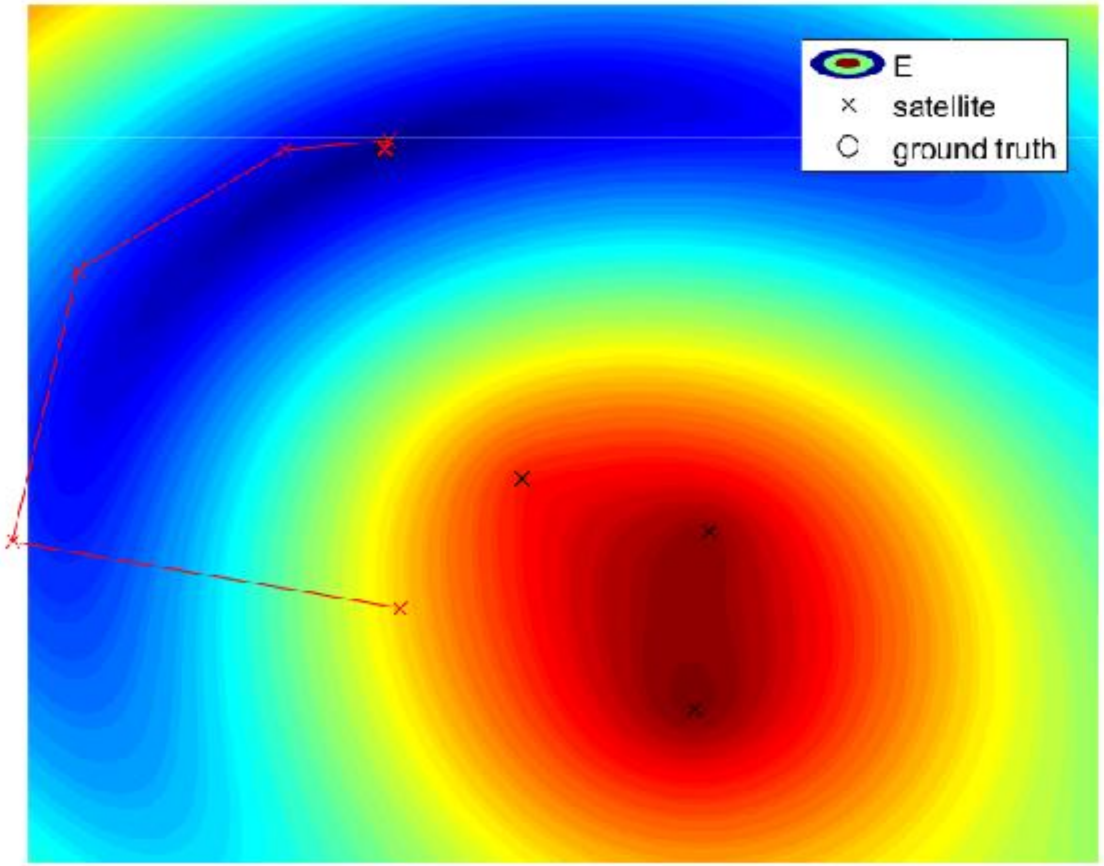
Example:
Localization using range data from beacons




$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

$$\Delta\mathbf{x} = \left( \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$




## Beacon.m

```
u = 2*(rand(3,2)-0.5);
x = [-0.5 1];

for i = 1 : size(u,1)
   d(i,1) = norm(x-u(i,:));
end

[x_grid, y_grid] = meshgrid(-1.5:0.01:1.5, -1.5:0.01:1.5);

E = zeros(size(x_grid));
for i = 1 : size(u,1)
   E = E + (sqrt((x_grid-u(i,1)).^2 +(y_grid-u(i,2)).^2)-d(i)).^2 ;
end
E = sqrt(E);

x0 = 2*(rand(2,1)-0.5);
for j = 1 : 10
   J = [];
   fx = [];
   for i = 1 : size(u,1)
      denom = norm(u(i,:)-x0');
      J = [J; (u(i,:)-x0')/denom];
      fx = [fx; norm(u(i,:)-x0')];
   end
   b = d;
   delta_x = -inv(J'*J)*J'*(b-fx);
   x0 = x0 + delta_x;
end
```