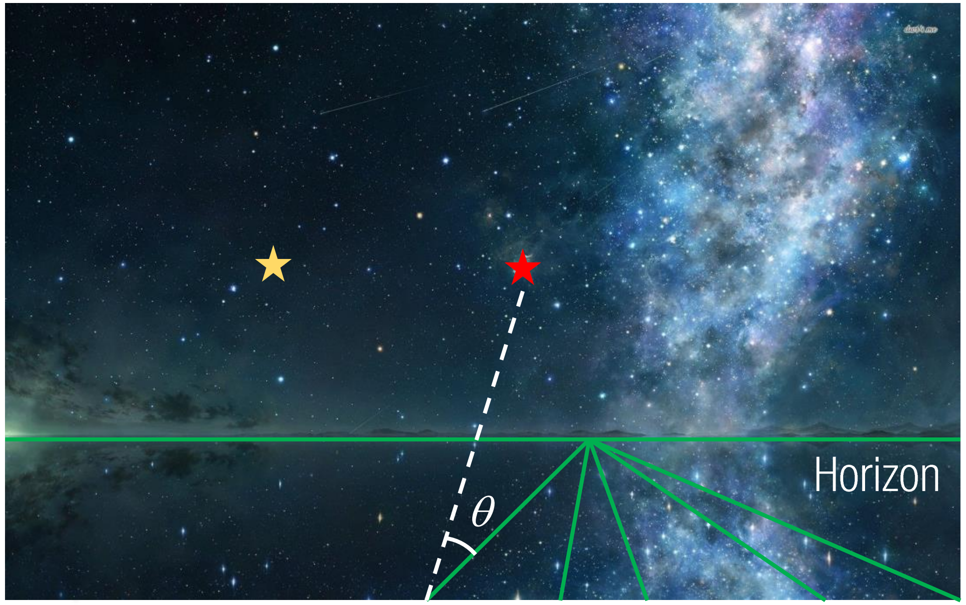
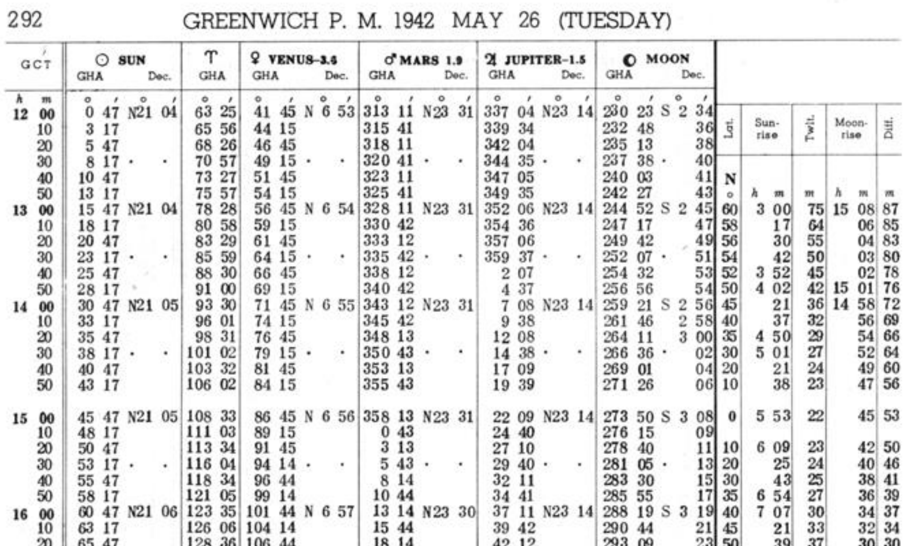
Where am I?
Perspective-n-Point

Announcement

- HW #5 deadline: April 13 (Thur)
- Paper presentation by Cheng.
- Paper presentation by Jingbin on next Thursday.

Recall: Celestial Navigation

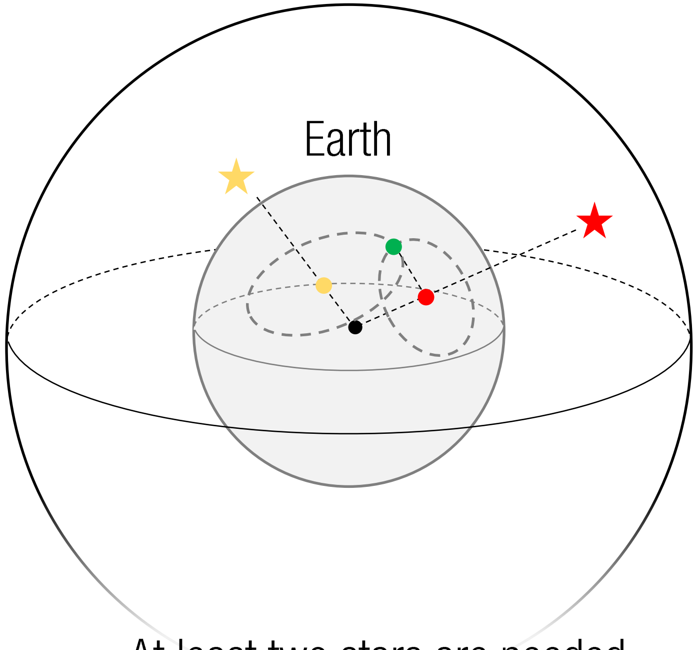




292 GREENWICH P. M. 1942 MAY 26 (TUESDAY)

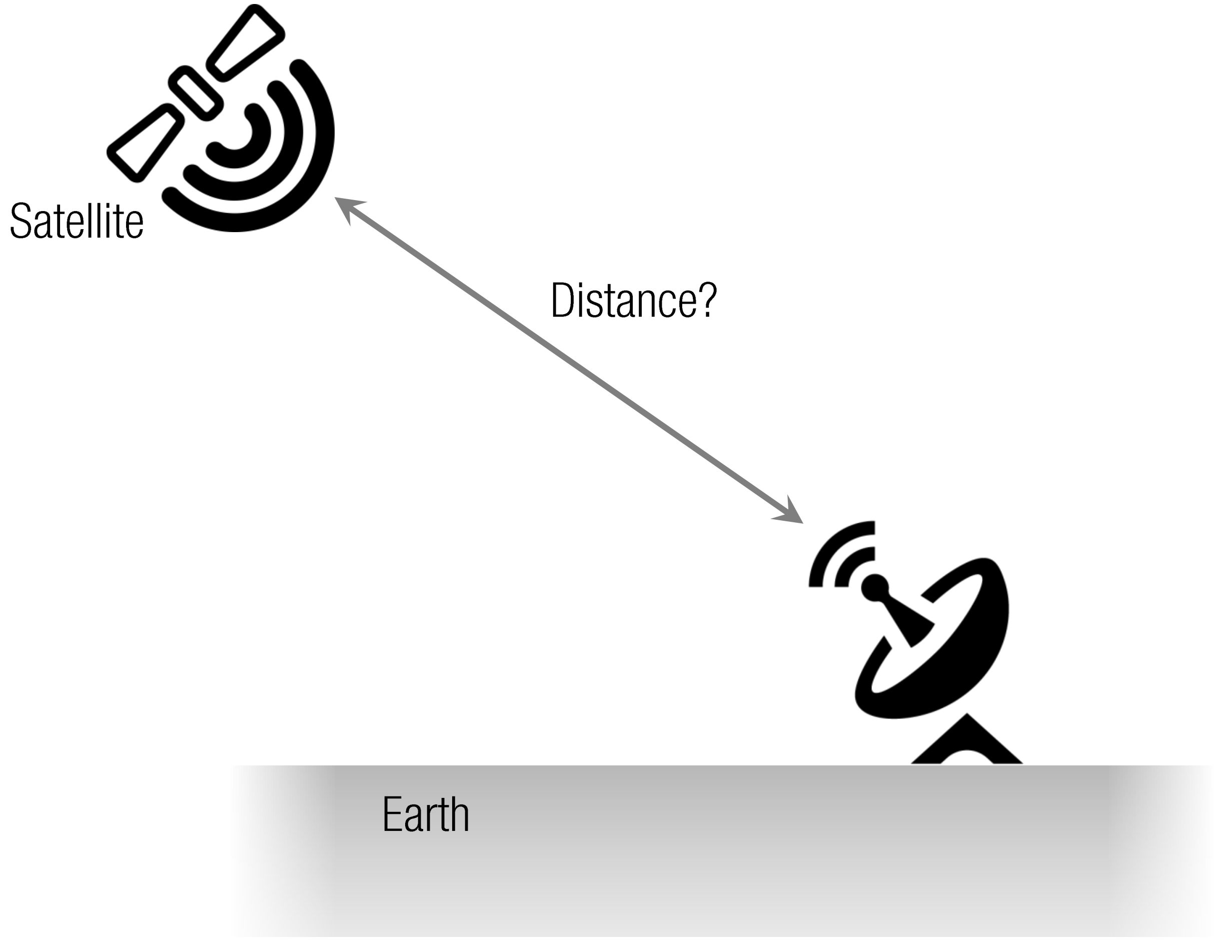
GCT	☉ SUN GHA	SUN Dec.	♈ GHA	♀ VENUS–3.4 GHA	VENUS Dec.	♂ MARS 1.9 GHA	MARS Dec.	♃ JUPITER–1.5 GHA	JUPITER Dec.	☾ MOON GHA	MOON Dec.	Lat.	Sun-rise	Twil.	Moon-rise	Diff.
h m	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′					
12 00	0 47	N21 04	63 25	41 45	N 6 53	313 11	N23 31	337 04	N23 14	230 23	S 2 34					
10	3 17		65 56	44 15		315 41		339 34		232 48	36					
20	5 47		68 26	46 45		318 11		342 04		235 13	38					
30	8 17	· ·	70 57	49 15	· ·	320 41	· ·	344 35	· ·	237 38	· 40					
40	10 47		73 27	51 45		323 11		347 05		240 03	41	N				
50	13 17		75 57	54 15		325 41		349 35		242 27	43	°	h m	m	h m	m
13 00	15 47	N21 04	78 28	56 45	N 6 54	328 11	N23 31	352 06	N23 14	244 52	S 2 45	60	3 00	75	15 08	87
10	18 17		80 58	59 15		330 42		354 36		247 17	47	58	17	64	06	85
20	20 47		83 29	61 45		333 12		357 06		249 42	49	56	30	55	04	83
30	23 17	· ·	85 59	64 15	· ·	335 42	· ·	359 37	· ·	252 07	· 51	54	42	50	03	80
40	25 47		88 30	66 45		338 12		2 07		254 32	53	52	3 52	45	02	78
50	28 17		91 00	69 15		340 42		4 37		256 56	54	50	4 02	42	15 01	76
14 00	30 47	N21 05	93 30	71 45	N 6 55	343 12	N23 31	7 08	N23 14	259 21	S 2 56	45	21	36	14 58	72
10	33 17		96 01	74 15		345 42		9 38		261 46	2 58	40	37	32	56	69
20	35 47		98 31	76 45		348 13		12 08		264 11	3 00	35	4 50	29	54	66
30	38 17	· ·	101 02	79 15	· ·	350 43	· ·	14 38	· ·	266 36	· 02	30	5 01	27	52	64
40	40 47		103 32	81 45		353 13		17 09		269 01	04	20	21	24	49	60
50	43 17		106 02	84 15		355 43		19 39		271 26	06	10	38	23	47	56
15 00	45 47	N21 05	108 33	86 45	N 6 56	358 13	N23 31	22 09	N23 14	273 50	S 3 08	0	5 53	22	45	53
10	48 17		111 03	89 15		0 43		24 40		276 15	09					
20	50 47		113 34	91 45		3 13		27 10		278 40	11	10	6 09	23	42	50
30	53 17	· ·	116 04	94 14	· ·	5 43	· ·	29 40	· ·	281 05	· 13	20	25	24	40	46
40	55 47		118 34	96 44		8 14		32 11		283 30	15	30	43	25	38	41
50	58 17		121 05	99 14		10 44		34 41		285 55	17	35	6 54	27	36	39
16 00	60 47	N21 06	123 35	101 44	N 6 57	13 14	N23 30	37 11	N23 14	288 19	S 3 19	40	7 07	30	34	37
10	63 17		126 06	104 14		15 44		39 42		290 44	21	45	21	33	32	34
20	65 47		128 36	106 44		18 14		42 12		293 09	23	50	39	37	30	30

Far far away: point at infinity



At least two stars are needed.

Where am I?
How Global Positioning System Works?



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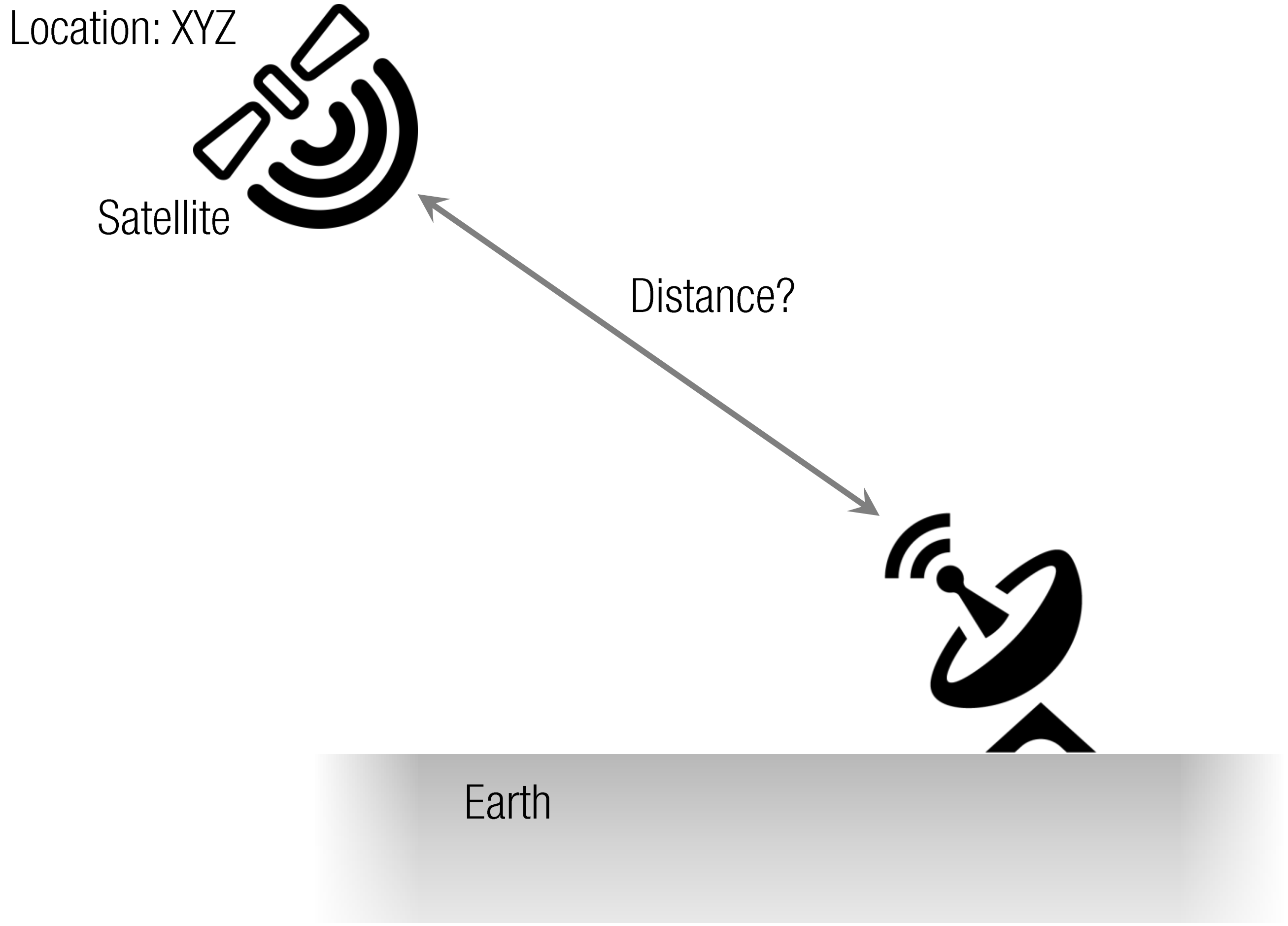

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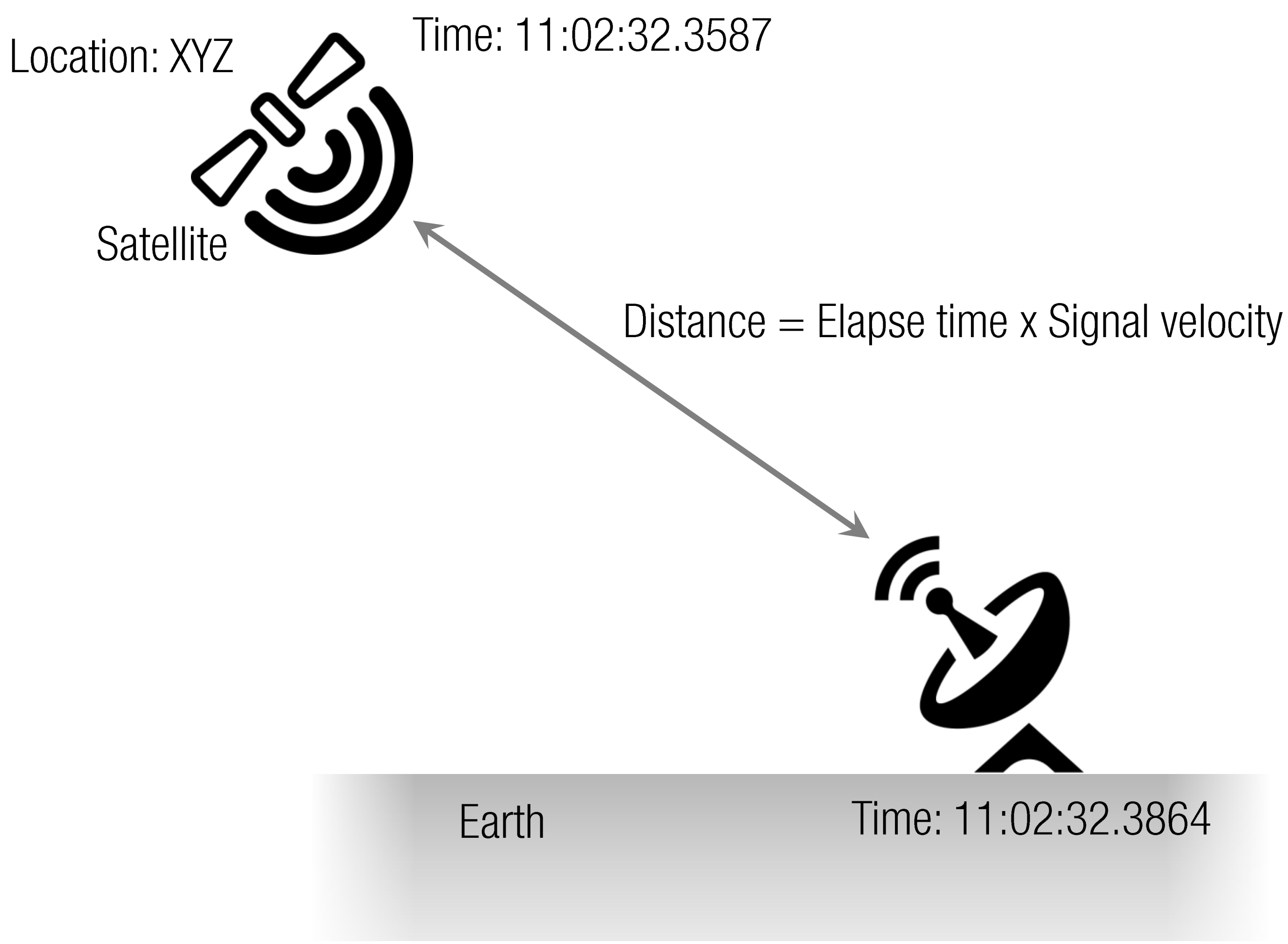

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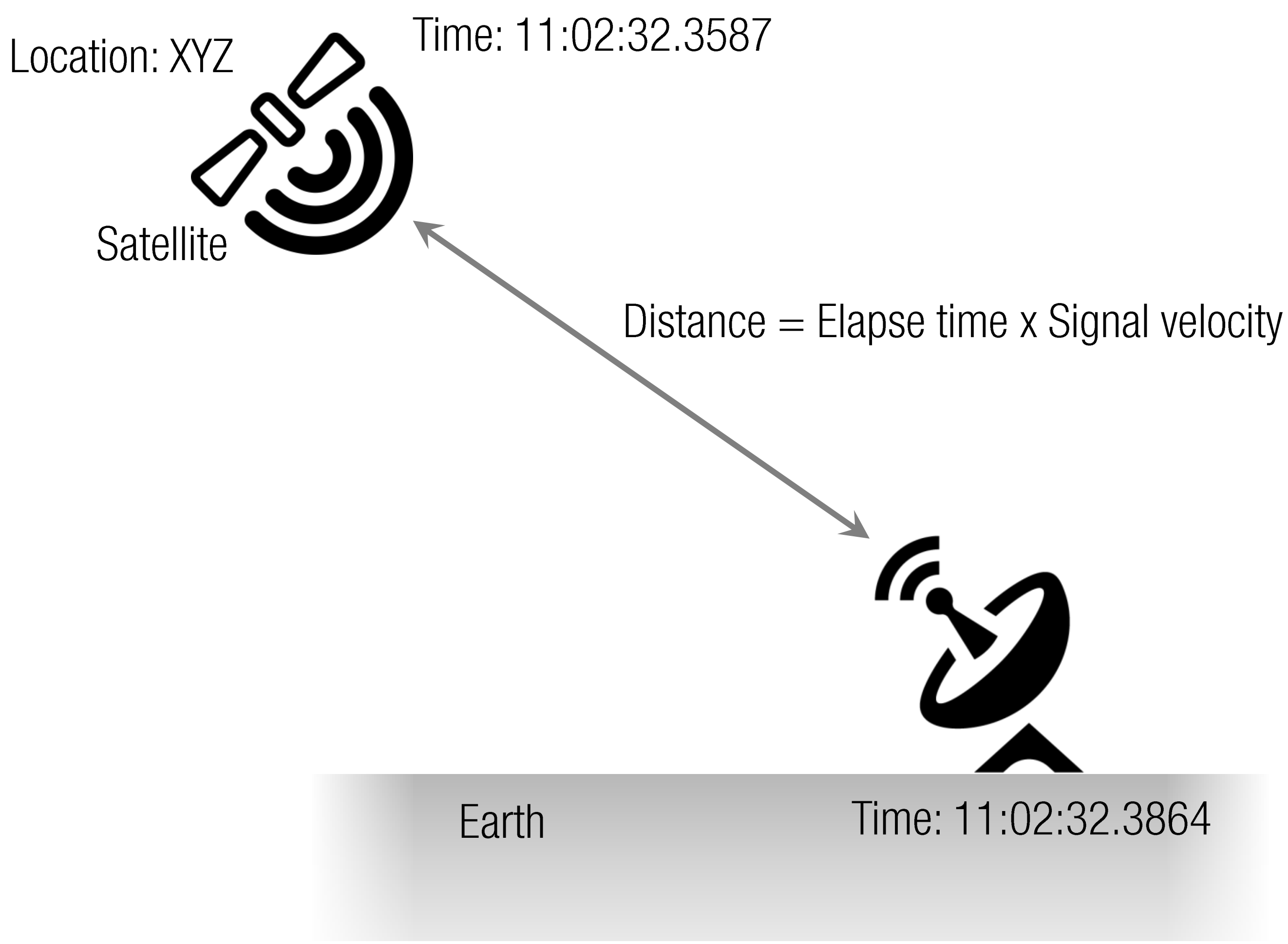

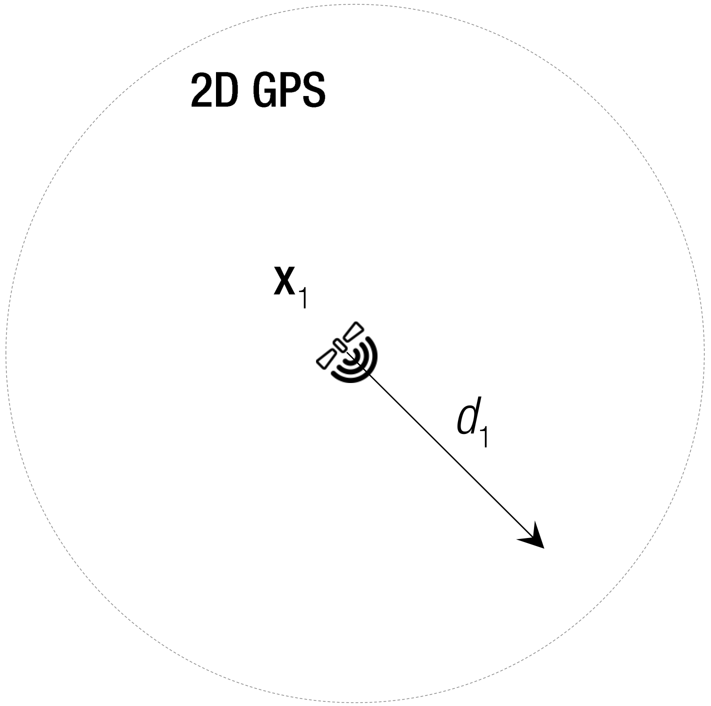

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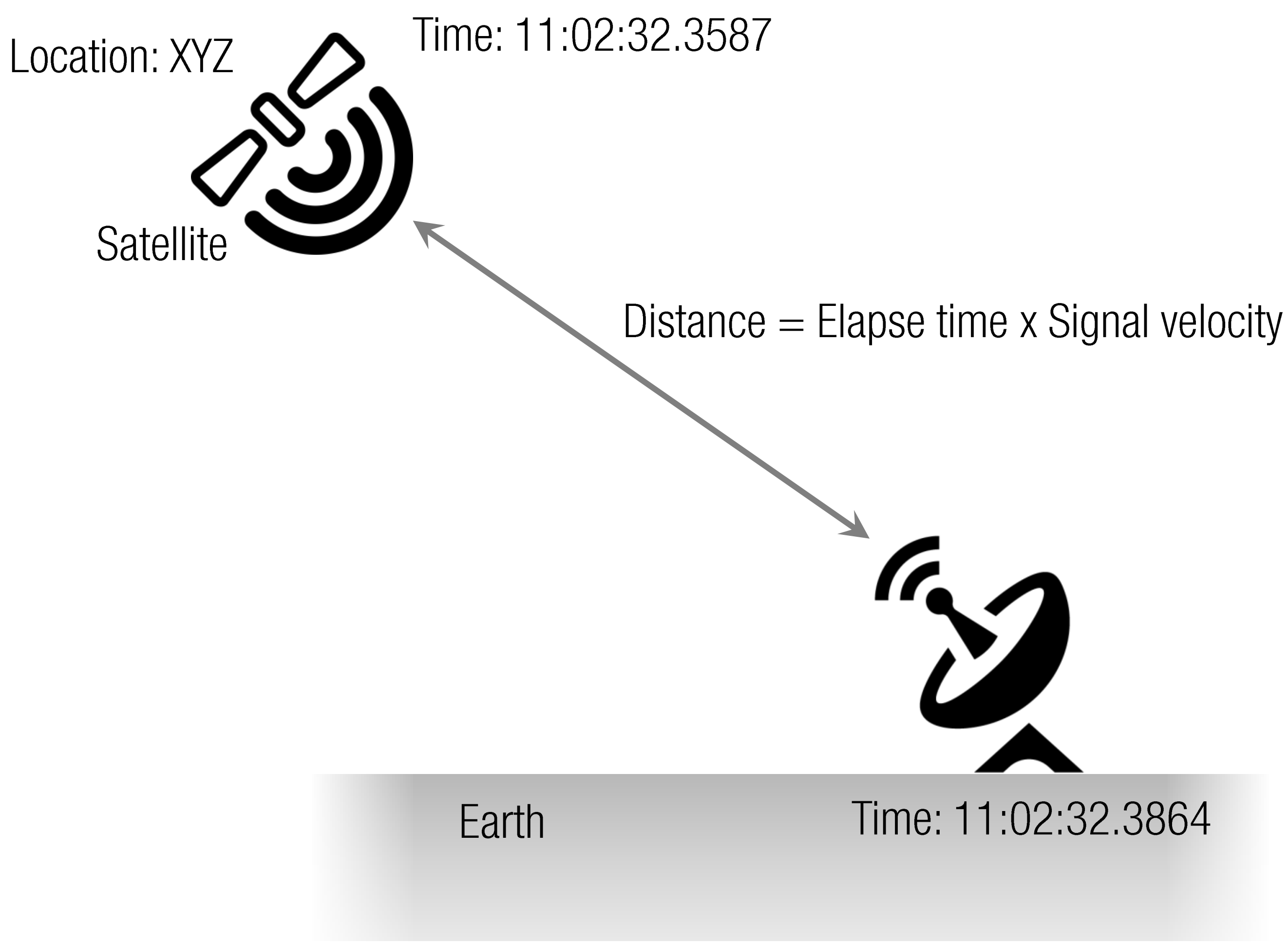

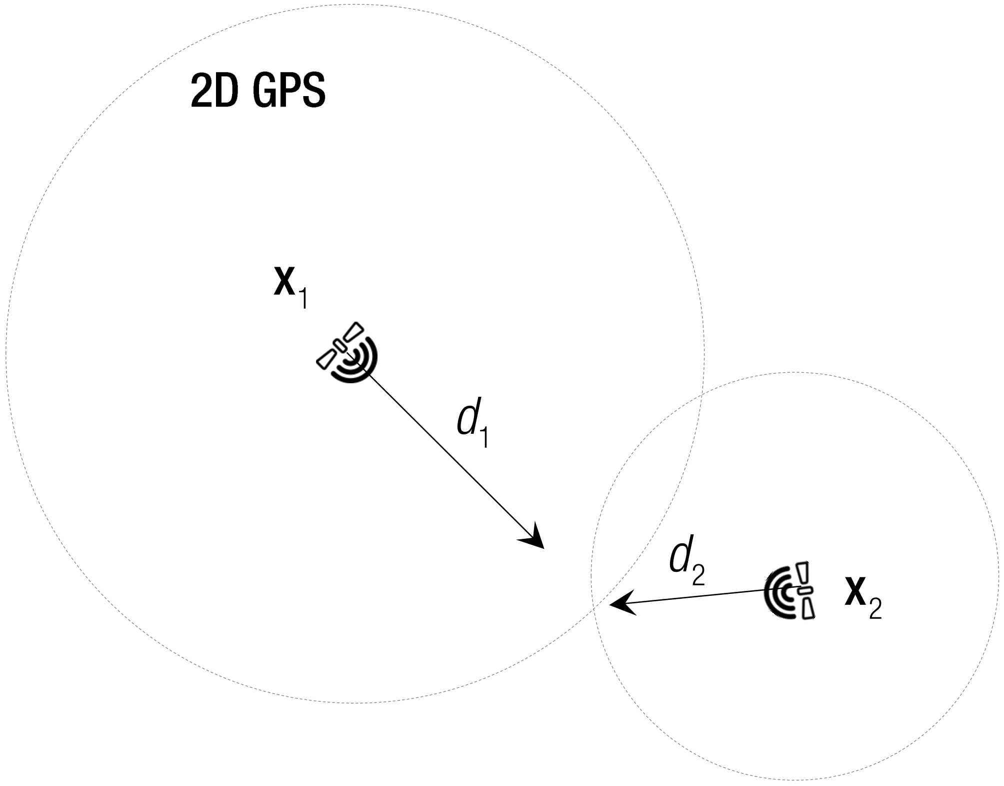

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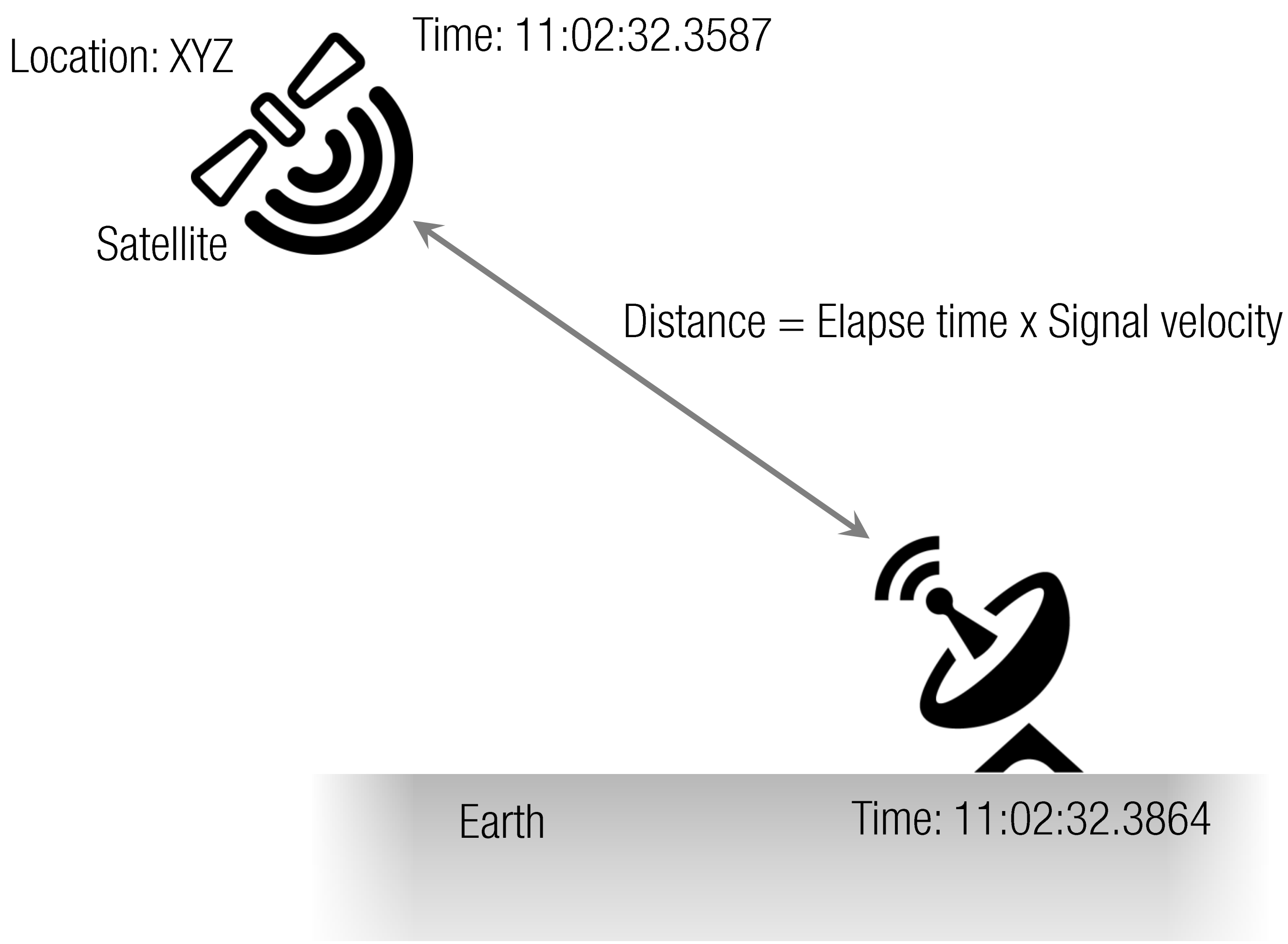

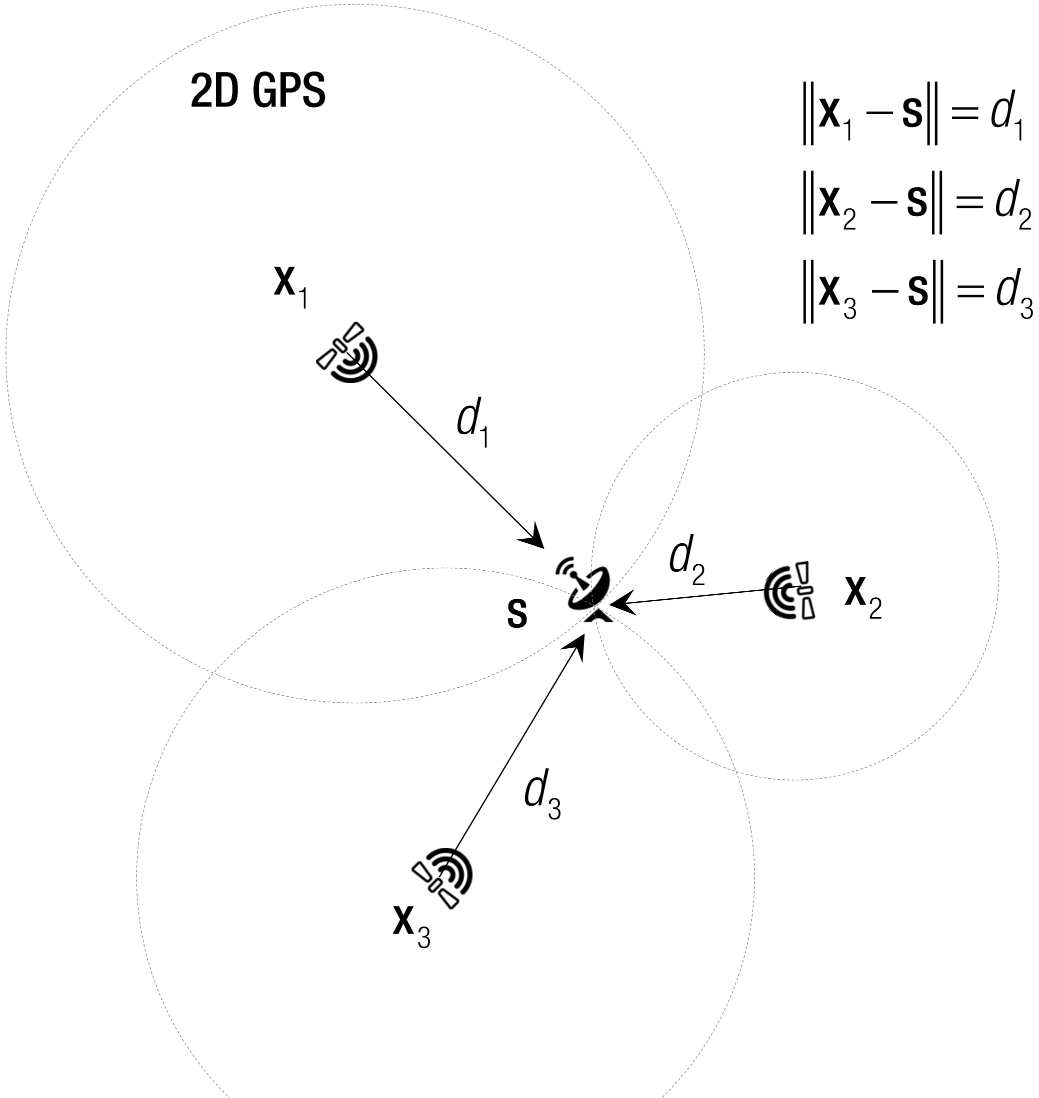

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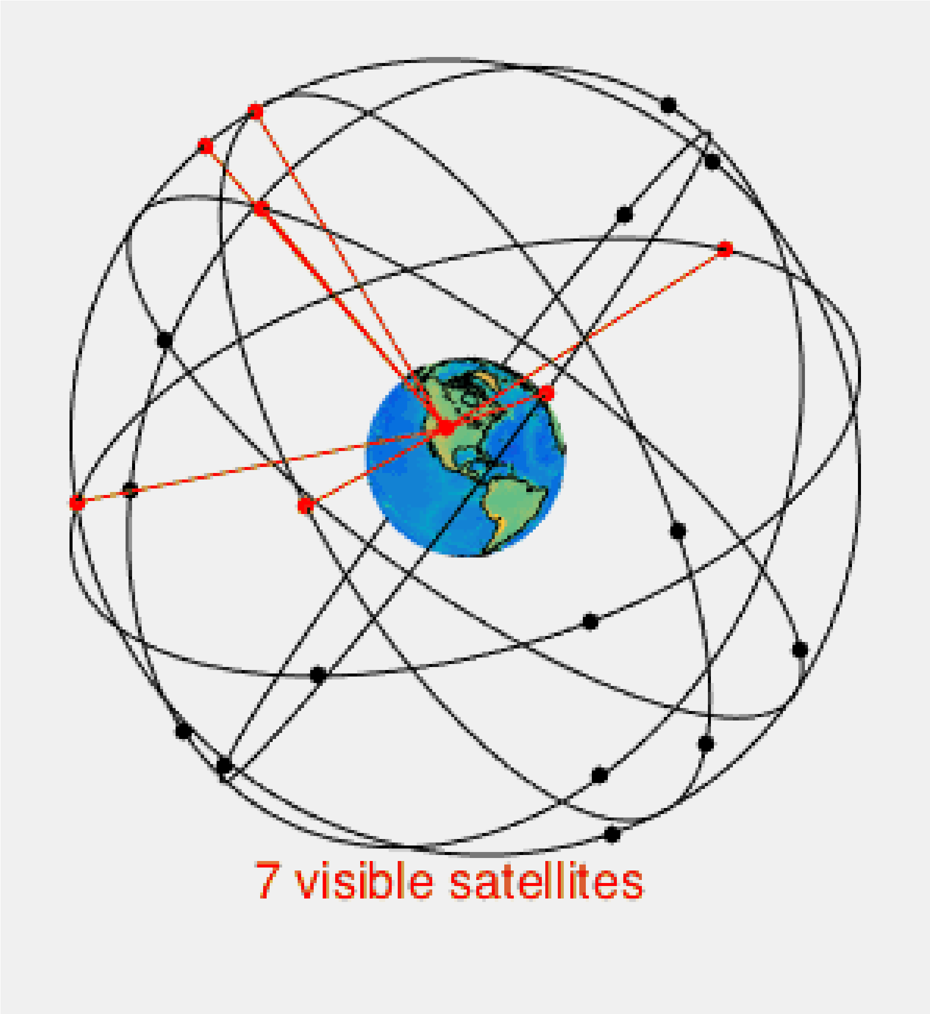

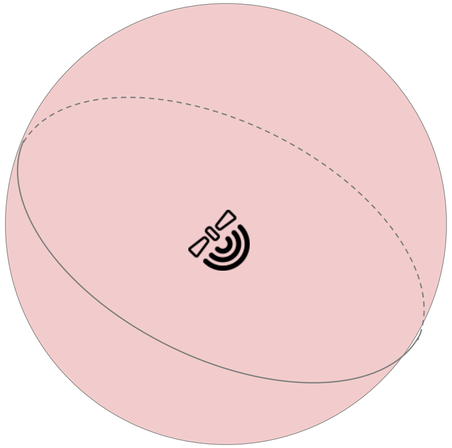

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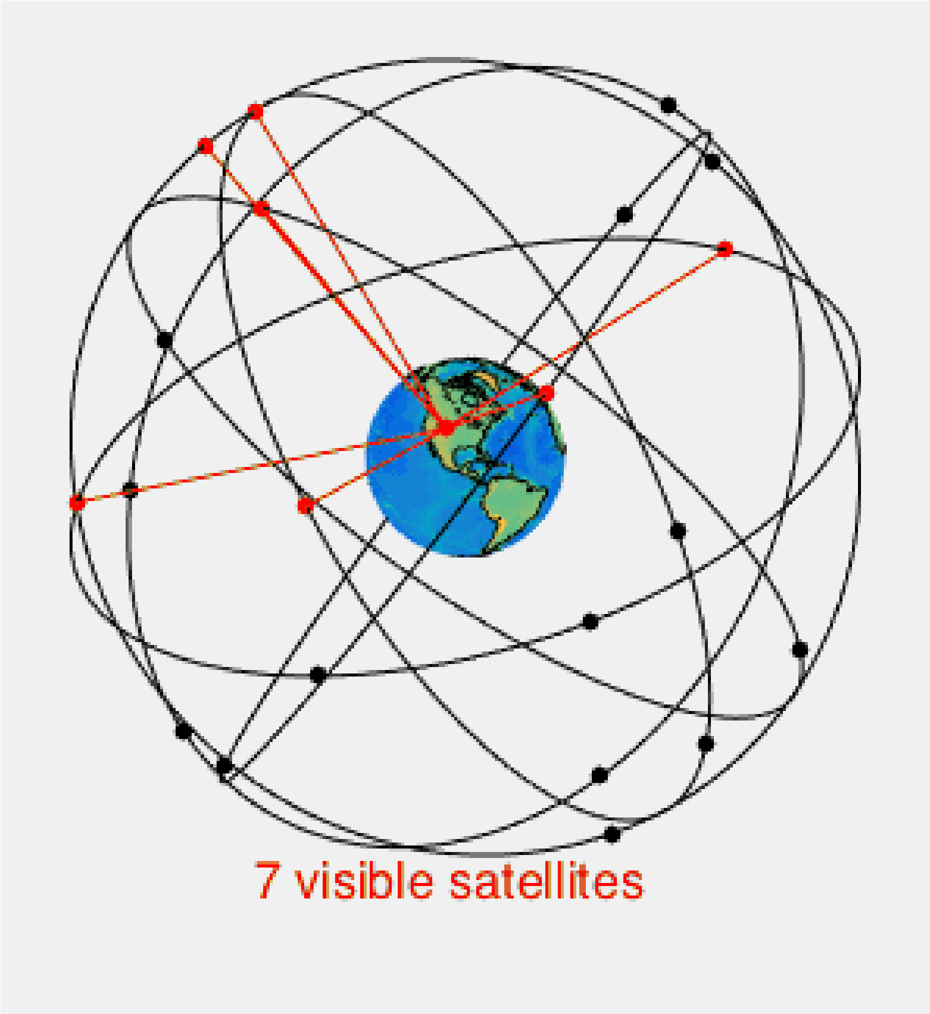

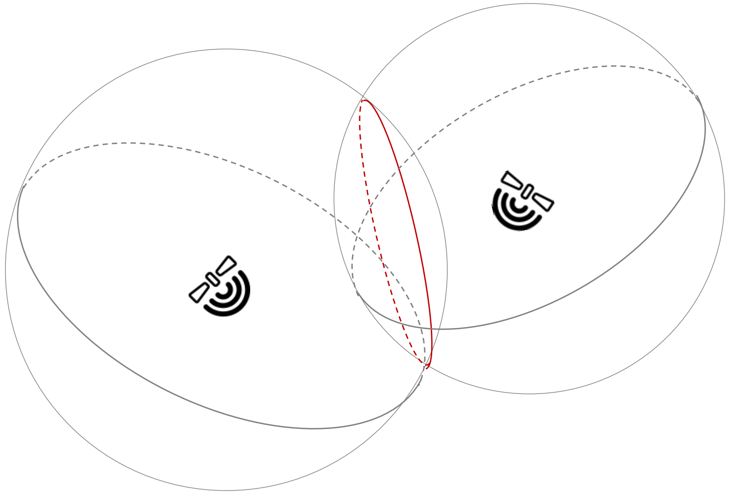

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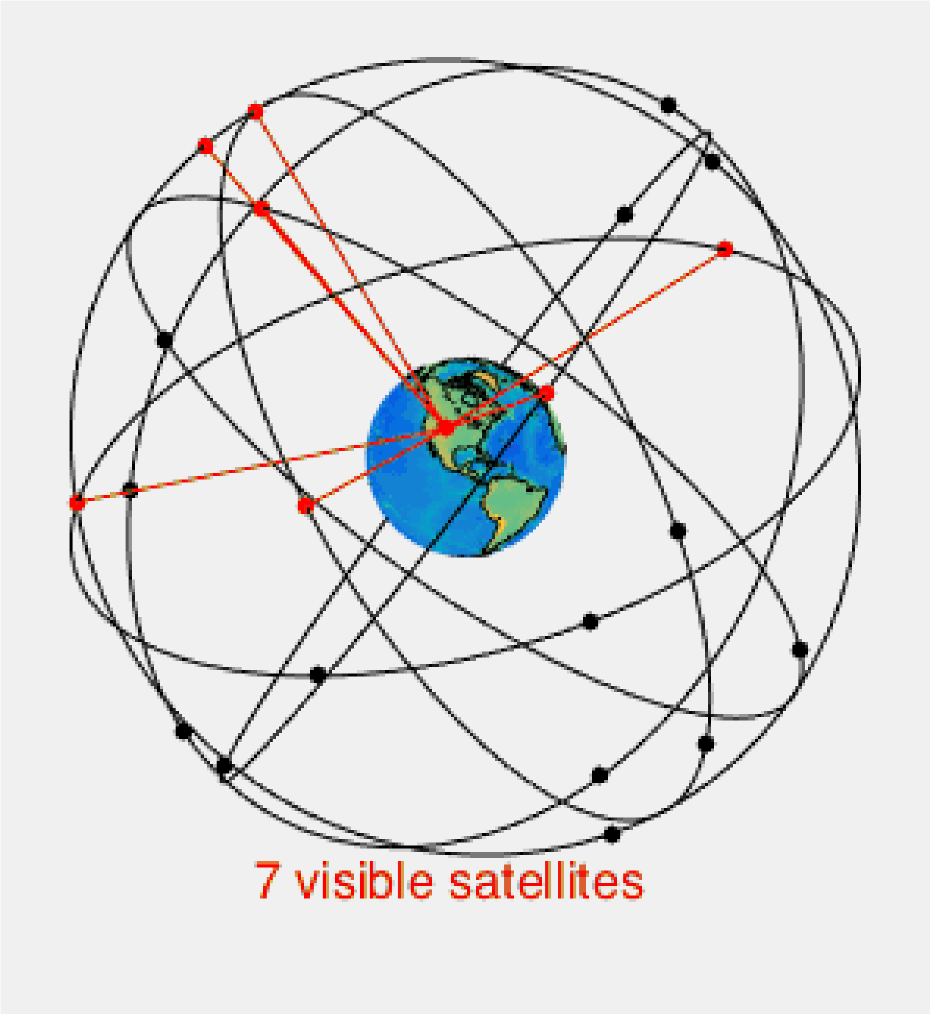

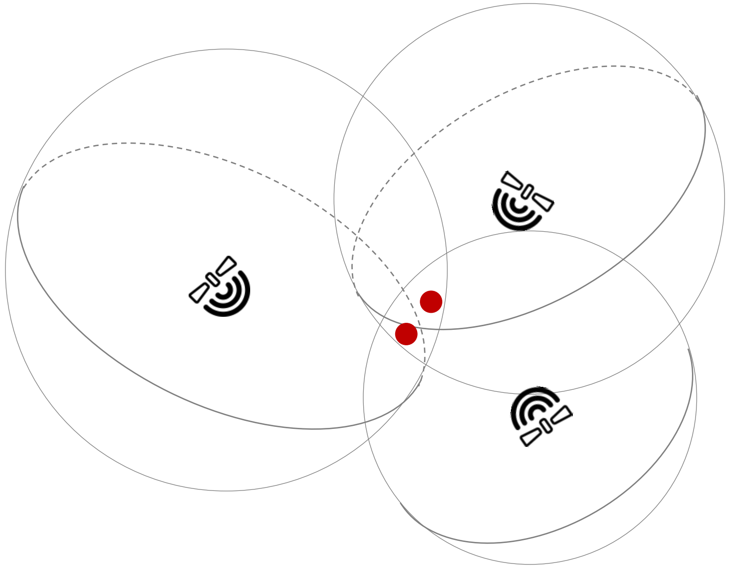

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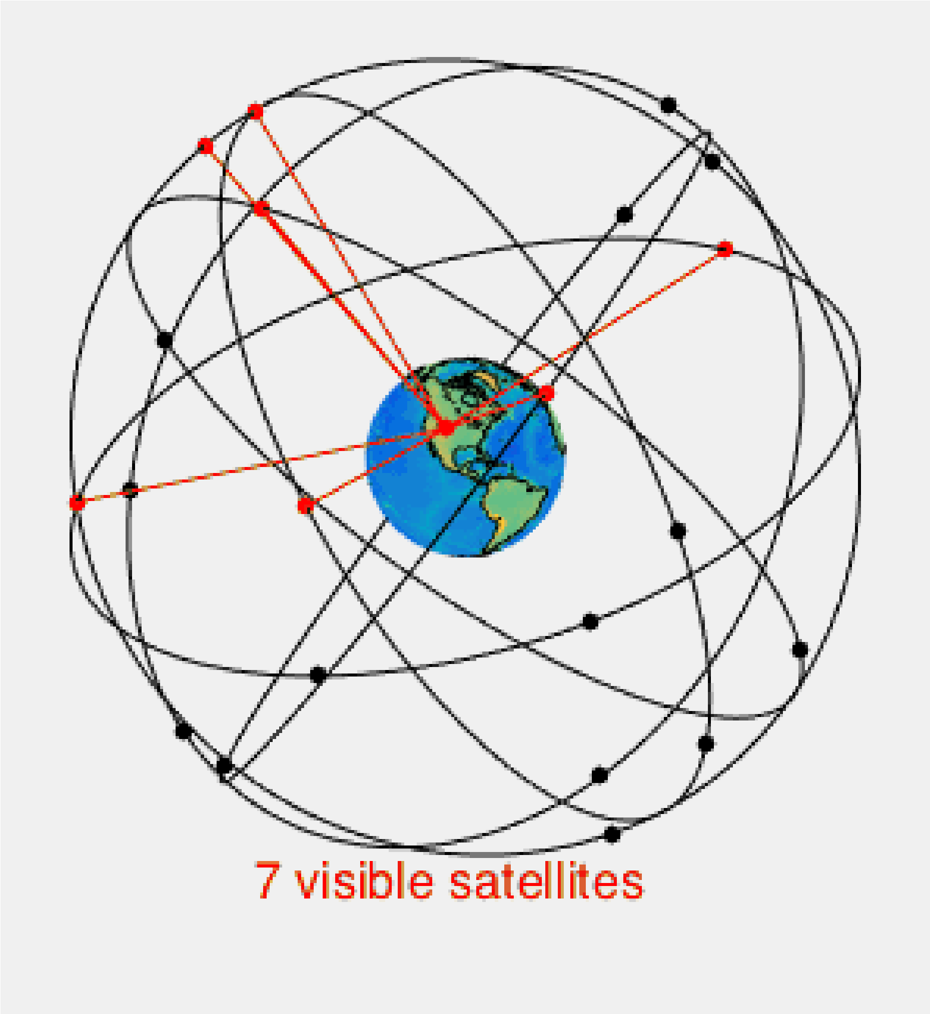

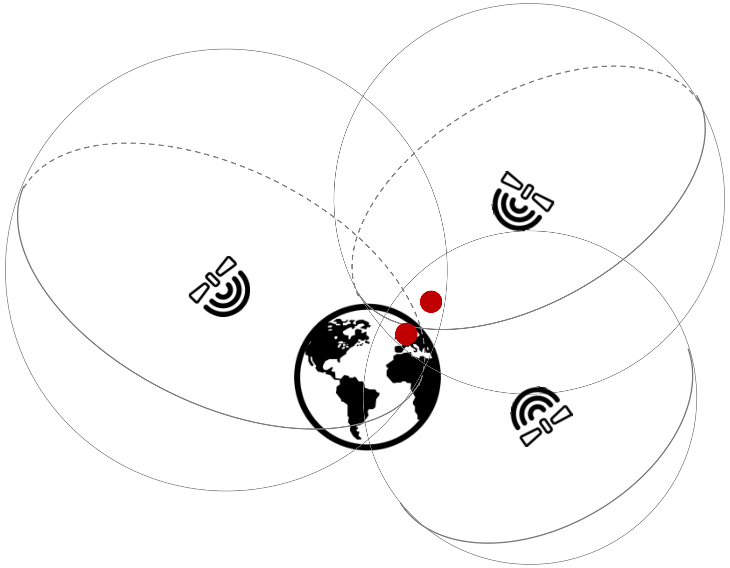

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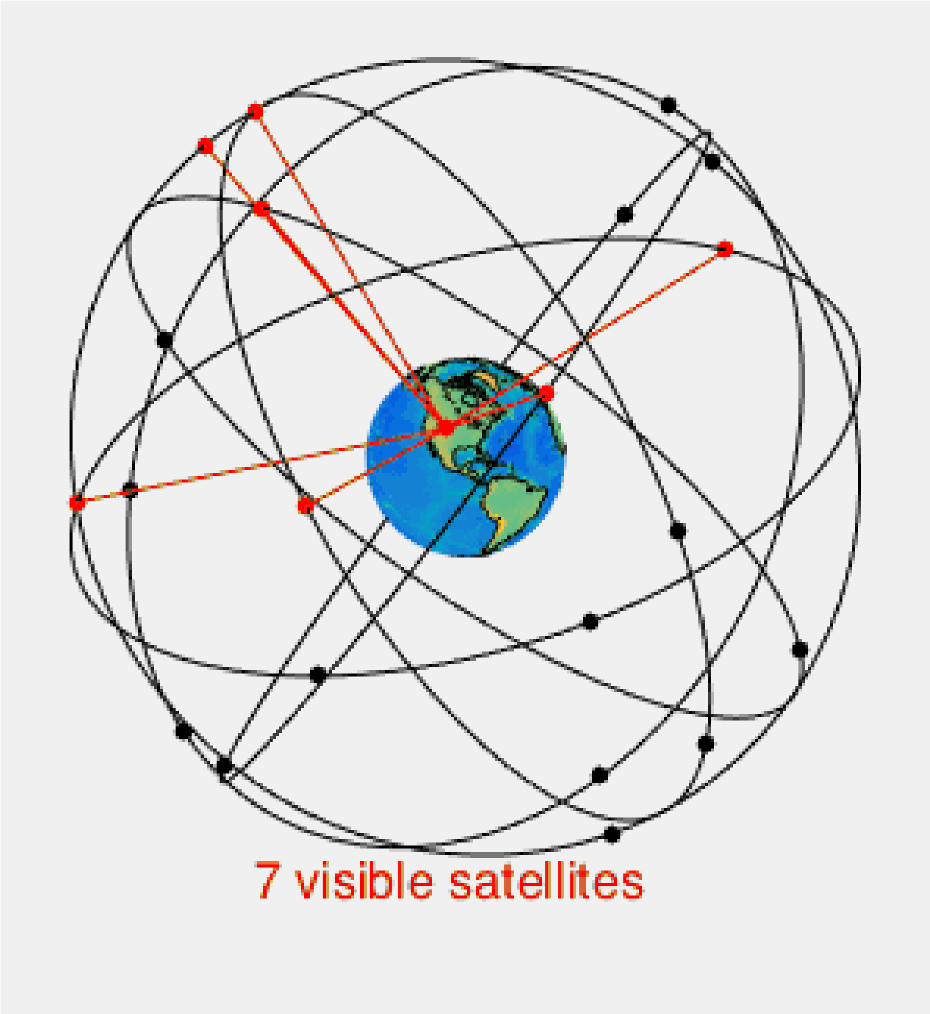
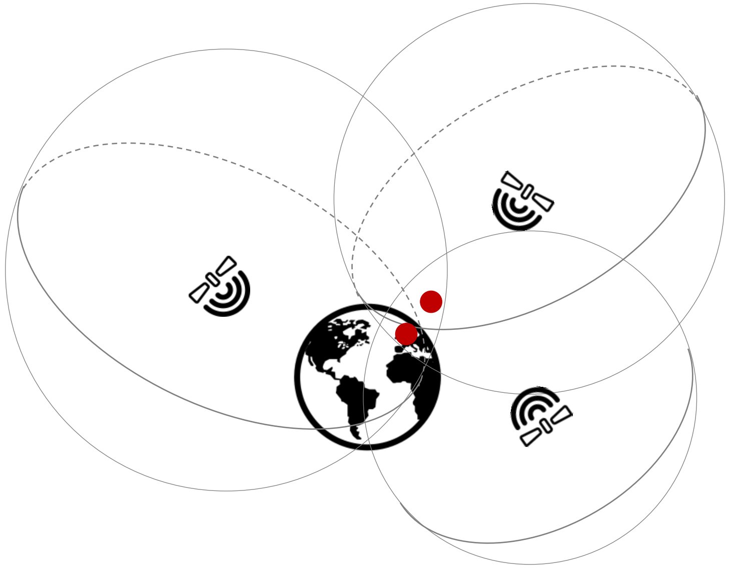
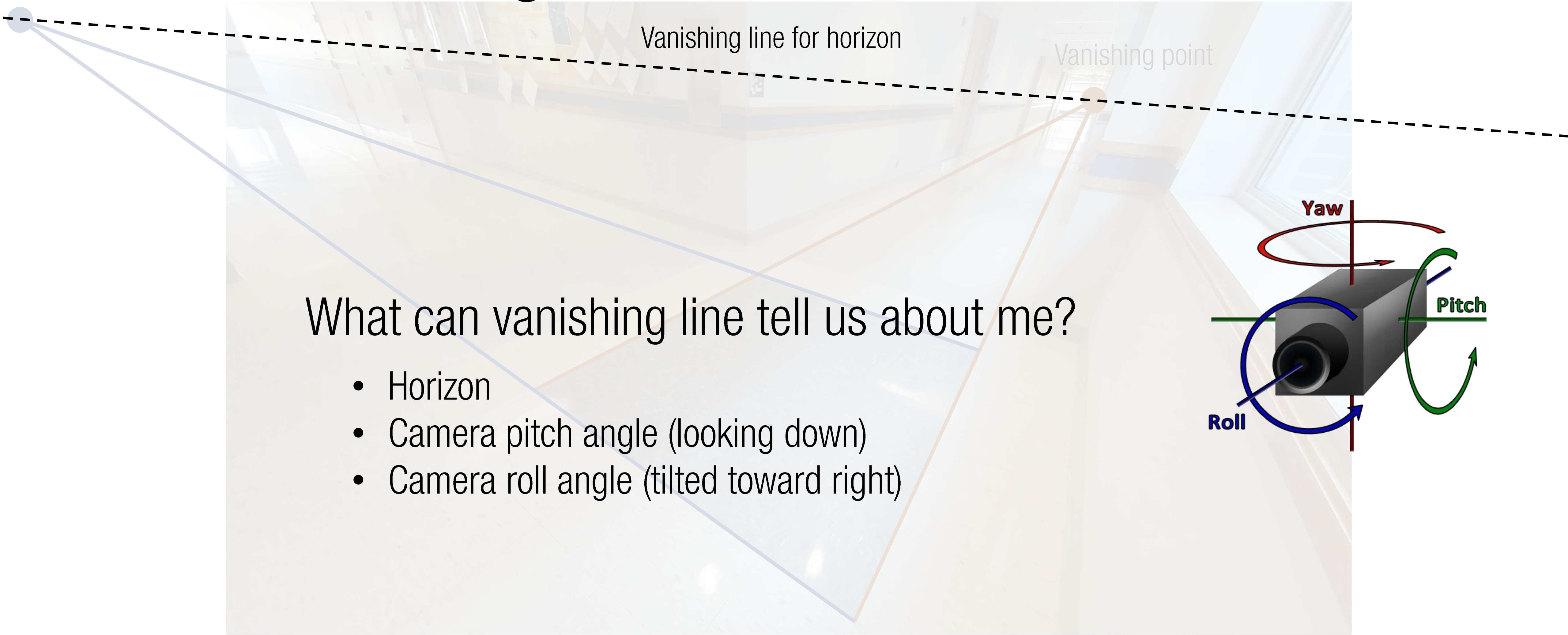




- Distance by time of flight (TOF)
- At least 4 satellites are visible (3+1 extra)
- Atomic clock synchronization

Recall: Vanishing Line

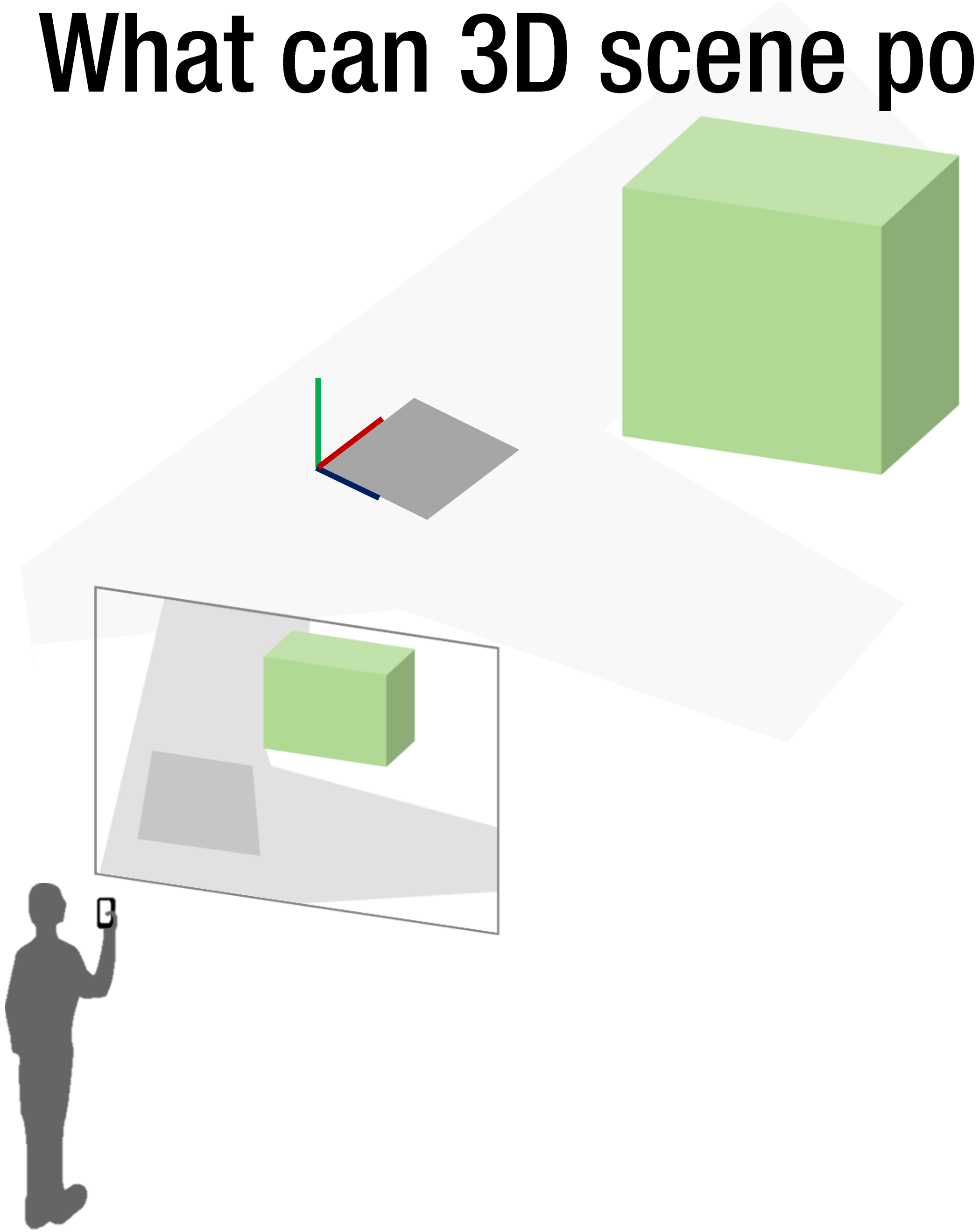


What can vanishing line tell us about me?

- Horizon
- Camera pitch angle (looking down)
- Camera roll angle (tilted toward right)

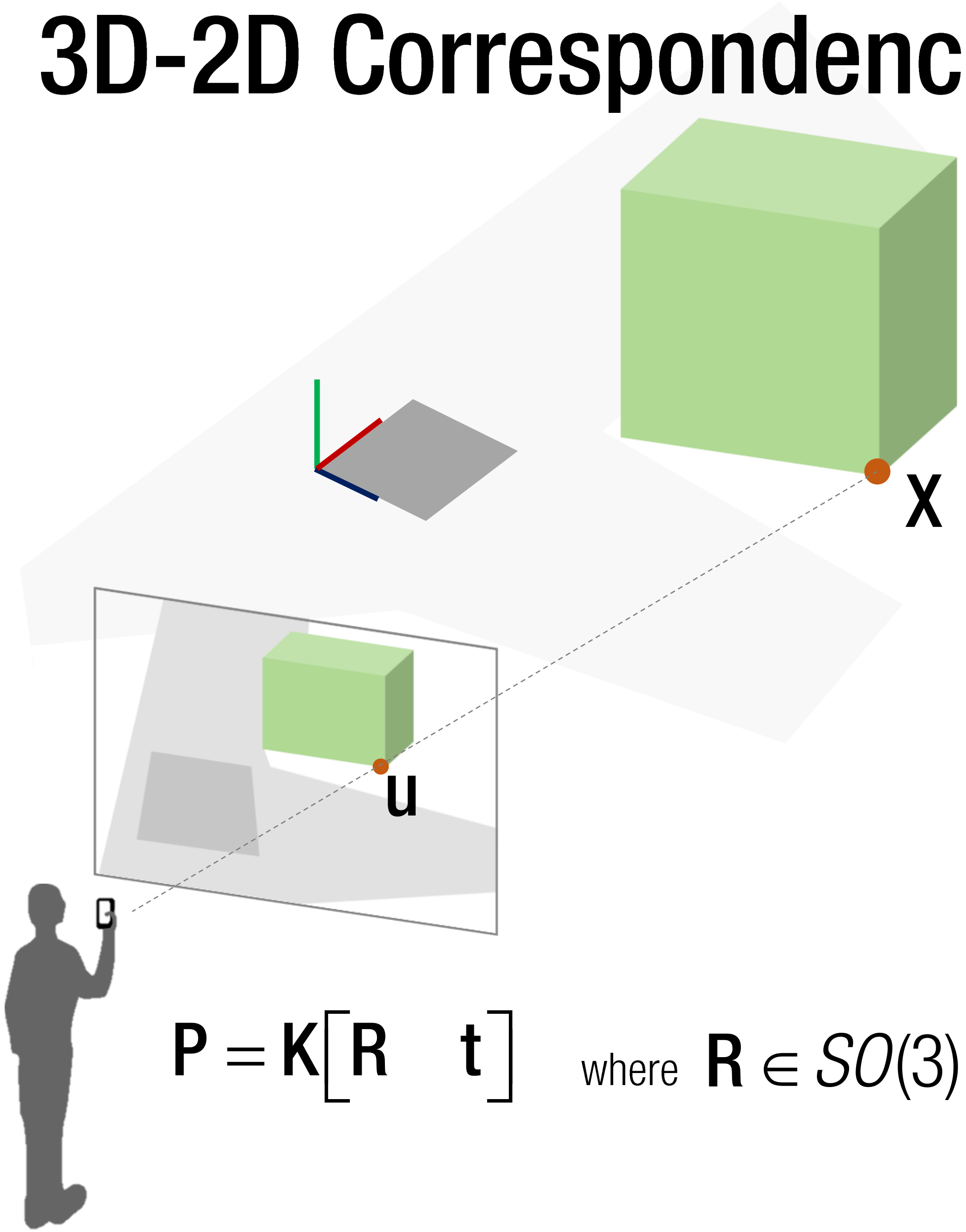
Keller Hall

What can 3D scene points tell us about?



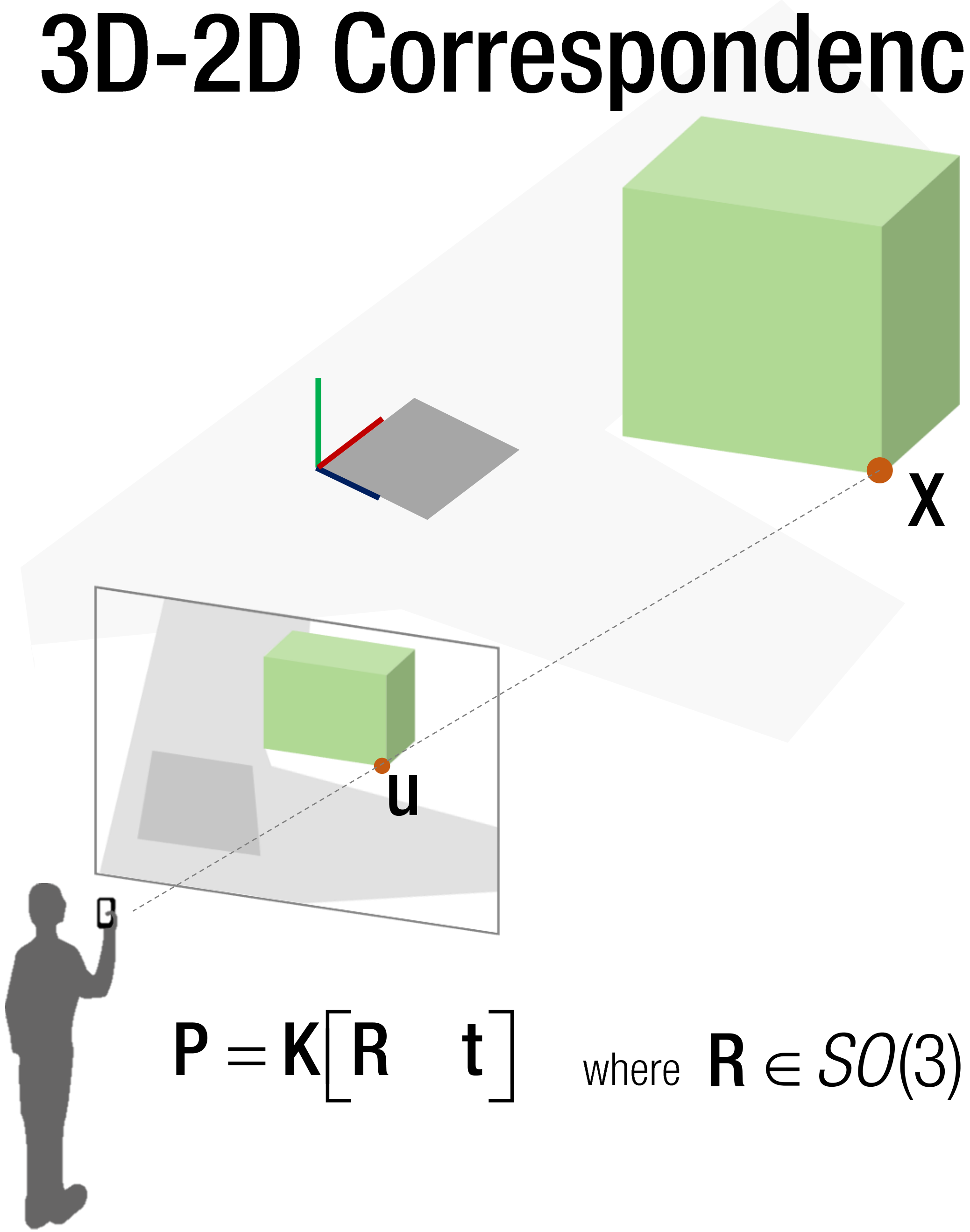
http://www.wired.com/gadgetlab/2010/07/camera-software-lets-you-see-into-the-past/

3D-2D Correspondence



$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

3D-2D Correspondence



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3D-2D correspondence: $\mathbf{u} \leftrightarrow \mathbf{X}$

$$\lambda \mathbf{u} = K\begin{bmatrix} R & t \end{bmatrix}\mathbf{X}$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

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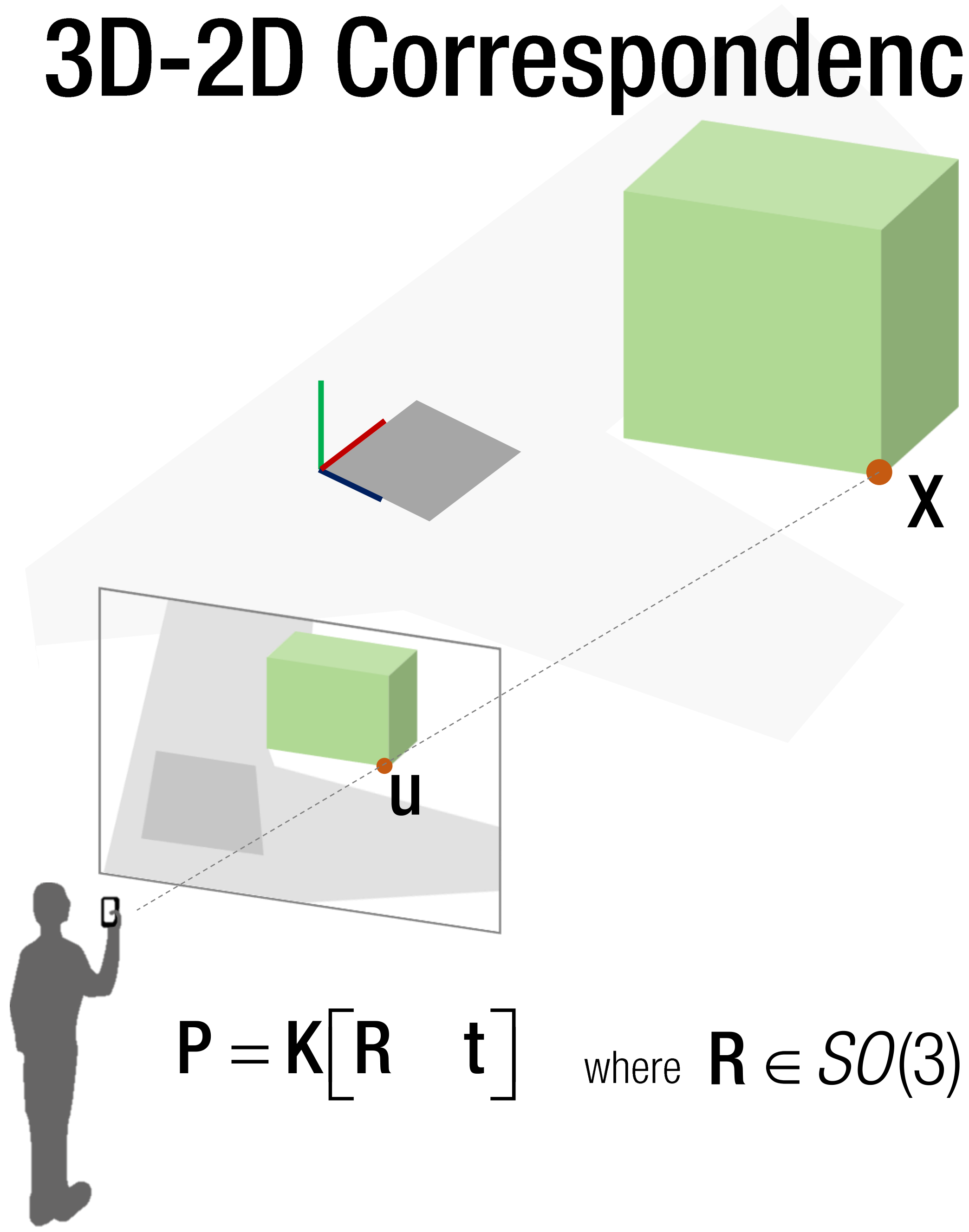

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Linear in camera matrix

$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

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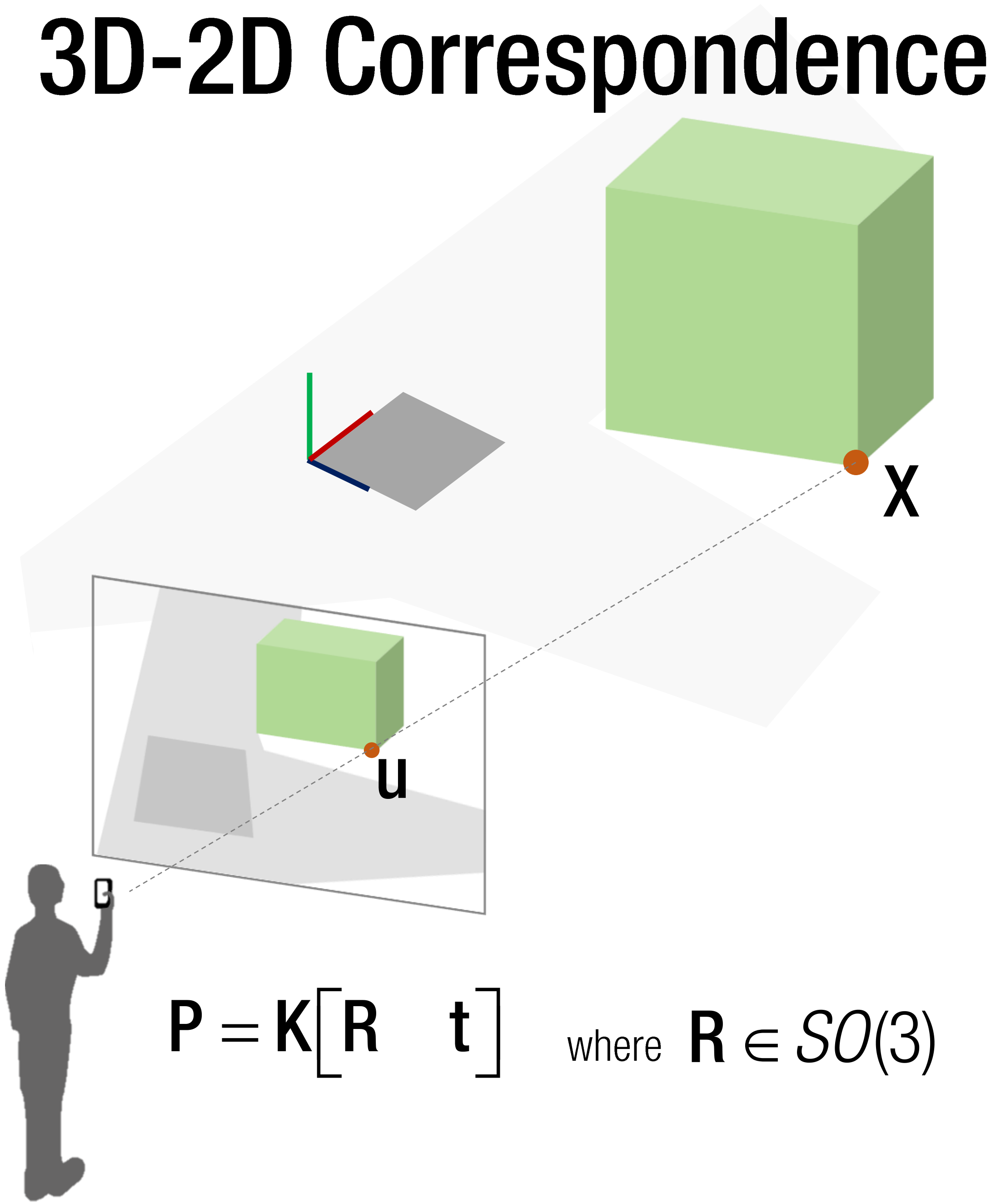

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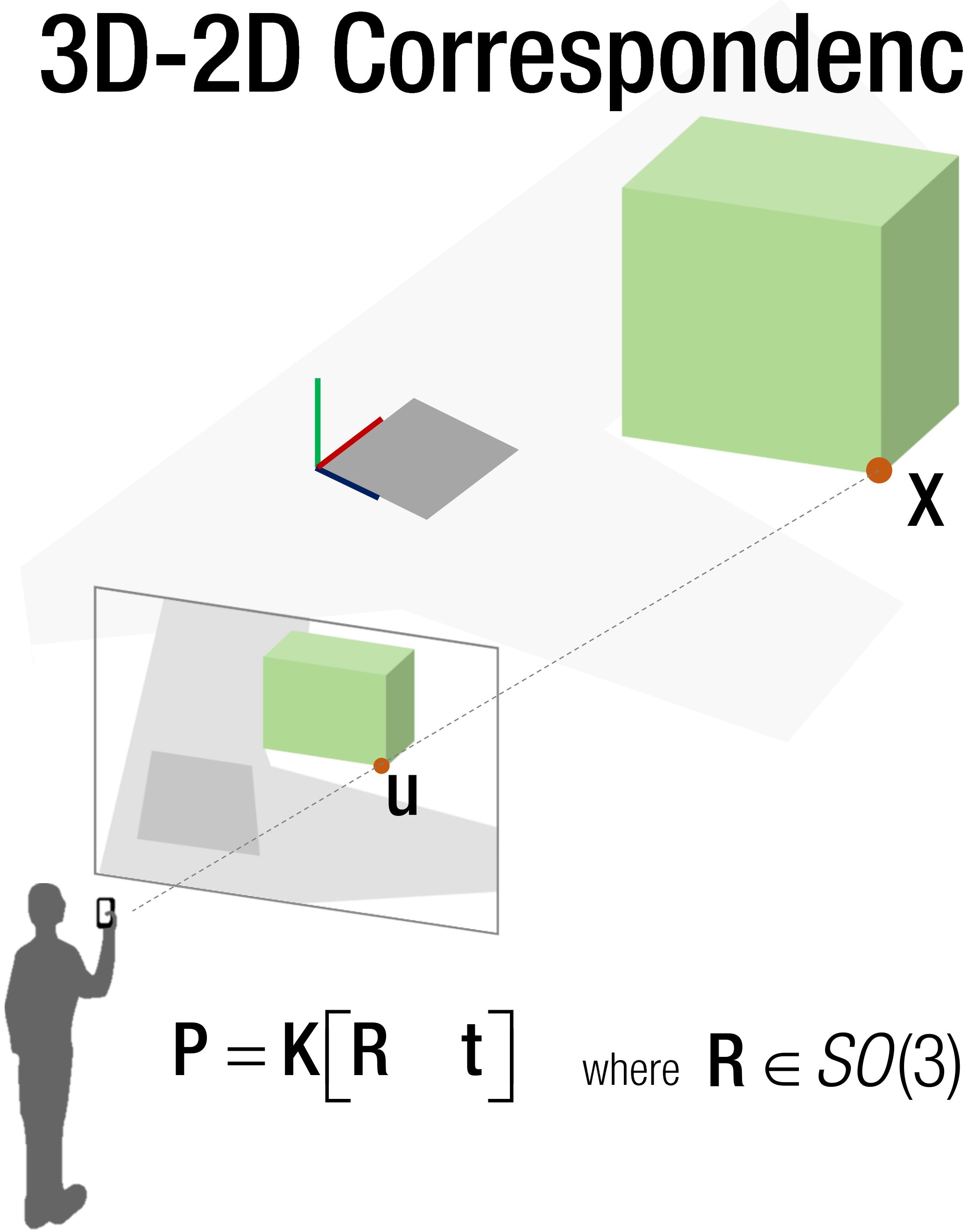
# of unknowns: 11 = 12 (3x4 matrix) – 1 (scale)
# of equations per correspondence: 2

3D-2D Correspondence



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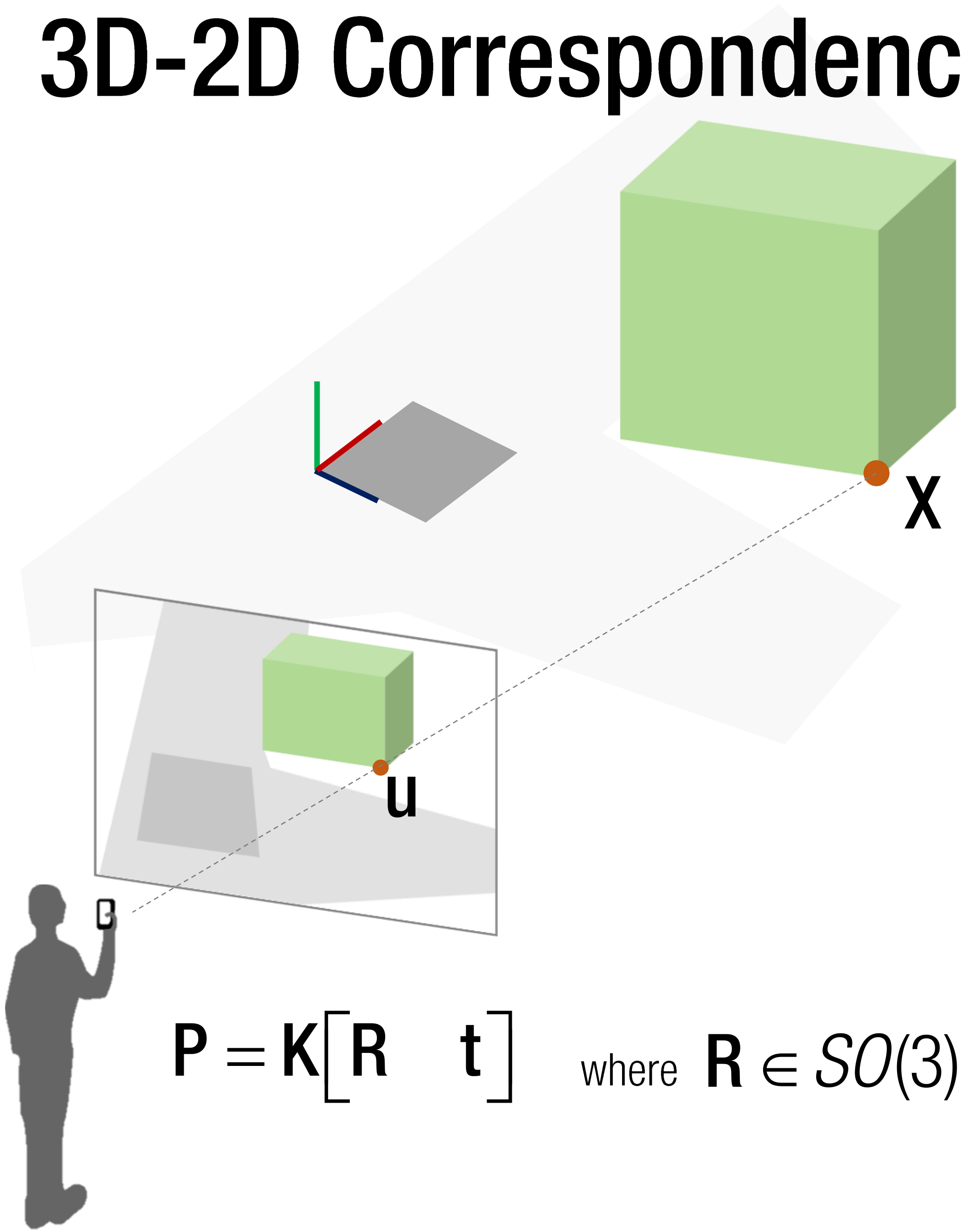

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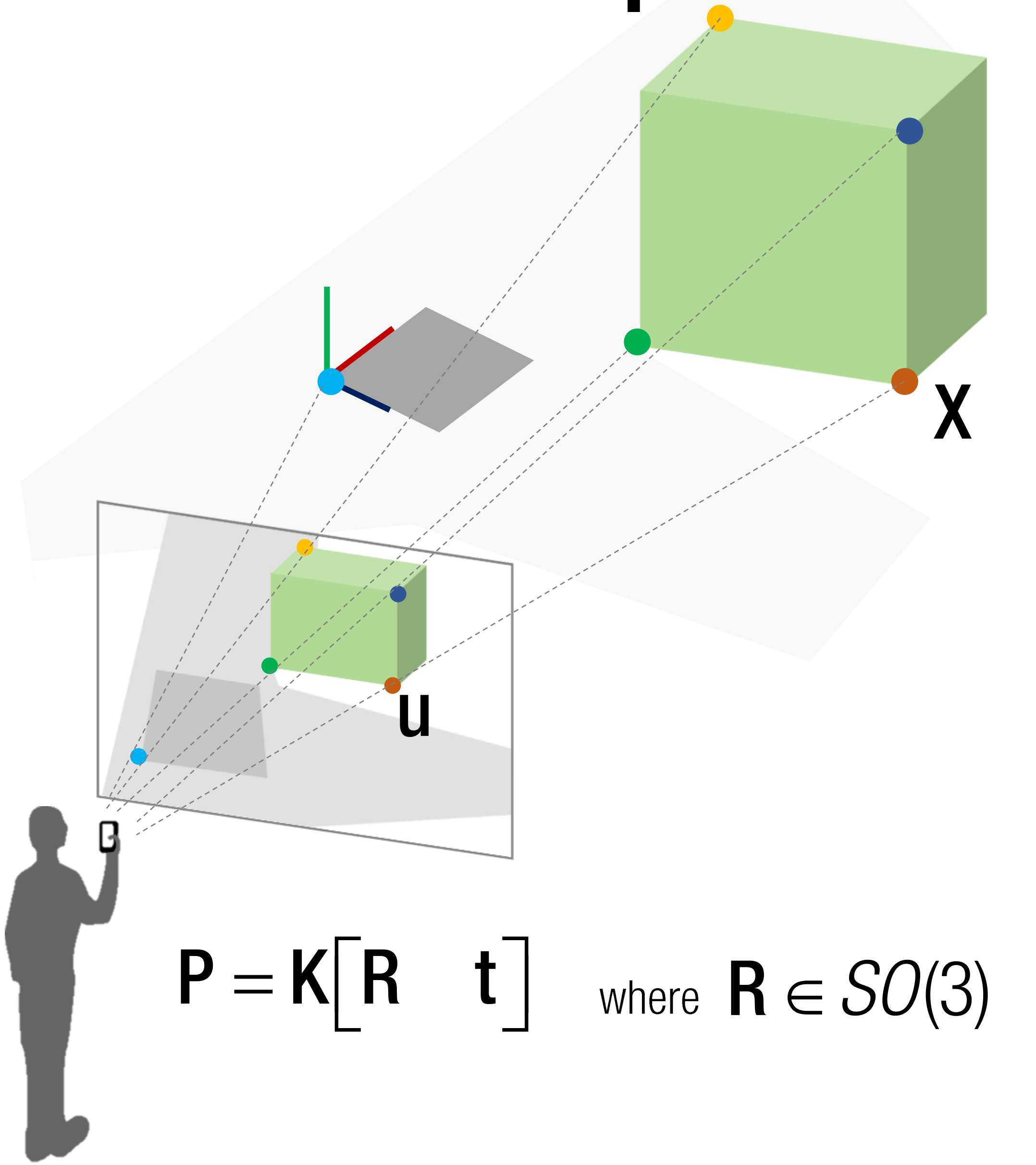
$$\begin{bmatrix} X & Y & Z & 1 & & & & & -u^x X & -u^x Y & -u^x Z & -u^x \\ & & & & X & Y & Z & 1 & -u^y X & -u^y Y & -u^y Z & -u^y \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2x12

3D-2D Correspondence



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$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & & & & & -u_1^x X & -u_1^x Y & -u_1^x Z & -u_1^x \\ & & & & X_1 & Y_1 & Z_1 & 1 & -u_1^y X & -u_1^y Y & -u_1^y Z & -u_1^y \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_m & Y_m & Z_m & 1 & & & & & -u_m^x X & -u_m^x Y & -u_m^x Z & -u_m^x \\ & & & & X_m & Y_m & Z_m & 1 & -u_m^y X & -u_m^y Y & -u_m^y Z & -u_m^y \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$2m$x12

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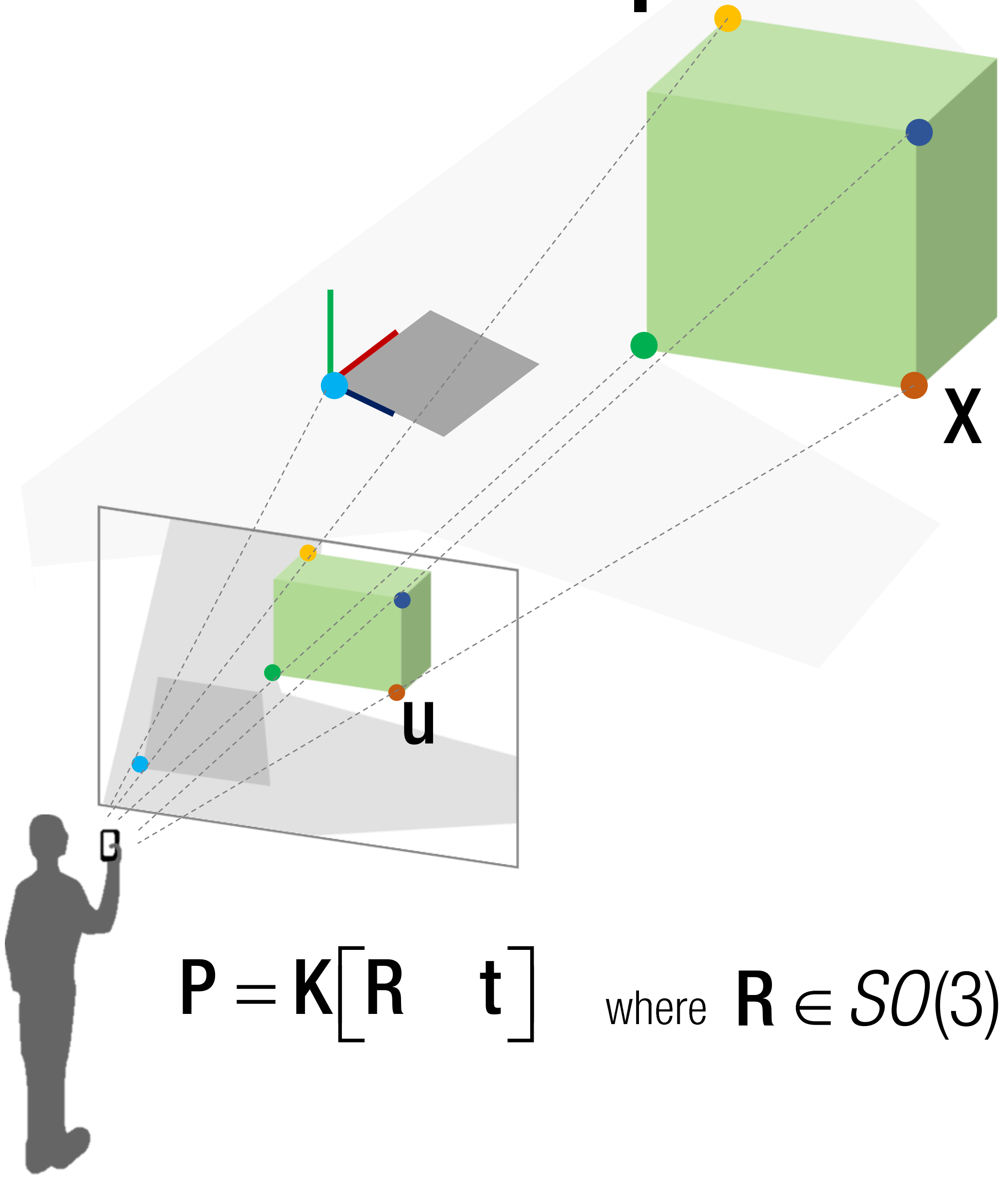

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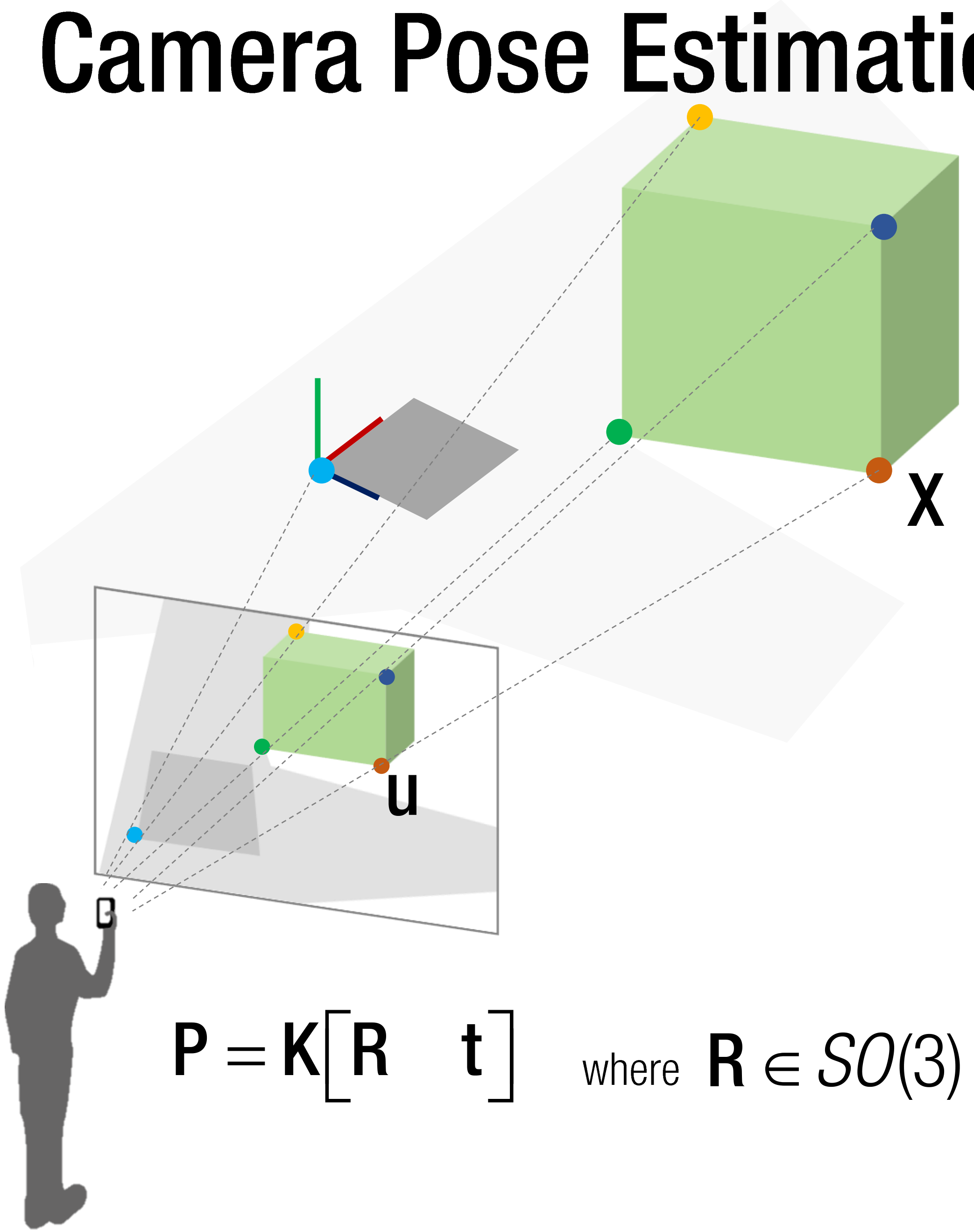
$2m$x12

Camera Pose Estimation



$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

If **K** is given,

$$\mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \gamma\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4\end{bmatrix}$$

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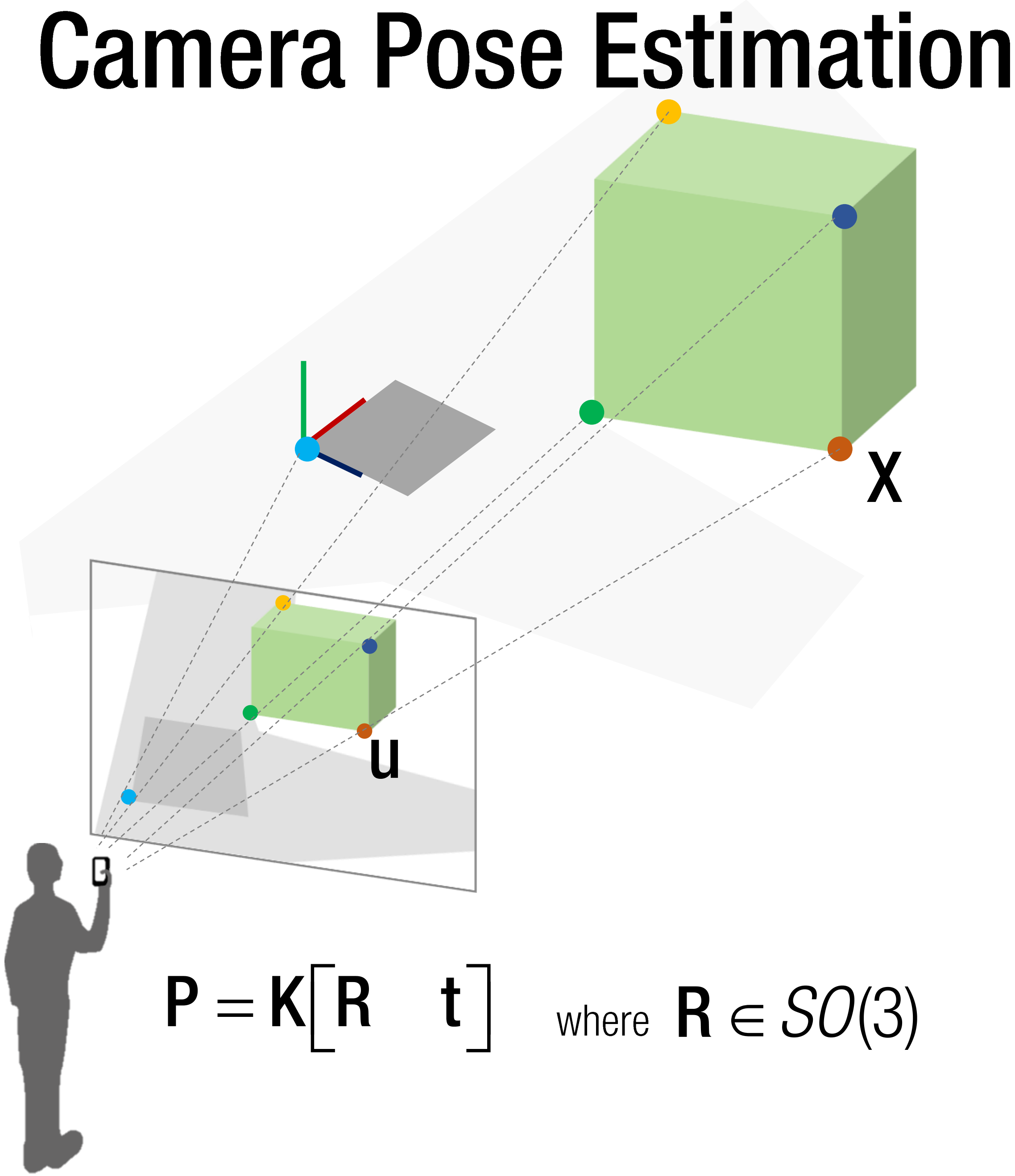

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If **K** is given,

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$$\longrightarrow \gamma R = K^{-1}\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$$

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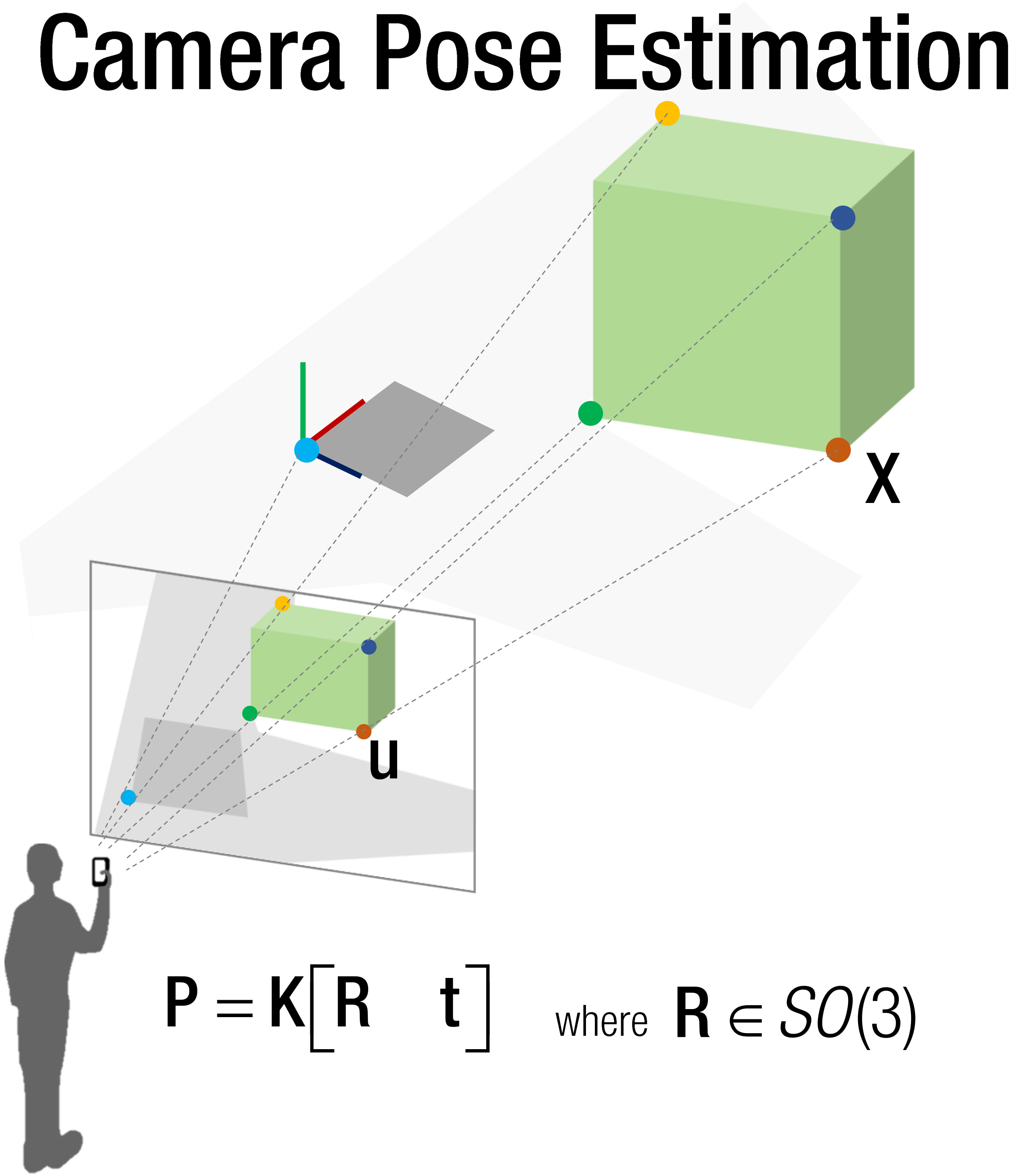

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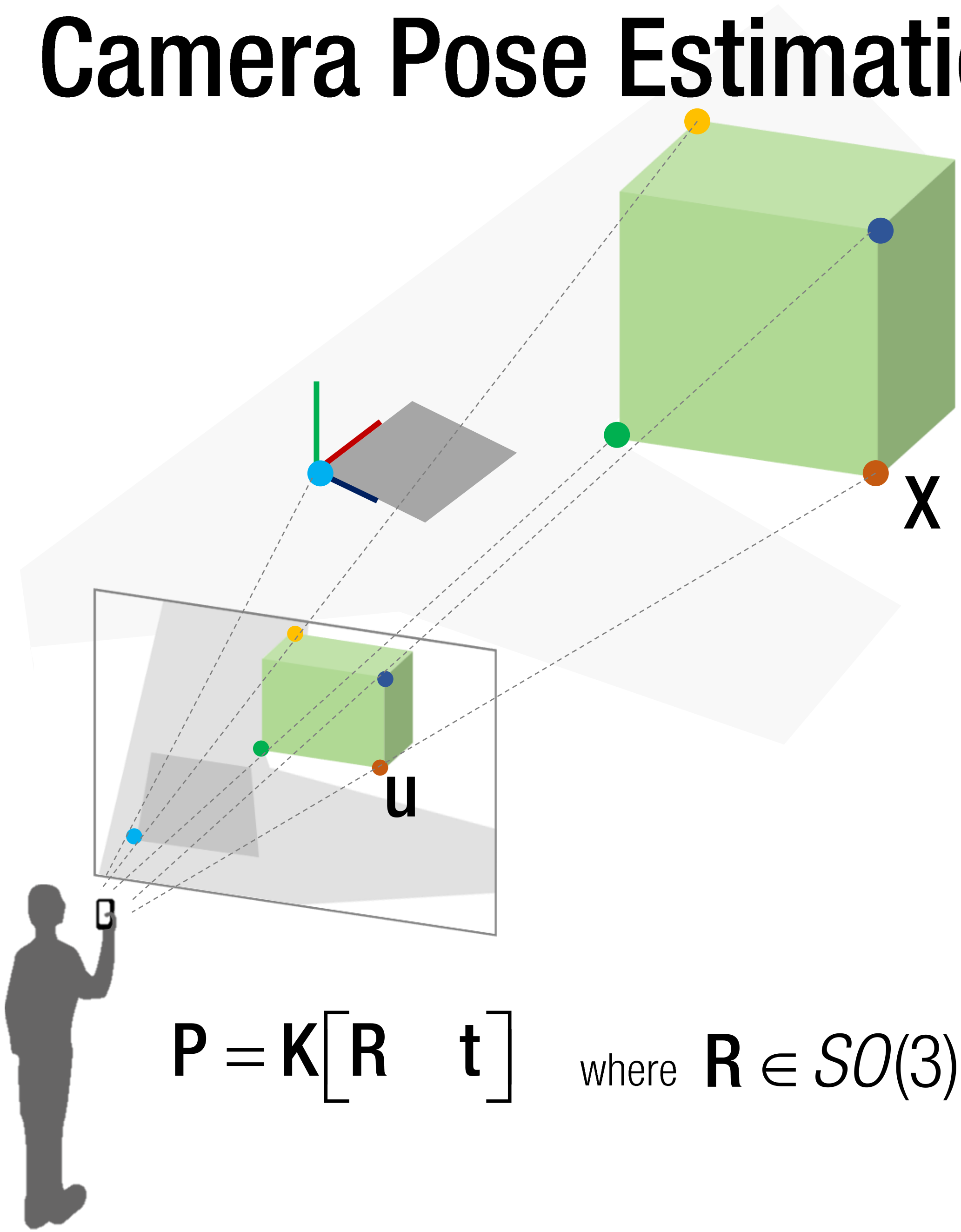
$$\longrightarrow \gamma\mathbf{R} = \mathbf{K}^{-1}\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix}$$

$$\mathbf{K}^{-1}\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix} = \mathbf{U}\begin{bmatrix}d_{11} & & \\ & d_{22} & \\ & & d_{33}\end{bmatrix}\mathbf{V}^{\mathsf{T}}$$

Camera Pose Estimation



$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

If **K** is given,

$$\mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \gamma\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4\end{bmatrix}$$

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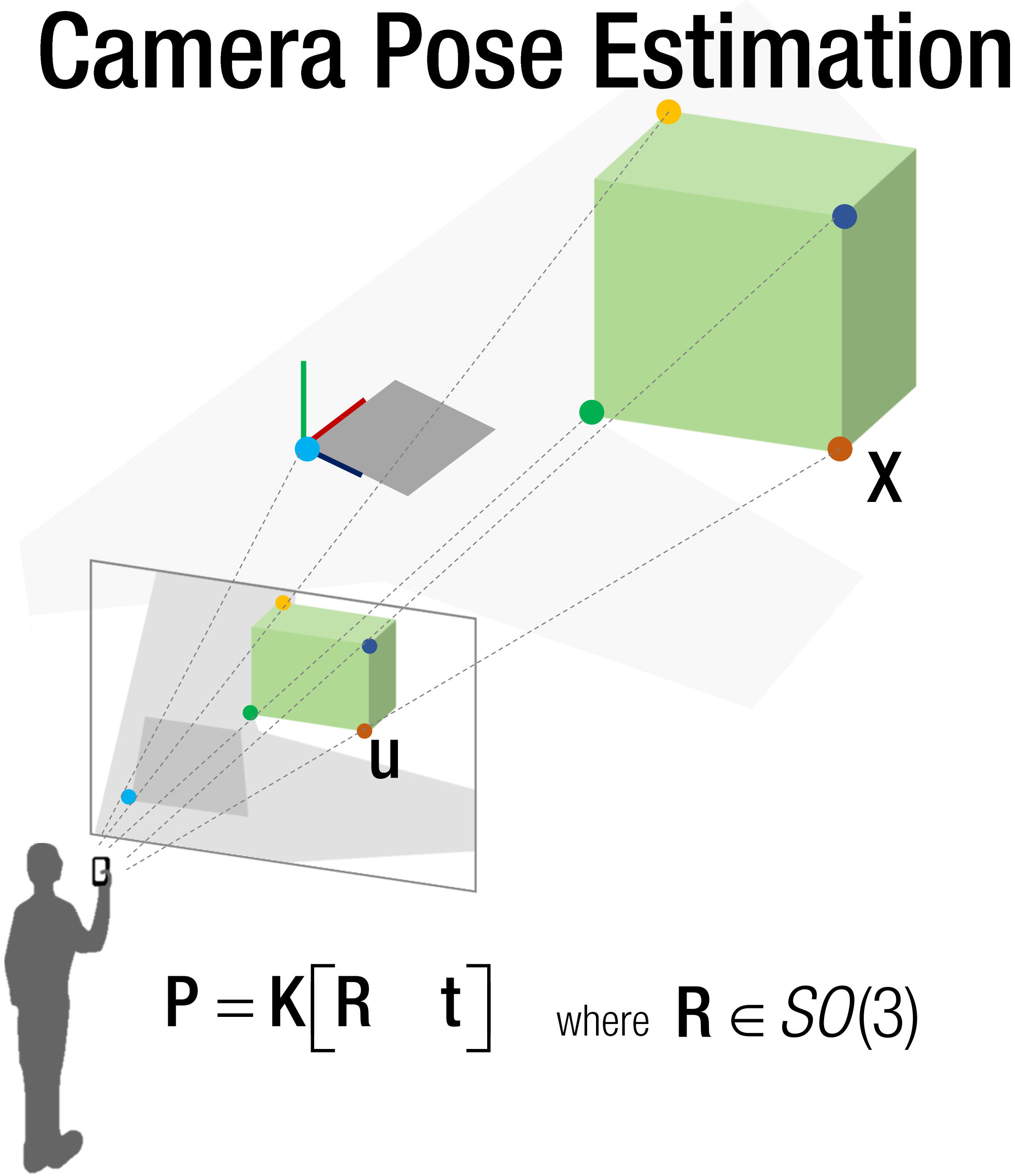
$$\longrightarrow \gamma \approx d_{11}$$

$\mathbf{R} = \mathbf{U}\mathbf{V}^{\mathrm{T}}$: SVD cleanup

Camera Pose Estimation



$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

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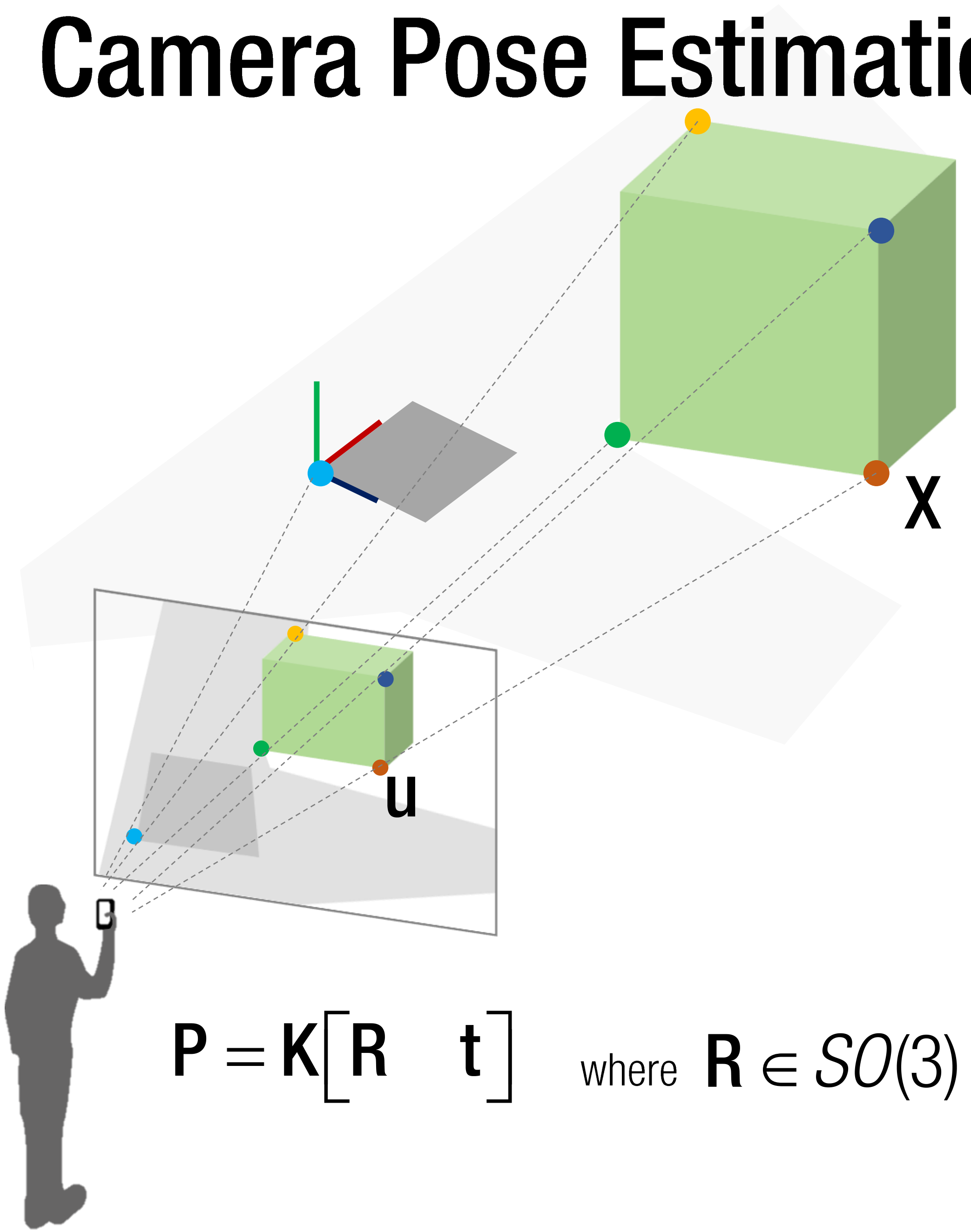
$$\mathbf{K}^{-1}\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix} = \mathbf{U}\begin{bmatrix}d_{11} & & \\ & d_{22} & \\ & & d_{33}\end{bmatrix}\mathbf{V}^{\mathsf{T}}$$

$$\longrightarrow \gamma \approx d_{11}$$

$$\mathbf{R} = \mathbf{U}\mathbf{V}^{\mathsf{T}}$$: SVD cleanup

$$\longrightarrow \mathbf{t} = \frac{\mathbf{K}^{-1}\mathbf{p}_4}{d_{11}}$$: Translation and scale recovery

Camera Pose Estimation



$$P = K\begin{bmatrix} R & t \end{bmatrix} \quad \text{where } R \in SO(3)$$

```
function [R t] = LinearPnP(X, u, K)
A = [];
for i = 1 : size(X,1)
   %% Build A matrix here
End

[u d v] = svd(A);
P = v(:,end);
P = [P(1:4)'; P(5:8)'; P(9:12)'];
R = inv(K)*P(:,1:3);
t = inv(K)*P(:,4);

% SVD clean up
[u d v] = svd(R);
R = u * v';
t = t/d(1,1);

if det(R) < 0
   R = -R;
   t = -t;
end
```

$$R = UV^T$$

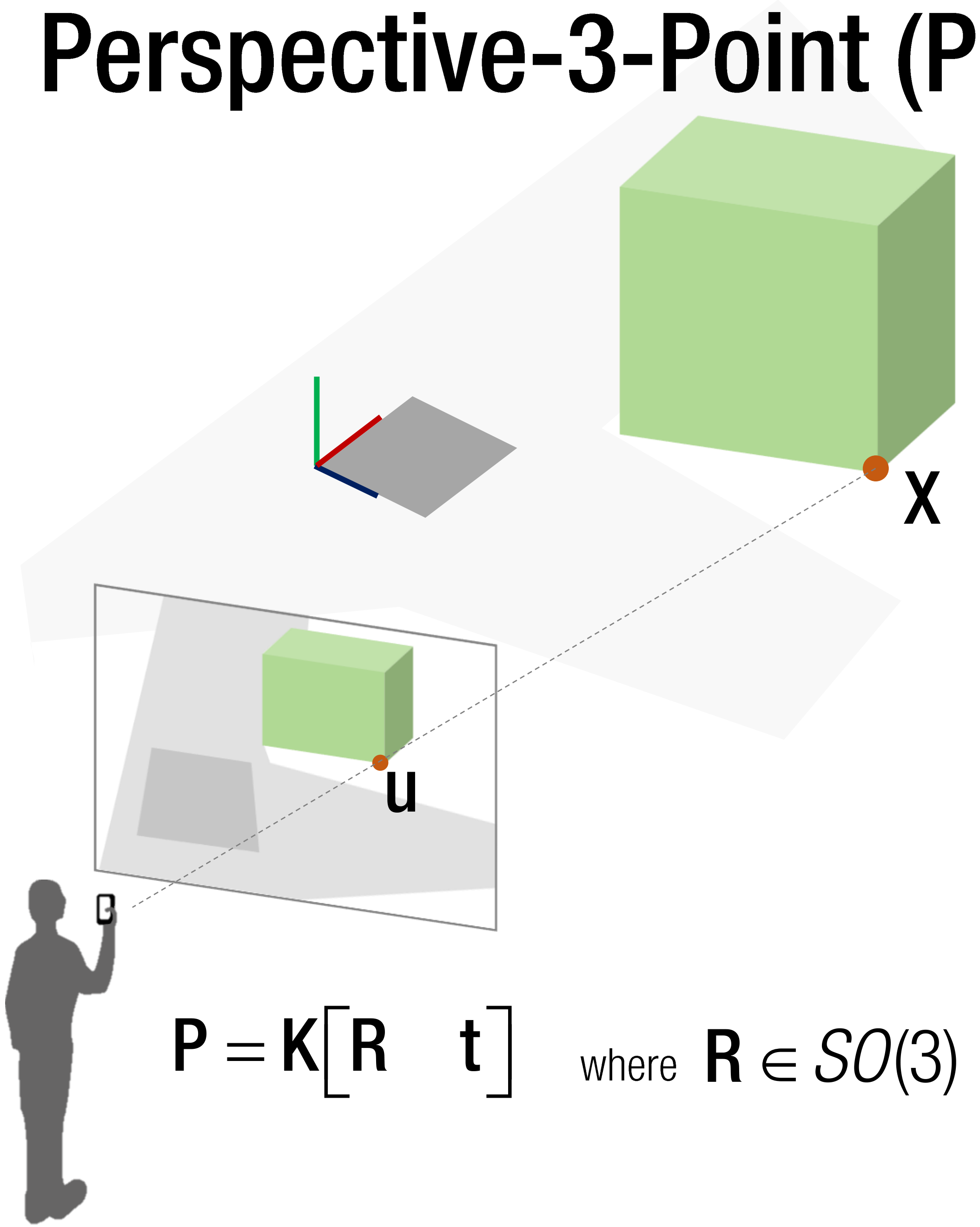
$$t = \frac{K^{-1}p_4}{d_{11}}$$

RANSAC Camera Pose Estimation

Algorithm 1 PnP RANSAC

$nInliers \leftarrow 0$
for $i = 1 : M$ **do**
    Choose 6 correspondences, $\mathbf{X}_r$ and $\mathbf{w}_r$, randomly from $\mathbf{X}$ and $\mathbf{w}$.
    $[\mathbf{R}_r, \mathbf{t}_r]$ = LinearPnP($\mathbf{X}_r$, $\mathbf{w}_r$, $\mathbf{K}$)
    Compute the number of inliers, n_r, with respect to $\mathbf{R}_r$, $\mathbf{t}_r$.
    if $n_r > nInliers$ **then**
        $nInliers \leftarrow n_r$
        $\mathbf{R} = \mathbf{R}_r$ and $\mathbf{t} = \mathbf{t}_r$
    end if
end for

Perspective-3-Point (P3P)



$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

3D-2D correspondence: $\mathbf{u} \leftrightarrow \mathbf{X}$

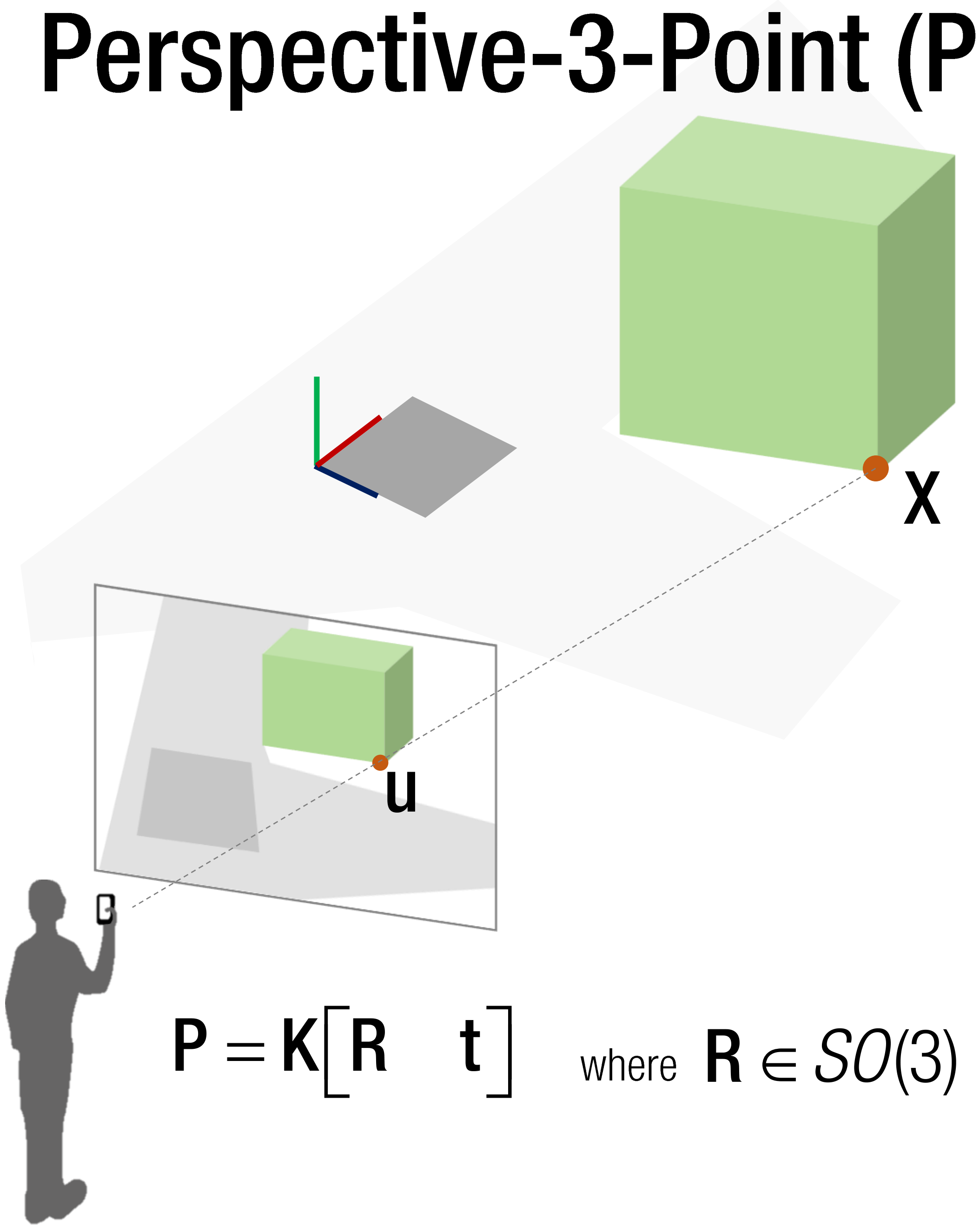
$$\lambda \mathbf{u} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Linear in camera matrix

$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

Perspective-3-Point (P3P)



$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$ where $\mathbf{R} \in SO(3)$

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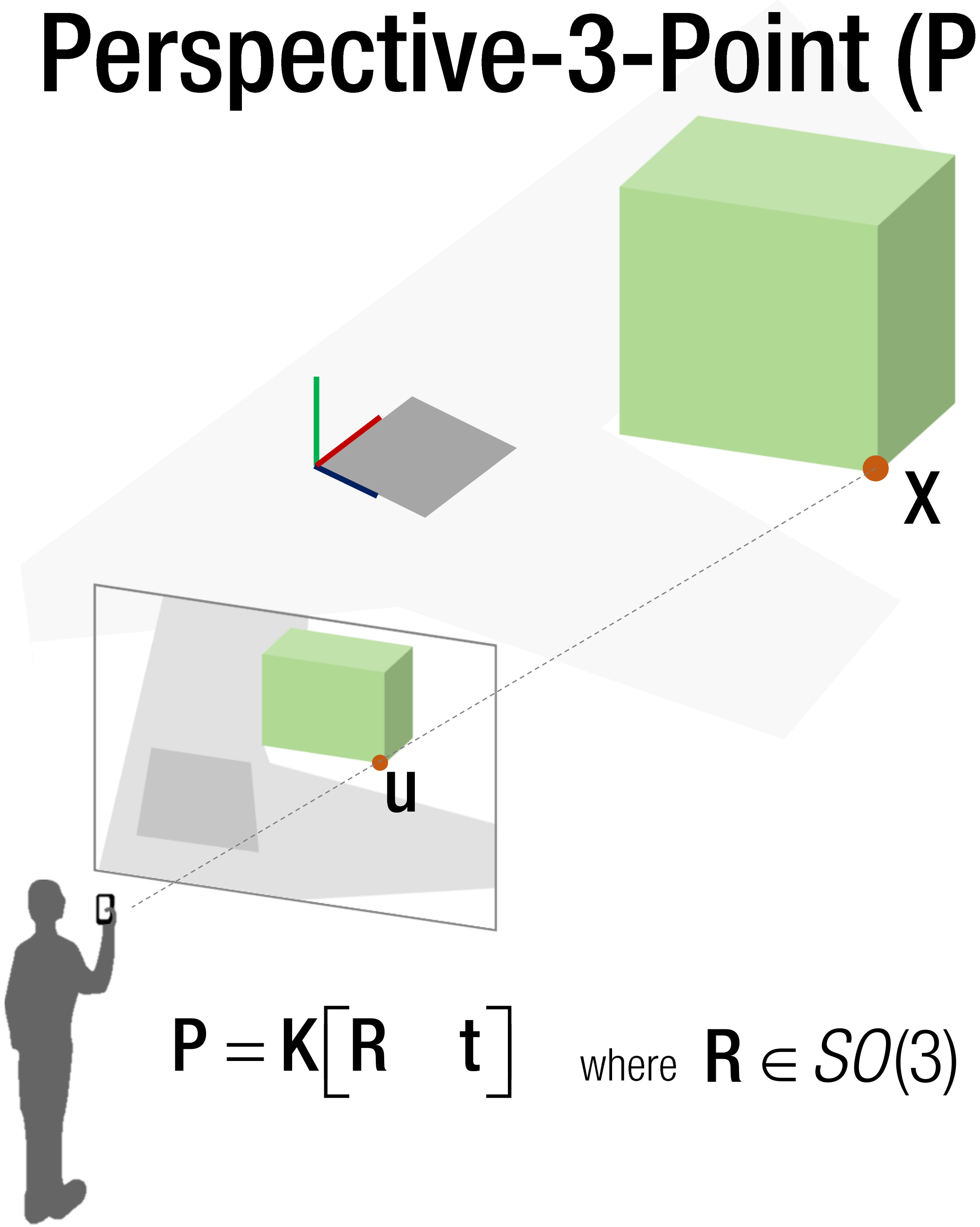
# of unknowns: 11 = 12 (3x4 matrix) – 1 (scale)

6 dof when **K** is known.

# of equations per correspondence: 2

Perspective-3-Point (P3P)



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3D-2D correspondence: $\mathbf{u} \leftrightarrow \mathbf{X}$

$$\lambda \mathbf{u} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

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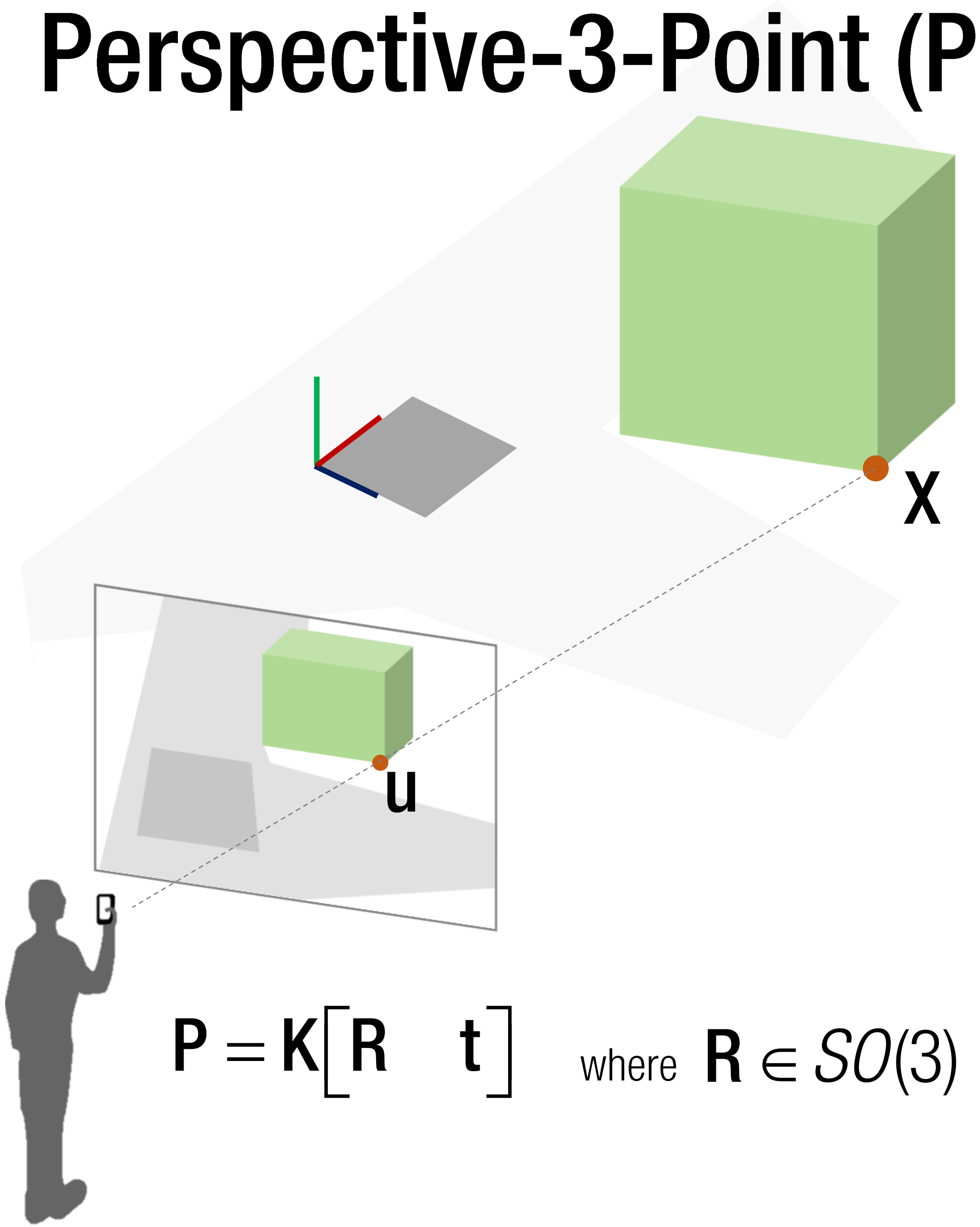
Perspective-3-Point (P3P)



$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} \quad \text{where } \mathbf{R} \in SO(3)$$

3D-2D correspondence: $\mathbf{u} \leftrightarrow \mathbf{X}$

$$\lambda \mathbf{u} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Linear in camera matrix

$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

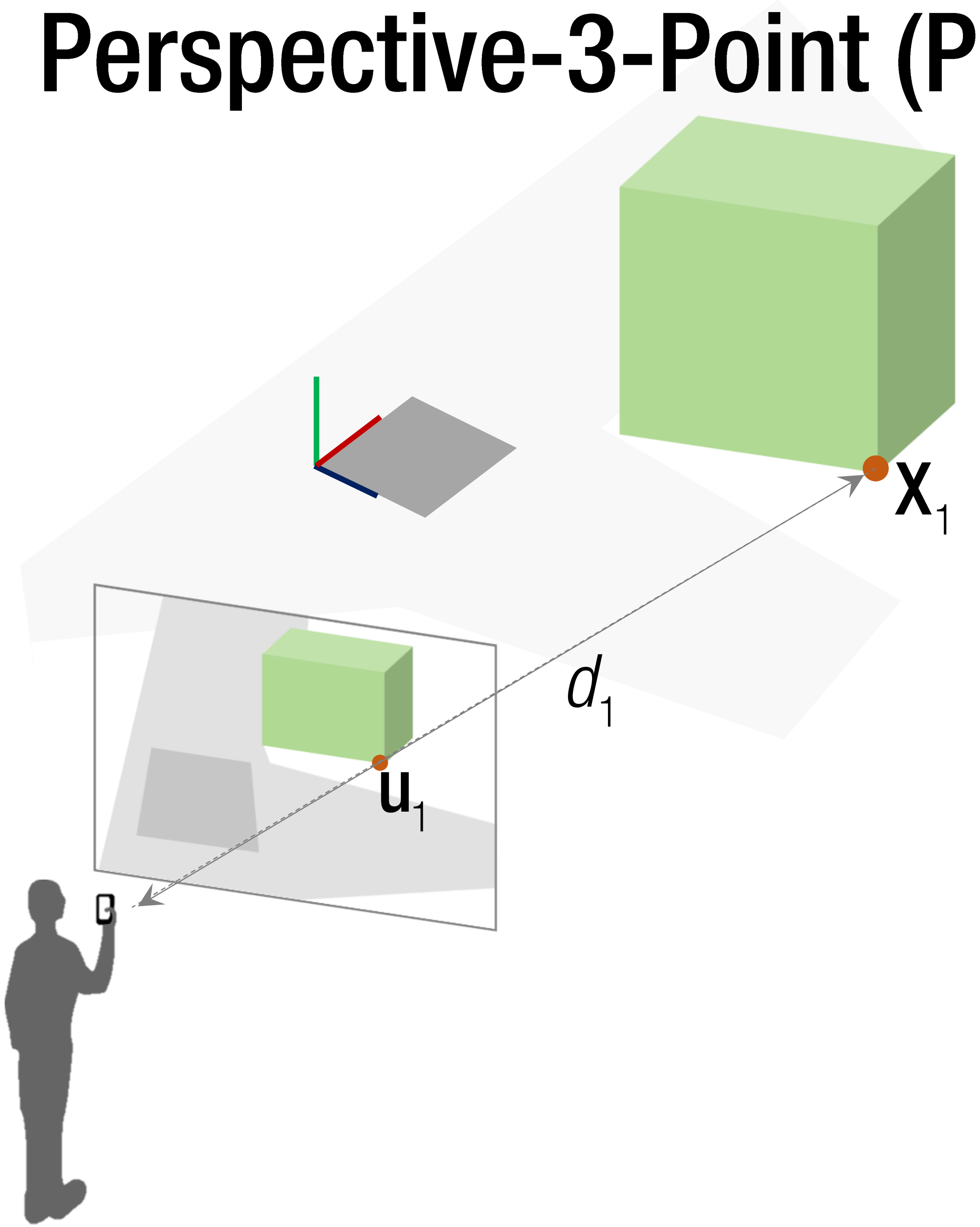
\# of unknowns: 11 = 12 (3x4 matrix) – 1 (scale)

6 dof when **K** is known.

\# of equations per correspondence: 2

3 correspondences should be enough.

Perspective-3-Point (P3P)

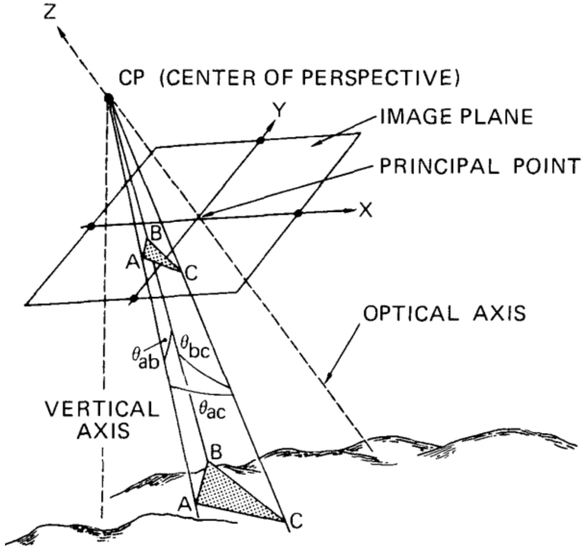


RANSAC with PnP

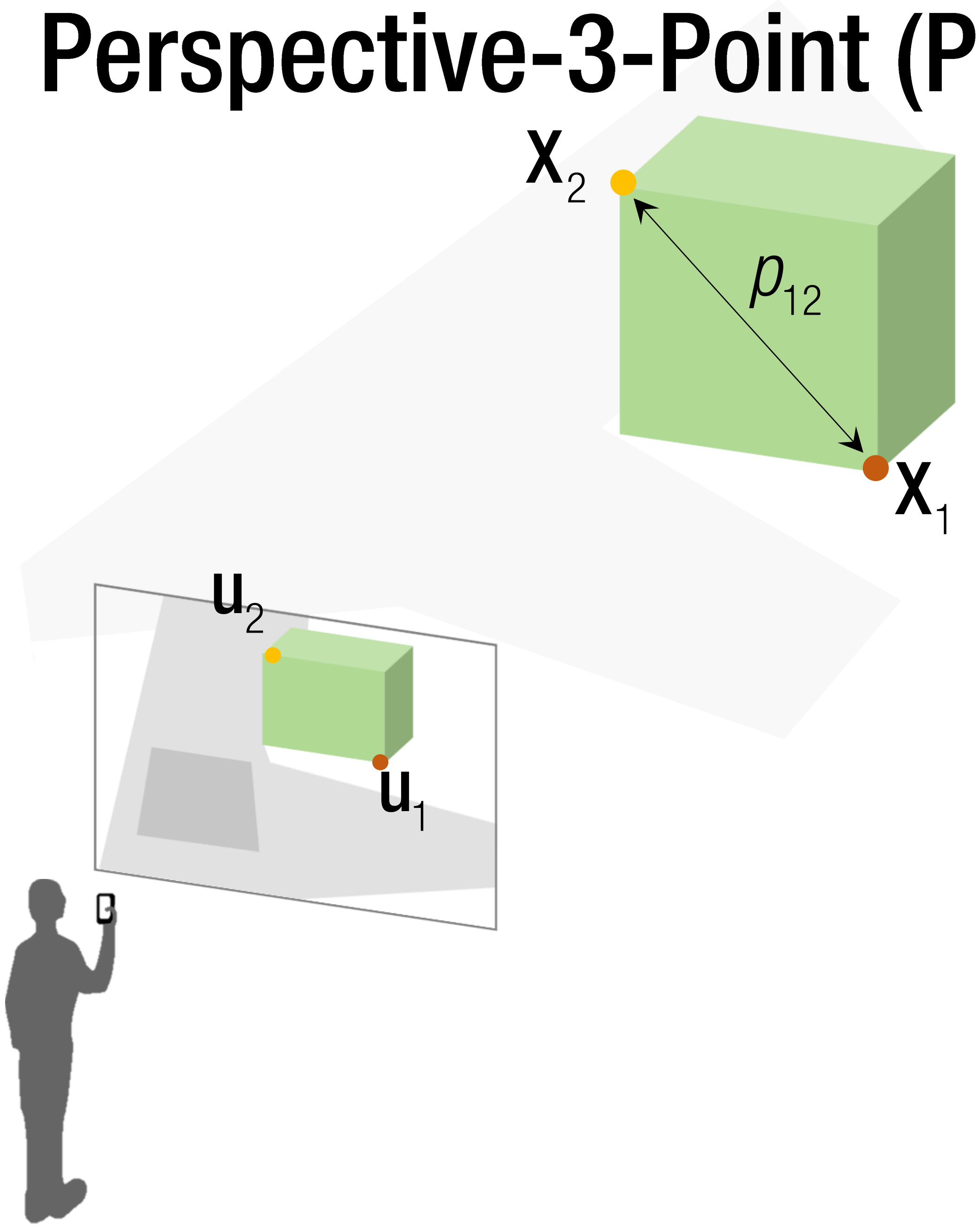
Graphics and Image Processing — J. D. Foley, Editor

Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography

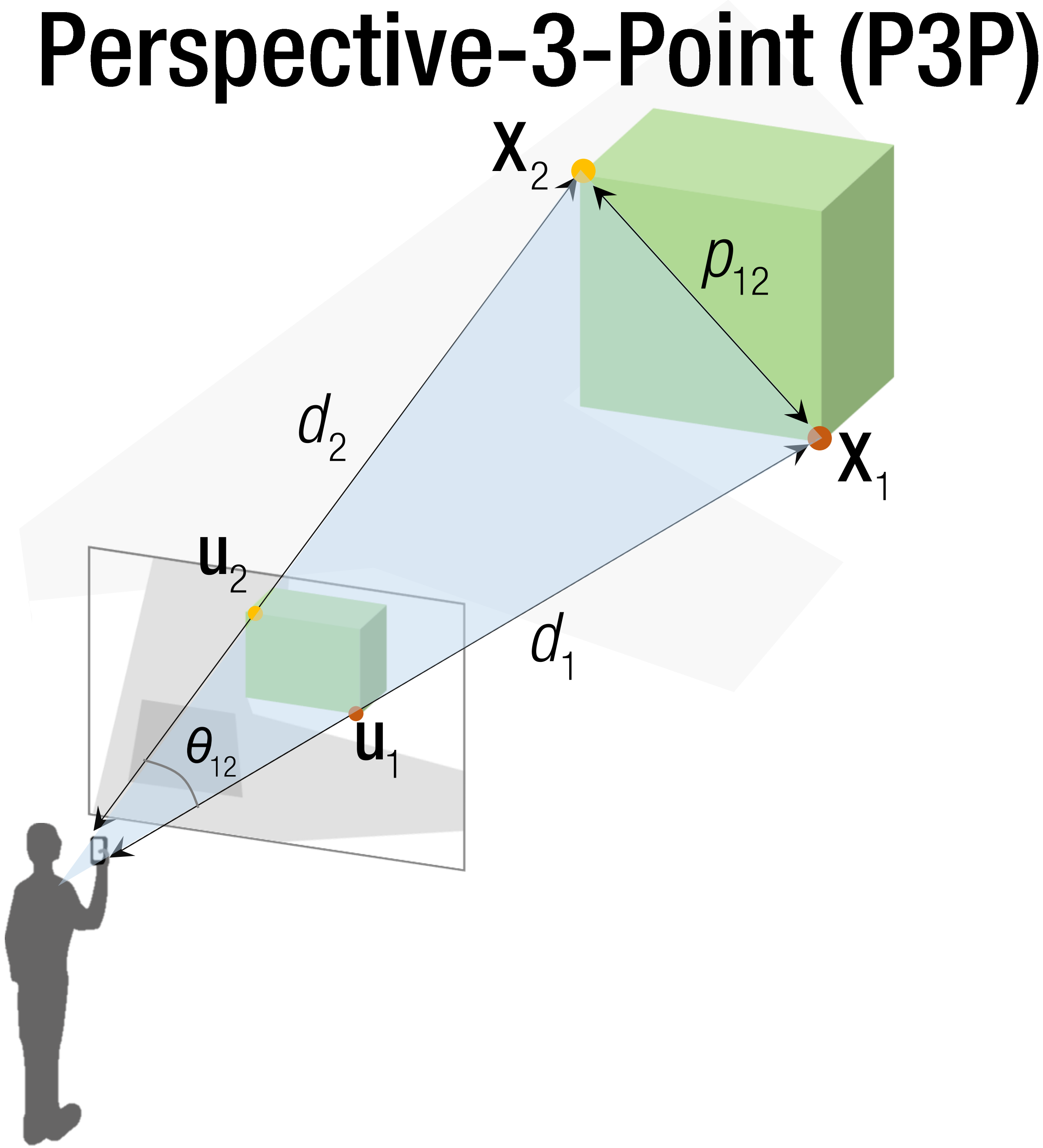
Martin A. Fischler and Robert C. Bolles
SRI International



Perspective-3-Point (P3P)



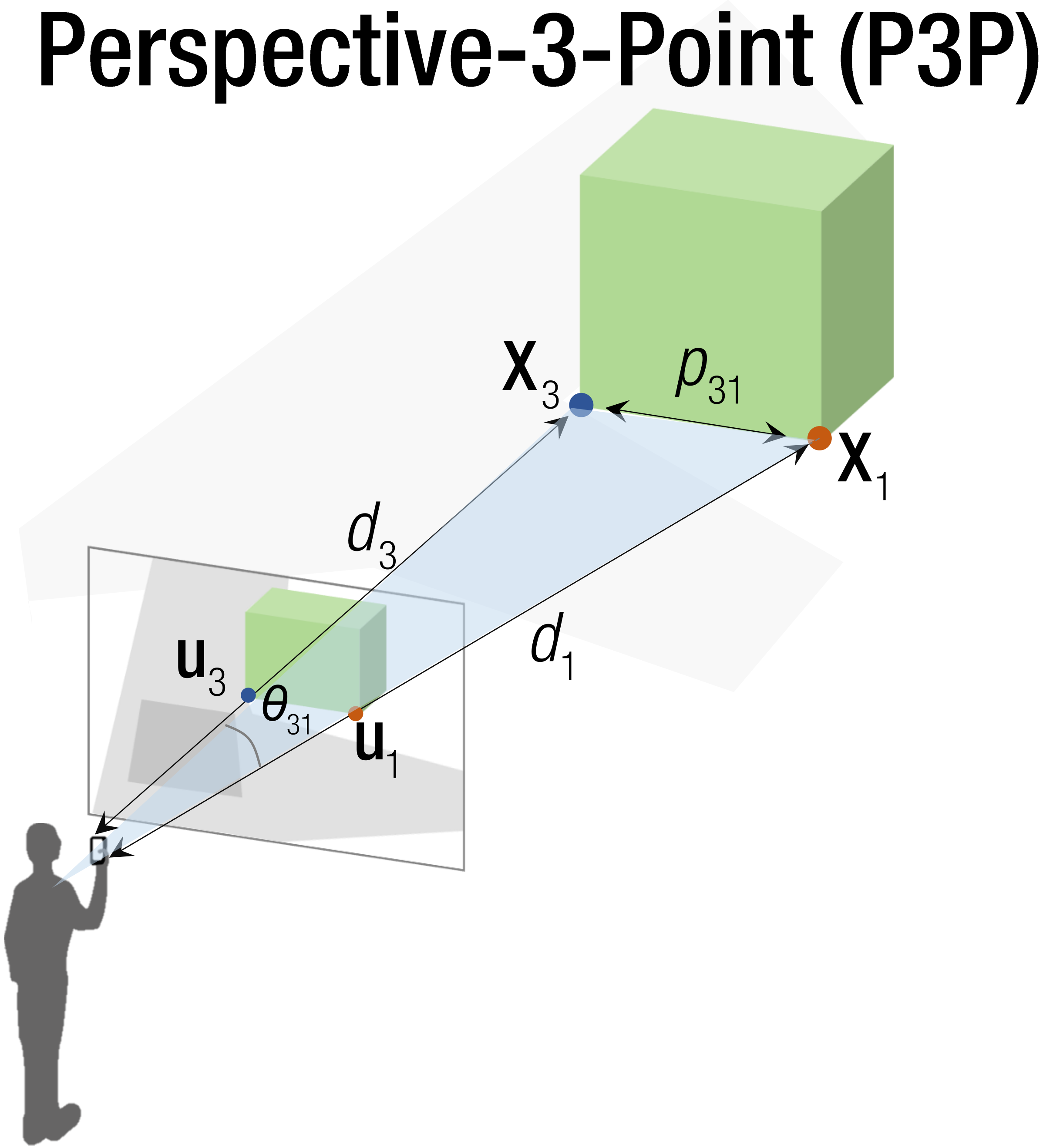
Perspective-3-Point (P3P)



2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2$$

Perspective-3-Point (P3P)

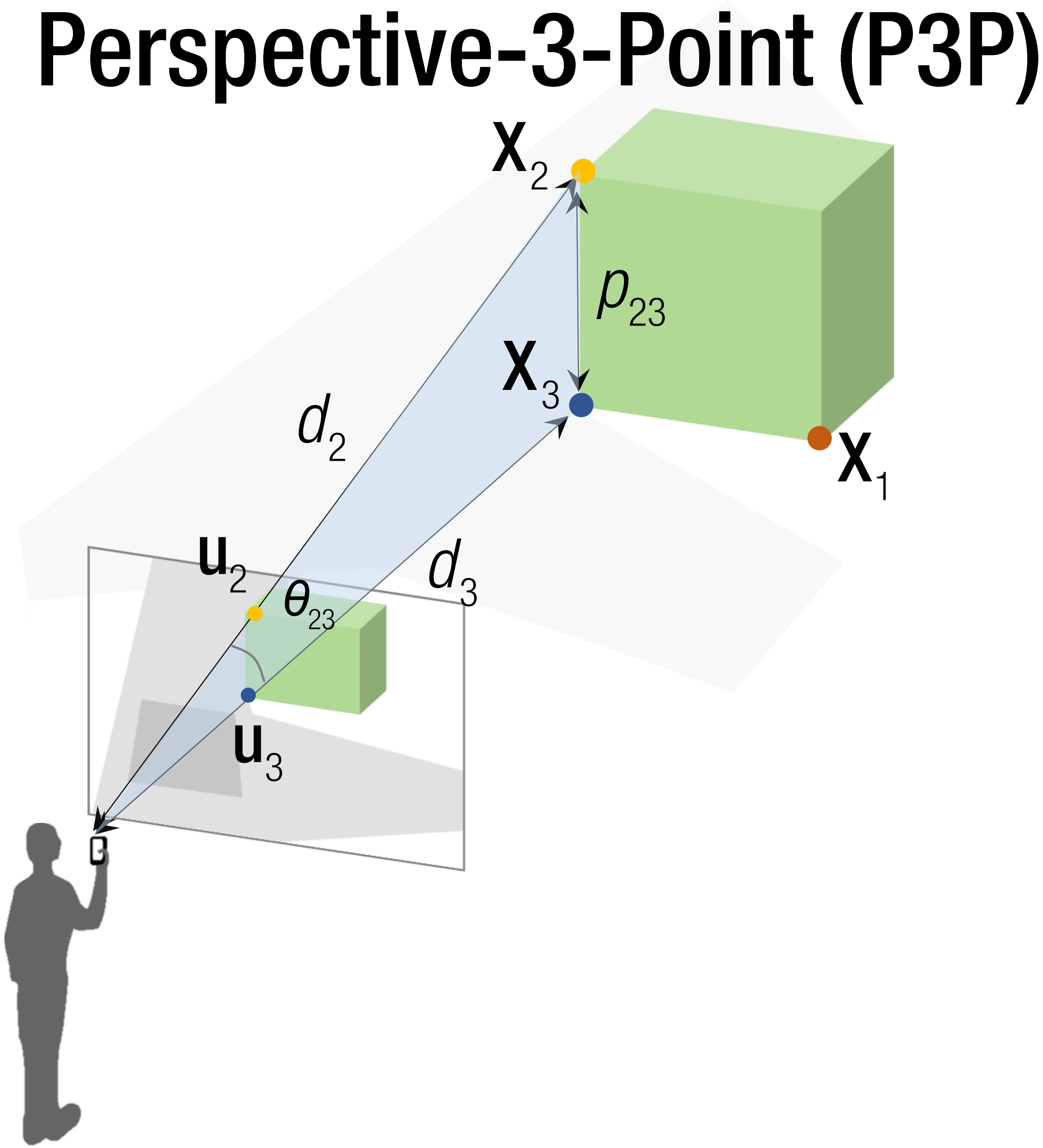


2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2$$

$$d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} = p_{31}^2$$

Perspective-3-Point (P3P)



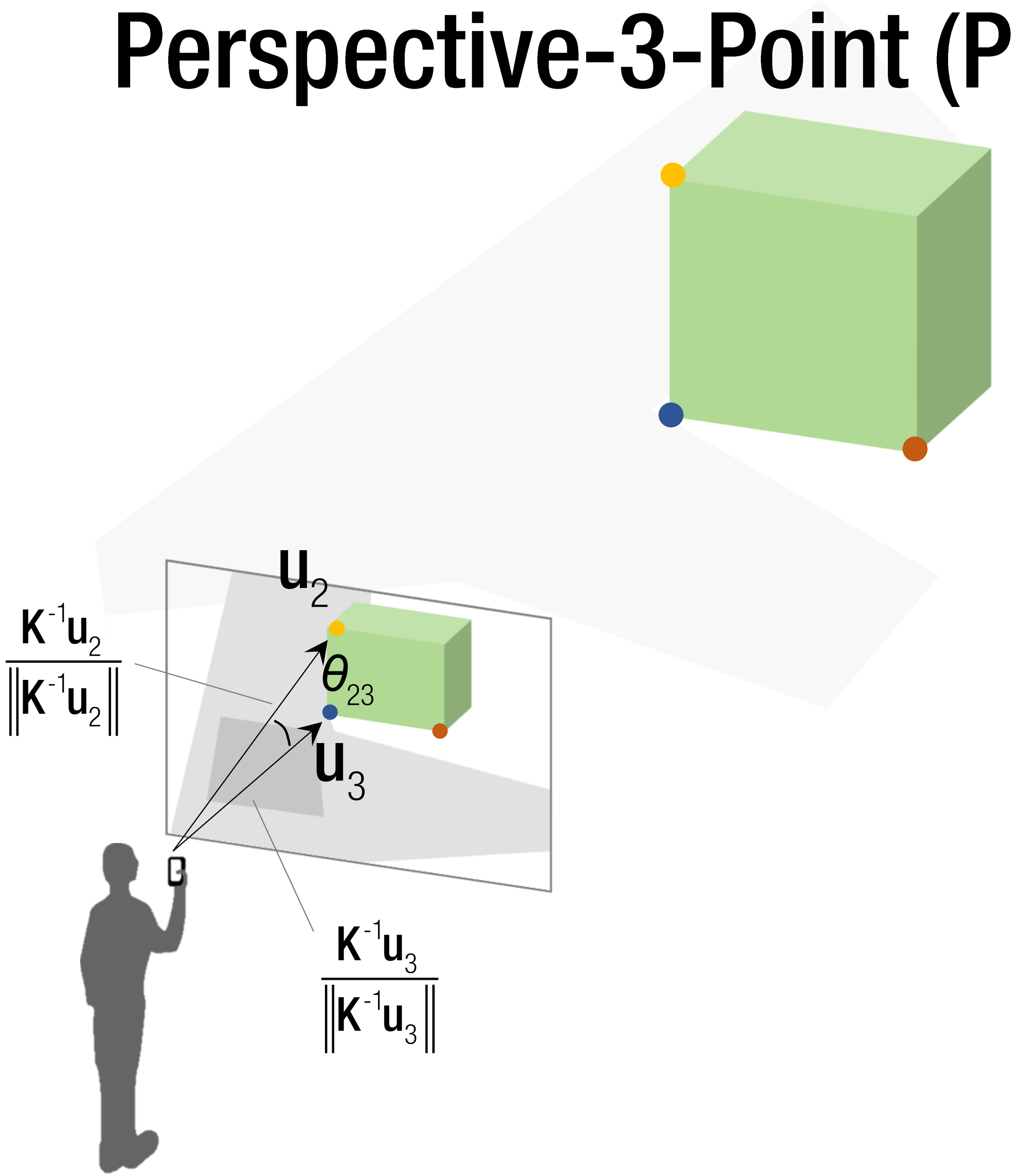
2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2$$
$$d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} = p_{31}^2$$
$$d_2^2 + d_3^2 - 2d_2d_3\cos\theta_{23} = p_{23}^2$$

3 equations

Unknowns: d_1, d_2, d_3

Perspective-3-Point (P3P)



2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1 d_2 \cos\theta_{12} = p_{12}^2$$
$$d_3^2 + d_1^2 - 2d_3 d_1 \cos\theta_{31} = p_{31}^2$$
$$d_2^2 + d_3^2 - 2d_2 d_3 \cos\theta_{23} = p_{23}^2$$

3 equations

Unknowns: d_1, d_2, d_3

Note:

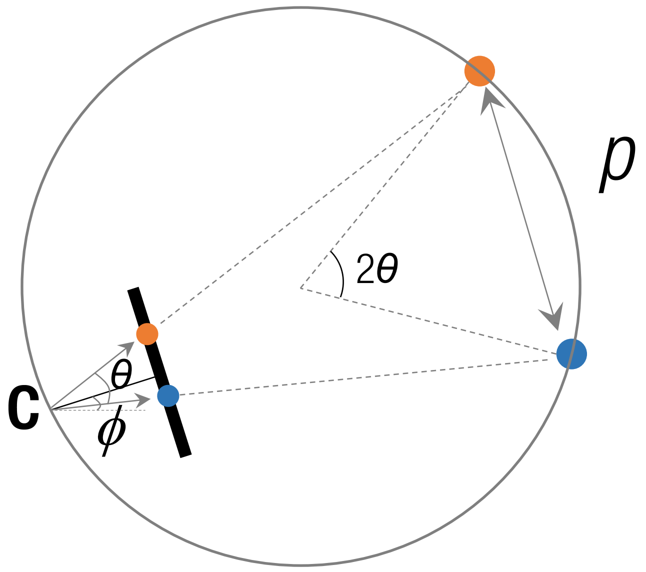
$$\cos\theta_{12} = \frac{\left(\mathbf{K}^{-1}\mathbf{u}_1\right)^{\mathrm{T}}\left(\mathbf{K}^{-1}\mathbf{u}_2\right)}{\left\|\mathbf{K}^{-1}\mathbf{u}_1\right\|\left\|\mathbf{K}^{-1}\mathbf{u}_2\right\|}$$

$$\cos\theta_{23} = \frac{\left(\mathbf{K}^{-1}\mathbf{u}_2\right)^{\mathrm{T}}\left(\mathbf{K}^{-1}\mathbf{u}_3\right)}{\left\|\mathbf{K}^{-1}\mathbf{u}_2\right\|\left\|\mathbf{K}^{-1}\mathbf{u}_3\right\|}$$

$$\cos\theta_{31} = \frac{\left(\mathbf{K}^{-1}\mathbf{u}_1\right)^{\mathrm{T}}\left(\mathbf{K}^{-1}\mathbf{u}_3\right)}{\left\|\mathbf{K}^{-1}\mathbf{u}_1\right\|\left\|\mathbf{K}^{-1}\mathbf{u}_3\right\|}$$

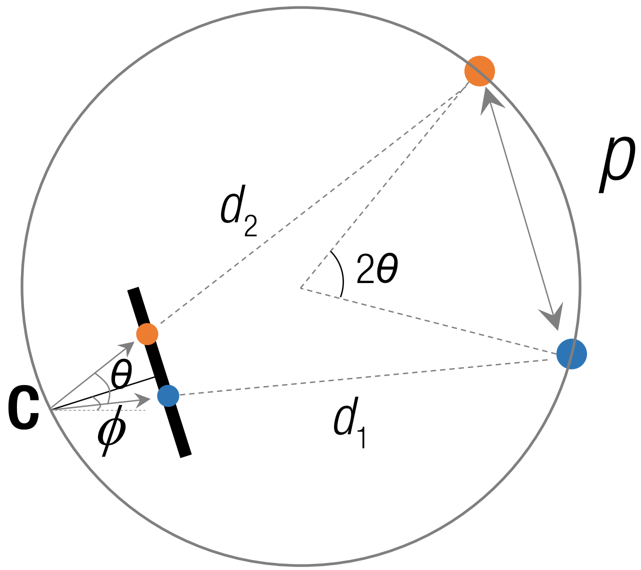
Geometric Interpretation: 1D Camera

Geometric Interpretation: 1D Camera



Property of inscribed angle

Geometric Interpretation: 1D Camera

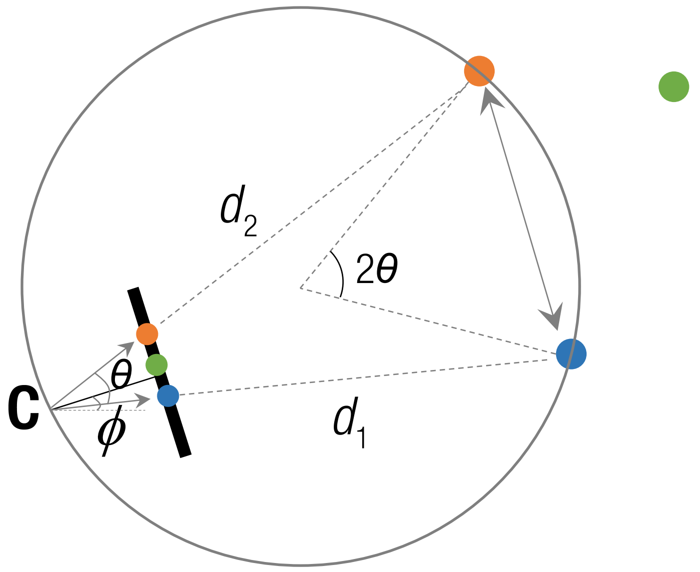


2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1 d_2 \cos\theta = p^2$$

Infinite number of solutions

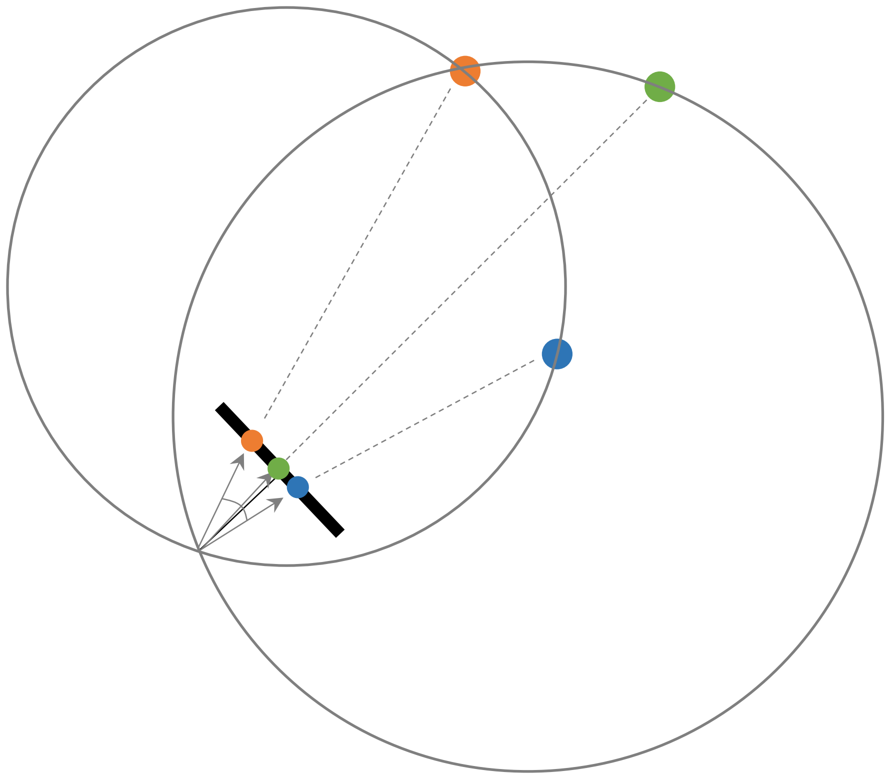
Geometric Interpretation: 1D Camera



2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1 d_2 \cos\boldsymbol{\theta} = p^2$$

Geometric Interpretation: 1D Camera

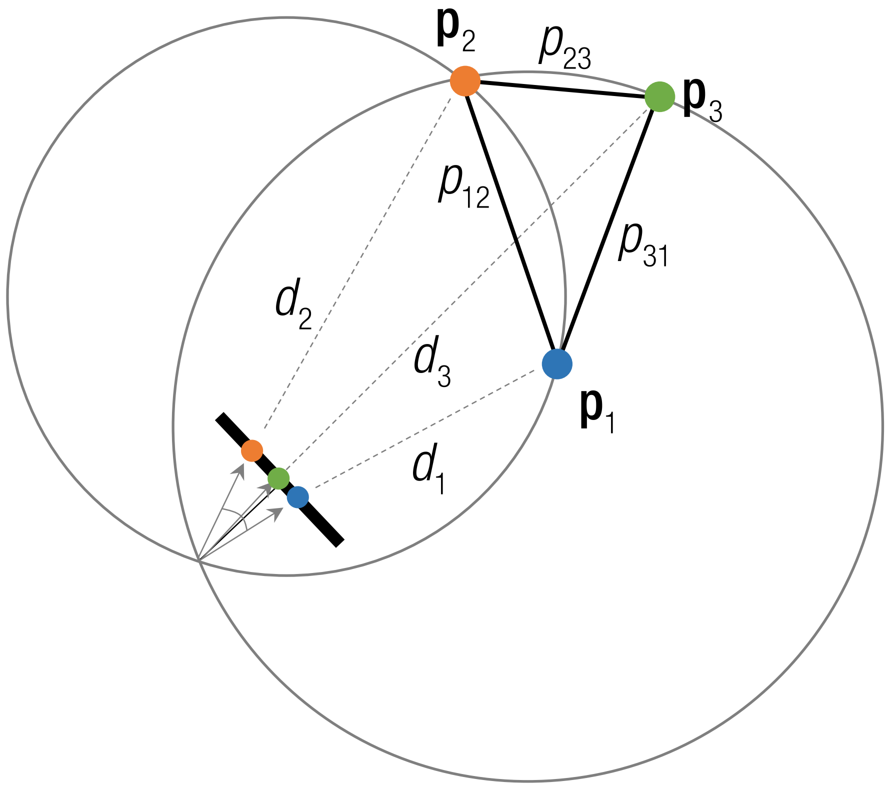


Finite number of solutions

2^{nd} Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta = p^2$$

Geometric Interpretation: 1D Camera



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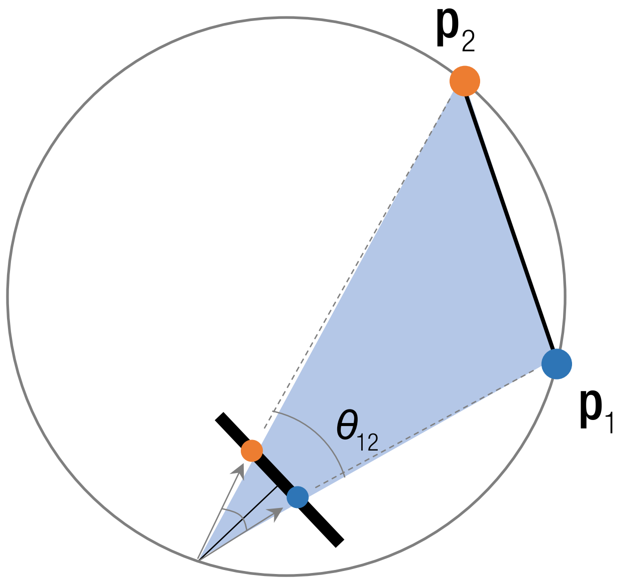
$$d_3^2 + d_1^2 - 2d_3 d_1 \cos\boldsymbol{\theta}_{31} = p_{31}^2$$

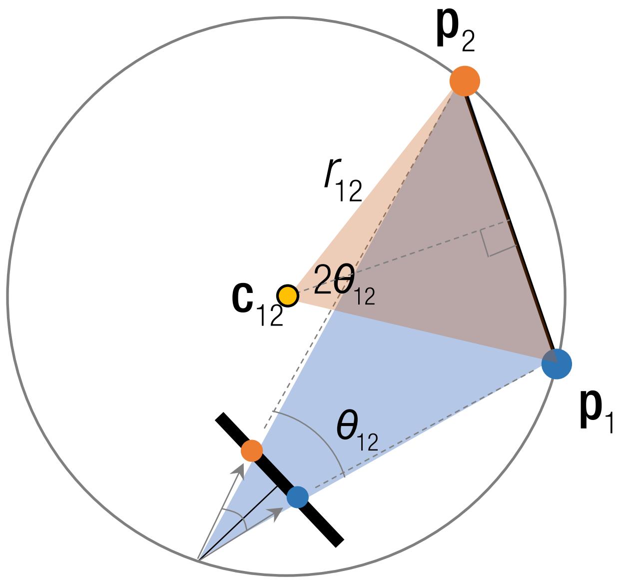
$$d_2^2 + d_3^2 - 2d_2 d_3 \cos\boldsymbol{\theta}_{23} = p_{23}^2$$

Finite number of solutions

Geometric Interpretation: 1D Camera



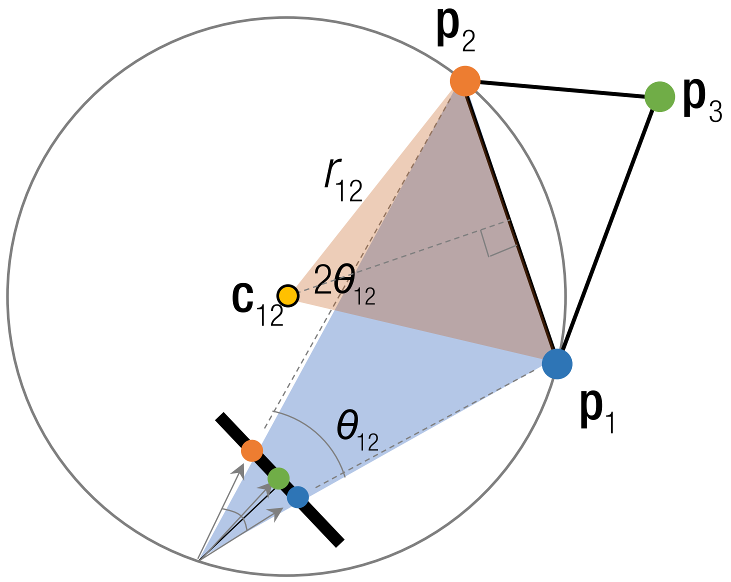
Geometric Interpretation: 1D Camera



$$\mathbf{c}_{12} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} \pm r_{12} \cos\theta_{12} \frac{\mathbf{p}_2 - \mathbf{p}_1}{\left\|\mathbf{p}_2 - \mathbf{p}_1\right\|}$$

where $r_{12} = \dfrac{\left\|\mathbf{p}_2 - \mathbf{p}_1\right\|}{2\sin\theta_{12}}$

Geometric Interpretation: 1D Camera



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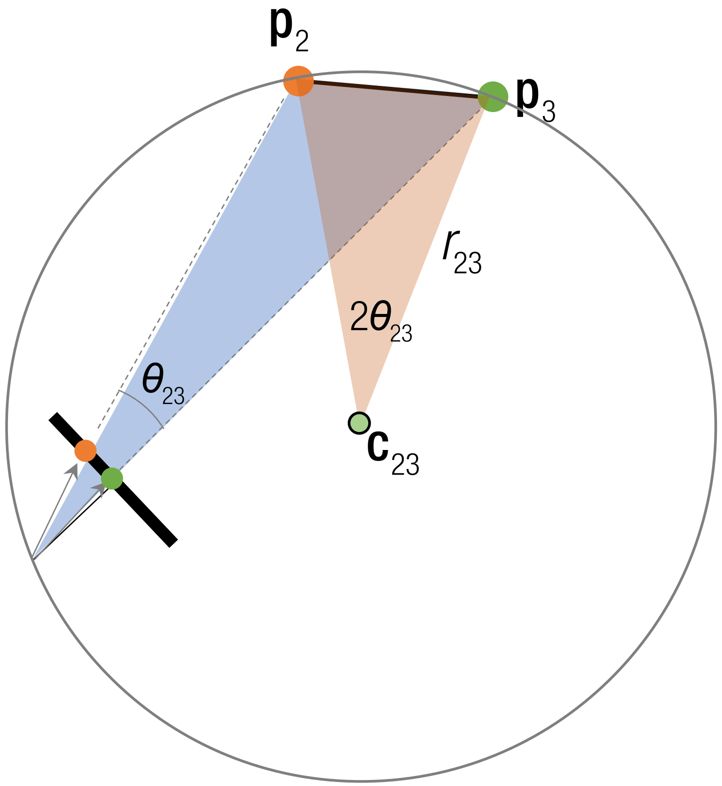
Geometric Interpretation: 1D Camera



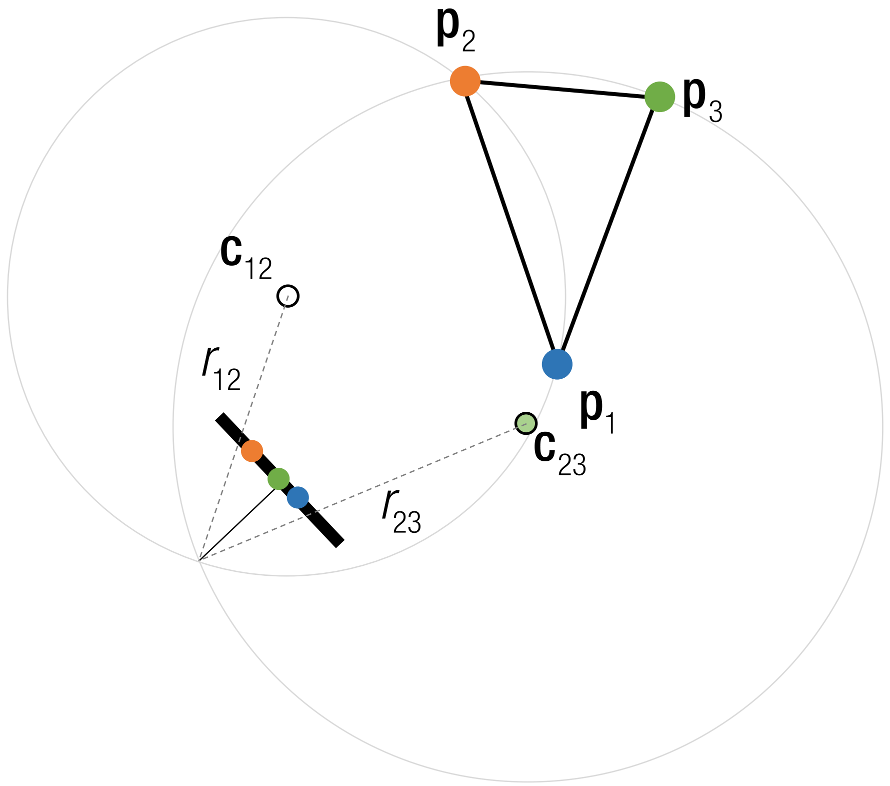
$$\mathbf{c}_{12} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} \pm r_{12} \cos\theta_{12} \frac{\mathbf{p}_2 - \mathbf{p}_1}{\|\mathbf{p}_2 - \mathbf{p}_1\|}$$

$$\text{where} \quad r_{12} = \frac{\|\mathbf{p}_2 - \mathbf{p}_1\|}{2\sin\theta_{12}}$$

$$\mathbf{c}_{23} = \frac{\mathbf{p}_3 + \mathbf{p}_2}{2} \pm r_{23} \cos\theta_{23} \frac{\mathbf{p}_3 - \mathbf{p}_2}{\|\mathbf{p}_3 - \mathbf{p}_2\|}$$

$$\text{where} \quad r_{23} = \frac{\|\mathbf{p}_3 - \mathbf{p}_2\|}{2\sin\theta_{23}}$$

Geometric Interpretation: 1D Camera



$$\mathbf{c}_{12} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} \pm r_{12} \cos\theta_{12} \frac{\mathbf{p}_2 - \mathbf{p}_1}{\left\| \mathbf{p}_2 - \mathbf{p}_1 \right\|}$$

where $r_{12} = \frac{\left\| \mathbf{p}_2 - \mathbf{p}_1 \right\|}{2 \sin\theta_{12}}$

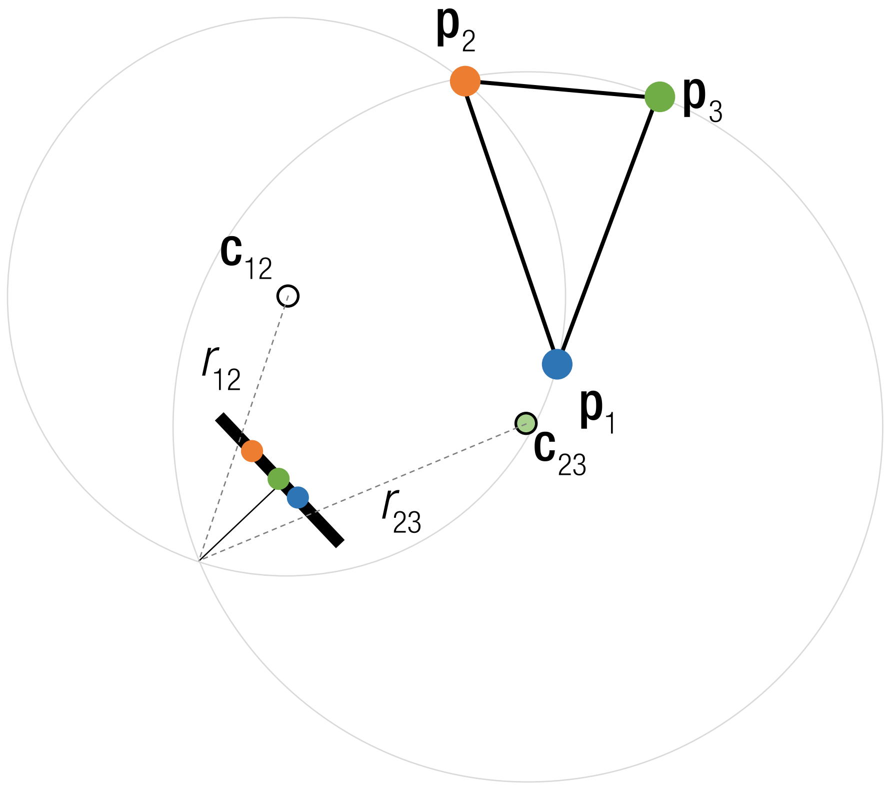
$$\mathbf{c}_{23} = \frac{\mathbf{p}_3 + \mathbf{p}_2}{2} \pm r_{23} \cos\theta_{23} \frac{\mathbf{p}_3 - \mathbf{p}_2}{\left\| \mathbf{p}_3 - \mathbf{p}_2 \right\|}$$

where $r_{23} = \frac{\left\| \mathbf{p}_3 - \mathbf{p}_2 \right\|}{2 \sin\theta_{23}}$

$$\left\| \mathbf{x} - \mathbf{c}_{12} \right\|^2 = r_{12}^2 \qquad \left\| \mathbf{x} - \mathbf{c}_{23} \right\|^2 = r_{23}^2$$

HW: Drive **x** and orientation.

Geometric Interpretation: Family of Solutions



$$\mathbf{c}_{12} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} \pm r_{12}\cos\theta_{12}\frac{\mathbf{p}_2 - \mathbf{p}_1}{\left\|\mathbf{p}_2 - \mathbf{p}_1\right\|}$$

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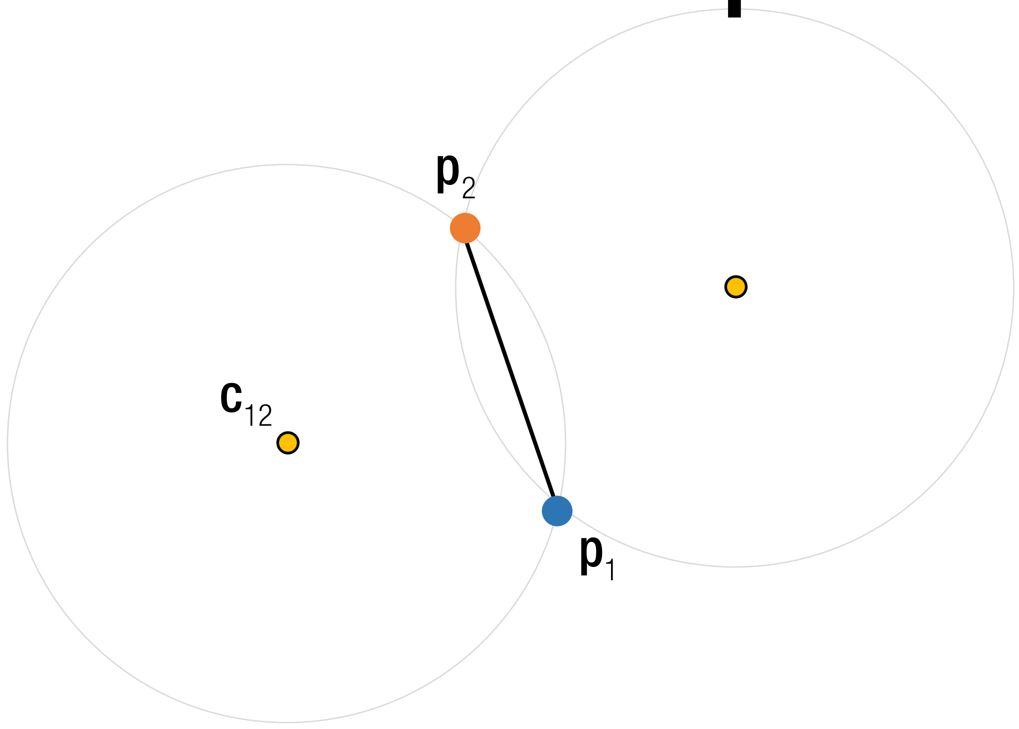
$$\mathbf{c}_{23} = \frac{\mathbf{p}_3 + \mathbf{p}_2}{2} \pm r_{23}\cos\theta_{23}\frac{\mathbf{p}_3 - \mathbf{p}_2}{\left\|\mathbf{p}_3 - \mathbf{p}_2\right\|}$$

where $r_{23} = \dfrac{\left\|\mathbf{p}_3 - \mathbf{p}_2\right\|}{2\sin\theta_{23}}$

$$\left\|\mathbf{x} - \mathbf{c}_{12}\right\|^2 = r_{12}^2 \qquad \left\|\mathbf{x} - \mathbf{c}_{23}\right\|^2 = r_{23}^2$$

HW: Drive **x** and orientation.

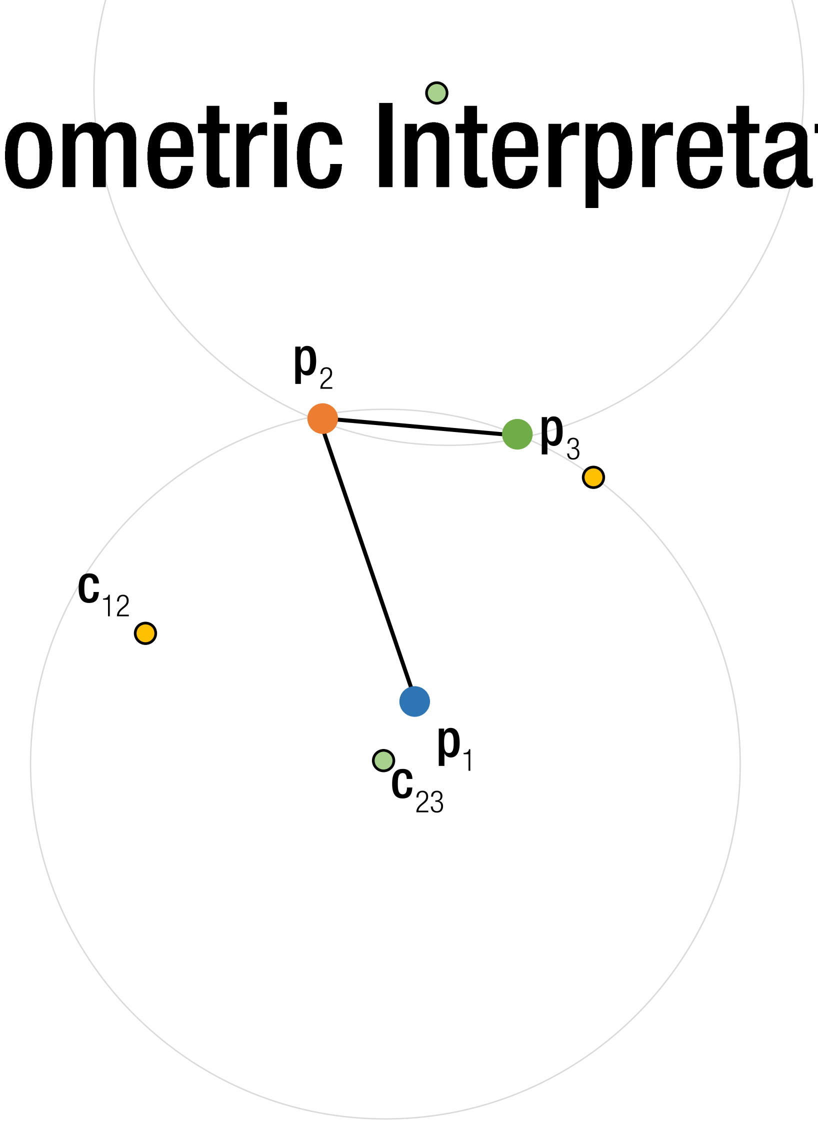
Geometric Interpretation: Family of Solutions



$$\mathbf{c}_{12} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} \pm r_{12} \cos\theta_{12} \frac{\mathbf{p}_2 - \mathbf{p}_1}{\|\mathbf{p}_2 - \mathbf{p}_1\|}$$

where $r_{12} = \dfrac{\|\mathbf{p}_2 - \mathbf{p}_1\|}{2\sin\theta_{12}}$

Geometric Interpretation: Family of Solutions



4 combinations

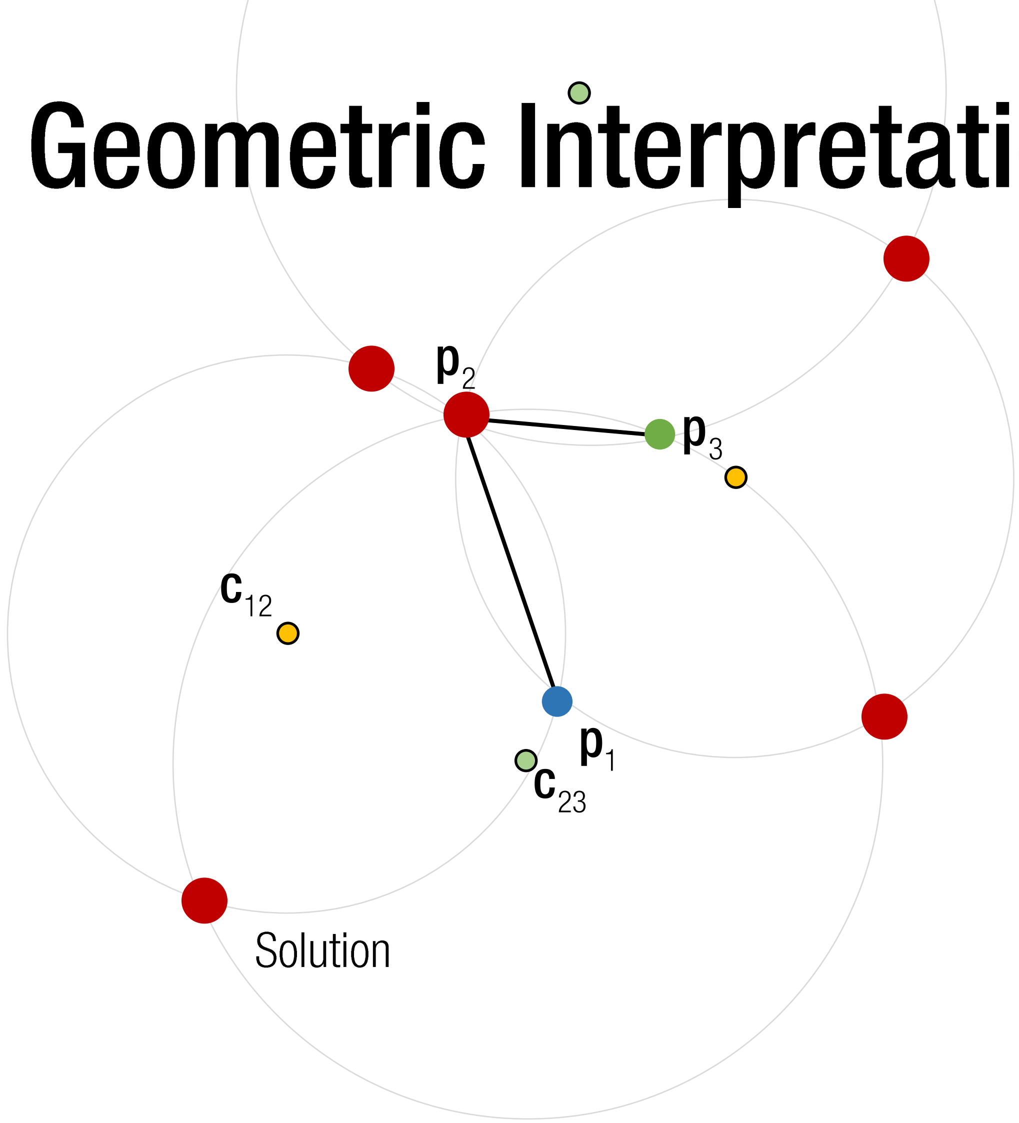
$$\mathbf{c}_{12} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} \pm r_{12}\cos\theta_{12}\frac{\mathbf{p}_2 - \mathbf{p}_1}{\left\|\mathbf{p}_2 - \mathbf{p}_1\right\|}$$

where $r_{12} = \dfrac{\left\|\mathbf{p}_2 - \mathbf{p}_1\right\|}{2\sin\theta_{12}}$

$$\mathbf{c}_{23} = \frac{\mathbf{p}_3 + \mathbf{p}_2}{2} \pm r_{23}\cos\theta_{23}\frac{\mathbf{p}_3 - \mathbf{p}_2}{\left\|\mathbf{p}_3 - \mathbf{p}_2\right\|}$$

where $r_{23} = \dfrac{\left\|\mathbf{p}_3 - \mathbf{p}_2\right\|}{2\sin\theta_{23}}$

Geometric Interpretation: Family of Solutions



$$\mathbf{c}_{12} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{2} \pm r_{12} \cos\theta_{12} \frac{\mathbf{p}_2 - \mathbf{p}_1}{\|\mathbf{p}_2 - \mathbf{p}_1\|}$$

where $r_{12} = \dfrac{\|\mathbf{p}_2 - \mathbf{p}_1\|}{2\sin\theta_{12}}$

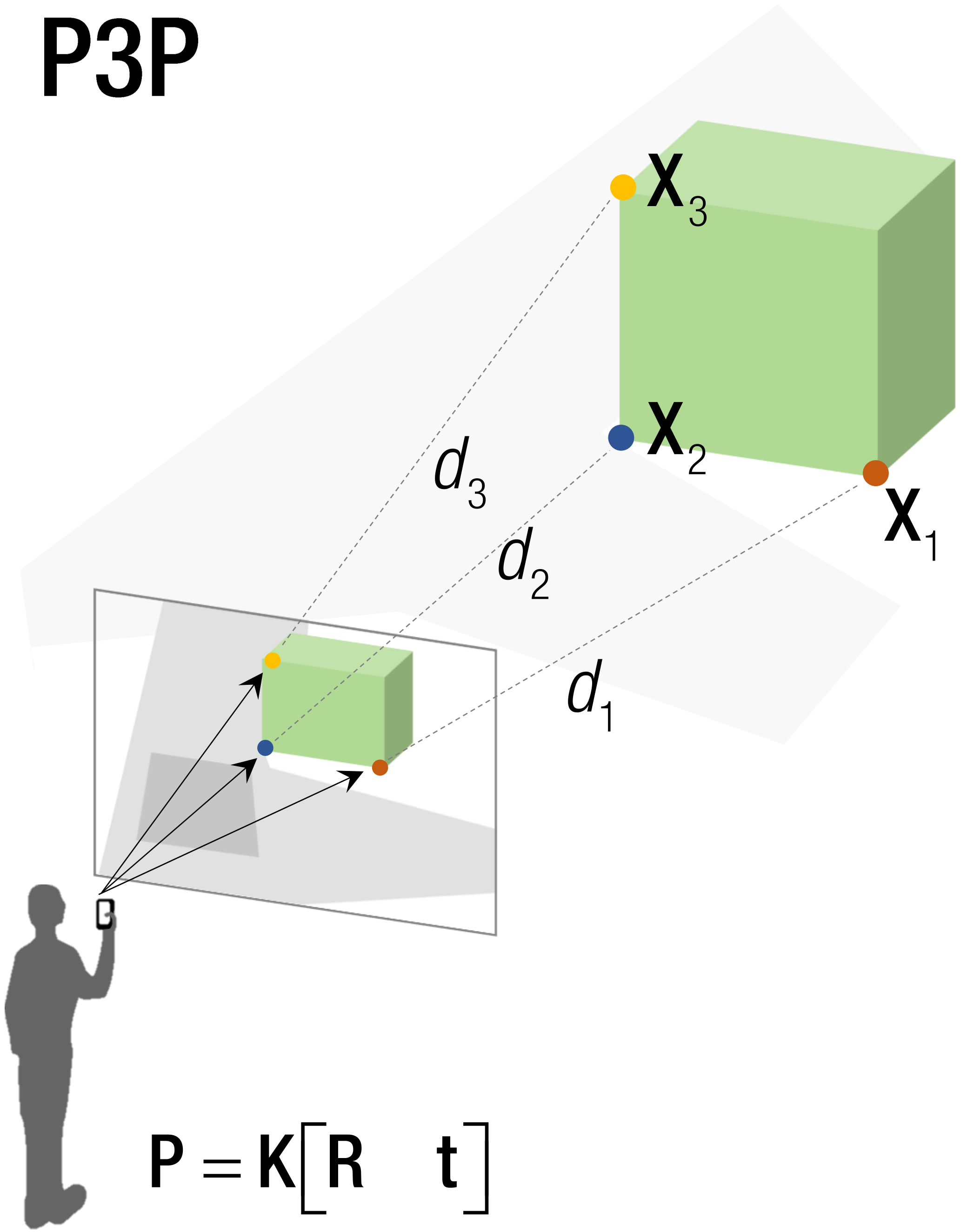
$$\mathbf{c}_{23} = \frac{\mathbf{p}_3 + \mathbf{p}_2}{2} \pm r_{23} \cos\theta_{23} \frac{\mathbf{p}_3 - \mathbf{p}_2}{\|\mathbf{p}_3 - \mathbf{p}_2\|}$$

where $r_{23} = \dfrac{\|\mathbf{p}_3 - \mathbf{p}_2\|}{2\sin\theta_{23}}$

4 combinations of circle centers

→ 4 solutions except for $\mathbf{p}_2$ ($\mathbf{p}_2$ is counted four times.).

P3P



2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2$$
$$d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} = p_{31}^2$$
$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{23} = p_{23}^2$$

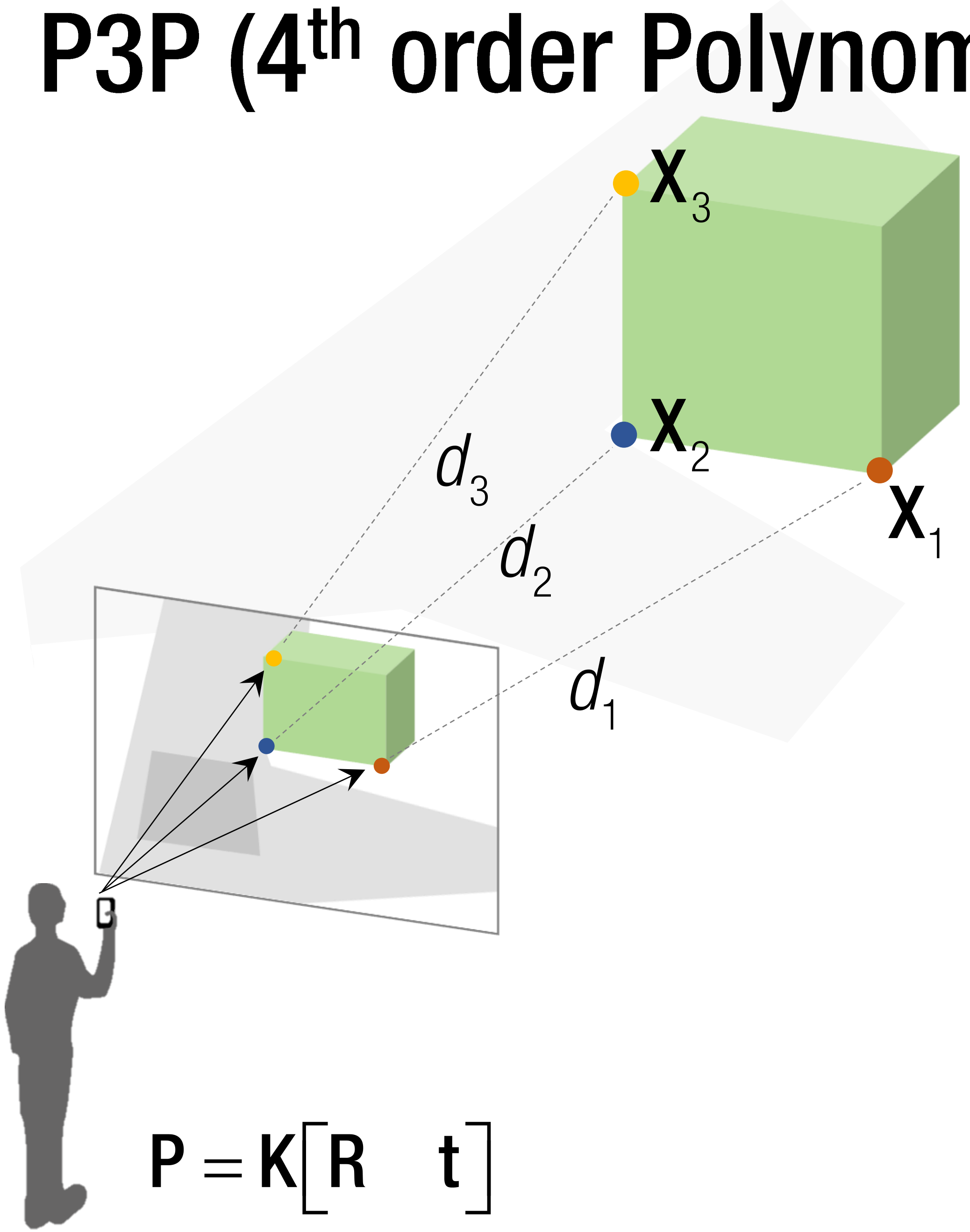
3 equations

The number of possible solutions: 8 = 2x2x2

$d_1 > 0 \quad d_2 > 0 \quad d_3 > 0$: 4 = 2x2x2/2

→ requires additional fourth point to verify the solution.

P3P (4th order Polynomial)



$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2$$
$$d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} = p_{31}^2$$
$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{23} = p_{23}^2$$

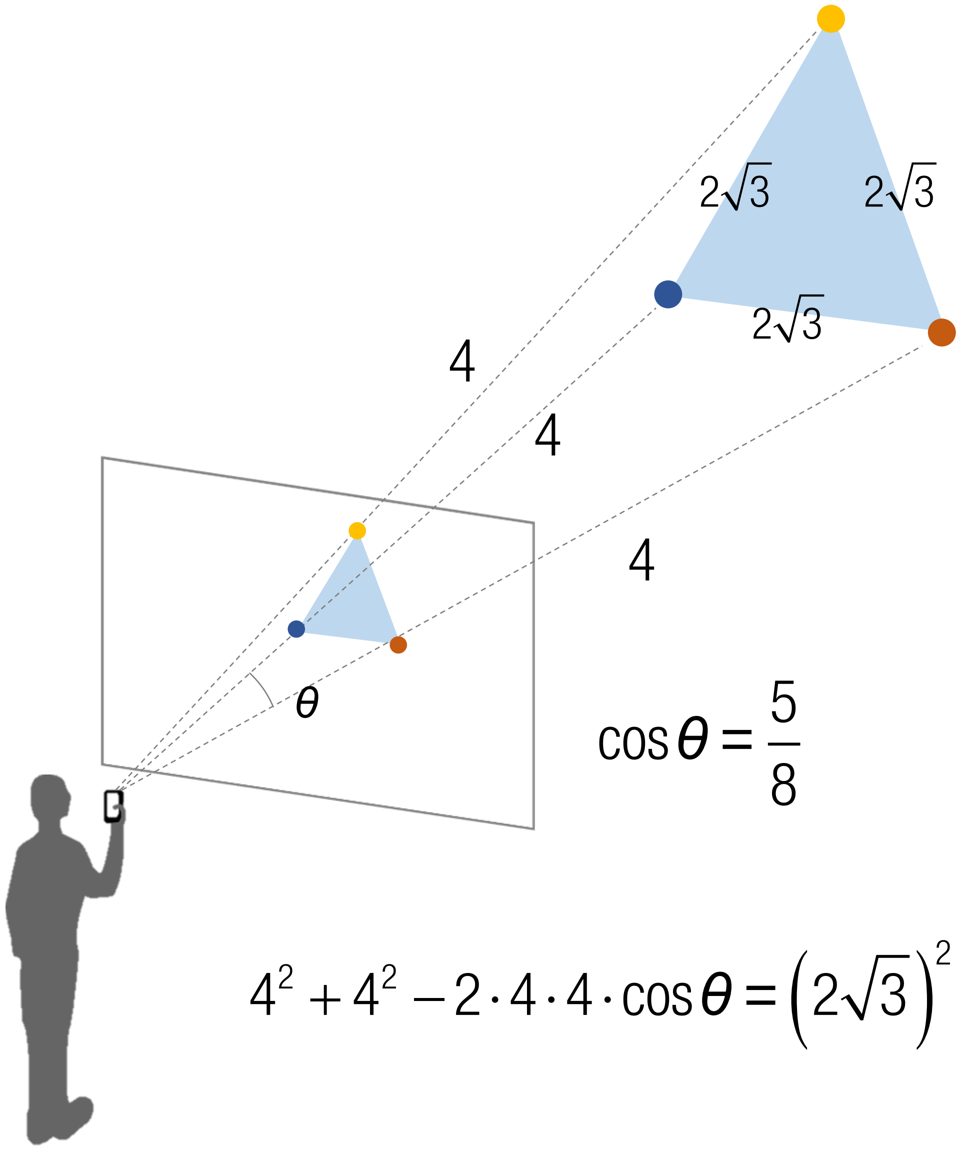
3 equations

4th order polynomial:

$$\longrightarrow a_4y^4 + a_3y^3 + a_2y^2 + a_1y + a_0 = 0$$

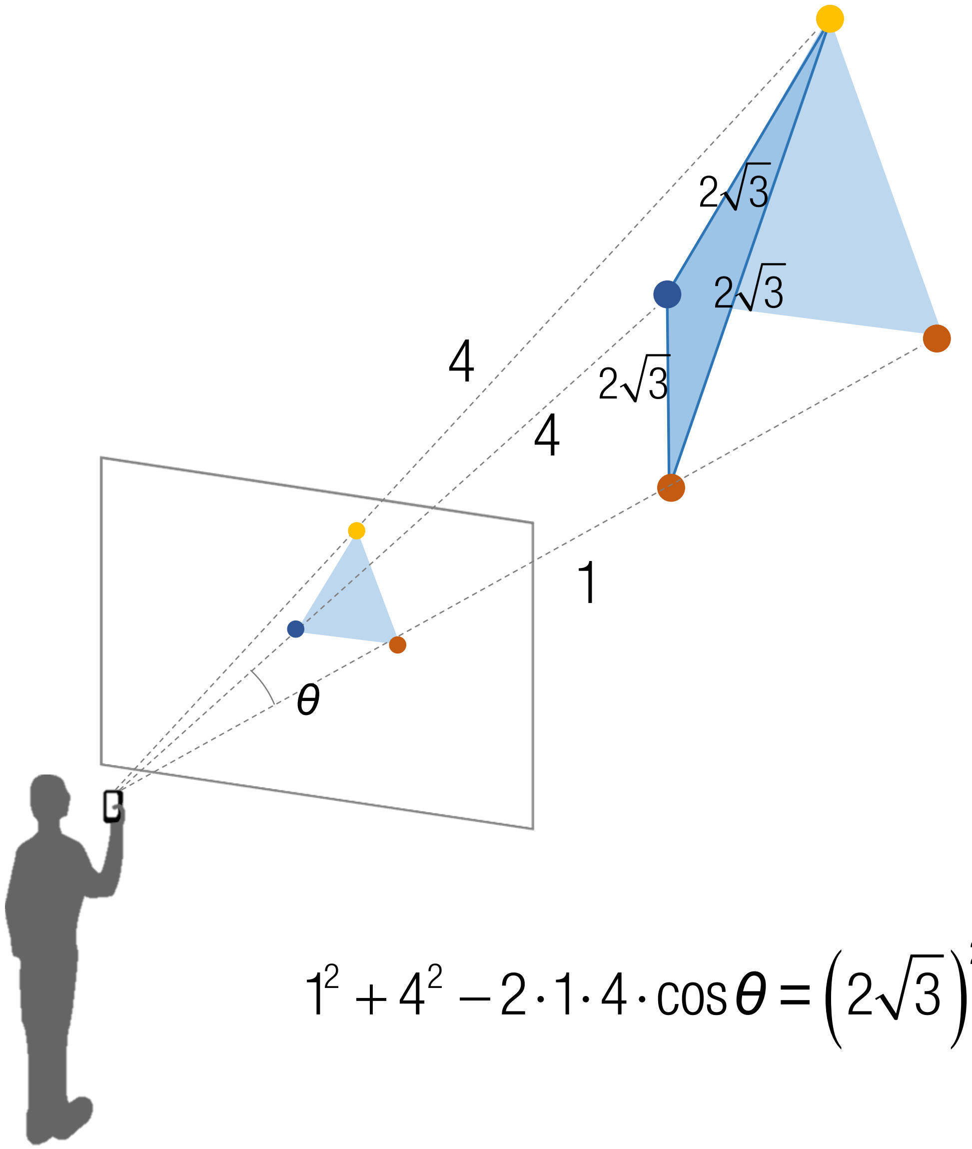
Closed form solutions exist.

Four Solution Example



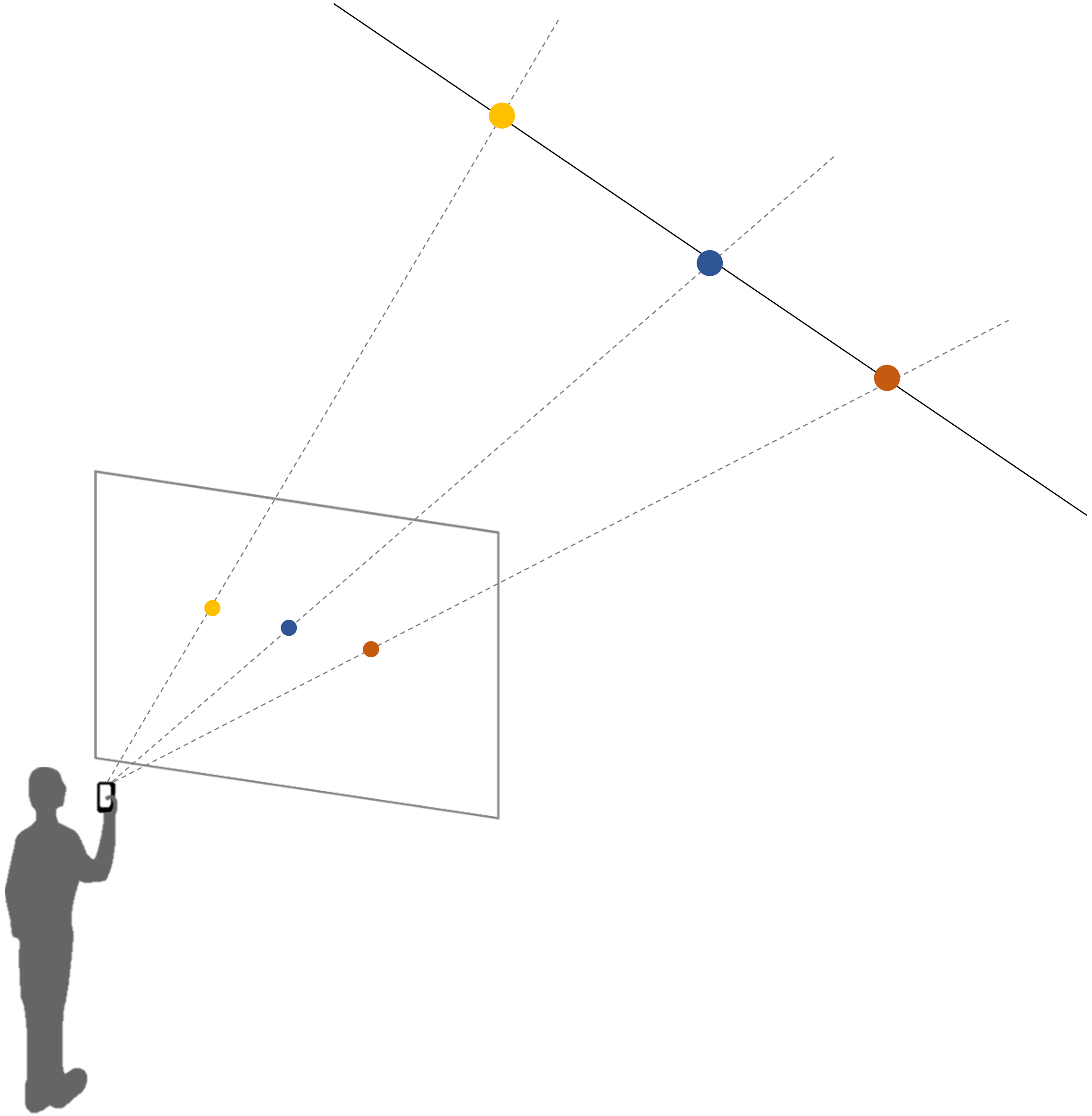
$$\cos\theta = \frac{5}{8}$$

$$4^2 + 4^2 - 2 \cdot 4 \cdot 4 \cdot \cos\theta = \left(2\sqrt{3}\right)^2$$

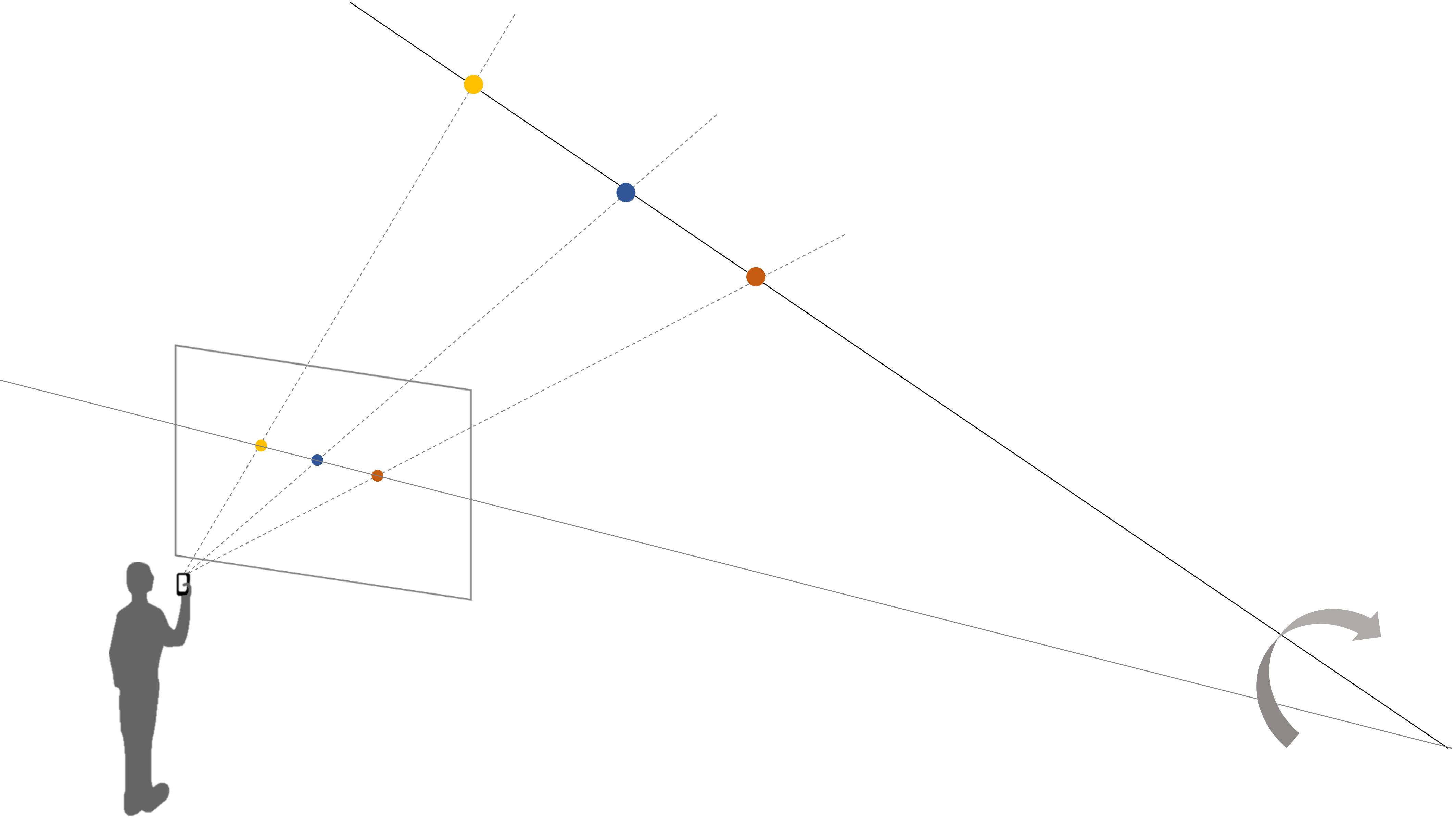


$$1^2 + 4^2 - 2 \cdot 1 \cdot 4 \cdot \cos\theta = \left(2\sqrt{3}\right)^2$$

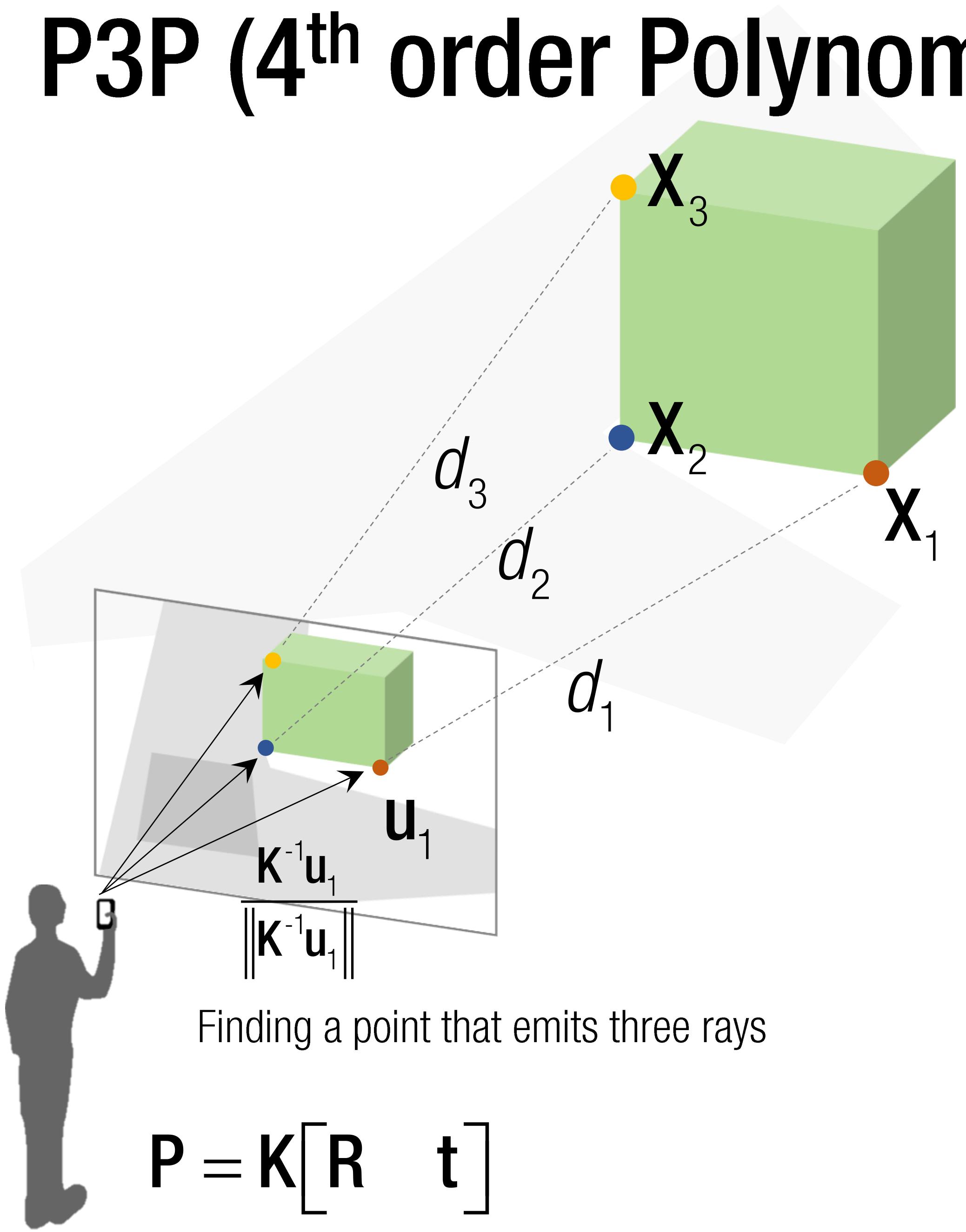
Ambiguity



Ambiguity (Colinear points)



P3P (4th order Polynomial)



Finding a point that emits three rays

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2$$
$$d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} = p_{31}^2$$
$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{23} = p_{23}^2$$

3 equations

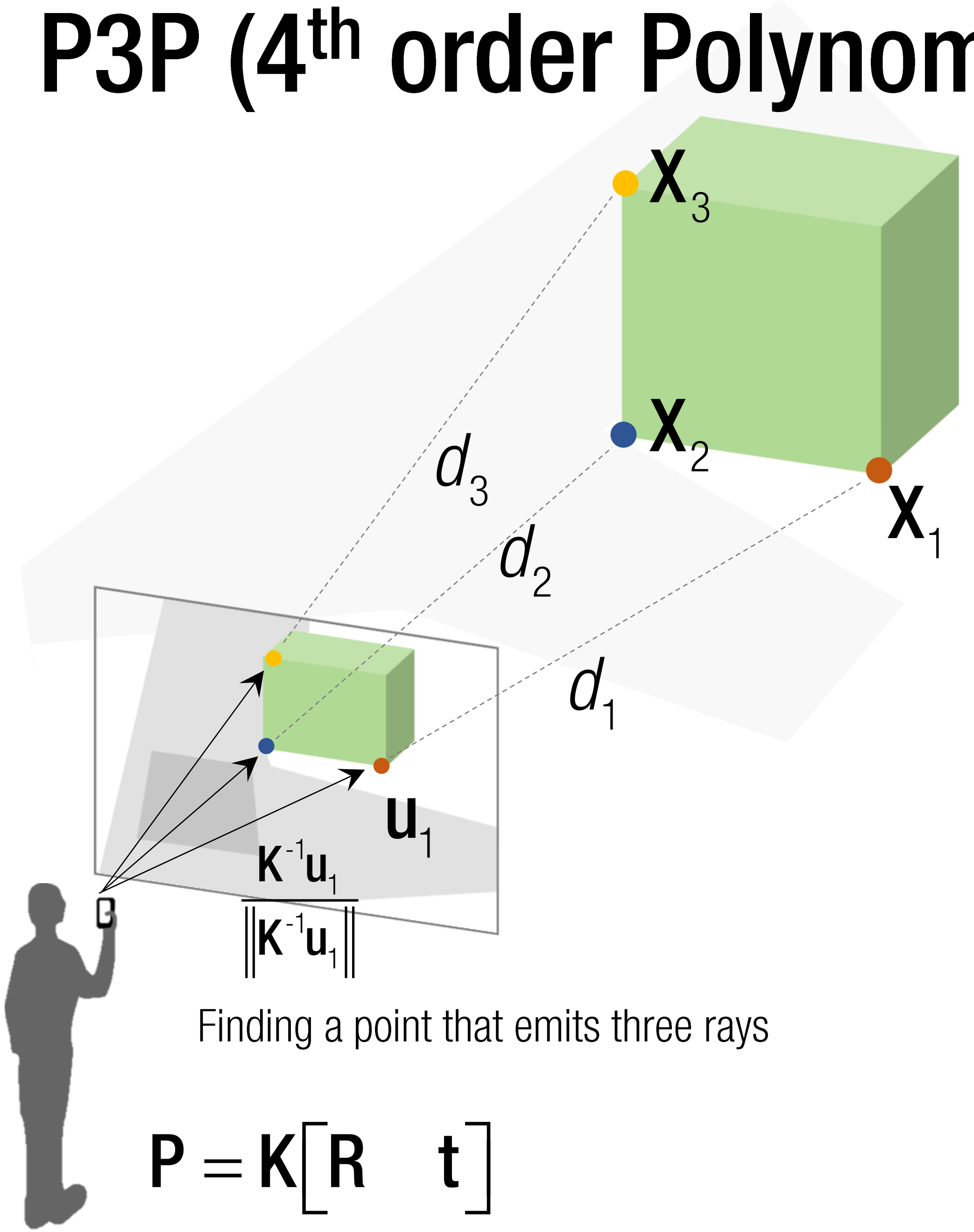
4th order polynomial:

→ $a_4y^4 + a_3y^3 + a_2y^2 + a_1y + a_0 = 0$

Closed form solutions exist.

→ Compute **t** using $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, d_1, d_2, and d_3.

P3P (4th order Polynomial)



Finding a point that emits three rays

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

2nd Cosine law:

$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2$$
$$d_3^2 + d_1^2 - 2d_3d_1\cos\theta_{31} = p_{31}^2$$
$$d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{23} = p_{23}^2$$

3 equations

4th order polynomial:

→ $a_4y^4 + a_3y^3 + a_2y^2 + a_1y + a_0 = 0$

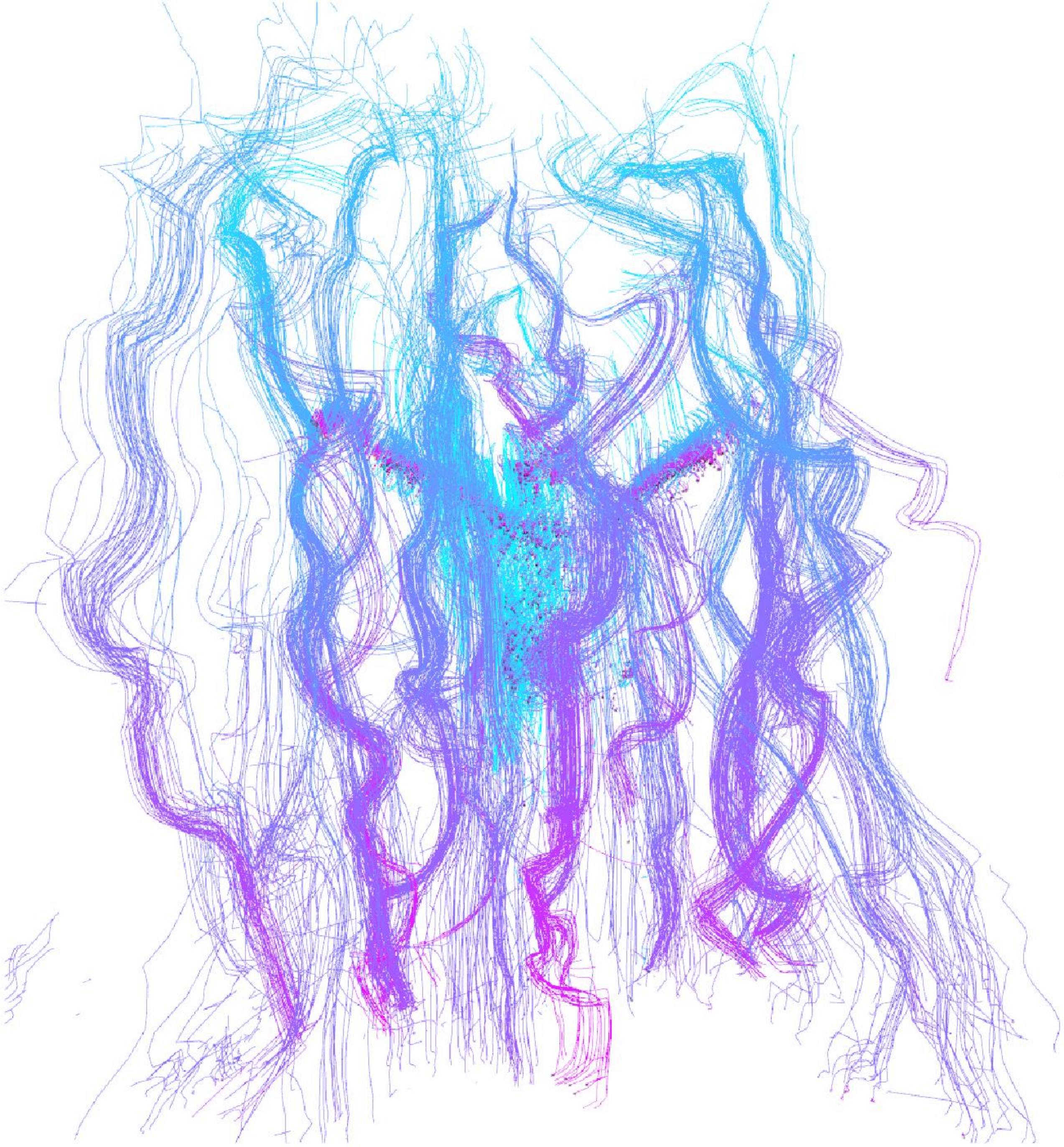
Closed form solutions exist.

→ Compute **t** using $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, d_1, d_2, and d_3.

→ $\begin{bmatrix}\tilde{\mathbf{X}}_1 & \tilde{\mathbf{X}}_2 & \tilde{\mathbf{X}}_3\end{bmatrix} = \mathbf{R}\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3\end{bmatrix}$

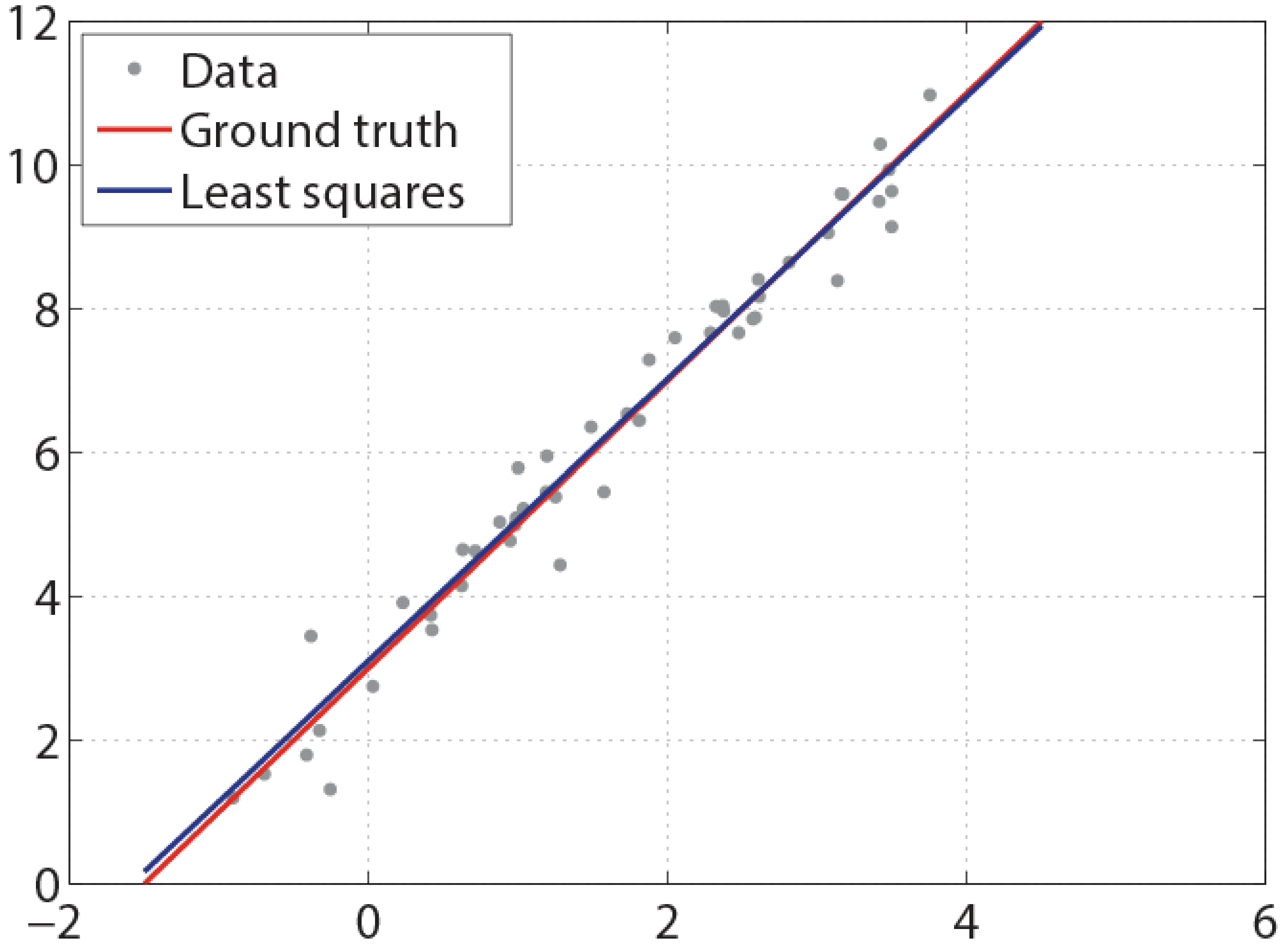
Rotation matrix computation

where $\tilde{\mathbf{X}}_1 = d_1\frac{\mathbf{K}^{-1}\mathbf{u}_1}{\|\mathbf{K}^{-1}\mathbf{u}_1\|}$ $\tilde{\mathbf{X}}_2 = d_2\frac{\mathbf{K}^{-1}\mathbf{u}_2}{\|\mathbf{K}^{-1}\mathbf{u}_2\|}$ $\tilde{\mathbf{X}}_3 = d_3\frac{\mathbf{K}^{-1}\mathbf{u}_3}{\|\mathbf{K}^{-1}\mathbf{u}_3\|}$



Nonlinear Estimation

Recall: Line Fitting (Ax=b)

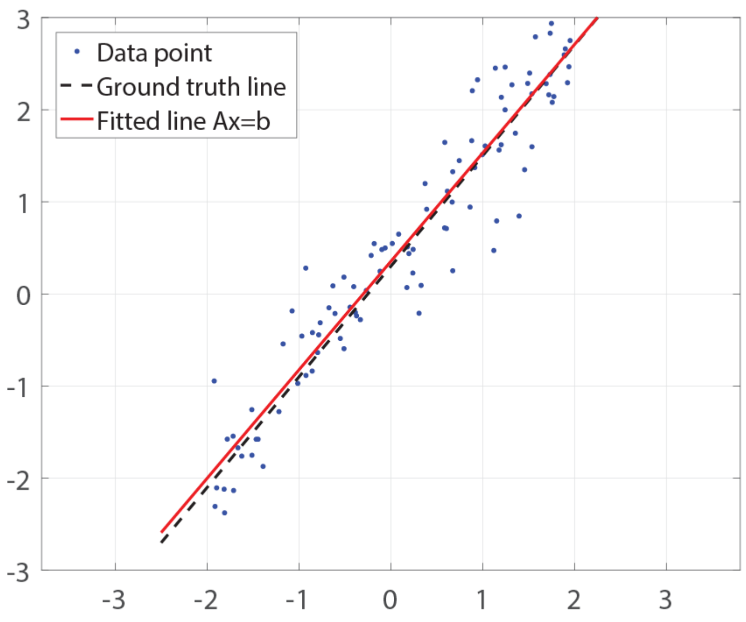


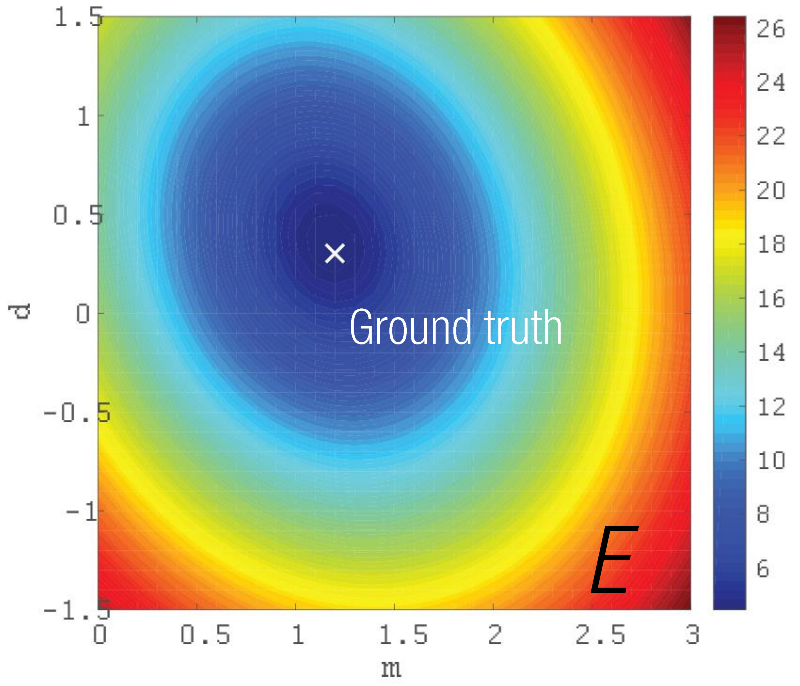
$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$\mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{A}^T \mathbf{b}$$

$$\mathbf{x} = \left[\mathbf{A}^T \mathbf{A} \right]^{-1} \mathbf{A}^T \mathbf{b}$$

Recall: Line Fitting (Ax=b)

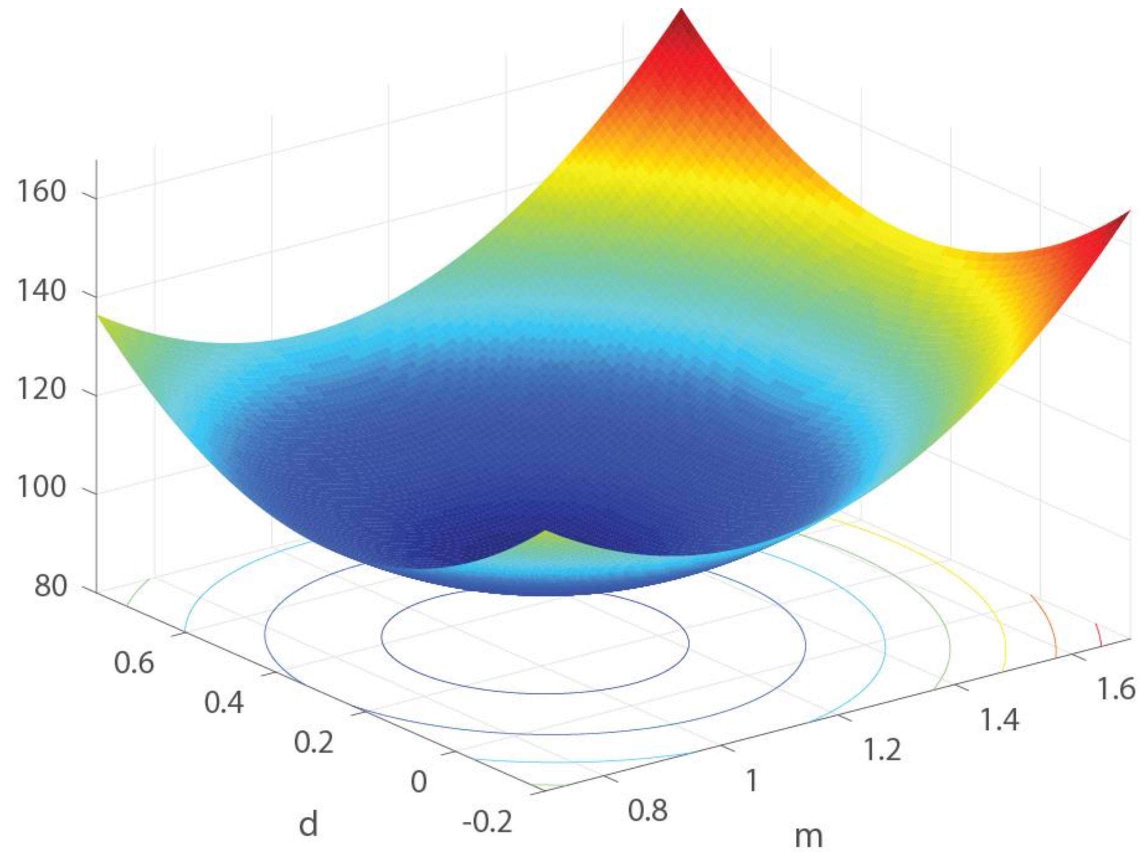




Error:

$$E = \left\| \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|^2$$

Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d \qquad \begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$

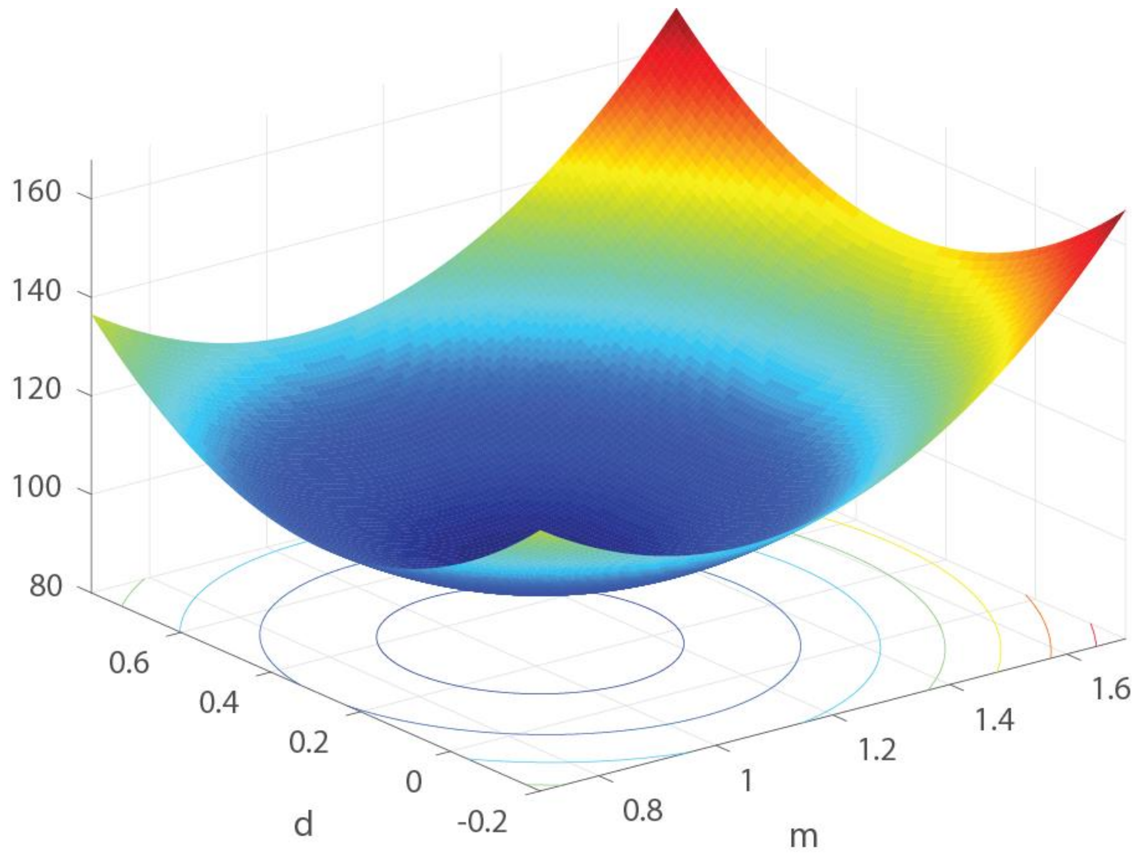
$$v_2 \approx mu_2 + d \quad \longrightarrow \quad \begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$

$$v_n \approx mu_n + d \qquad \begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$



$v_1 \approx mu_1 + d$

$v_2 \approx mu_2 + d$

$v_n \approx mu_n + d$

$\rightarrow$

$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$

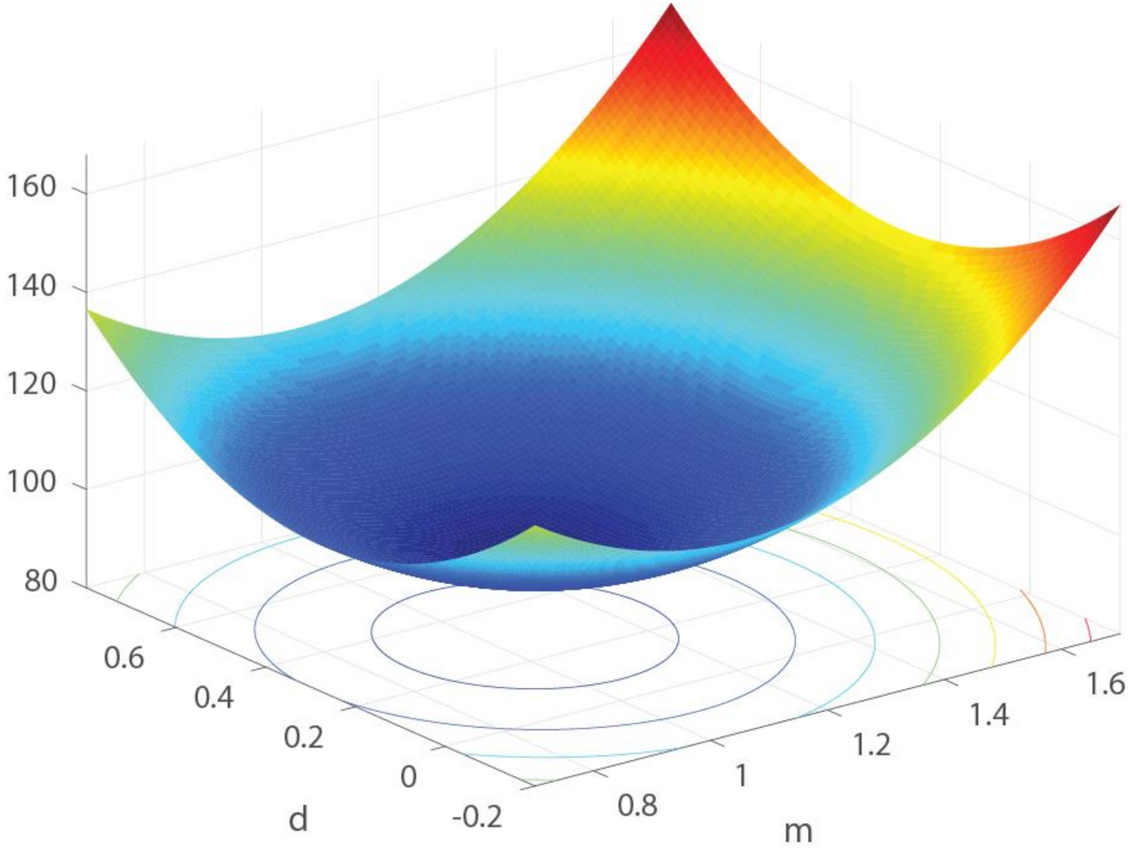
$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$

$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$

$\rightarrow$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$v_1 \approx mu_1 + d$

$v_2 \approx mu_2 + d$

$v_n \approx mu_n + d$

→ $\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$, $\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$, $\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$

→ $\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$

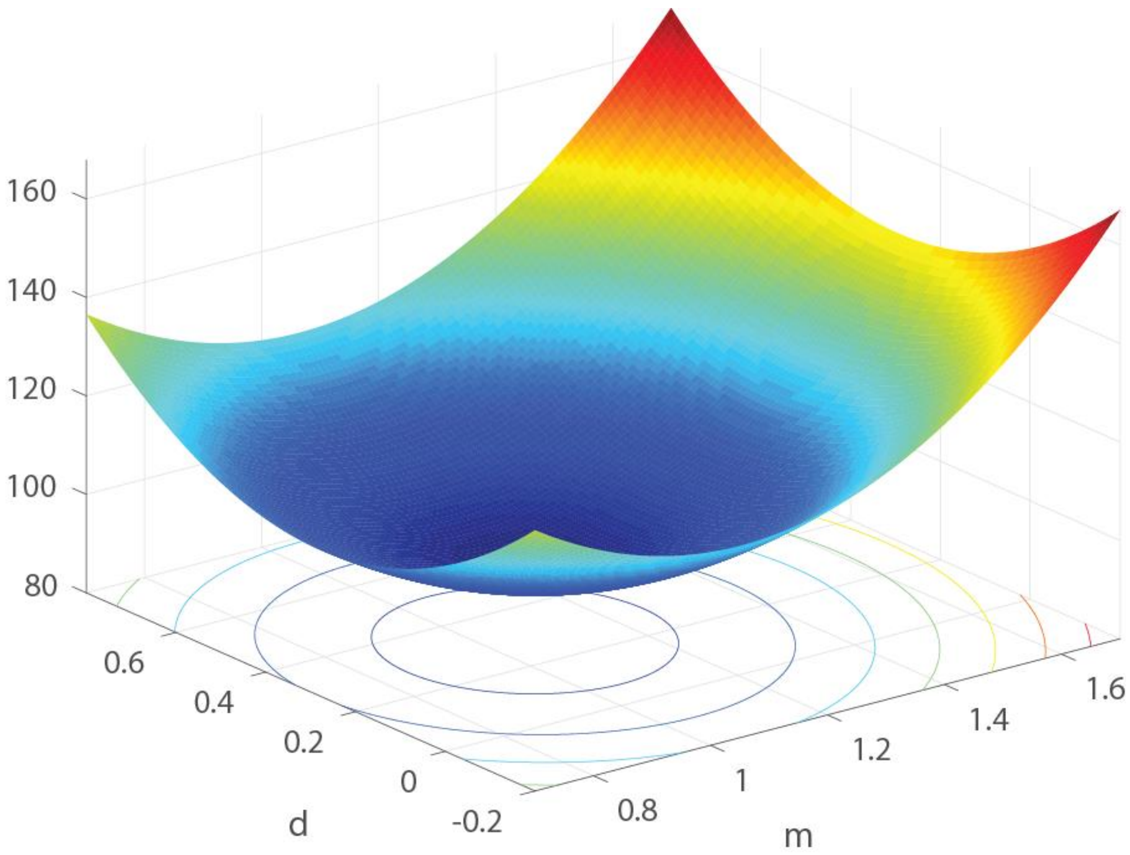
We can't invert **A**.

Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$E = (\mathbf{Ax} - \mathbf{b})^{\mathrm{T}} (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$



Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

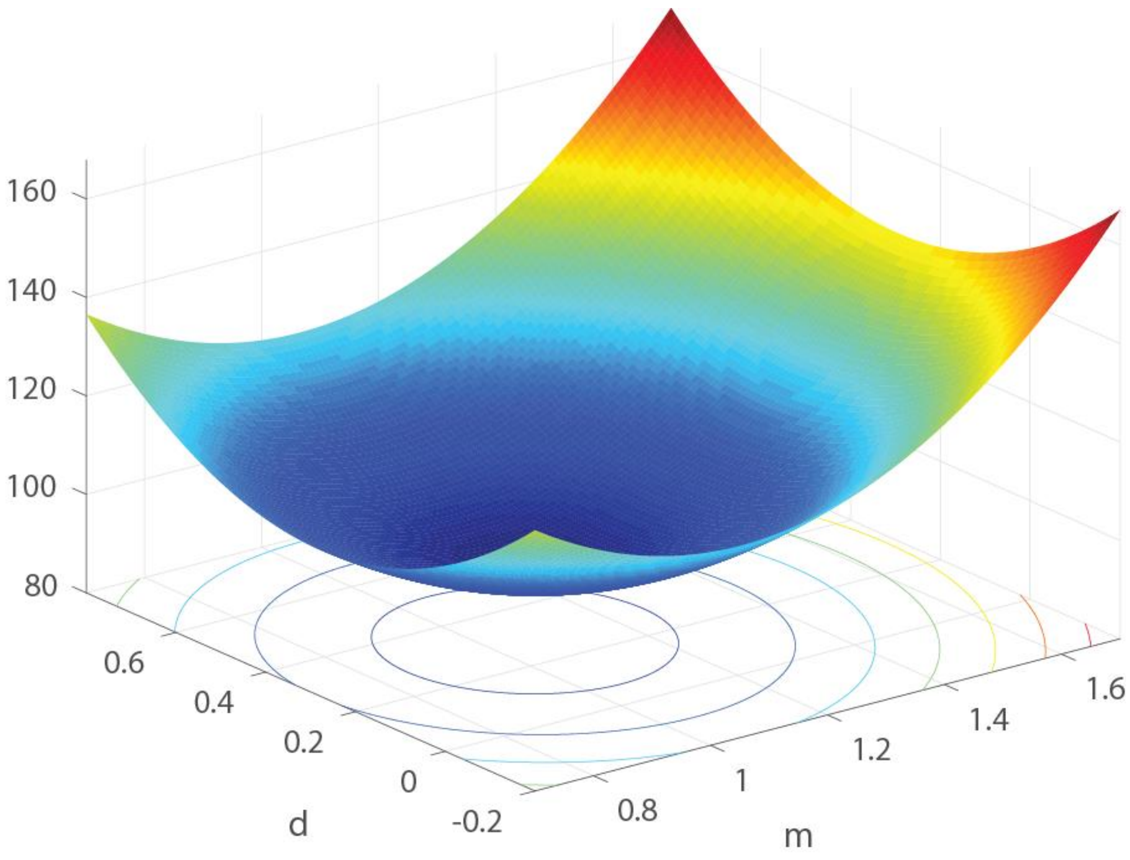
$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A}\mathbf{x} \approx \mathbf{b}$$

$$E = (\mathbf{Ax} - \mathbf{b})^{\mathsf{T}}(\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \quad ?$$



Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

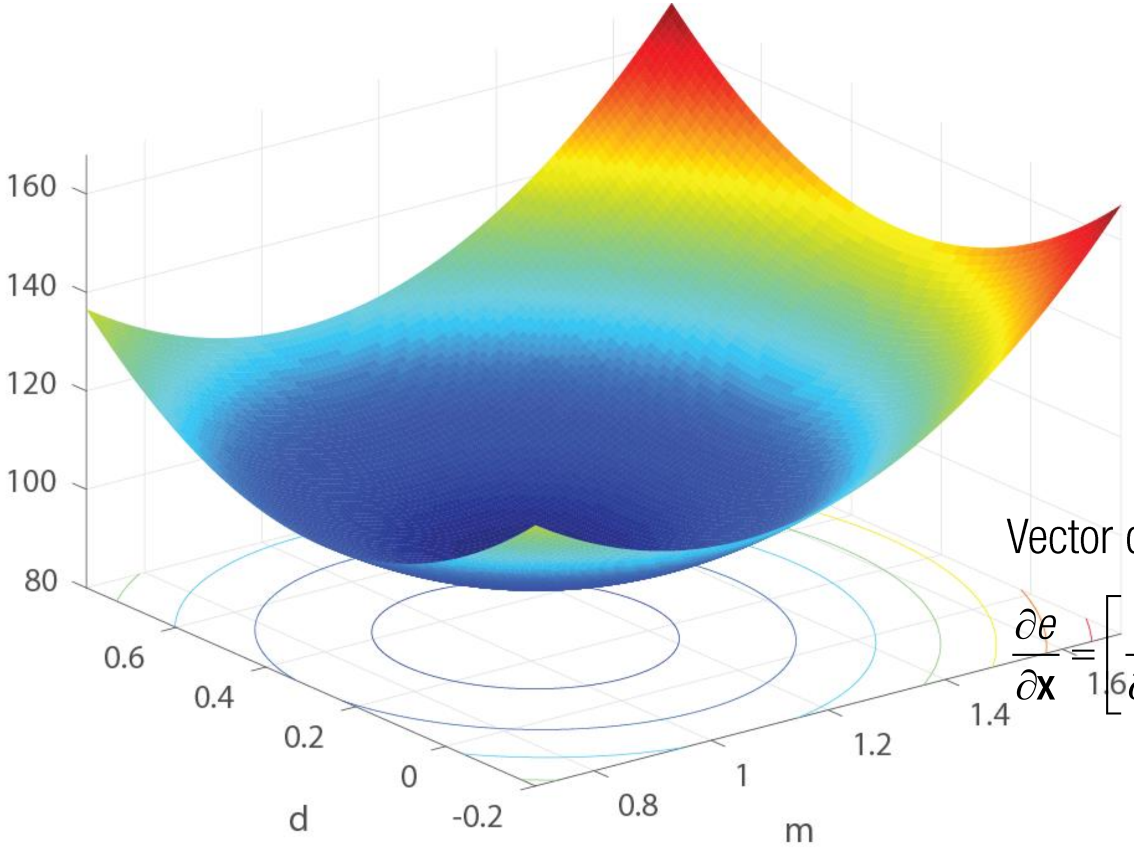
$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A} \quad \mathbf{x} \approx \mathbf{b}$$

$$E = \left(\mathbf{Ax} - \mathbf{b}\right)^{\mathsf{T}} \left(\mathbf{Ax} - \mathbf{b}\right) = \left\|\mathbf{Ax} - \mathbf{b}\right\|^2$$

$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \quad ?$$

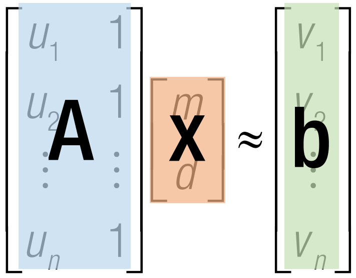


Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

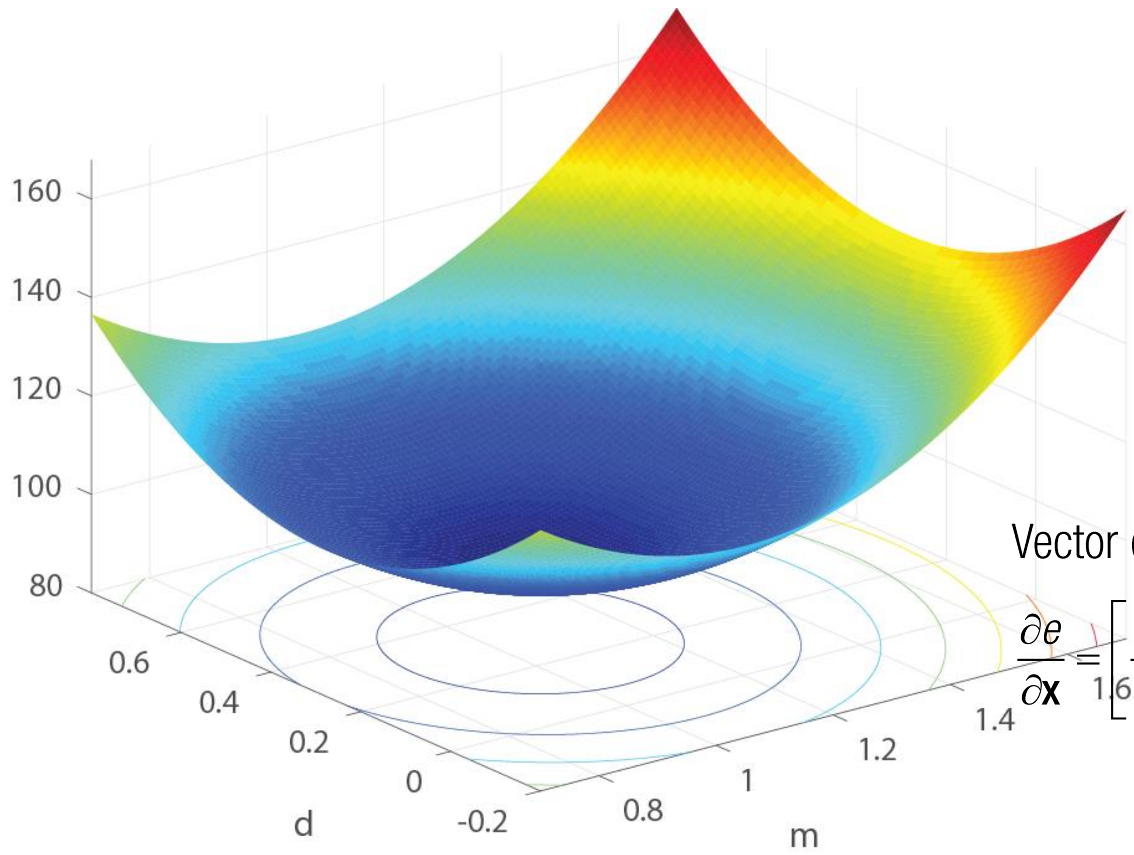
Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$



$$E = (\mathbf{Ax} - \mathbf{b})^{\mathsf{T}} (\mathbf{Ax} - \mathbf{b}) = \left\| \mathbf{Ax} - \mathbf{b} \right\|^2$$

$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$



$$\frac{\partial E}{\partial \mathbf{x}} = \qquad ?$$

Vector derivative:

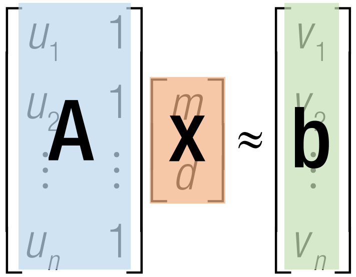
$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathbf{c}^{\mathsf{T}}\mathbf{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^{\mathsf{T}}\mathbf{x}}{\partial \mathbf{x}} =$$

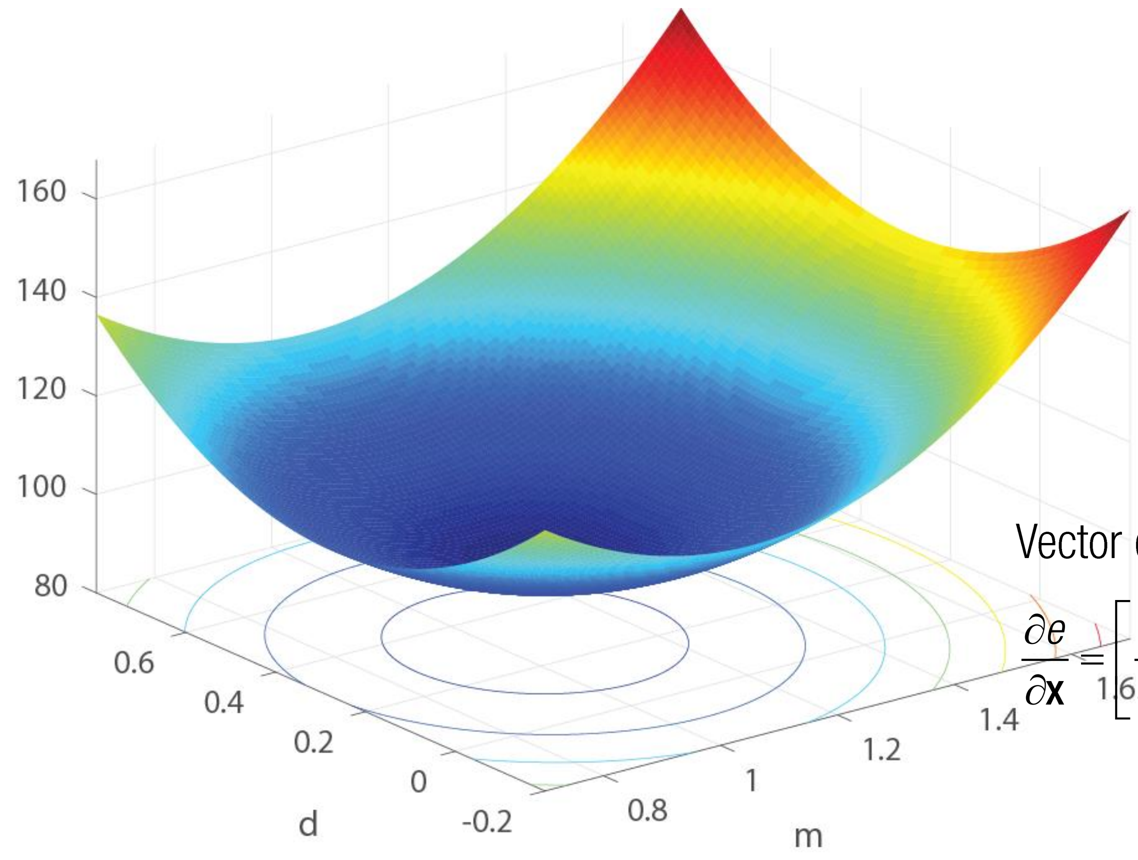
Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$



$$E = (\mathbf{Ax} - \mathbf{b})^{\mathsf{T}} (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$
$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \quad ?$$



Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

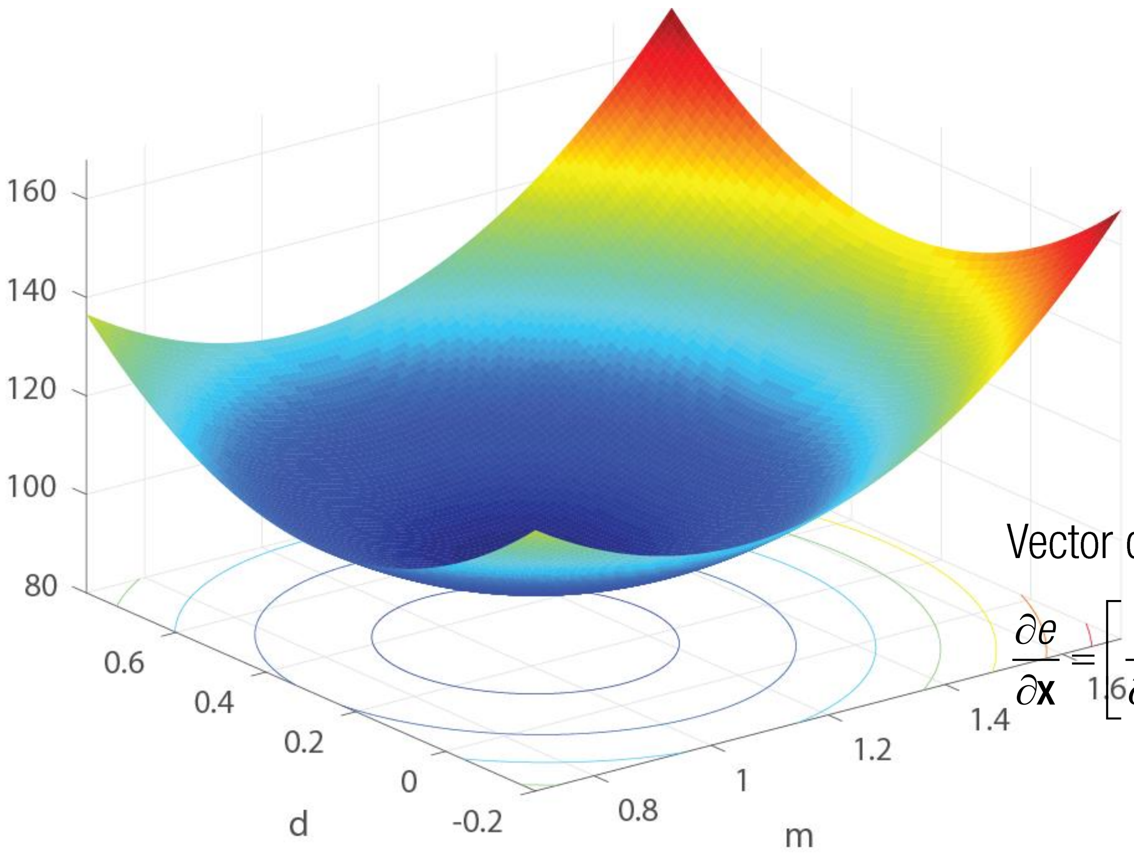
Ex) $e = \mathbf{c}^{\mathsf{T}}\mathbf{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^{\mathsf{T}}\mathbf{x}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}} (c_1 x_1 + \cdots + c_n x_n)$$

Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathbf{A}\,\mathbf{x} \approx \mathbf{b}$$



$$E = (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \quad ?$$

Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathbf{c}^\top \mathbf{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^\top \mathbf{x}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}(c_1 x_1 + \cdots + c_n x_n)$$

$$= \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$$

Recall: Line Fitting

Total error: $E=\sum_{i=1}^{n}\left(v_i-(mu_i+d)\right)^2$

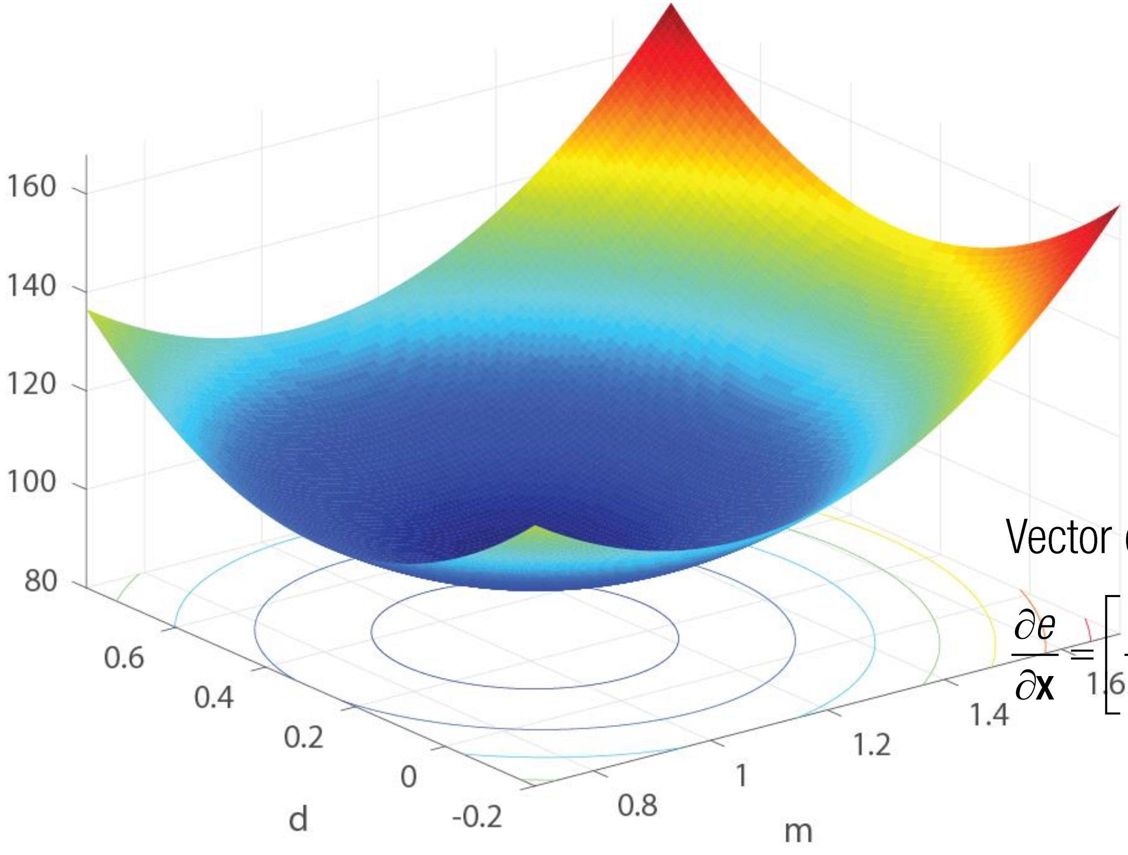
$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}\begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A}\quad\mathbf{x}\approx\mathbf{b}$$

$$E=\left(\mathbf{Ax}-\mathbf{b}\right)^{\mathsf T}\left(\mathbf{Ax}-\mathbf{b}\right)=\left\|\mathbf{Ax}-\mathbf{b}\right\|^2$$

$$=\mathbf{x}^{\mathsf T}\mathbf{A}^{\mathsf T}\mathbf{Ax}-2\mathbf{x}^{\mathsf T}\mathbf{A}^{\mathsf T}\mathbf{b}+\mathbf{b}^{\mathsf T}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}}=2\mathbf{A}^{\mathsf T}\mathbf{Ax}-2\mathbf{A}^{\mathsf T}\mathbf{b}=\mathbf{0}$$



Vector derivative:

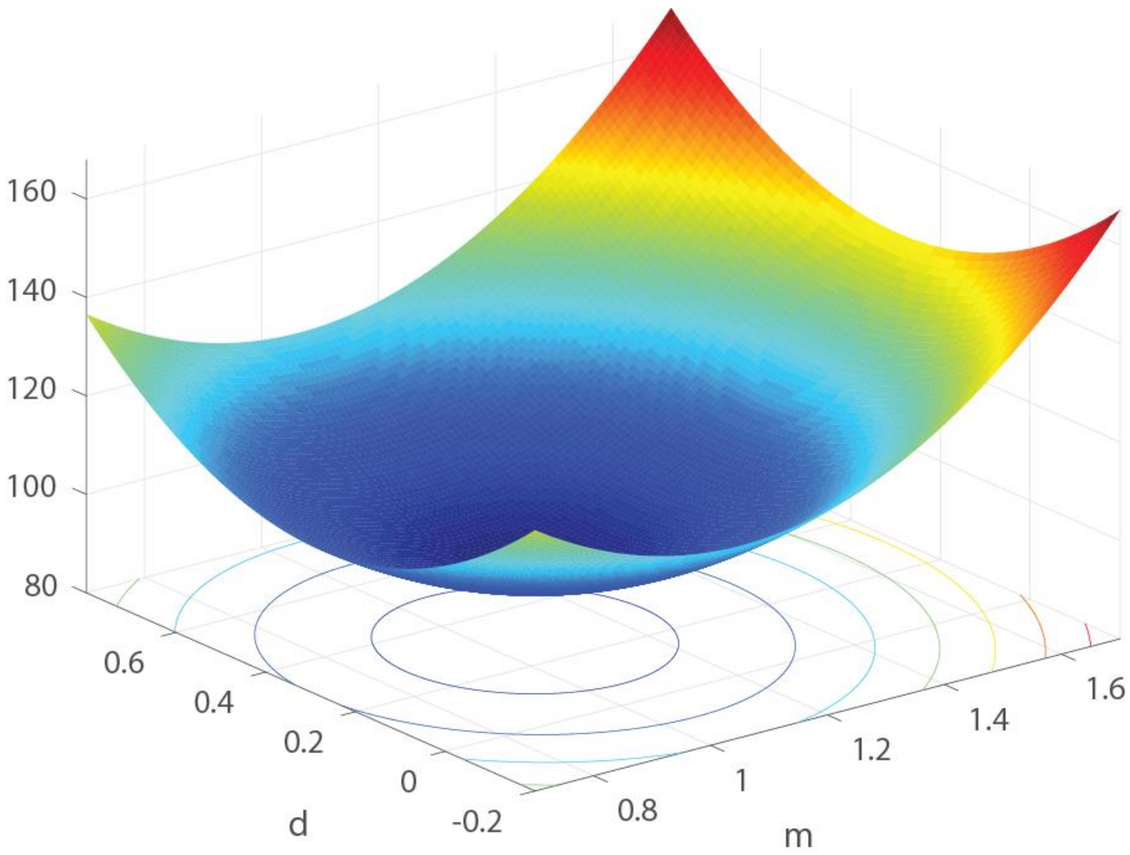
$$\frac{\partial e}{\partial \mathbf{x}}=\begin{bmatrix}\frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n}\end{bmatrix}$$

Ex) $e=\mathbf{c}^{\mathsf T}\mathbf{x}=\begin{bmatrix} c_1 & \cdots & c_n\end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix}$

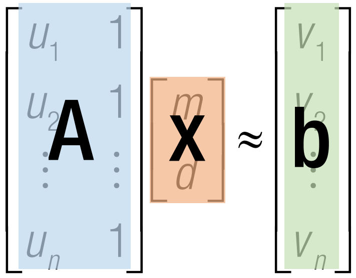
$$\frac{\partial \mathbf{c}^{\mathsf T}\mathbf{x}}{\partial \mathbf{x}}=\frac{\partial}{\partial \mathbf{x}}\left(c_1x_1+\cdots+c_nx_n\right)$$

$$=\begin{bmatrix} c_1 & \cdots & c_n\end{bmatrix}$$

Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$



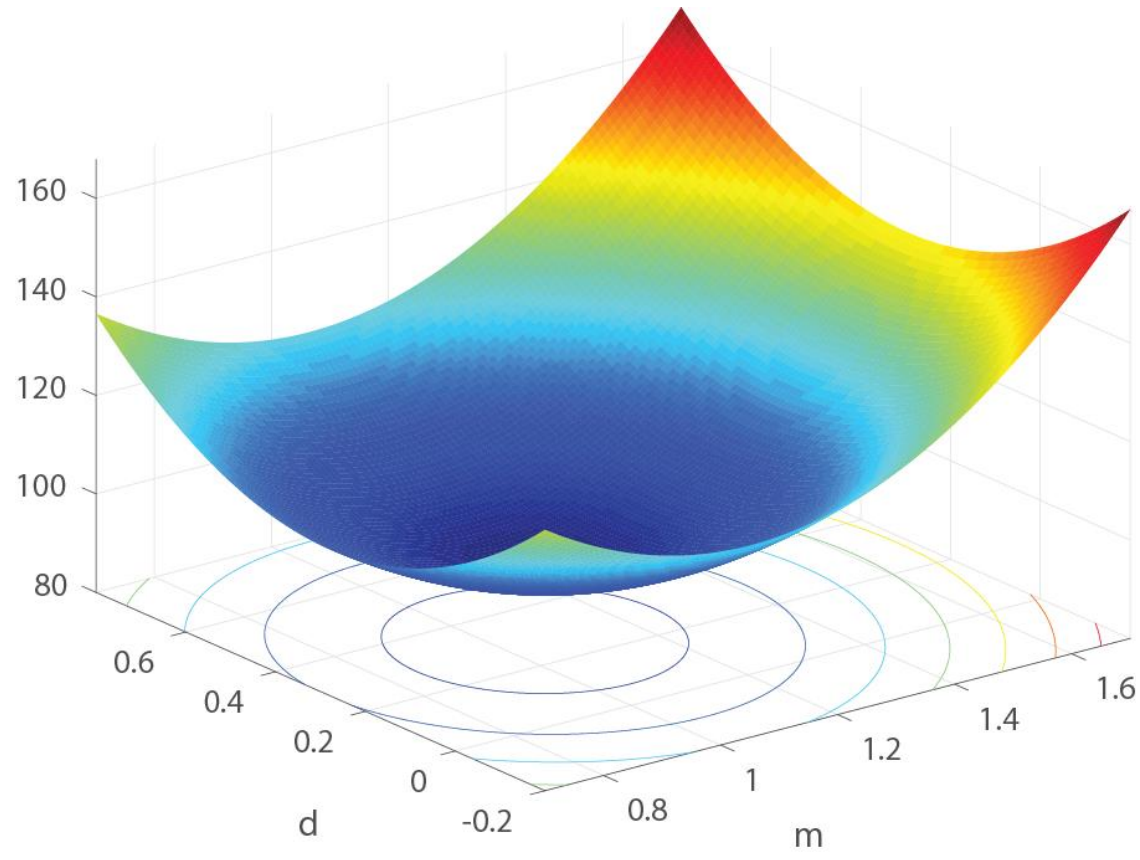
$$E = (\mathbf{Ax} - \mathbf{b})^{\mathrm{T}} (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{b} + \mathbf{b}^{\mathrm{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{A}^{\mathrm{T}}\mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d) \right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$E = (\mathbf{Ax} - \mathbf{b})^{\mathsf{T}} (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

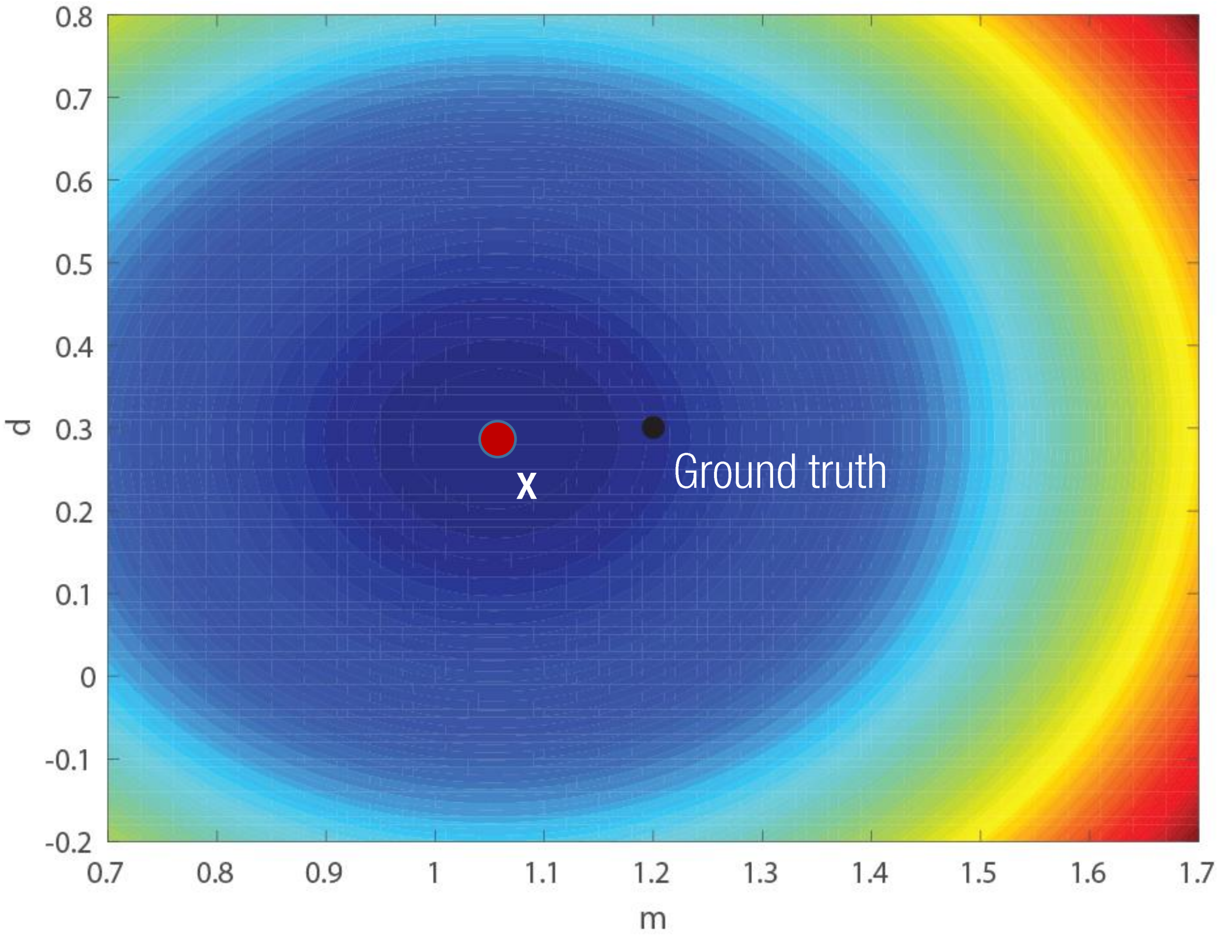
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{A}^{\mathsf{T}}\mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^{\mathsf{T}}\mathbf{Ax} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$$

$$\mathbf{A}^{\mathsf{T}} \; \mathbf{A} \; \mathbf{x} = \mathbf{A}^{\mathsf{T}} \; \mathbf{b}$$

Normal equation

Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} (v_i - (mu_i + d))^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A} \mathbf{x} \approx \mathbf{b}$$

$$E = (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x} - 2\mathbf{x}^T \mathbf{A}^T \mathbf{b} + \mathbf{b}^T \mathbf{b}$$

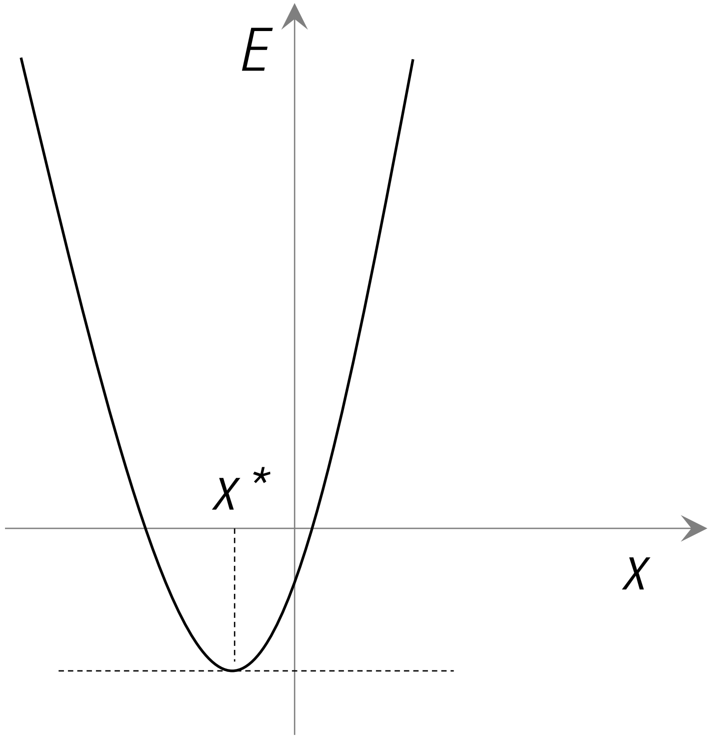
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^T \mathbf{A} \mathbf{x} - 2\mathbf{A}^T \mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{A}^T \mathbf{b}$$

$$\mathbf{A}^T \; \mathbf{A} \; \mathbf{x} = \mathbf{A}^T \; \mathbf{b}$$

$$\mathbf{x} = \left[\mathbf{A}^T \; \mathbf{A} \right]^{-1} \mathbf{A}^T \; \mathbf{b}$$

Linear System Recap



$$\left.\frac{dE}{dx}\right|_{x=x^*} = 0$$

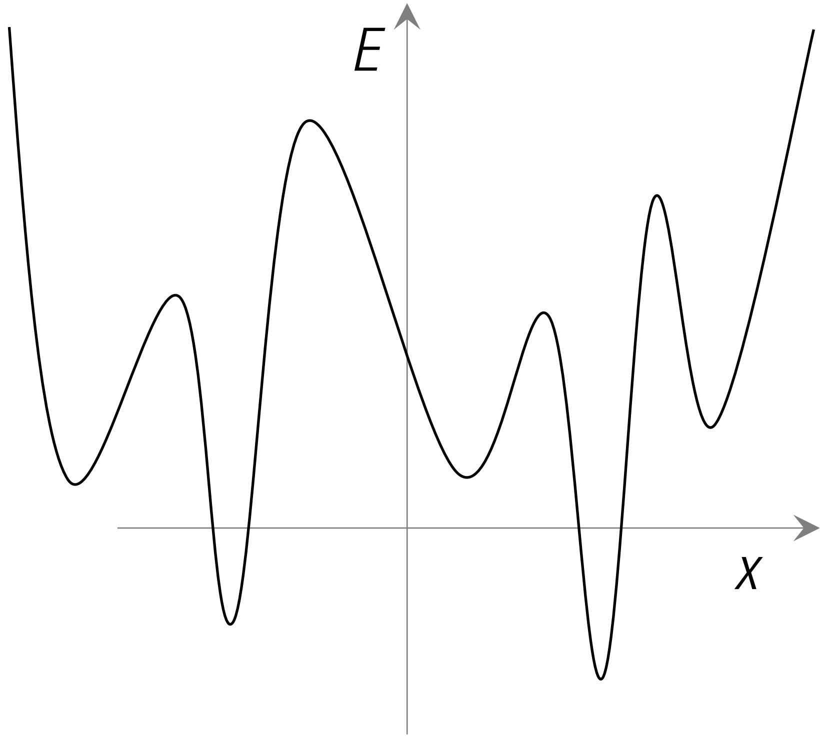
$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$\mathbf{x} = \left[\mathbf{A}^T \mathbf{A} \right]^{-1} \mathbf{A}^T \mathbf{b}$$

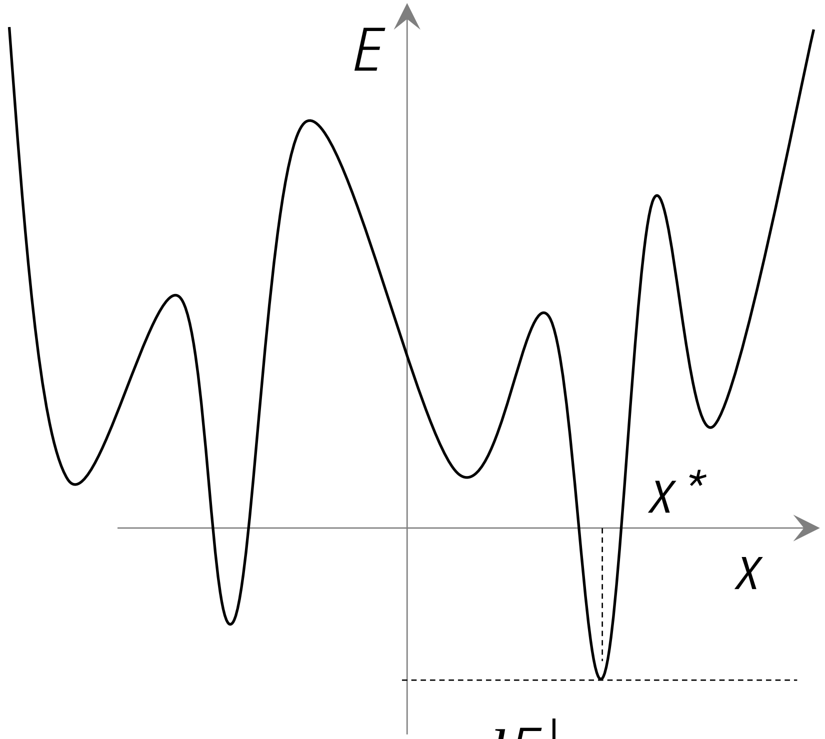
- Has the global solution
- Has the closed form solution (non-iterative solve)
- Is solved efficiently ($O(n^2)$)
- Does not require an initialization

Nonlinear System

$$f(\mathbf{x}) = \mathbf{b}$$



Nonlinear System



$$\left.\frac{dE}{dx}\right|_{x=x^*} = 0$$

$$f(\mathbf{x}) = \mathbf{b}$$

$$E = \left\| f(\mathbf{x}) - \mathbf{b} \right\|^2$$

Nonlinear System

$$f(\mathbf{x}) = \mathbf{b}$$

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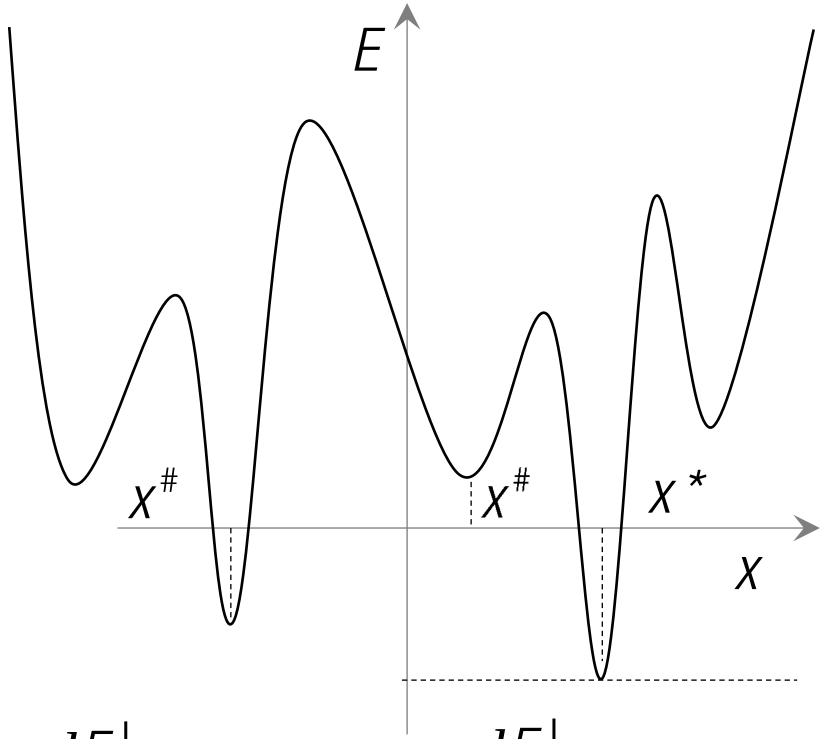
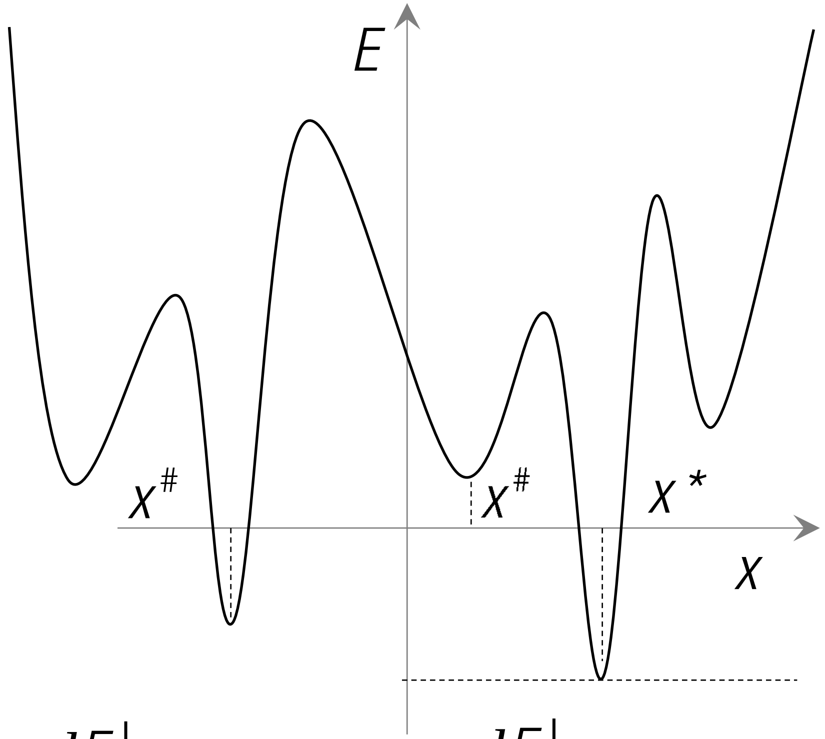


$$\left.\frac{dE}{dx}\right|_{x=x^{\#}} = 0 \qquad \left.\frac{dE}{dx}\right|_{x=x^{*}} = 0$$

Nonlinear System



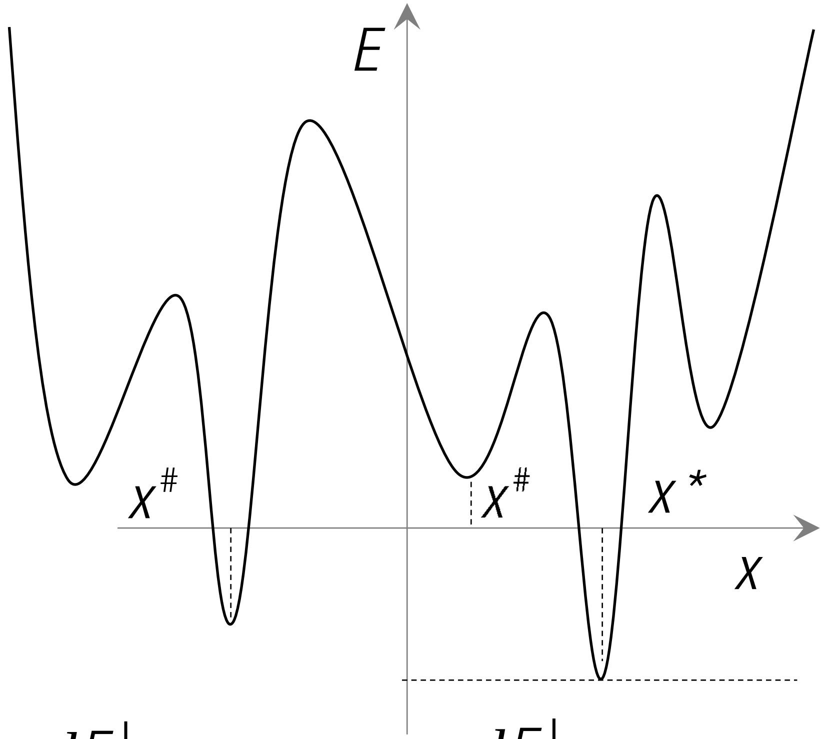
$$\left.\frac{dE}{dx}\right|_{x=x^{\#}} = 0 \qquad \left.\frac{dE}{dx}\right|_{x=x^{*}} = 0$$

$$f(\mathbf{x}) = \mathbf{b}$$

$$E = \left\| f(\mathbf{x}) - \mathbf{b} \right\|^2$$

$$E = \left(f(\mathbf{x}) - \mathbf{b}\right)^{\mathsf{T}} \left(f(\mathbf{x}) - \mathbf{b}\right)$$
$$= f(\mathbf{x})^{\mathsf{T}} f(\mathbf{x}) - 2f(\mathbf{x})^{\mathsf{T}} \mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

Nonlinear System



$$\left.\frac{dE}{dx}\right|_{x=x^{\#}}=0 \qquad \left.\frac{dE}{dx}\right|_{x=x^{*}}=0$$

$$f(\mathbf{x})=\mathbf{b}$$

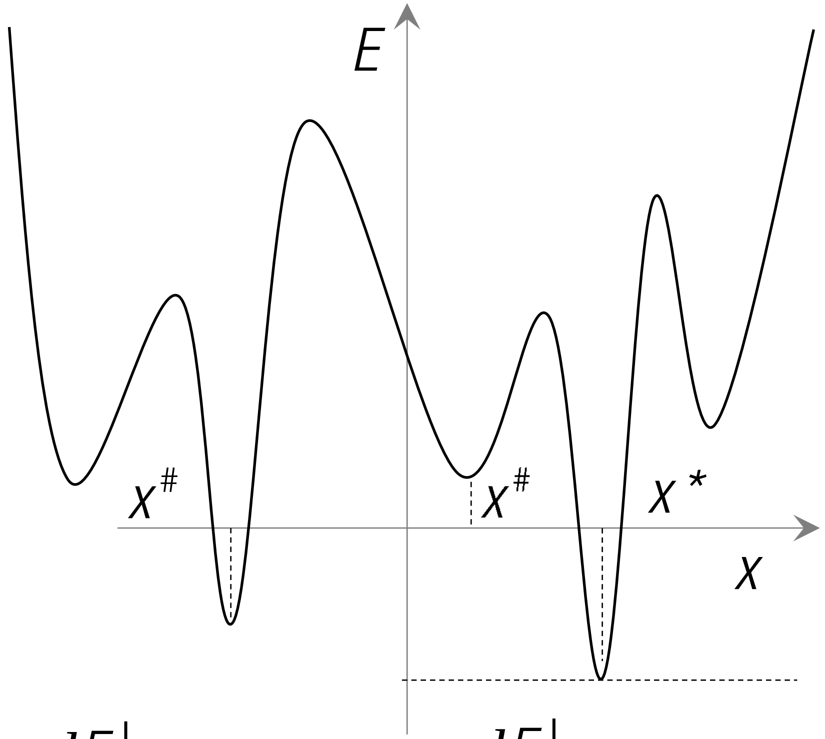
$$E=\left\|f(\mathbf{x})-\mathbf{b}\right\|^2$$

$$E=\left(f(\mathbf{x})-\mathbf{b}\right)^{\mathrm{T}}\left(f(\mathbf{x})-\mathbf{b}\right)$$

$$=f(\mathbf{x})^{\mathrm{T}}f(\mathbf{x})-2f(\mathbf{x})^{\mathrm{T}}\mathbf{b}+\mathbf{b}^{\mathrm{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}}=2\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}}f(\mathbf{x})-2\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}}\mathbf{b}=\mathbf{0}$$

Nonlinear System



$$\left.\frac{dE}{dx}\right|_{x=x^{\#}}=0 \qquad \left.\frac{dE}{dx}\right|_{x=x^{*}}=0$$

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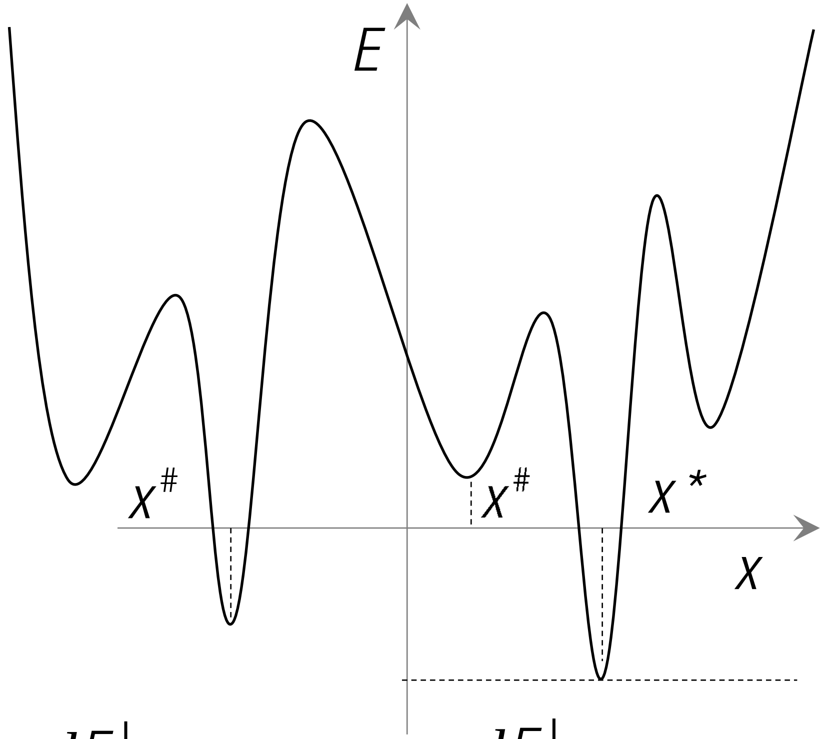
$$\frac{\partial E}{\partial \mathbf{x}}=2\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}}f(\mathbf{x})-2\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}}\mathbf{b}=\mathbf{0}$$

where $$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}=\begin{bmatrix}\frac{\partial f_1}{\partial \mathbf{x}_1} & \cdots & \frac{\partial f_1}{\partial \mathbf{x}_n}\\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial \mathbf{x}_n} & \cdots & \frac{\partial f_m}{\partial \mathbf{x}_n}\end{bmatrix}$$: Jacobian

Nonlinear System



$$\left.\frac{dE}{dx}\right|_{x=x^{\#}}=0 \qquad \left.\frac{dE}{dx}\right|_{x=x^{*}}=0$$

$$f(\mathbf{x})=\mathbf{b}$$

$$E=\left\|f(\mathbf{x})-\mathbf{b}\right\|^2$$

$$E=\left(f(\mathbf{x})-\mathbf{b}\right)^{\mathsf{T}}\left(f(\mathbf{x})-\mathbf{b}\right)$$
$$=f(\mathbf{x})^{\mathsf{T}}f(\mathbf{x})-2f(\mathbf{x})^{\mathsf{T}}\mathbf{b}+\mathbf{b}^{\mathsf{T}}\mathbf{b}$$

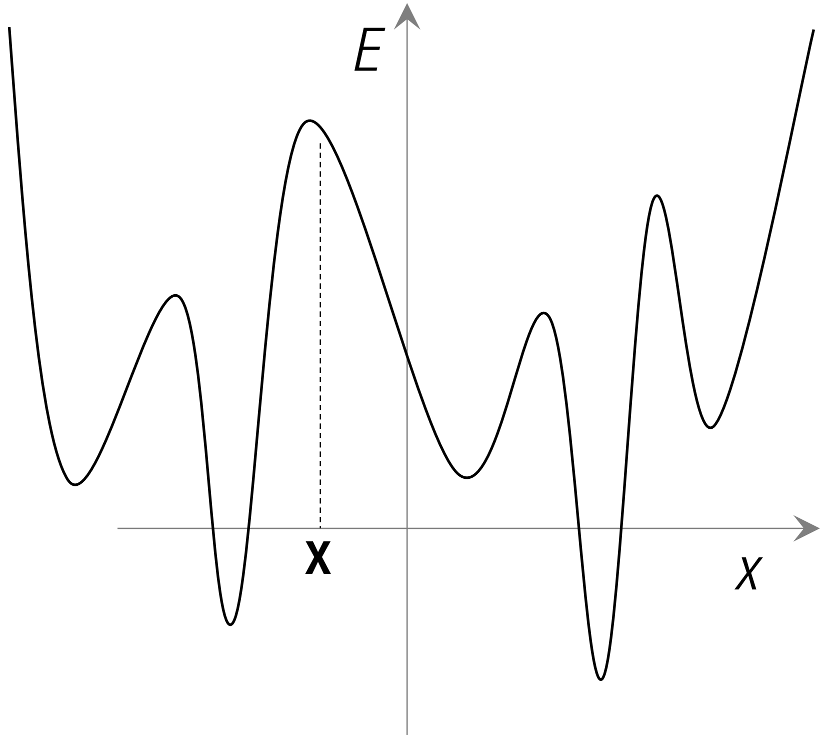
$$\frac{\partial E}{\partial \mathbf{x}}=2\frac{\partial f(\mathbf{x})^{\mathsf{T}}}{\partial \mathbf{x}}f(\mathbf{x})-2\frac{\partial f(\mathbf{x})^{\mathsf{T}}}{\partial \mathbf{x}}\mathbf{b}=\mathbf{0}$$

Find **x** such that the following equation is satisfied:

$$\frac{\partial f(\mathbf{x})^{\mathsf{T}}}{\partial \mathbf{x}}f(\mathbf{x})=\frac{\partial f(\mathbf{x})^{\mathsf{T}}}{\partial \mathbf{x}}\mathbf{b}$$

How?

Nonlinear System



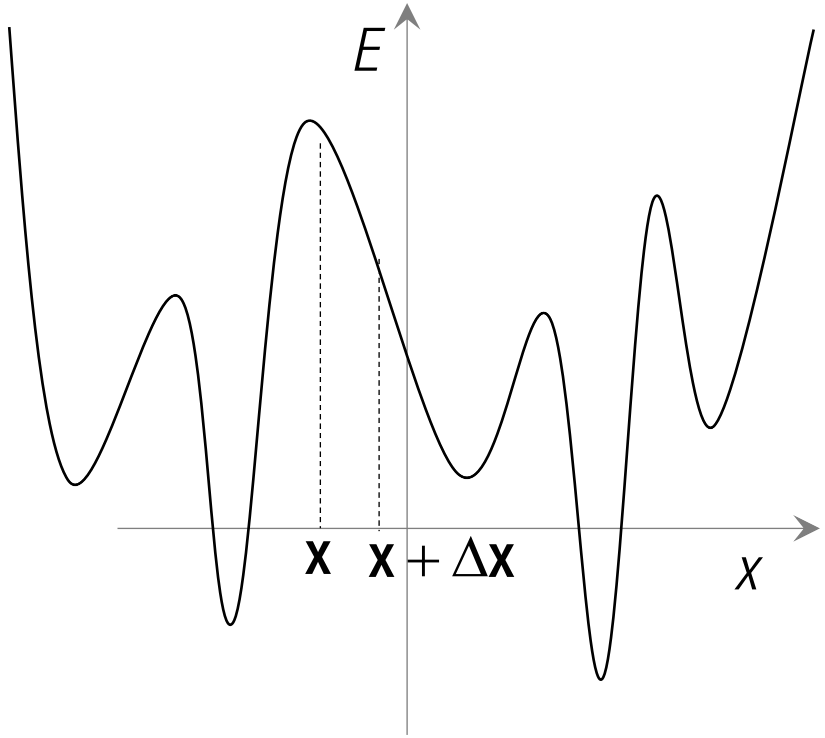
Find **x** such that the following equation is satisfied:

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} f(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{b}$$

How?

Strategy: Given **x**,

Nonlinear System



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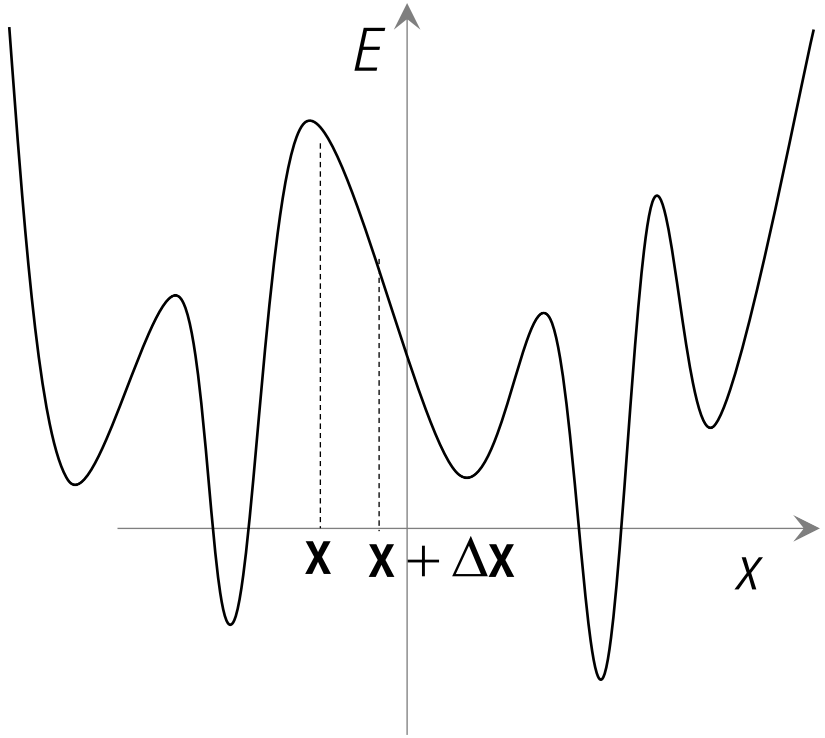
How?

Strategy: Given **x,** move $\Delta\mathbf{x}$ such that $E(\mathbf{x}+\Delta\mathbf{x}) \leq E(\mathbf{x})$

Nonlinear System



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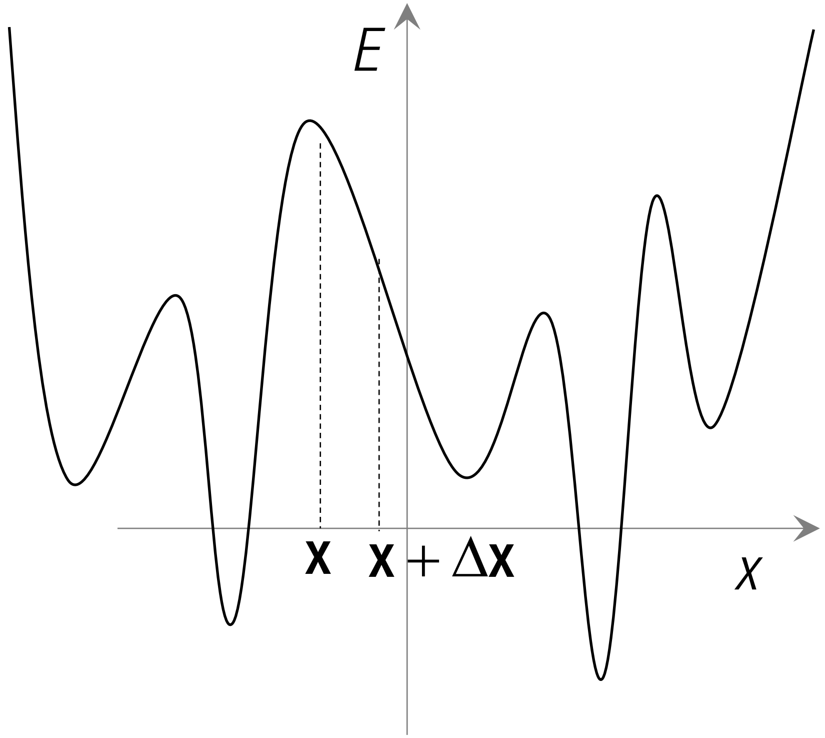
How?

Strategy: Given **x,** move $\Delta\mathbf{x}$ such that $E(\mathbf{x}+\Delta\mathbf{x}) \leq E(\mathbf{x})$

Taylor expansion:

$$f(\mathbf{x}+\Delta\mathbf{x}) =$$

Nonlinear System



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How?

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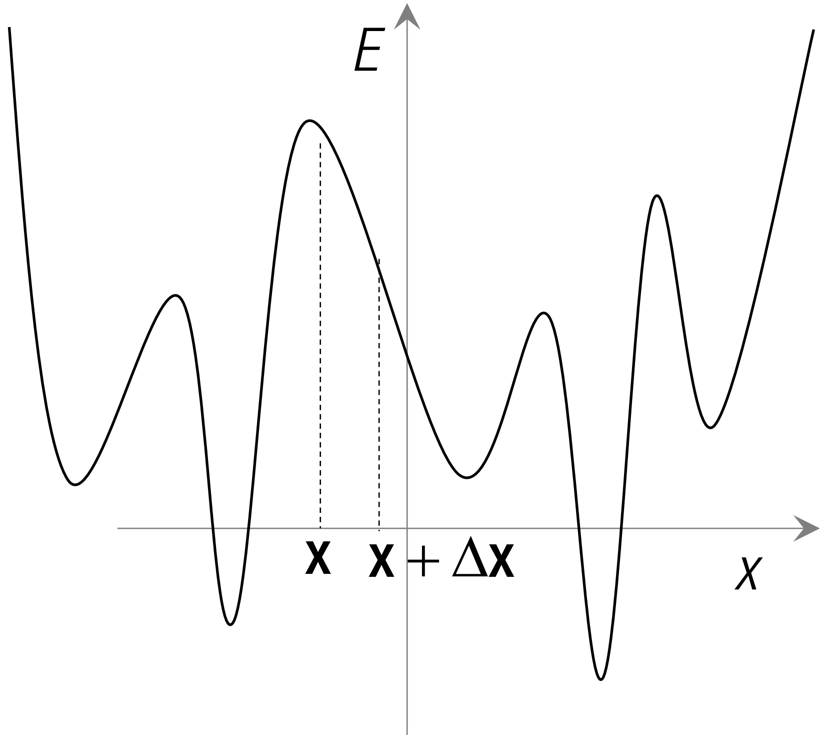
Taylor expansion:

$$f(\mathbf{x}+\Delta\mathbf{x}) = f(\mathbf{x}+\Delta\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x} + \mathrm{H.O.T.}$$

Nonlinear System



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How?

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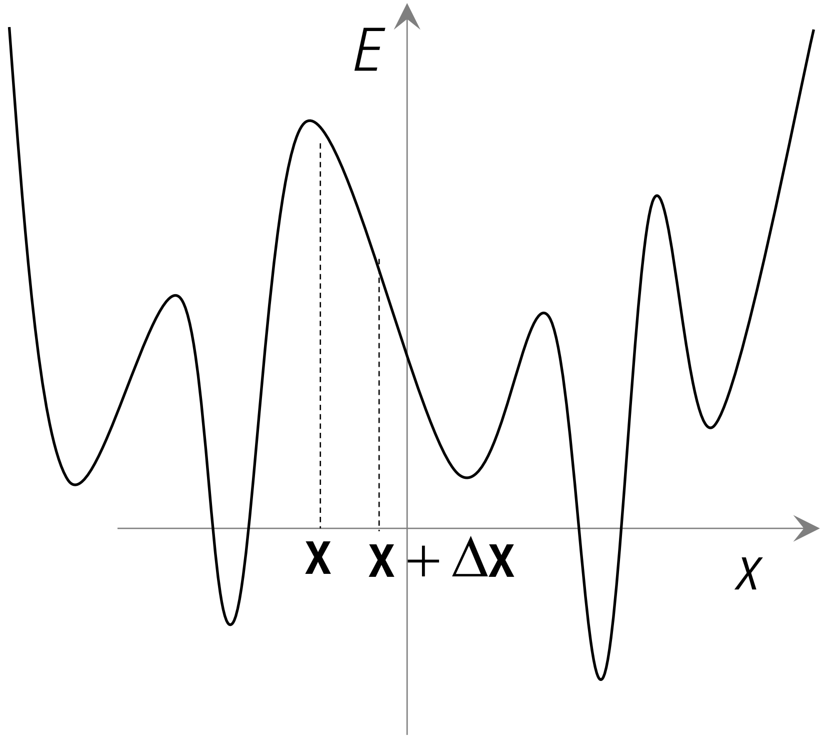
Taylor expansion:

$$f(\mathbf{x}+\Delta\mathbf{x}) = f(\mathbf{x}+\Delta\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x} + \text{H.O.T.}$$

$$\rightarrow \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(f(\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x} \right) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{b}$$

Nonlinear System



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How?

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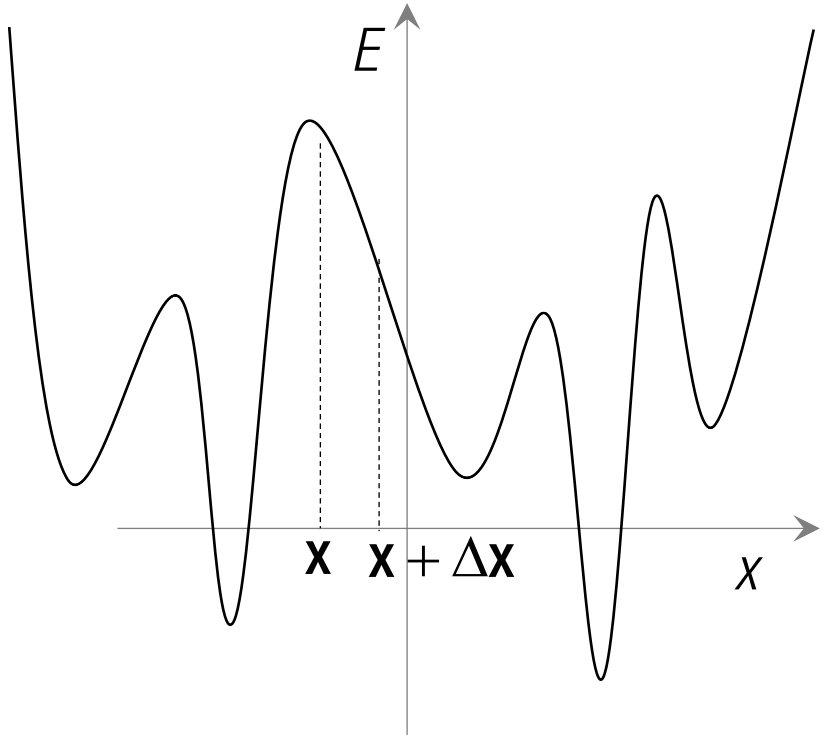
Taylor expansion:

$$f(\mathbf{x}+\Delta\mathbf{x}) = f(\mathbf{x}+\Delta\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x} + \text{H.O.T.}$$

$$\rightarrow \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(f(\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x} \right) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \mathbf{b}$$

$$\rightarrow \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

Nonlinear System



Cf.) $\mathbf{x} = \left[\mathbf{A}^{T}\mathbf{A}\right]^{-1}\mathbf{A}^{T}\mathbf{b}$

Find **x** such that the following equation is satisfied:

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{T} f(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{T} \mathbf{b}$$

How?

Strategy: Given **x,** move $\Delta\mathbf{x}$ such that $E(\mathbf{x}+\Delta\mathbf{x}) \leq E(\mathbf{x})$

Taylor expansion:

$$f(\mathbf{x}+\Delta\mathbf{x}) = f(\mathbf{x}+\Delta\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x} + \text{H.O.T.}$$

$$\rightarrow \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{T}\left(f(\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x}\right) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{T}\mathbf{b}$$

$$\rightarrow \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{T}\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{T}\left(\mathbf{b} - f(\mathbf{x})\right)$$