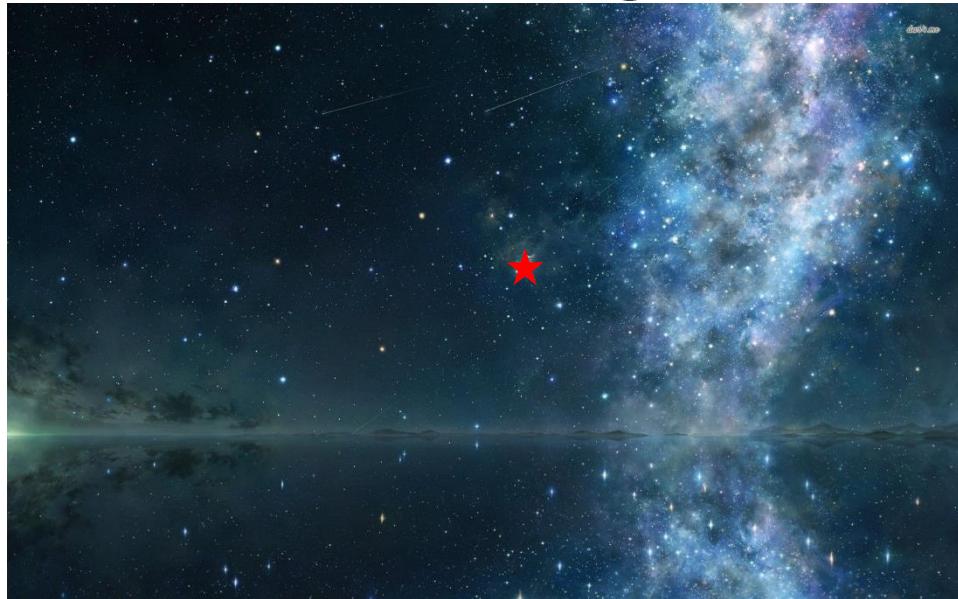


Projective Line

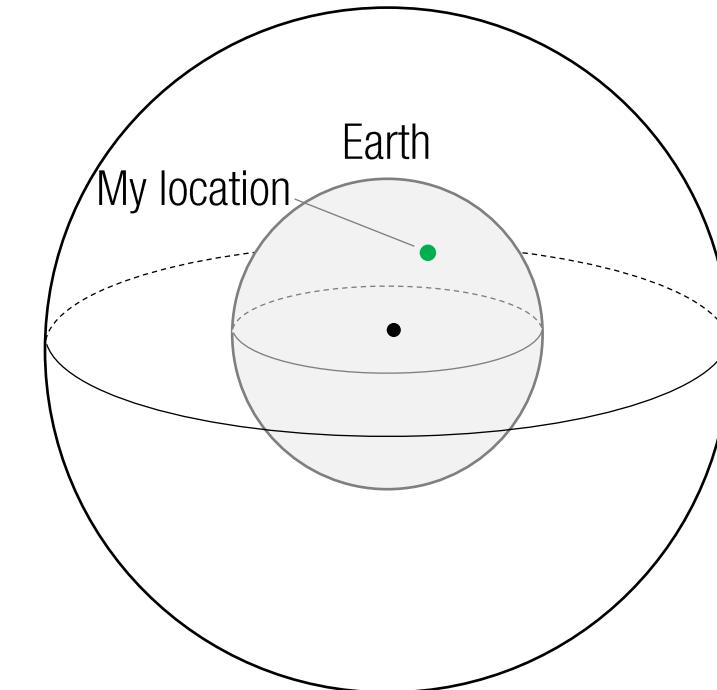


Where am I?

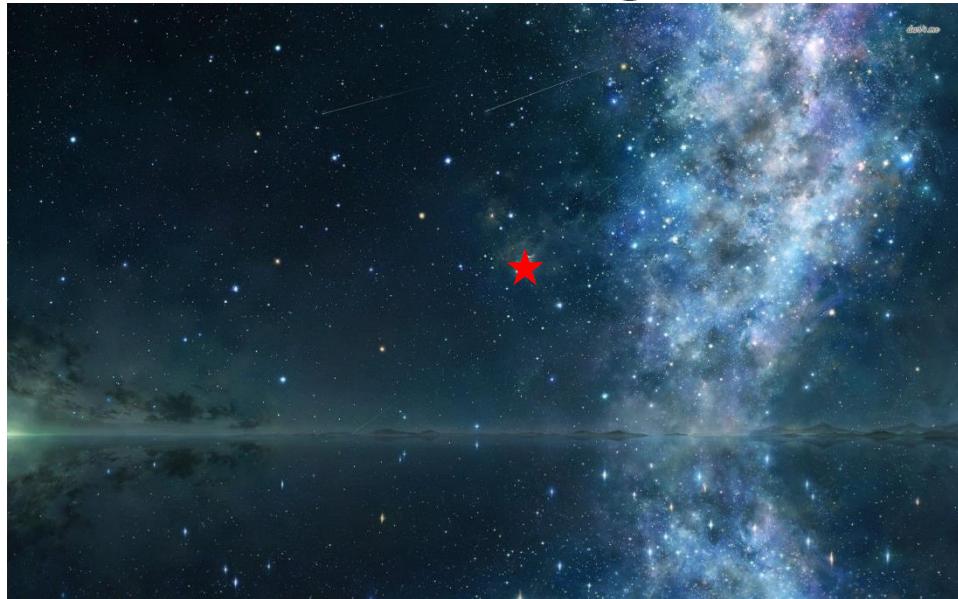
Celestial Navigation



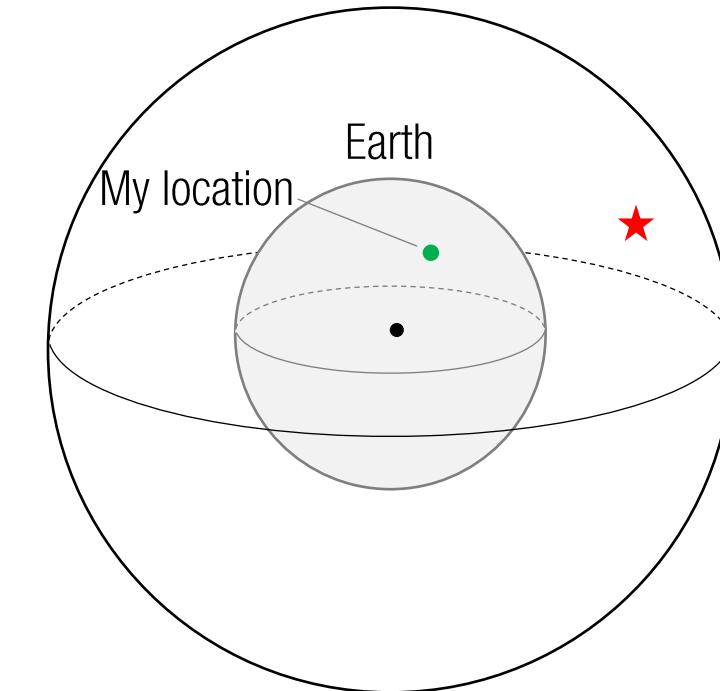
Far far away: point at infinity



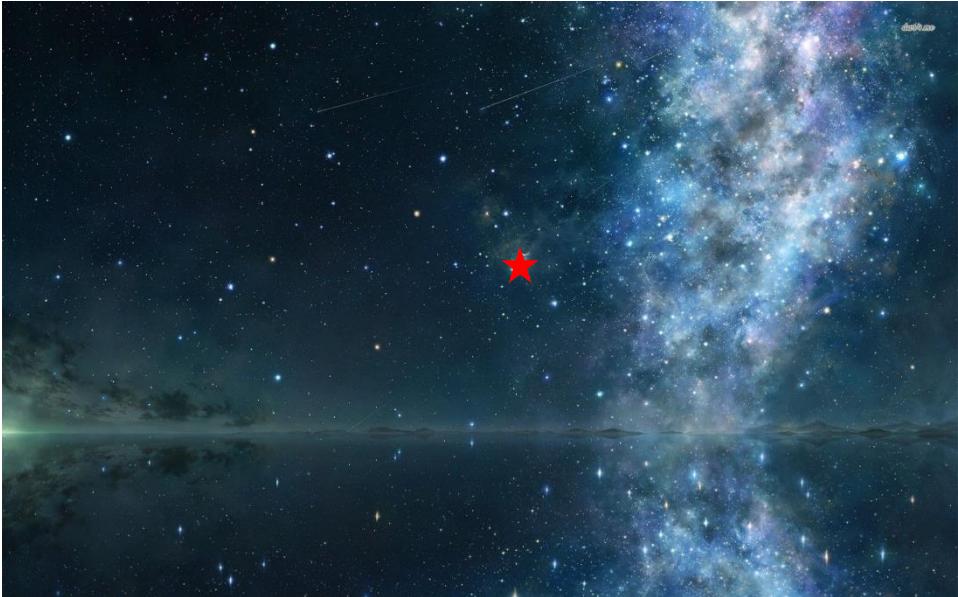
Celestial Navigation



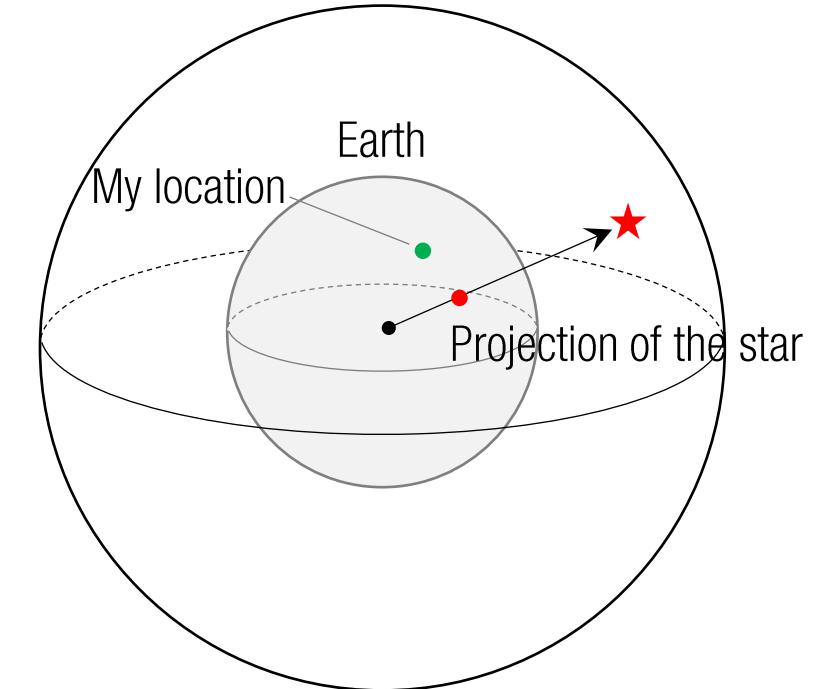
Far far away: point at infinity



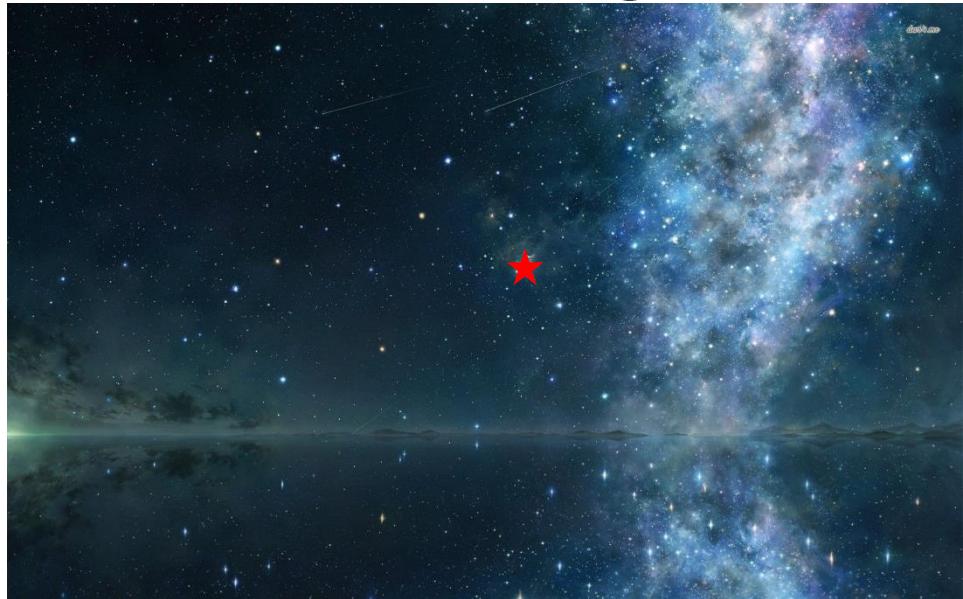
Celestial Navigation



Far far away: point at infinity



Celestial Navigation

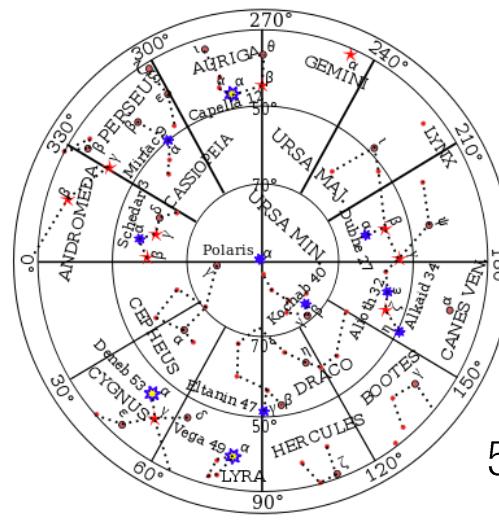
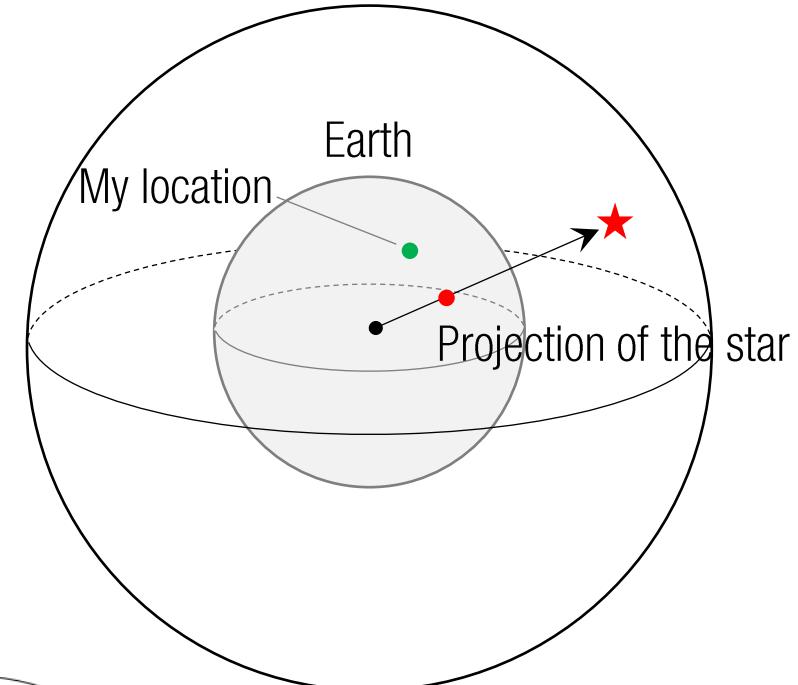


292

GREENWICH P. M. 1942 MAY 26 (TUESDAY)

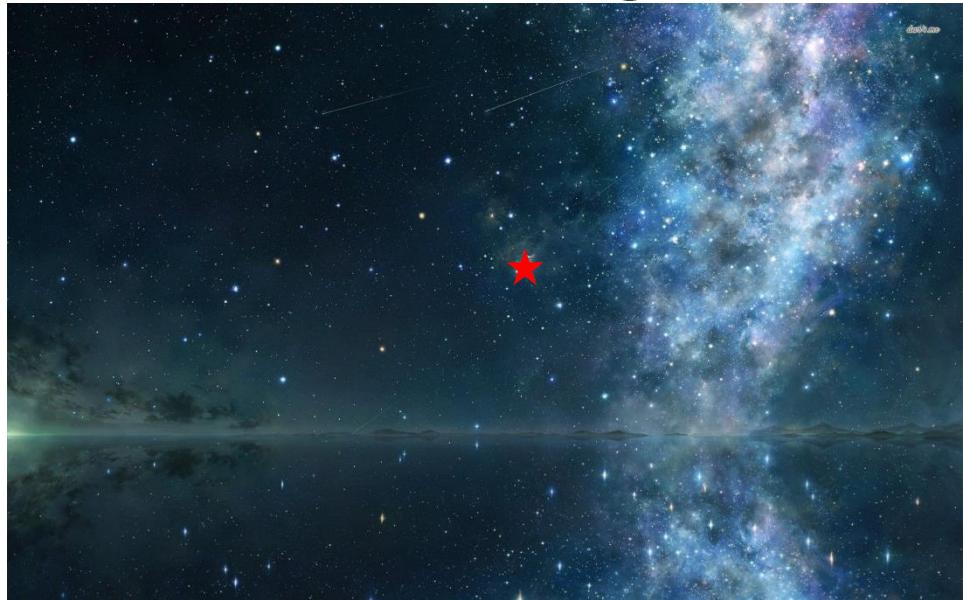
GCT h m	SUN	T	VENUS-1.5	MARS 1.5	JUPITER-1.5	MOON	
	GHA Dec.	GHA	GHA Dec.	GHA Dec.	GHA Dec.	GHA Dec.	
12 00	0 47 N21 04	63 25	41 45 N 6 53	313 11 N23 31	337 04 N23 14	230 23 S 2 34	
10	3 17	65 56	44 15	315 41	339 34	232 48	36
20	5 47	68 26	46 45	318 11	342 04	235 13	38
30	8 17	70 57	49 15	320 41	344 35	237 38	40
40	10 47	73 27	51 45	323 11	347 05	240 03	41
50	13 17	75 57	54 15	325 41	349 35	242 27	43
13 00	15 47 N21 04	78 28	56 45 N 6 54	328 11 N23 31	352 06 N23 14	244 52 S 2 45	60
10	8 58	80 58	59 15	330 42	354 36	247 17	47
20	20 47	83 29	61 45	333 12	357 06	249 42	49
30	23 17	85 59	64 15	335 42	359 37	252 07	51
40	25 47	88 30	66 45	338 12	2 07	254 32	53
50	28 17	91 00	69 15	340 42	4 37	256 56	54
14 00	30 47 N21 05	93 30	71 45 N 6 55	343 12 N23 31	7 08 N23 14	259 21 S 2 56	45
10	33 17	96 01	74 15	345 42	9 38	261 46	2
20	35 47	98 31	76 45	348 13	12 08	264 11	3 00
30	38 17	101 02	79 15	350 43	14 38	266 36	35
40	40 47	103 32	81 45	353 13	17 09	269 01	04
50	43 17	106 02	84 15	355 43	19 39	271 26	10
15 00	45 47 N21 05	108 33	86 45 N 6 56	358 13 N23 31	22 09 N23 14	273 50 S 3 08	0
10	48 17	111 03	89 15	0 43	24 40	276 15	09
20	50 47	113 34	91 45	3 13	27 10	278 40	11
30	53 17	116 04	94 14	5 43	29 40	281 05	13
40	55 47	118 34	96 44	8 14	32 11	283 30	15
50	58 17	121 05	99 14	10 44	34 41	285 55	17
16 00	60 47 N21 06	123 35	101 44 N 6 57	13 14 N23 30	37 11 N23 14	288 19 S 3 19	40
10	63 17	126 06	104 14	15 44	39 42	290 44	21
20	65 47	128 36	106 44	18 14	42 12	293 09	23

Far far away: point at infinity



57 stars for navigation

Celestial Navigation

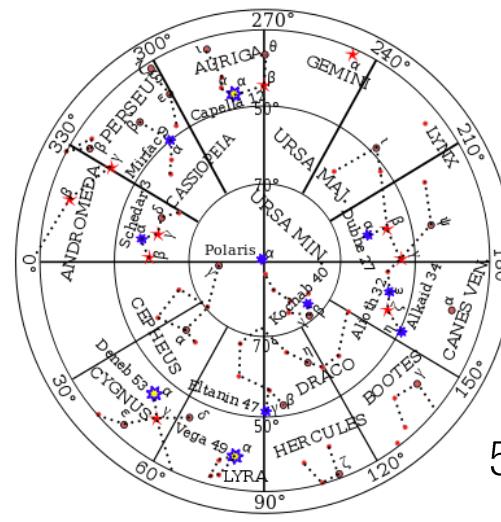
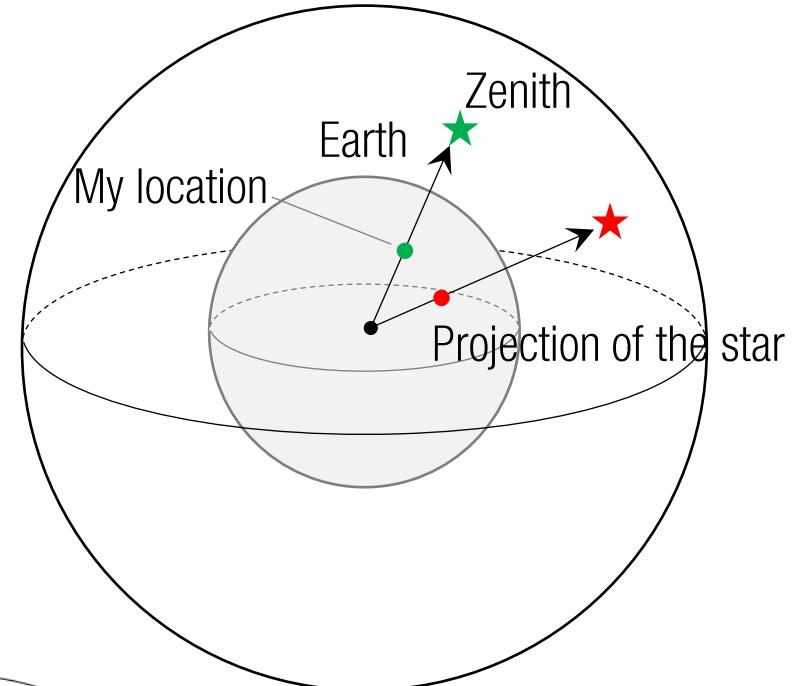


292

GREENWICH P. M. 1942 MAY 26 (TUESDAY)

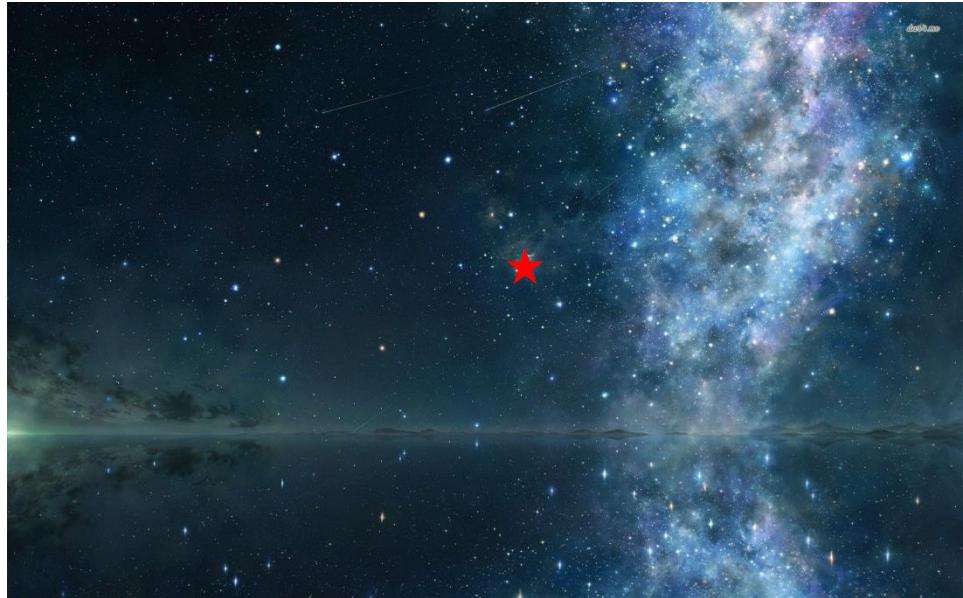
GCT h m	SUN	T	VENUS-1.5	MARS 1.5	JUPITER-1.5	MOON	
	GHA Dec.	GHA	GHA Dec.	GHA Dec.	GHA Dec.	GHA Dec.	
12 00	0 47 N21 04	63 25	41 45 N 6 53	313 11 N23 31	337 04 N23 14	230 23 S 2 34	
10	3 17	65 56	44 15	315 41	339 34	232 48	36
20	5 47	68 26	46 45	318 11	342 04	235 13	38
30	8 17	70 57	49 15	320 41	344 35	237 38	40
40	10 47	73 27	51 45	323 11	347 05	240 03	41
50	13 17	75 57	54 15	325 41	349 35	242 27	43
13 00	15 47 N21 04	78 28	56 45 N 6 54	328 11 N23 31	352 06 N23 14	244 52 S 2 45	60
10	18 17	80 58	59 15	330 42	354 36	247 17	47
20	20 47	83 29	61 45	333 12	357 06	249 42	49
30	23 17	85 59	64 15	335 42	359 37	252 07	51
40	25 47	88 30	66 45	338 12	2 07	254 32	53
50	28 17	91 00	69 15	340 42	4 37	256 56	54
14 00	30 47 N21 05	93 30	71 45 N 6 55	343 12 N23 31	7 08 N23 14	259 21 S 2 56	45
10	33 17	96 01	74 15	345 42	9 38	261 46	2
20	35 47	98 31	76 45	348 13	12 08	264 11	3 00
30	38 17	101 02	79 15	350 43	14 38	266 36	35
40	40 47	103 32	81 45	353 13	17 09	269 01	04
50	43 17	106 02	84 15	355 43	19 39	271 26	10
15 00	45 47 N21 05	108 33	86 45 N 6 56	358 13 N23 31	22 09 N23 14	273 50 S 3 08	0
10	48 17	111 03	89 15	0 43	24 40	276 15	09
20	50 47	113 34	91 45	3 13	27 10	278 40	11
30	53 17	116 04	94 14	5 43	29 40	281 05	13
40	55 47	118 34	96 44	8 14	32 11	283 30	15
50	58 17	121 05	99 14	10 44	34 41	285 55	17
16 00	60 47 N21 06	123 35	101 44 N 6 57	13 14 N23 30	37 11 N23 14	288 19 S 3 19	40
10	63 17	126 06	104 14	15 44	39 42	290 44	21
20	65 47	128 36	106 44	18 14	42 12	293 09	23

Far far away: point at infinity



57 stars for navigation

Celestial Navigation

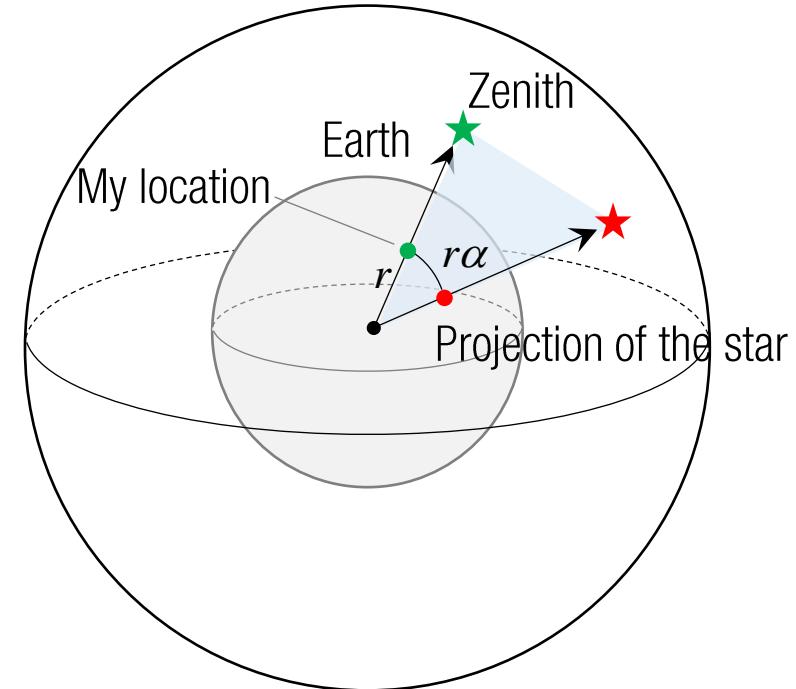


292

GREENWICH P. M. 1942 MAY 26 (TUESDAY)

GCT h m	SUN	T	VENUS-1.5	MARS-1.5	JUPITER-1.5	MOON					
	GHA Dec.	GHA	GHA Dec.	GHA Dec.	GHA Dec.	GHA	Lat.	Sun-rise	Twil.	Moon-rise	Dini.
12 00	0 47 N21 04	63 25	41 45 N 6 53	313 11 N23 31	337 04 N23 14	230 23 S 2 34					
10	3 17	65 56	44 15	315 41	339 34	232 48	36				
20	5 47	68 26	46 45	318 11	342 04	235 13	38				
30	8 17	70 57	49 15	320 41	344 35	237 38	40				
40	10 47	73 27	51 45	323 11	347 05	240 03	41				
50	13 17	75 57	54 15	325 41	349 35	242 27	43				
13 00	15 47 N21 04	78 28	56 45 N 6 54	328 11 N23 31	352 06 N23 14	244 52 S 2 45	60	3 00	75	15 08	87
10	18 17	80 58	59 15	330 42	354 36	247 17	47	58	17	64	06 53
20	20 47	83 29	61 45	333 12	357 06	249 42	49	56	30	55	04 53
30	23 17	85 59	64 15	335 42	359 37	252 07	51	54	42	50	03 80
40	25 47	88 30	66 45	338 12	2 07	254 32	53	52	3 52	45	02 78
50	28 17	91 00	69 15	340 42	4 37	256 56	54	50	4 02	42	15 01 76
14 00	30 47 N21 05	93 30	71 45 N 6 55	343 12 N23 31	7 08 N23 14	259 21 S 2 56	45	21	36	14 58	72
10	33 17	96 01	74 15	345 42	9 38	261 46	2 58	40	37	32	56 69
20	35 47	98 31	76 45	348 13	12 08	264 11	3 00	35	4 50	29	54 66
30	38 17	101 02	79 15	350 43	14 38	266 36	02	30	5 01	27	52 64
40	40 47	103 32	81 45	353 13	17 09	269 01	04	20	21	24	49 60
50	43 17	106 02	84 15	355 43	19 39	271 26	06	10	38	23	47 56
15 00	45 47 N21 05	108 33	86 45 N 6 56	358 13 N23 31	22 09 N23 14	273 50 S 3 08	0	5 53	22	45	53
10	48 17	111 03	89 15	0 43	24 40	276 15	09				
20	50 47	113 34	91 45	3 13	27 10	278 40	11	10	6 09	23	42 50
30	53 17	116 04	94 14	5 43	29 40	281 05	13	20	25	24	40 46
40	55 47	118 34	96 44	8 14	32 11	283 30	15	30	43	25	38 41
50	58 17	121 05	99 14	10 44	34 41	285 55	17	35	6 54	27	36 39
16 00	60 47 N21 06	123 35	101 44 N 6 57	13 14 N23 30	37 11 N23 14	288 19 S 3 19	40	7 07	30	34	37
10	63 17	126 06	104 14	15 44	39 42	290 44	21	45	21	33	32 34
20	65 47	128 36	106 44	18 14	42 12	293 09	23	50	39	37	30 30

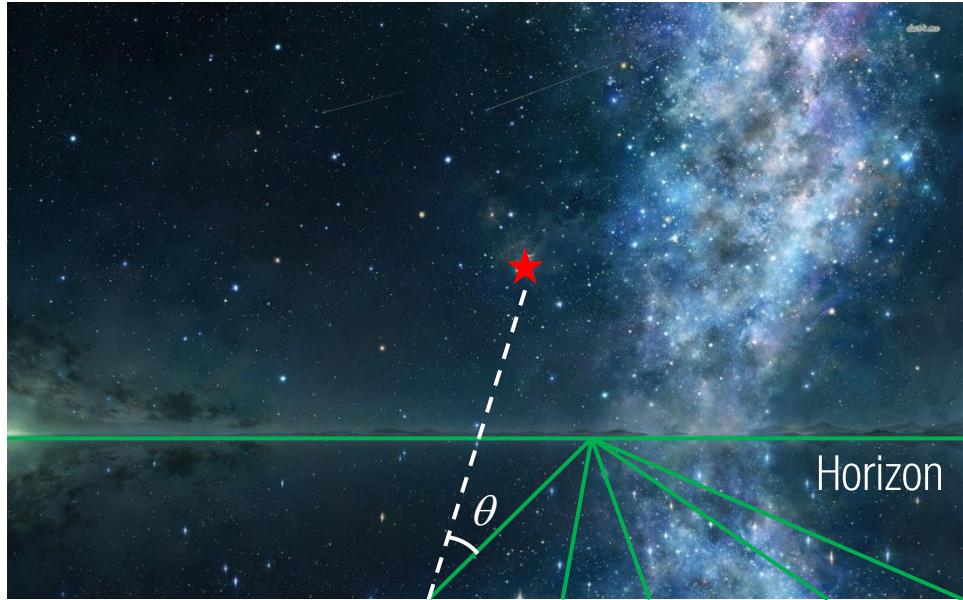
Far far away: point at infinity



r : radius of earth

α : angle between stars

Celestial Navigation

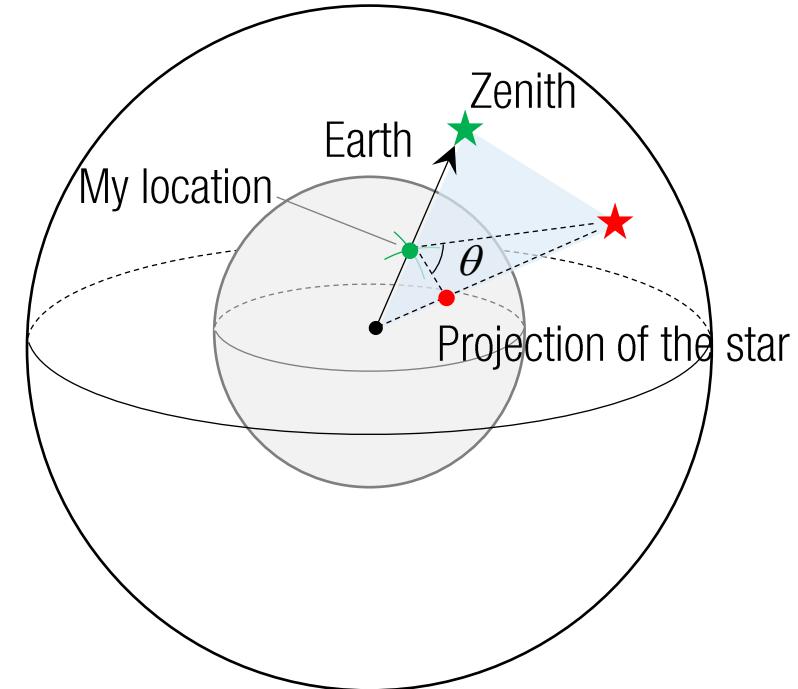


292

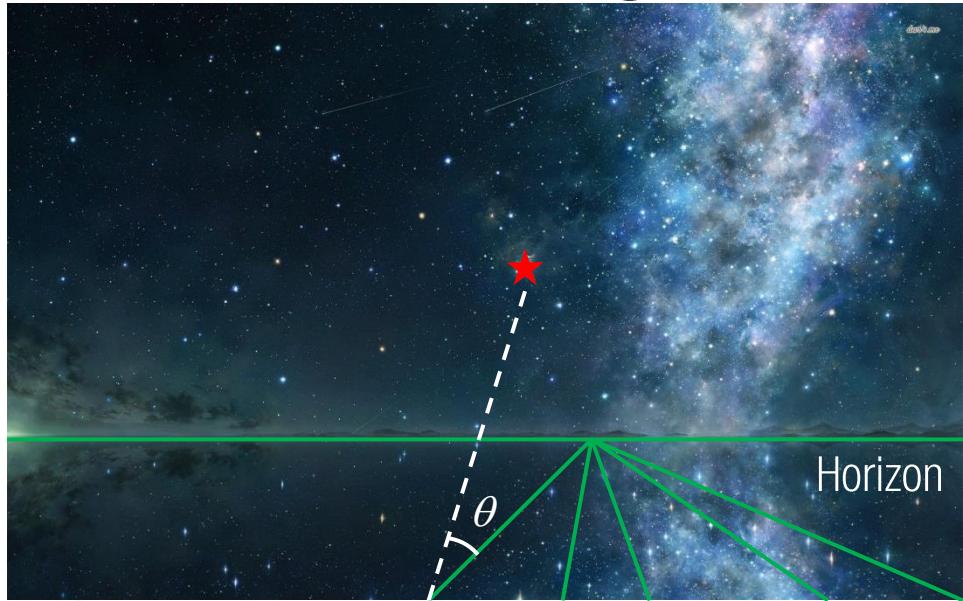
GREENWICH P. M. 1942 MAY 26 (TUESDAY)

GCT h m	SUN GHA Dec.	T GHA	VENUS-1.5 GHA Dec.	MARS-1.5 GHA Dec.	JUPITER-1.5 GHA Dec.	MOON GHA Dec.					
							Lat. N	Sun- rise h m	Twil. m	Moon- rise h m	Dini. m
12 00	0 47 N21 04	63 25	41 45 N 6 53	313 11 N23 31	337 04 N23 14	230 23 S 2 34					
10	3 17	65 56	44 15	315 41	339 34	232 48	36				
20	5 47	68 26	46 45	318 11	342 04	235 13	38				
30	8 17	70 57	49 15	320 41	344 35	237 38	40				
40	10 47	73 27	51 45	323 11	347 05	240 03	41				
50	13 17	75 57	54 15	325 41	349 35	242 27	43				
13 00	15 47 N21 04	78 28	56 45 N 6 54	328 11 N23 31	352 06 N23 14	244 52 S 2 45	60	3 00	75	15 08	87
10	18 17	80 58	59 15	330 42	354 36	247 17	47	58	17	64	06 53
20	20 47	83 29	61 45	333 12	357 06	249 42	49	56	30	55	04 53
30	23 17	85 59	64 15	335 42	359 37	252 07	51	54	42	50	03 80
40	25 47	88 30	66 45	338 12	2 07	254 32	53	52	3 52	45	02 78
50	28 17	91 00	69 15	340 42	4 37	256 56	54	50	4 02	42	15 01 76
14 00	30 47 N21 05	93 30	71 45 N 6 55	343 12 N23 31	7 08 N23 14	259 21 S 2 56	45	21	36	14 58	72
10	33 17	96 01	74 15	345 42	9 38	261 46	2 58	40	37	32	56 69
20	35 47	98 31	76 45	348 13	12 08	264 11	3 00	35	4 50	29	54 66
30	38 17	101 02	79 15	350 43	14 38	266 36	02	30	5 01	27	52 64
40	40 47	103 32	81 45	353 13	17 09	269 01	04	20	21	24	49 60
50	43 17	106 02	84 15	355 43	19 39	271 26	06	10	38	23	47 56
15 00	45 47 N21 05	108 33	86 45 N 6 56	358 13 N23 31	22 09 N23 14	273 50 S 3 08	0	5 53	22	45	53
10	48 17	111 03	89 15	0 43	24 40	276 15	09				
20	50 47	113 34	91 45	3 13	27 10	278 40	11	10	6 09	23	42 50
30	53 17	116 04	94 14	5 43	29 40	281 05	13	20	25	24	40 46
40	55 47	118 34	96 44	8 14	32 11	283 30	15	30	43	25	38 41
50	58 17	121 05	99 14	10 44	34 41	285 55	17	35	6 54	27	36 39
16 00	60 47 N21 06	123 35	101 44 N 6 57	13 14 N23 30	37 11 N23 14	288 19 S 3 19	40	7 07	30	34	37
10	63 17	126 06	104 14	15 44	39 42	290 44	21	45	21	33	32 34
20	65 47	128 36	106 44	18 14	42 12	293 09	23	50	39	37	30 30

Far far away: point at infinity



Celestial Navigation

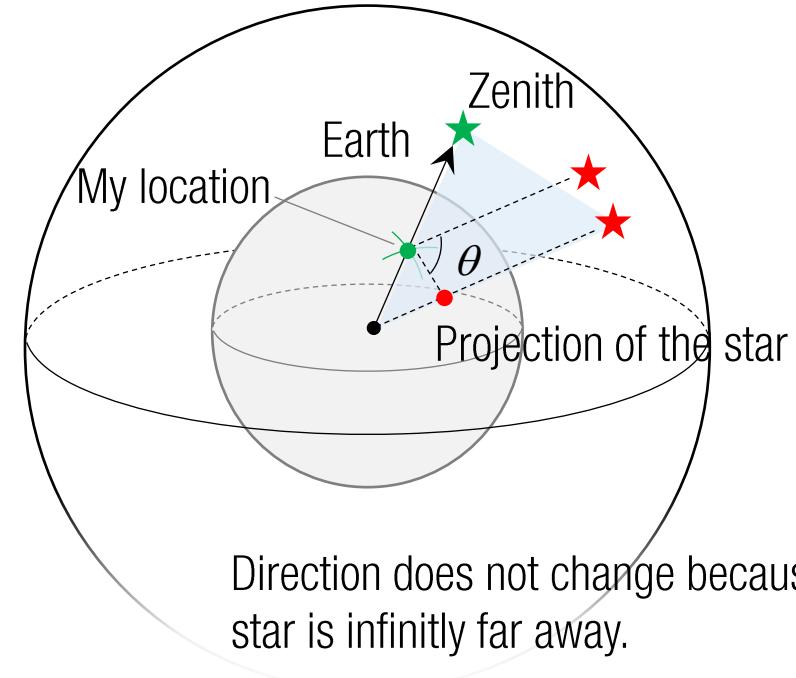


292

GREENWICH P. M. 1942 MAY 26 (TUESDAY)

GCT h m	SUN GHA Dec.	T GHA	VENUS-1.5 GHA Dec.	MARS 1.5 GHA Dec.	JUPITER-1.5 GHA Dec.	MOON GHA Dec.					
							Lat.	Sun- rise	Twi- light	Moon- rise	Dini
12 00	0 47 N21 04	63 25	41 45 N 6 53	313 11 N23 31	337 04 N23 14	230 23 S 2 34					
10	3 17	65 56	44 15	315 41	339 34	232 48	36				
20	5 47	68 26	46 45	318 11	342 04	235 13	38				
30	8 17	70 57	49 15	320 41	344 35	237 38	40				
40	10 47	73 27	51 45	323 11	347 05	240 03	41				
50	13 17	75 57	54 15	325 41	349 35	242 27	43				
13 00	15 47 N21 04	78 28	56 45 N 6 54	328 11 N23 31	352 06 N23 14	244 52 S 2 45	60	3 00	75	15 08	87
10	18 17	80 58	59 15	330 42	353 36	247 17	47	17	64	06	85
20	20 47	83 29	61 45	333 12	357 06	249 42	49	56	30	55	04 53
30	23 17	85 59	64 15	335 42	359 37	252 07	51	54	42	50	03 80
40	25 47	88 30	66 45	338 12	2 07	254 32	53	52	3 52	45	02 78
50	28 17	91 00	69 15	340 42	4 37	256 56	54	50	4 02	42	15 01 76
14 00	30 47 N21 05	93 30	71 45 N 6 55	343 12 N23 31	7 08 N23 14	259 21 S 2 56	45	21	36	14 58	72
10	33 17	96 01	74 15	345 42	9 38	261 46	2 58	40	37	32	56 69
20	35 47	98 31	76 45	348 13	12 08	264 11	3 00	35	4 50	29	54 66
30	38 17	101 02	79 15	350 43	14 38	266 36	02	30	5 01	27	52 64
40	40 47	103 32	81 45	353 13	17 09	269 01	04	20	21	24	49 60
50	43 17	106 02	84 15	355 43	19 39	271 26	06	10	38	23	47 56
15 00	45 47 N21 05	108 33	86 45 N 6 56	358 13 N23 31	22 09 N23 14	273 50 S 3 08	0	5 53	22	45	53
10	48 17	111 03	89 15	0 43	24 40	276 15	09				
20	50 47	113 34	91 45	3 13	27 10	278 40	11	10	6 09	23	42 50
30	53 17	116 04	94 14	5 43	29 40	281 05	13	20	25	24	40 46
40	55 47	118 34	96 44	8 14	32 11	283 30	15	30	43	25	38 41
50	58 17	121 05	99 14	10 44	34 41	285 55	17	35	6 54	27	36 39
16 00	60 47 N21 06	123 35	101 44 N 6 57	13 14 N23 30	37 11 N23 14	288 19 S 3 19	40	7 07	30	34	37
10	63 17	126 06	104 14	15 44	39 42	290 44	21	45	21	33	32 34
20	65 47	128 36	106 44	18 14	42 12	293 09	23	50	39	37	30 30

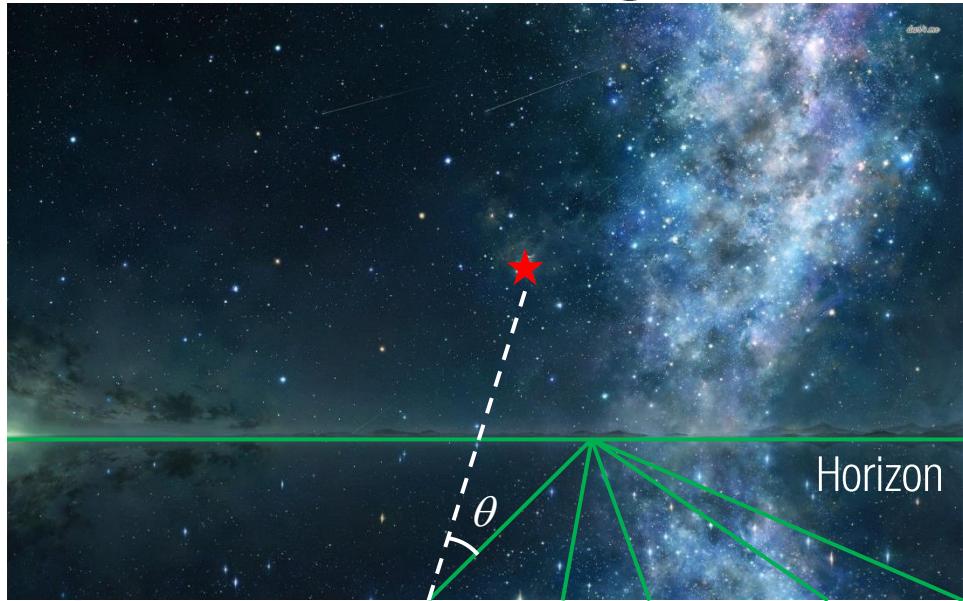
Far far away: point at infinity



Direction does not change because the star is infinitely far away.



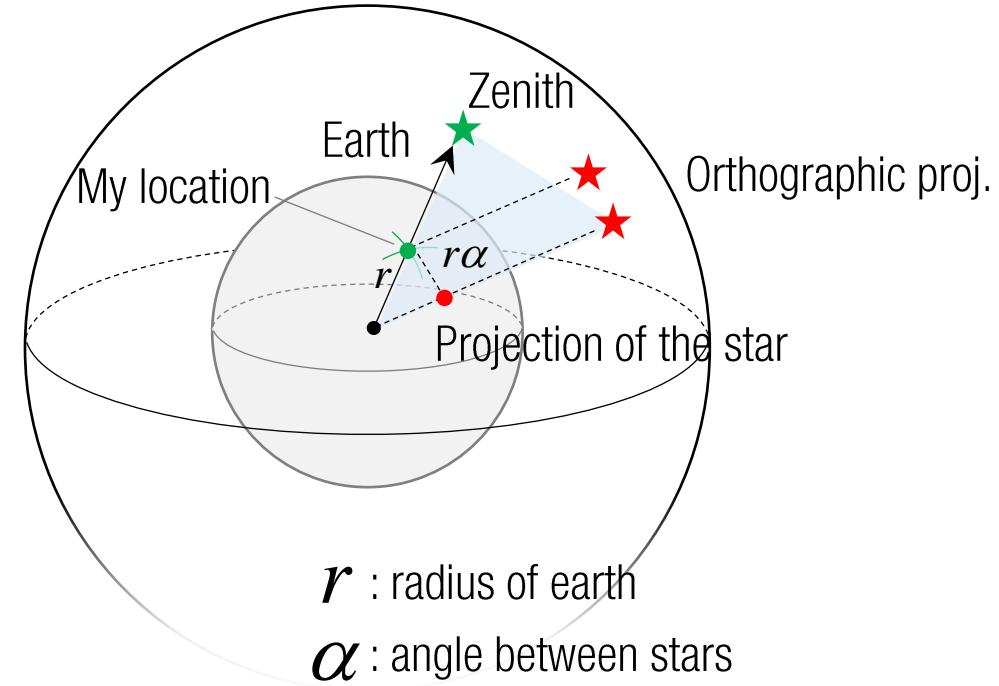
Celestial Navigation



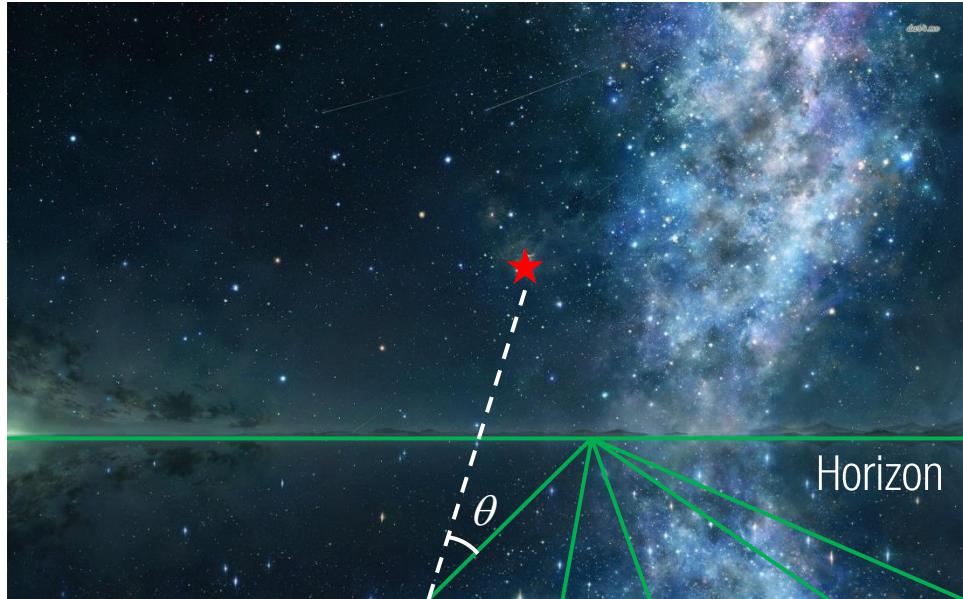
292 GREENWICH P. M. 1942 MAY 26 (TUESDAY)

GCT	SUN	T	VENUS-1.5	MARS-1.5	JUPITER-1.5	MOON	
	GHA Dec.	GHA	GHA Dec.	GHA Dec.	GHA Dec.	GHA	
12 00	0 47 N21 04	63 25	41 45 N 6 53	313 11 N23 31	373 04 N23 14	230 23 S 2 34	
10	3 17	65 56	44 15	315 41	339 34	232 48	36
20	5 47	68 26	46 45	318 11	342 04	235 13	38
30	8 17	70 57	49 15	320 41	344 35	237 38	40
40	10 47	73 27	51 45	323 11	347 05	240 03	41
50	13 17	75 57	54 15	325 41	349 35	242 27	43
13 00	15 47 N21 04	78 28	56 45 N 6 54	328 11 N23 31	352 06 N23 14	244 52 S 2 45	60
10	18 17	80 58	59 15	330 42	354 36	247 17	47
20	20 47	83 29	61 45	333 12	357 06	249 42	49
30	23 17	85 59	64 15	335 42	359 37	252 07	51
40	25 47	88 30	66 45	338 12	2 07	254 32	53
50	28 17	91 00	69 15	340 42	4 37	256 56	54
14 00	30 47 N21 05	93 30	71 45 N 6 55	343 12 N23 31	7 08 N23 14	259 21 S 2 56	45
10	33 17	96 01	74 15	345 42	9 38	261 46	22
20	35 47	98 31	76 45	348 13	12 08	264 11	30
30	38 17	101 02	79 15	350 43	14 38	266 36	40
40	40 47	103 32	81 45	353 13	17 09	269 01	42
50	43 17	106 02	84 15	355 43	19 39	271 26	46
15 00	45 47 N21 05	108 33	86 45 N 6 56	358 13 N23 31	22 09 N23 14	273 50 S 3 08	0
10	48 17	111 03	89 15	0 43	24 40	276 15	09
20	50 47	113 34	91 45	3 13	27 10	278 40	11
30	53 17	116 04	94 14	5 43	29 40	281 05	13
40	55 47	118 34	96 44	8 14	32 11	283 30	15
50	58 17	121 05	99 14	10 44	34 41	285 55	17
16 00	60 47 N21 06	123 35	101 44 N 6 57	13 14 N23 30	37 11 N23 14	288 19 S 3 19	40
10	63 17	126 06	104 14	15 44	39 42	290 44	21
20	65 47	128 36	106 44	18 14	42 12	293 09	23

Far far away: point at infinity



Celestial Navigation

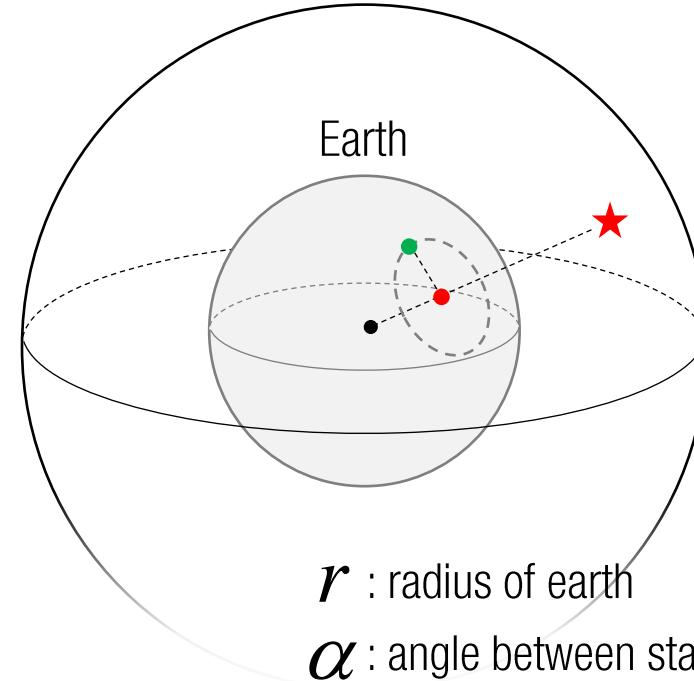


292

GREENWICH P. M. 1942 MAY 26 (TUESDAY)

GCT	SUN	T	VENUS-1.5	MARS-1.5	JUPITER-1.5	MOON	
	GHA	Dec.	GHA	Dec.	GHA	Dec.	GHA
12 00	0 47 N21 04	63 25	41 45 N 6 53	313 11 N 23 31	337 04 N23 14	230 23 S 2 34	
10	3 17	65 56	44 15	315 41	339 34	232 48	36
20	5 47	68 26	46 45	318 11	342 04	235 13	38
30	8 17	70 57	49 15	320 41	344 35	237 38	40
40	10 47	73 27	51 45	323 11	347 05	240 03	41
50	13 17	75 57	54 15	325 41	349 35	242 27	43
13 00	15 47 N21 04	78 28	56 45 N 6 54	328 11 N23 31	352 06 N23 14	244 52 S 2 45	60
10	18 17	80 58	59 15	330 42	354 36	247 17	47
20	20 47	83 29	61 45	333 12	357 06	249 42	49
30	23 17	85 59	64 15	335 42	359 37	252 07	51
40	25 47	88 30	66 45	338 12	2 07	254 32	53
50	28 17	91 00	69 15	340 42	4 37	256 56	54
14 00	30 47 N21 05	93 30	71 45 N 6 55	343 12 N23 31	7 08 N23 14	259 21 S 2 56	45
10	33 17	96 01	74 15	345 42	9 38	261 46	22
20	35 47	98 31	76 45	348 13	12 08	264 11	3 00
30	38 17	101 02	79 15	350 43	14 38	266 36	35
40	40 47	103 32	81 45	353 13	17 09	269 01	40
50	43 17	106 02	84 15	355 43	19 39	271 26	46
15 00	45 47 N21 05	108 33	86 45 N 6 56	358 13 N23 31	22 09 N23 14	273 50 S 3 08	0
10	48 17	111 03	89 15	0 43	24 40	276 15	09
20	50 47	113 34	91 45	3 13	27 10	278 40	11
30	53 17	116 04	94 14	5 43	29 40	281 05	13
40	55 47	118 34	96 44	8 14	32 11	283 30	15
50	58 17	121 05	99 14	10 44	34 41	285 55	17
16 00	60 47 N21 06	123 35	101 44 N 6 57	13 14 N23 30	37 11 N23 14	288 19 S 3 19	40
10	63 17	126 06	104 14	15 44	39 42	290 44	21
20	65 47	128 36	106 44	18 14	42 12	293 09	23

Far far away: point at infinity

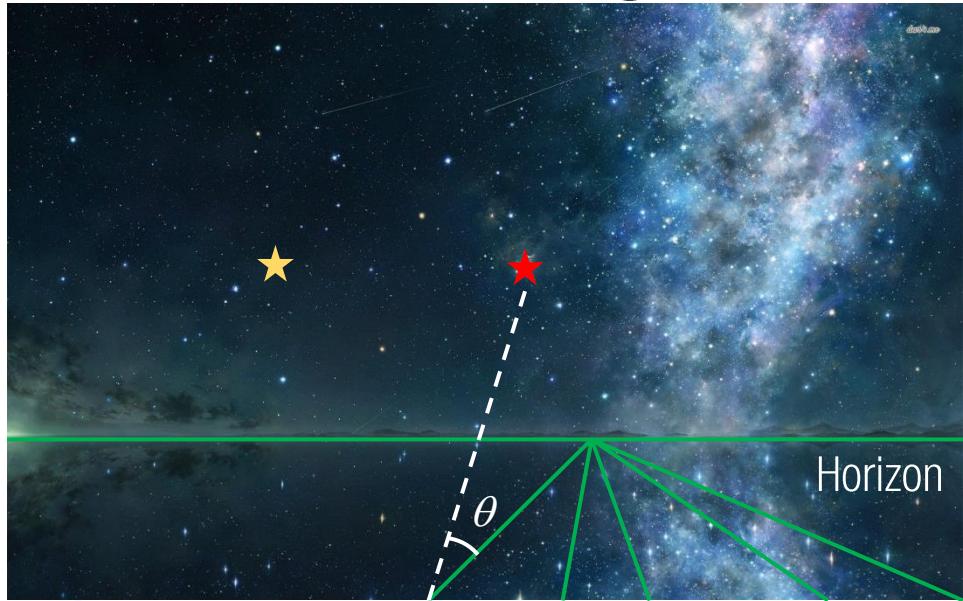


r : radius of earth

α : angle between stars



Celestial Navigation

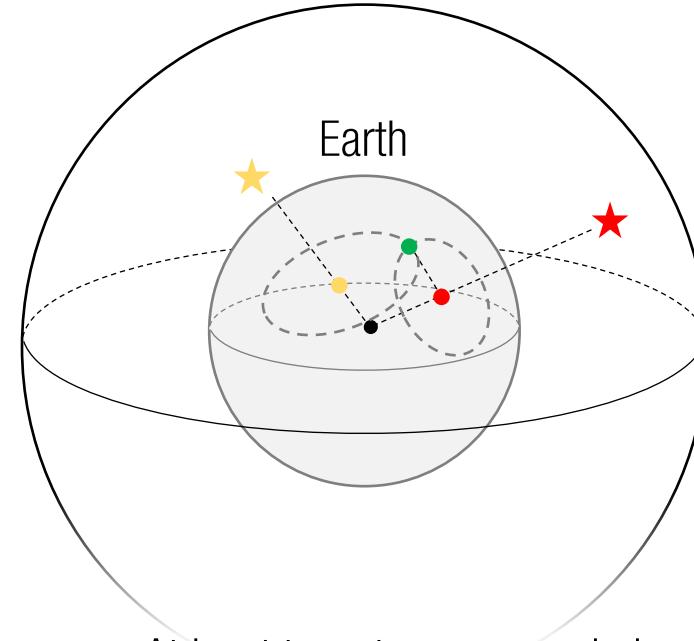


292

GREENWICH P. M. 1942 MAY 26 (TUESDAY)

GCT h m	SUN	T	VENUS-1.5	MARS-1.5	JUPITER-1.5	MOON					
	GHA Dec.	GHA	GHA Dec.	GHA Dec.	GHA Dec.	GHA	Lat.	Sun-rise	Twil.	Moon-rise	Dini.
12 00	0 47 N21 04	63 25	41 45 N 6 53	313 11 N23 31	337 04 N23 14	230 23 S 2 34					
10 3 17	65 56 44 15	315 41	339 34	232 48	336						
20 5 47	68 26 46 45	318 11	342 04	235 13	338						
30 8 17	70 57 49 15	320 41	344 35	237 38	40						
40 10 47	73 27 51 45	323 11	347 05	240 03	41						
50 13 17	75 57 54 15	325 41	349 35	242 27	43						
13 00	15 47 N21 04	78 28 56 45 N 6 54	328 11 N23 31	352 06 N23 14	244 52 S 2 45	60	3 00	75	15 08	08	87
10 18 17	80 58 59 15	330 42	354 36	247 17	47	58	17	64	06	53	
20 20 47	83 29 61 45	333 12	357 06	249 42	49	56	30	55	04	53	
30 23 17	85 59 64 15	335 42	359 37	252 07	51	54	42	50	03	80	
40 25 47	88 30 66 45	338 12	2 07	254 32	53	52	3 52	45	02	78	
50 28 17	91 00 69 15	340 42	4 37	256 56	54	50	4 02	42	15 01	76	
14 00	30 47 N21 05	93 30 71 45 N 6 55	343 12 N23 31	7 08 N23 14	259 21 S 2 56	45	21	36	14 58	72	
10 33 17	96 01 74 15	345 42	9 38	261 46	2 58	40	37	32	56	69	
20 35 47	98 31 76 45	348 13	12 08	264 11	3 00	35	4 50	29	54	66	
30 38 17	101 02 79 15	350 43	14 38	266 36	02	30	5 01	27	52	64	
40 40 47	103 32 81 45	353 13	17 09	269 01	04	20	21	24	49	60	
50 43 17	106 02 84 15	355 43	19 39	271 26	06	10	38	23	47	56	
15 00	45 47 N21 05	108 33 86 45 N 6 56	358 13 N23 31	22 09 N23 14	273 50 S 3 08	0	5 53	22	45	53	
10 48 17	111 03 89 15	0 43	24 40	276 15	09						
20 50 47	113 34 91 45	3 13	27 10	278 40	11	10	6 09	23	42	50	
30 53 17	116 04 94 14	5 43	29 40	281 05	13	20	25	24	40	46	
40 55 47	118 34 96 44	8 14	32 11	283 30	15	30	43	25	38	41	
50 58 17	121 05 99 14	10 44	34 41	285 55	17	35	6 54	27	36	39	
16 00	60 47 N21 06	123 35 101 44 N 6 57	13 14 N23 30	37 11 N23 14	288 19 S 3 19	40	7 07	30	34	37	
10 63 17	126 06 104 14	15 44	39 42	290 44	21	45	21	33	32	34	
20 65 47	128 36 106 44	18 14	42 12	293 09	23	50	39	37	30	30	

Far far away: point at infinity

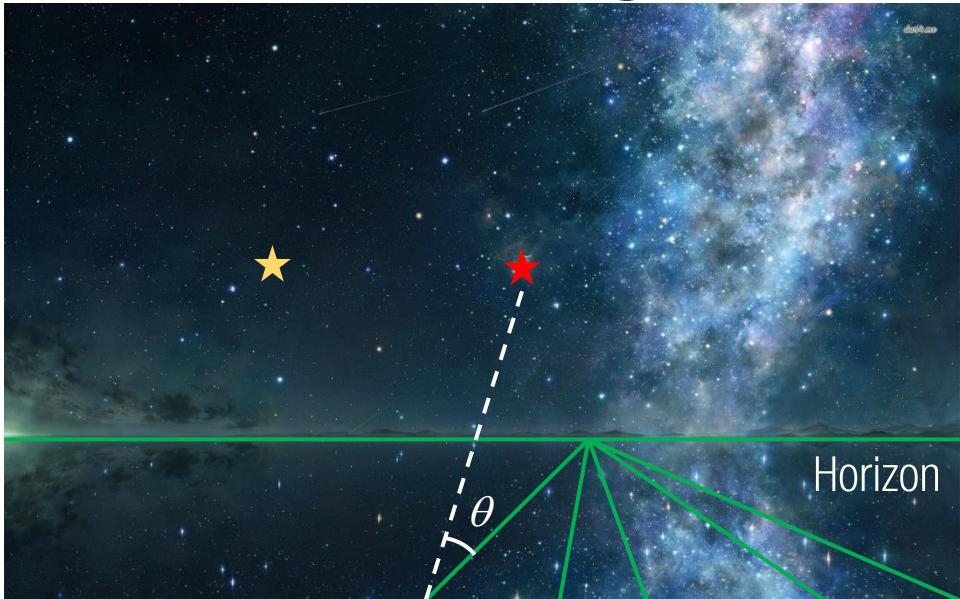


At least two stars are needed.

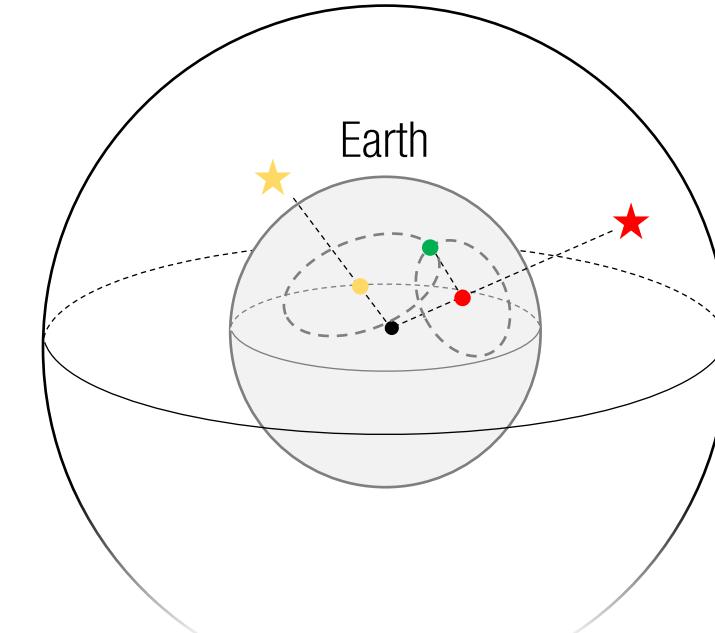


Sextant

Celestial Navigation



Far far away: point at infinity



At least two stars are needed.

Two points at infinity (vanishing points) tells us about where I am.

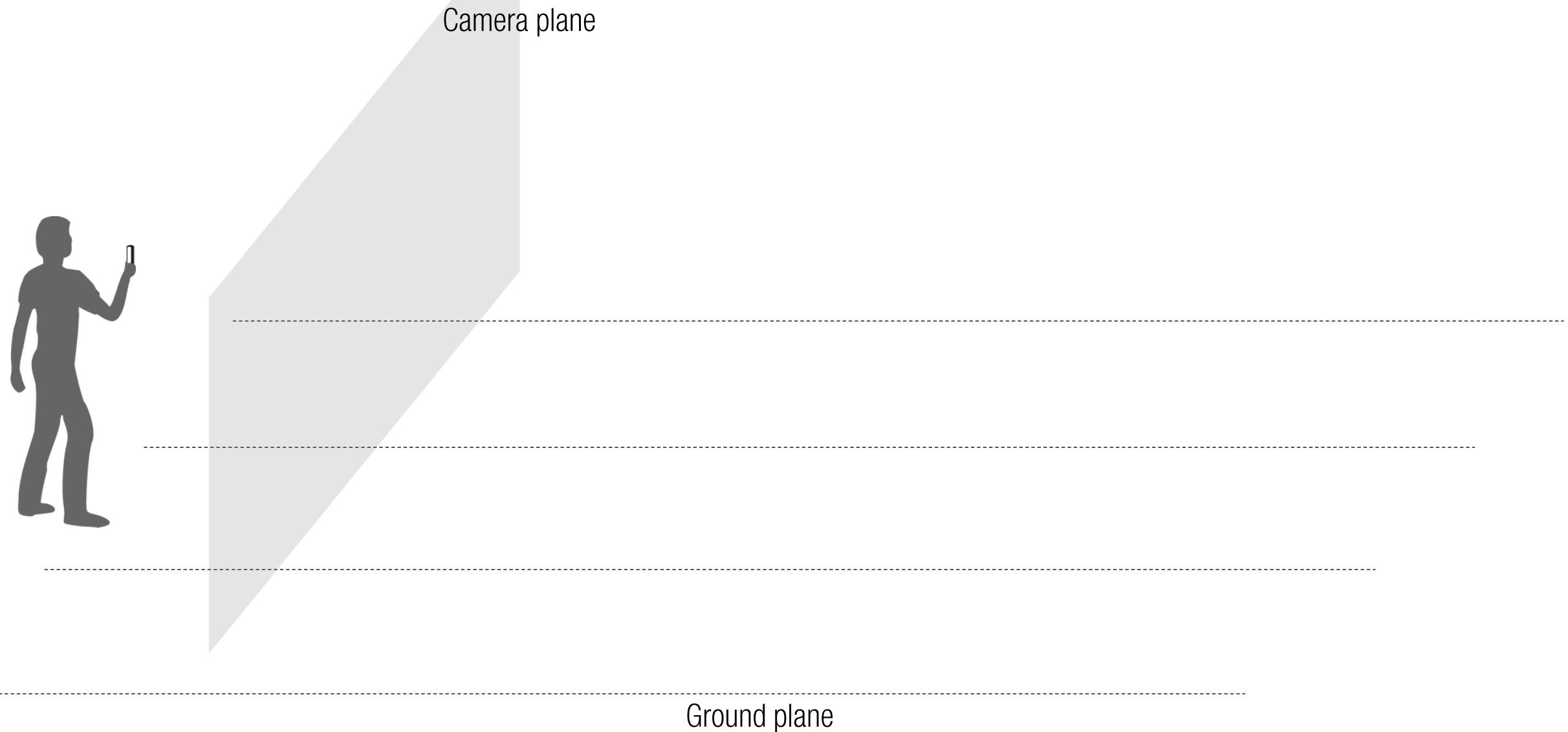
MLPS-St. Paul International Airport



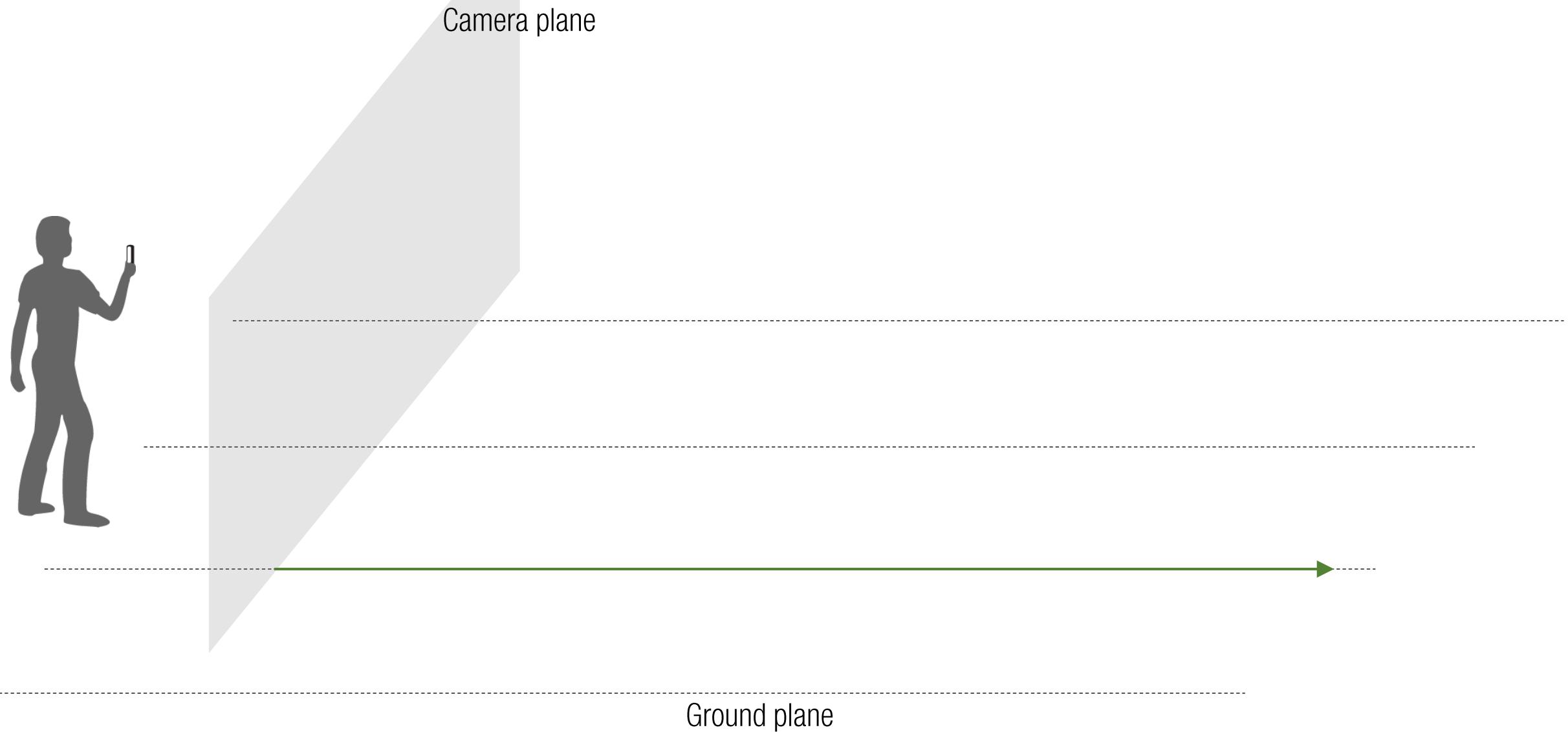
Parallel lines in 3D converge to a point in the image.

Indoor point at infinity

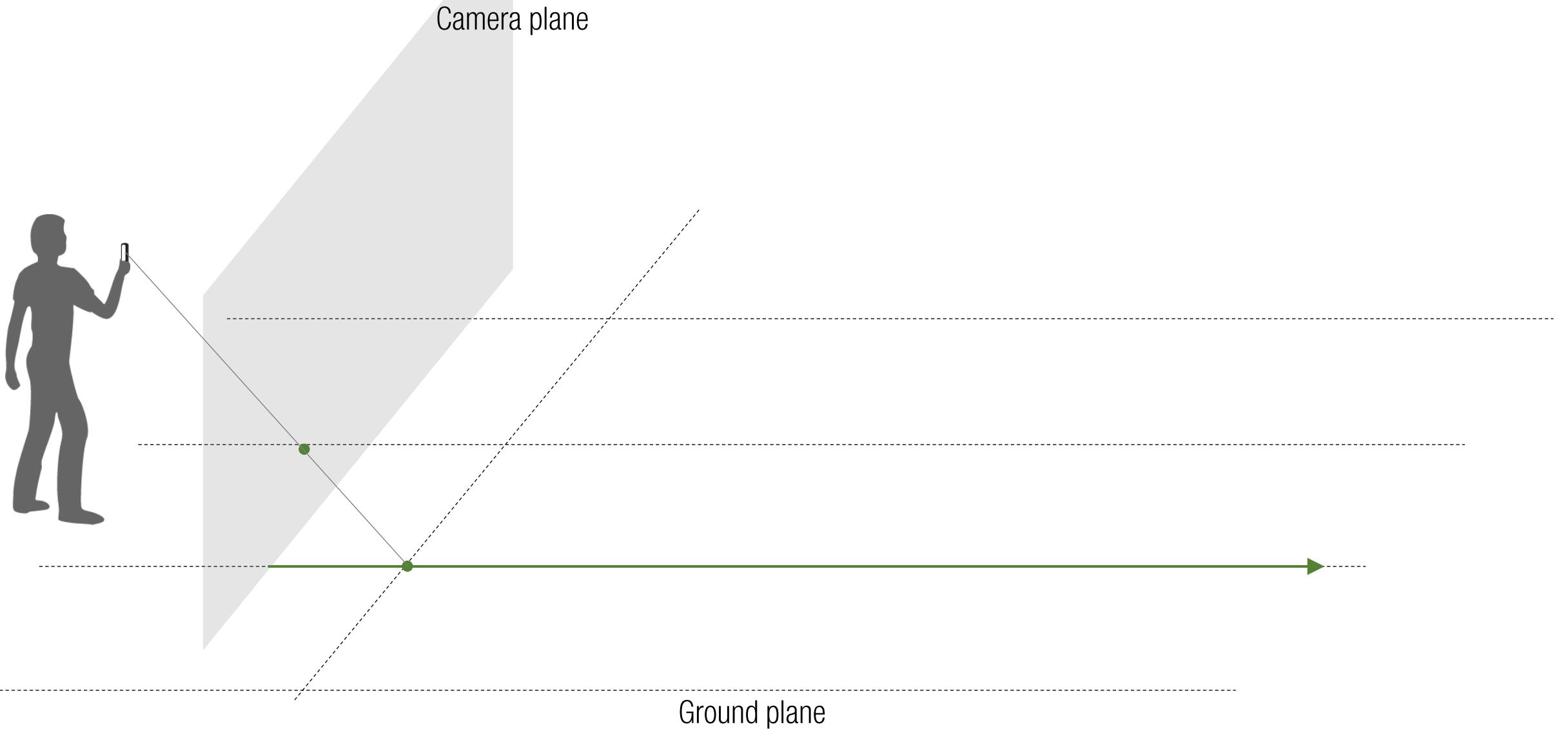
3D Parallel Line Projection



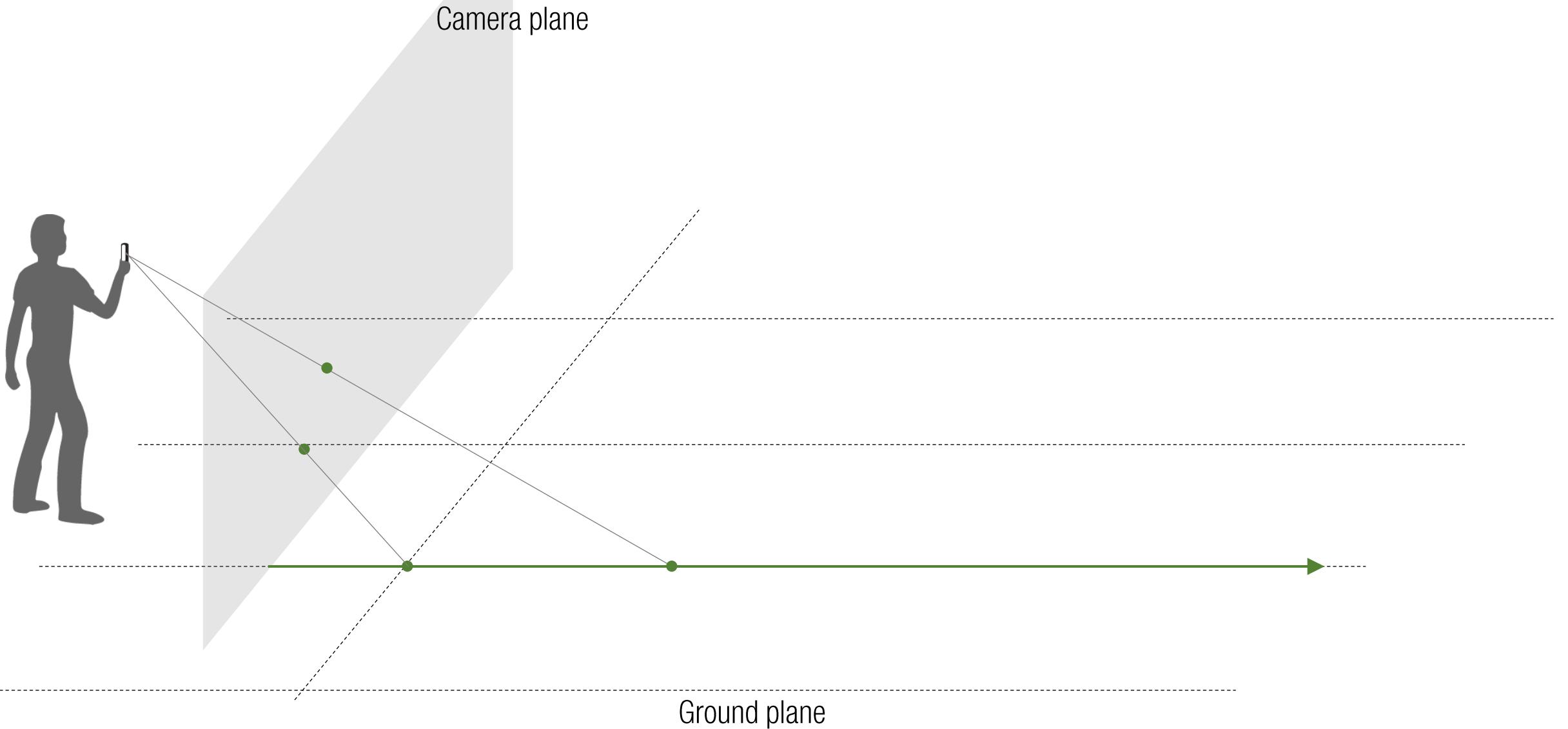
3D Parallel Line Projection



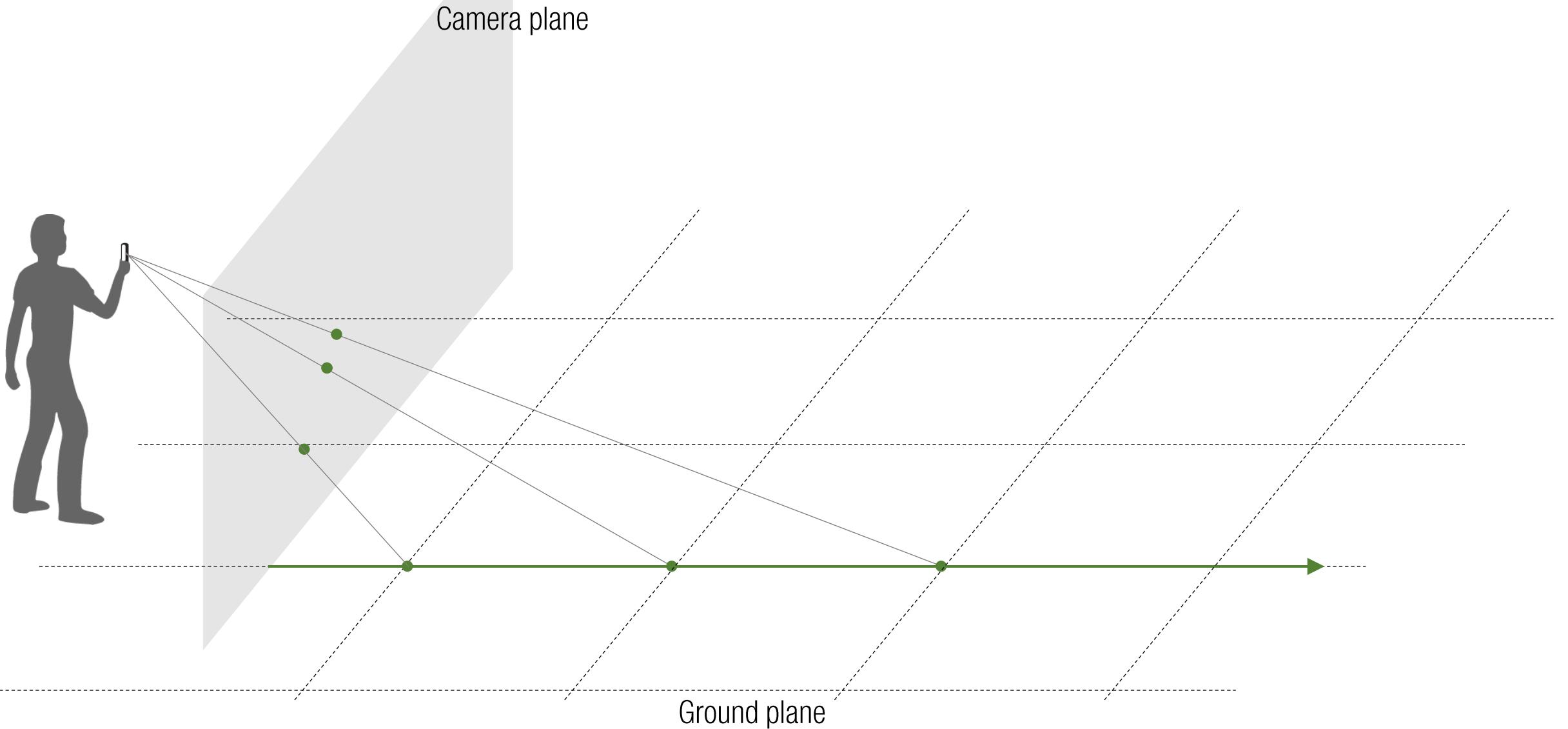
3D Parallel Line Projection



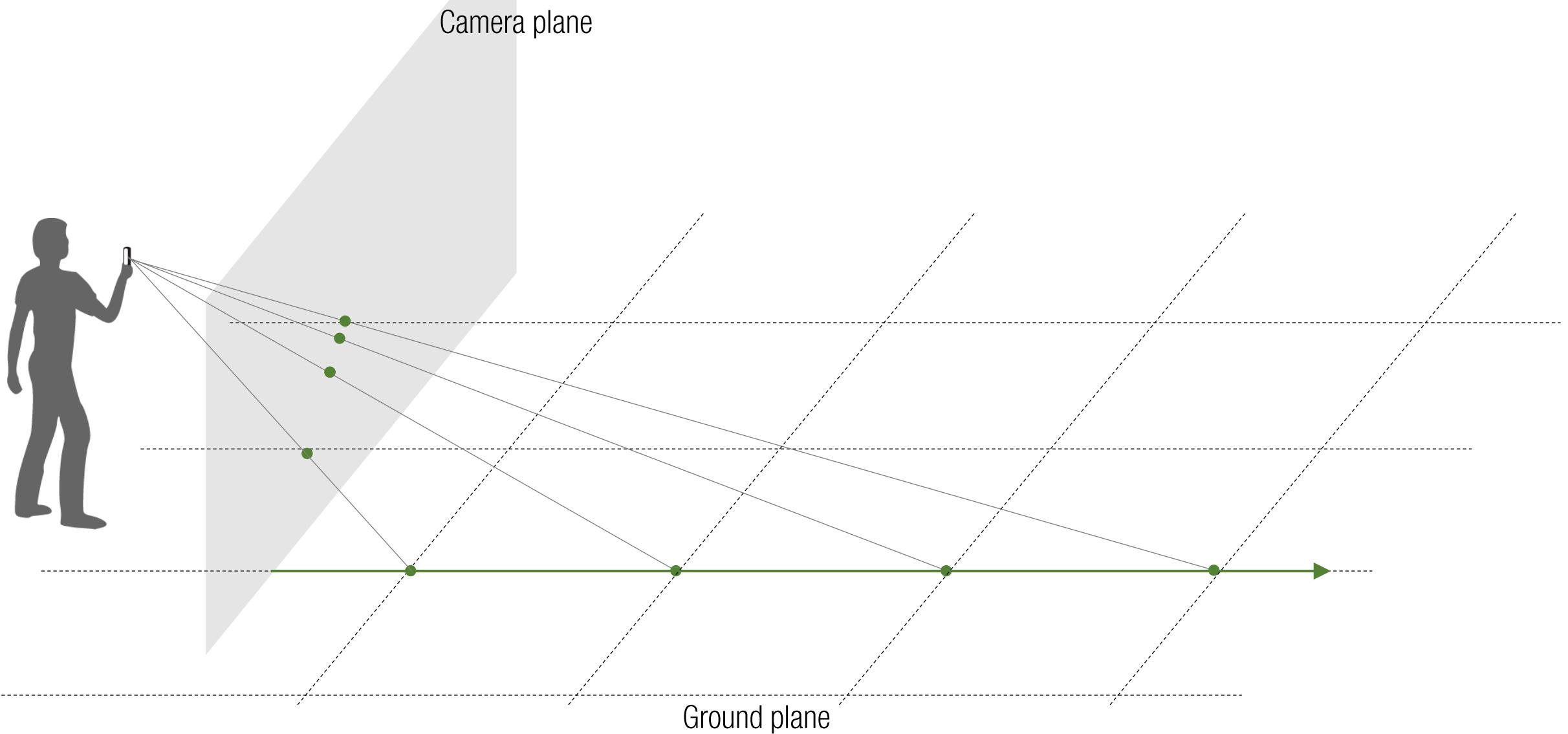
3D Parallel Line Projection



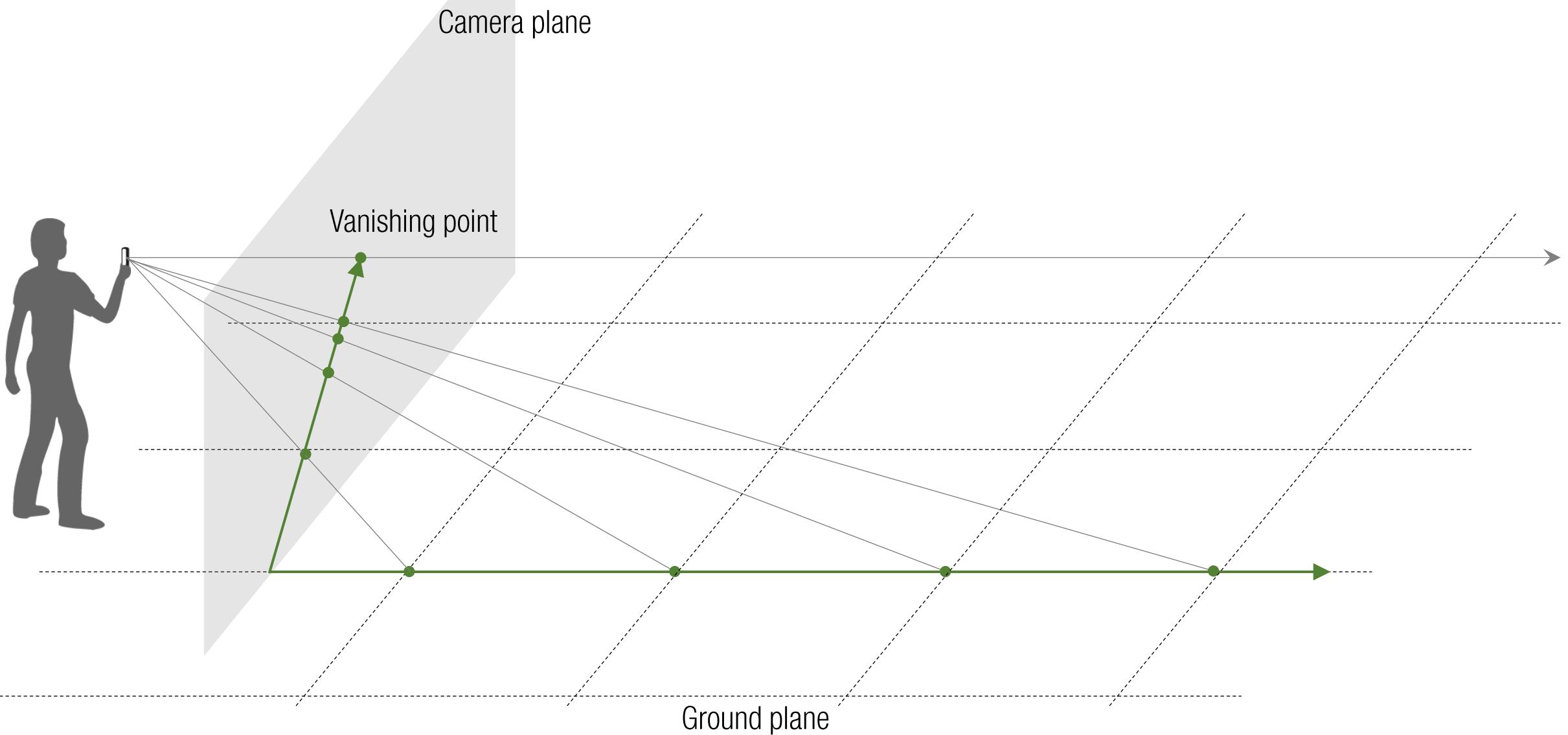
3D Parallel Line Projection



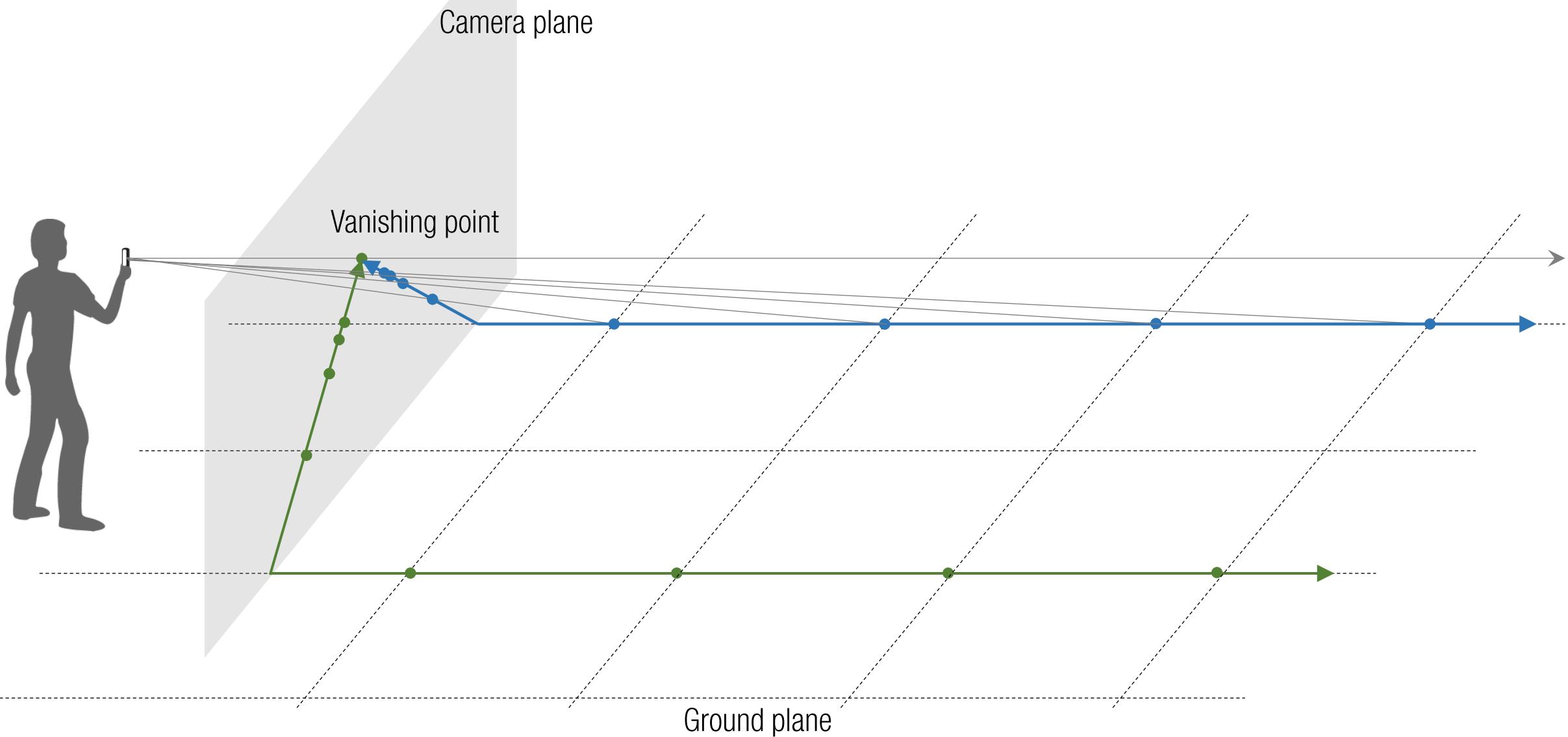
3D Parallel Line Projection



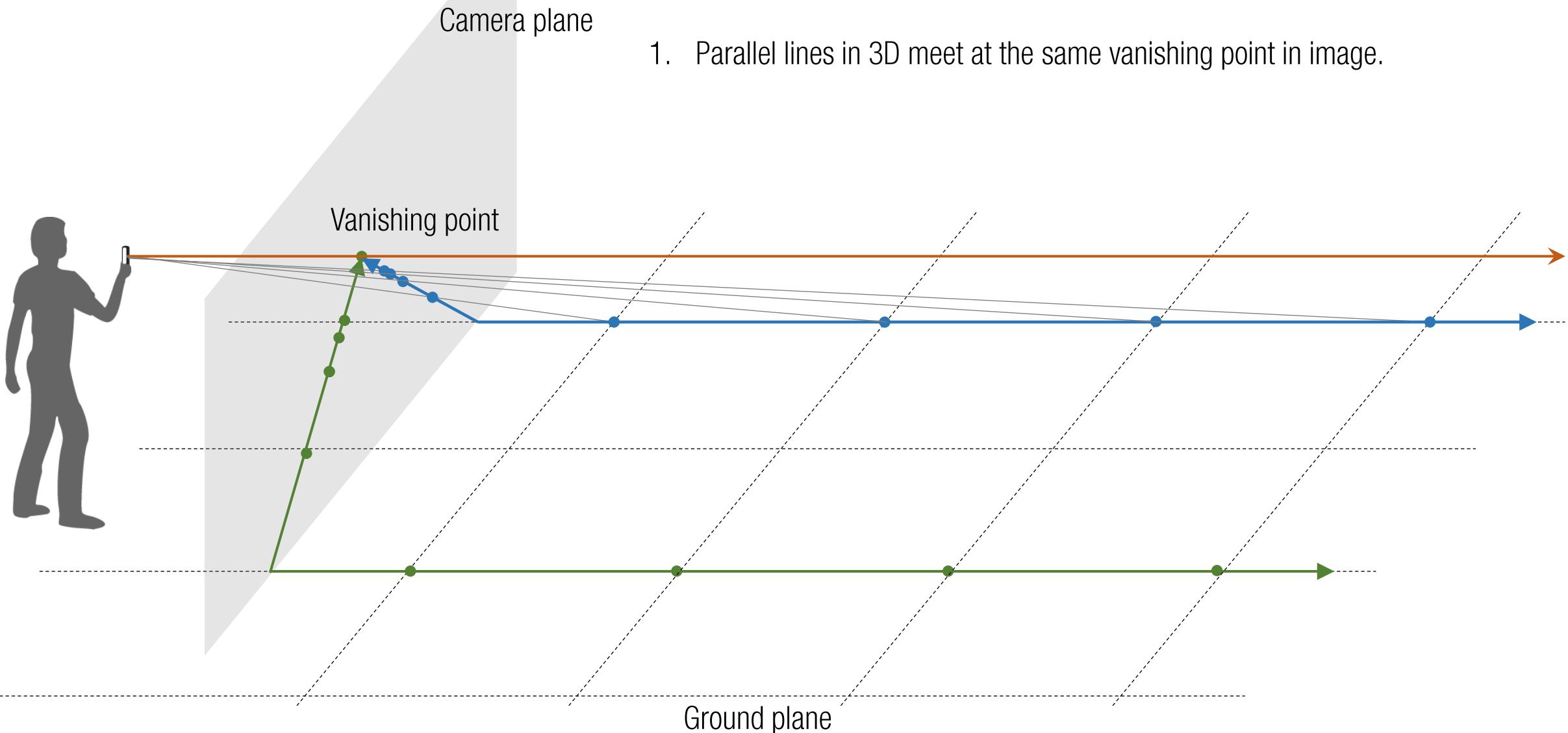
3D Parallel Line Projection



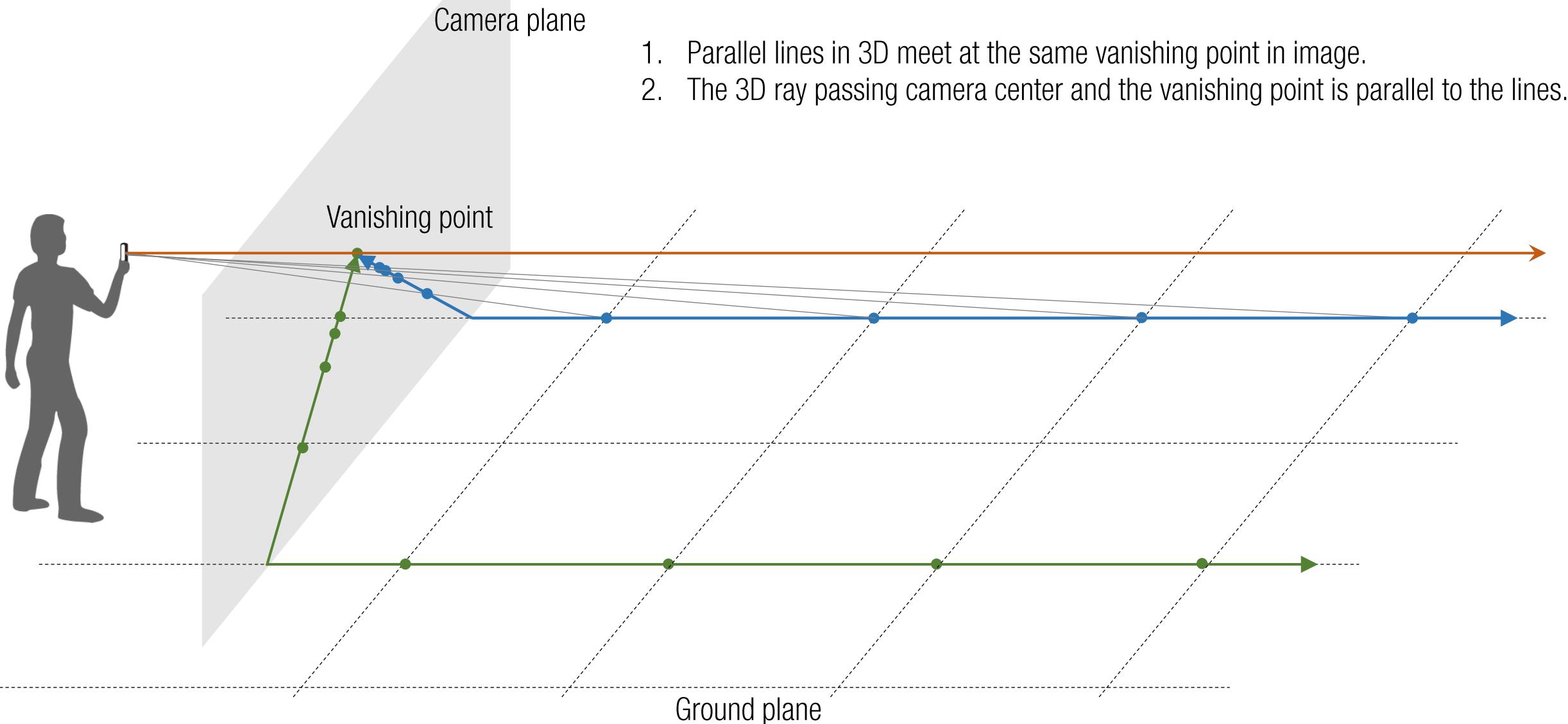
3D Parallel Line Projection



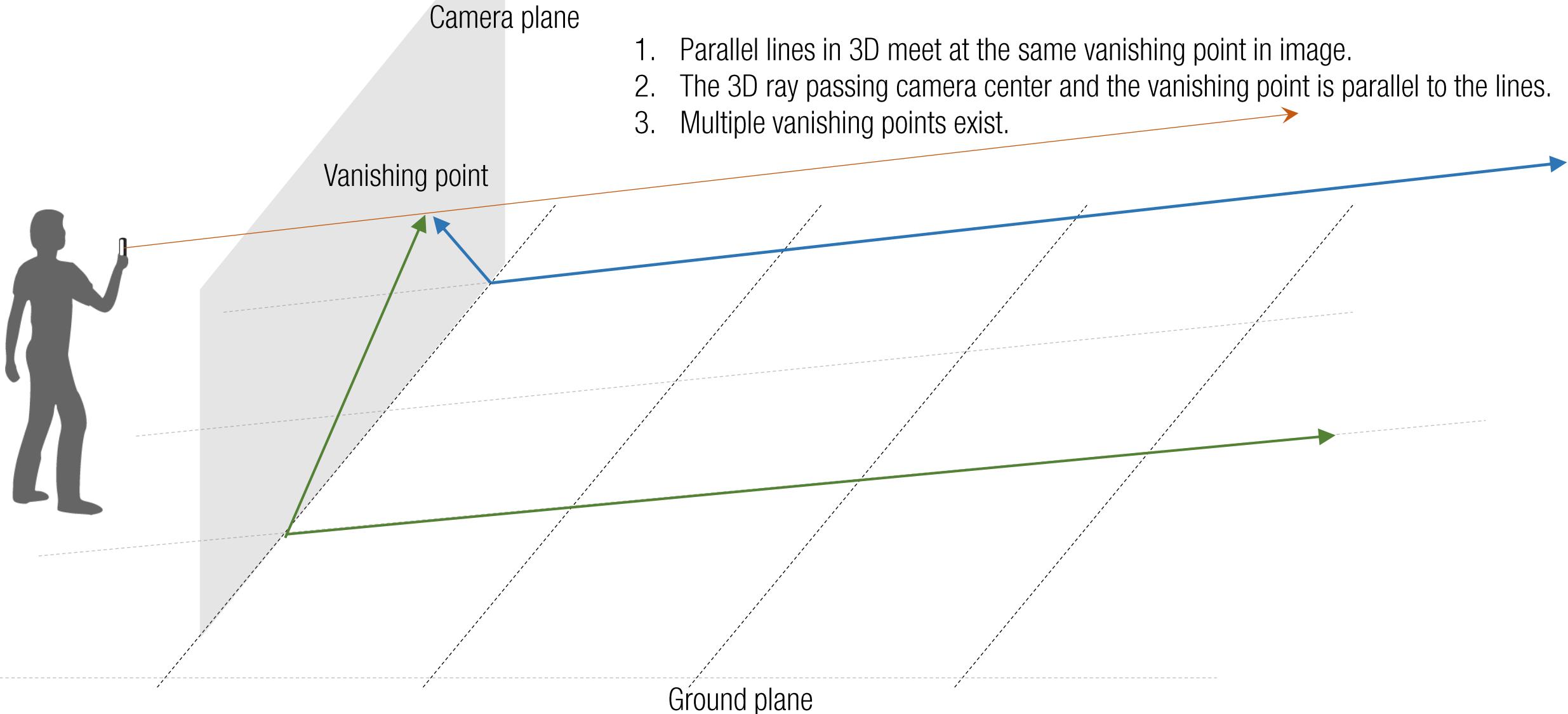
3D Parallel Line Projection



3D Parallel Line Projection



Vanishing Point





Keller Hall



Keller Hall

Vanishing point



Keller Hall

Vanishing point



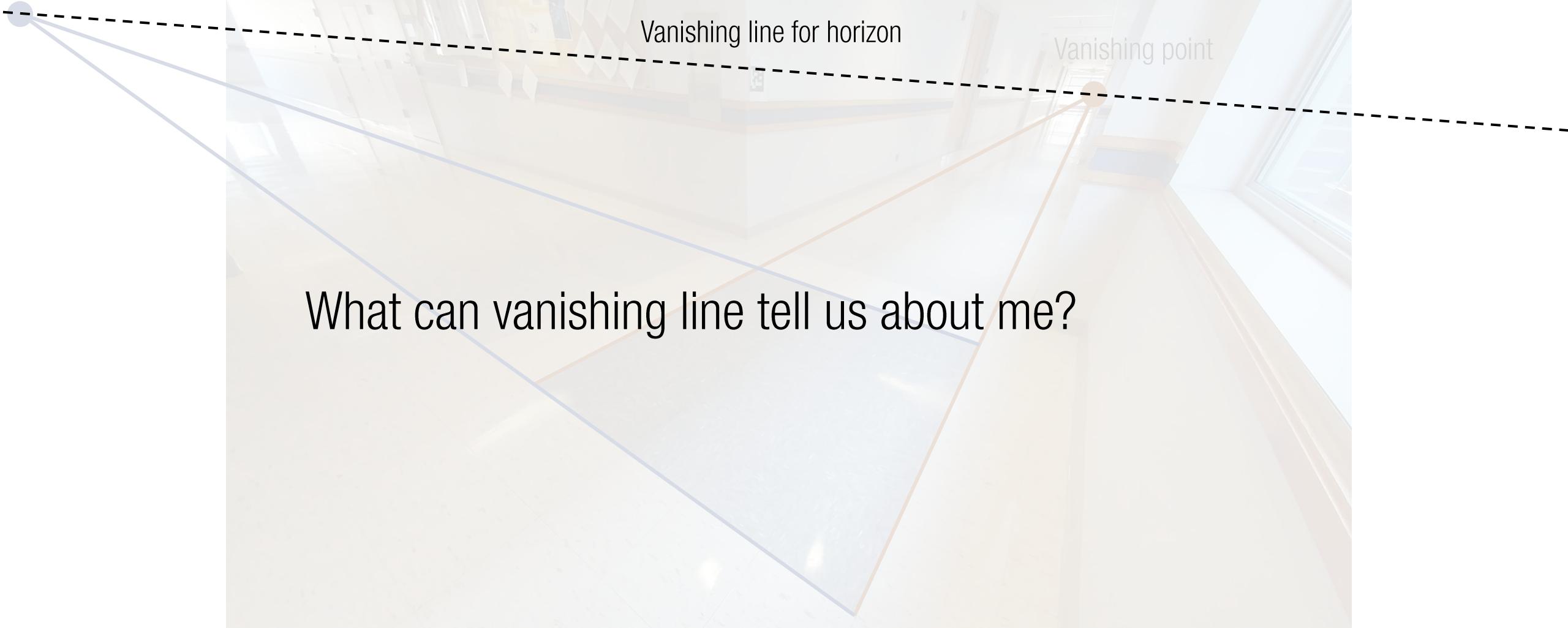
Vanishing line for horizon

Vanishing point

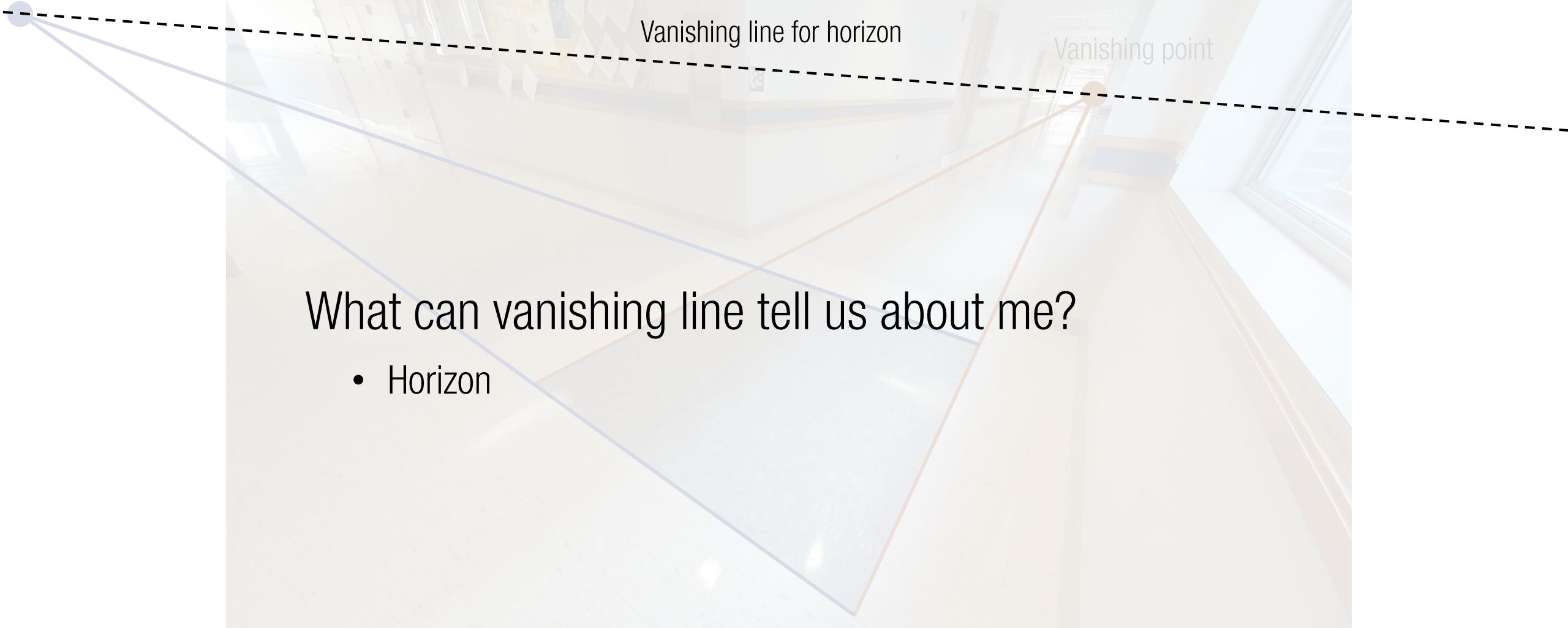


Keller Hall

Vanishing point



Vanishing point



Vanishing point

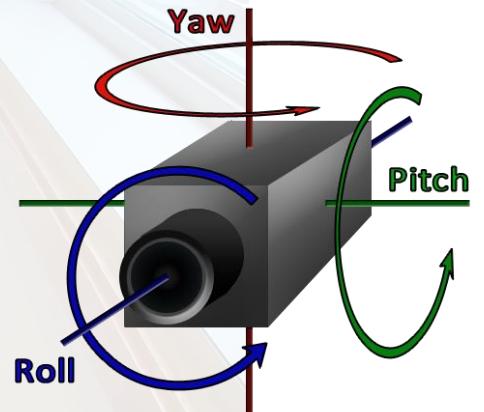


Vanishing line for horizon

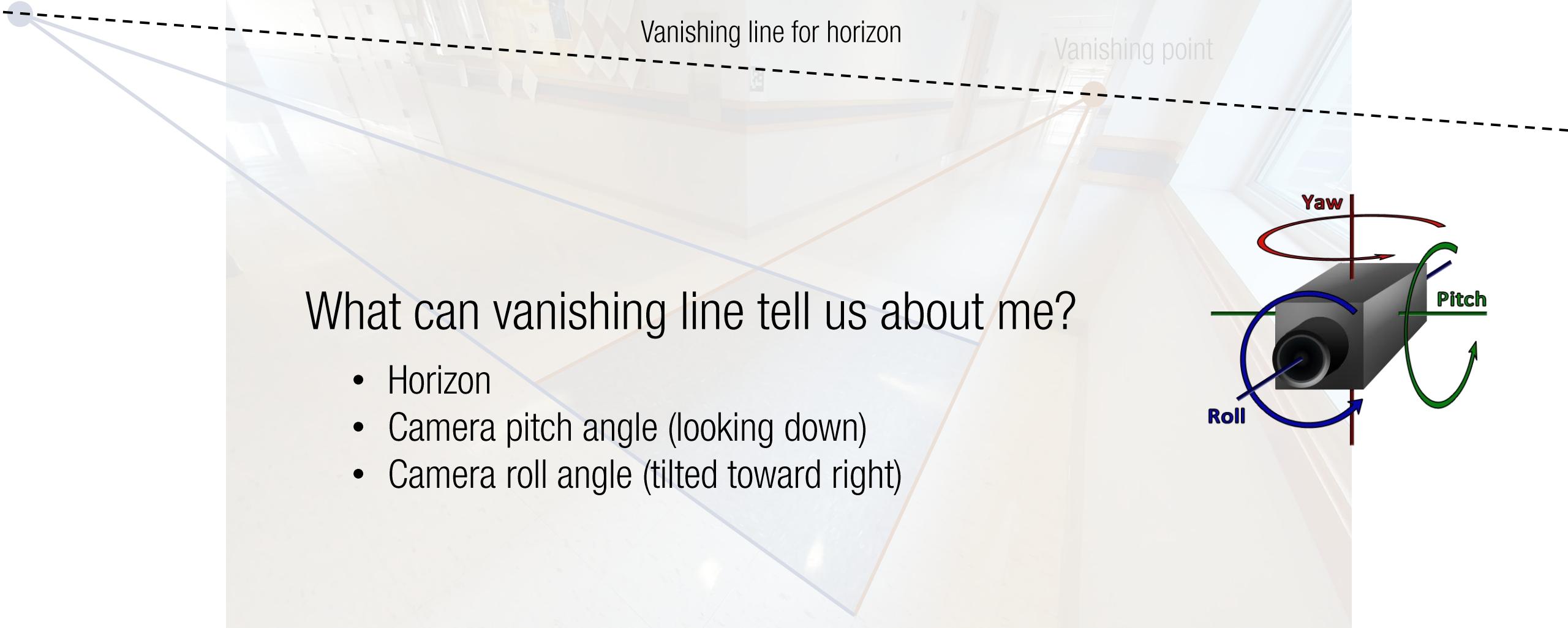
Vanishing point

What can vanishing line tell us about me?

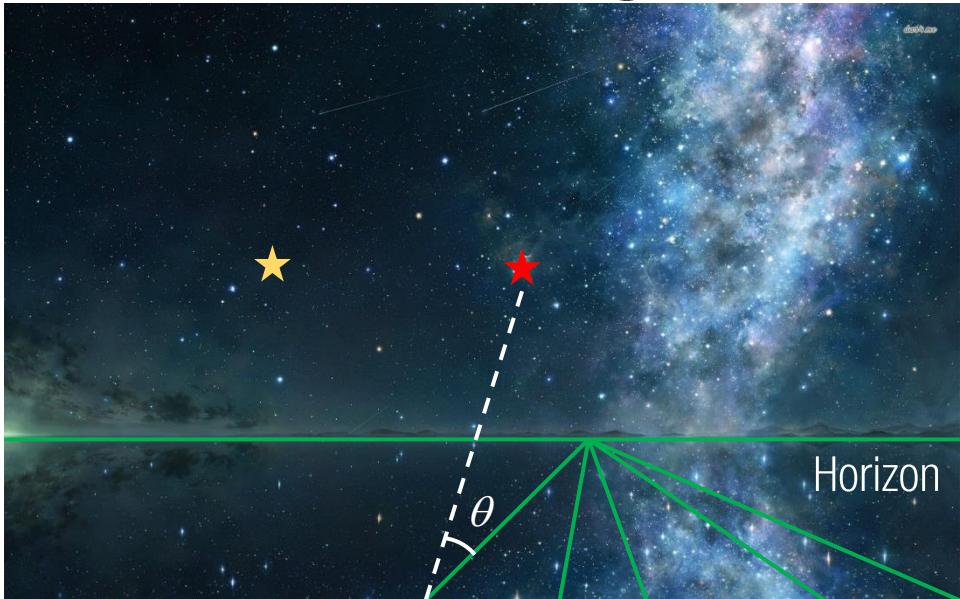
- Horizon
- Camera pitch angle (looking down)



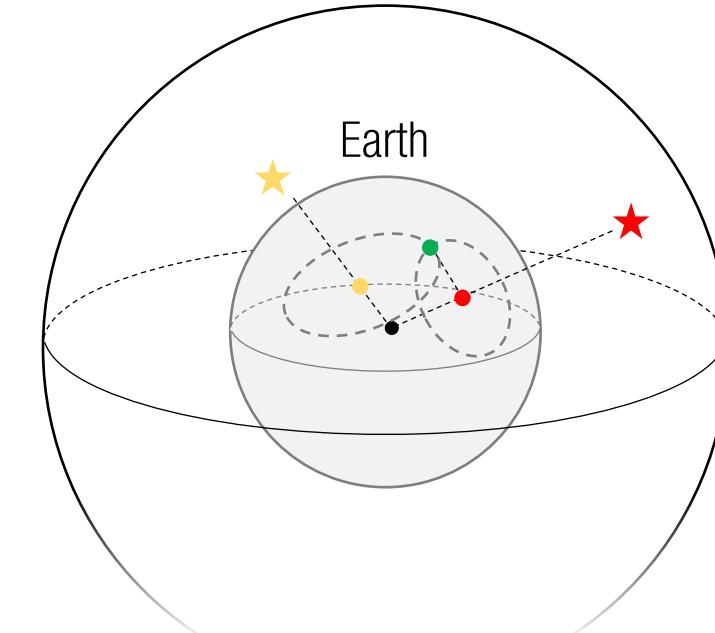
Vanishing point



Celestial Navigation



Far far away: point at infinity



At least two stars are needed.

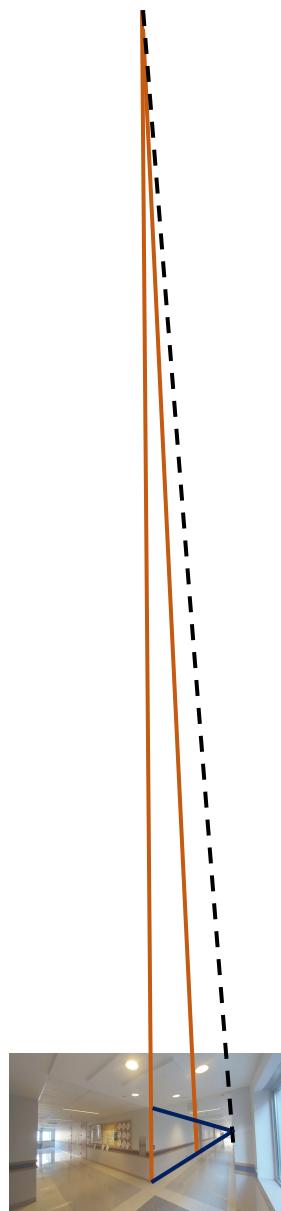
Two points at infinity (vanishing points) tells us about where I am.



Parallel 3D planes share the vanishing line.



Different plane produces different vanishing line.



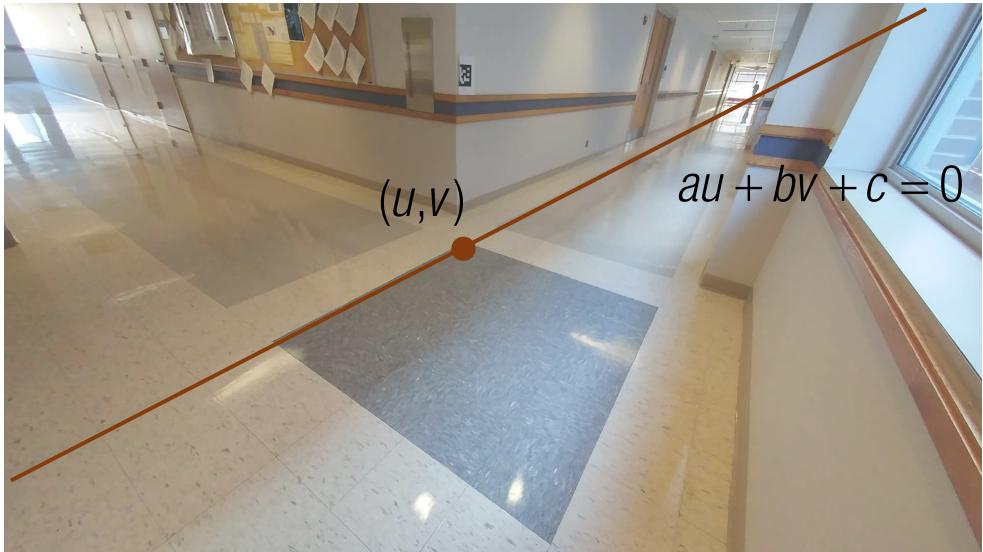
Different plane produces different vanishing line.

How to compute a vanishing point?



Different plane produces different vanishing line.

Point-Line in Image

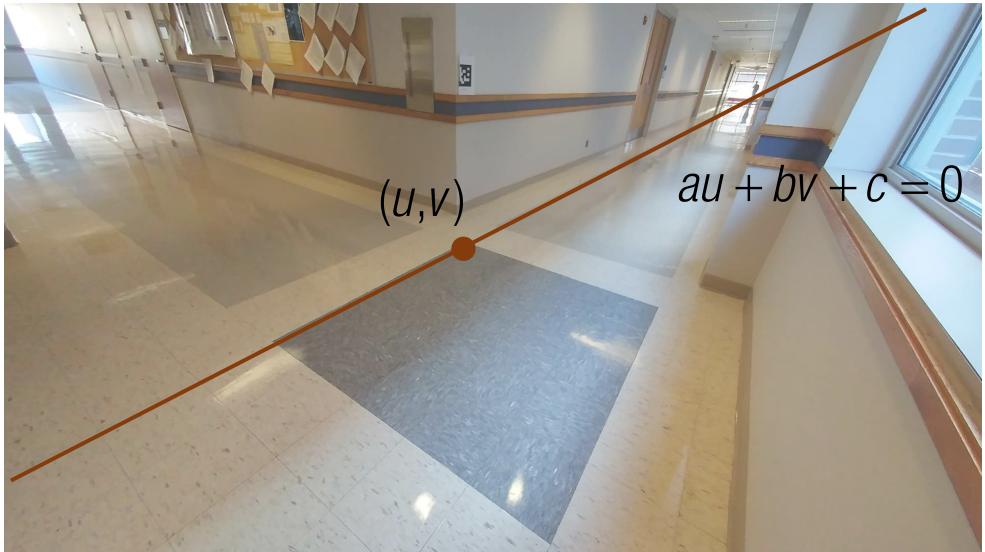


A 2D line passing through 2D point (u,v) :

$$au + bv + c = 0$$

Line parameter: (a,b,c)

Point-Line in Image



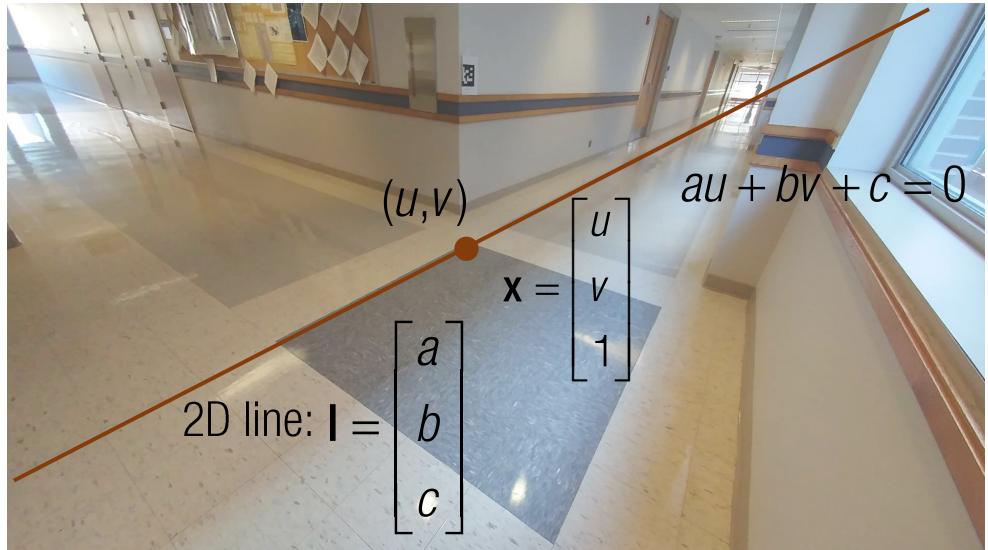
A 2D line passing through 2D point (u,v) :

$$au + bv + c = 0$$

Line parameter: (a,b,c)

$$au + bv + c = 0 \rightarrow \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{l}^T \mathbf{x} = 0$$

Point-Line in Image



A 2D line passing through 2D point (u, v) :

$$au + bv + c = 0$$

Line parameter: (a, b, c)

$$au + bv + c = 0 \rightarrow \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{l}^T \mathbf{x} = 0$$

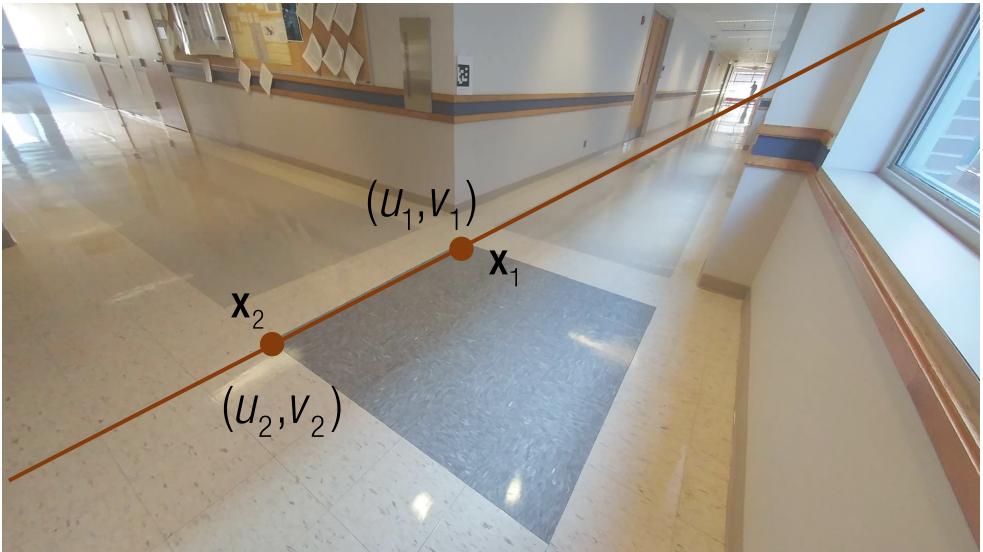
where $\mathbf{x} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}^T$ and $\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}^T$

2D point Line parameter

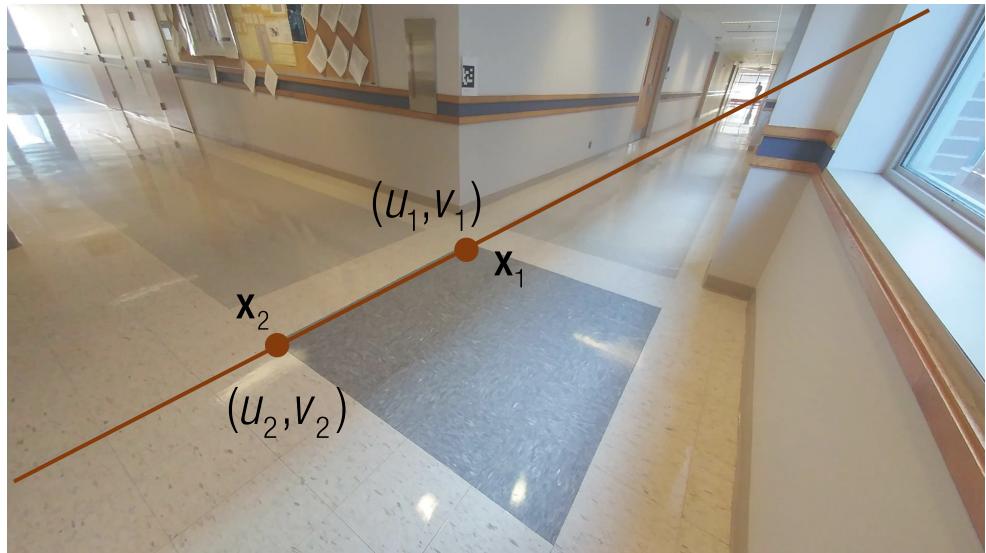
Point-Point in Image

A 2D line passing through two 2D points:

$$au_1 + bv_1 + c = 0 \quad au_2 + bv_2 + c = 0$$



Point-Point in Image



A 2D line passing through two 2D points:

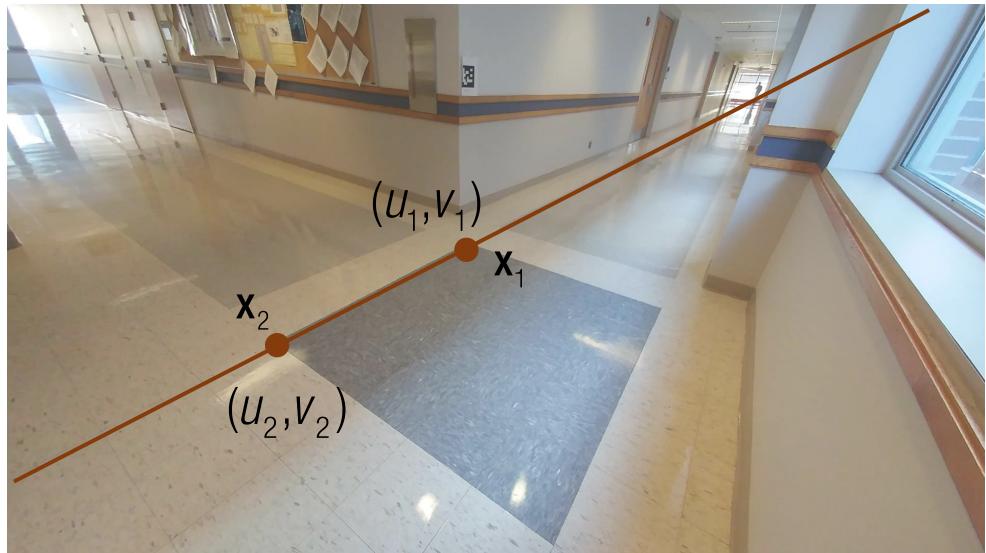
$$au_1 + bv_1 + c = 0 \quad au_2 + bv_2 + c = 0$$

$$\mathbf{x}_1^T \mathbf{I} = 0$$

$$\mathbf{x}_2^T \mathbf{I} = 0$$

where $\mathbf{x}_1 = \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix}$ $\mathbf{x}_2 = \begin{bmatrix} u_2 \\ v_2 \\ 1 \end{bmatrix}$ $\mathbf{I} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Point-Point in Image



A 2D line passing through two 2D points:

$$au_1 + bv_1 + c = 0 \quad au_2 + bv_2 + c = 0$$

$$\mathbf{x}_1^T \mathbf{l} = 0$$

$$\mathbf{x}_2^T \mathbf{l} = 0$$

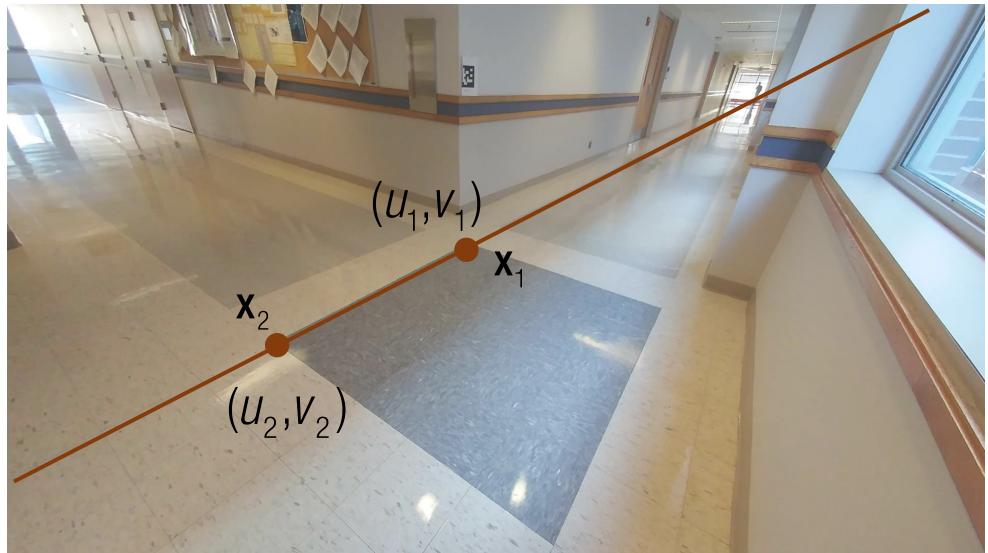
where $\mathbf{x}_1 = \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix}$ $\mathbf{x}_2 = \begin{bmatrix} u_2 \\ v_2 \\ 1 \end{bmatrix}$ $\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\rightarrow \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{bmatrix} \mathbf{l} = \mathbf{0}$$

$$\begin{array}{c|c} \mathbf{A} & \mathbf{l} \\ \hline & \\ \hline & \end{array} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3x2

Point-Point in Image



A 2D line passing through two 2D points:

$$au_1 + bv_1 + c = 0 \quad au_2 + bv_2 + c = 0$$

$$\mathbf{x}_1^T \mathbf{l} = 0$$

$$\mathbf{x}_2^T \mathbf{l} = 0$$

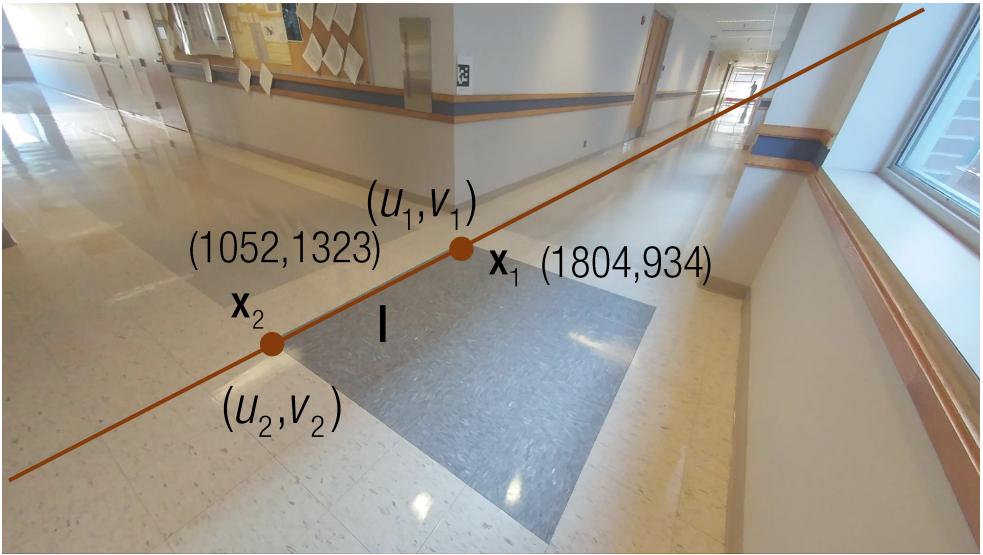
where $\mathbf{x}_1 = \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix}$ $\mathbf{x}_2 = \begin{bmatrix} u_2 \\ v_2 \\ 1 \end{bmatrix}$ $\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\rightarrow \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{bmatrix} \mathbf{l} = \mathbf{0}$$

$$\underbrace{\mathbf{A}}_{3 \times 2} \begin{bmatrix} \mathbf{l} \\ \mathbf{0} \end{bmatrix} = \mathbf{0} \rightarrow \mathbf{l} = \text{null}\left(\begin{bmatrix} \mathbf{A} \end{bmatrix}\right) \quad \text{or} \quad \mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$

Point-Point in Image

$x_1 = [1804; 934; 1];$
 $x_2 = [1052; 1323; 1];$

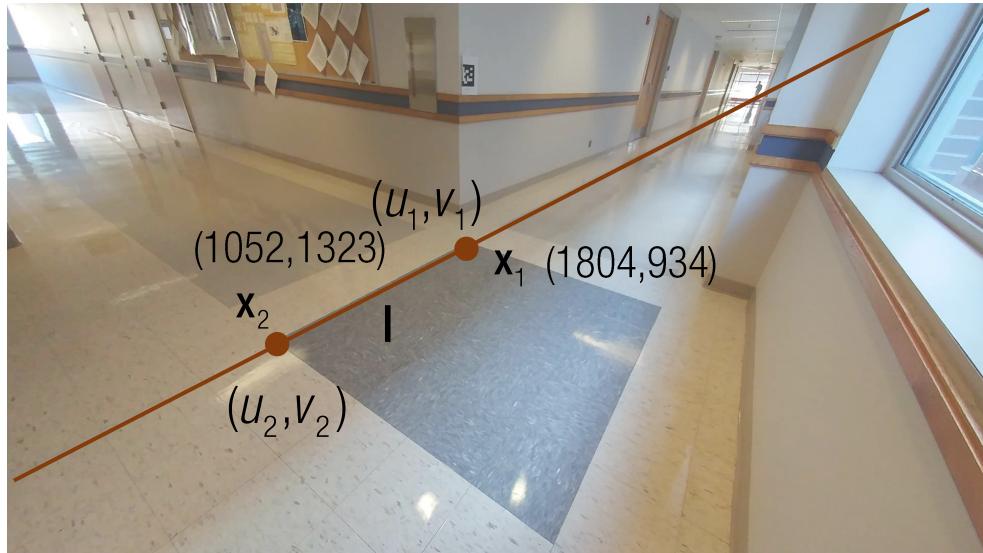


Point-Point in Image

GetLineFromTwoPoints.m

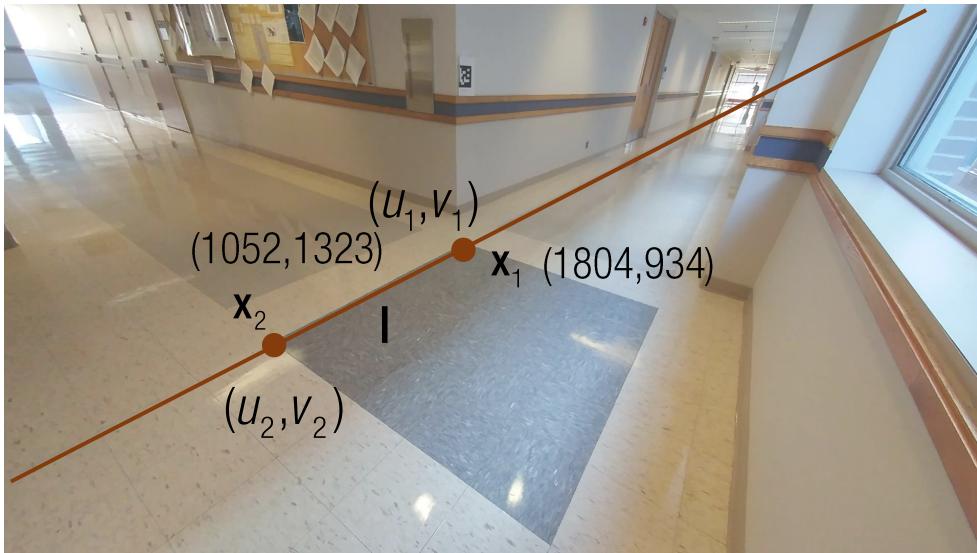
```
x1 = [1804;934;1];  
x2 = [1052;1323;1];
```

```
| = Vec2Skew(x1)*x2;  
Cross product
```



Point-Point in Image

GetLineFromTwoPoints.m

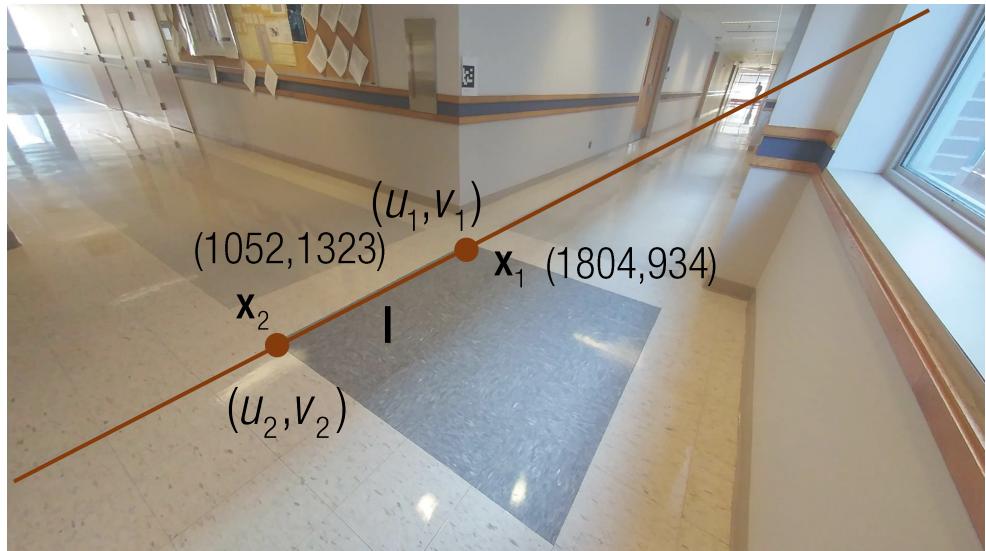


```
x1 = [1804;934;1];  
x2 = [1052;1323;1];
```

```
| = Vec2Skew(x1)*x2;  
Cross product
```

```
| =  
-389  
-752  
1404124
```

Point-Point in Image



GetLineFromTwoPoints.m

```
x1 = [1804;934;1];  
x2 = [1052;1323;1];
```

I = Vec2Skew(x1)*x2;

Cross product

I =

```
-389  
-752  
1404124
```

Cross product with skew-symmetric matrix representation:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [\mathbf{a}]_\times \mathbf{b} \end{aligned}$$

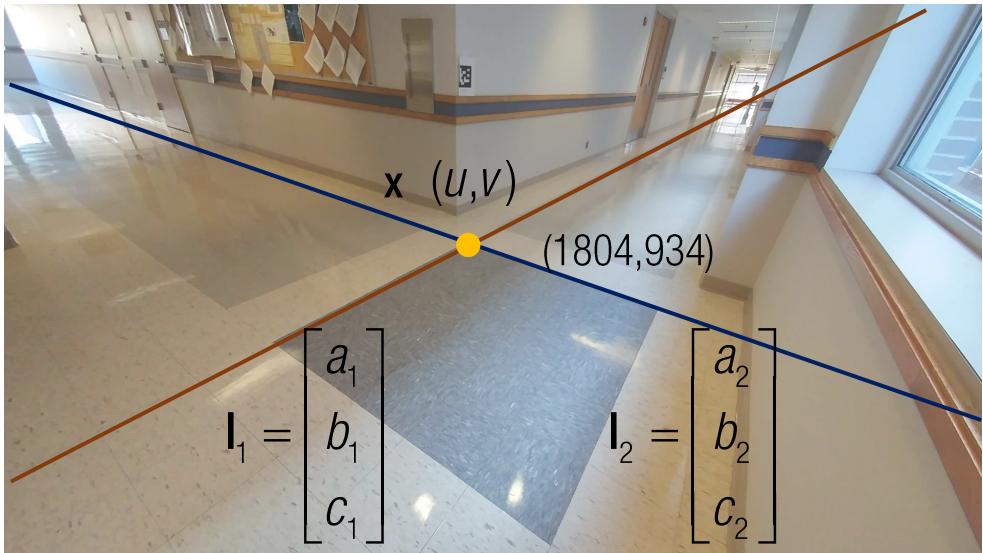
Vec2Skew.m

```
function skew = Vec2Skew(v)  
skew = [0 -v(3) v(2);  
v(3) 0 -v(1);  
-v(2) v(1) 0];
```

Line-Line in Image

Two 2D lines in an image intersect at a 2D point:

$$a_1u + b_1v + c_1 = 0 \quad a_2u + b_2v + c_2 = 0$$



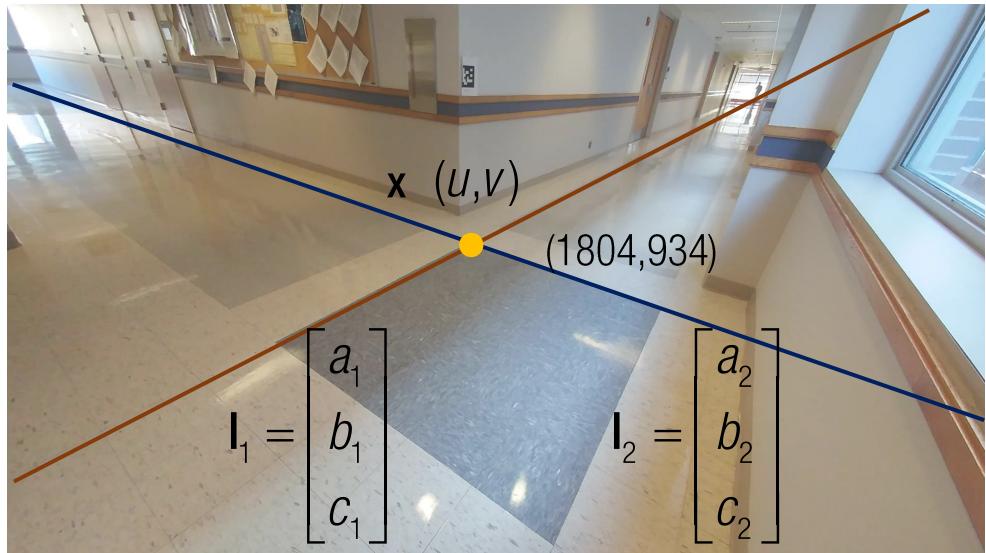
Line-Line in Image

Two 2D lines in an image intersect at a 2D point:

$$a_1u + b_1v + c_1 = 0 \quad a_2u + b_2v + c_2 = 0$$

$$\mathbf{l}_1^T \mathbf{x} = 0 \quad \mathbf{l}_2^T \mathbf{x} = 0$$

where $\mathbf{x} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$ $\mathbf{l}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ $\mathbf{l}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$



Line-Line in Image

Two 2D lines in an image intersect at a 2D point:

$$a_1u + b_1v + c_1 = 0 \quad a_2u + b_2v + c_2 = 0$$

$$\mathbf{l}_1^T \mathbf{x} = 0 \quad \mathbf{l}_2^T \mathbf{x} = 0$$

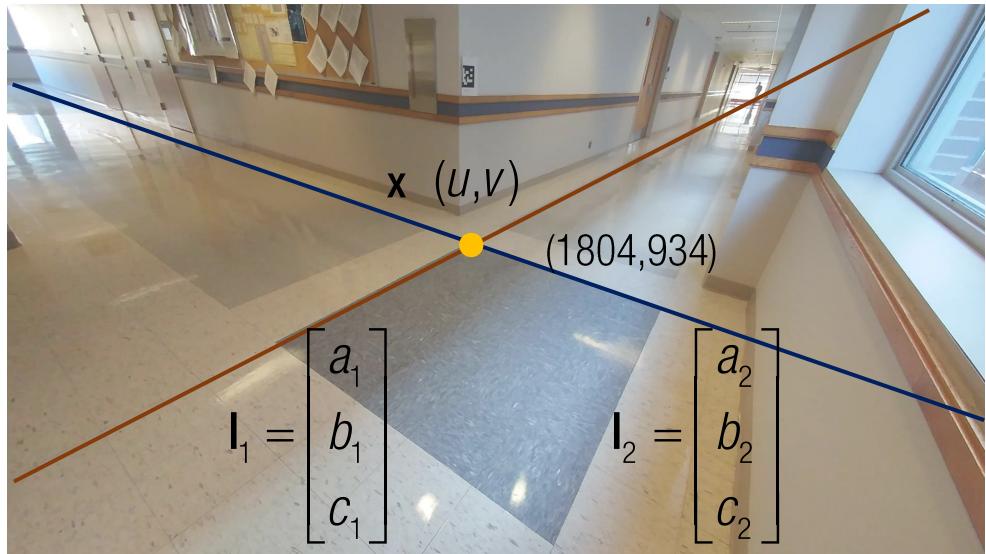
where $\mathbf{x} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$ $\mathbf{l}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ $\mathbf{l}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} \mathbf{l}_1^T \\ \mathbf{l}_2^T \end{bmatrix} \mathbf{x} = \mathbf{0}$$

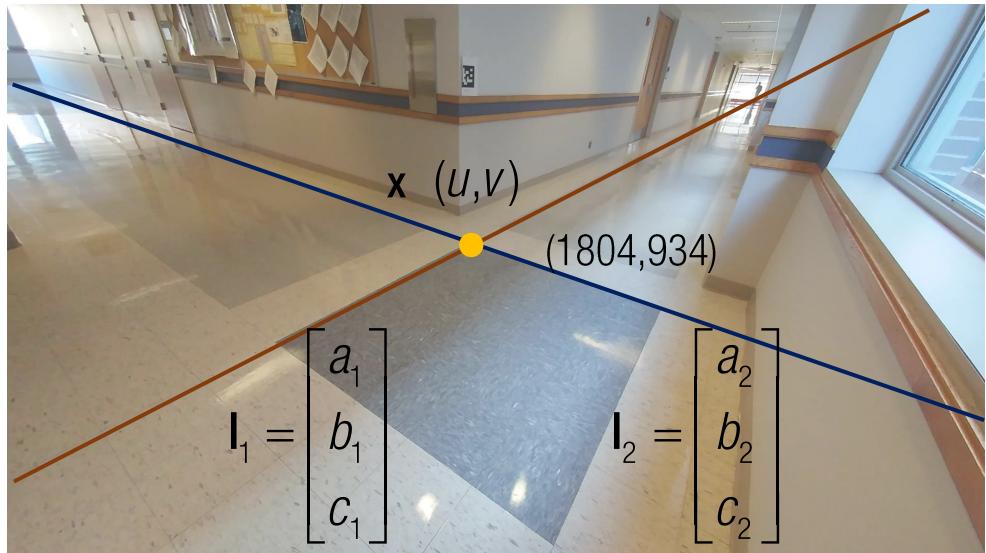
$$\underbrace{\mathbf{A} \quad \mathbf{|}}_{3 \times 2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \mathbf{|} = \text{null}\left(\begin{array}{c|c} & \mathbf{A} \end{array}\right)$$

3x2

or $\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$



Line-Line in Image



GetPointFromTwoLines.m

```
|1 = [-398;-752;1404124];  
|2 = [310;-924;303790];  
x = Vec2Skew(|1)*|2;  
x = x/x(3)
```

$x =$

1779.0 similar to $(1804, 934)$
925.6
1

2D Point and Line Duality



The 2D line joining two points:

$$l = \mathbf{x}_1 \times \mathbf{x}_2$$

The intersection between two lines:

$$\mathbf{x} = l_1 \times l_2$$

Given any formula, we can switch the meaning of point and line to get another formula.

2D Point and Line Duality



The 2D line joining two points:

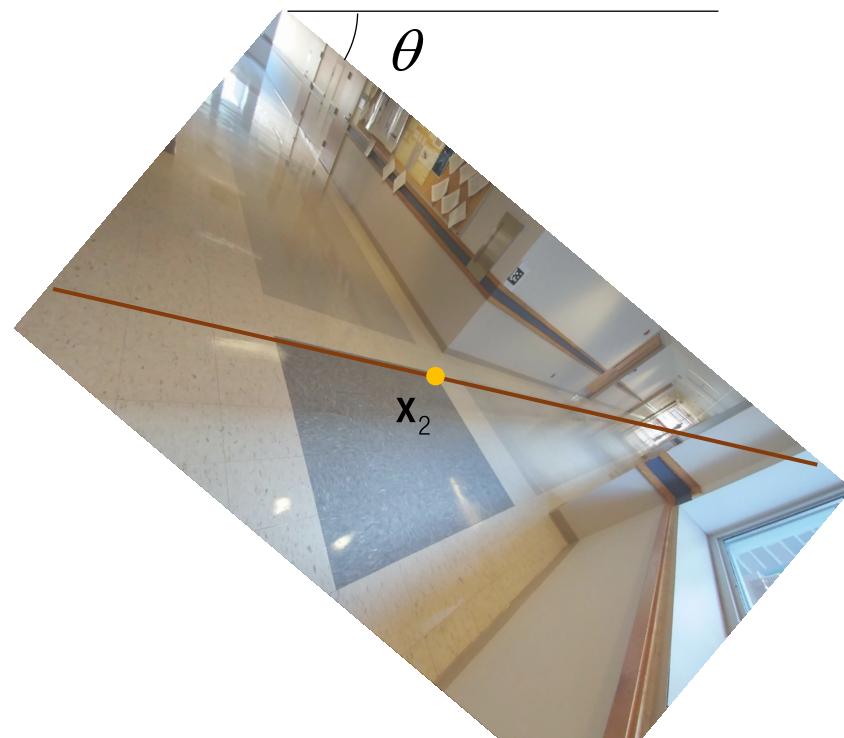
$$l = x_1 \times x_2$$

The intersection between two lines:

$$x = l_1 \times l_2$$

Given any formula, we can switch the meaning of point and line to get another formula.

$$x_2 = Tx_1 \leftrightarrow l_2 = T^T l_1 \quad T: \text{Transformation}$$



2D Point and Line Duality



The 2D line joining two points:

$$l = x_1 \times x_2$$

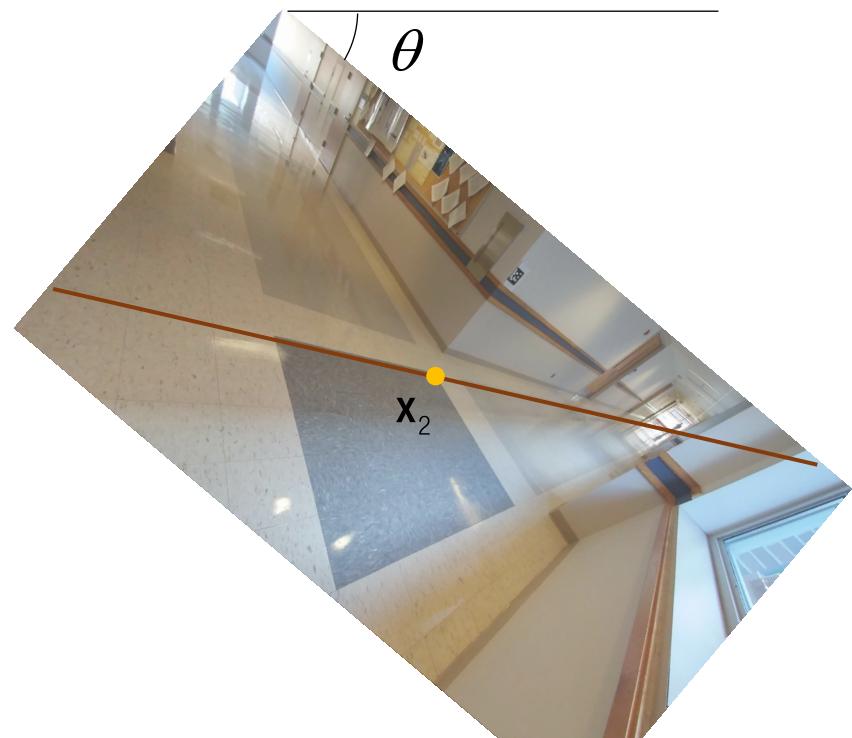
The intersection between two lines:

$$x = l_1 \times l_2$$

Given any formula, we can switch the meaning of point and line to get another formula.

$$x_2 = Tx_1 \leftrightarrow l_2 = T^T l_1 \quad T: \text{Transformation}$$

$$\therefore l_1^T x_1 = 0$$



2D Point and Line Duality



The 2D line joining two points:

$$l = x_1 \times x_2$$

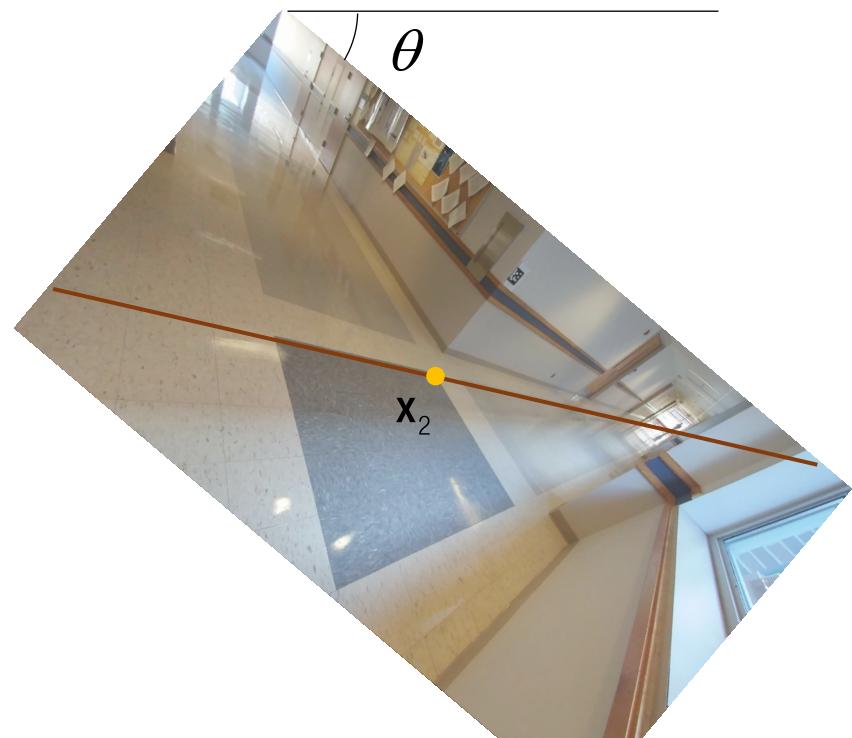
The intersection between two lines:

$$x = l_1 \times l_2$$

Given any formula, we can switch the meaning of point and line to get another formula.

$$x_2 = Tx_1 \leftrightarrow l_2 = T^{-T}l_1 \quad T: \text{Transformation}$$

$$\therefore l_1^T x_1 = (l_1^T T^{-1})(Tx_1) = 0$$



2D Point and Line Duality



The 2D line joining two points:

$$l = x_1 \times x_2$$

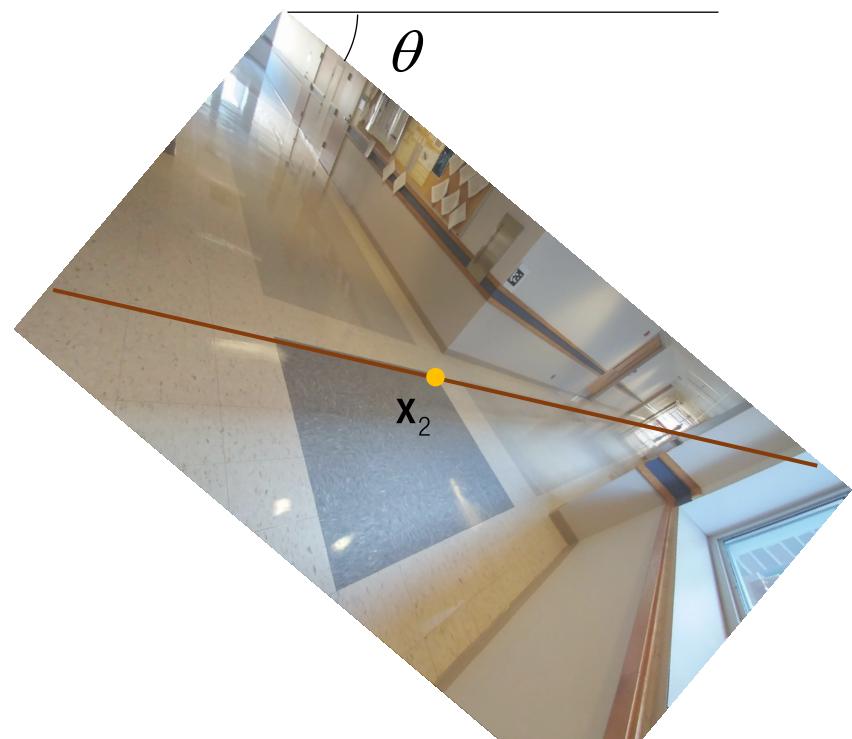
The intersection between two lines:

$$x = l_1 \times l_2$$

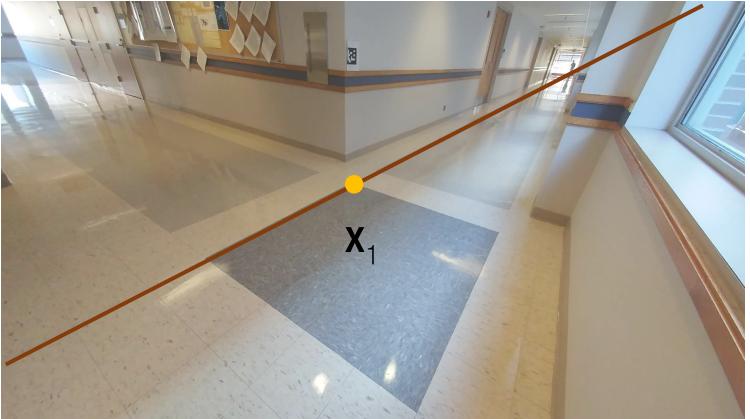
Given any formula, we can switch the meaning of point and line to get another formula.

$$x_2 = Tx_1 \leftrightarrow l_2 = T^{-T}l_1 \quad T: \text{Transformation}$$

$$\therefore l_1^T x_1 = (l_1^T T^{-1})(Tx_1) = (T^{-T}l_1)^T (Tx_1) = l_2^T x_2$$



2D Point and Line Duality



The 2D line joining two points:

$$l = x_1 \times x_2$$

The intersection between two lines:

$$x = l_1 \times l_2$$

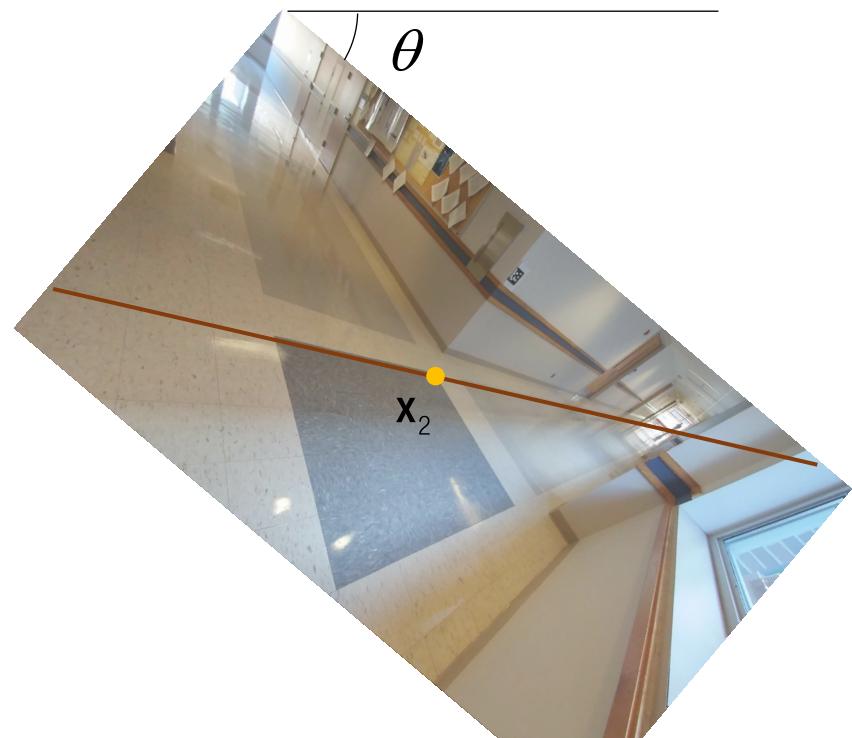
Given any formula, we can switch the meaning of point and line to get another formula.

$$x_2 = Tx_1 \leftrightarrow l_2 = T^{-T}l_1 \quad T: \text{Transformation}$$

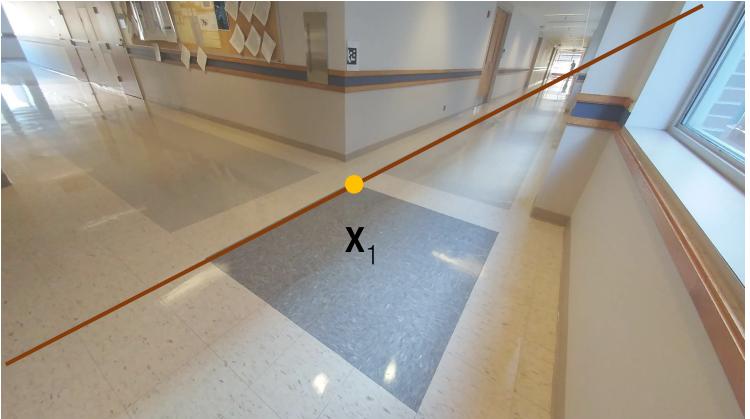
$$\therefore l_1^T x_1 = (l_1^T T^{-1})(Tx_1) = (T^{-T}l_1)^T (Tx_1) = l_2^T x_2$$

$$x_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} x_1 \longleftrightarrow$$

?



2D Point and Line Duality



The 2D line joining two points:

$$l = \mathbf{x}_1 \times \mathbf{x}_2$$

The intersection between two lines:

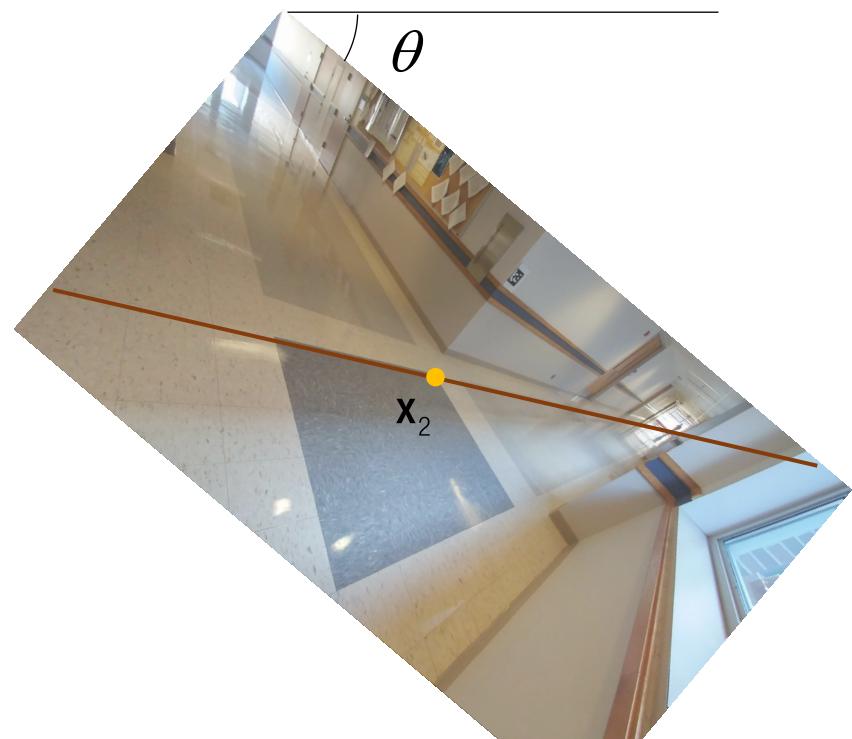
$$\mathbf{x} = l_1 \times l_2$$

Given any formula, we can switch the meaning of point and line to get another formula.

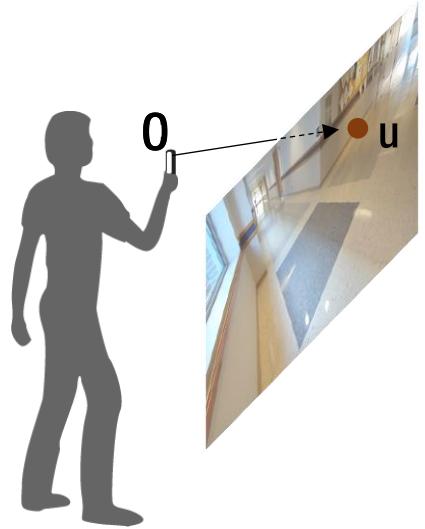
$$\mathbf{x}_2 = T\mathbf{x}_1 \leftrightarrow l_2 = T^{-T}l_1 \quad T: \text{Transformation}$$

$$\therefore l_1^T \mathbf{x}_1 = (l_1^T T^{-1})(T\mathbf{x}_1) = (T^{-T}l_1)^T (T\mathbf{x}_1) = l_2^T \mathbf{x}_2$$

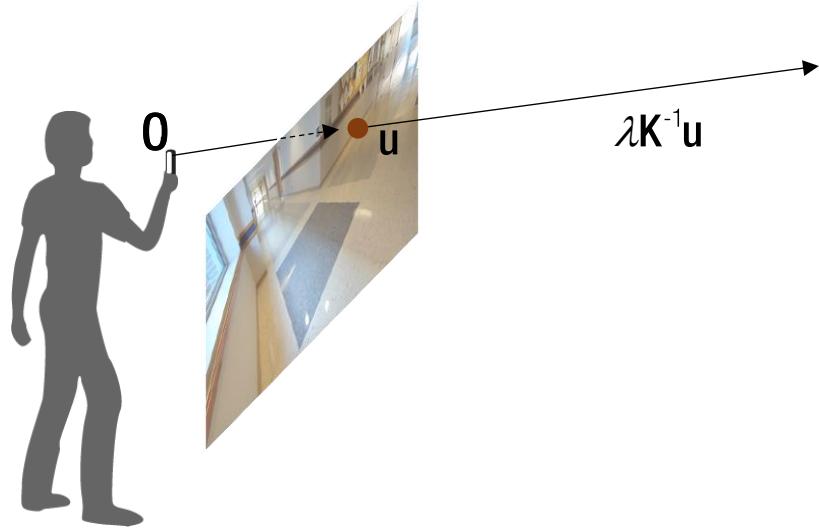
$$\mathbf{x}_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_1 \longleftrightarrow l_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-T} \mathbf{l}_1 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{l}_1$$



Geometric Interpretation (Point)



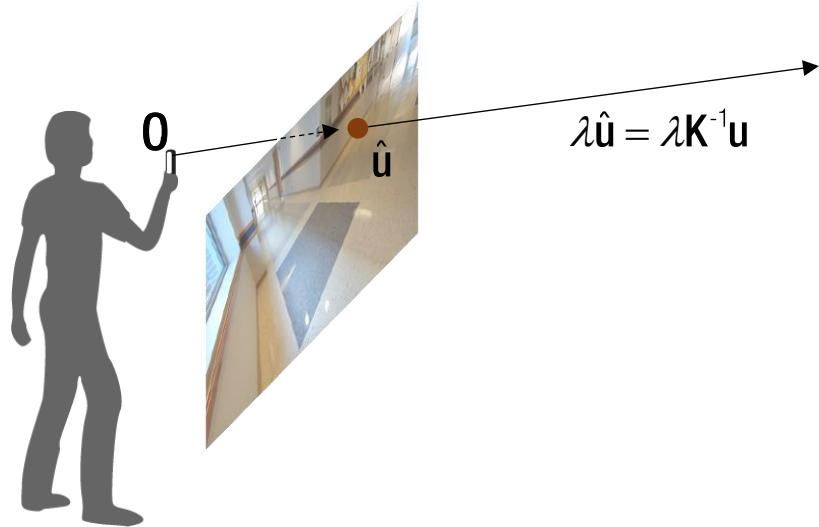
Geometric Interpretation (Point)



Geometric Interpretation (Point)

Normalized coordinate:

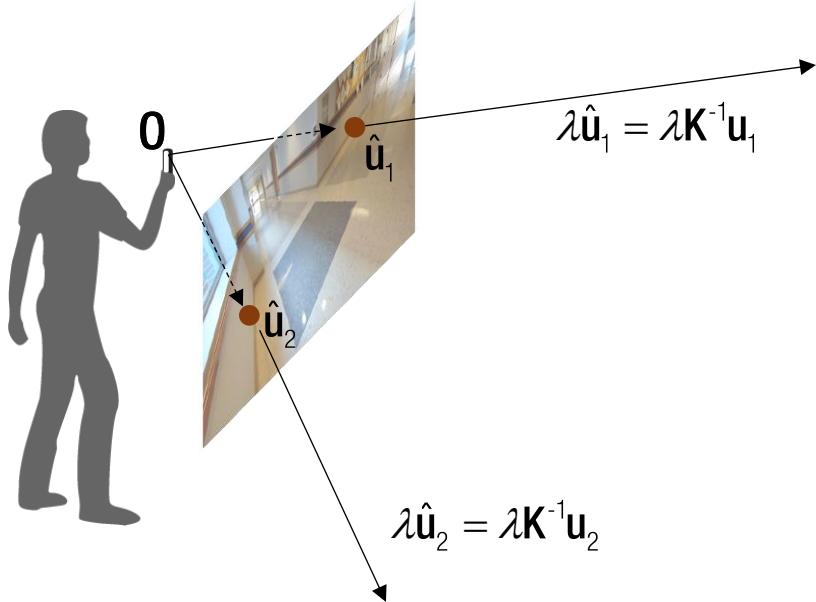
$$\hat{\mathbf{u}} = \mathbf{K}^{-1}\mathbf{u}$$



Geometric Interpretation (Point)

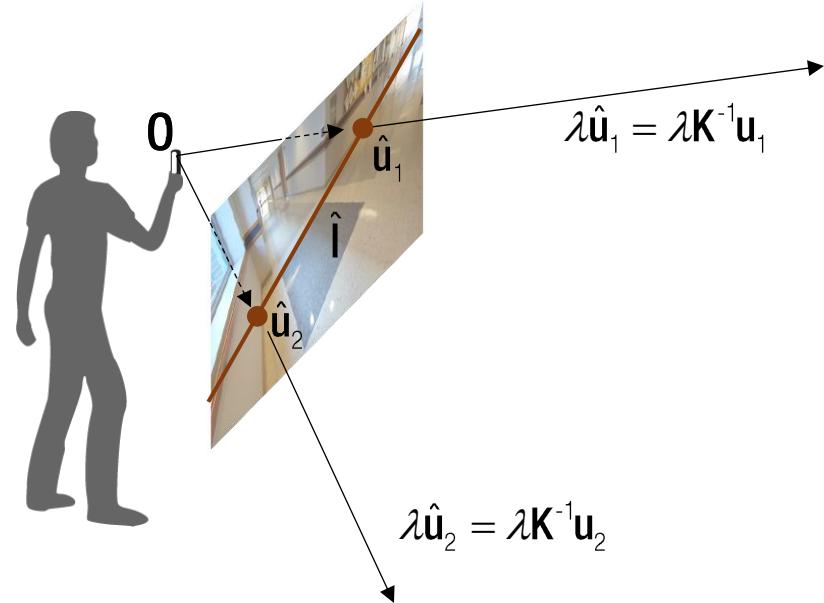
Normalized coordinate:

$$\hat{\mathbf{u}}_1 = \mathbf{K}^{-1}\mathbf{u}_1 \quad \hat{\mathbf{u}}_2 = \mathbf{K}^{-1}\mathbf{u}_2$$



Geometric Interpretation (Line)

Normalized coordinate:



$$\hat{u}_1 = K^{-1}u_1 \quad \hat{u}_2 = K^{-1}u_2$$

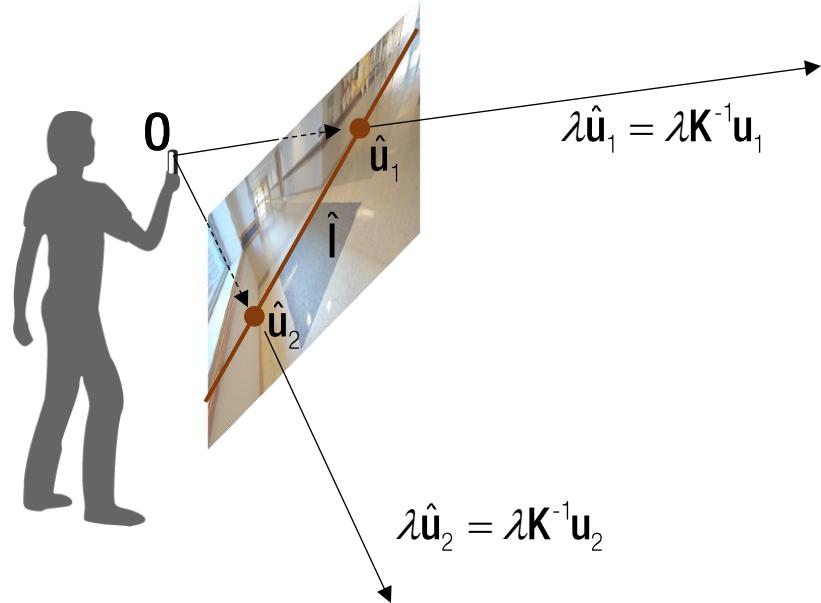
$$\longrightarrow \hat{l} = \hat{u}_1 \times \hat{u}_2$$

where $\hat{l} = ?$



Geometric Interpretation (Line)

Normalized coordinate:



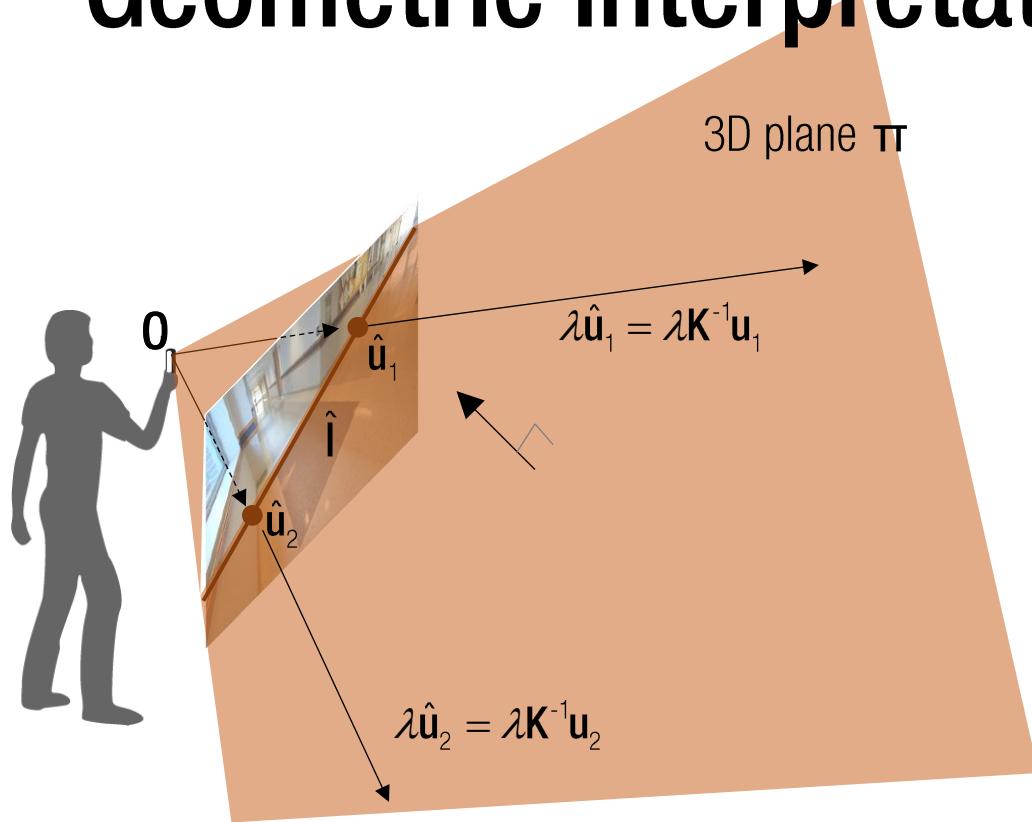
$$\hat{u}_1 = K^{-1}u_1 \quad \hat{u}_2 = K^{-1}u_2$$

$$\longrightarrow \hat{l} = \hat{u}_1 \times \hat{u}_2$$

$$\text{where } \hat{l} = (K^{-1})^T l = K^T l \text{ due to duality}$$



Geometric Interpretation (Line)



Normalized coordinate:

$$\hat{\mathbf{u}}_1 = \mathbf{K}^{-1}\mathbf{u}_1 \quad \hat{\mathbf{u}}_2 = \mathbf{K}^{-1}\mathbf{u}_2$$

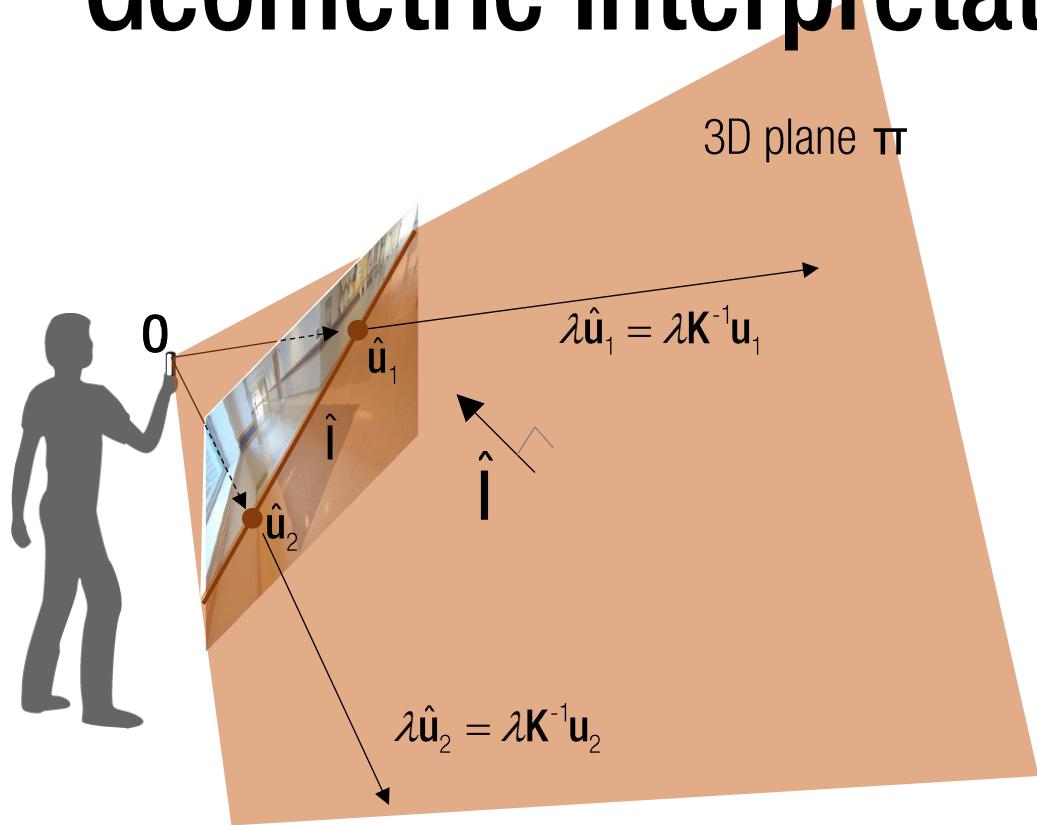
$$\longrightarrow \hat{\mathbf{l}} = \hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2$$

$$\text{where } \hat{\mathbf{l}} = (\mathbf{K}^{-1})^T \mathbf{l} = \mathbf{K}^T \mathbf{l} \text{ due to duality}$$

A 2D line in an image defines to a 3D plane passing the camera center:

$$\hat{\mathbf{l}} \rightarrow \pi$$

Geometric Interpretation (Line)



Normalized coordinate:

$$\hat{u}_1 = K^{-1}u_1 \quad \hat{u}_2 = K^{-1}u_2$$

$$\rightarrow \hat{l} = \hat{u}_1 \times \hat{u}_2$$

$$\text{where } \hat{l} = (K^{-1})^T l = K^T l \text{ due to duality}$$

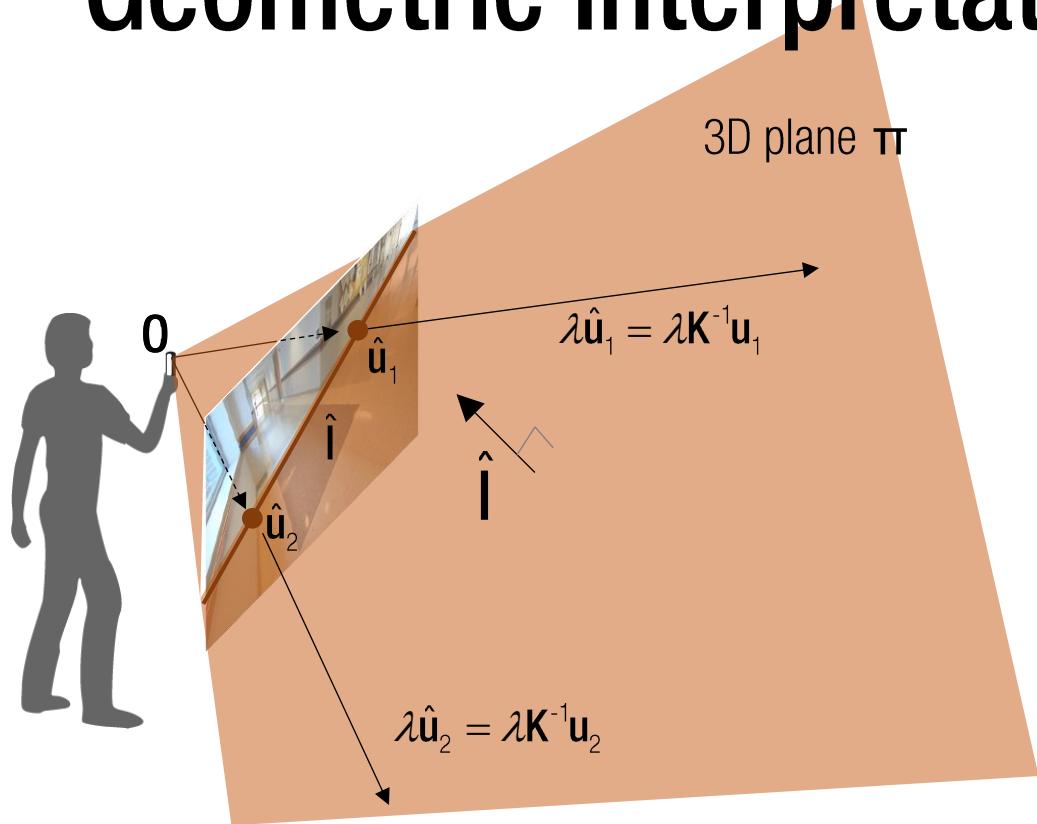
A 2D line in an image defines to a 3D plane passing the camera center:

$$\hat{l} \rightarrow \pi$$

Plane normal:

$$? = \lambda \hat{l}$$

Geometric Interpretation (Line)



Normalized coordinate:

$$\hat{\mathbf{u}}_1 = \mathbf{K}^{-1}\mathbf{u}_1 \quad \hat{\mathbf{u}}_2 = \mathbf{K}^{-1}\mathbf{u}_2$$

$$\longrightarrow \hat{\mathbf{i}} = \hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2$$

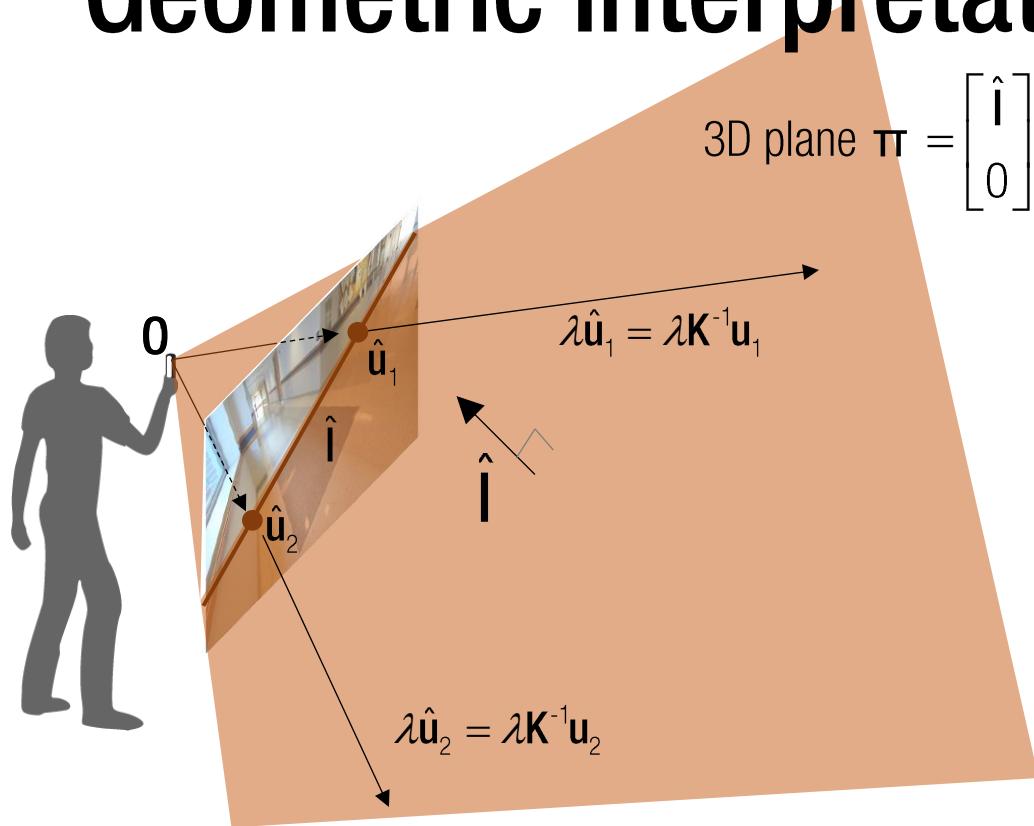
$$\text{where } \hat{\mathbf{i}} = (\mathbf{K}^{-1})^T \mathbf{l} = \mathbf{K}^T \mathbf{l} \text{ due to duality}$$

A 2D line in an image defines to a 3D plane passing the camera center:

$$\hat{\mathbf{i}} \rightarrow \pi$$

Plane normal: $(\lambda_1 \hat{\mathbf{u}}_1) \times (\lambda_2 \hat{\mathbf{u}}_2) = \lambda \hat{\mathbf{i}}$

Geometric Interpretation (Line)



Normalized coordinate:

$$\hat{\mathbf{u}}_1 = \mathbf{K}^{-1}\mathbf{u}_1 \quad \hat{\mathbf{u}}_2 = \mathbf{K}^{-1}\mathbf{u}_2$$

$$\longrightarrow \hat{\mathbf{i}} = \hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2$$

$$\text{where } \hat{\mathbf{i}} = (\mathbf{K}^{-1})^T \mathbf{i} = \mathbf{K}^T \mathbf{i} \text{ due to duality}$$

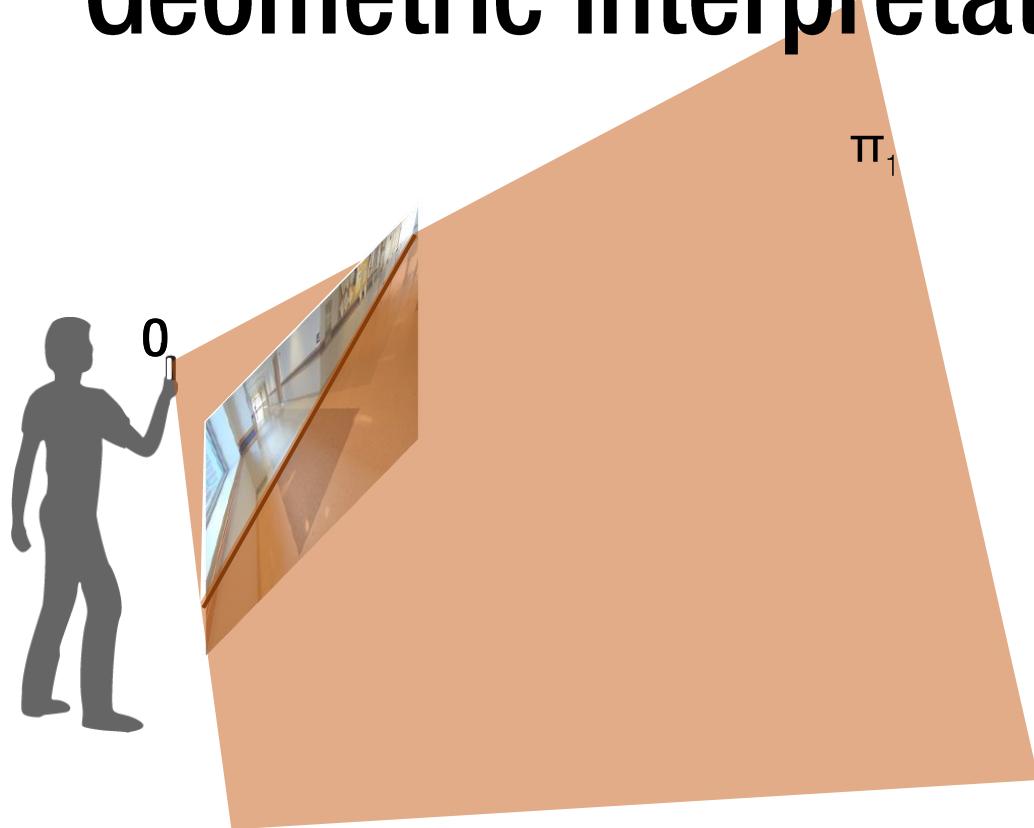
A 2D line in an image defines to a 3D plane passing the camera center:

$$\hat{\mathbf{i}} \rightarrow \pi$$

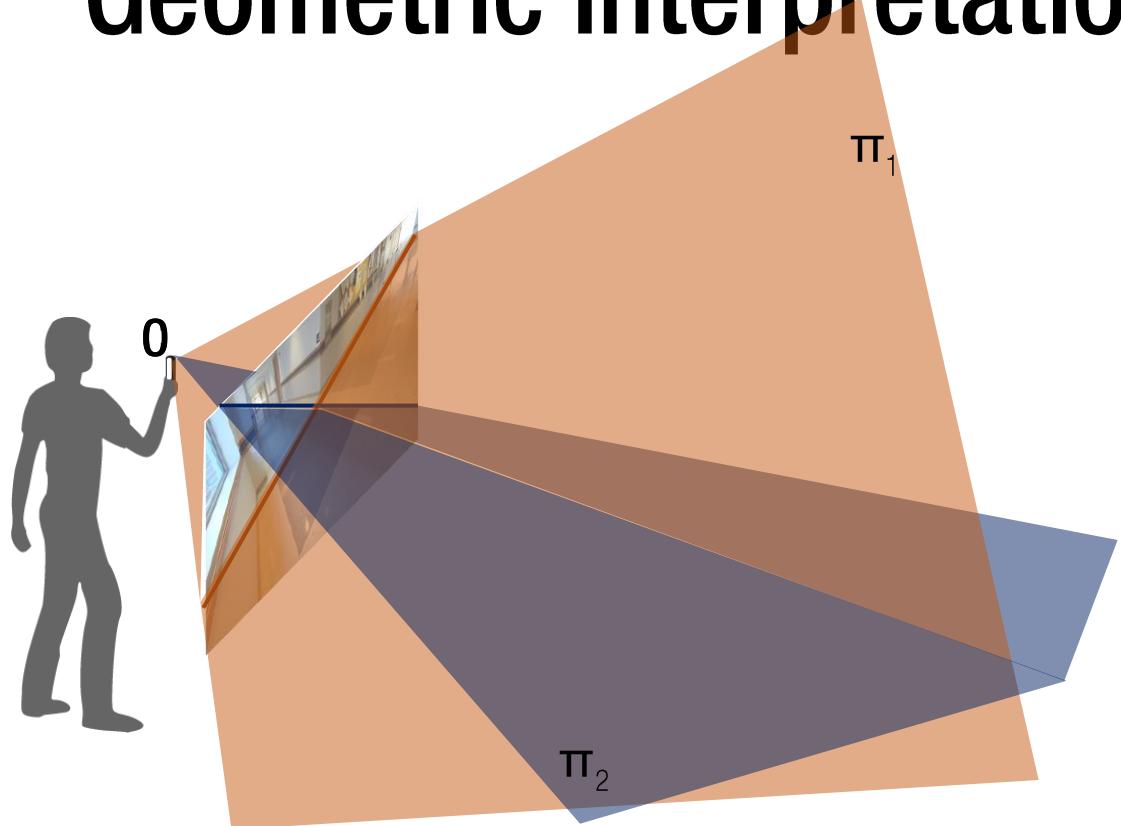
Plane normal: $(\lambda_1 \hat{\mathbf{u}}_1) \times (\lambda_2 \hat{\mathbf{u}}_2) = \lambda \hat{\mathbf{i}}$

$$\therefore \pi = \begin{bmatrix} \hat{\mathbf{i}} \\ 0 \end{bmatrix}$$

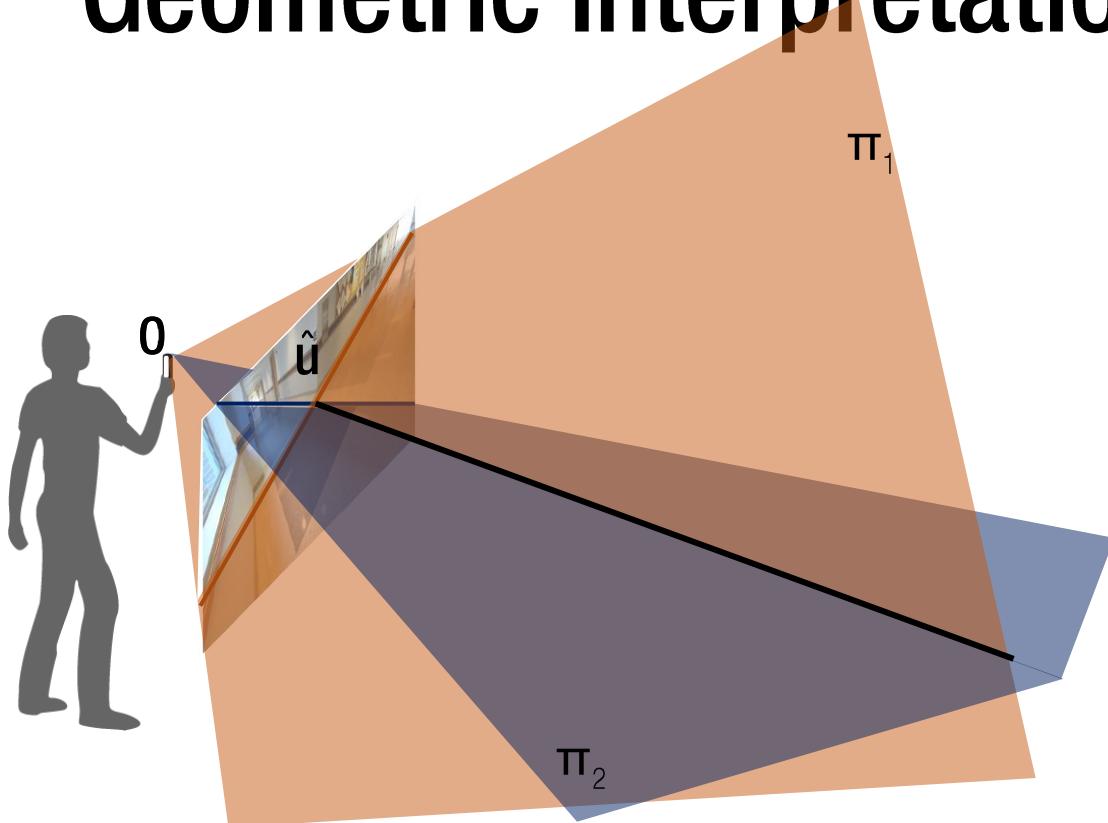
Geometric Interpretation (Line-Line)



Geometric Interpretation (Line-Line)

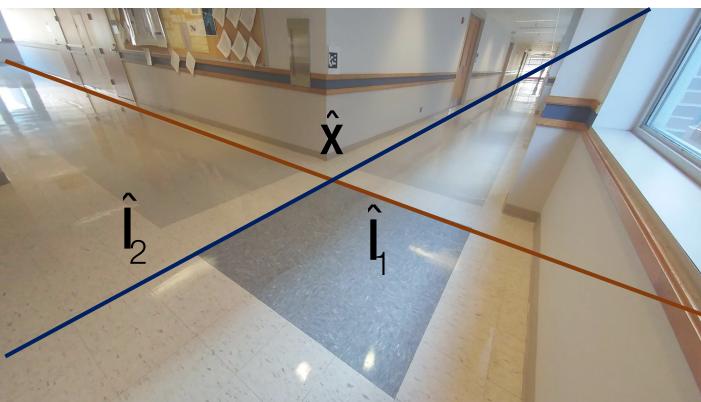


Geometric Interpretation (Line-Line)

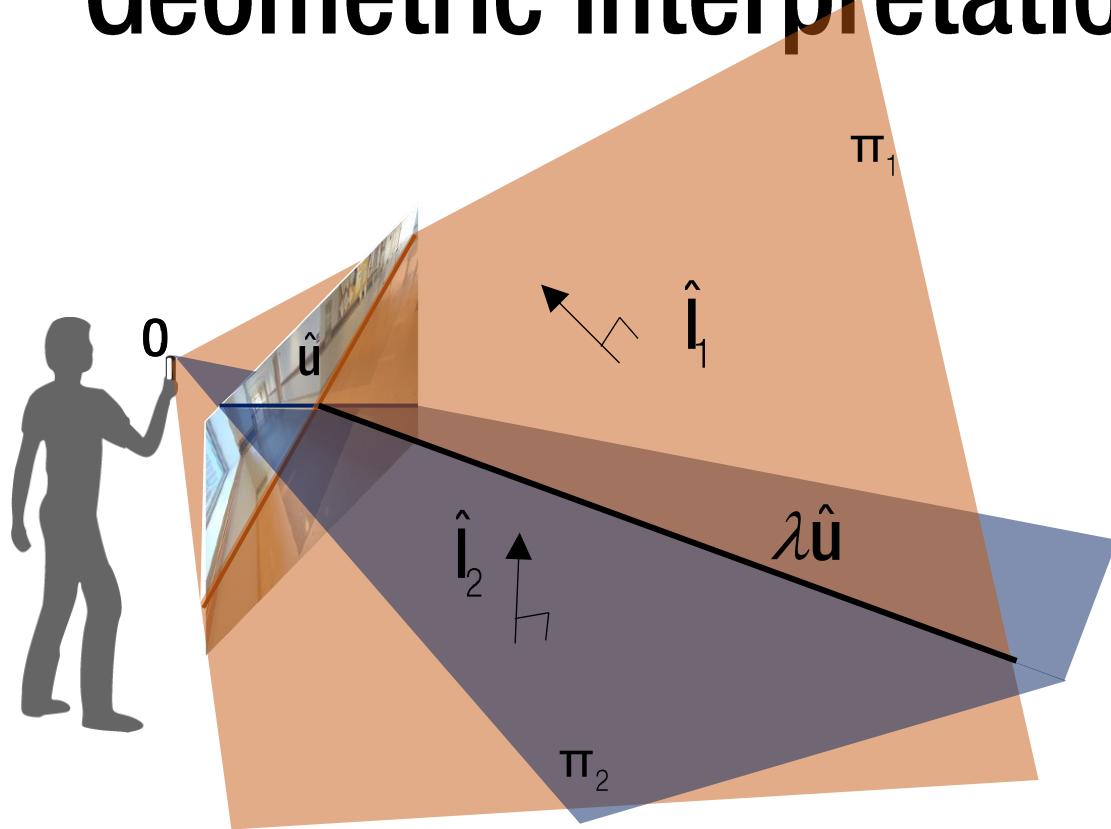


2D lines in an image intersect a 2D point corresponding to a 3D ray:

$$\hat{\mathbf{u}} = \hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2$$



Geometric Interpretation (Line-Line)

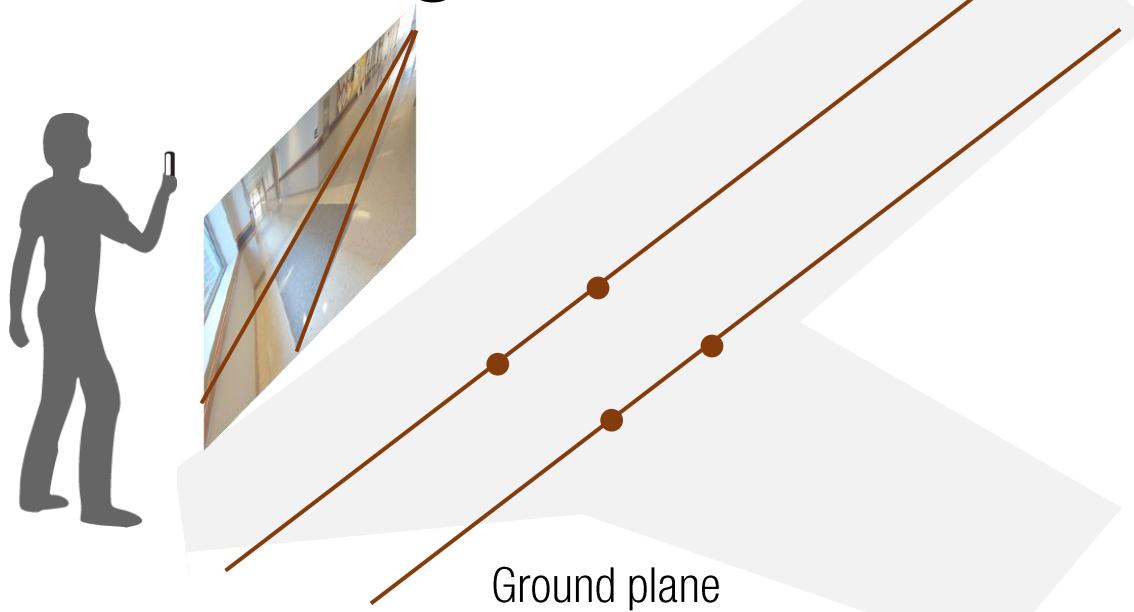


2D lines in an image intersect a 2D point corresponding to a 3D ray:

$$\hat{u} = \hat{i}_1 \times \hat{i}_2$$

: the 3D ray is perpendicular to two plane normals.

Vanishing Point



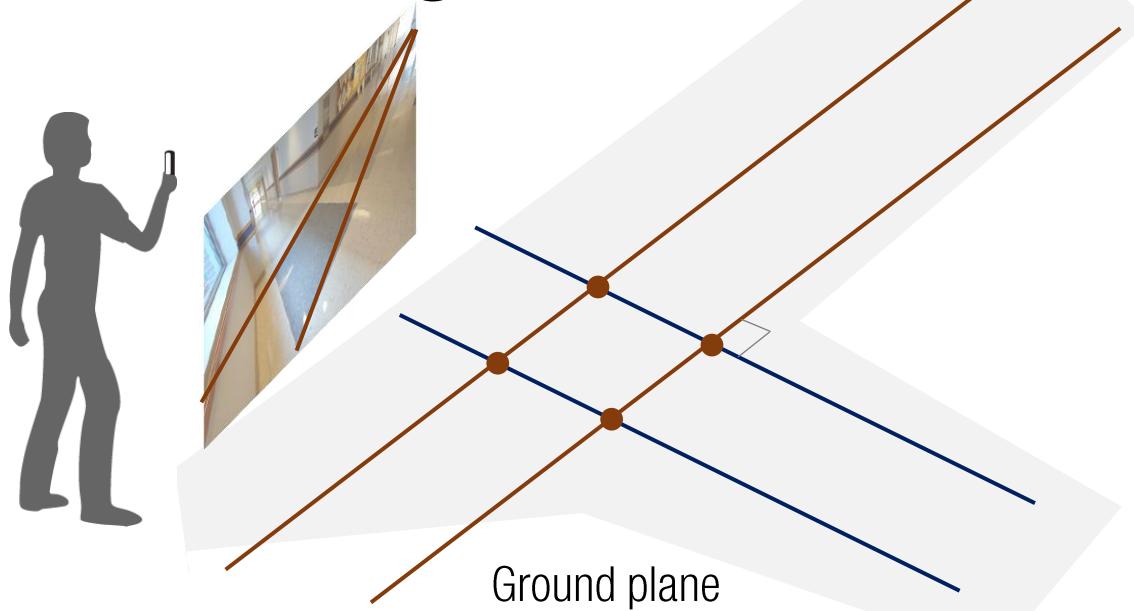
Parallel lines:

$$\mathbf{l}_{11} = \mathbf{u}_4 \times \mathbf{u}_3$$

$$\mathbf{l}_{12} = \mathbf{u}_1 \times \mathbf{u}_2$$



Vanishing Point



Ground plane

Parallel lines:

$$l_{11} = \mathbf{u}_4 \times \mathbf{u}_3$$

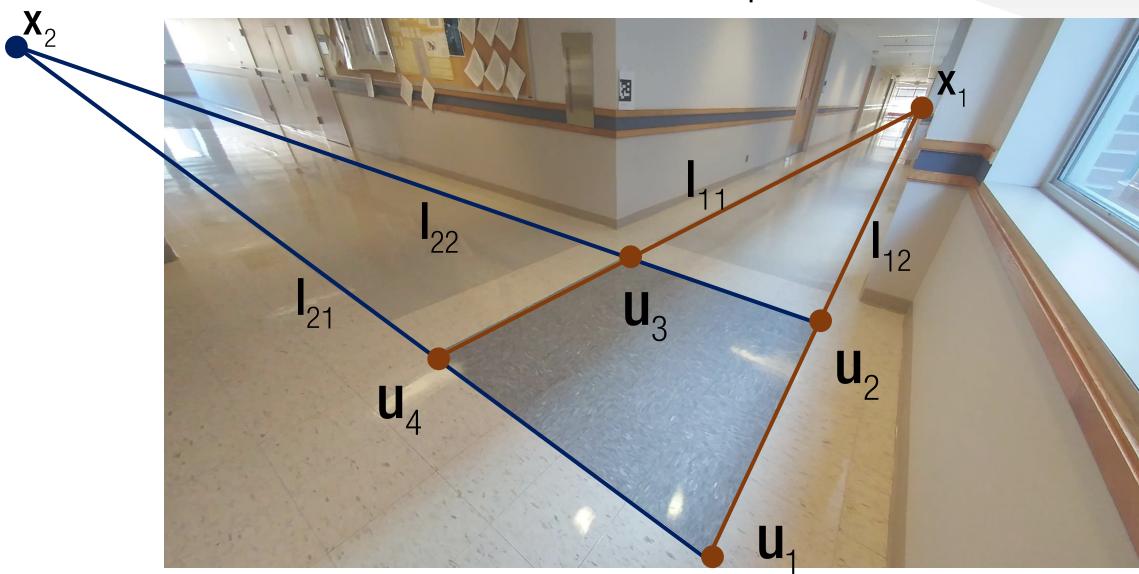
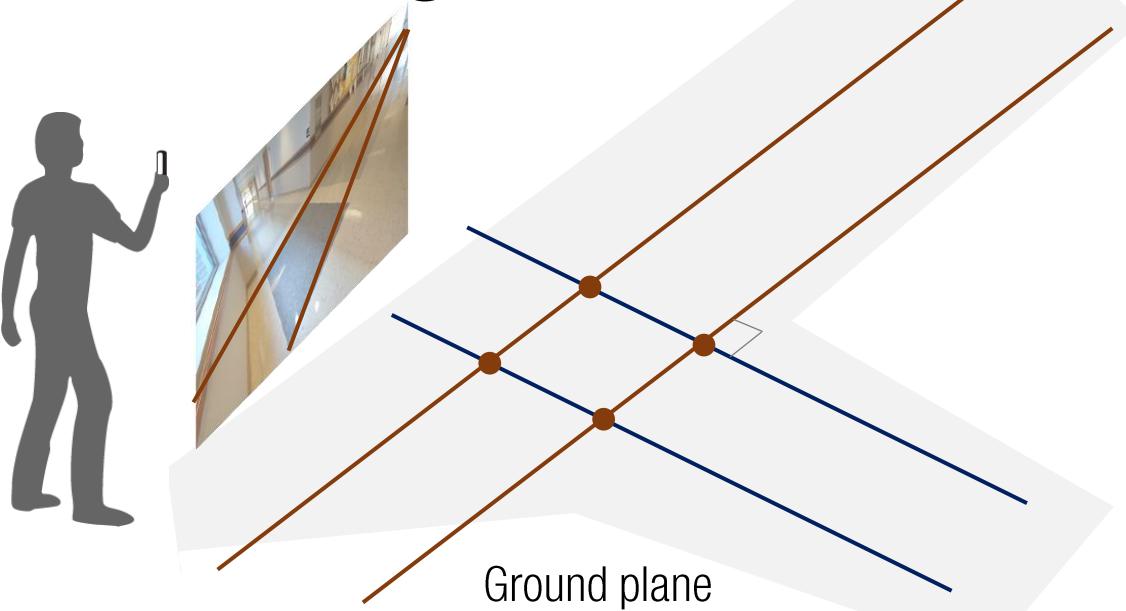
$$l_{12} = \mathbf{u}_1 \times \mathbf{u}_2$$

$$l_{21} = \mathbf{u}_4 \times \mathbf{u}_1$$

$$l_{22} = \mathbf{u}_3 \times \mathbf{u}_4$$



Vanishing Point



Parallel lines:

$$l_{11} = \mathbf{u}_4 \times \mathbf{u}_3$$

$$l_{12} = \mathbf{u}_1 \times \mathbf{u}_2$$

$$l_{21} = \mathbf{u}_4 \times \mathbf{u}_1$$

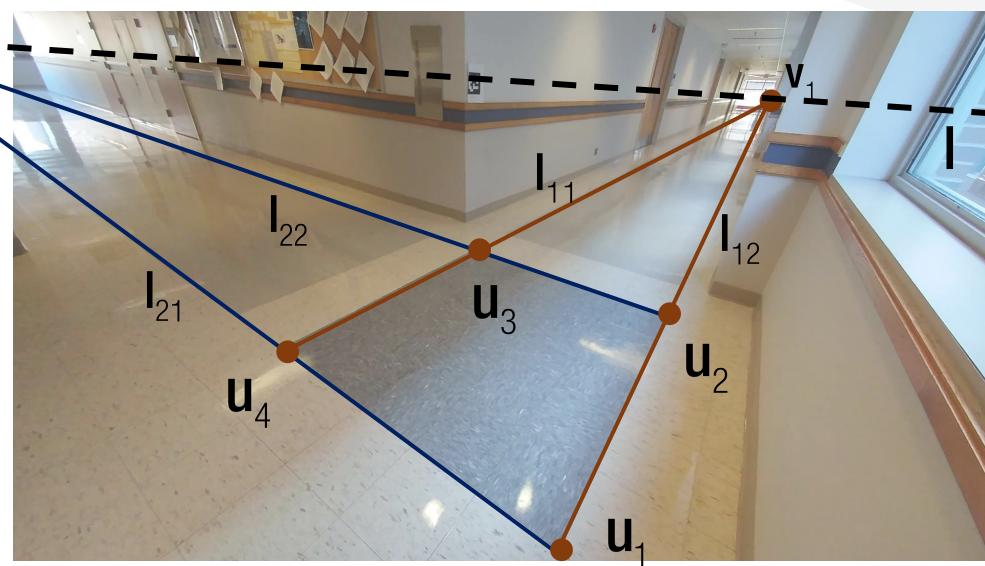
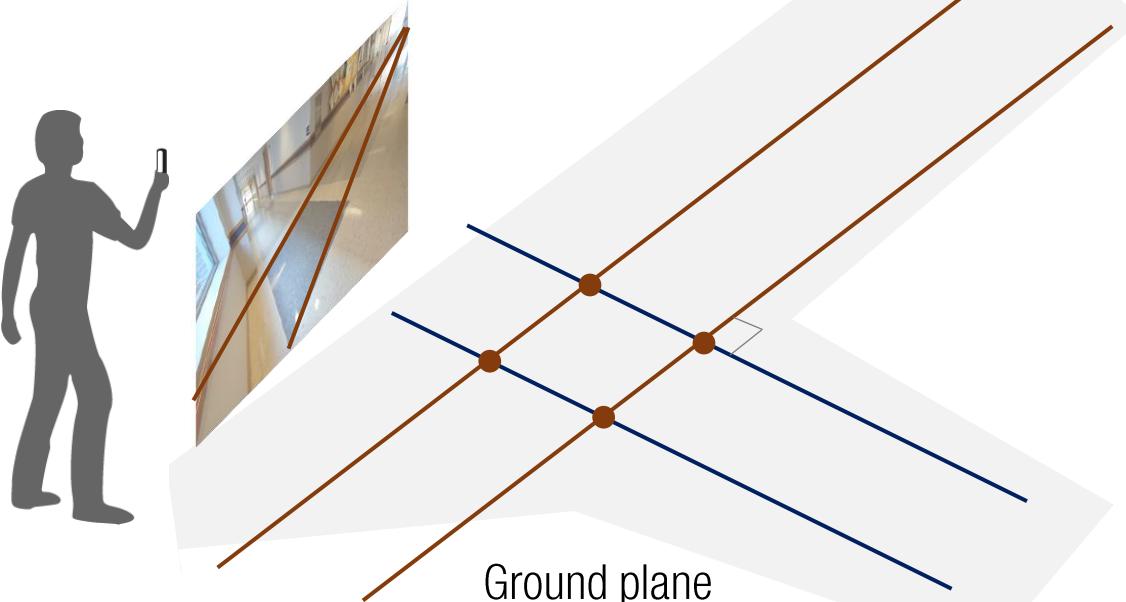
$$l_{22} = \mathbf{u}_3 \times \mathbf{u}_4$$

Vanishing points:

$$\mathbf{x}_1 = l_{11} \times l_{12}$$

$$\mathbf{x}_2 = l_{21} \times l_{22}$$

Vanishing Point



Parallel lines:

$$l_{11} = \mathbf{u}_4 \times \mathbf{u}_3$$

$$l_{12} = \mathbf{u}_1 \times \mathbf{u}_2$$

$$l_{21} = \mathbf{u}_4 \times \mathbf{u}_1$$

$$l_{22} = \mathbf{u}_3 \times \mathbf{u}_4$$

Vanishing points:

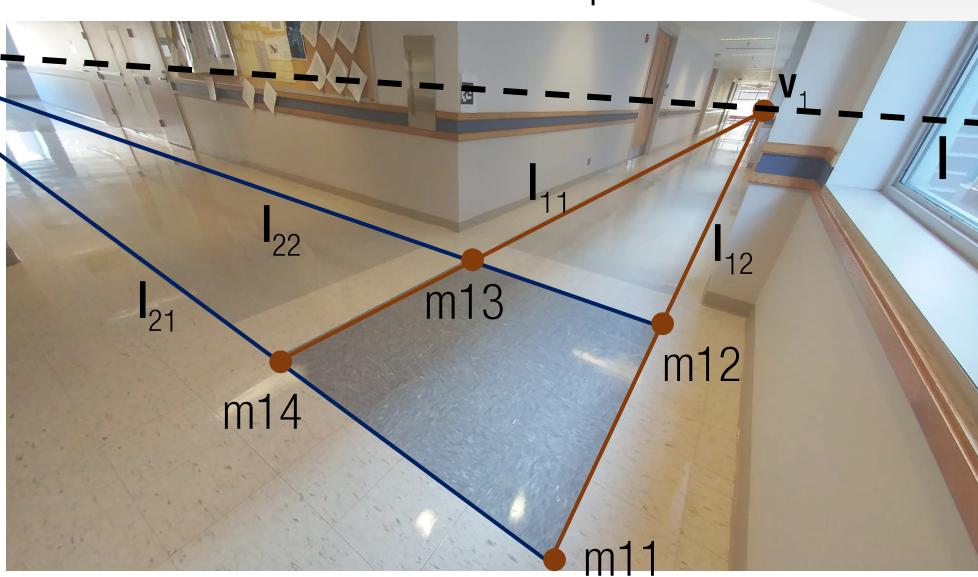
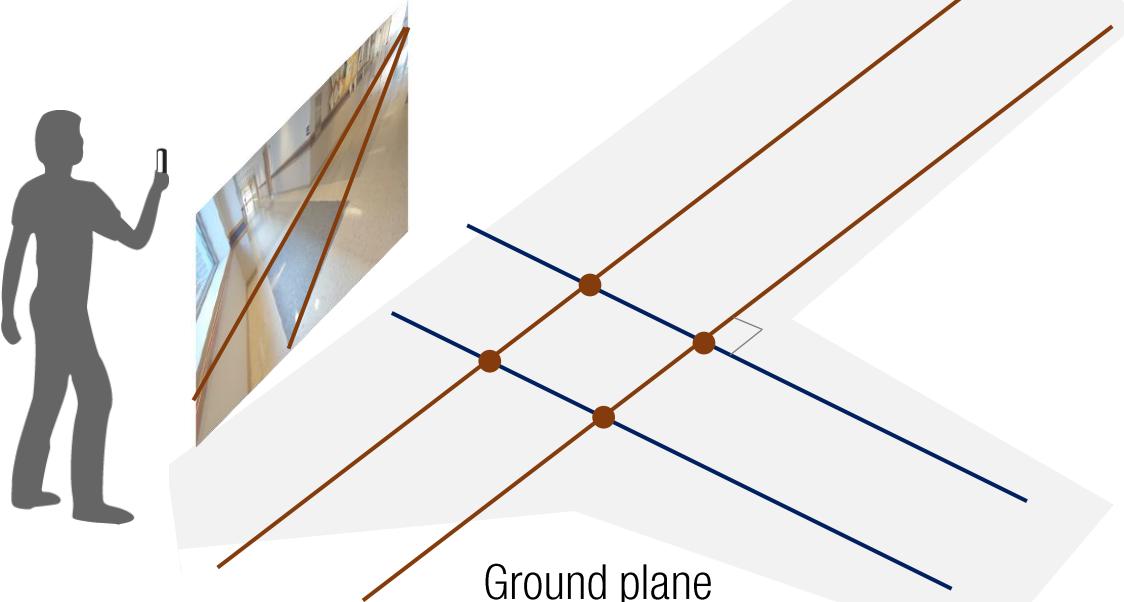
$$\mathbf{v}_1 = l_{11} \times l_{12}$$

$$\mathbf{v}_2 = l_{21} \times l_{22}$$

Vanishing line:

$$\mathbf{l} = \mathbf{v}_1 \times \mathbf{v}_2$$

Vanishing Point



```
l11 = GetLineFromTwoPoints(m11,m12);  
l12 = GetLineFromTwoPoints(m13,m14);
```

```
l21 = GetLineFromTwoPoints(m21,m22);  
l22 = GetLineFromTwoPoints(m23,m24);
```

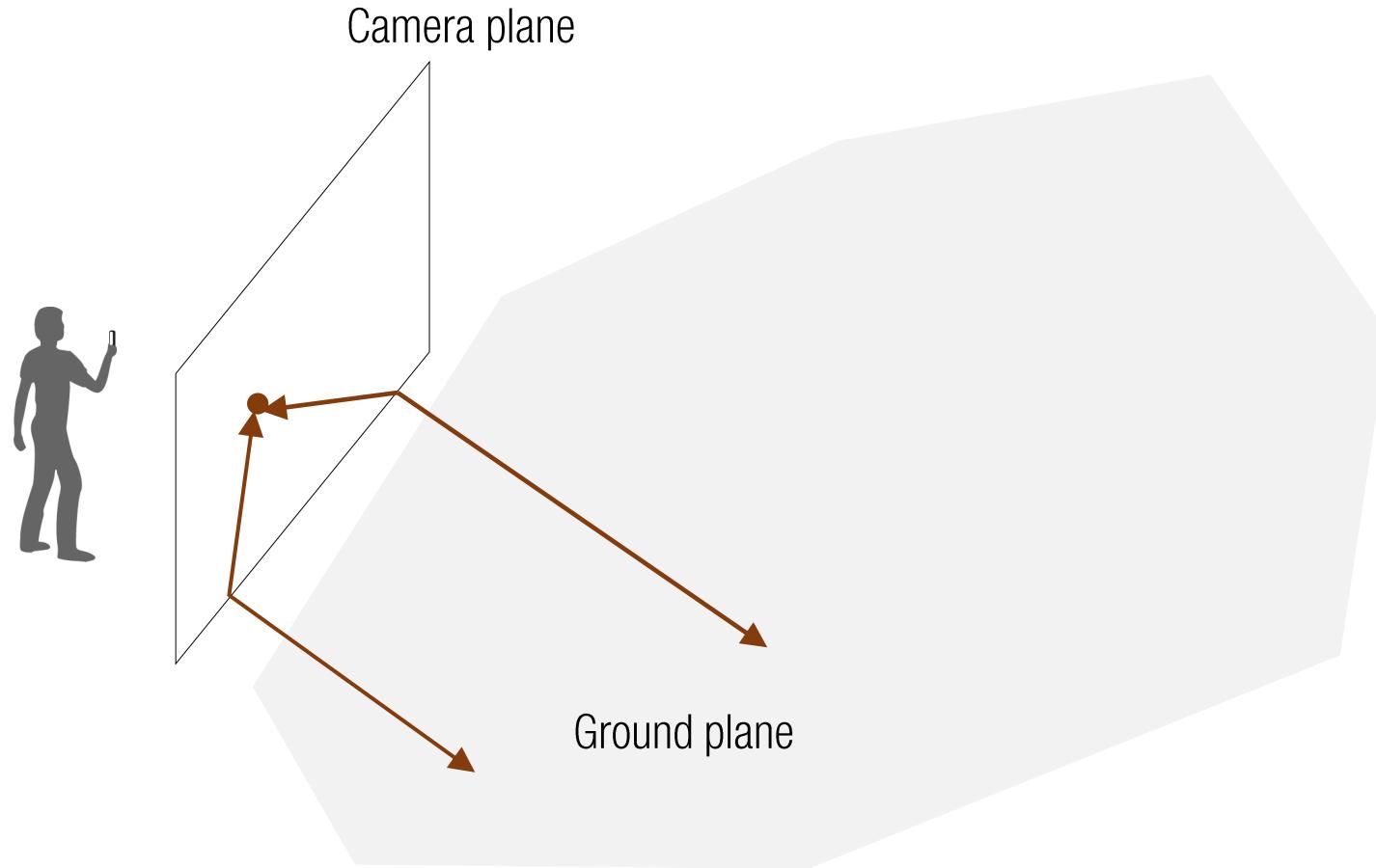
```
v1 = GetPointFromTwoLines(l11,l12);  
v2 = GetPointFromTwoLines(l21,l22);
```

```
vanishing_line = GetLineFromTwoPoints(x1, x2);
```

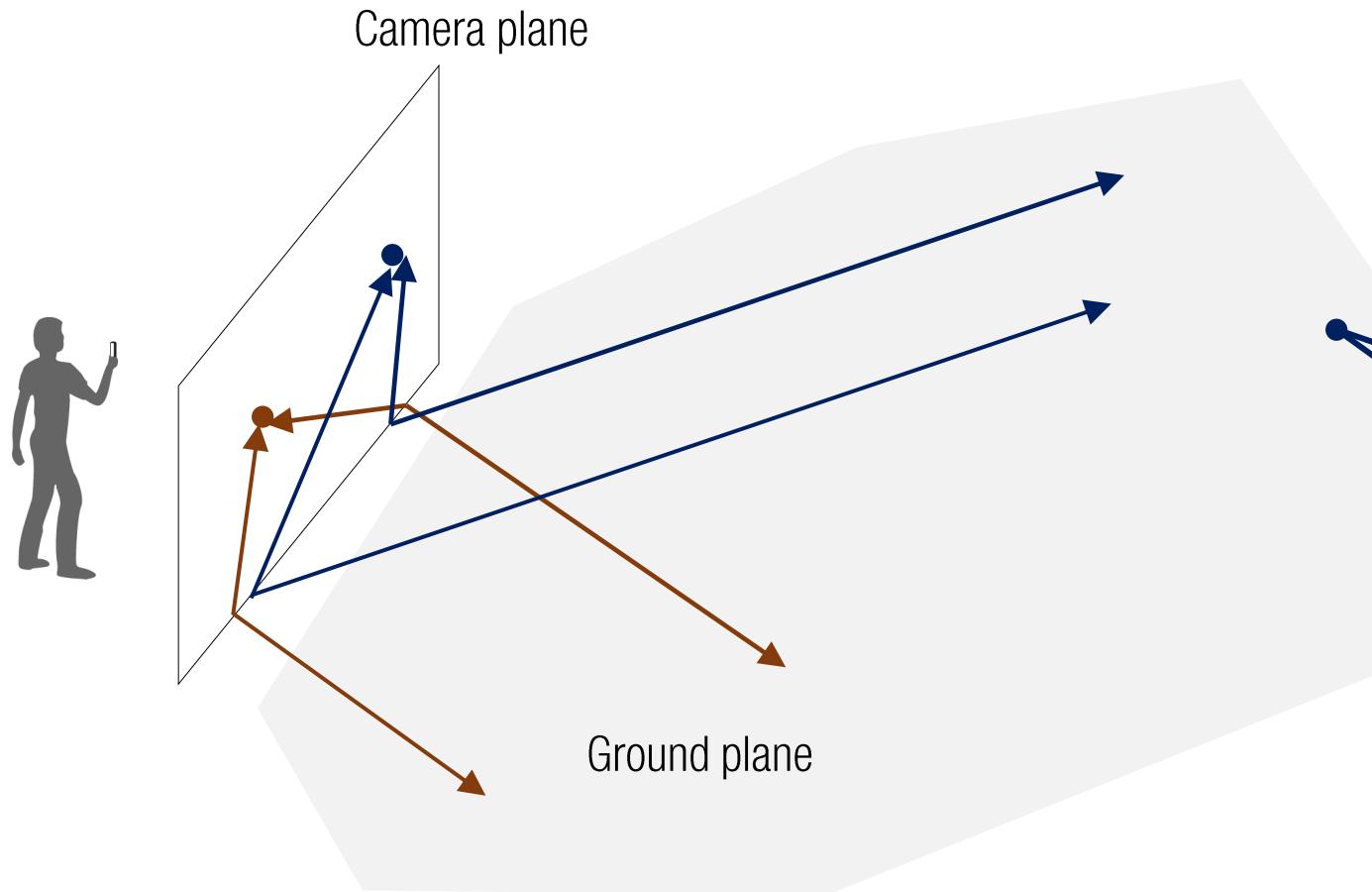
ComputeVanishingLine.m



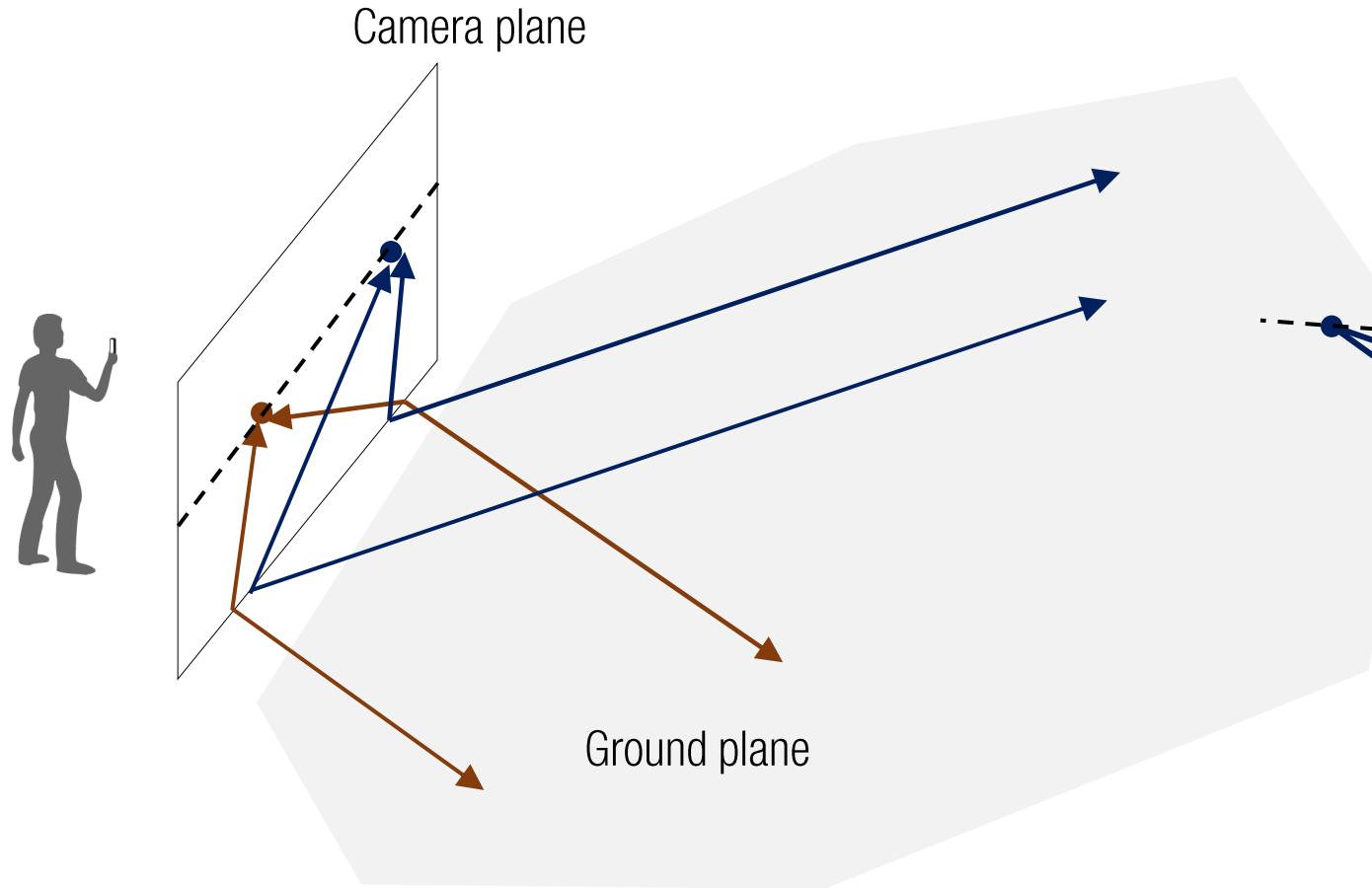
Geometric Interpretation of Vanishing Line



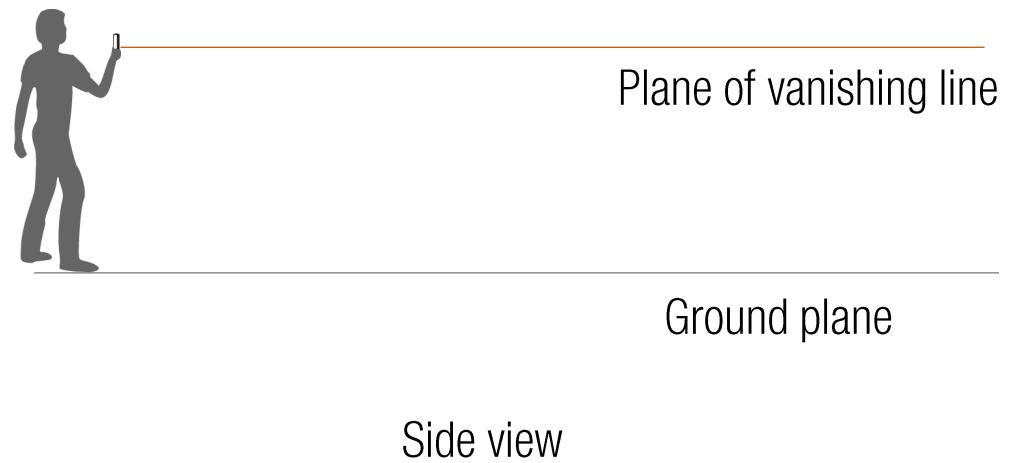
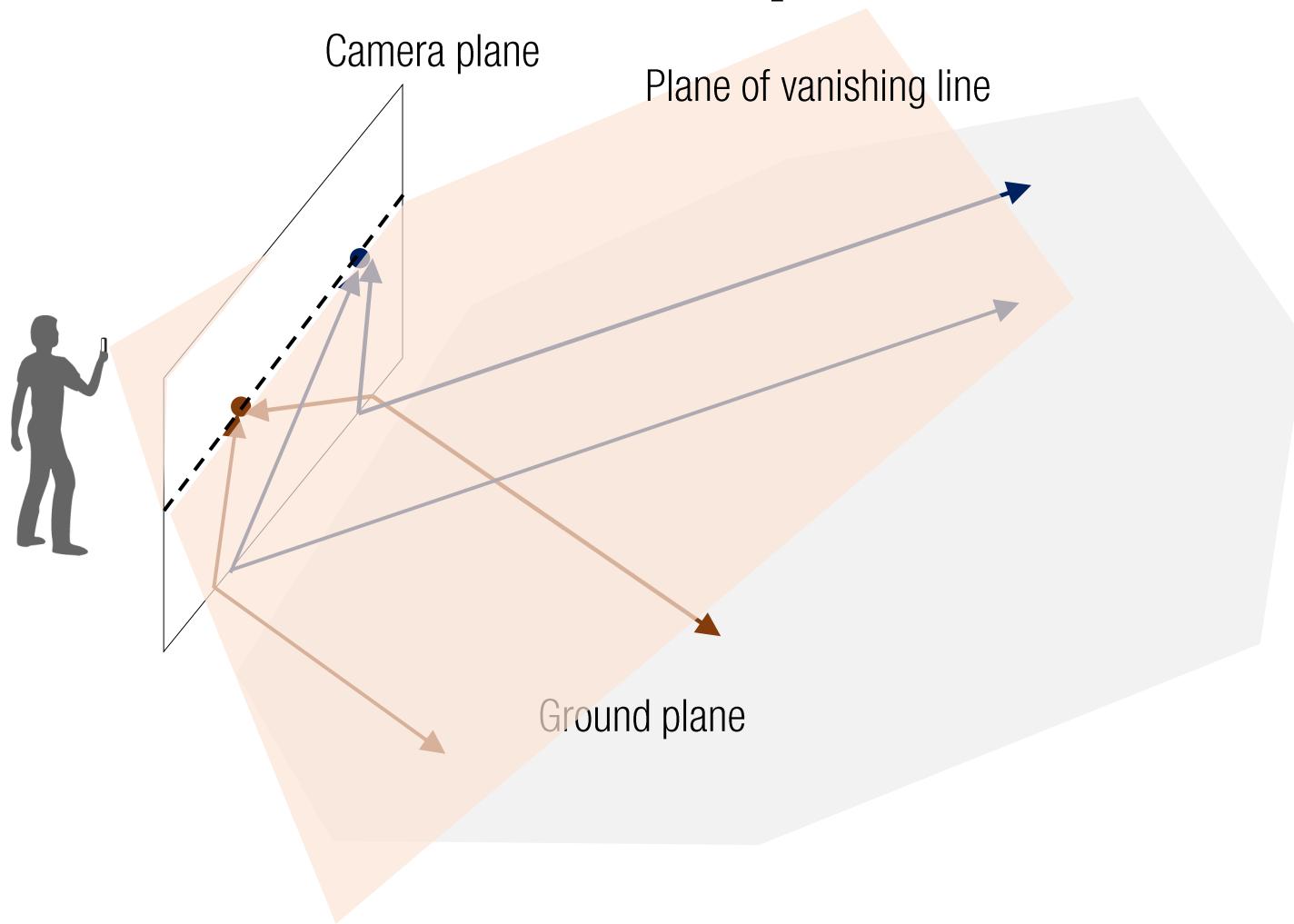
Geometric Interpretation of Vanishing Line



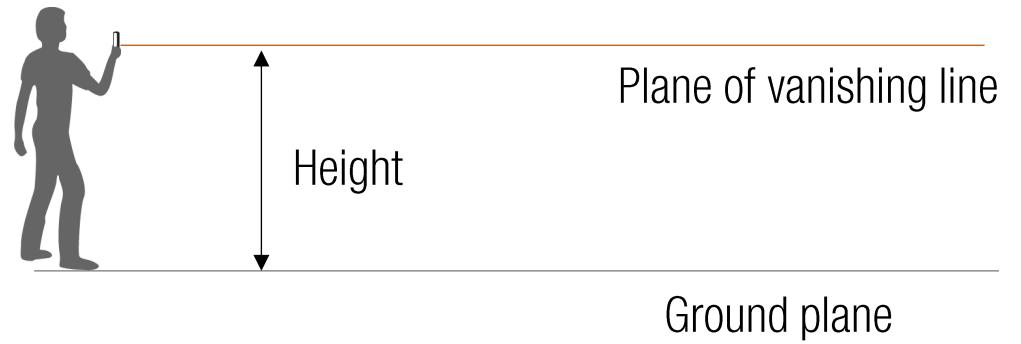
Geometric Interpretation of Vanishing Line



Geometric Interpretation of Vanishing Line



Geometric Interpretation of Vanishing Line



Side view

Where was I (how high)?



Where was I (how high)?



Taken from my hotel room (6th floor)

Taken from beach

Where was I (how high)?



Taken from my hotel room (6th floor)

Taken from beach

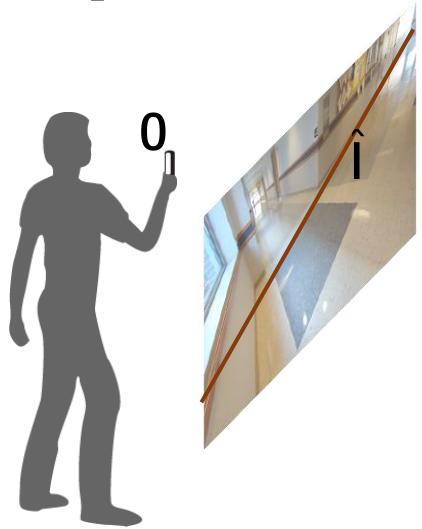


First person video



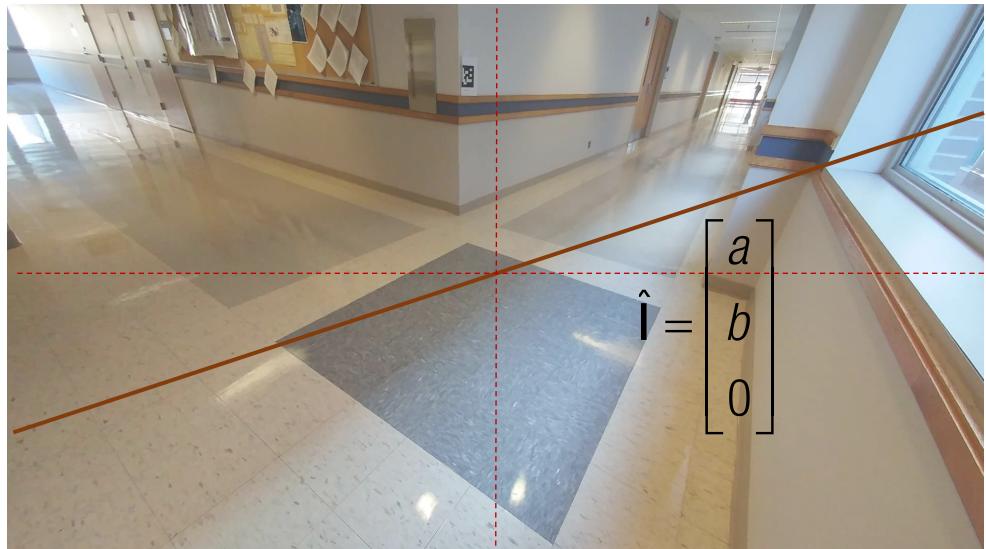
Cylindrical projection

Special Case

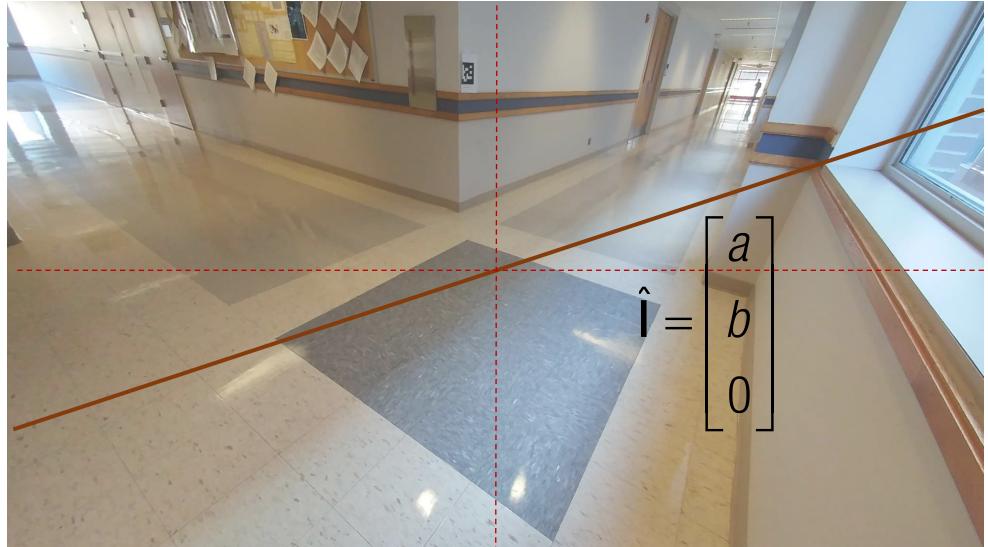
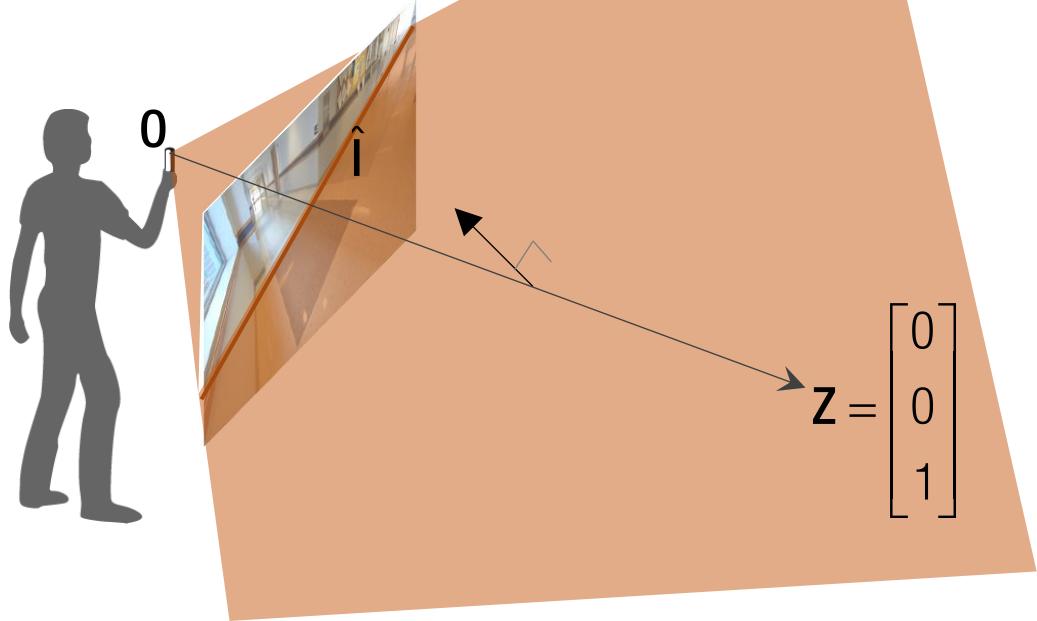


When the last element of line, \mathbf{l} is zero:

- the line passes through the center of image.



Special Case

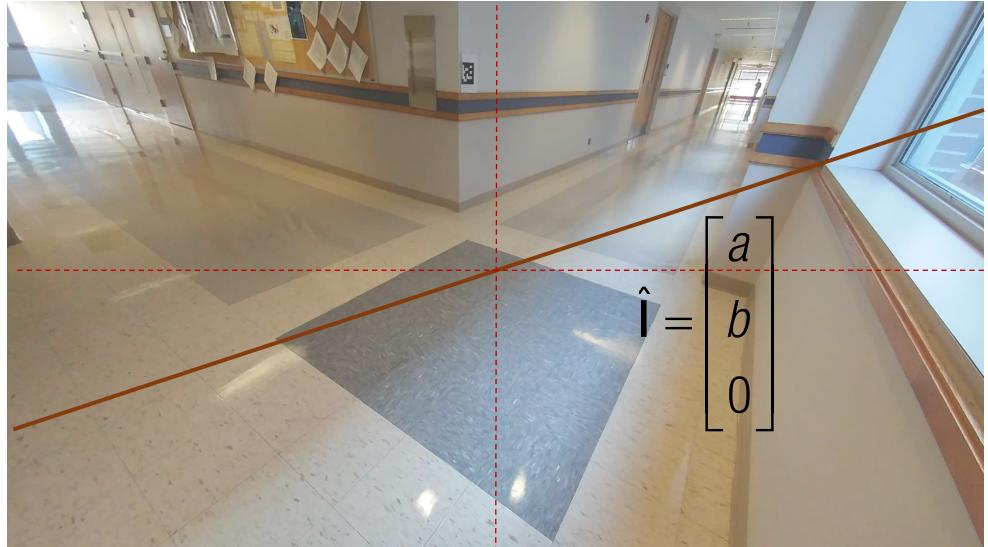
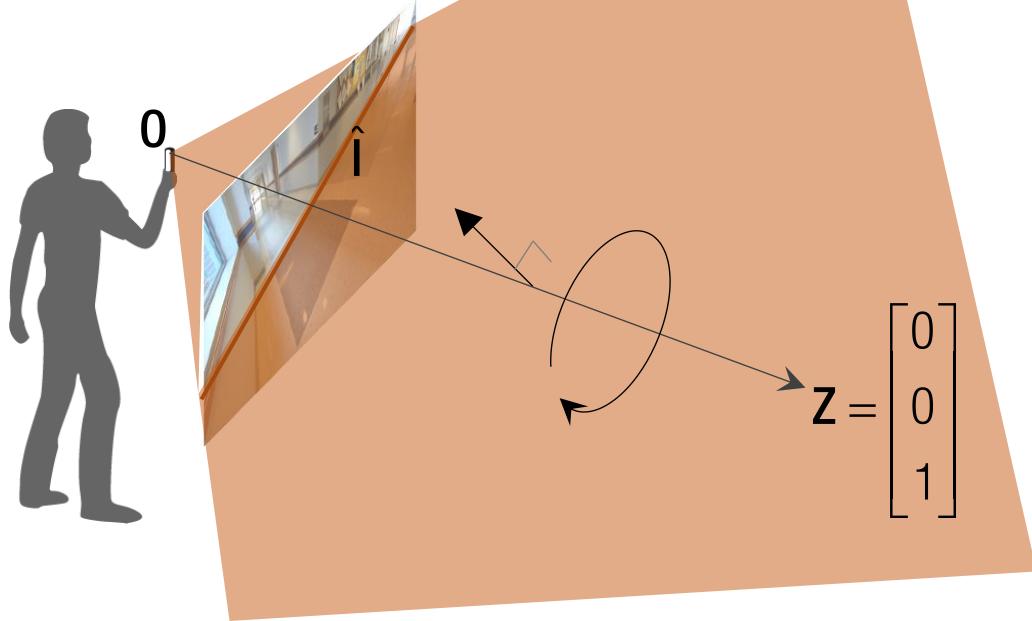


When the last element of line, \mathbf{l} is zero:

- the line passes through the center of image.
- the surface normal of the 3D plane is perpendicular to Z axis

$$[a \ b \ 0] \mathbf{z} = 0$$

Special Case



When the last element of line, \mathbf{i} is zero:

- the line passes through the center of image.
- the surface normal of the 3D plane is perpendicular to Z axis
- the 3D plane is a plane rotating around Z axis

$$[a \quad b \quad 0] \mathbf{z} = 0$$

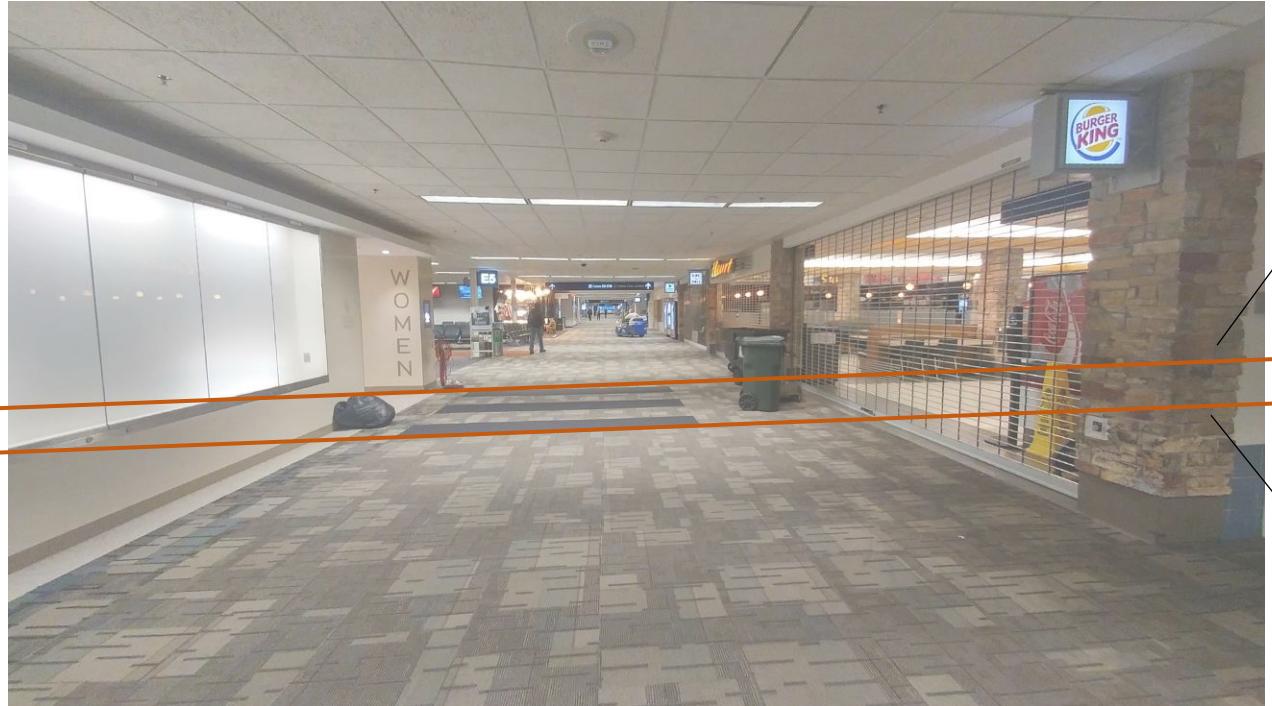
Special Case: 2D Point at Infinity



$$\mathbf{l}_2 = \begin{bmatrix} a \\ b \\ c_2 \end{bmatrix}$$

$$\mathbf{l}_1 = \begin{bmatrix} a \\ b \\ c_1 \end{bmatrix}$$

Special Case: 2D Point at Infinity



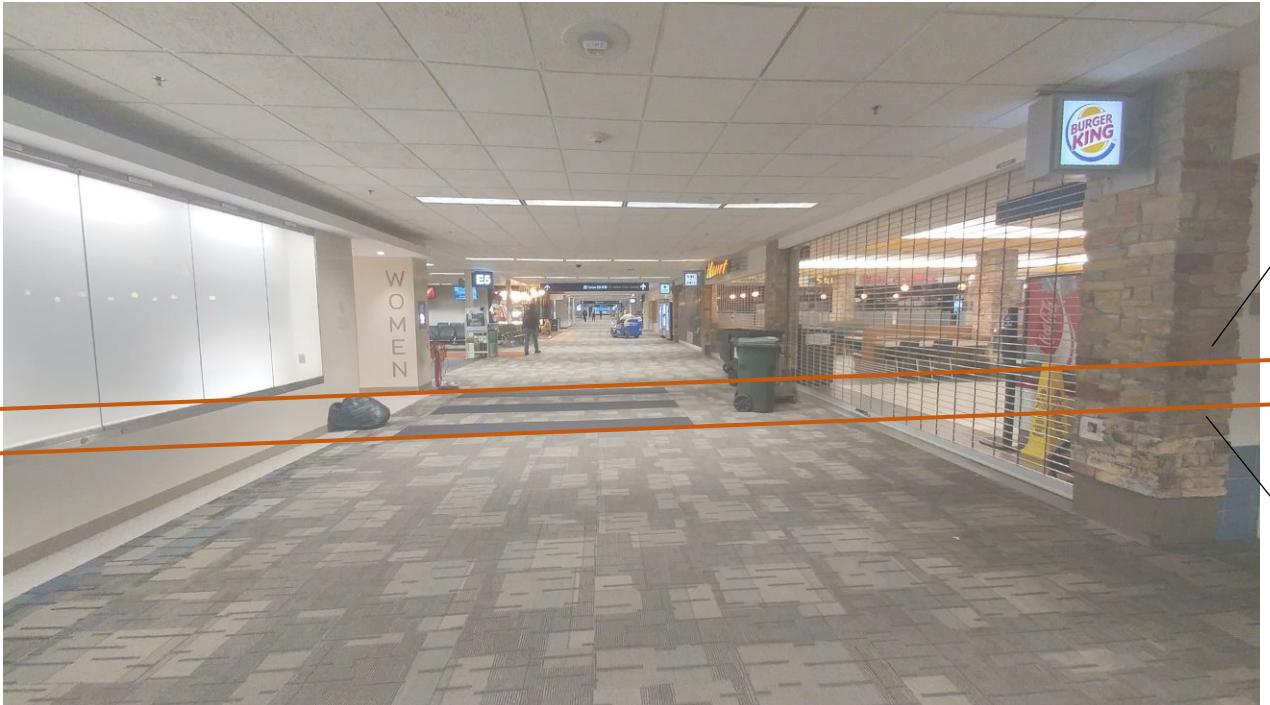
$$\mathbf{l}_2 = \begin{bmatrix} a \\ b \\ c_2 \end{bmatrix}$$

$$\mathbf{l}_1 = \begin{bmatrix} a \\ b \\ c_1 \end{bmatrix}$$

The intersection of two parallel lines in an image:

$$\mathbf{u} = \mathbf{l}_1 \times \mathbf{l}_2 = (c_2 - c_1) \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

Special Case: 2D Point at Infinity



$$\mathbf{l}_2 = \begin{bmatrix} a \\ b \\ c_2 \end{bmatrix}$$

$$\mathbf{l}_1 = \begin{bmatrix} a \\ b \\ c_1 \end{bmatrix}$$

The intersection of two parallel lines in an image:

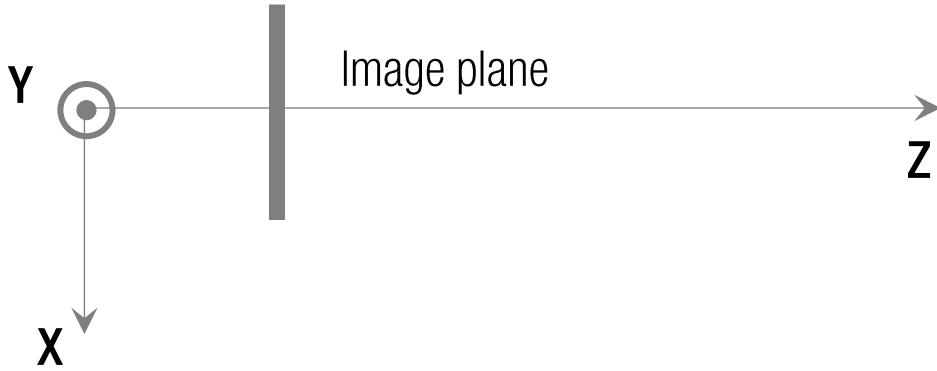
$$\mathbf{u} = \mathbf{l}_1 \times \mathbf{l}_2 = (c_2 - c_1) \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

This point does not correspond to a finite point because:

$$\mathbf{u} = \begin{bmatrix} -b / 0 \\ a / 0 \\ 0 / 0 \end{bmatrix}$$

Special Case: 2D Point at Infinity

Top view



The intersection of two parallel lines in an image:

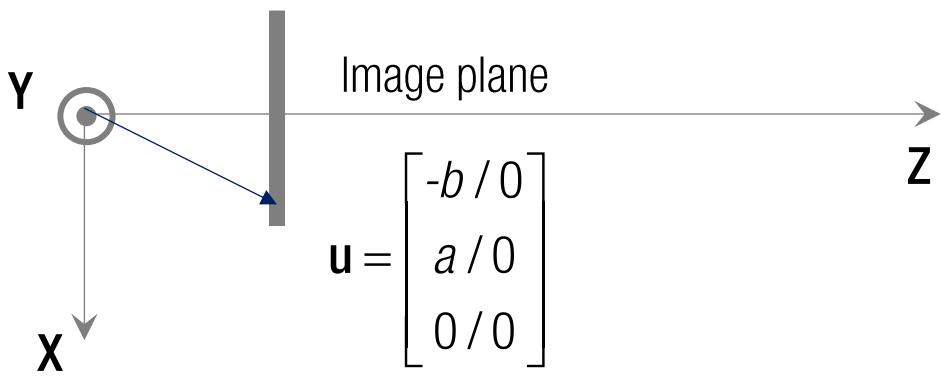
$$\mathbf{u} = \mathbf{l}_1 \times \mathbf{l}_2 = (c_2 - c_1) \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

This point does not correspond to a finite point because:

$$\mathbf{u} = \begin{bmatrix} -b / 0 \\ a / 0 \\ 0 / 0 \end{bmatrix}$$

Special Case: 2D Point at Infinity

Top view



The intersection of two parallel lines in an image:

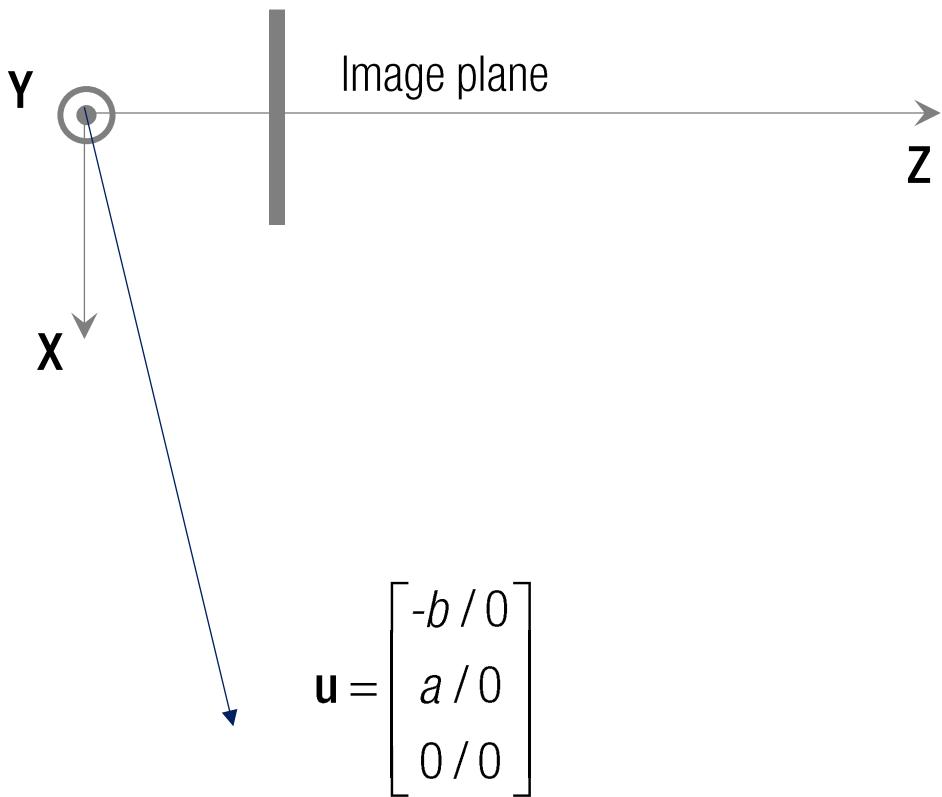
$$\mathbf{u} = \mathbf{l}_1 \times \mathbf{l}_2 = (c_2 - c_1) \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

This point does not correspond to a finite point because:

$$\mathbf{u} = \begin{bmatrix} -b/0 \\ a/0 \\ 0/0 \end{bmatrix}$$

Special Case: 2D Point at Infinity

Top view



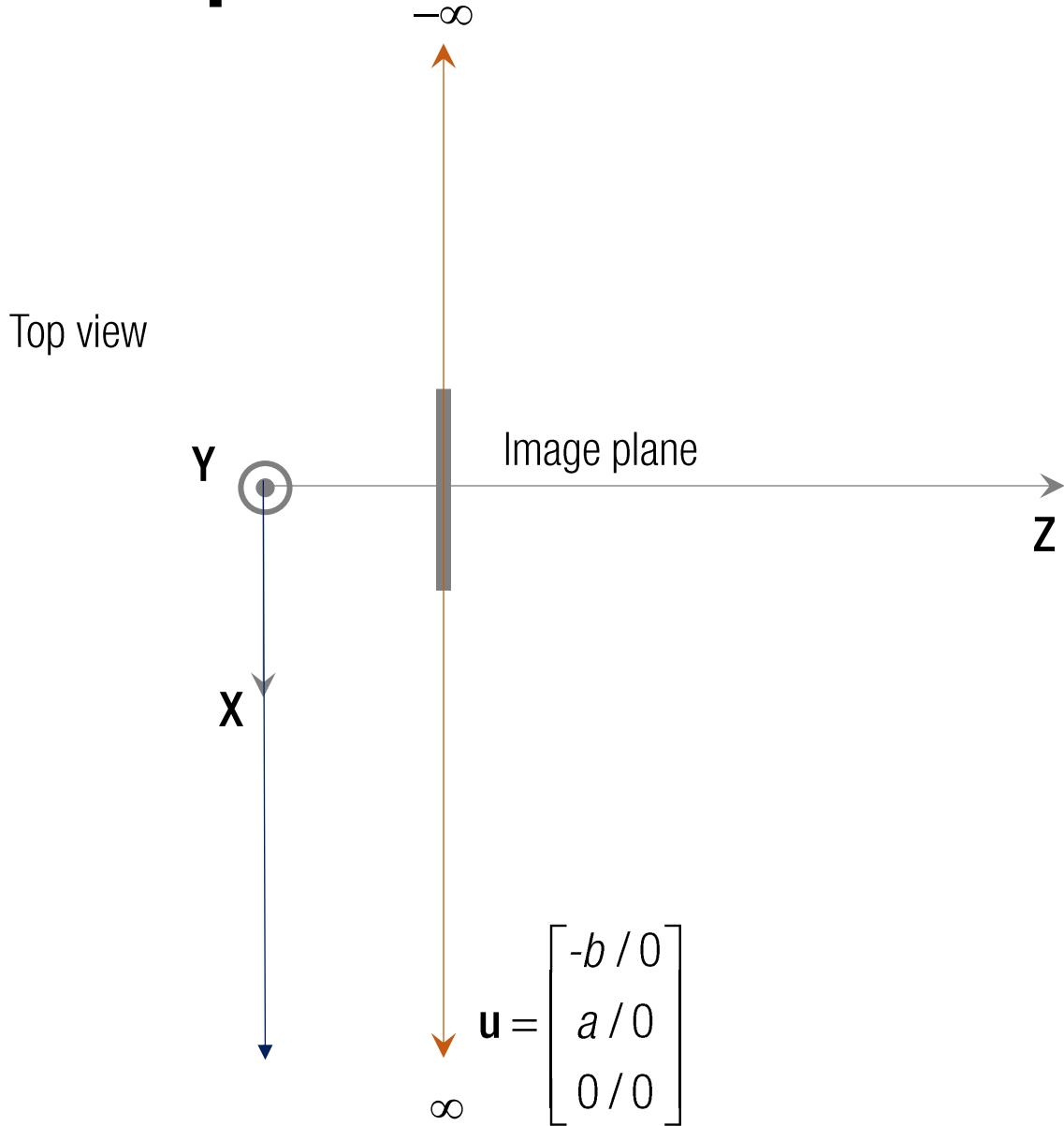
The intersection of two parallel lines in an image:

$$\mathbf{u} = \mathbf{l}_1 \times \mathbf{l}_2 = (c_2 - c_1) \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

This point does not correspond to a finite point because:

$$\mathbf{u} = \begin{bmatrix} -b/0 \\ a/0 \\ 0/0 \end{bmatrix}$$

Special Case: 2D Point at Infinity



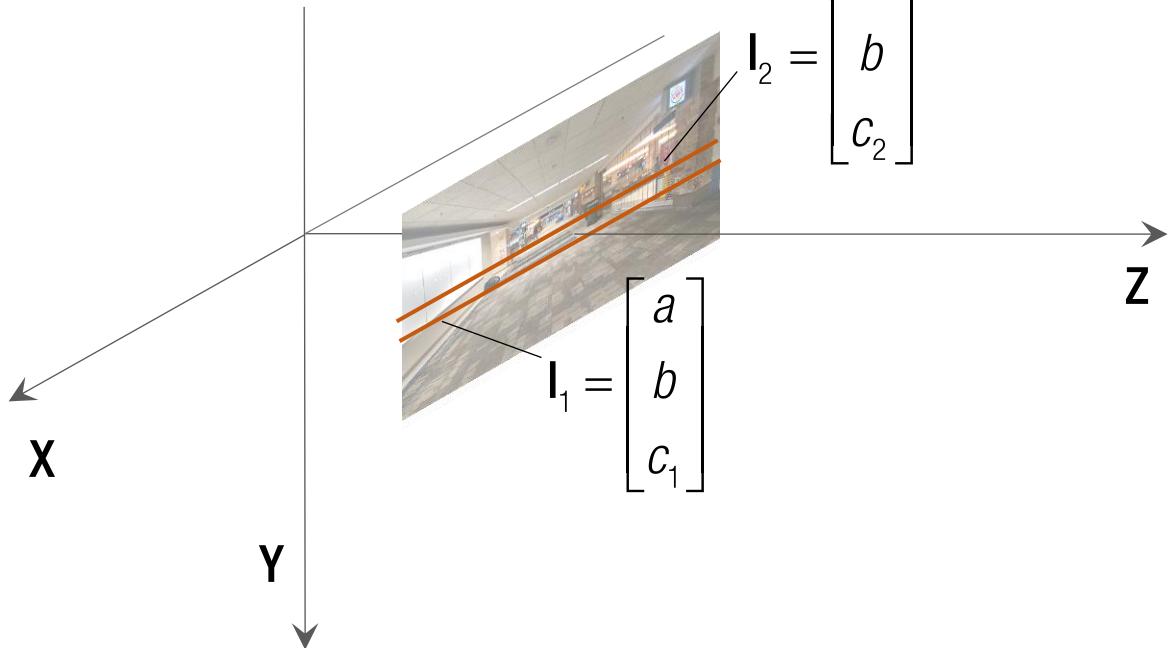
The intersection of two parallel lines in an image:

$$\mathbf{u} = \mathbf{l}_1 \times \mathbf{l}_2 = (c_2 - c_1) \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

This point does not correspond to a finite point because:

$$\mathbf{u} = \begin{bmatrix} -b/0 \\ a/0 \\ 0/0 \end{bmatrix}$$

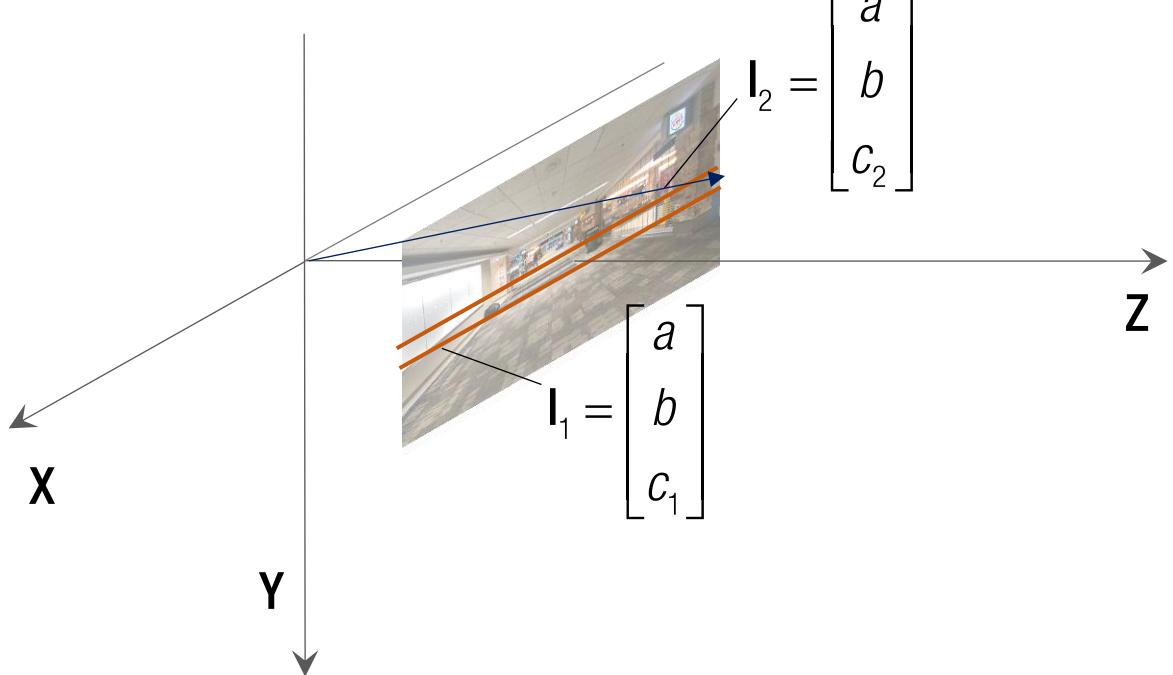
Point and Line at Infinity



Parallel lines intersect at the point at infinity:

$$\text{Point at infinity: } \mathbf{u}_\infty = \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

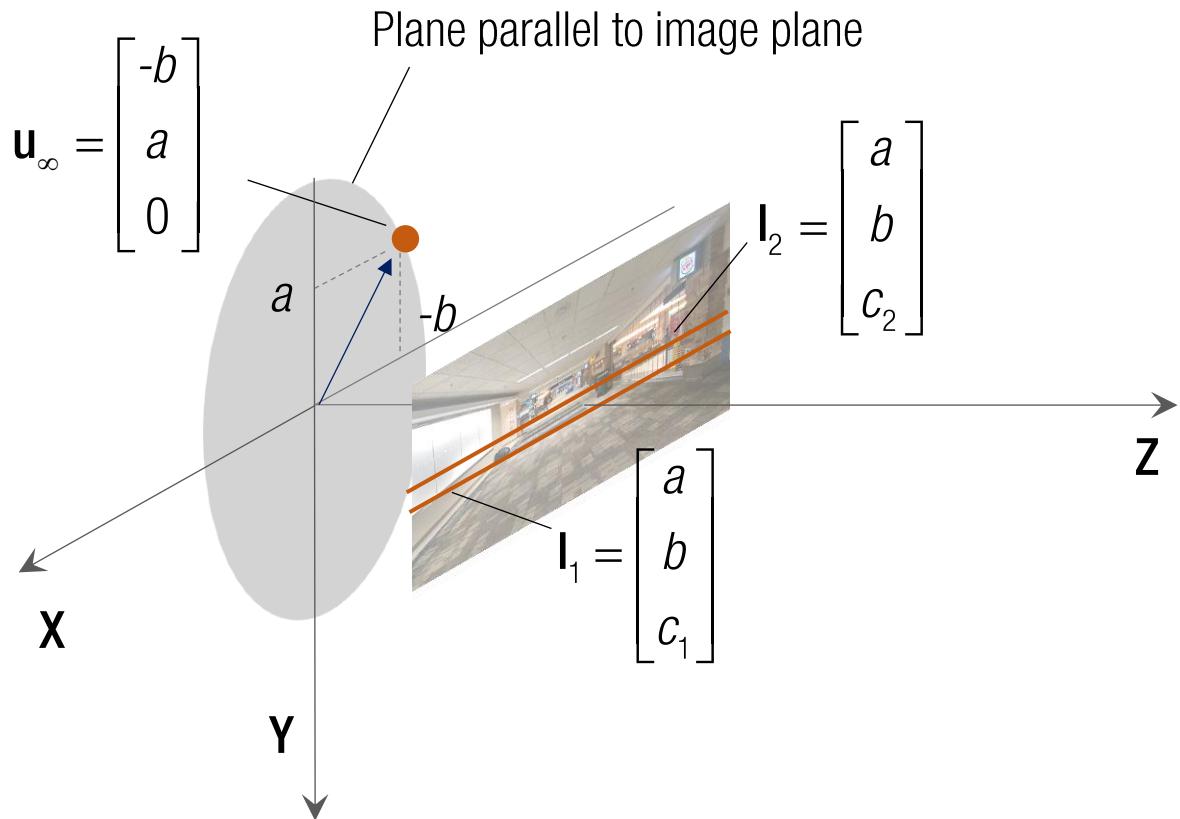
Point and Line at Infinity



Parallel lines intersect at the point at infinity:

$$\text{Point at infinity: } \mathbf{u}_{\infty} = \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

Point and Line at Infinity

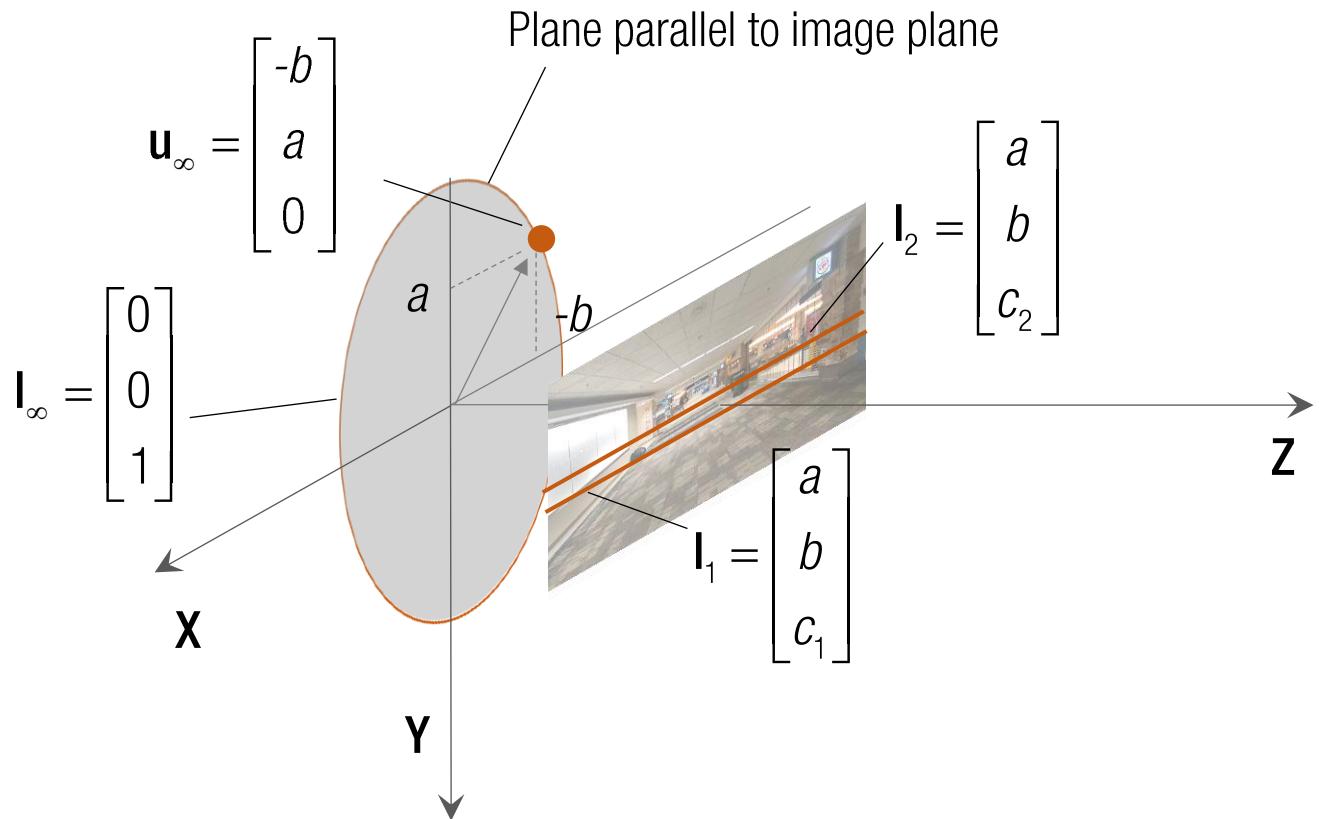


Parallel lines intersect at the point at infinity:

$$\text{Point at infinity: } \mathbf{u}_\infty = \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

A point at infinity corresponds to a direction.

Point and Line at Infinity



Parallel lines intersect at the point at infinity:

$$\text{Point at infinity: } \mathbf{u}_\infty = \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

A point at infinity corresponds to a direction.

All points at infinity lie in the line at infinity:

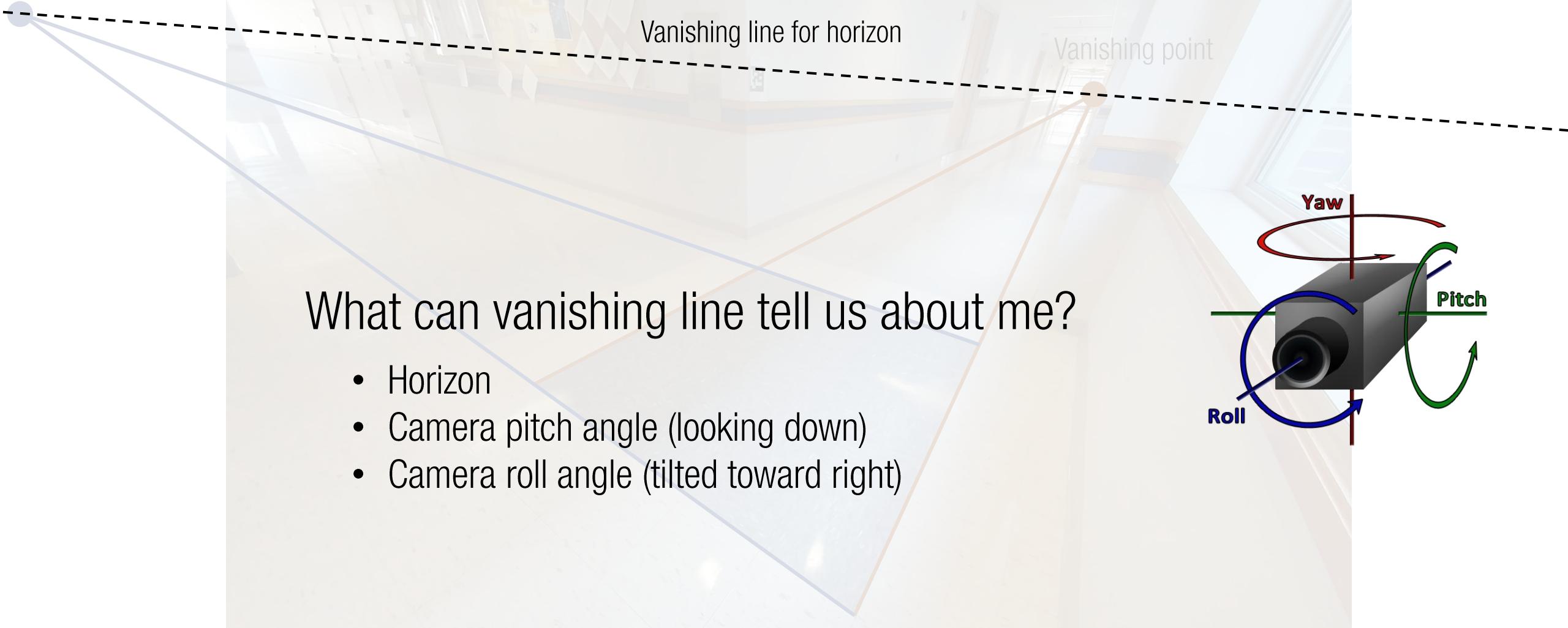
$$\mathbf{l}_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{Z} \quad \because \mathbf{u}_\infty^\top \mathbf{l}_\infty = 0$$

where the surface normal is parallel to the Z axis.

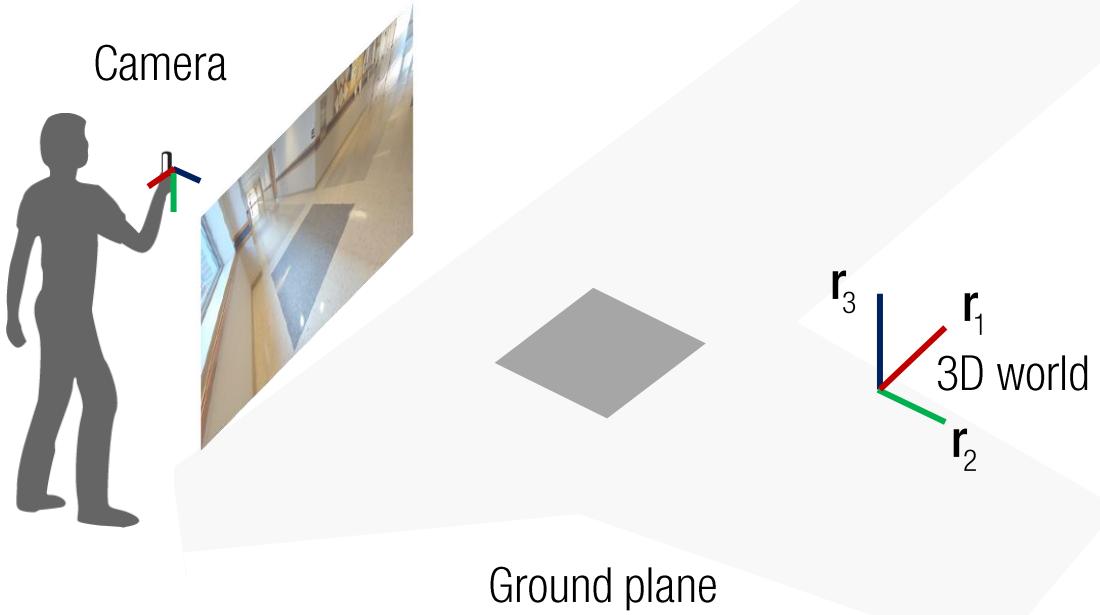
Where am I? Using Vanishing Points



Vanishing point



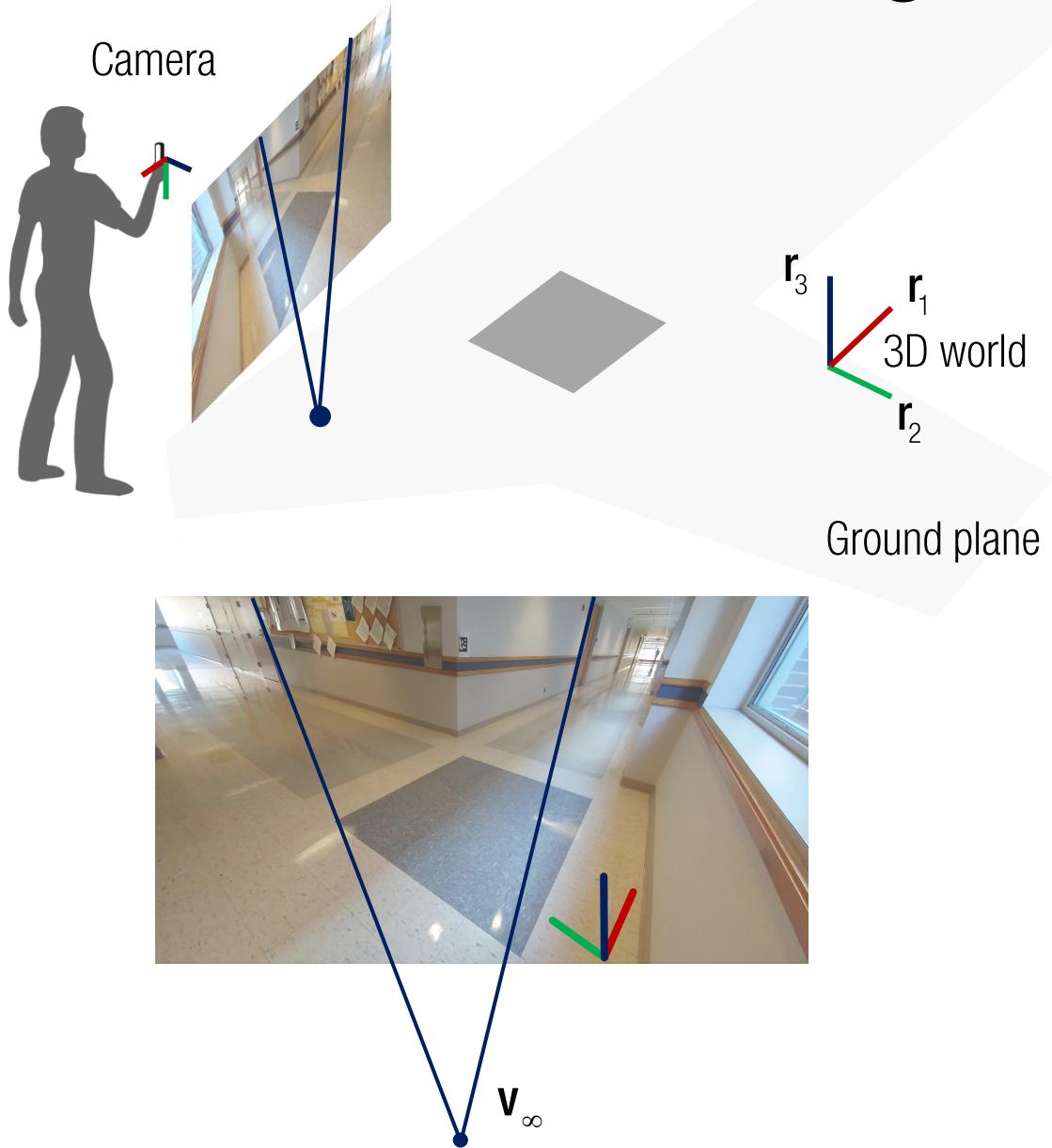
Where am I w.r.t. Ground Plane?



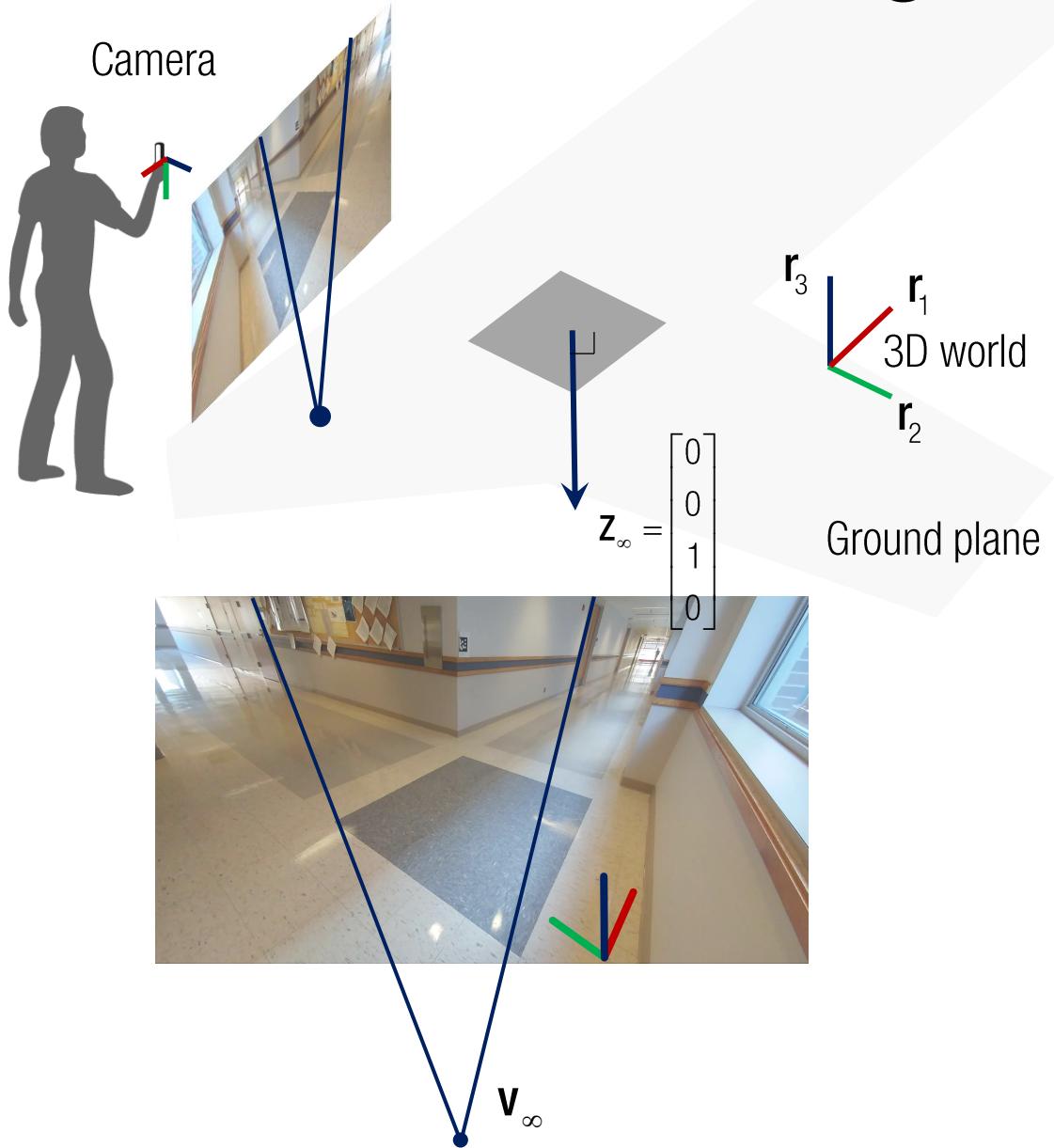
$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t}_w^c \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How to compute?

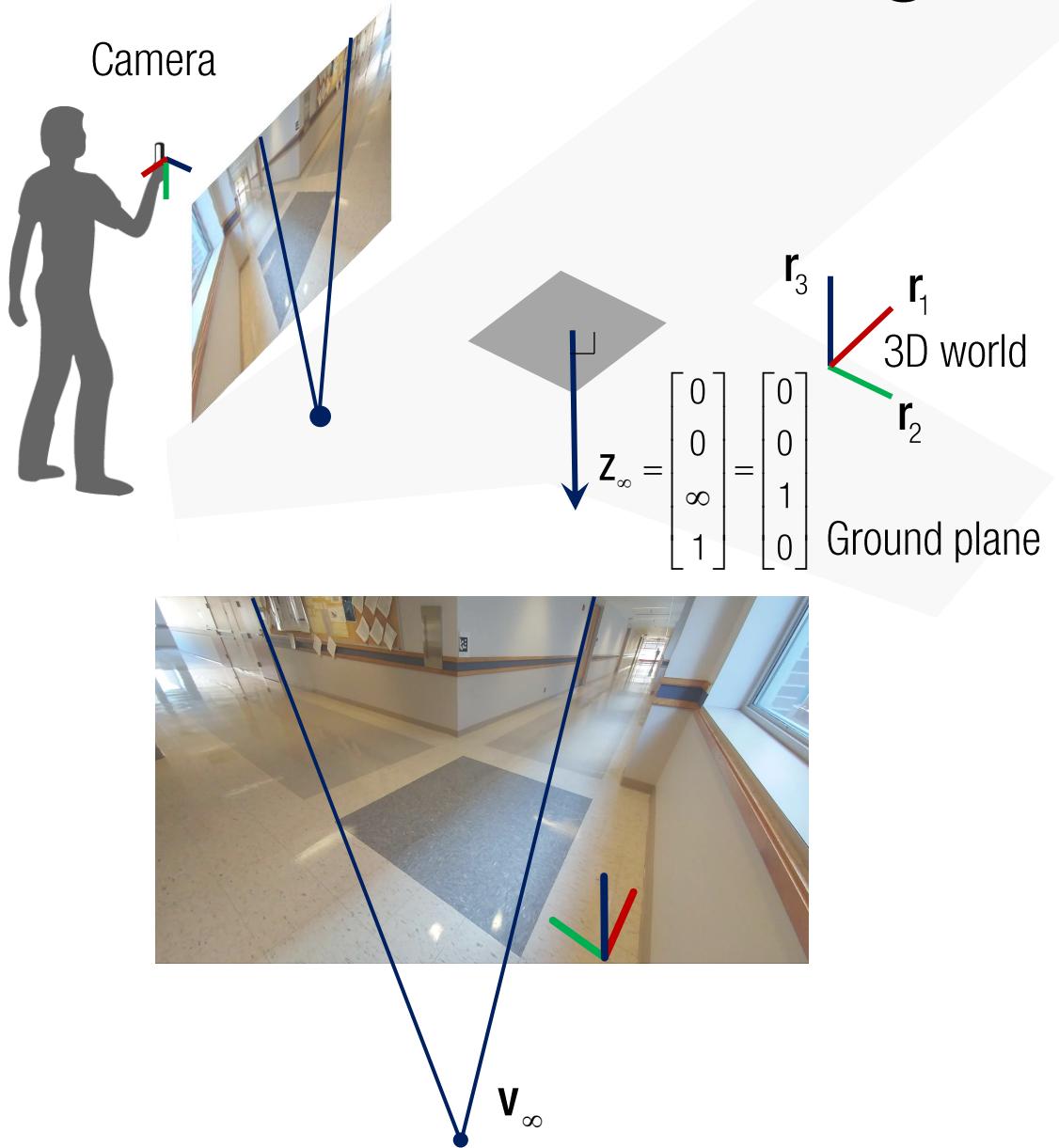
What can a Vanishing Point tell us about?



What can a Vanishing Point tell us about?

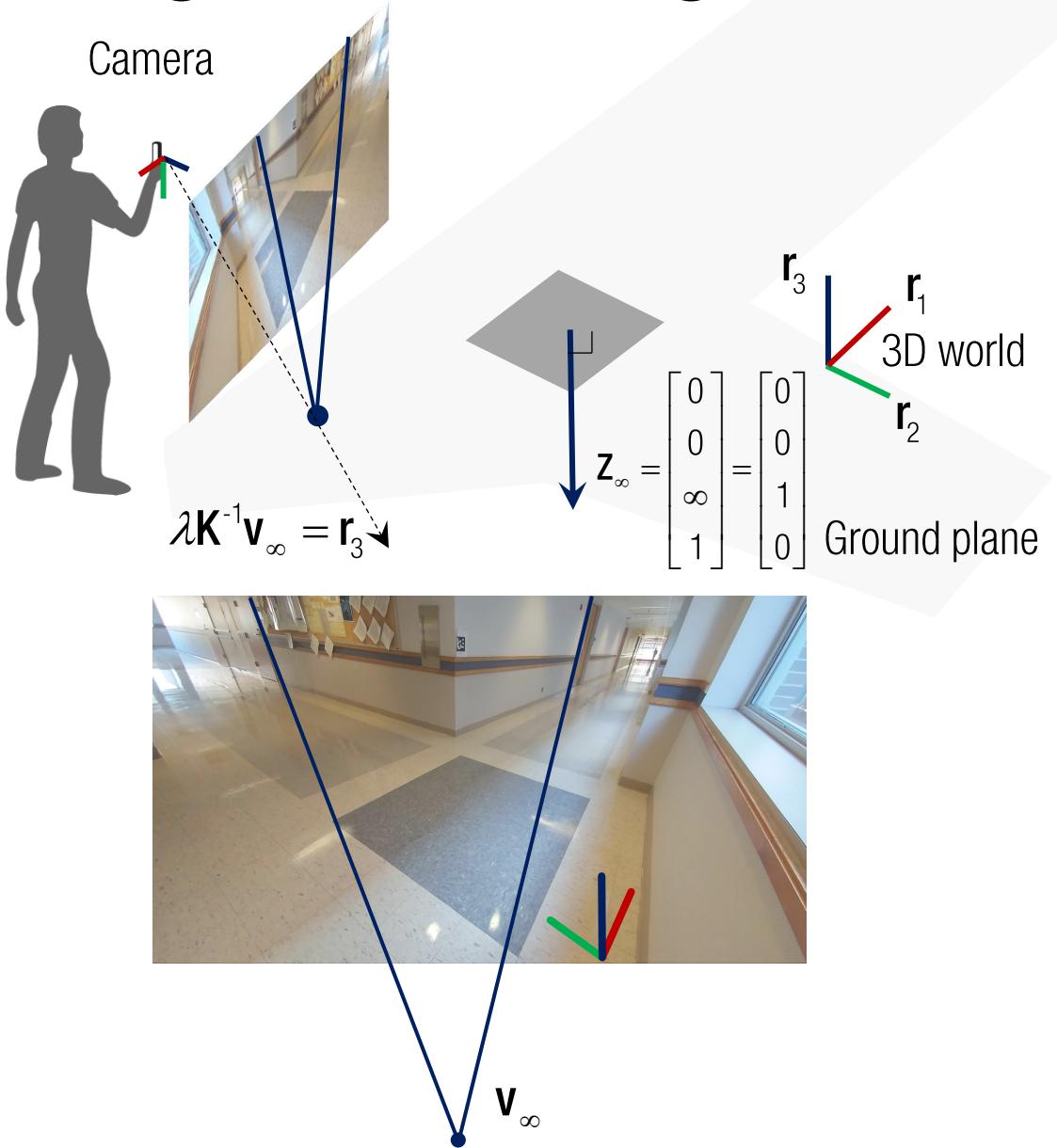


What can a Vanishing Point tell us about?



$$\lambda v_\infty = K \begin{bmatrix} r_1 & r_2 & r_3 & t_w^C \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

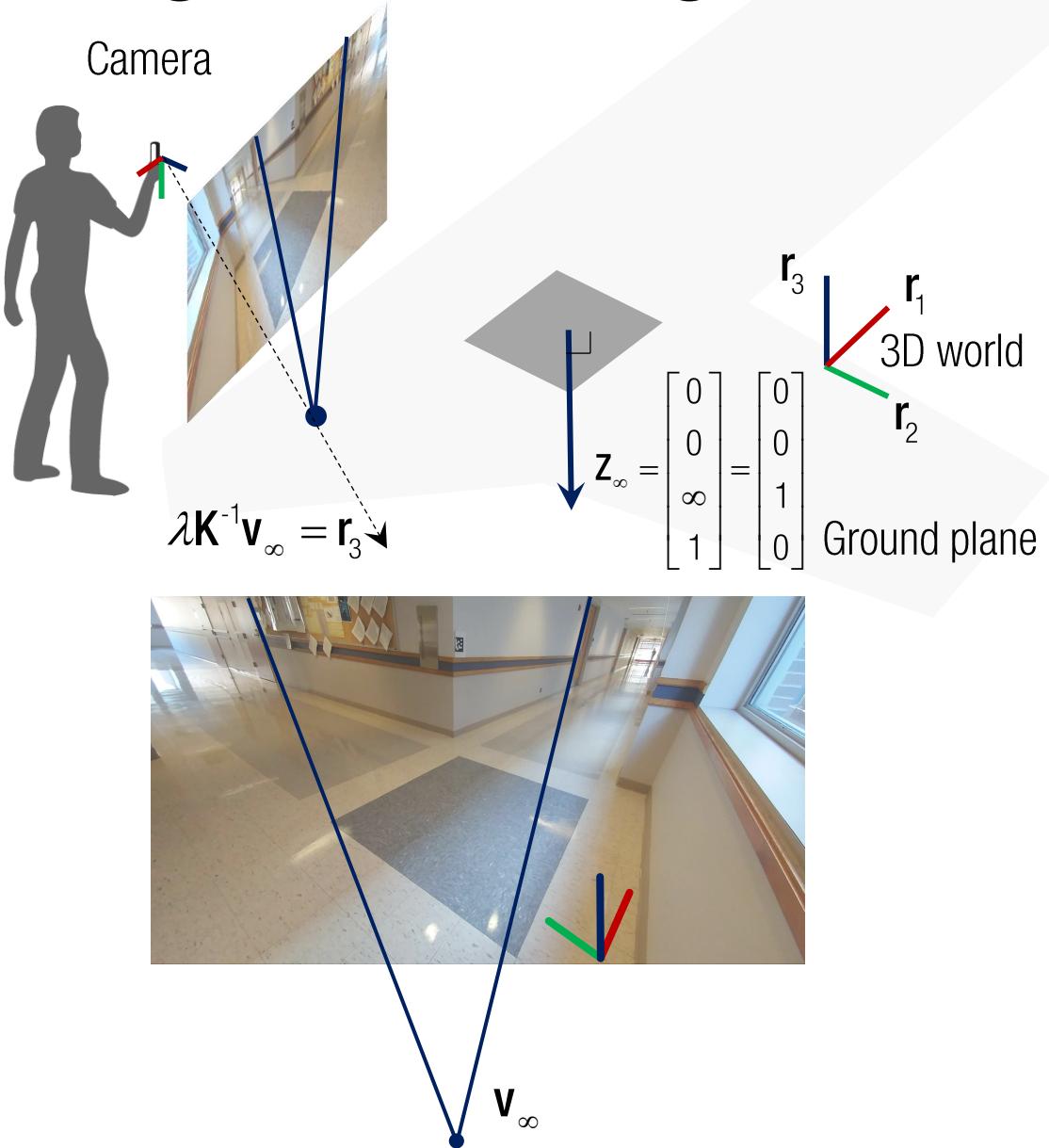
Single Vanishing Point



$$\lambda v_\infty = K \begin{bmatrix} r_1 & r_2 & r_3 & t_w^C \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda v_\infty = Kr_3$$

Single Vanishing Point



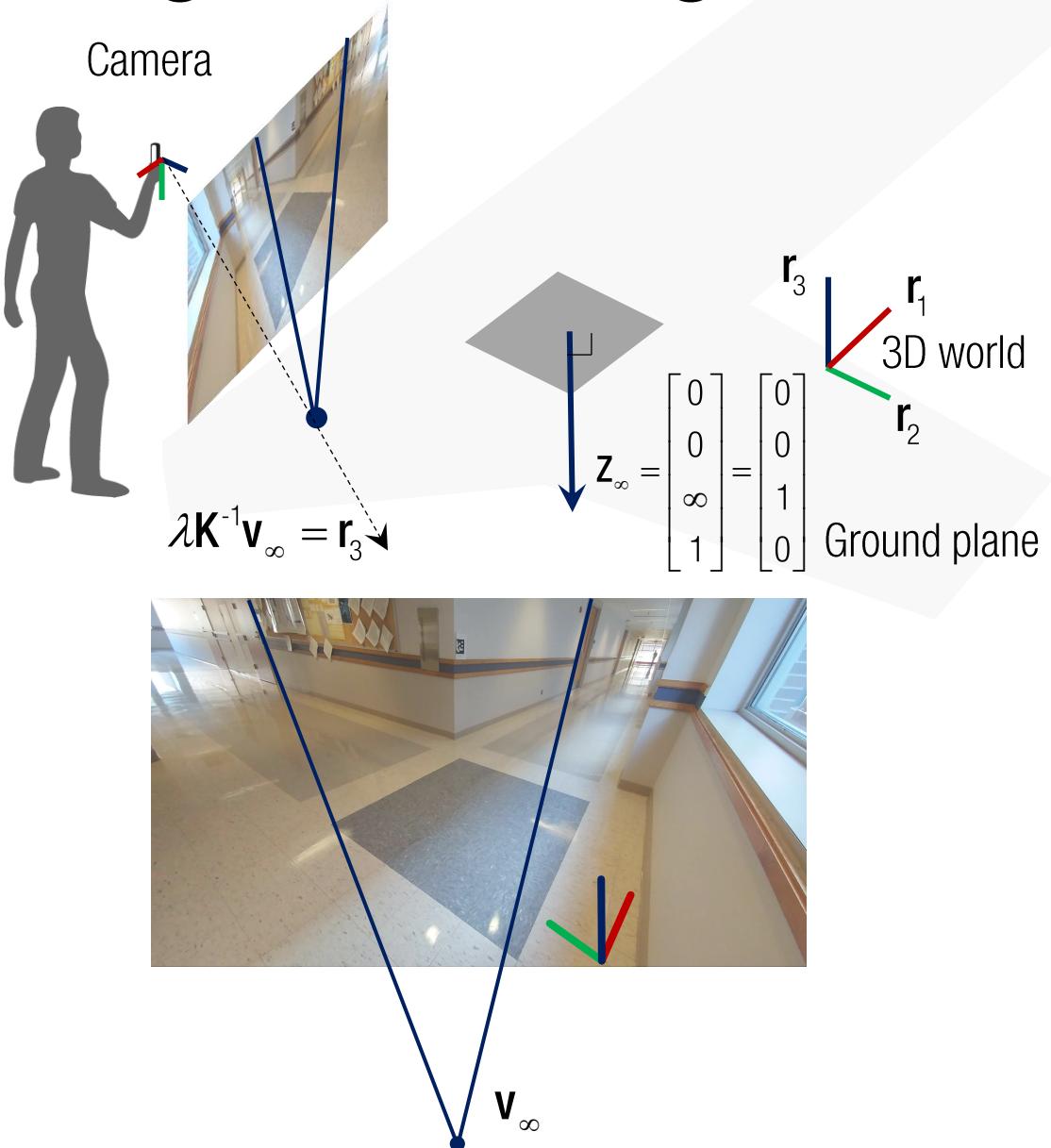
$$\lambda v_\infty = K \begin{bmatrix} r_1 & r_2 & r_3 & t_w^C \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda v_\infty = Kr_3$$

$$\rightarrow r_3 = \frac{K^{-1}v_\infty}{\|K^{-1}v_\infty\|} \quad \text{because } r_3 \text{ is a unit vector.}$$

Z vanishing point tells us about the surface normal of the ground plane

Single Vanishing Point



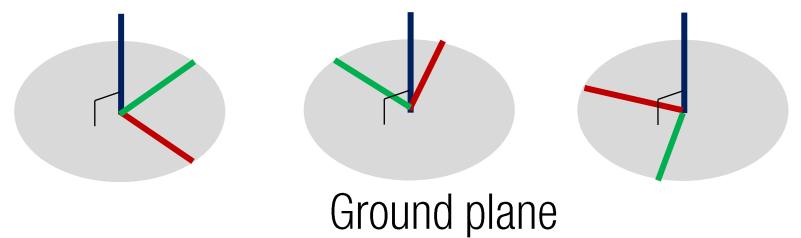
$$\lambda \mathbf{v}_\infty = \mathbf{K} \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t}_w^C \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda \mathbf{v}_\infty = \mathbf{K} \mathbf{r}_3$$

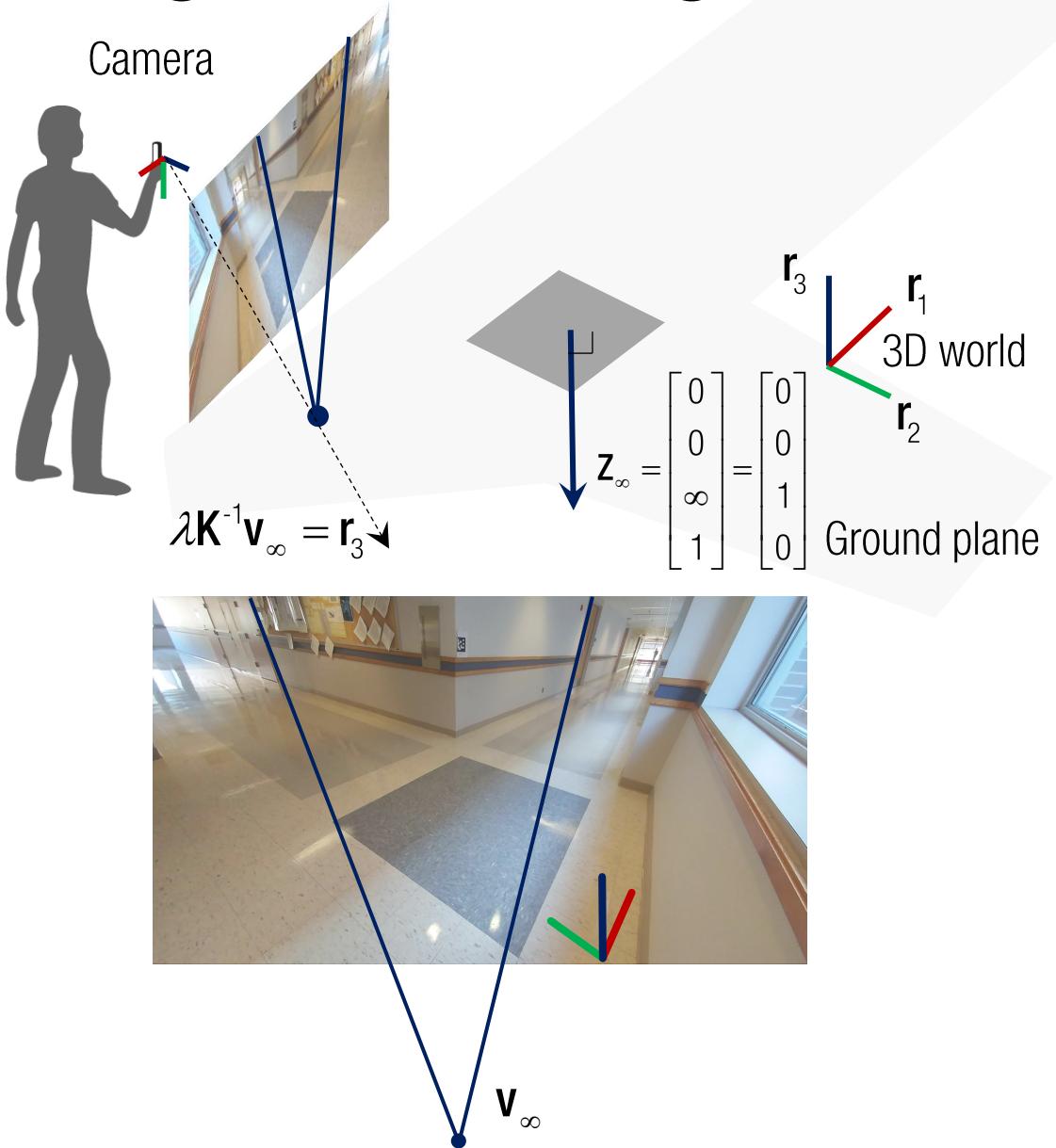
$$\rightarrow \mathbf{r}_3 = \frac{\mathbf{K}^{-1} \mathbf{v}_\infty}{\|\mathbf{K}^{-1} \mathbf{v}_\infty\|} \quad \text{because } \mathbf{r}_3 \text{ is a unit vector.}$$

Z vanishing point tells us about the surface normal of the ground plane

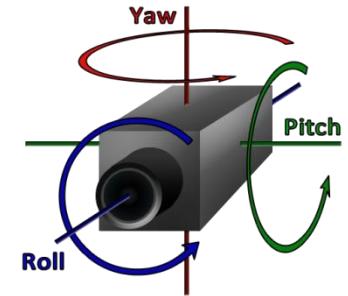
Rotation ambiguity



Single Vanishing Point

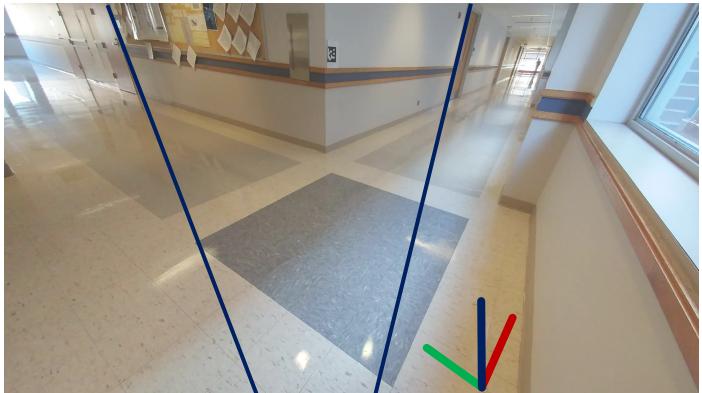
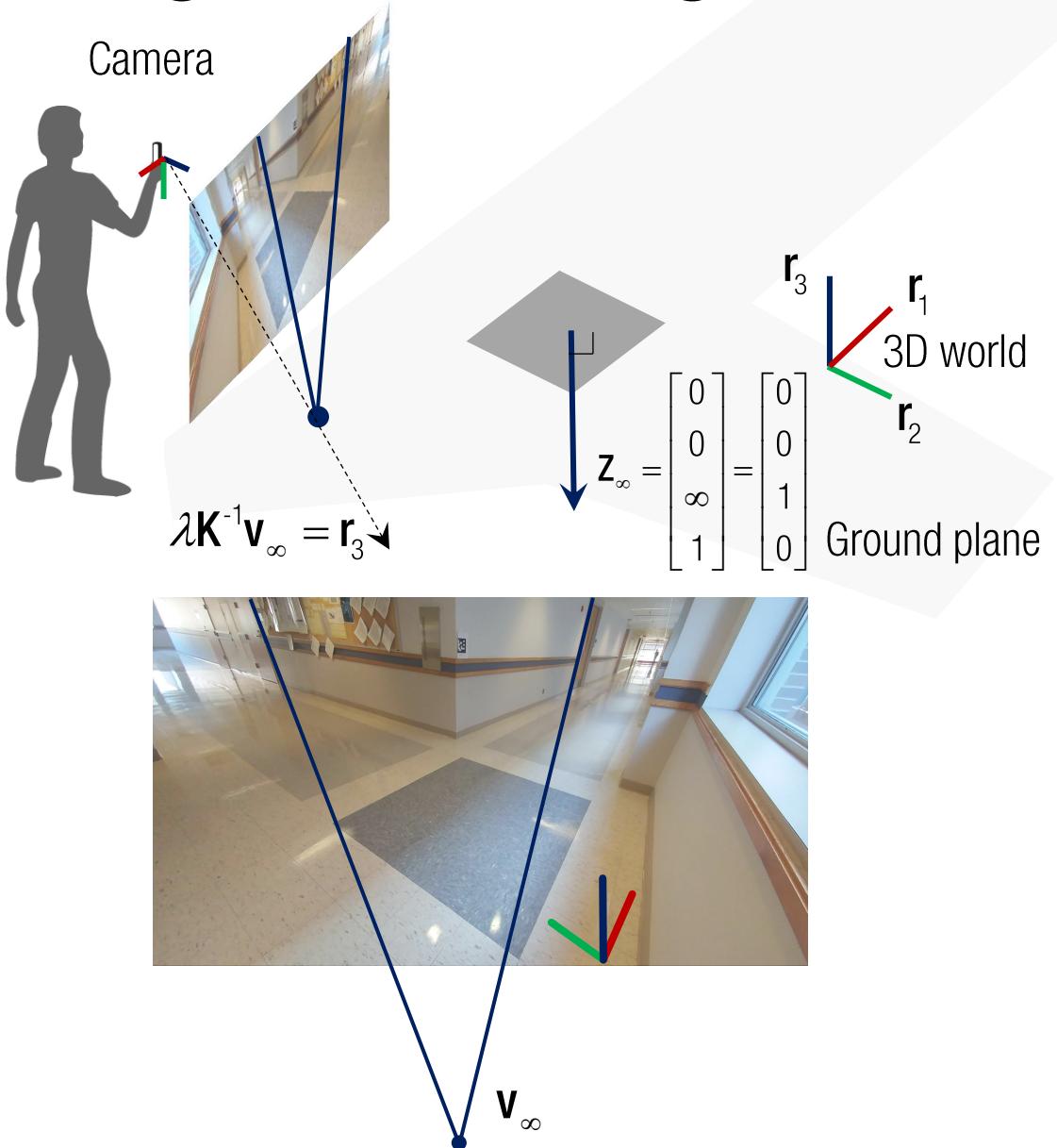


$$\mathbf{r}_3 = \frac{\mathbf{K}^{-1}\mathbf{v}_\infty}{\|\mathbf{K}^{-1}\mathbf{v}_\infty\|}$$

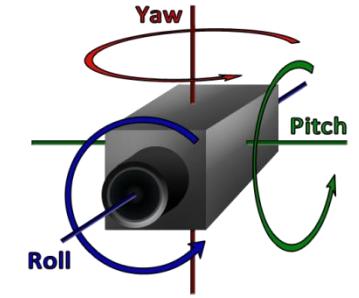


Roll and pitch angle can be computed by the ground plane.

Single Vanishing Point



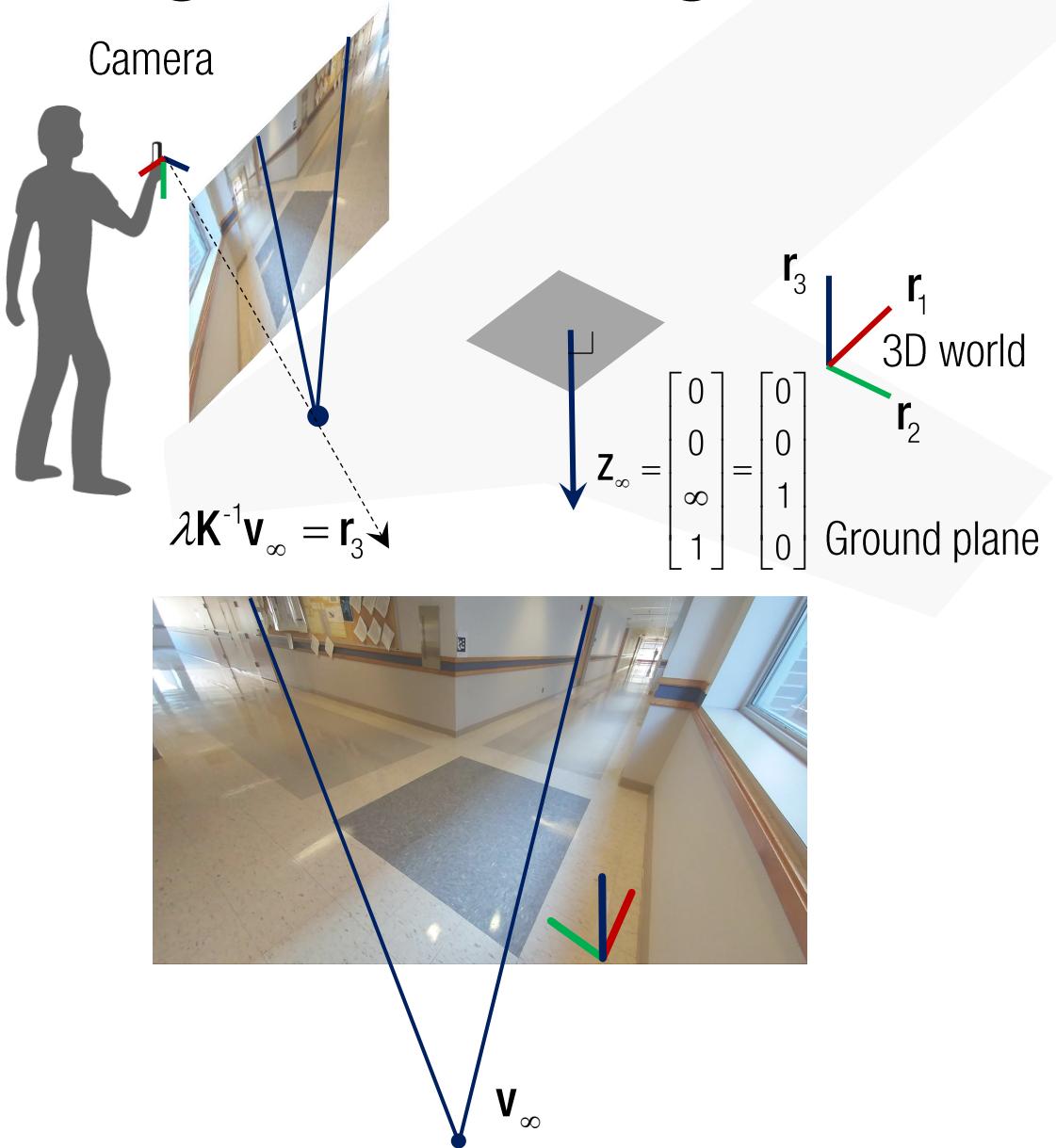
$$r_3 = \frac{K^{-1}v_\infty}{\|K^{-1}v_\infty\|}$$



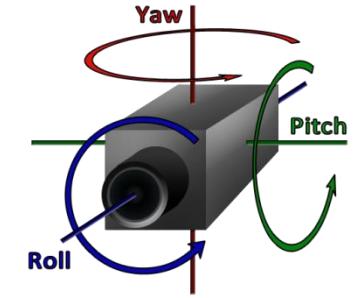
Roll and pitch angle can be computed by the ground plane.

Yaw: rotation about z axis: $R_{yaw} = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Single Vanishing Point



$$\mathbf{r}_3 = \frac{\mathbf{K}^{-1} \mathbf{v}_\infty}{\|\mathbf{K}^{-1} \mathbf{v}_\infty\|}$$

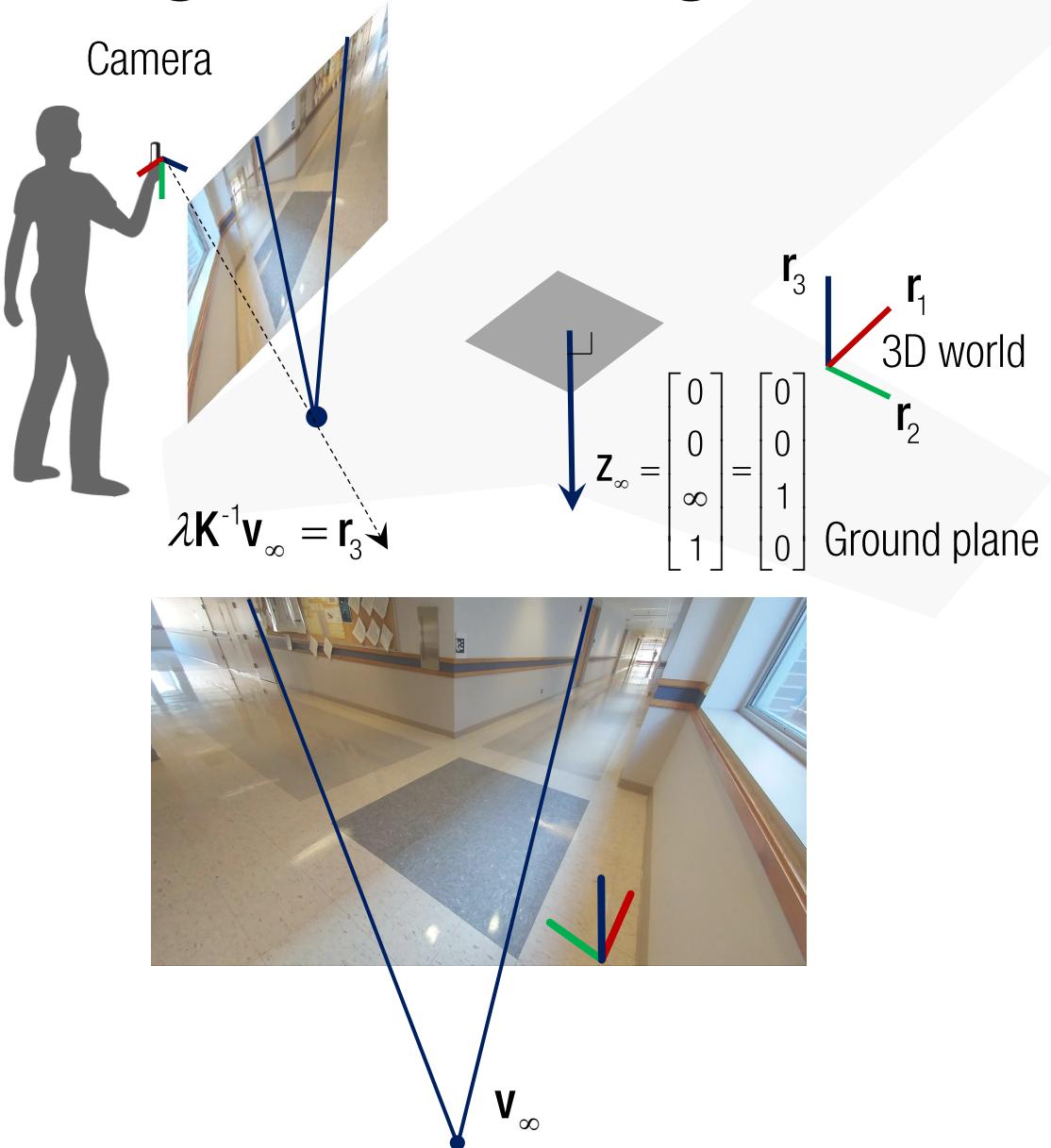


Roll and pitch angle can be computed by the ground plane.

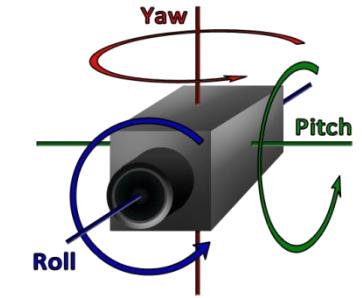
Yaw: rotation about z axis: $\mathbf{R}_{yaw} = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Pitch: rotation about y axis: $\mathbf{R}_{pitch} = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$

Single Vanishing Point



$$\mathbf{r}_3 = \frac{\mathbf{K}^{-1} \mathbf{v}_\infty}{\|\mathbf{K}^{-1} \mathbf{v}_\infty\|}$$



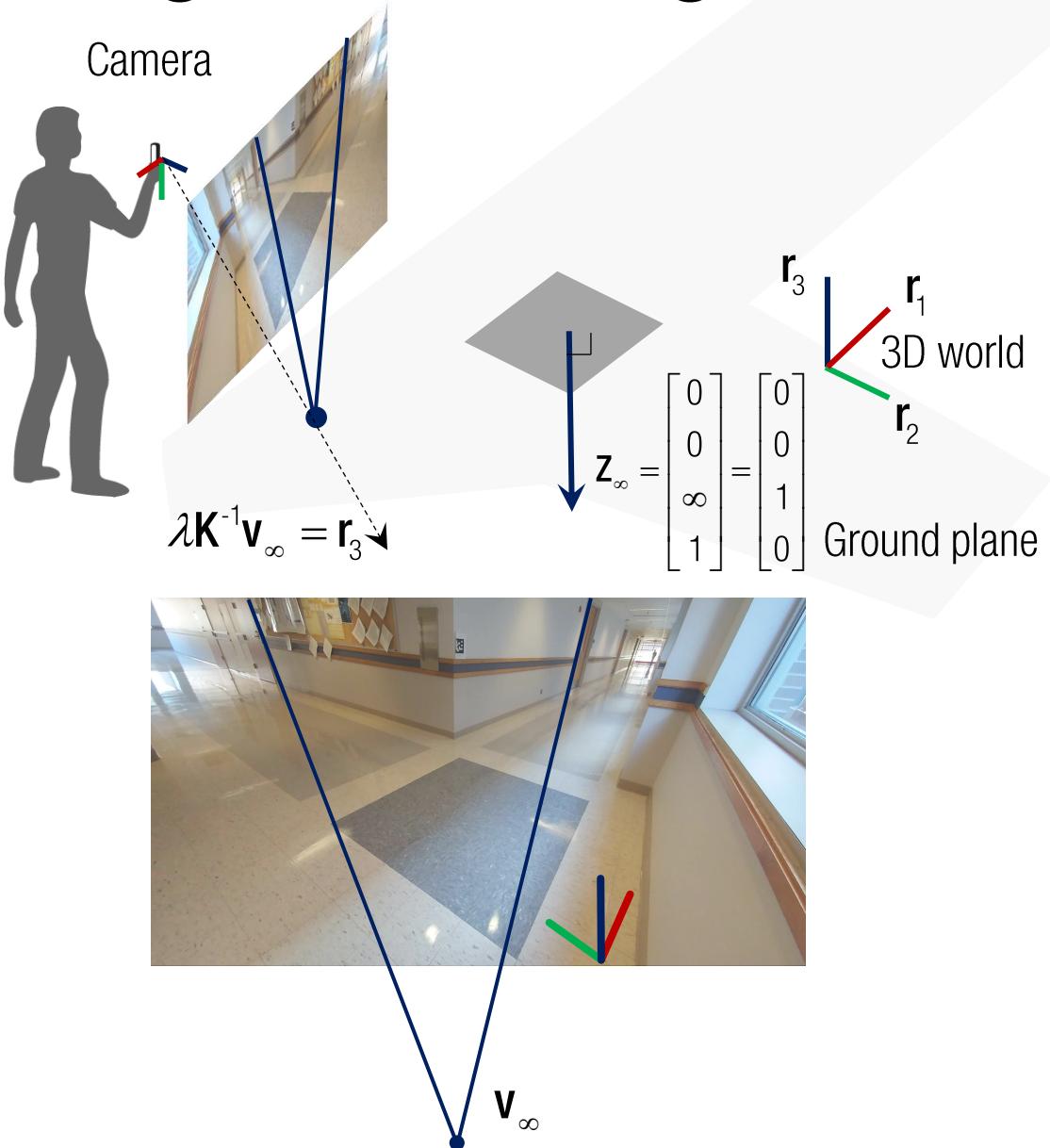
Roll and pitch angle can be computed by the ground plane.

Yaw: rotation about z axis: $\mathbf{R}_{yaw} = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

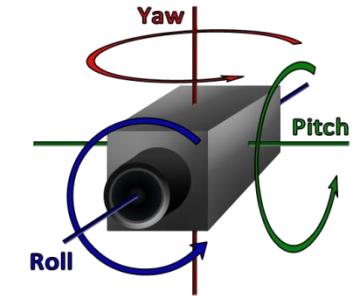
Pitch: rotation about y axis: $\mathbf{R}_{pitch} = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$

Roll: rotation about x axis: $\mathbf{R}_{roll} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & -\sin \gamma \\ 0 & \sin \gamma & \cos \gamma \end{bmatrix}$

Single Vanishing Point



$$r_3 = \frac{K^{-1}v_\infty}{\|K^{-1}v_\infty\|}$$



Roll and pitch angle can be computed by the ground plane.

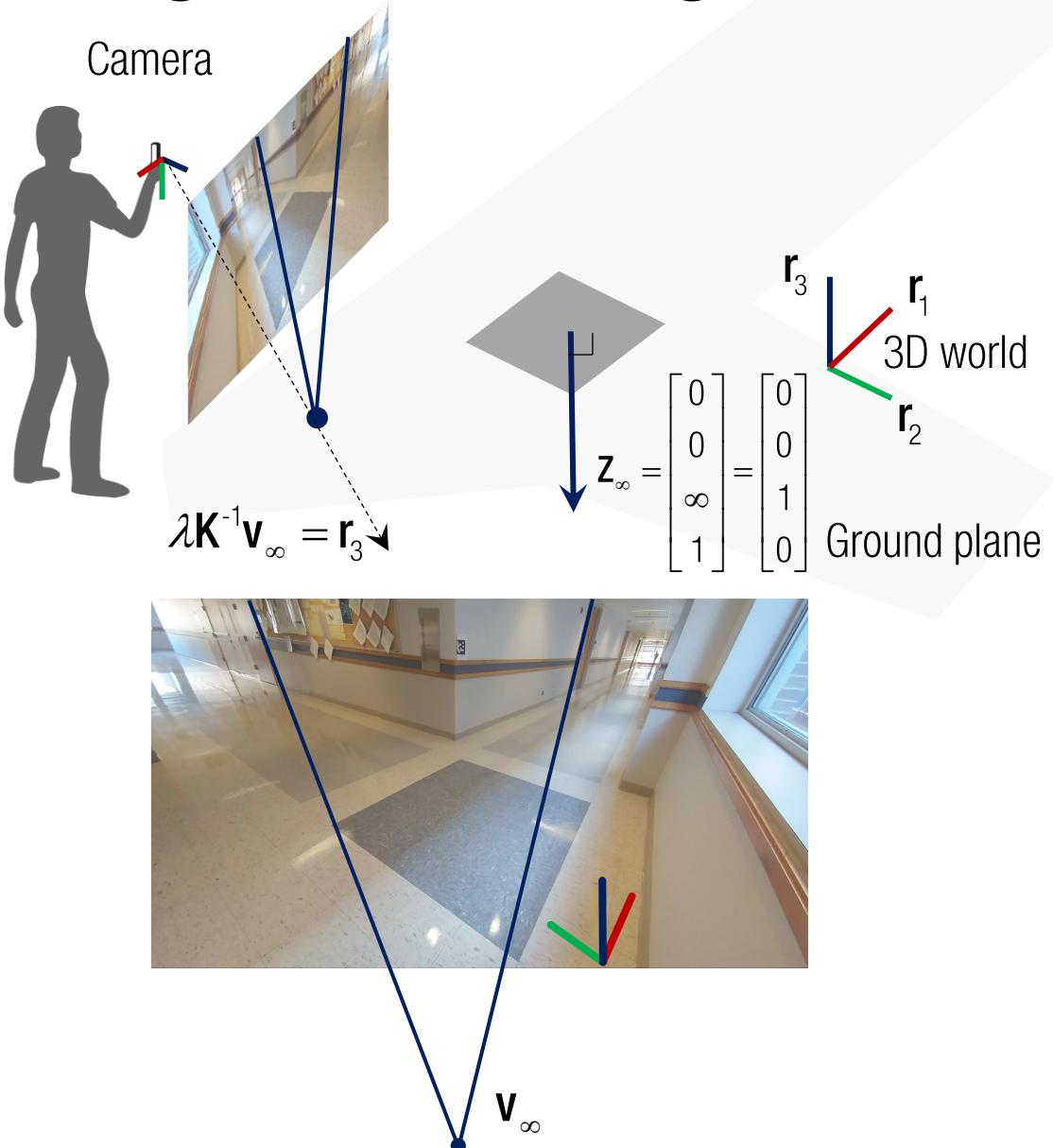
$$\text{Yaw: rotation about z axis: } R_{\text{yaw}} = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Pitch: rotation about y axis: } R_{\text{pitch}} = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

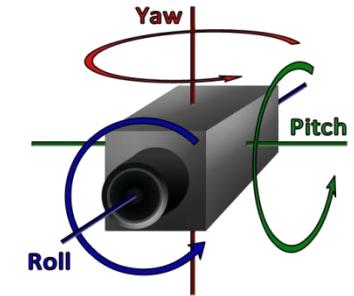
$$\text{Roll: rotation about x axis: } R_{\text{roll}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & -\sin \gamma \\ 0 & \sin \gamma & \cos \gamma \end{bmatrix}$$

$$\longrightarrow [r_1 \ r_2 \ r_3] = (R_{\text{yaw}} R_{\text{pitch}} R_{\text{roll}})^T = \begin{bmatrix} \bullet & \bullet & -\sin \beta \\ \bullet & \bullet & \cos \beta \sin \gamma \\ \bullet & \bullet & \cos \beta \cos \gamma \end{bmatrix}$$

Single Vanishing Point



$$\mathbf{r}_3 = \frac{\mathbf{K}^{-1} \mathbf{v}_\infty}{\|\mathbf{K}^{-1} \mathbf{v}_\infty\|}$$



Roll and pitch angle can be computed by the ground plane.

$$[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3] = (\mathbf{R}_{\text{yaw}} \mathbf{R}_{\text{pitch}} \mathbf{R}_{\text{roll}})^T = \begin{bmatrix} \bullet & \bullet & -\sin \beta \\ \bullet & \bullet & \cos \beta \sin \gamma \\ \bullet & \bullet & \cos \beta \cos \gamma \end{bmatrix}$$

Pitch: $\beta = \tan^{-1} \left(-\frac{\mathbf{r}_{31}}{\sqrt{\mathbf{r}_{32}^2 + \mathbf{r}_{33}^2}} \right)$

Roll: $\gamma = \tan^{-1} \frac{\mathbf{r}_{32}}{\mathbf{r}_{33}}$

Single Vanishing Point (Exercise)

ComputeCameraUsingVanishingPoint.m

```
f = 1224;  
K = [f 0 size(im,2)/2;  
      0 f size(im,1)/2;  
      0 0 1];
```

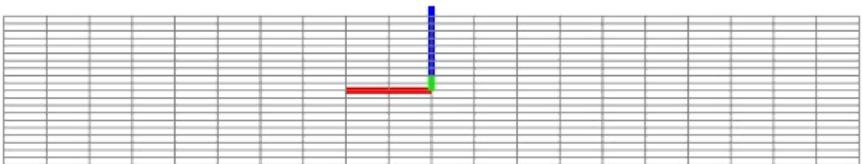
```
m1 = [2563;25;1];  
m2 = [2439;545;1];  
m3 = [571;25;1];  
m4 = [723;498;1];
```

```
l1 = GetLineFromTwoPoints(m1,m2);  
l2 = GetLineFromTwoPoints(m3,m4);
```

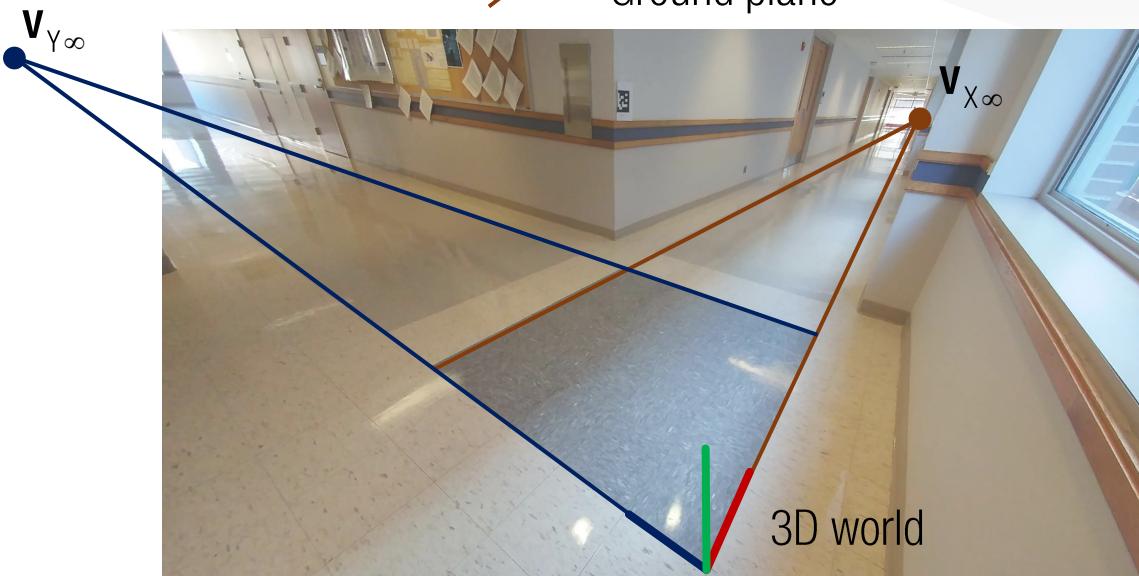
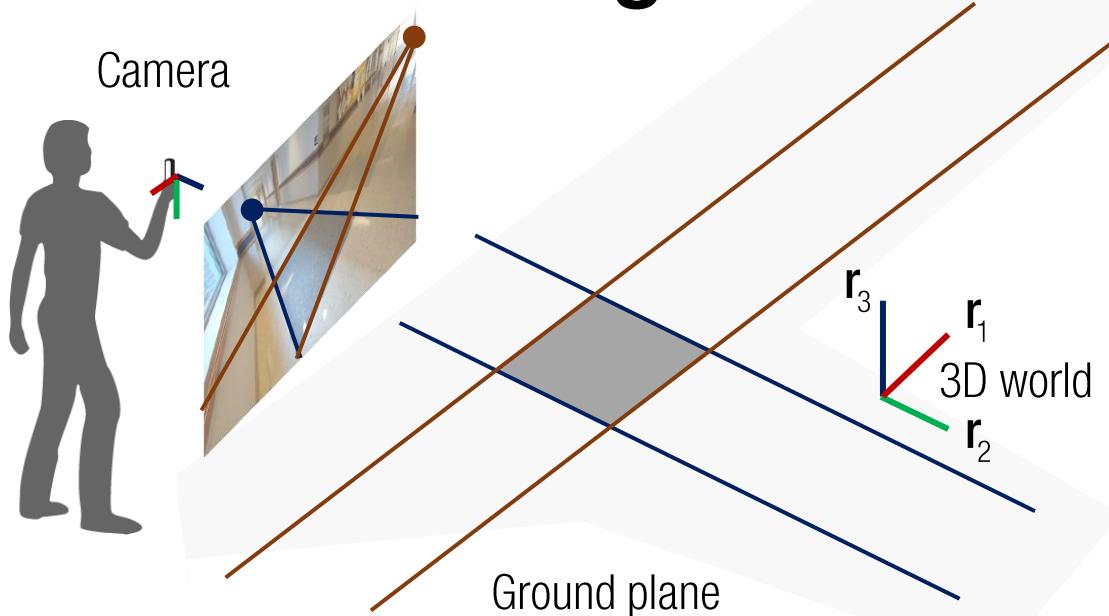
```
v1 = GetPointFromTwoLines(l1,l2);  
v1 = v1/v1(3);
```

```
r3 = inv(K)*v1/norm(inv(K)*v1)  
pitch = atan(-r3(1)/norm(r3(2:3)))  
roll = atan(r3(2)/r3(3))
```

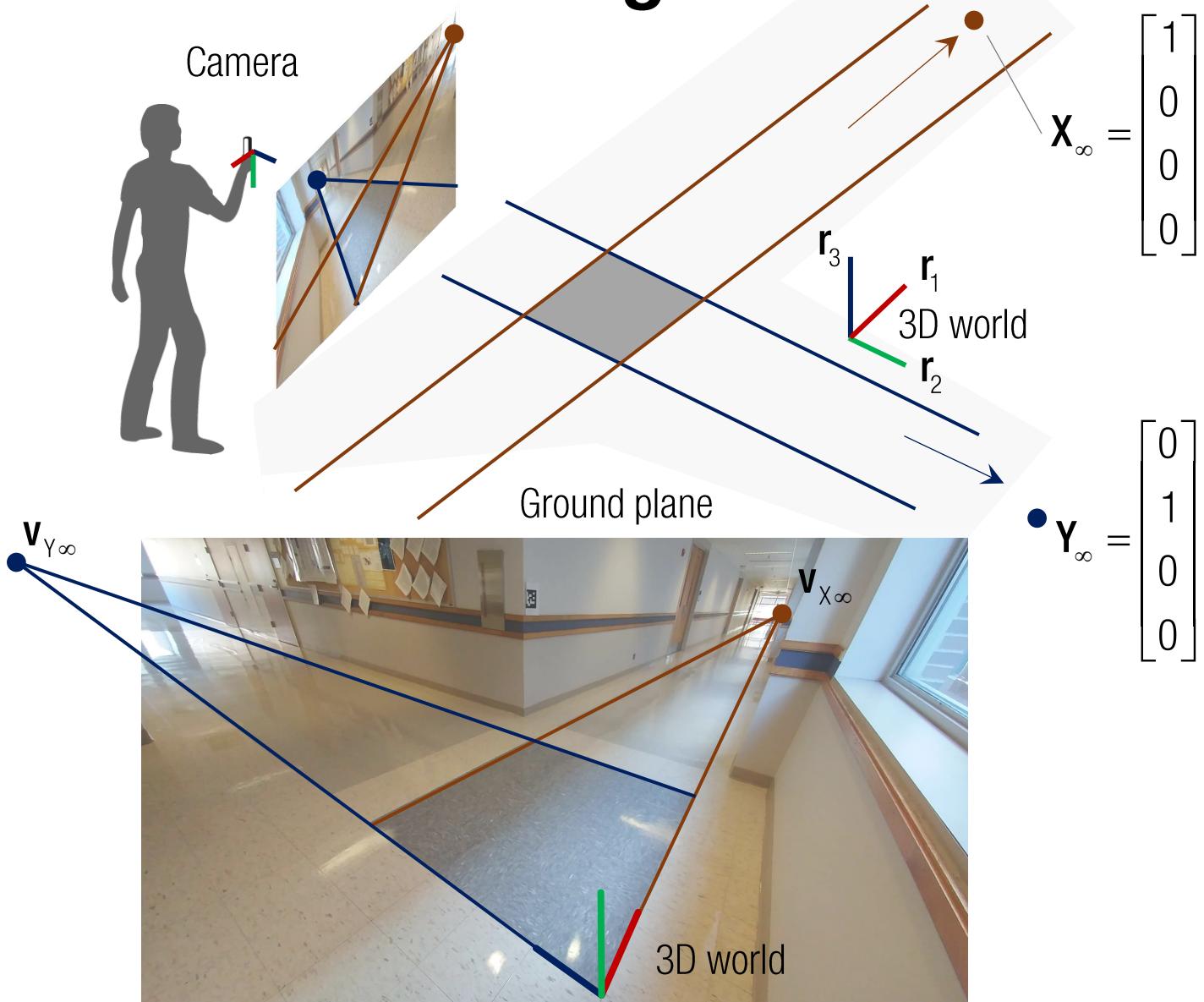
```
r3 =  
-0.0736  
0.8959  
0.4381  
  
pitch =  
0.0736  
  
roll =  
1.1160
```



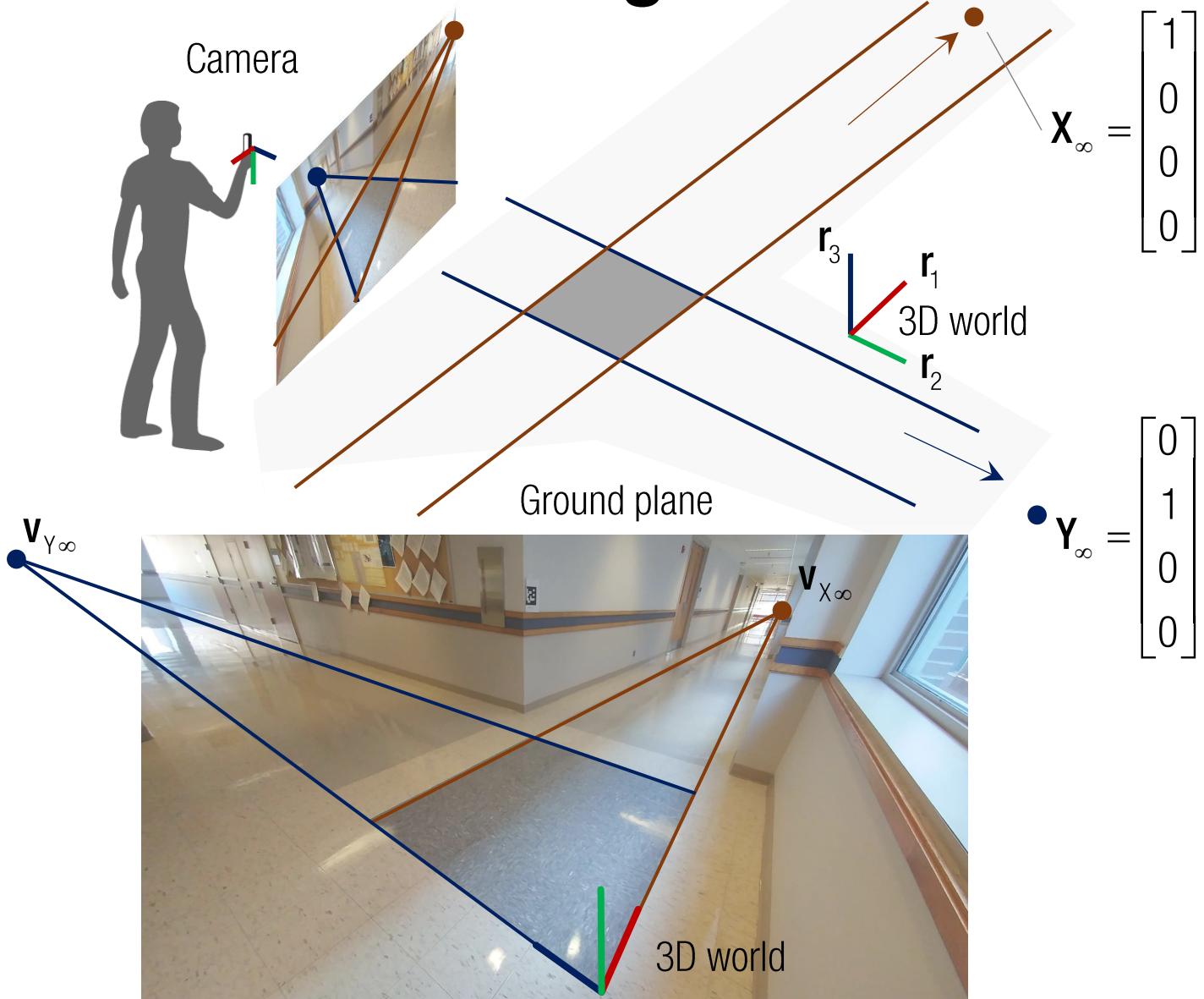
Two Vanishing Points



Two Vanishing Points



Two Vanishing Points

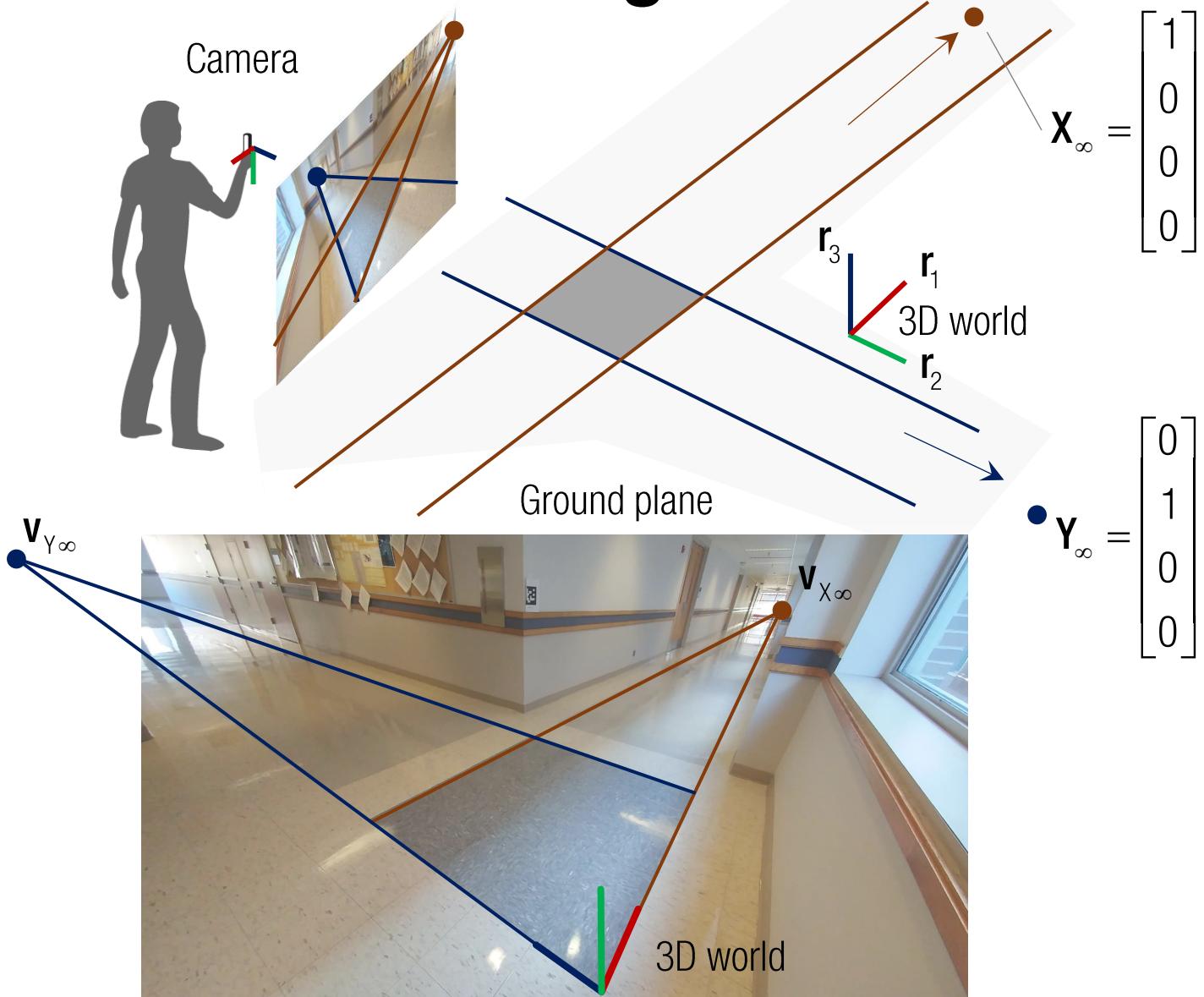


$$x_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_\infty = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}\lambda v_{x\infty} &= K \begin{bmatrix} r_1 & r_2 & r_3 & t_w^c \end{bmatrix} x_\infty \\ &= Kr_1\end{aligned}$$

Two Vanishing Points



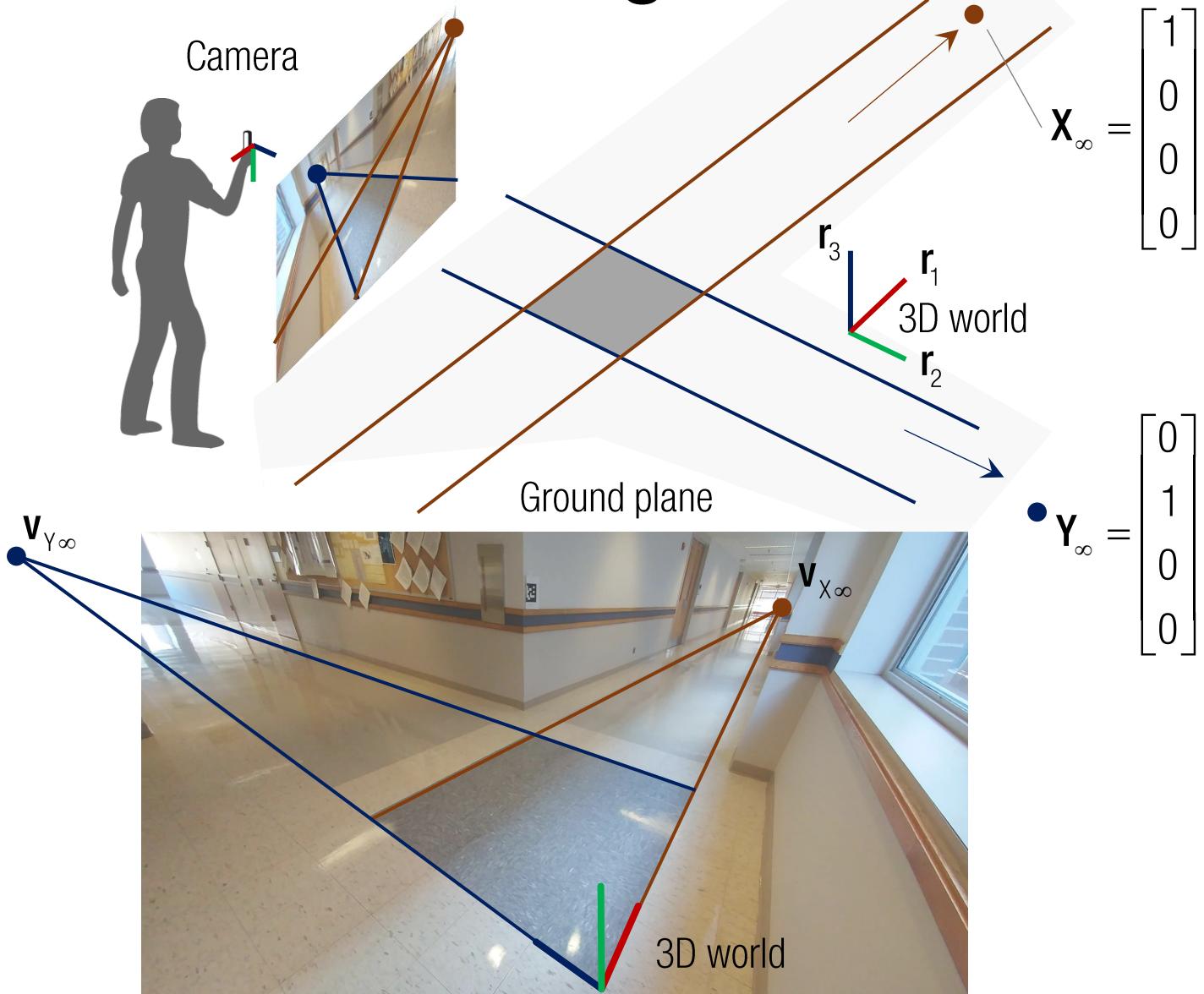
$$x_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_\infty = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda v_{X\infty} = K \begin{bmatrix} r_1 & r_2 & r_3 & t_w^c \end{bmatrix} x_\infty \\ = Kr_1$$

$$\lambda v_{Y\infty} = K \begin{bmatrix} r_1 & r_2 & r_3 & t_w^c \end{bmatrix} y_\infty \\ = Kr_2$$

Two Vanishing Points



$$x_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_\infty = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

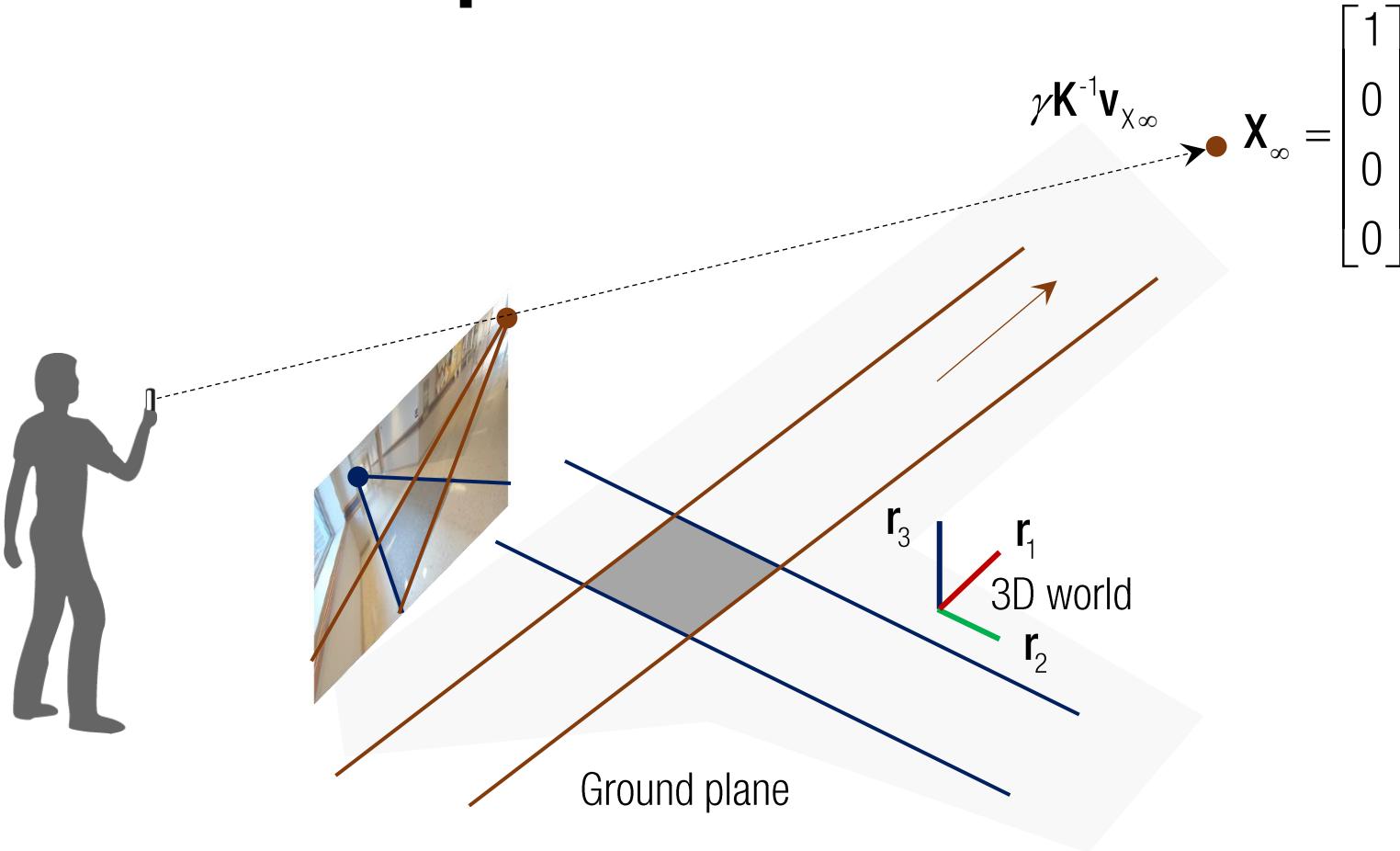
$$\lambda v_{X_\infty} = K \begin{bmatrix} r_1 & r_2 & r_3 & t_w^C \end{bmatrix} x_\infty \\ = Kr_1$$

$$\lambda v_{Y_\infty} = K \begin{bmatrix} r_1 & r_2 & r_3 & t_w^C \end{bmatrix} y_\infty \\ = Kr_2$$

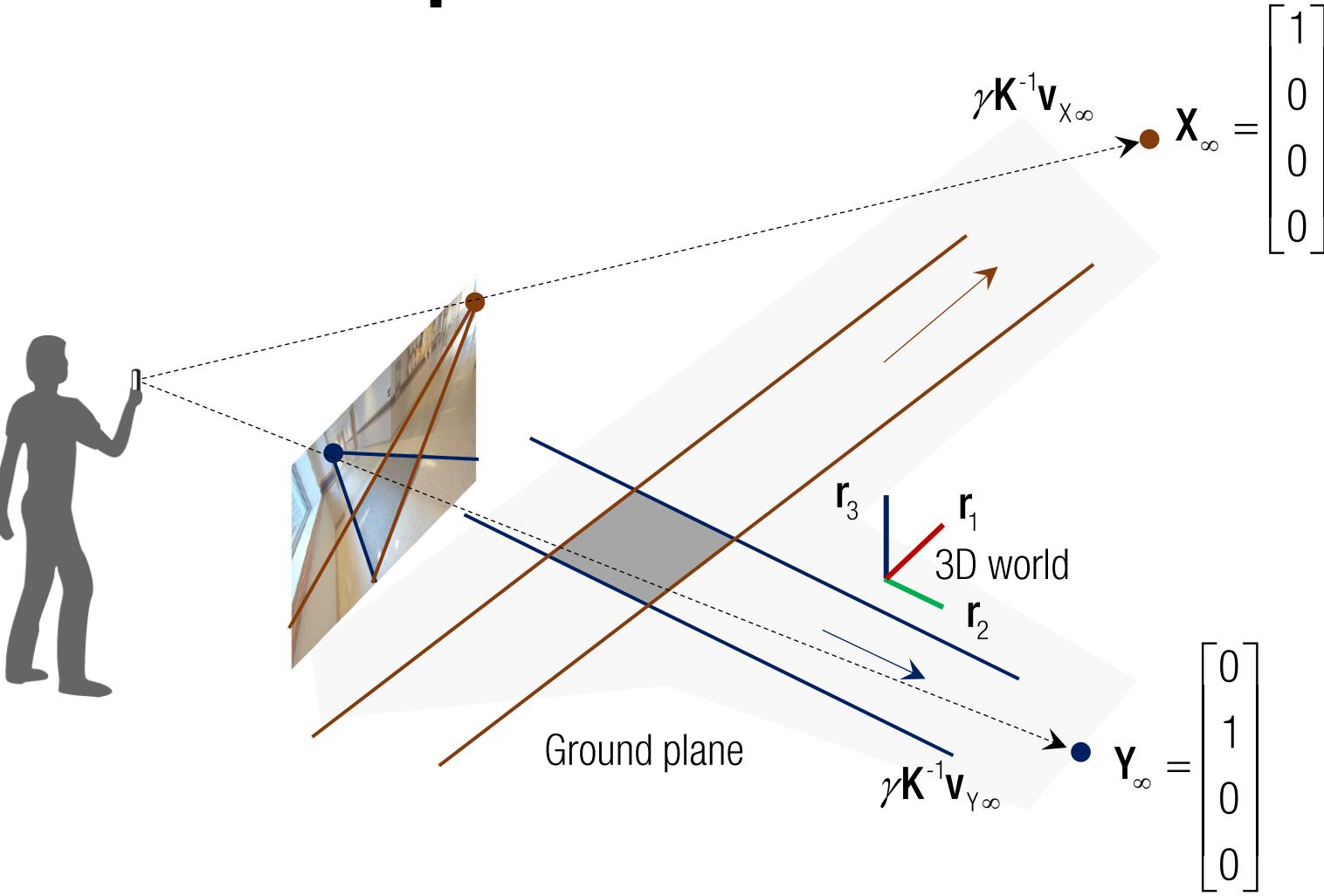
$$\rightarrow r_1 = \frac{K^{-1}v_{X_\infty}}{\|K^{-1}v_{X_\infty}\|}, \quad r_2 = \frac{K^{-1}v_{Y_\infty}}{\|K^{-1}v_{Y_\infty}\|}$$

$r_3 = r_1 \times r_2$: Orthogonality constraint

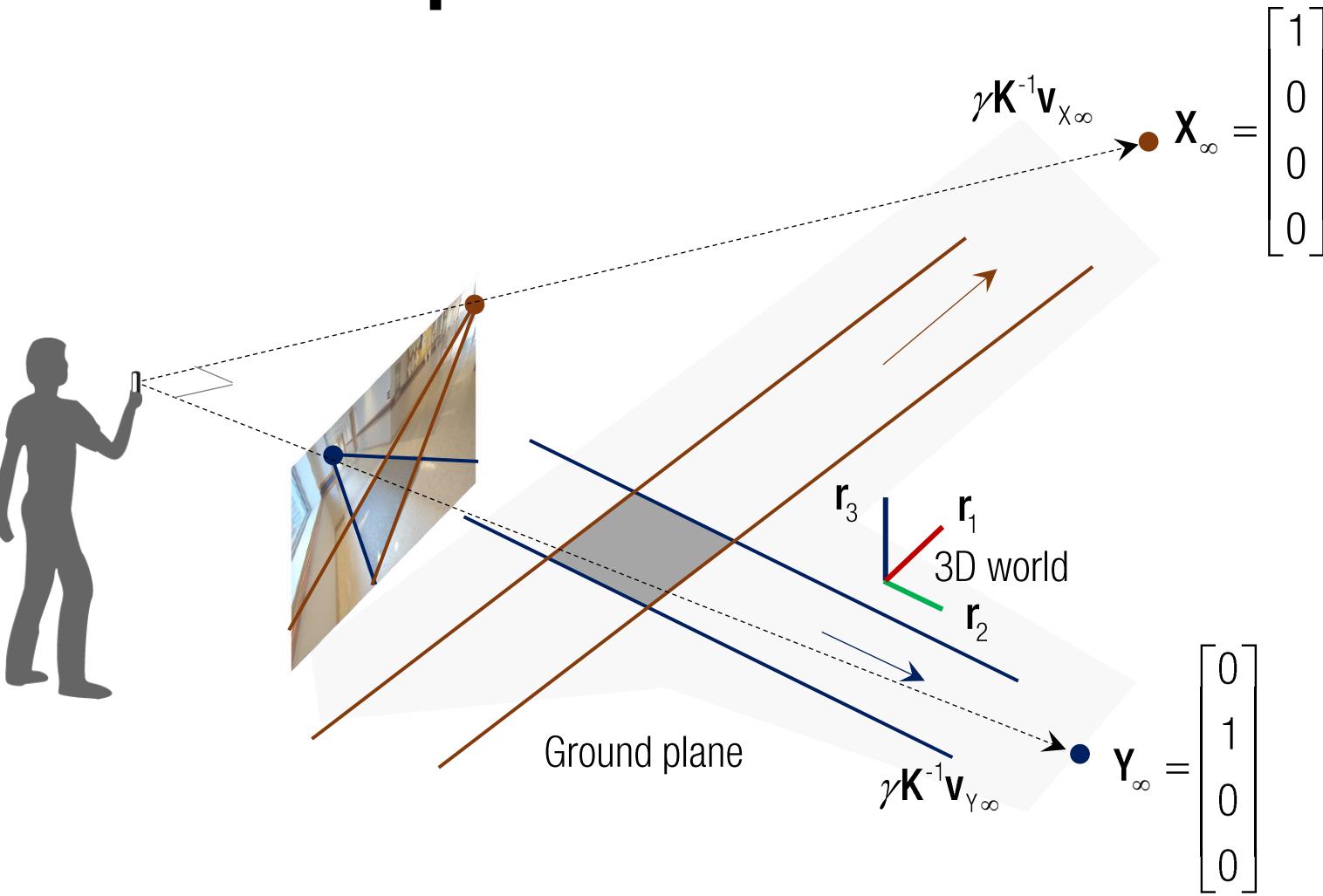
Geometric Interpretation



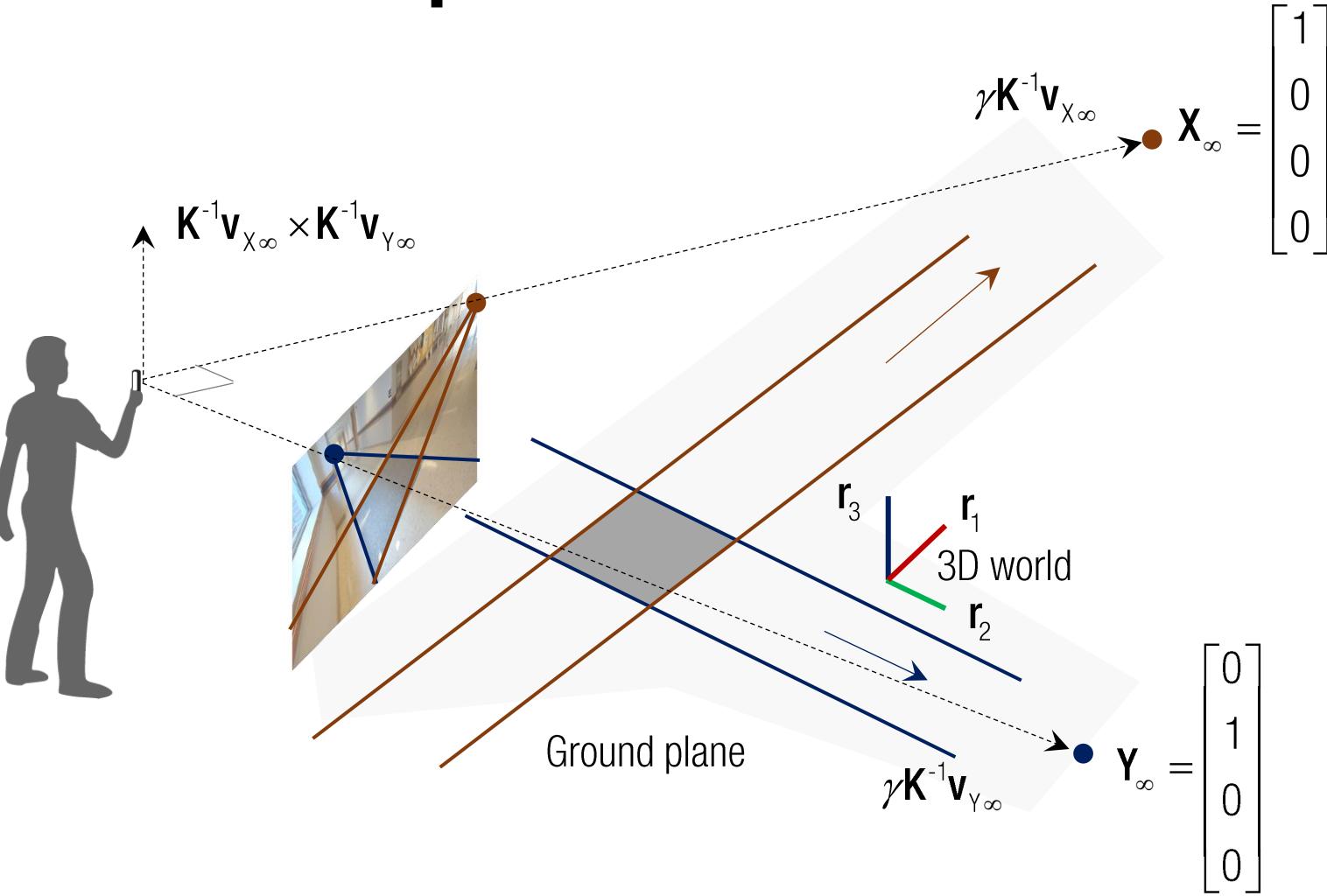
Geometric Interpretation



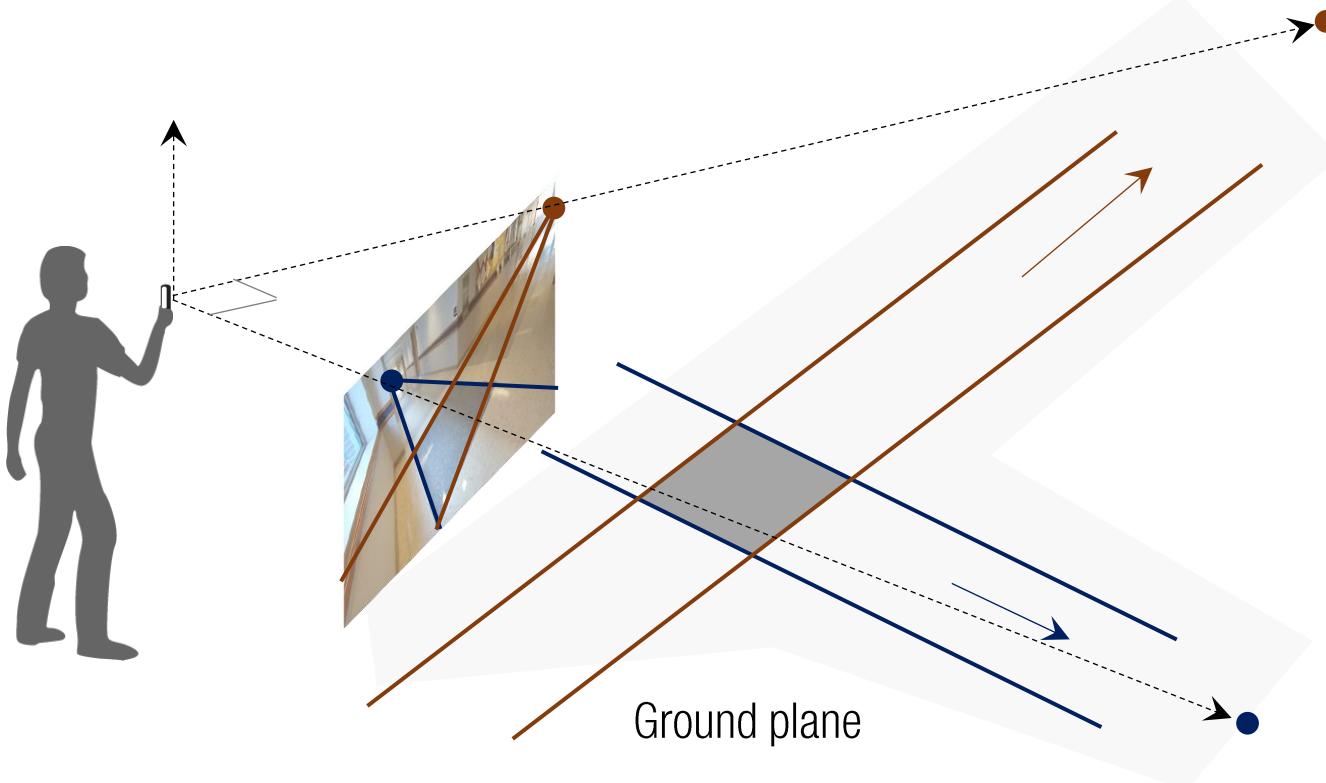
Geometric Interpretation



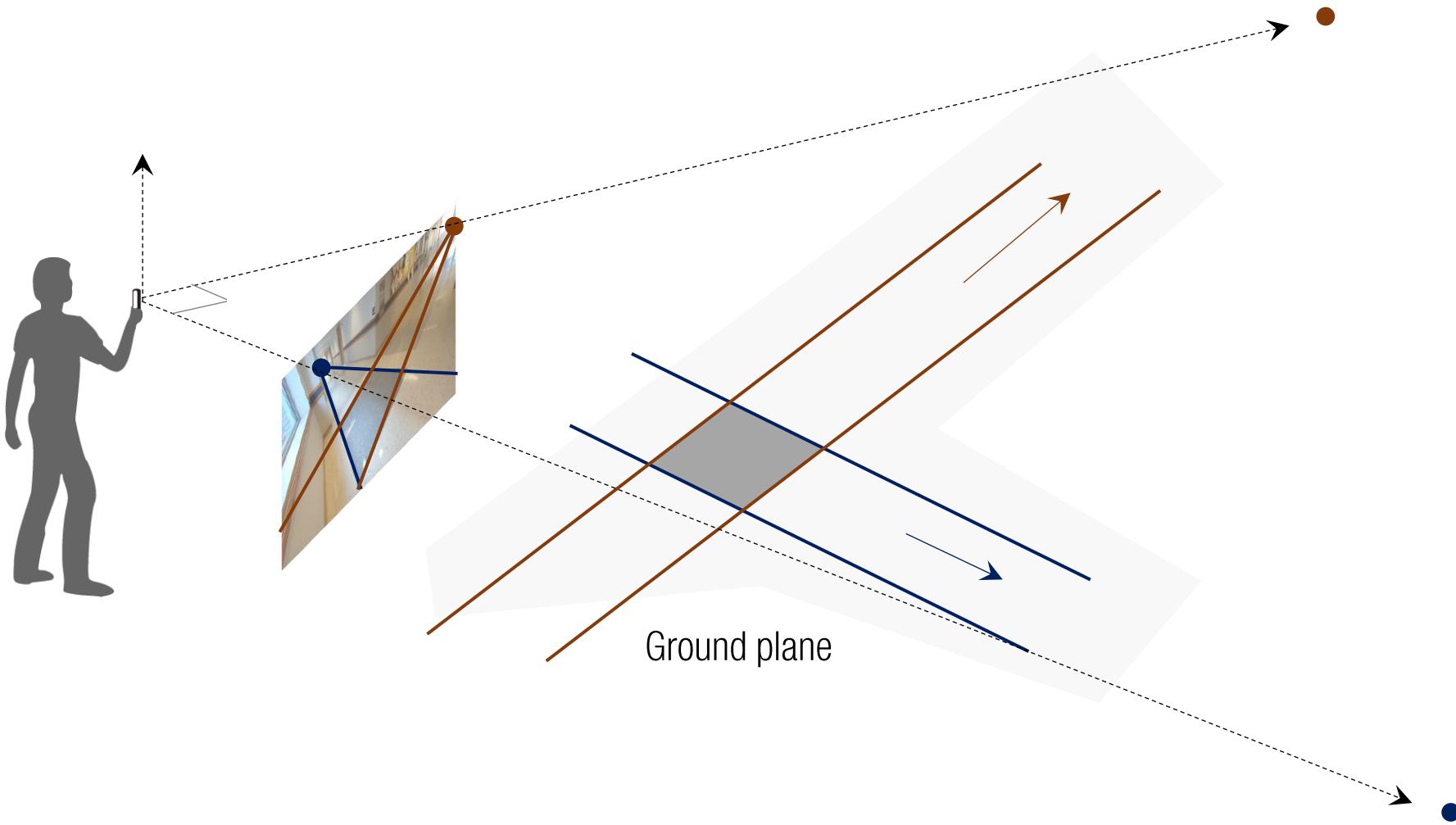
Geometric Interpretation



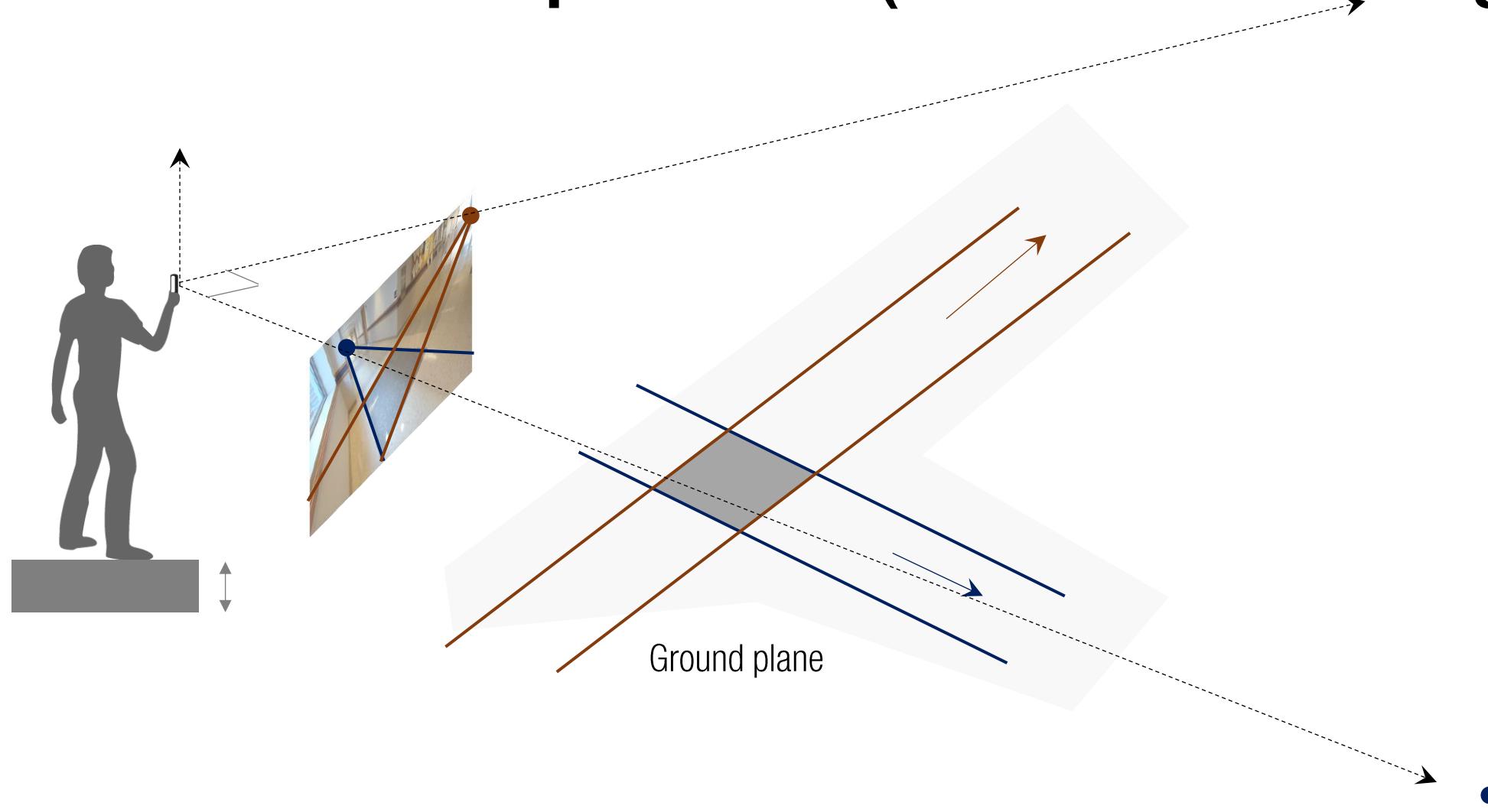
Geometric Interpretation (Translation Ambiguity)



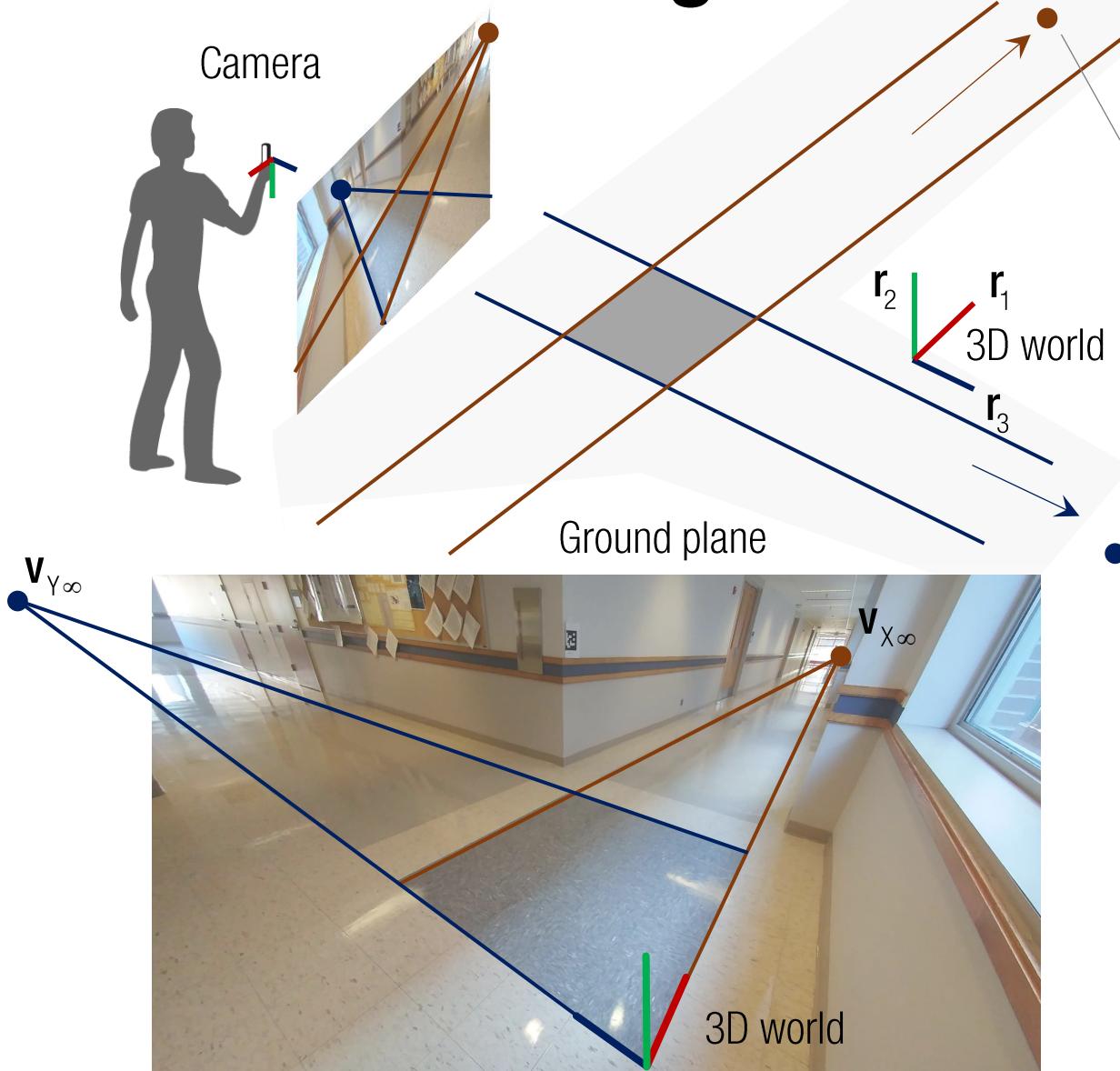
Geometric Interpretation (Translation Ambiguity)



Geometric Interpretation (Translation Ambiguity)



Two Vanishing Points



ComputeCameraUsingTwoVanishingPoints.m

```
f = 4000;
K = [f 0 size(im,2)/2;
      0 f size(im,1)/2;
      0 0 1];
```

```
|l11 = GetLineFromTwoPoints(m11,m12);
l12 = GetLineFromTwoPoints(m13,m14);
```

```
|l21 = GetLineFromTwoPoints(m21,m22);
l22 = GetLineFromTwoPoints(m23,m24);
```

```
v1 = GetPointFromTwoLines(l11,l12);
v2 = GetPointFromTwoLines(l21,l22);
```

```
r1 = inv(K)*v1/norm(inv(K)*v1);
r2 = inv(K)*v2/norm(inv(K)*v2);
```

```
r3 = Vec2Skew(r1)*r2;
```

R = Not orthogonal matrix!

$$0.2448 \quad -0.5178 \quad 0.0424$$

$$-0.1737 \quad -0.1960 \quad -0.6978$$

$$0.9539 \quad 0.8327 \quad -0.1379$$

$\det(R) =$

$$0.5077$$

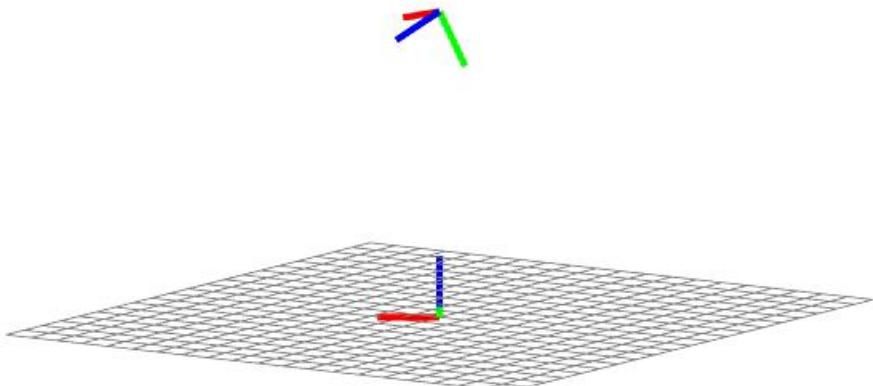
$R' * R =$

$$0.3299 \quad 0.0294 \quad -0.2036$$

$$0.0294 \quad 0.5555 \quad -0.2327$$

$$-0.2036 \quad -0.2327 \quad 1.6224$$

Two Vanishing Points



ComputeCameraUsingTwoVanishingPoints.m

```
f = 1224; % Change focal length
```

```
K = [f 0 size(im,2)/2;  
      0 f size(im,1)/2;  
      0 0 1];
```

```
|l11 = GetLineFromTwoPoints(m11,m12);  
|l12 = GetLineFromTwoPoints(m13,m14);
```

```
|l21 = GetLineFromTwoPoints(m21,m22);  
|l22 = GetLineFromTwoPoints(m23,m24);
```

```
v1 = GetPointFromTwoLines(l11,l12);  
v2 = GetPointFromTwoLines(l21,l22);
```

```
r1 = inv(K)*v1/norm(inv(K)*v1);  
r2 = inv(K)*v2/norm(inv(K)*v2);
```

```
r3 = Vec2Skew(r1)*r2;
```

Orthogonal matrix!

```
R =  
0.5846 -0.8496 0.0508  
-0.4149 -0.3216 -0.8367  
0.6972 0.4180 -0.5405
```

```
det(R) =  
0.9948
```

```
R'*R =  
1.0662 -0.0118 0.0250  
-0.0118 0.9757 0.0285  
0.0250 0.0285 0.9530
```