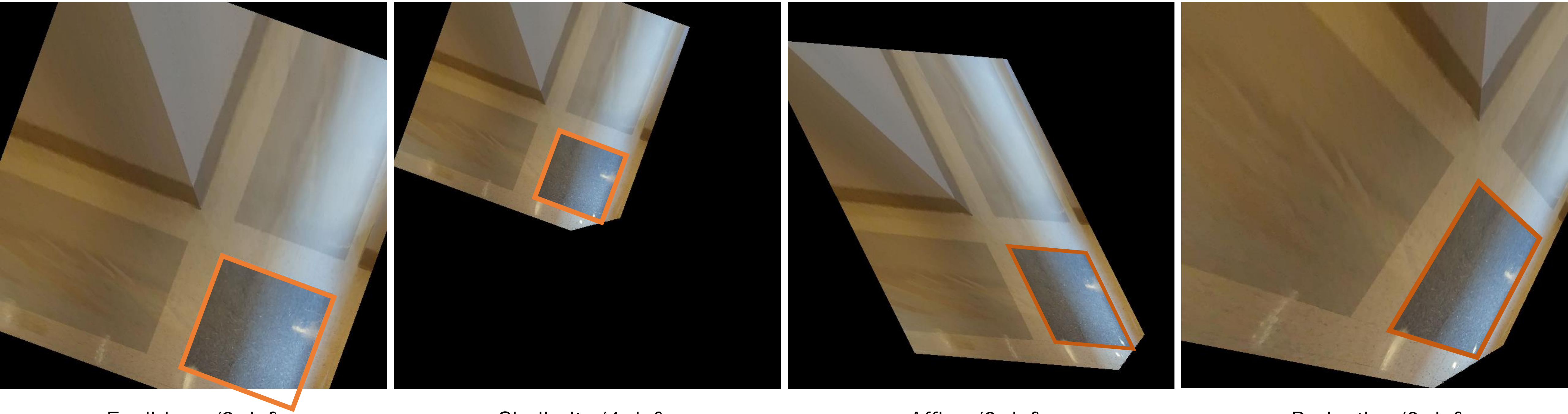
Linear Parameter Estimation

Announcement

- HW #2 deadline: Thursday

- HW #1 grading done.

Hierarchy of Transformations



Euclidean (3 dof)

- Length
- Angle
- Area

$$\begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ & & 1 \end{bmatrix}$$

Similarity (4 dof)

- Length ratio
- Angle

$$\begin{bmatrix} \alpha\cos\theta & -\alpha\sin\theta & t_x \\ \alpha\sin\theta & \alpha\cos\theta & t_y \\ & & 1 \end{bmatrix}$$

Affine (6 dof)

- Parallelism
- Ratio of area
- Ratio of length

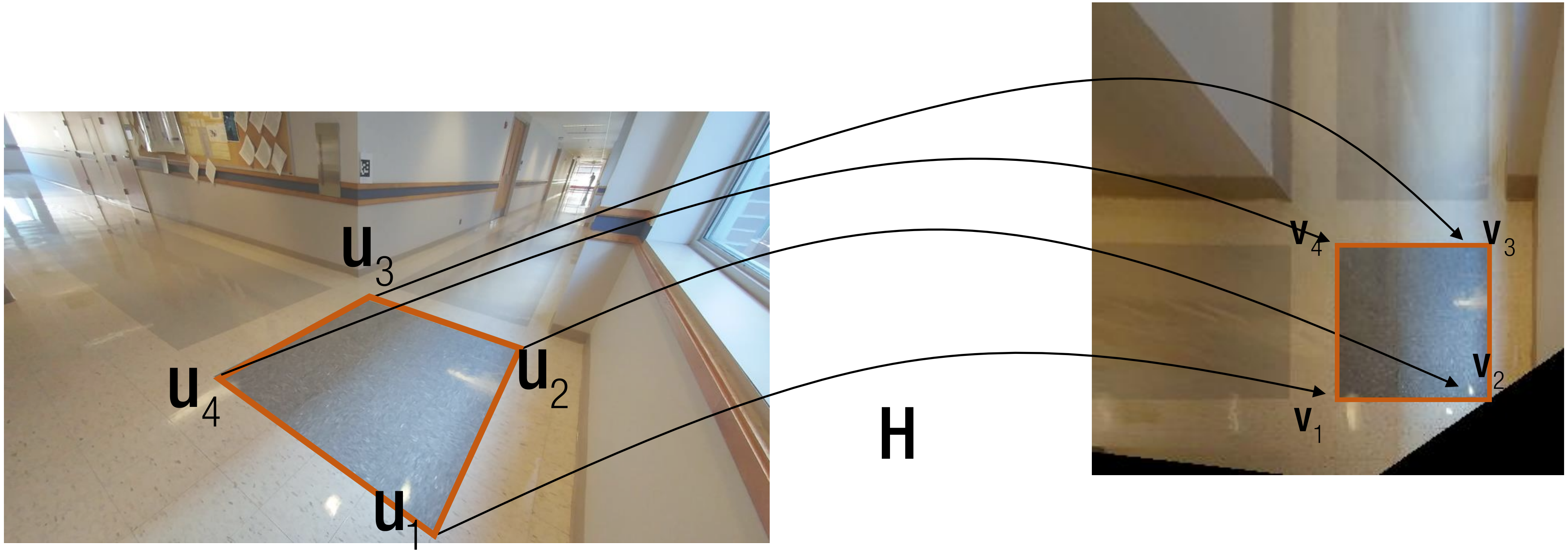
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Projective (8 dof)

- Cross ratio
- Concurrency
- Colinearity

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix}$$

Fun with Homography



Fun with Homography

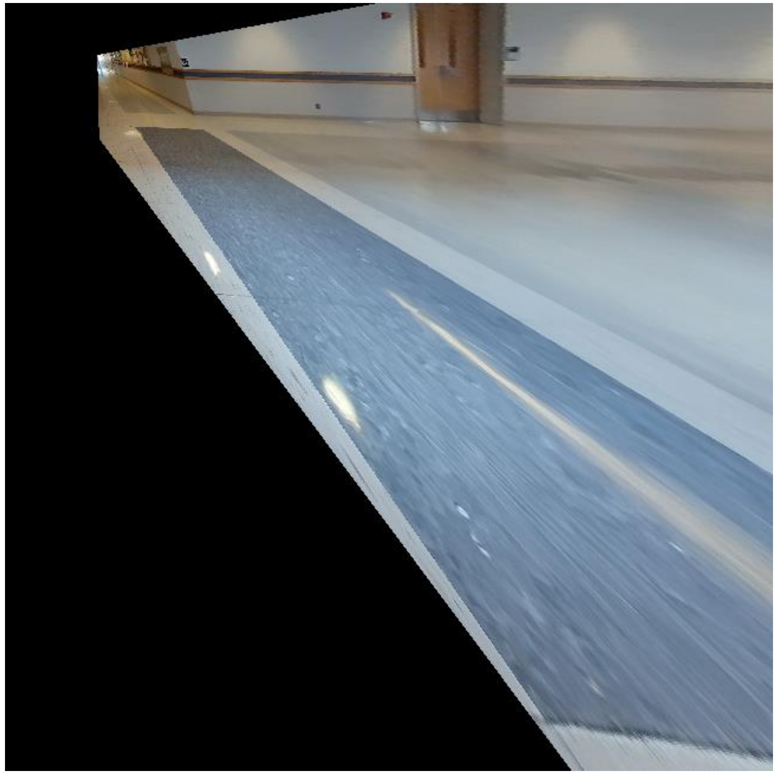


Image Inpainting

Image Inpainting

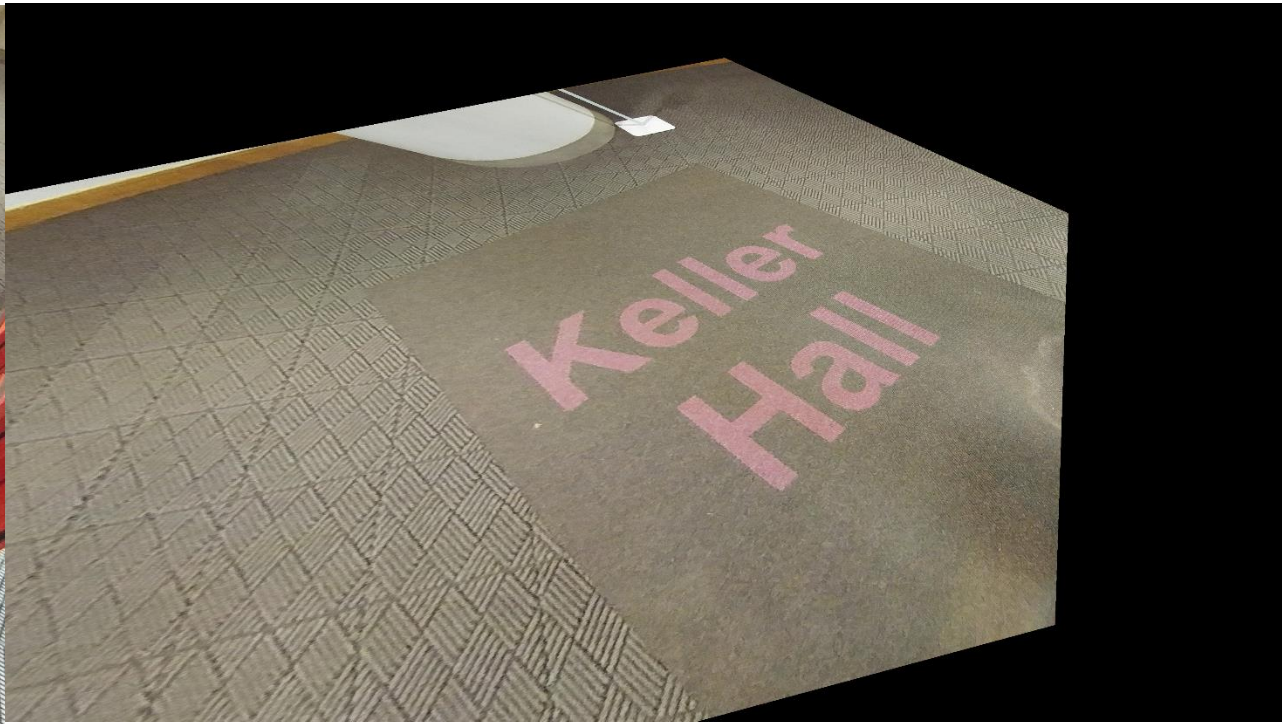
Image Transform via 3D Plane

Keller entrance left

Keller entrance right

Image Transform via 3D Plane

Keller entrance left



Right image to left

Image Transform via 3D Plane

Image Transform via 3D Plane

Left image to right

Right image to left

Image Transform via 3D Plane

Image Transform by Pure 3D Rotation

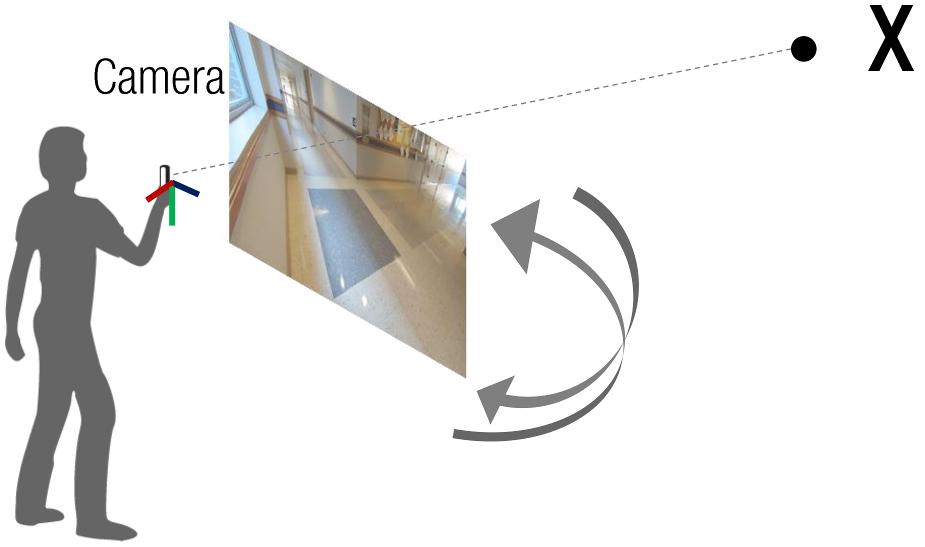


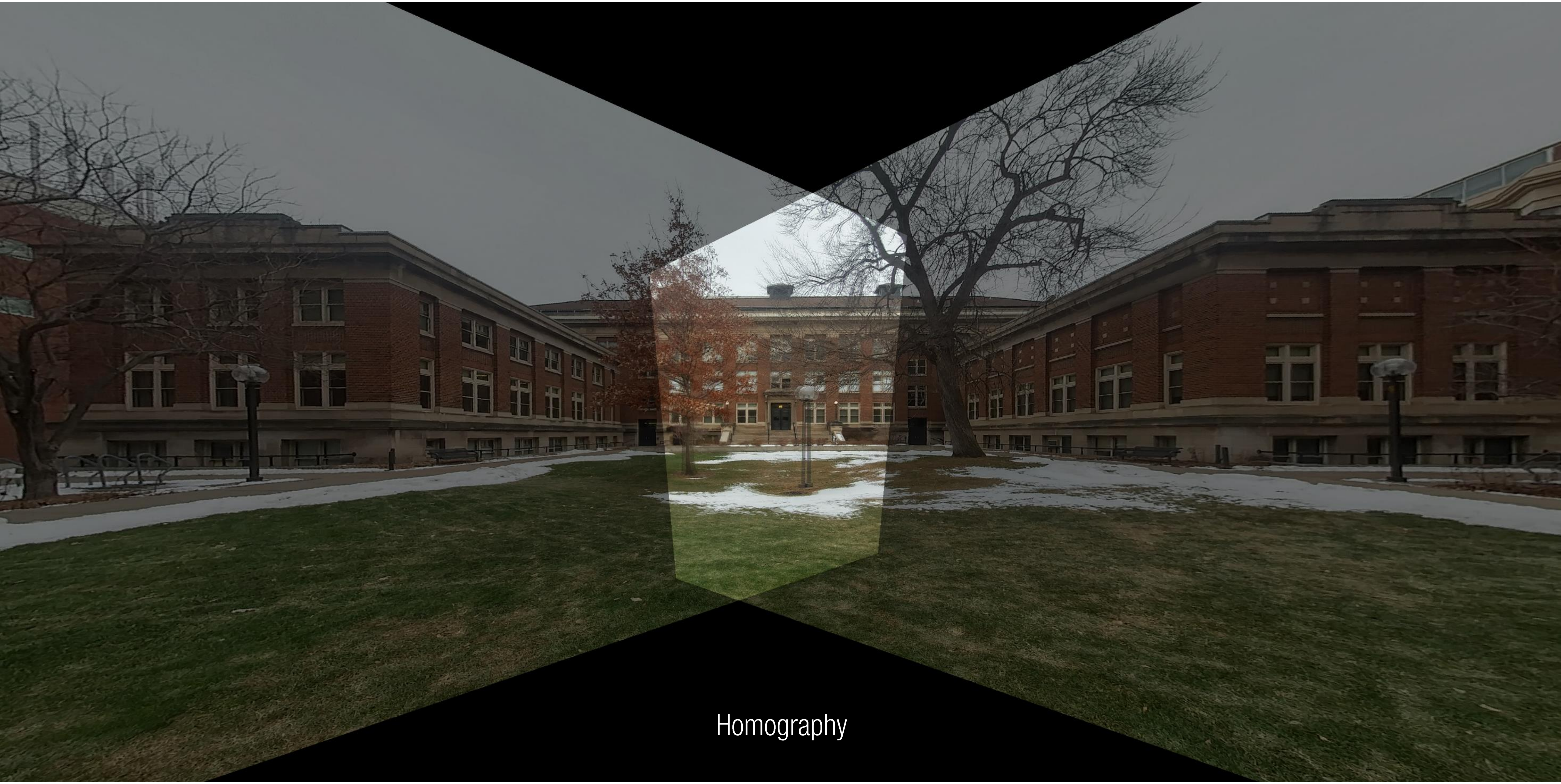
Image Transform by Pure 3D Rotation

$$\lambda \mathbf{u} = \mathbf{H}\mathbf{v}$$

Lind Hall Left

Lind Hall Right

Euclidean Transform (Translation)



Homography

Image Panorama (Cylindrical Projection)

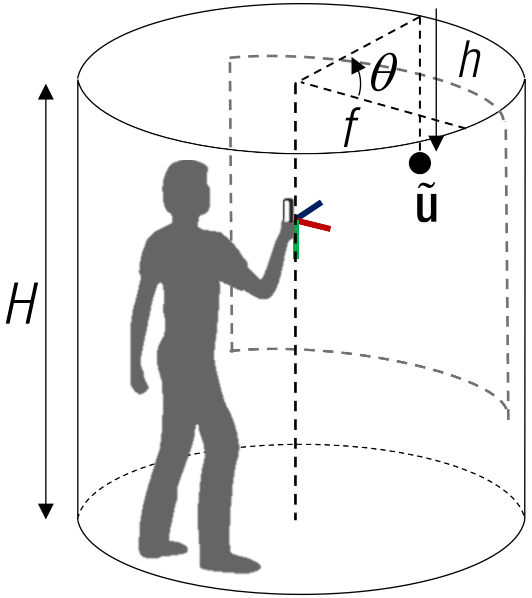
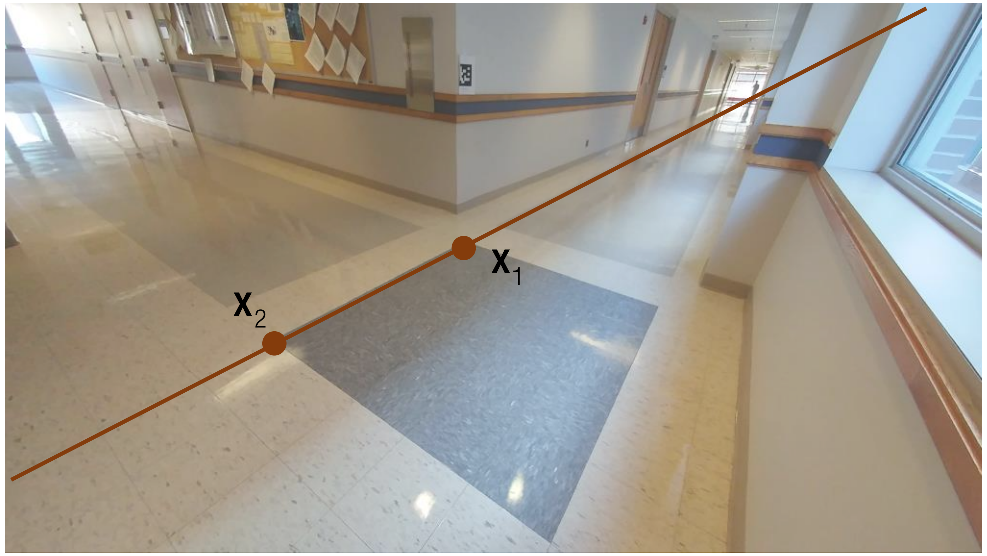


Image Panorama (Cylindrical Projection)

Linear Parameter Estimation

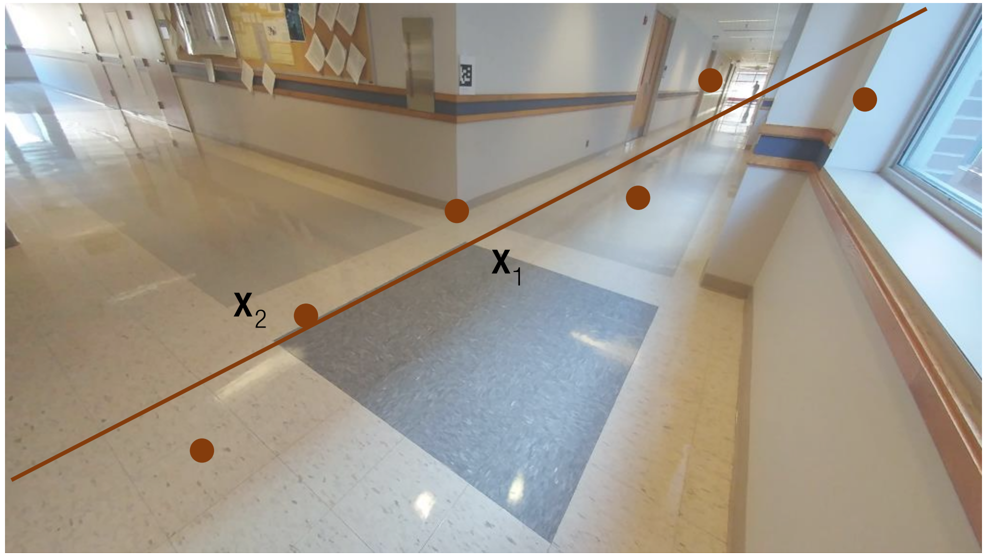
Point-Point in Image

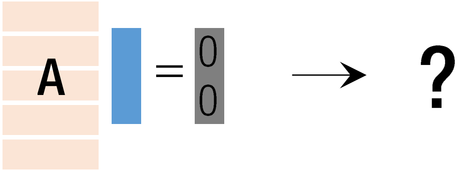


$$\begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{bmatrix} \mathbf{l} = \mathbf{0}$$

$$\underbrace{\mathbf{A}}_{3\text{x}2} \; \blacksquare = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \; \blacksquare = \text{null}\left(\mathbf{A} \right) \quad \text{or} \quad \mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$

Line Fitting

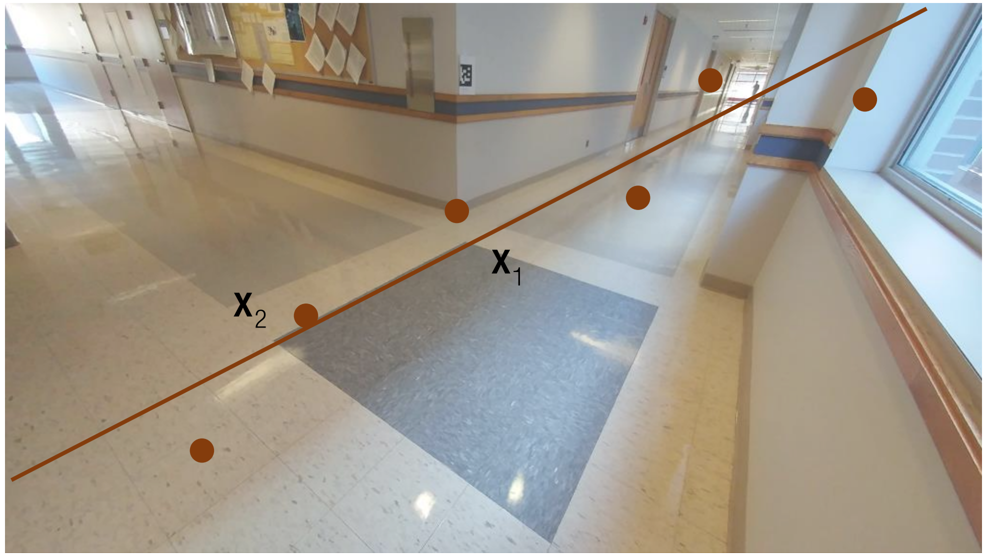




Line Fitting

Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: (a, b, c)



Line Fitting

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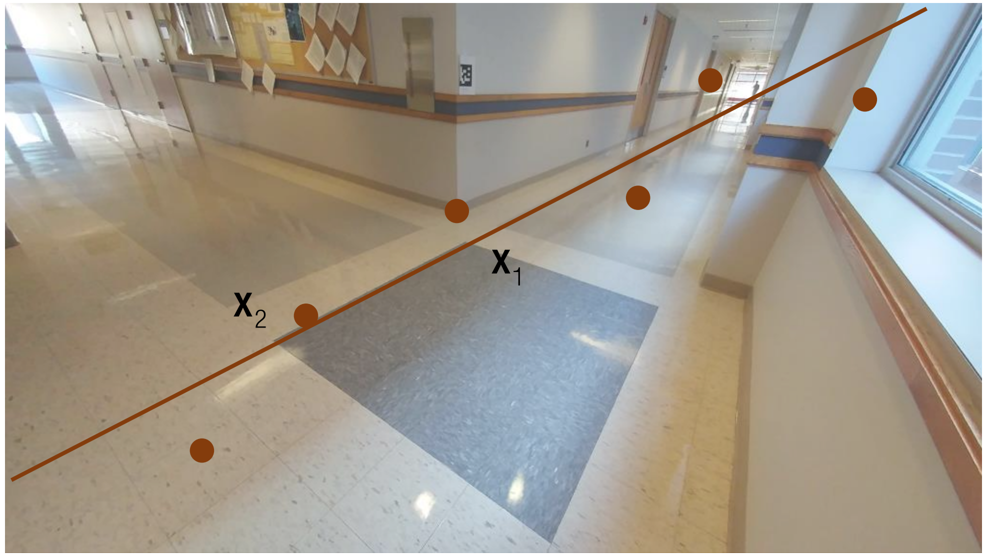


$$\longrightarrow \quad au_1 + bv_1 + c \approx 0$$

$$au_2 + bv_2 + c \approx 0$$

$$\vdots$$

$$au_n + bv_n + c \approx 0$$

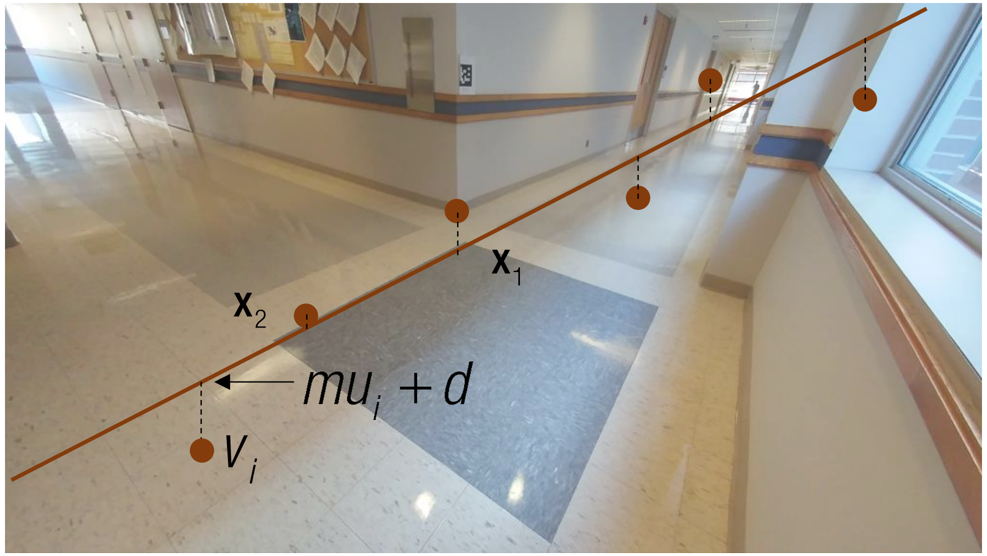
Line Fitting

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m},\underline{d})$

slope y-intercept

$$\longrightarrow \quad v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$\vdots$$
$$v_n \approx mu_n + d$$

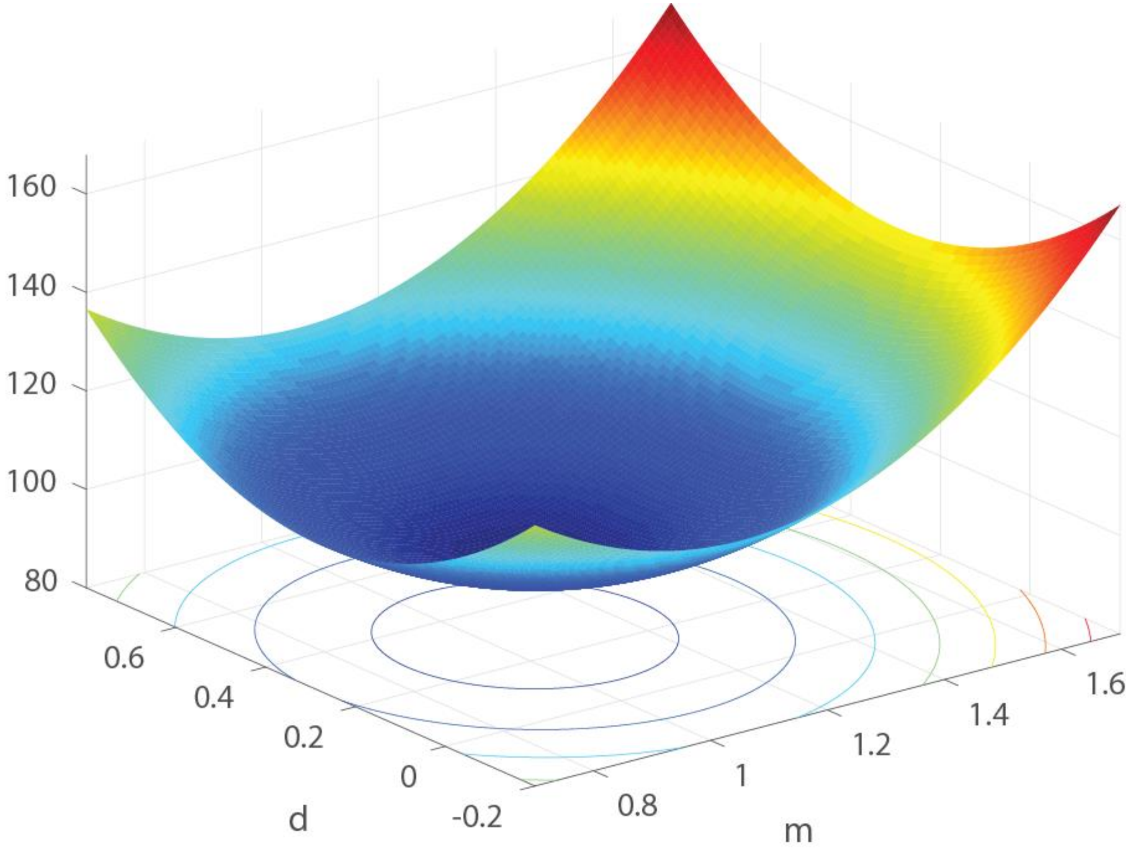


Error: $e_i = v_i - (mu_i + d)$

$$e_1 = v_1 - (mu_1 + d)$$
$$e_2 = v_2 - (mu_2 + d)$$

$$e_n = v_n - (mu_n + d)$$

Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d) \right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$v_1 \approx mu_1 + d$

$v_2 \approx mu_2 + d$

$v_n \approx mu_n + d$

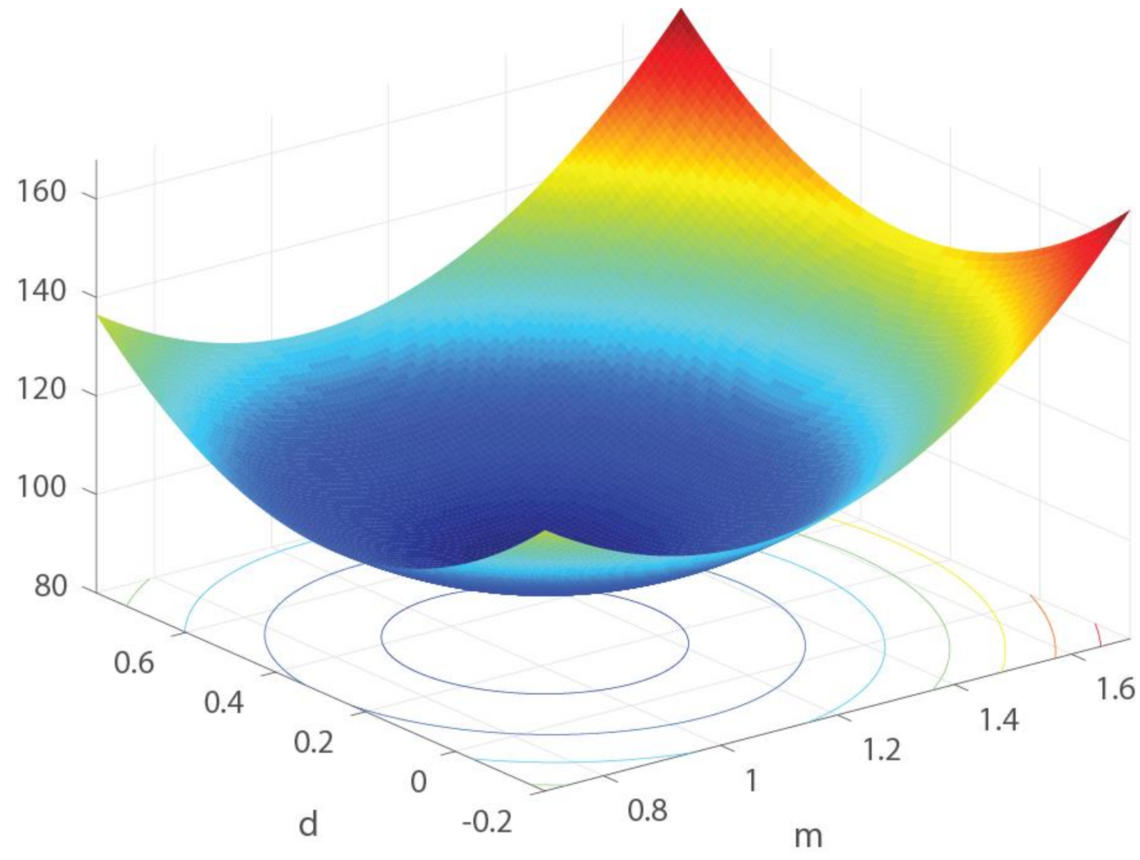
→

$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$

$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$

$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$

Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$v_n \approx mu_n + d$$

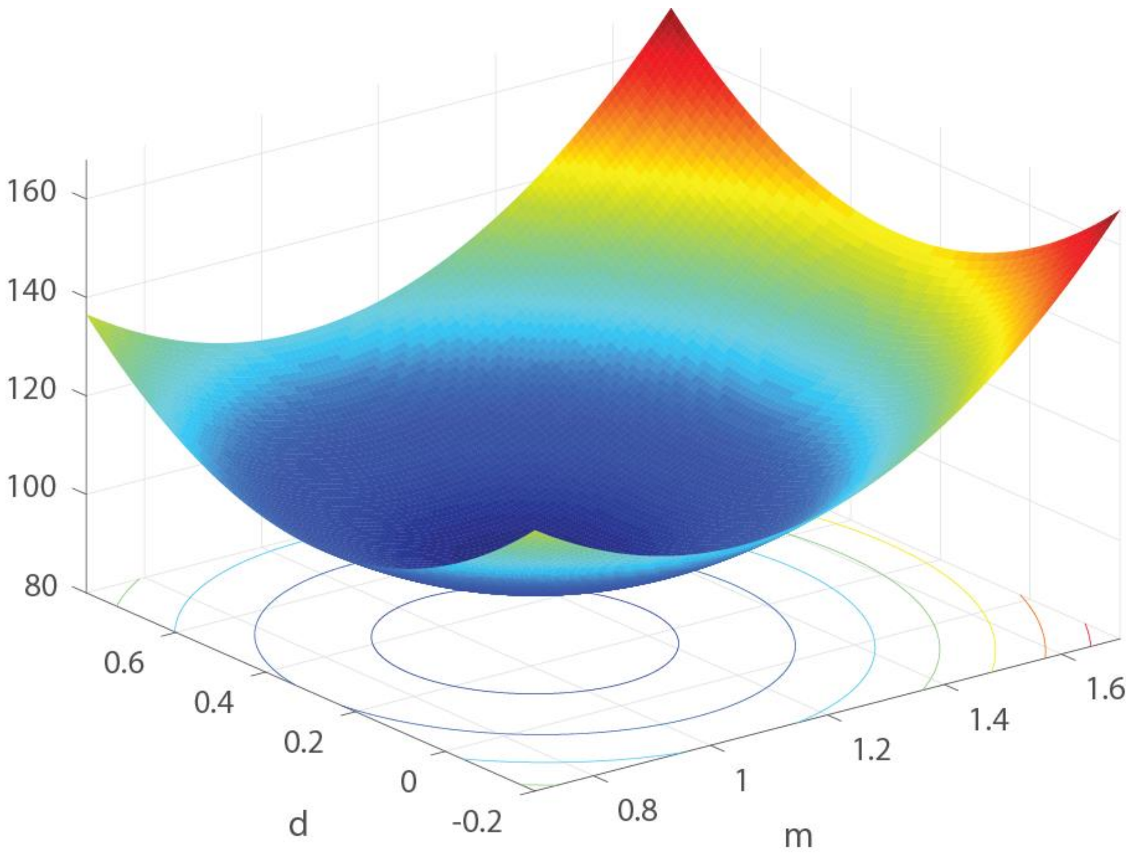
→

$$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$
$$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$
$$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

→

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d \qquad \begin{bmatrix} u_1 & 1 \end{bmatrix}\begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$

$$v_2 \approx mu_2 + d \quad \longrightarrow \quad \begin{bmatrix} u_2 & 1 \end{bmatrix}\begin{bmatrix} m \\ d \end{bmatrix} \approx v_2 \quad \longrightarrow \quad \underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$v_n \approx mu_n + d \qquad \begin{bmatrix} u_n & 1 \end{bmatrix}\begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

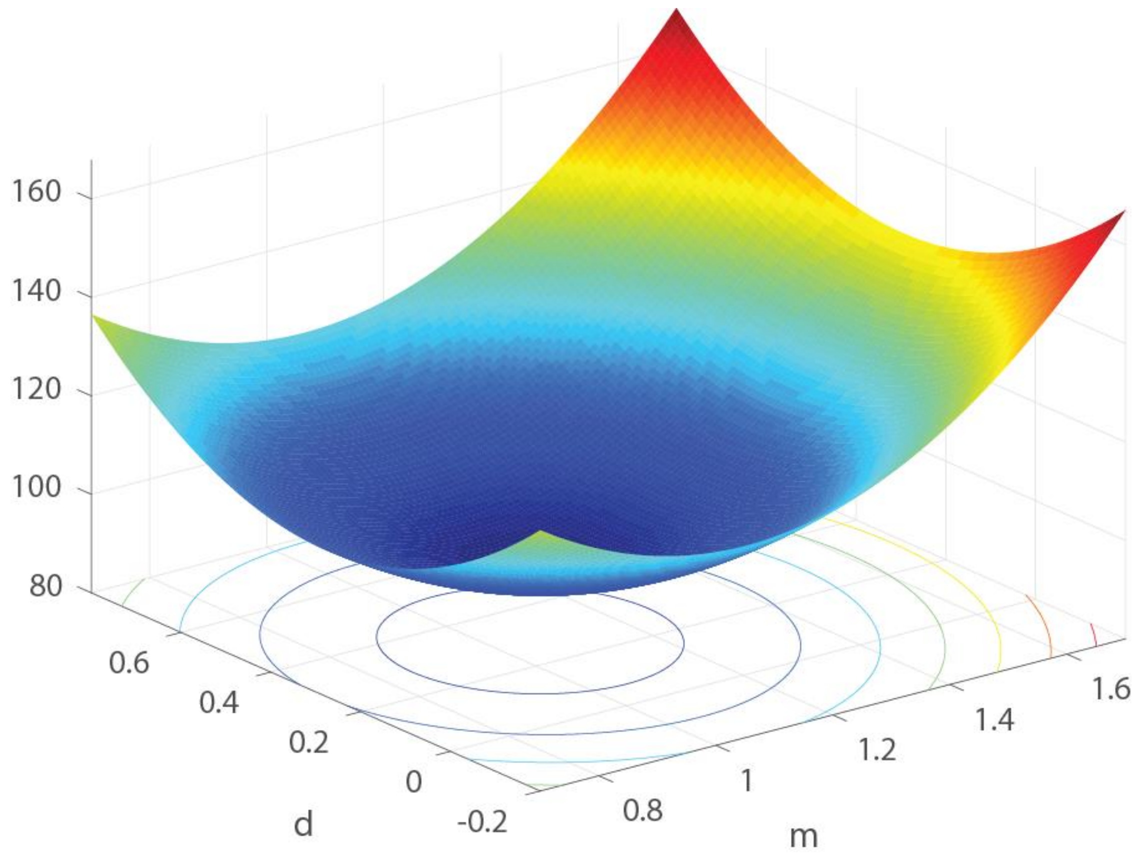
We can't invert **A**.

Line Fitting (Ax=b)

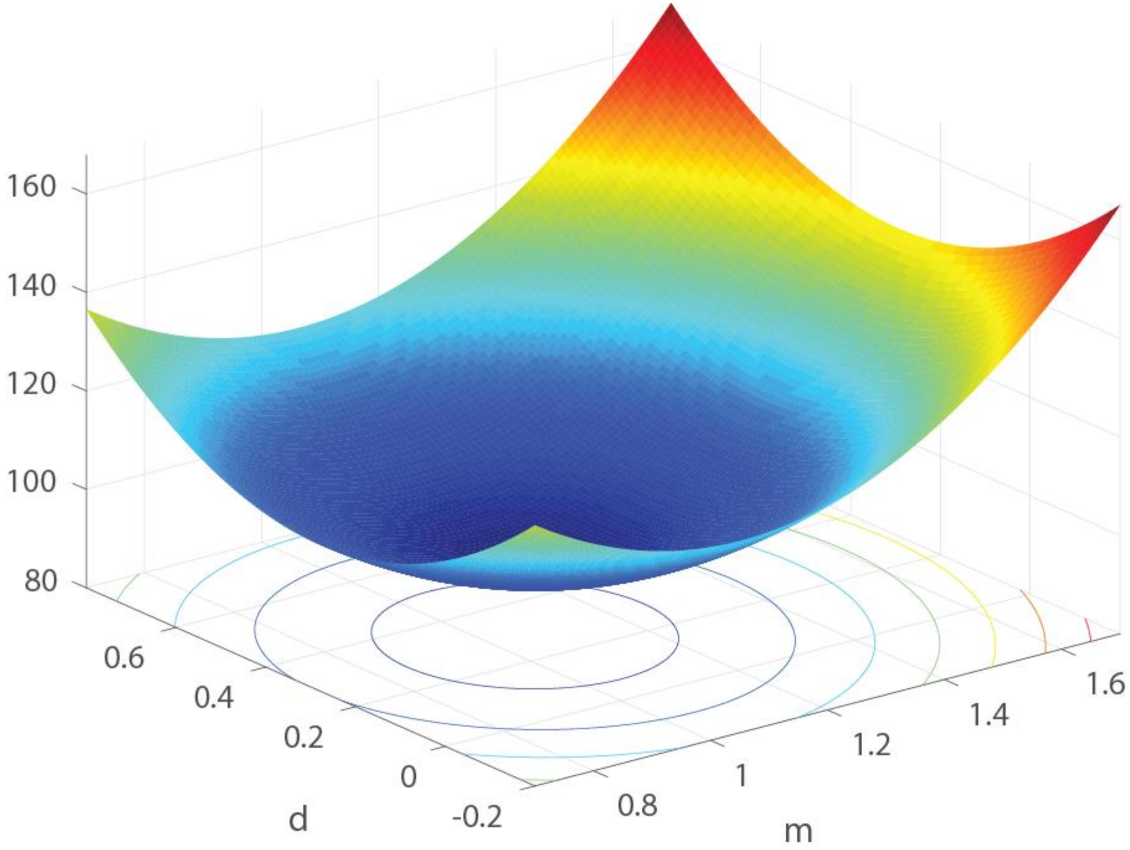
Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$E = (\mathbf{Ax} - \mathbf{b})^{\mathrm{T}} (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$



Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$E = (\mathbf{Ax}-\mathbf{b})^{\mathrm{T}}(\mathbf{Ax}-\mathbf{b}) = \|\mathbf{Ax}-\mathbf{b}\|^2$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{b} + \mathbf{b}^{\mathrm{T}}\mathbf{b}$$

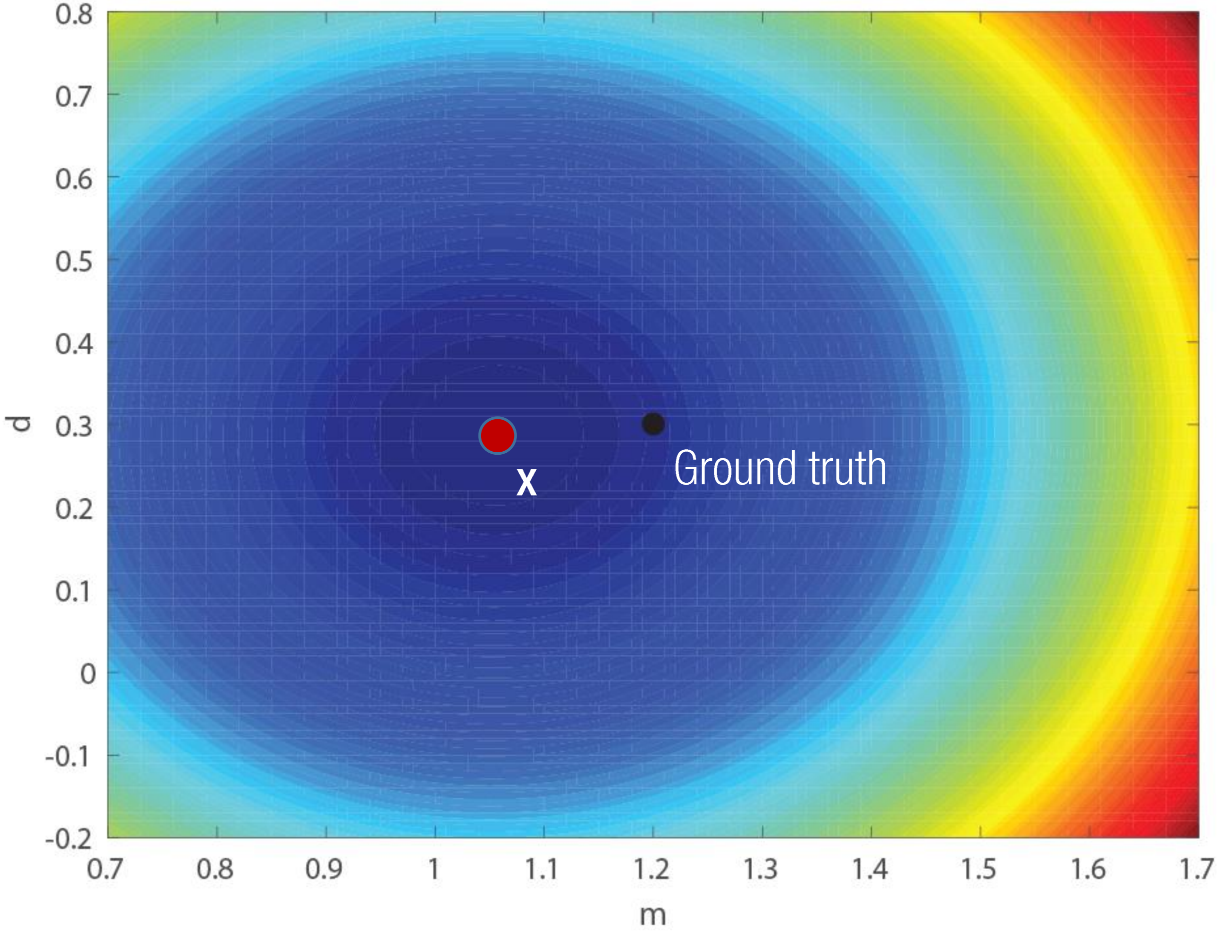
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{A}^{\mathrm{T}}\mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

$$\mathbf{A}^{\mathrm{T}} \; \mathbf{A} \; \mathbf{x} = \mathbf{A}^{\mathrm{T}} \; \mathbf{b}$$

Normal equation

Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A}\,\mathbf{x} \approx \mathbf{b}$$

$$E = \left(\mathbf{Ax} - \mathbf{b}\right)^{\mathrm{T}}\left(\mathbf{Ax} - \mathbf{b}\right) = \left\|\mathbf{Ax} - \mathbf{b}\right\|^2$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{b} + \mathbf{b}^{\mathrm{T}}\mathbf{b}$$

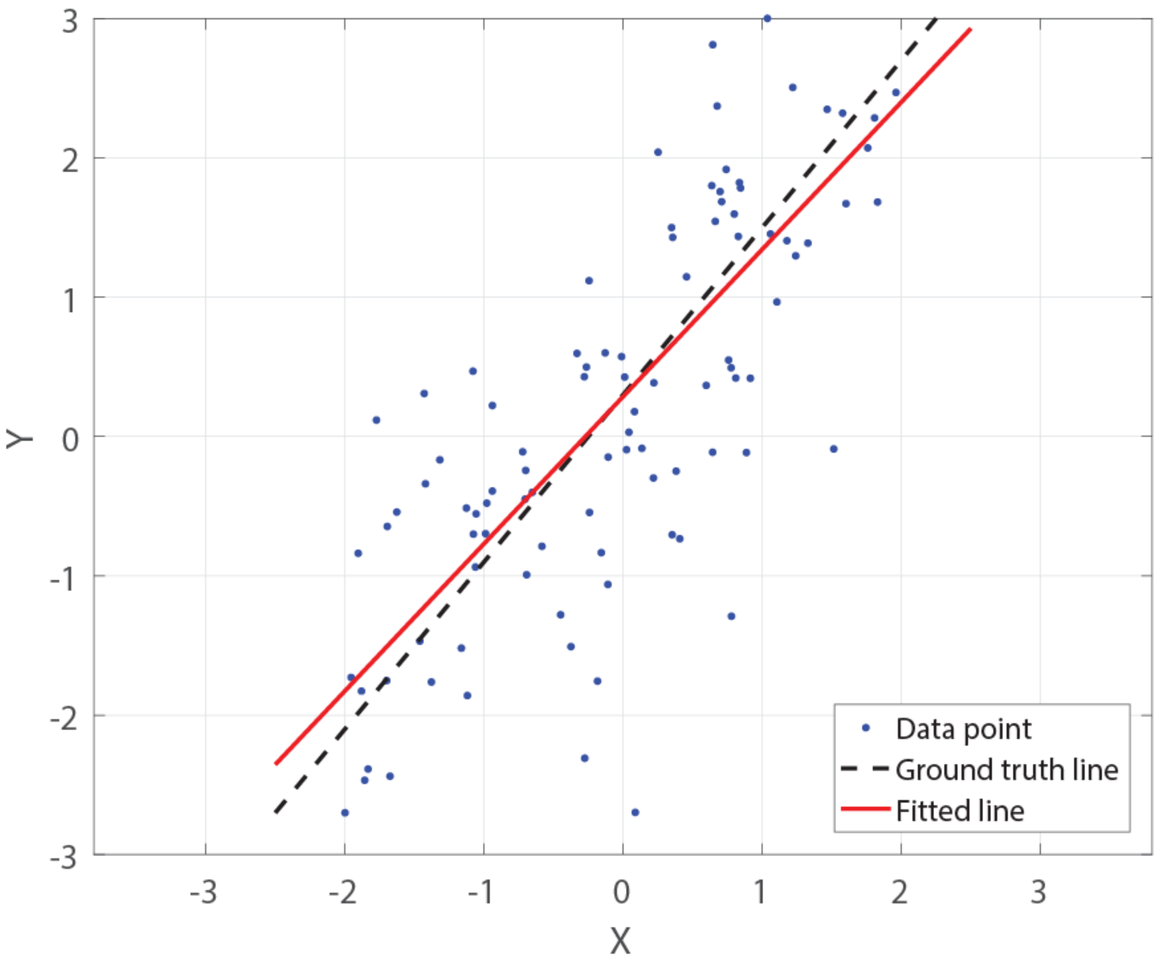
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{A}^{\mathrm{T}}\mathbf{b} = 0$$

$$\longrightarrow \mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

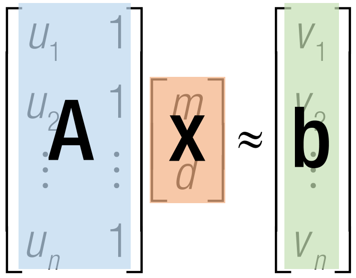
$$\mathbf{A}^{\mathrm{T}}\ \mathbf{A}\ \mathbf{x} = \mathbf{A}^{\mathrm{T}}\ \mathbf{b}$$

$$\mathbf{x} = \left[\mathbf{A}^{\mathrm{T}}\ \mathbf{A}\right]^{-1} \mathbf{A}^{\mathrm{T}}\ \mathbf{b}$$

Line Fitting (Ax=b)



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

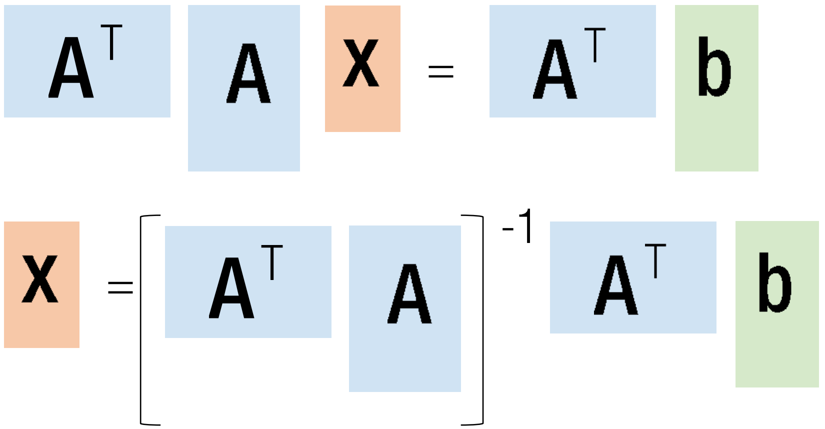


$$E = (\mathbf{Ax} - \mathbf{b})^T(\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

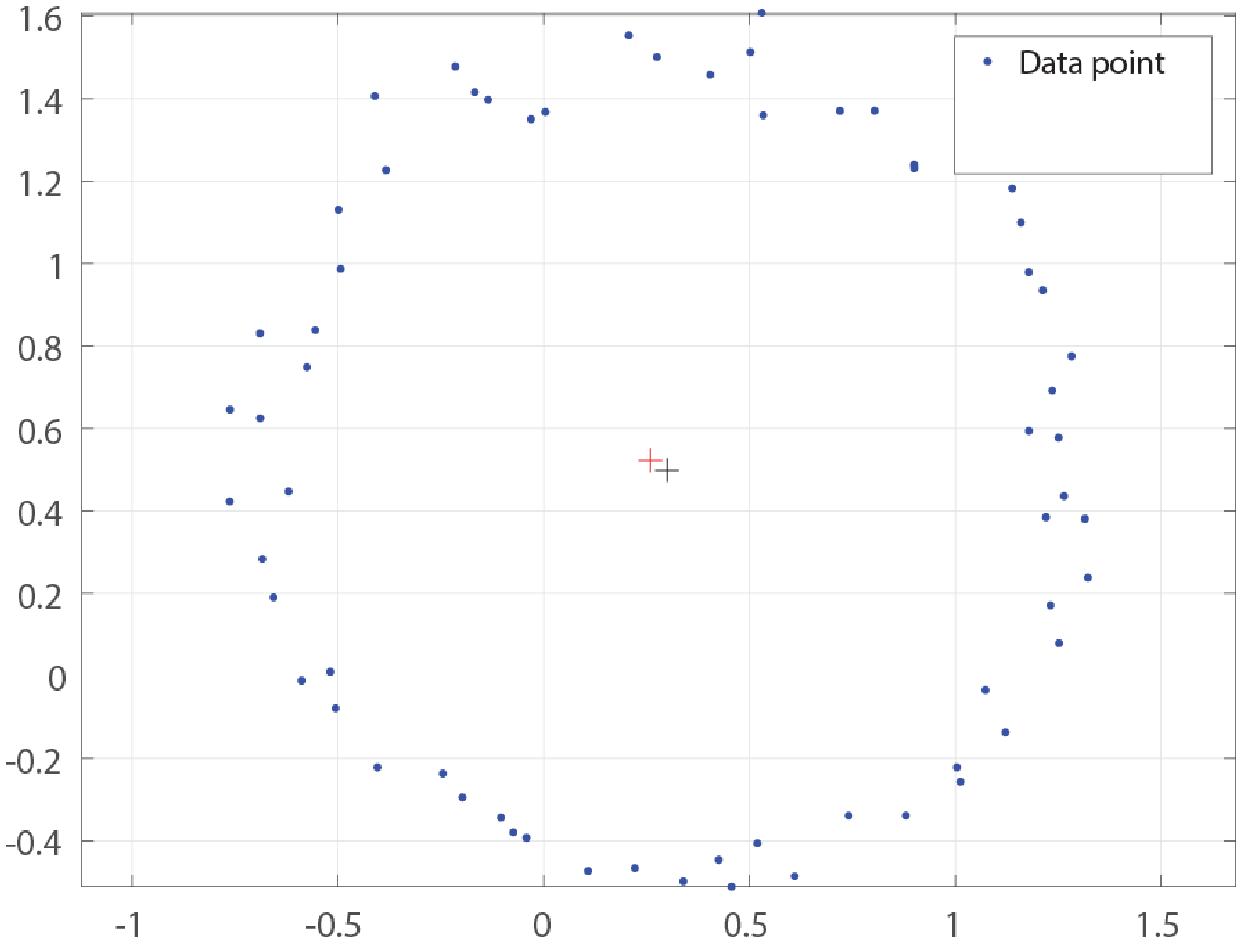
$$= \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} - 2\mathbf{x}^T\mathbf{A}^T\mathbf{b} + \mathbf{b}^T\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^T\mathbf{Ax} - 2\mathbf{A}^T\mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$$



Circle Fitting (Ax=b)



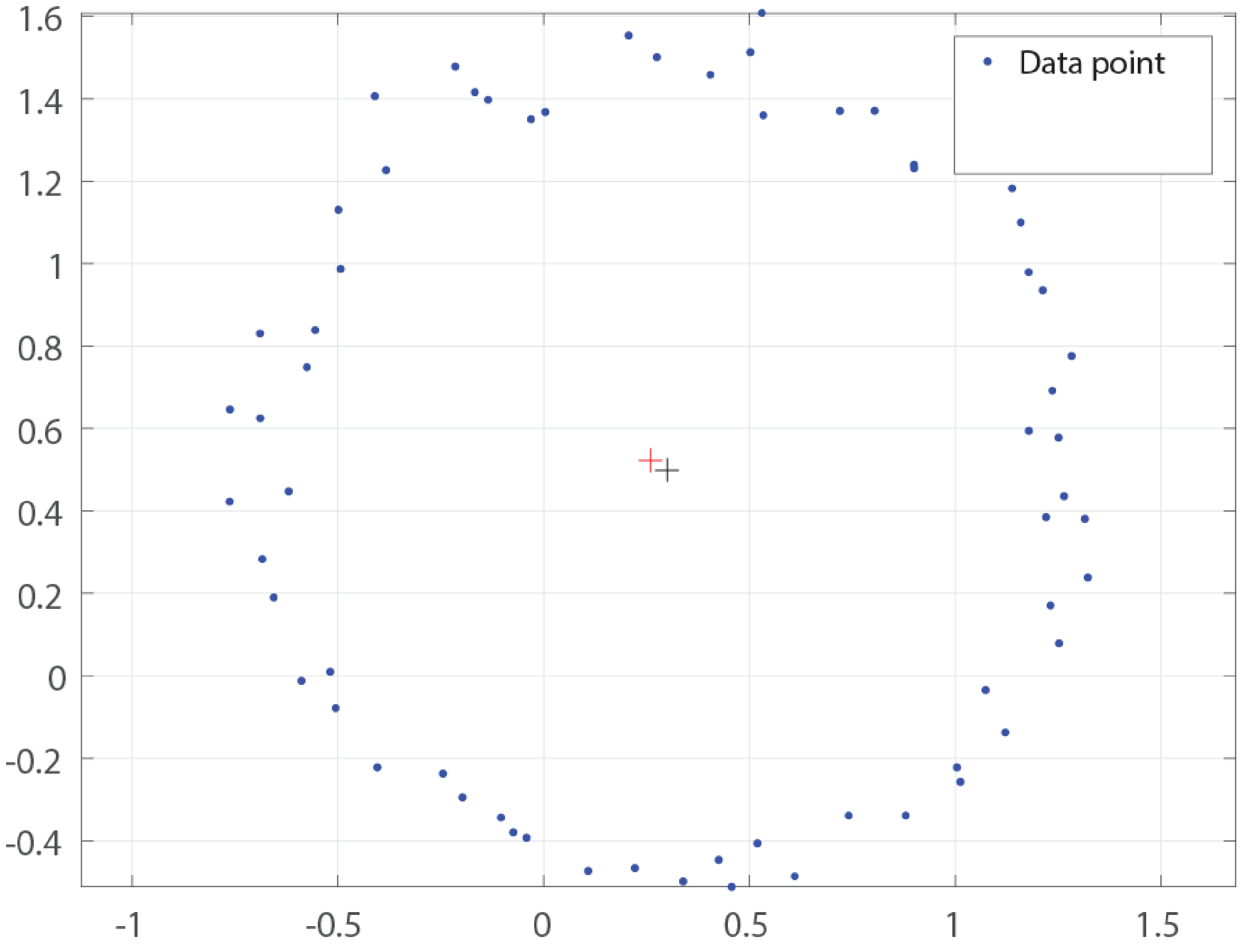
$$(x_1 - c_x)^2 + (y_1 - c_y)^2 = r^2$$

$$\vdots$$

$$(x_n - c_x)^2 + (y_n - c_y)^2 = r^2$$

Unkowns: c_x, c_y, r

Circle Fitting (Ax=b)

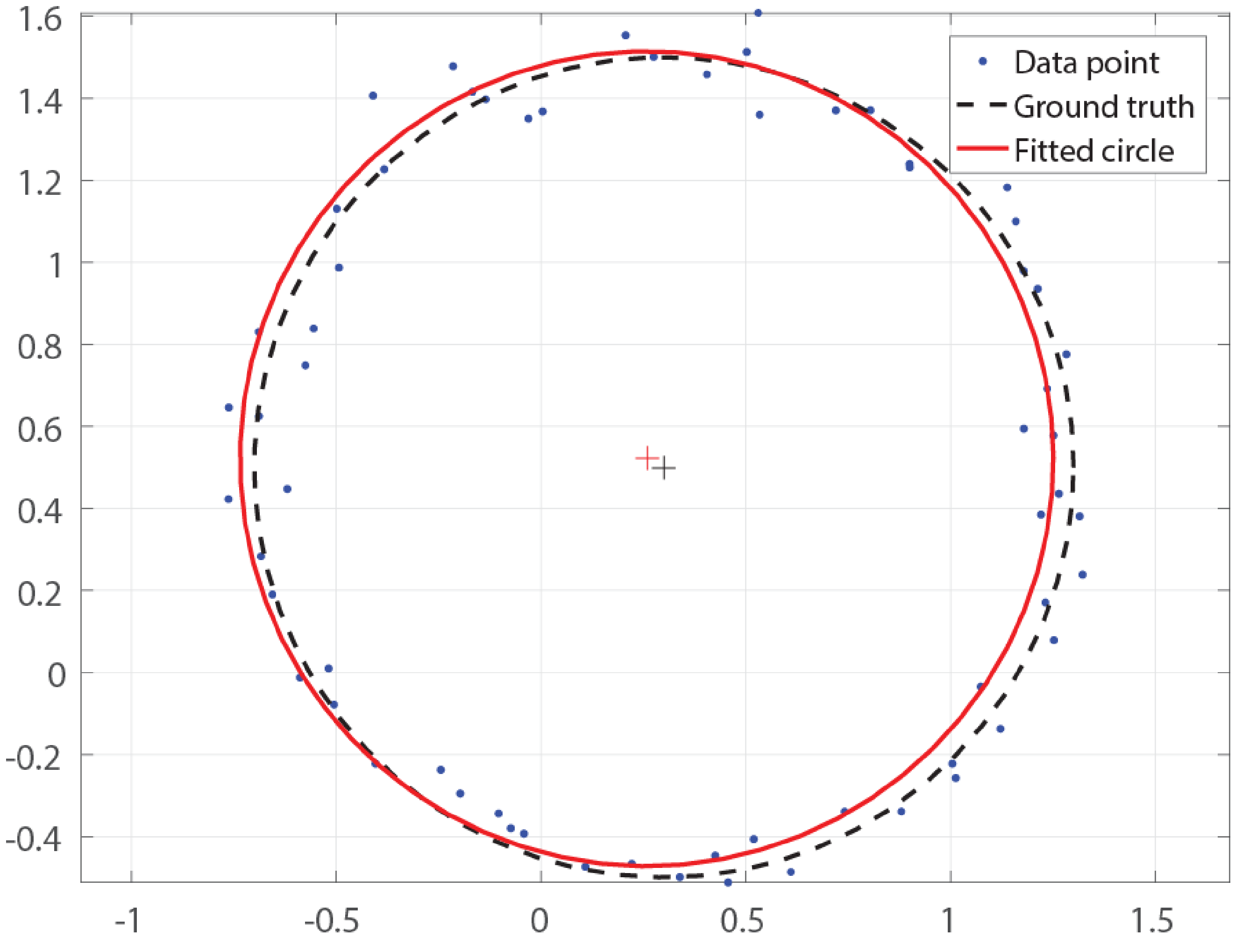


$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

Circle Fitting (Ax=b)



$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

$$\downarrow$$

$$x_i^2 - x_1^2 - 2c_x\left(x_i - x_1\right) + y_i^2 - y_1^2 - 2\left(y_i - y_1\right)c_y = 0$$

$$\downarrow$$

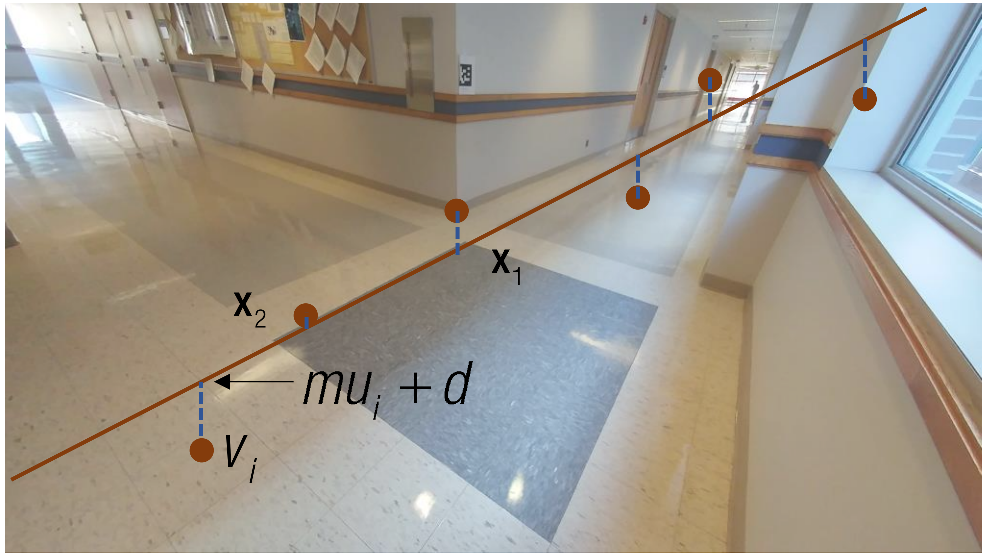
$$\underbrace{\begin{bmatrix} 2\left(x_2 - x_1\right) & 2\left(y_2 - y_1\right) \\ \vdots & \vdots \\ 2\left(x_n - x_1\right) & 2\left(y_n - y_1\right) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} c_x \\ c_y \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} x_2^2 - x_1^2 + y_2^2 - y_1^2 \\ \vdots \\ x_n^2 - x_1^2 + y_n^2 - y_1^2 \end{bmatrix}}_{\mathbf{b}}$$

Line Fitting (Ax=b)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m},\underline{d})$

slope y-intercept



$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$\vdots$$
$$v_n \approx mu_n + d$$

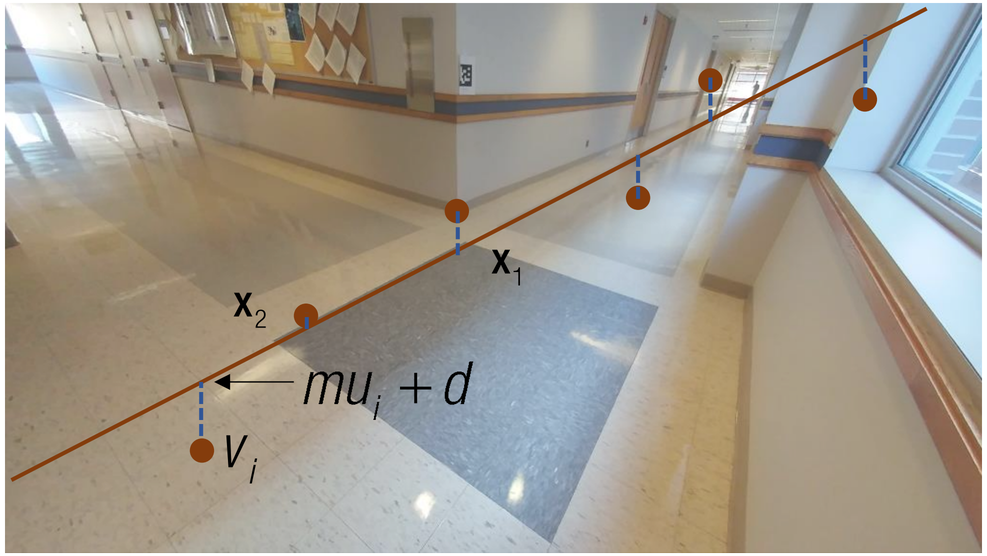
$$\mathbf{Ax} = \mathbf{b}$$

Line Fitting (Ax=b)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m},\underline{d})$

slope y-intercept



$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \longrightarrow \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

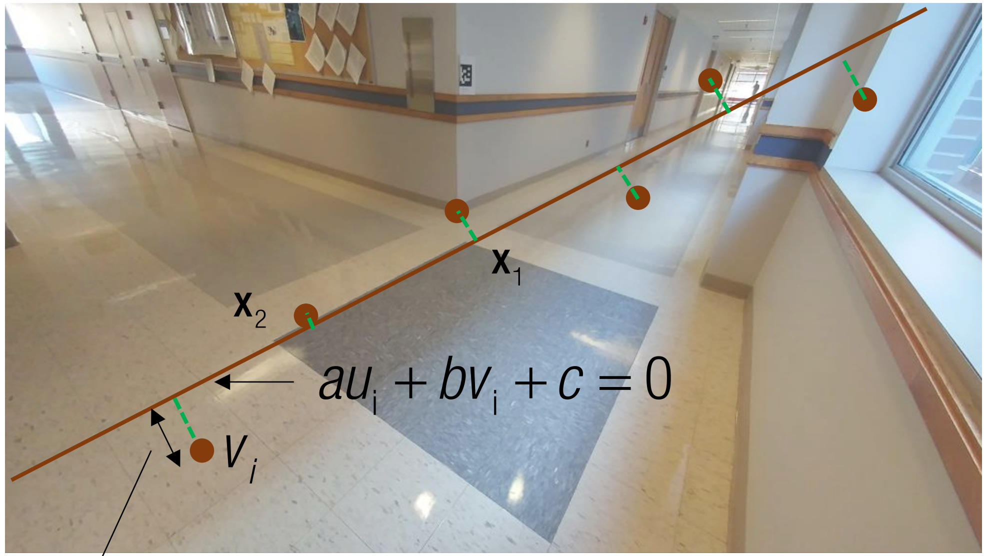
$$\mathbf{Ax} = \mathbf{b}$$

What is different?

Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

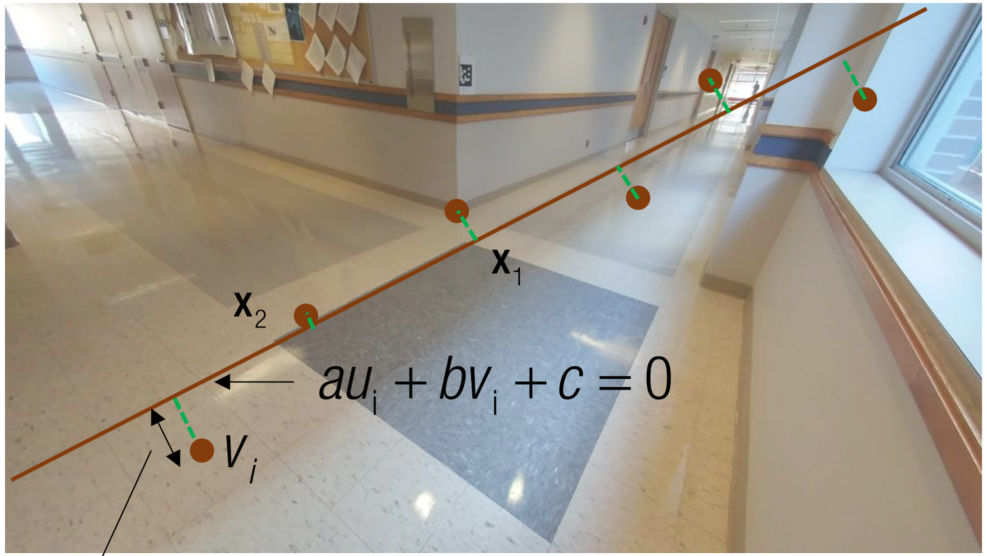
$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \longrightarrow \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$\mathbf{Ax} = \mathbf{0}$ $\qquad$ $\mathbf{Ax} = \mathbf{b}$

Line Fitting (Ax=0)

Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: (a, b, c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

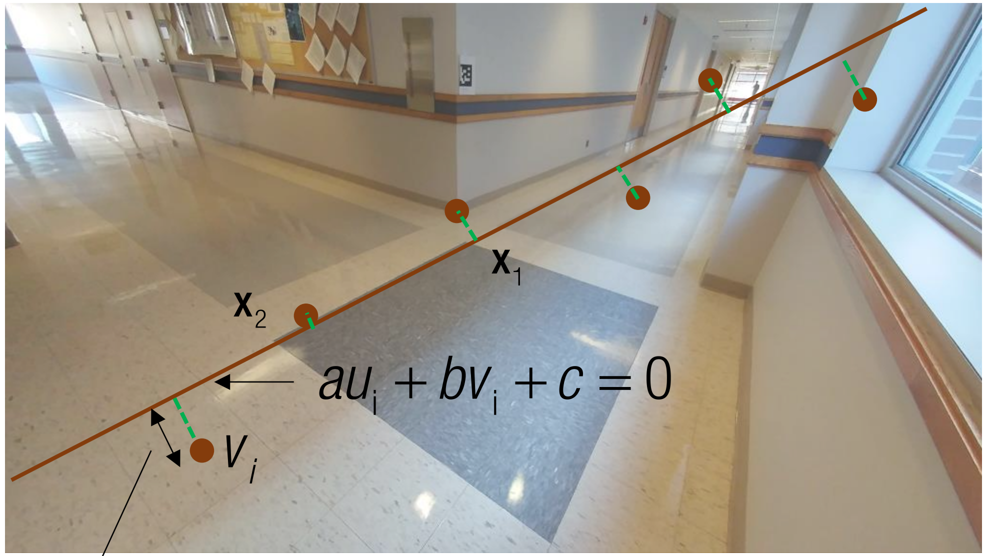
$\mathbf{Ax} = \mathbf{0}$ $\qquad\qquad$ $\mathbf{Ax} = \mathbf{b}$

Trivial solution: $\mathbf{x} = \mathbf{0}$

Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \longrightarrow \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$\mathbf{Ax} = \mathbf{0}$ $\mathbf{Ax} = \mathbf{b}$

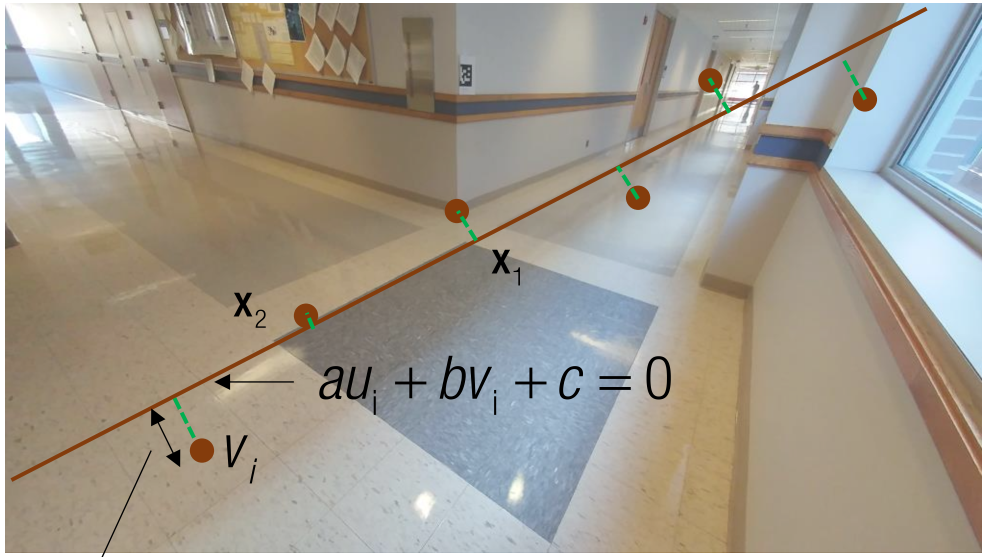
Trivial solution: $\mathbf{x} = \mathbf{0}$

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$$

Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$\mathbf{Ax} = \mathbf{0}$ $\qquad$ $\mathbf{Ax} = \mathbf{b}$

Trivial solution: $\mathbf{x} = \mathbf{0}$

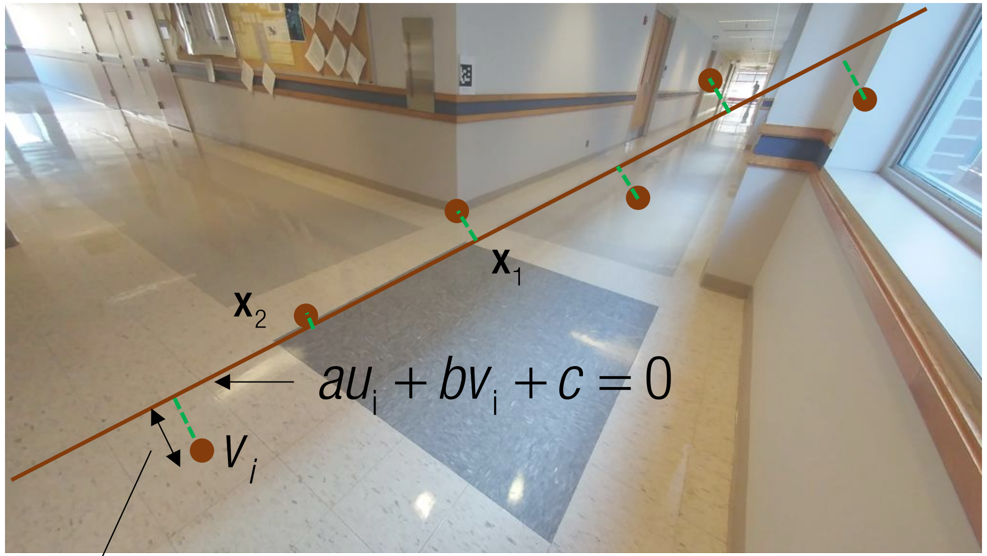
$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$ subject to $\|\mathbf{x}\| = 1$

Condition to avoid the trivial solution

Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \longrightarrow \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$\mathbf{Ax} = \mathbf{0}$ $\qquad$ $\mathbf{Ax} = \mathbf{b}$

Trivial solution: $\mathbf{x} = \mathbf{0}$

$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$ $\qquad$ subject to $\|\mathbf{x}\| = 1$

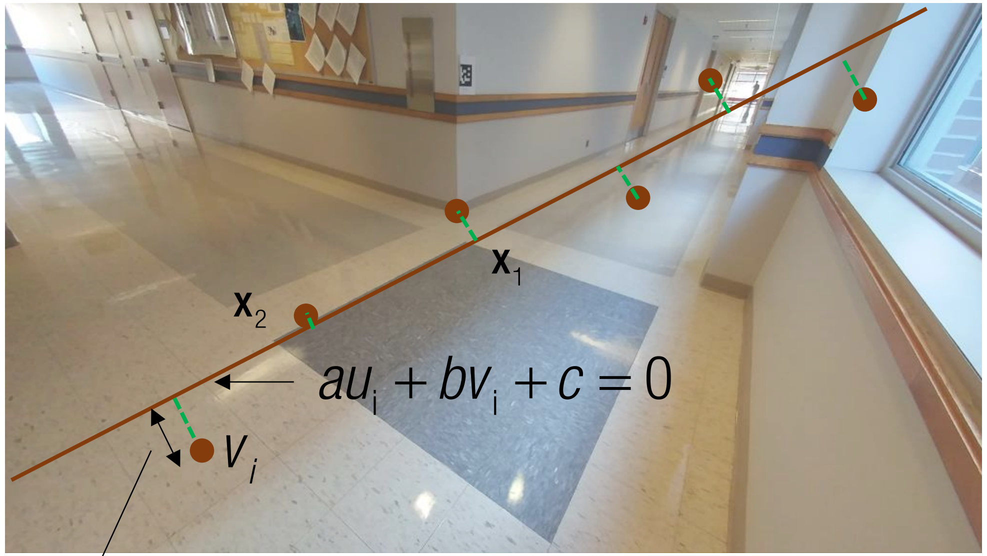
Condition to avoid the trivial solution

How to solve?

Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



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$\mathbf{Ax} = \mathbf{0}$ $\qquad$ $\mathbf{Ax} = \mathbf{b}$

Trivial solution: $\mathbf{x} = \mathbf{0}$

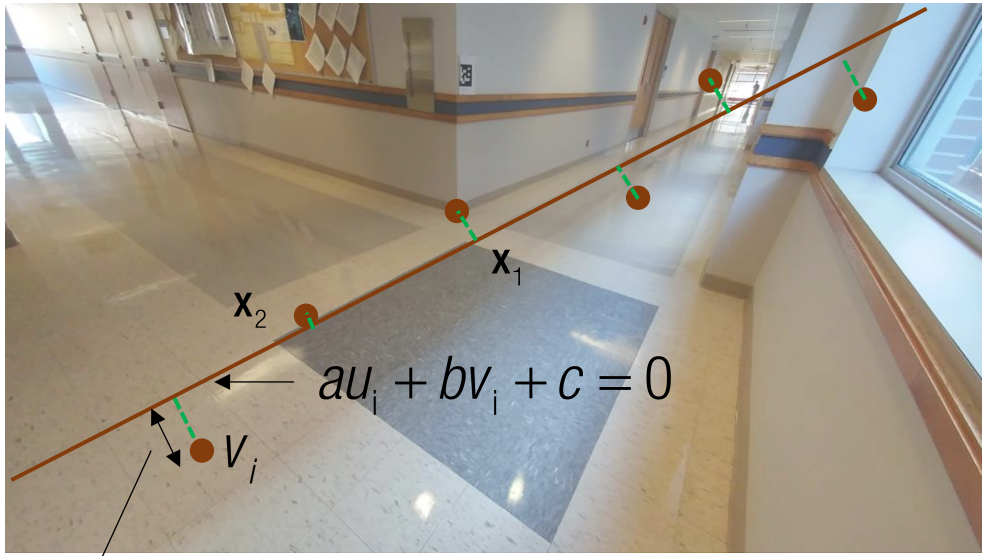
$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \qquad \text{subject to} \quad \|\mathbf{x}\| = 1$$

If A is mxn matrix where m < n,

Line Fitting (Ax=0)

Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: (a, b, c)



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Trivial solution: $\mathbf{x} = \mathbf{0}$

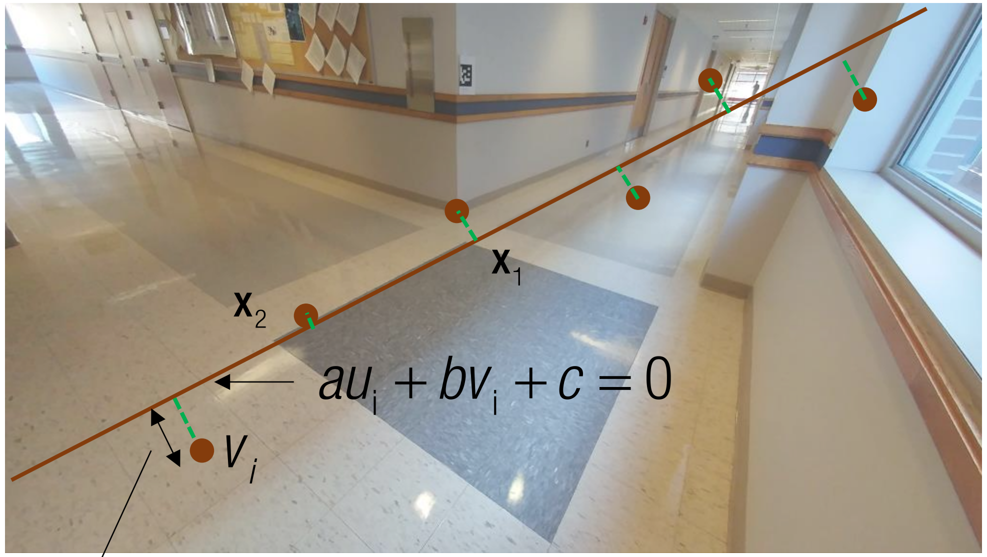
$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \quad \text{subject to} \quad \|\mathbf{x}\| = 1$$

If A is mxn matrix where m < n, $= \text{null}(\mathbf{A})$

Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$\mathbf{Ax} = \mathbf{0}$ $\qquad$ $\mathbf{Ax} = \mathbf{b}$

Trivial solution: $\mathbf{x} = \mathbf{0}$

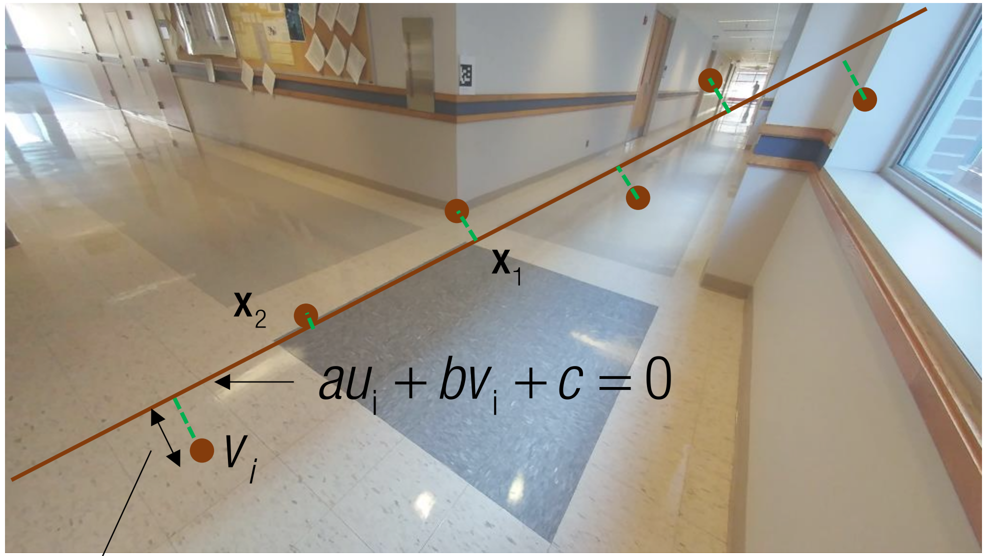
$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \qquad \text{subject to} \quad \|\mathbf{x}\| = 1$$

If A is mxn matrix where m > n,

Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

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$\mathbf{Ax} = \mathbf{0}$ $\qquad$ $\mathbf{Ax} = \mathbf{b}$

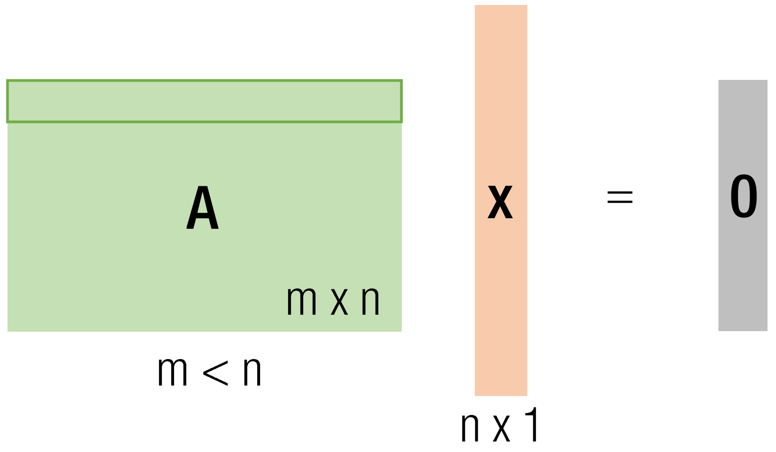
Trivial solution: $\mathbf{x} = \mathbf{0}$

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \qquad \text{subject to} \quad \|\mathbf{x}\| = 1$$

If A is mxn matrix where m > n, ▮ = approx.null(**A**)

Nullspace

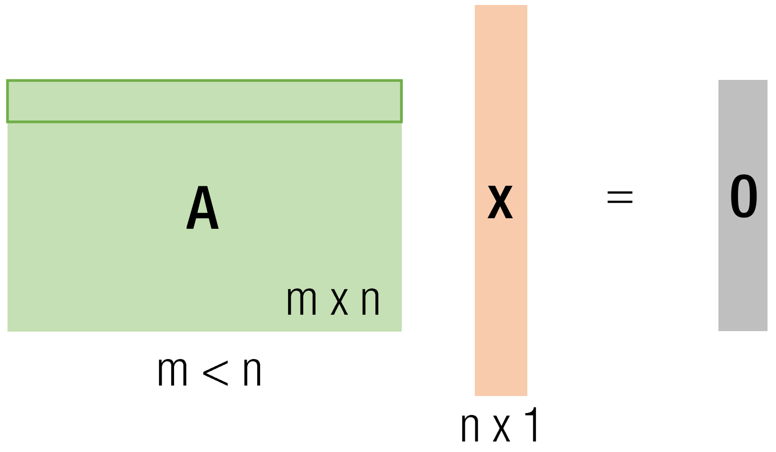
eqs < # unknowns



Nullspace

eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.



$$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$$

More SVD

1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1

14x14

More SVD

1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1

14x14

=

Column space

1
1
1
1
1
1
1
1
1
1
1
1
1
1

Row space

1	1	1	1	1	1	1	1	1	1	1	1	1	1

More SVD

Column space

Row space

$$
\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{bmatrix}
= \boxed{14}
\begin{bmatrix} 1/\sqrt{14} \\ \vdots \\ 1/\sqrt{14} \end{bmatrix}
\begin{bmatrix} 1/\sqrt{14} & \cdots & 1/\sqrt{14} \end{bmatrix}
$$

14x14

More SVD

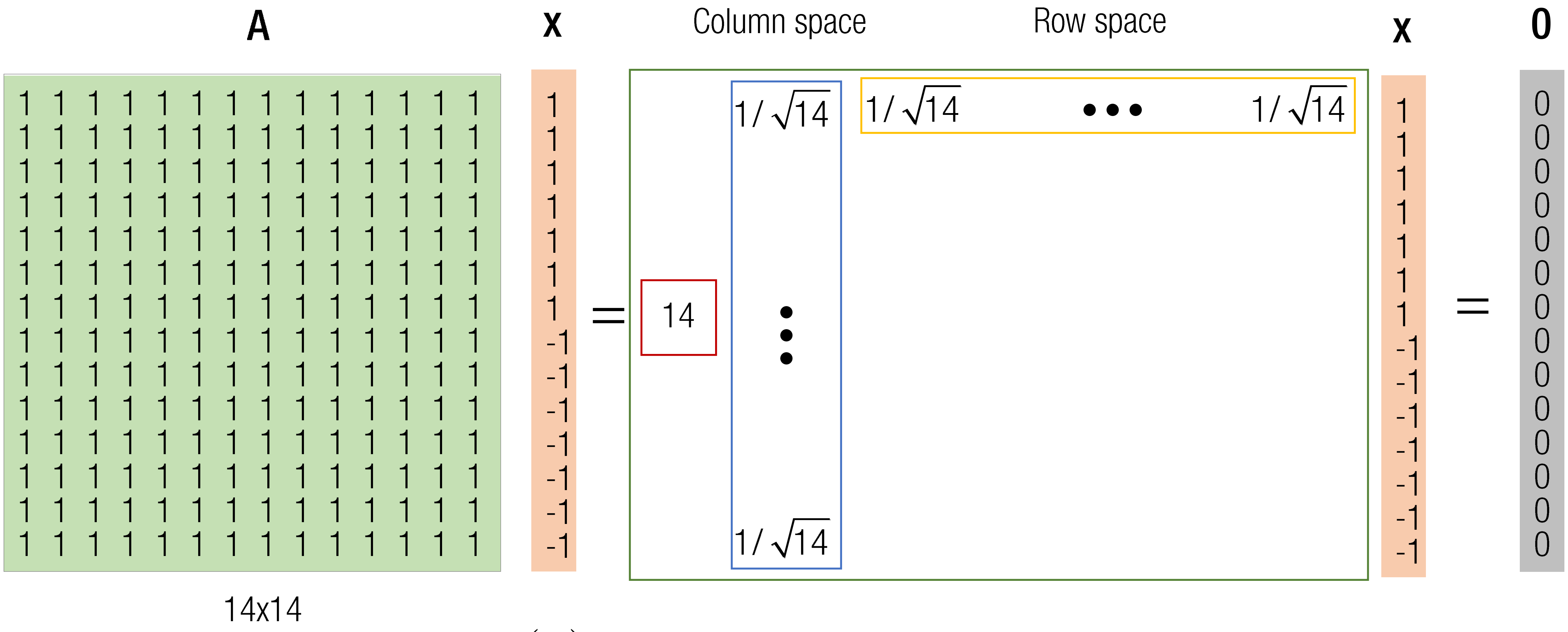
$$\underset{14\times14}{\mathbf{A}}\;\mathbf{x}=\mathbf{0}$$

$$
\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{bmatrix}
\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\-1\\-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix}
=
\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}
$$

14x14

$$\mathbf{x}=\mathrm{null}(\mathbf{A})$$

More SVD



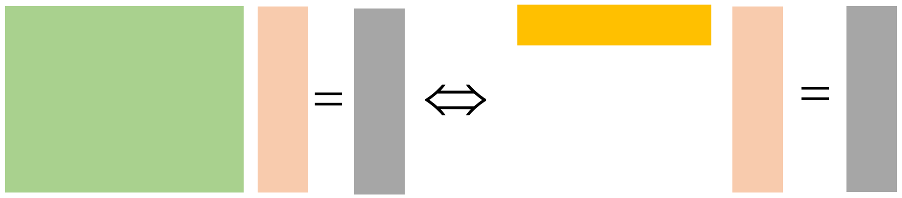
$$\mathbf{x} = \text{null}(\mathbf{A})$$

More SVD

Column space — Row space

$$\underbrace{\begin{bmatrix}1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1&1&1&1&1&1&1\end{bmatrix}}_{\mathbf{A},\ 14\times14}\underbrace{\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\-1\\-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix}}_{\mathbf{x}} = 14\begin{bmatrix}1/\sqrt{14}\\ \vdots\\ 1/\sqrt{14}\end{bmatrix}\begin{bmatrix}1/\sqrt{14} & \cdots & 1/\sqrt{14}\end{bmatrix}\underbrace{\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\-1\\-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}}_{\mathbf{0}}$$

$$\mathbf{x} = \text{null}(\mathbf{A})$$



More SVD

1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1

14x14

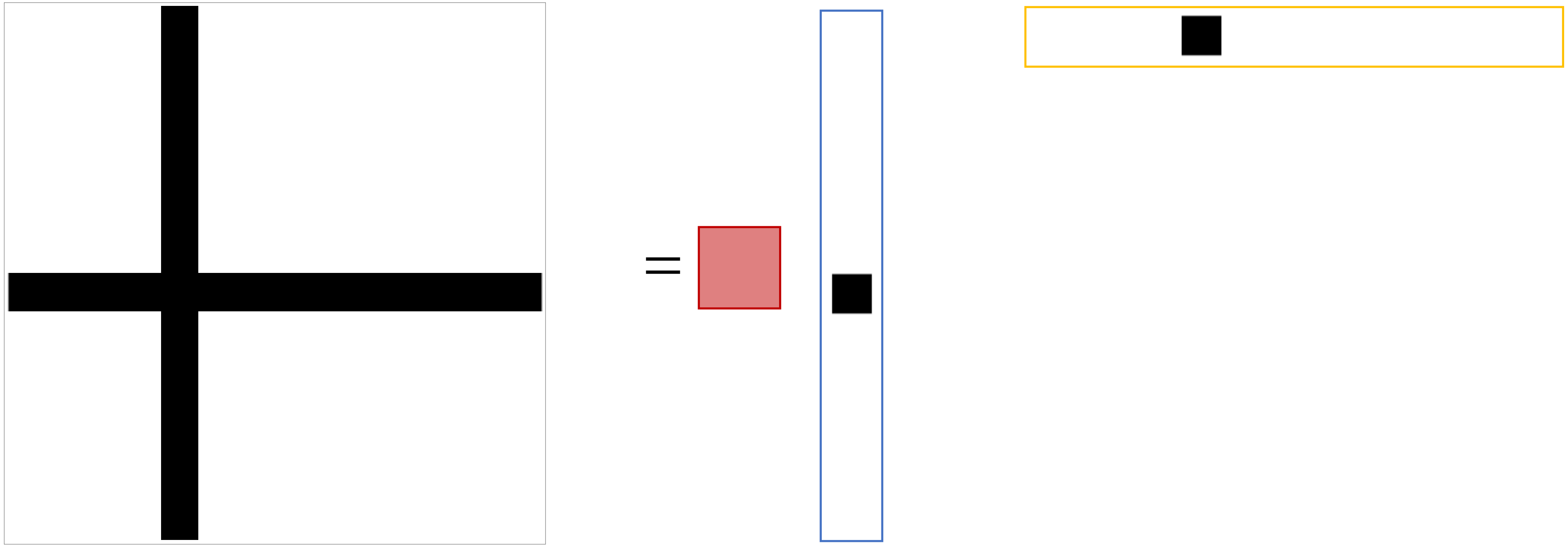
More SVD

1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1

14x14

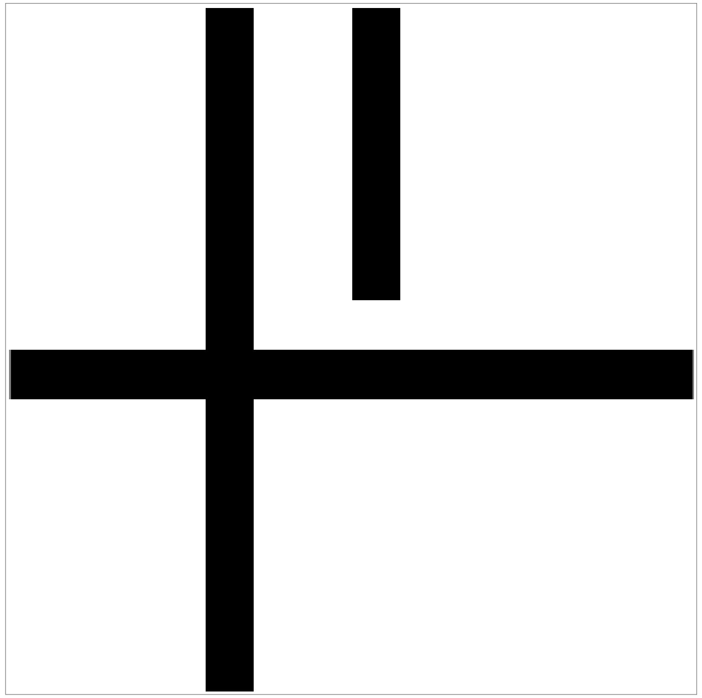
$$= \boxed{13} \begin{bmatrix} 1/\sqrt{13} \\ \vdots \\ 1/\sqrt{13} \end{bmatrix} \begin{bmatrix} 1/\sqrt{13} & \cdots & 1/\sqrt{13} \end{bmatrix}$$

More SVD



14x14

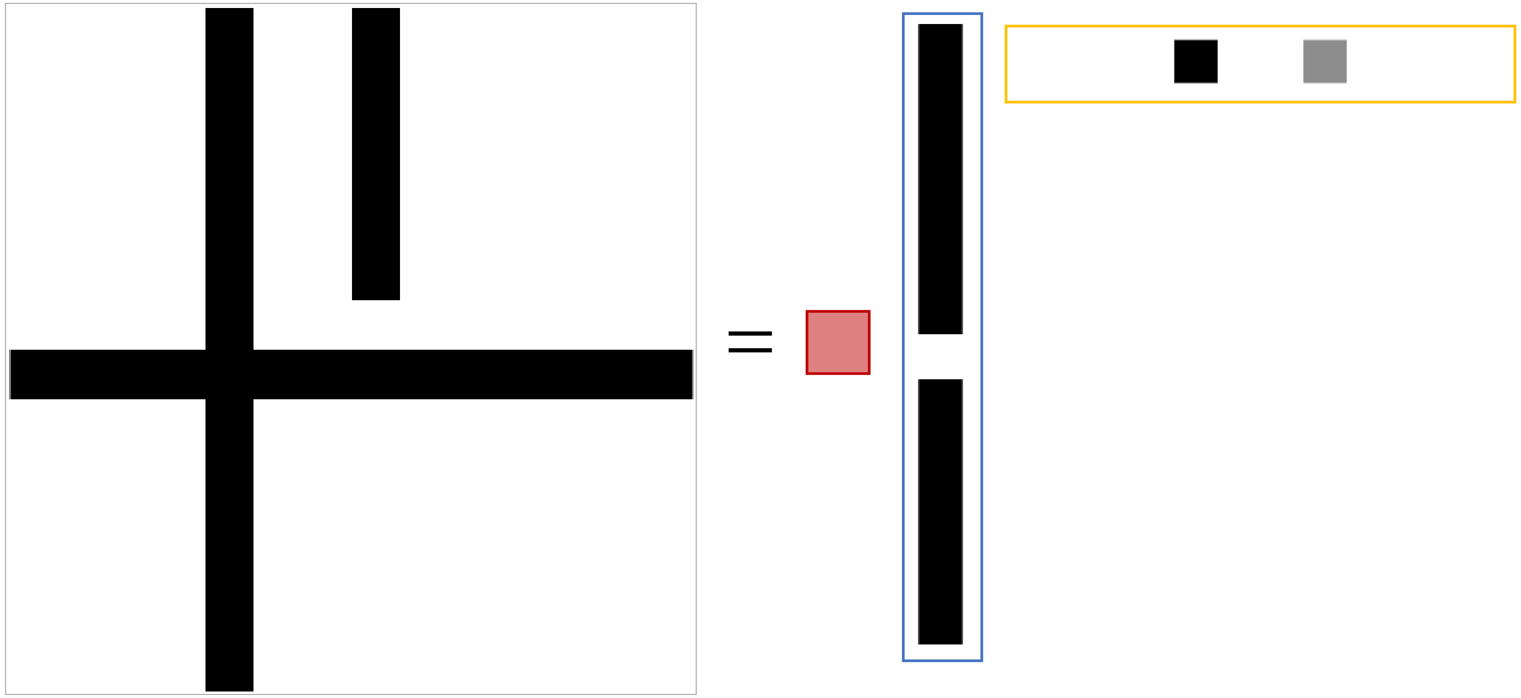
More SVD



=

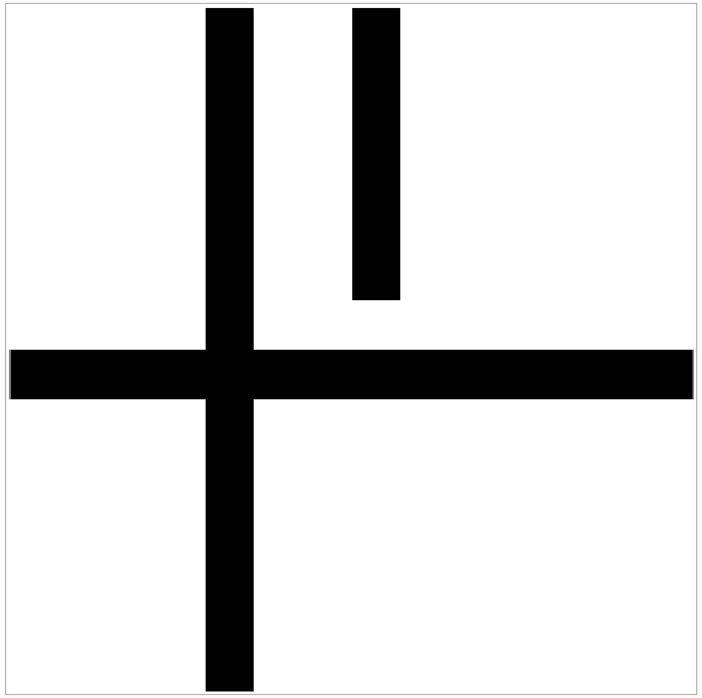
14x14

More SVD

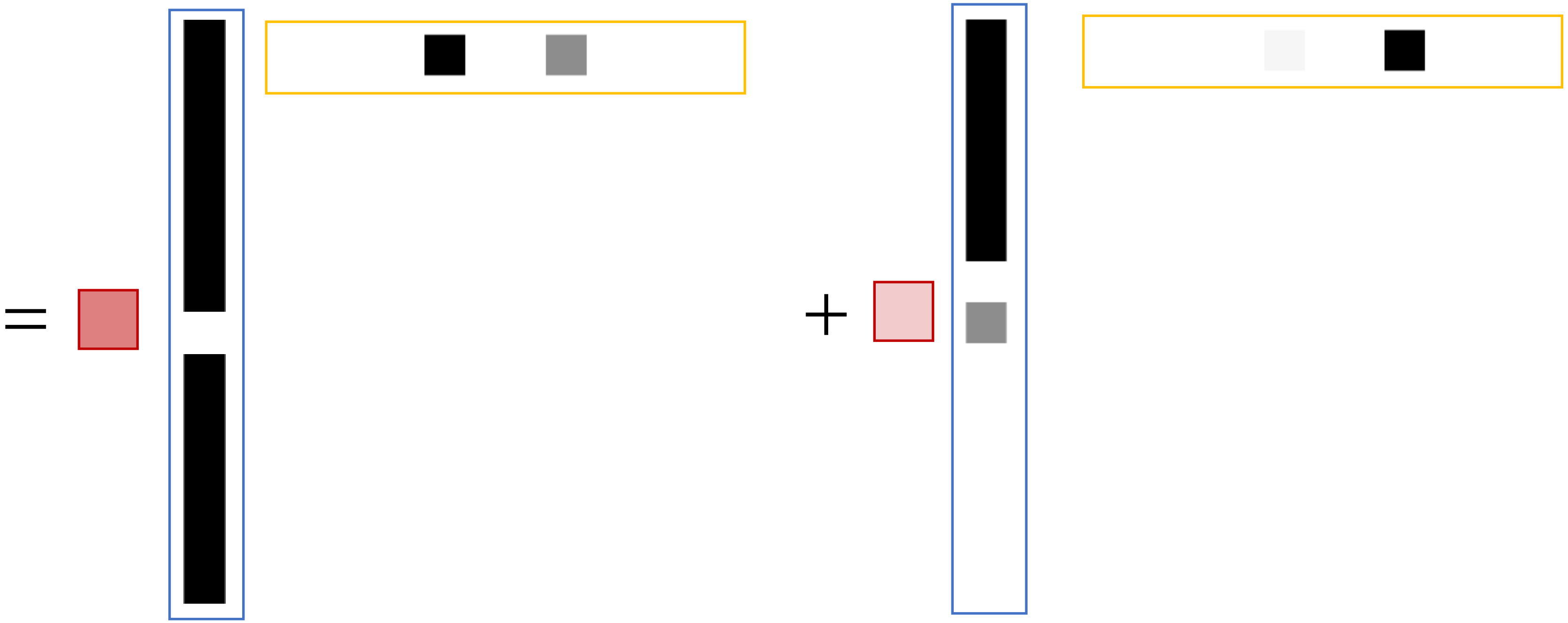


14x14

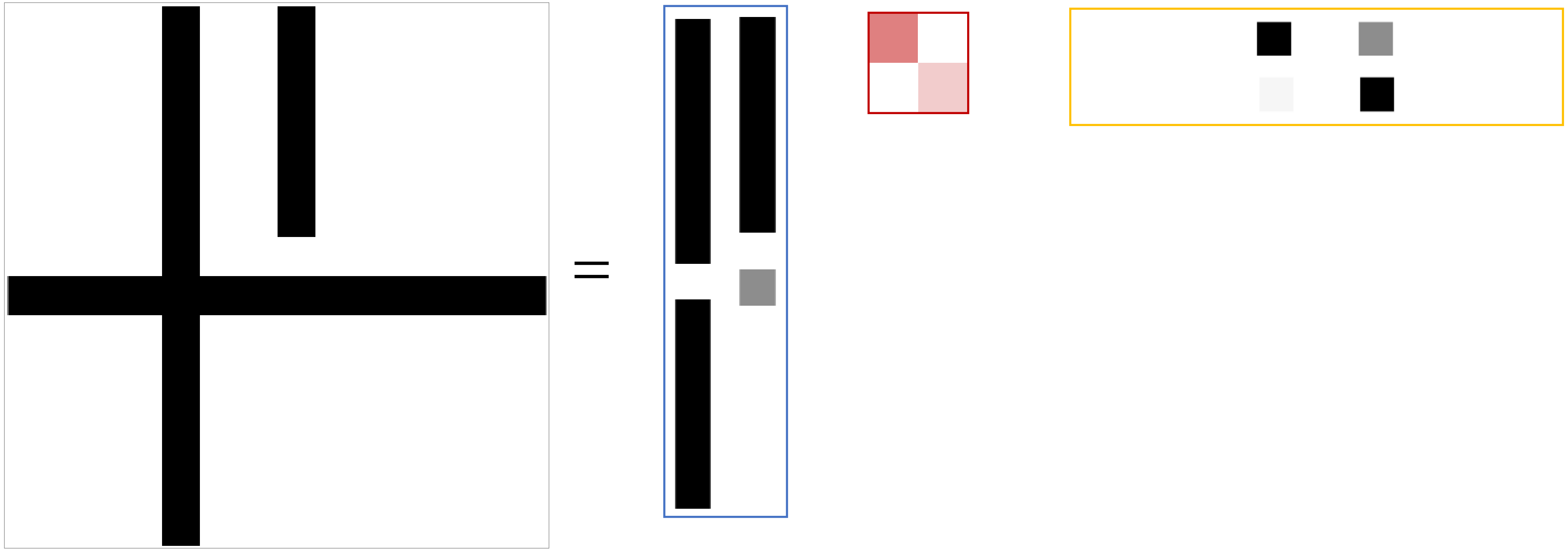
More SVD



14x14



More SVD

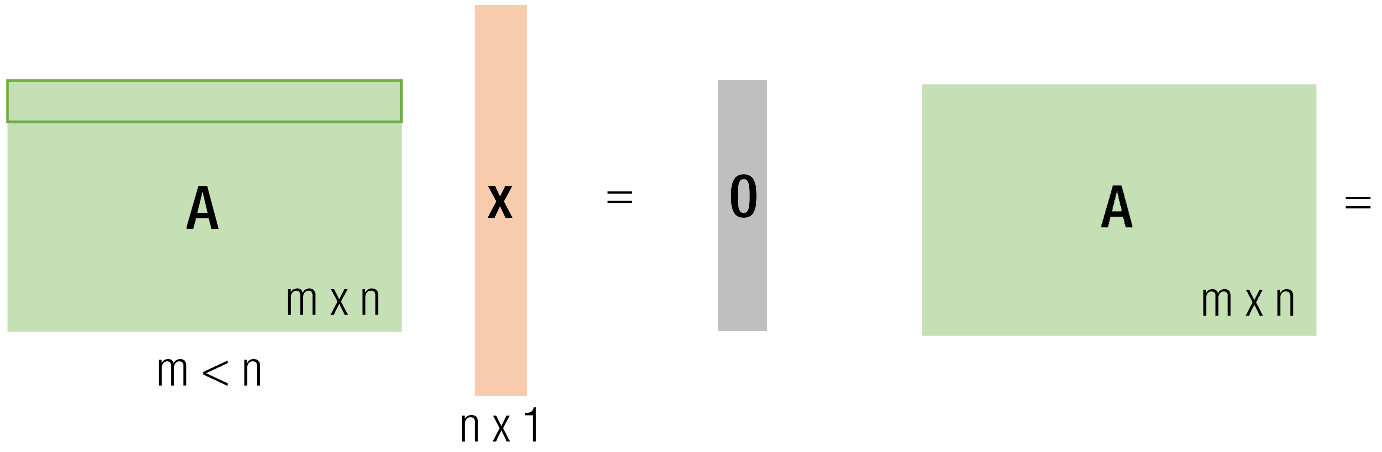


14x14

Singular Value Decomposition (SVD)

\# eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.

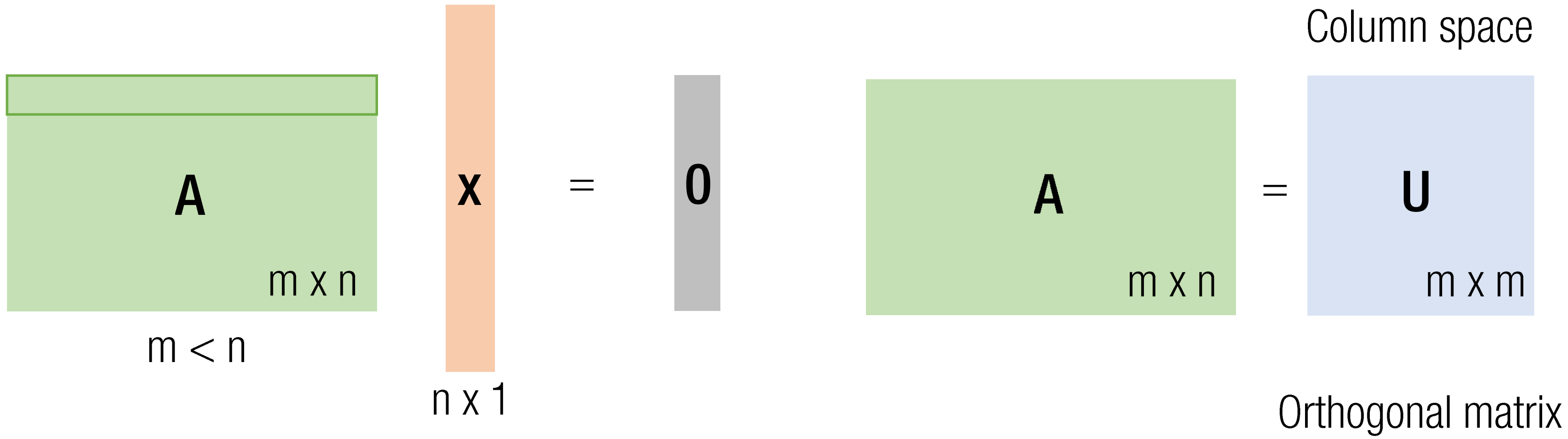


$\rightarrow \mathbf{x} = \text{null}(\mathbf{A})$

Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.

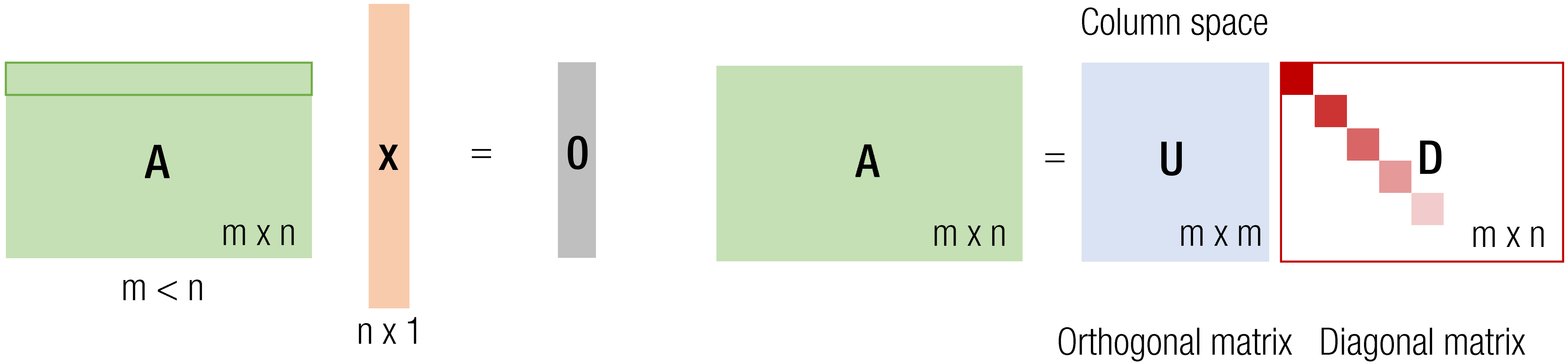


$\longrightarrow$ $\mathbf{x} = \text{null}(\mathbf{A})$

Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.

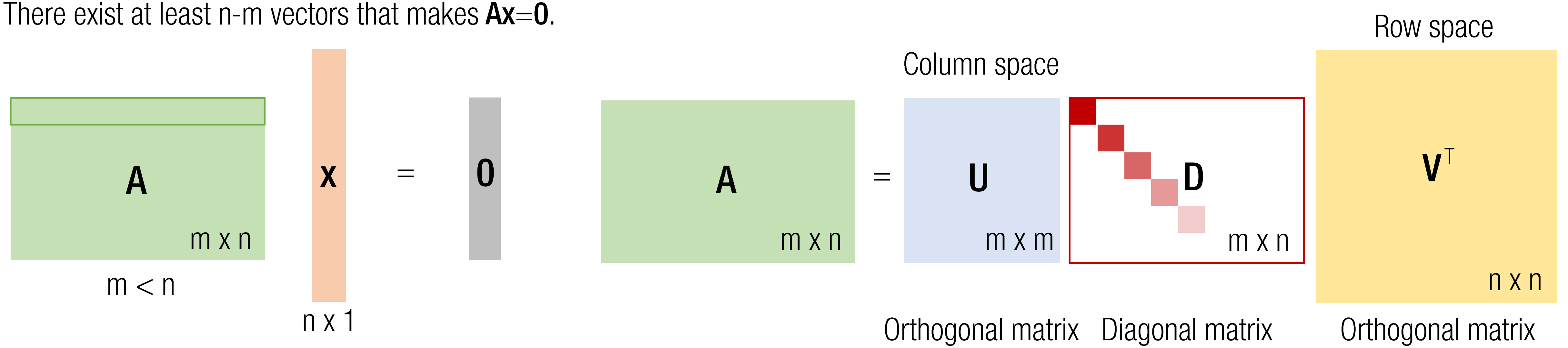


$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$

Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.

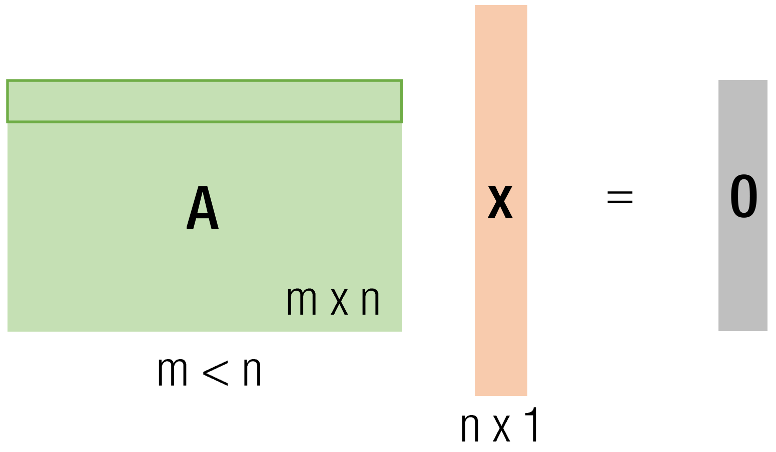


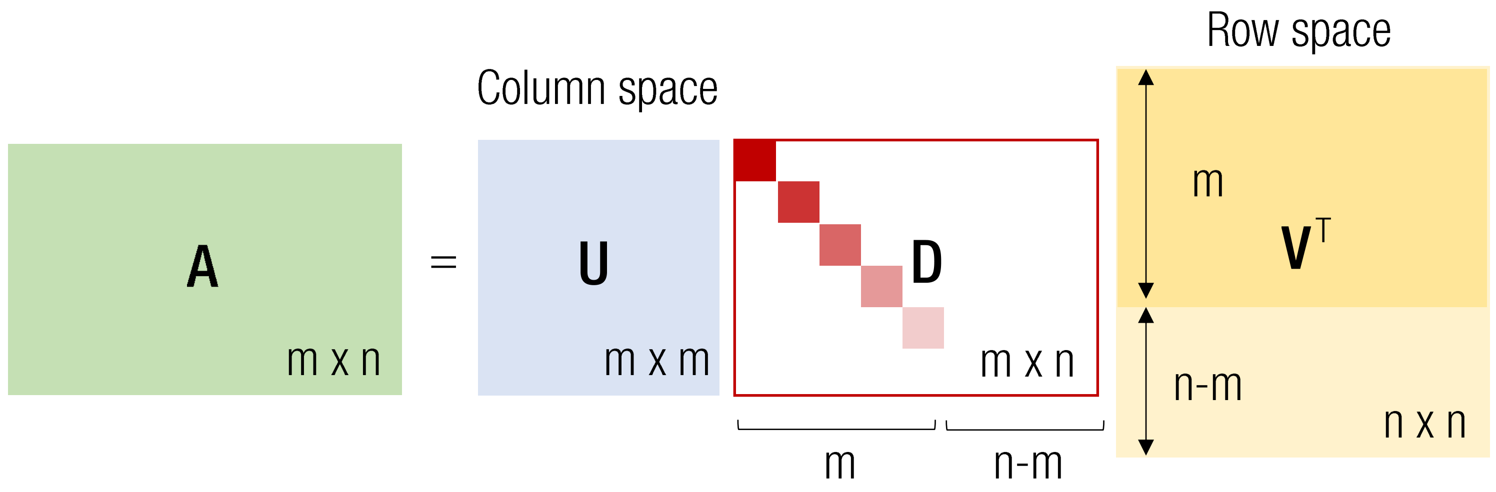
$\longrightarrow \mathbf{x} = \mathrm{null}(\mathbf{A})$

Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.



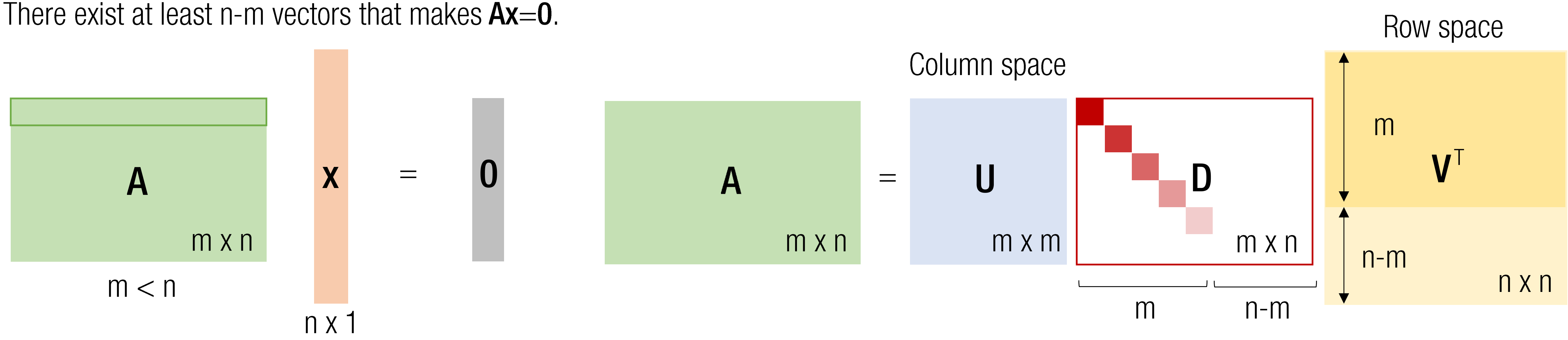


$$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$$

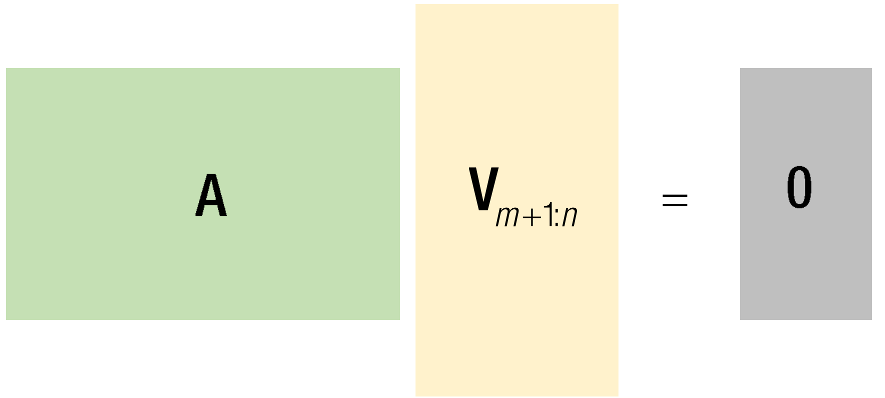
Singular Value Decomposition (SVD)

\# eqs < # unknowns

There exist at least n-m vectors that makes $\mathbf{Ax}=\mathbf{0}$.



$$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$$

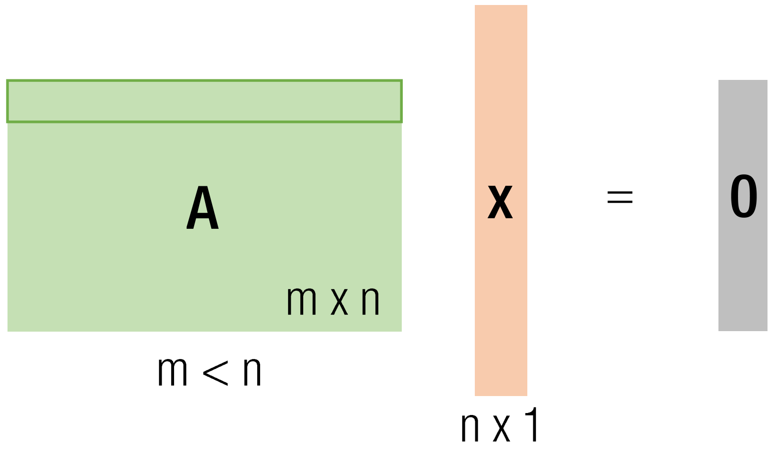


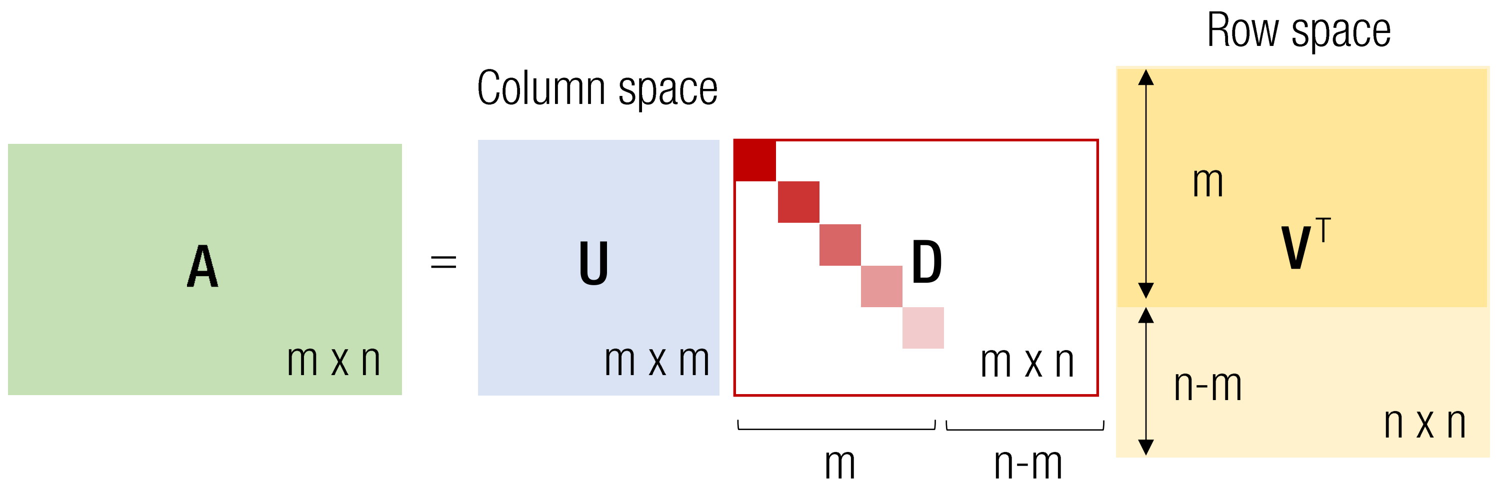
$$\mathbf{V}_{m+1:n} = \text{null}(\mathbf{A})$$

Singular Value Decomposition (SVD)

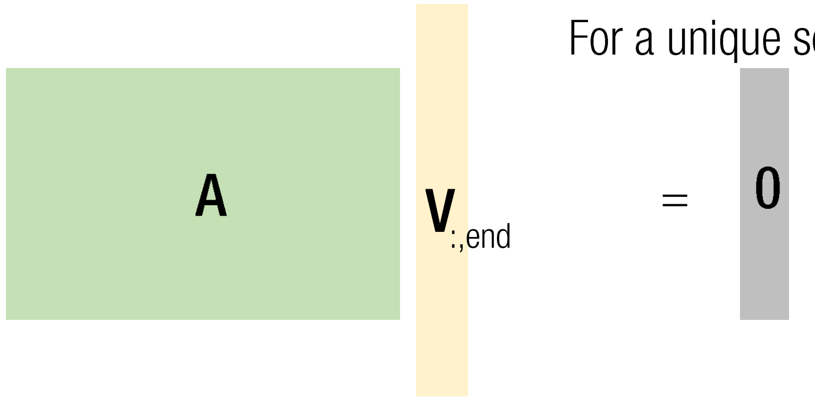
\# eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.





$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$



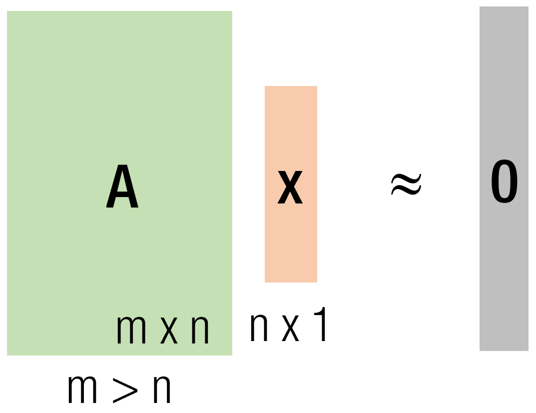
For a unique solution, **A** should be m x (m+1)

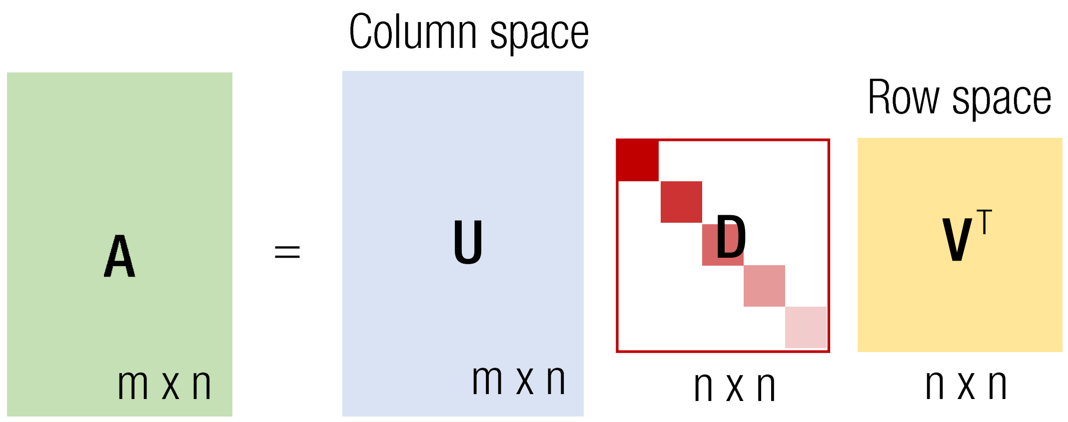
$$\mathbf{V}_{:end} = \text{null}(\mathbf{A})$$

Singular Value Decomposition (SVD)

eqs > # unknowns

There exist no nullspace of **A**.



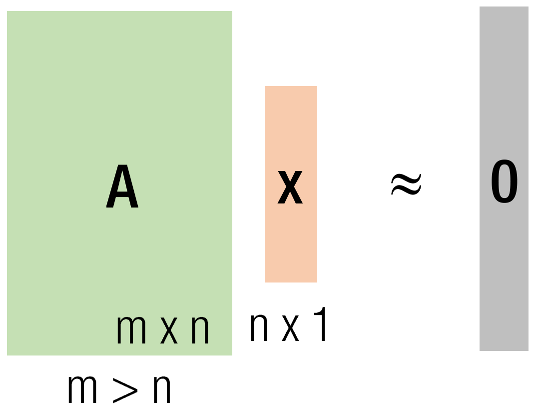


$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \text{ subject to } \|\mathbf{x}\| = 1$$

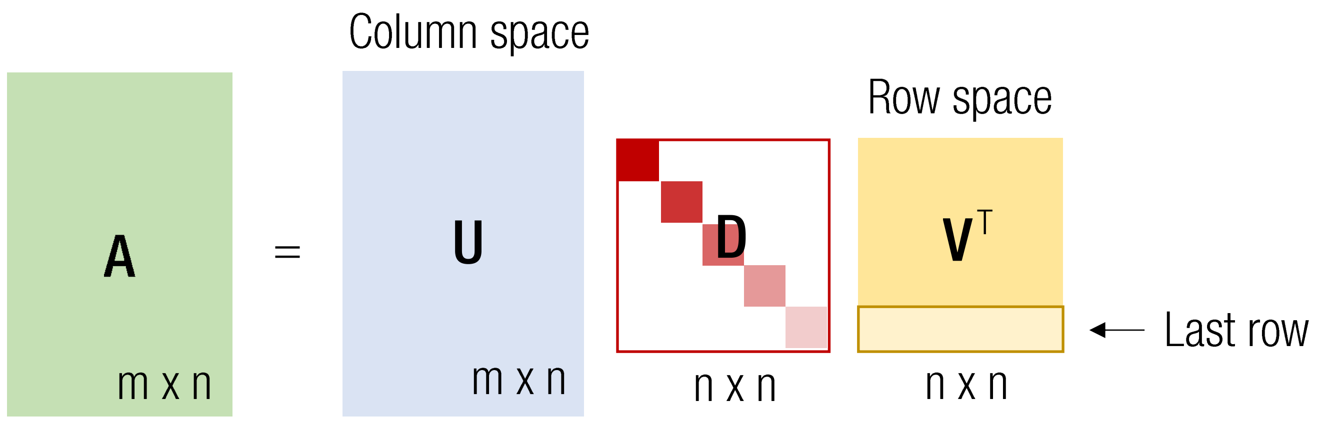
Singular Value Decomposition (SVD)

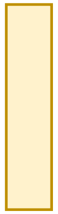
eqs > # unknowns

There exist no nullspace of **A**.

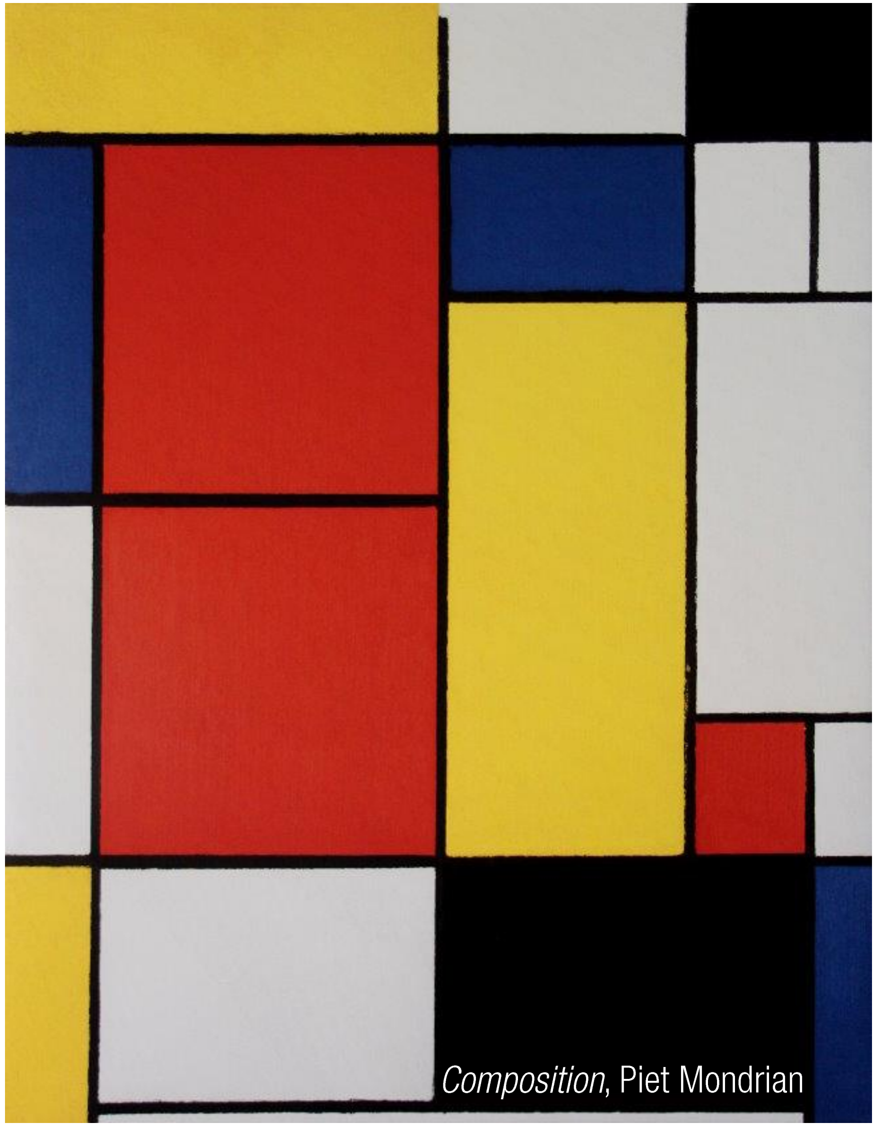


$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \text{ subject to } \|\mathbf{x}\| = 1$$

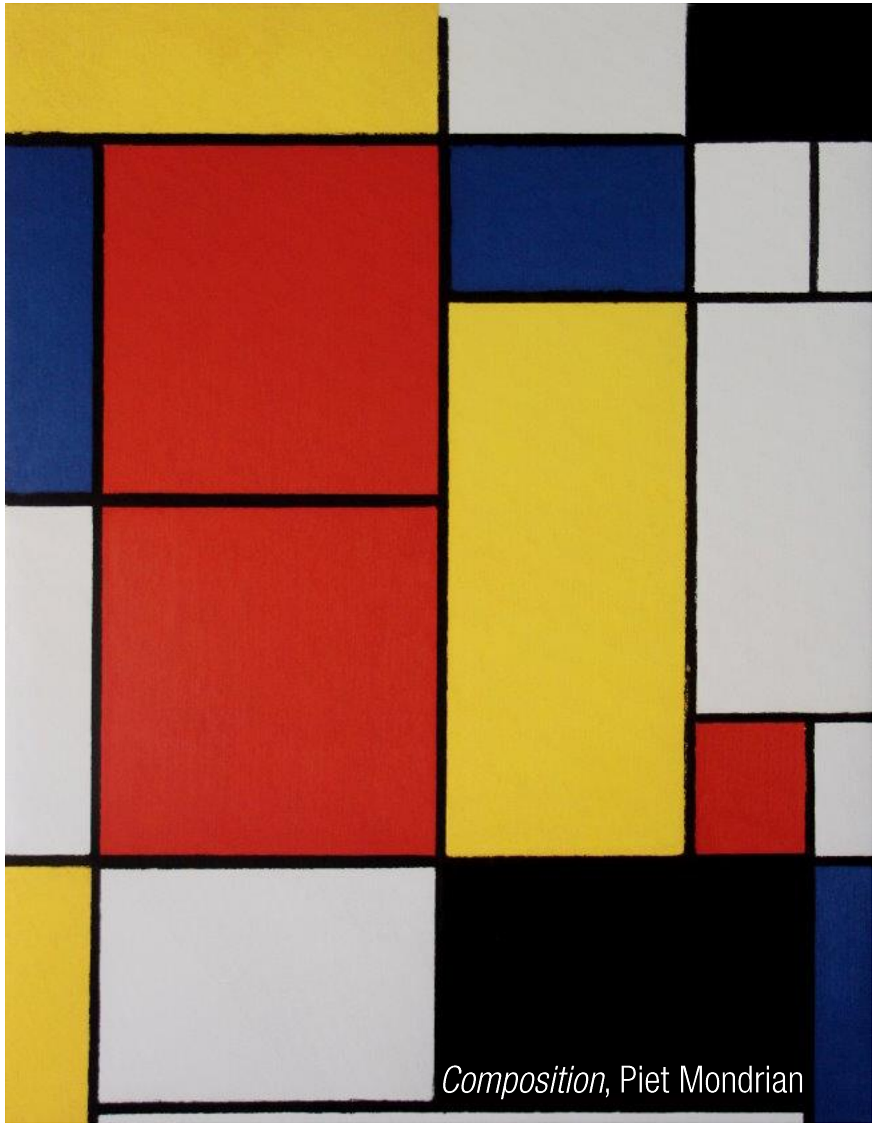


Approximated nullspace of **A**:  $\mathbf{V}_{:,end}$

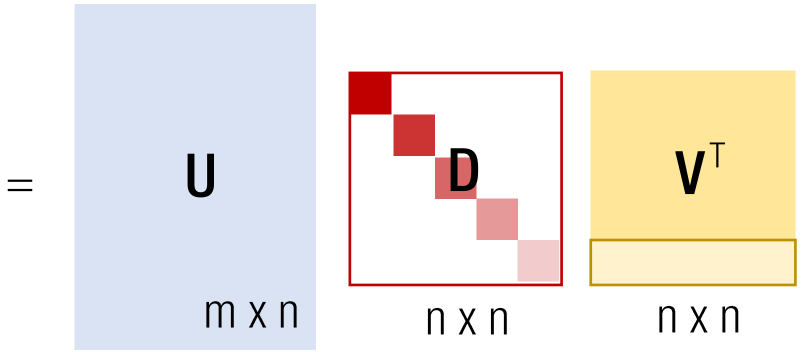
Mondrian Painting SVD


Composition, Piet Mondrian

Mondrian Painting SVD

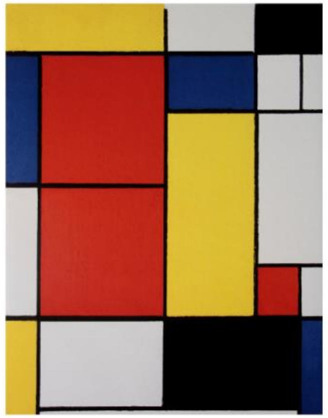


Composition, Piet Mondrian

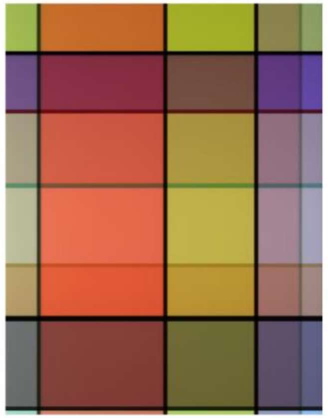


Mondrian Painting SVD Approximation

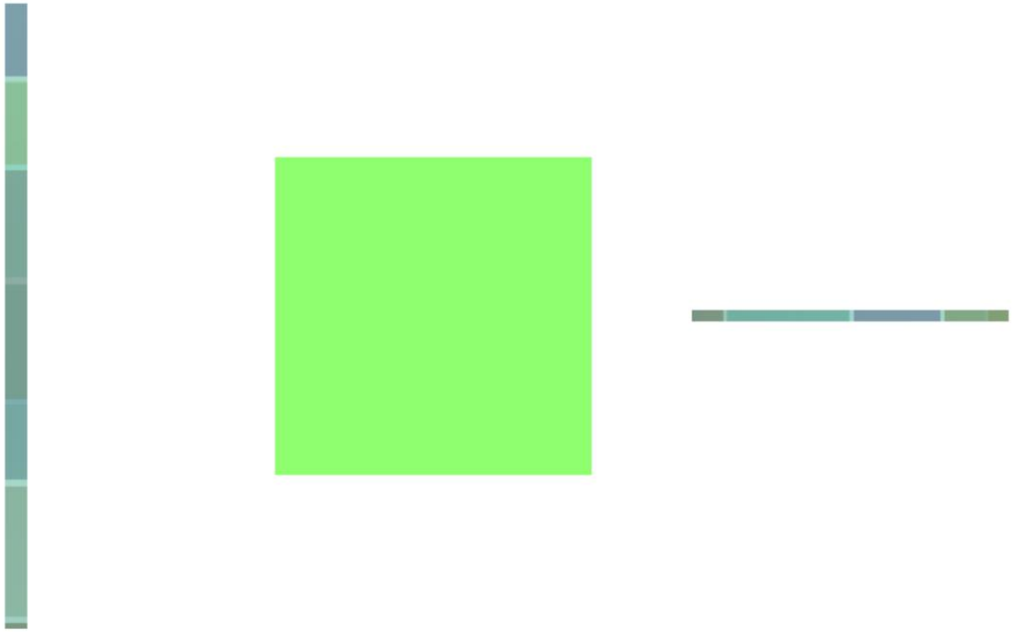
MondrianSVD.m



Ground truth



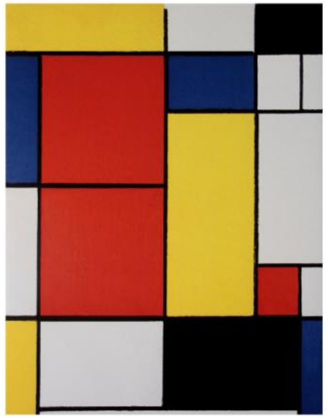
Number of basis: 1



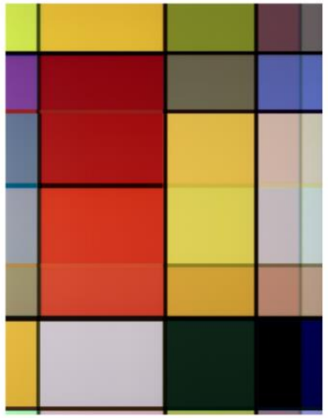
$$A = U \quad D \quad V^T$$

Mondrian Painting SVD Approximation

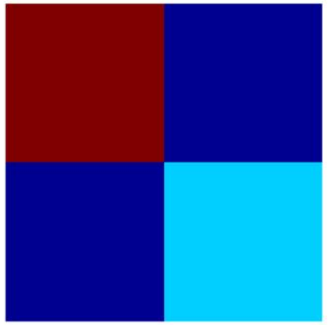
MondrianSVD.m



Ground truth



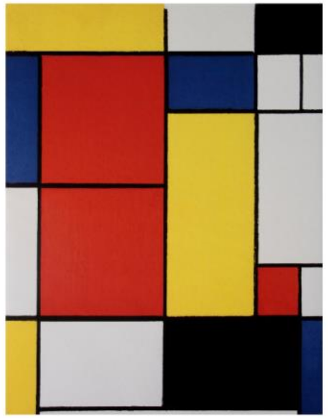
Number of basis: 2



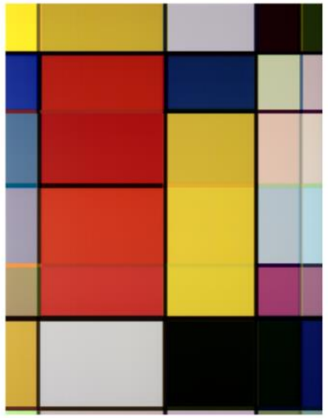
$$A = U \; D \; V^{T}$$

Mondrian Painting SVD Approximation

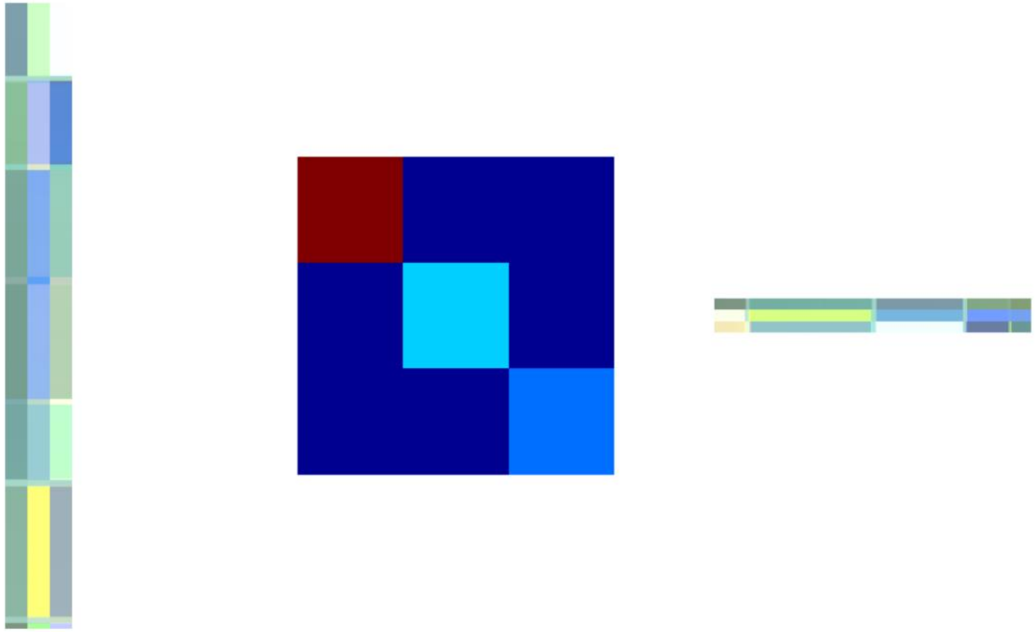
MondrianSVD.m



Ground truth



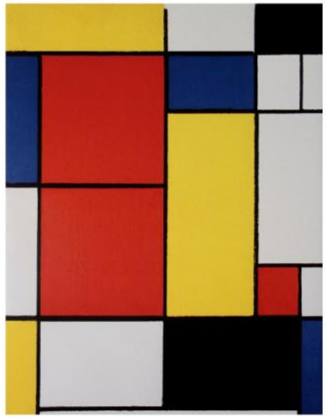
Number of basis: 3



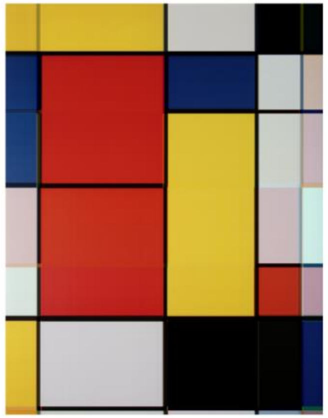
$$A = U \quad D \quad V^T$$

Mondrian Painting SVD Approximation

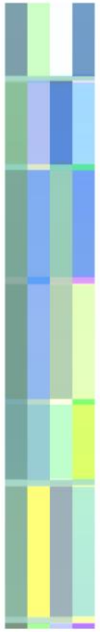
MondrianSVD.m

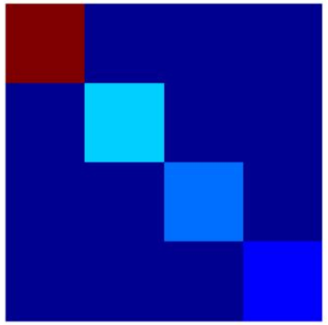


Ground truth



Number of basis: 4

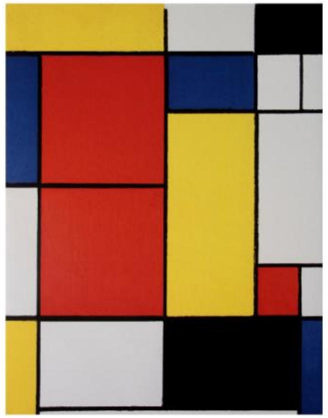




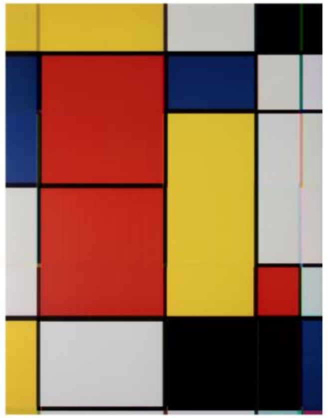
$$A = U \quad D \quad V^T$$

Mondrian Painting SVD Approximation

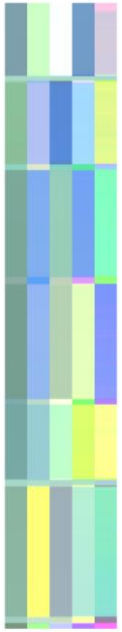
MondrianSVD.m

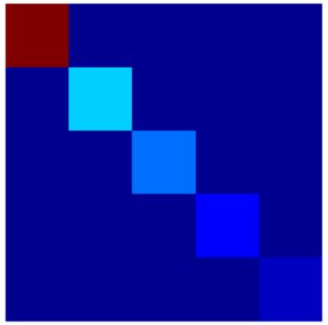


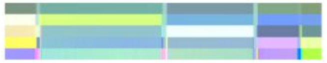
Ground truth



Number of basis: 5

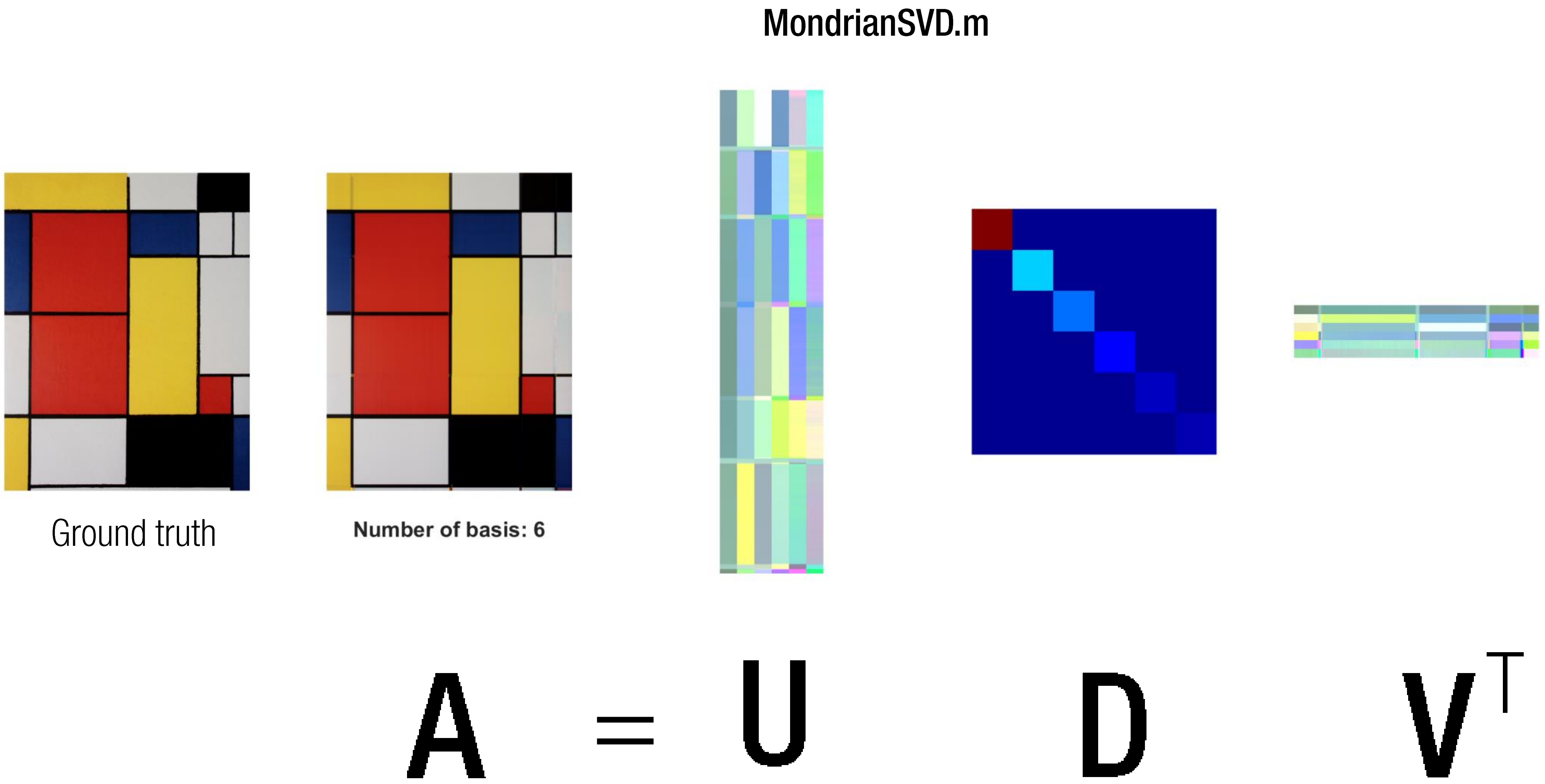






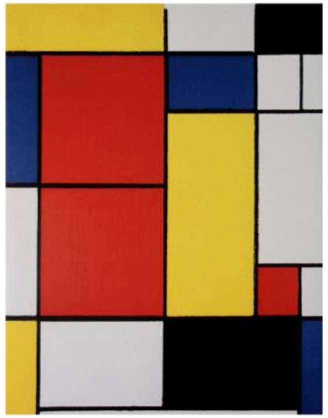
$$A = U \quad D \quad V^T$$

Mondrian Painting SVD Approximation

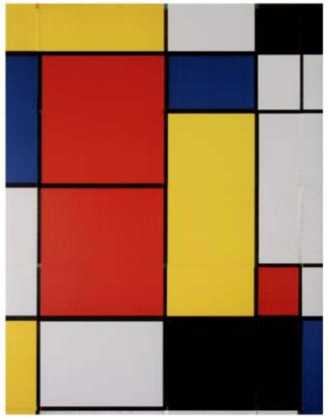


Mondrian Painting SVD Approximation

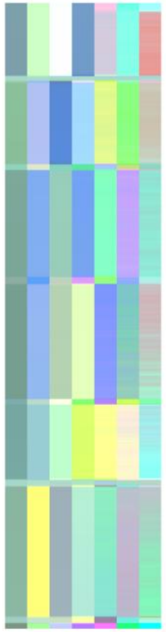
MondrianSVD.m

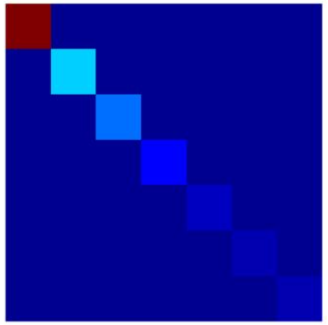


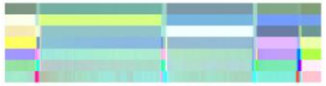
Ground truth



Number of basis: 7



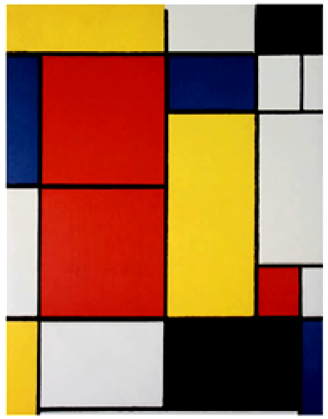




$$A = U \quad D \quad V^T$$

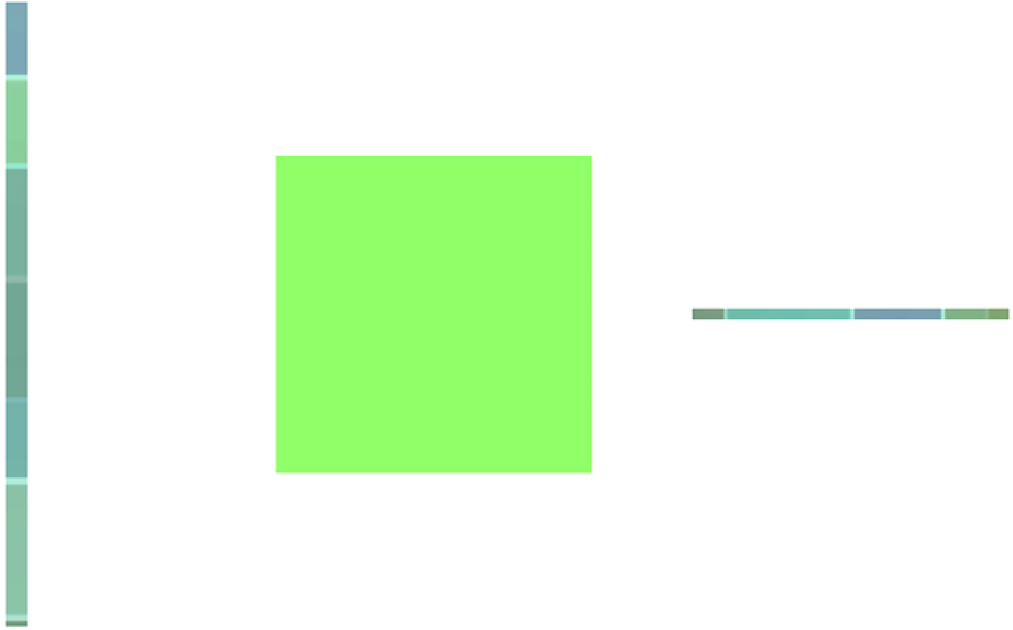
Mondrian Painting SVD Approximation

MondrianSVD.m



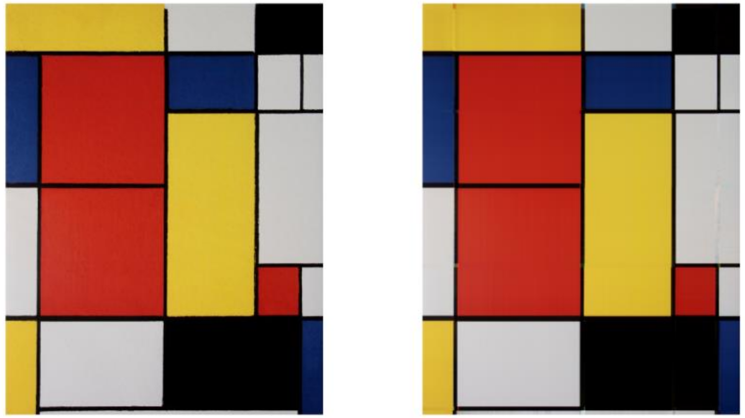
Ground truth

Number of basis: 1



$$A = U \quad D \quad V^T$$

Reconstruction Error

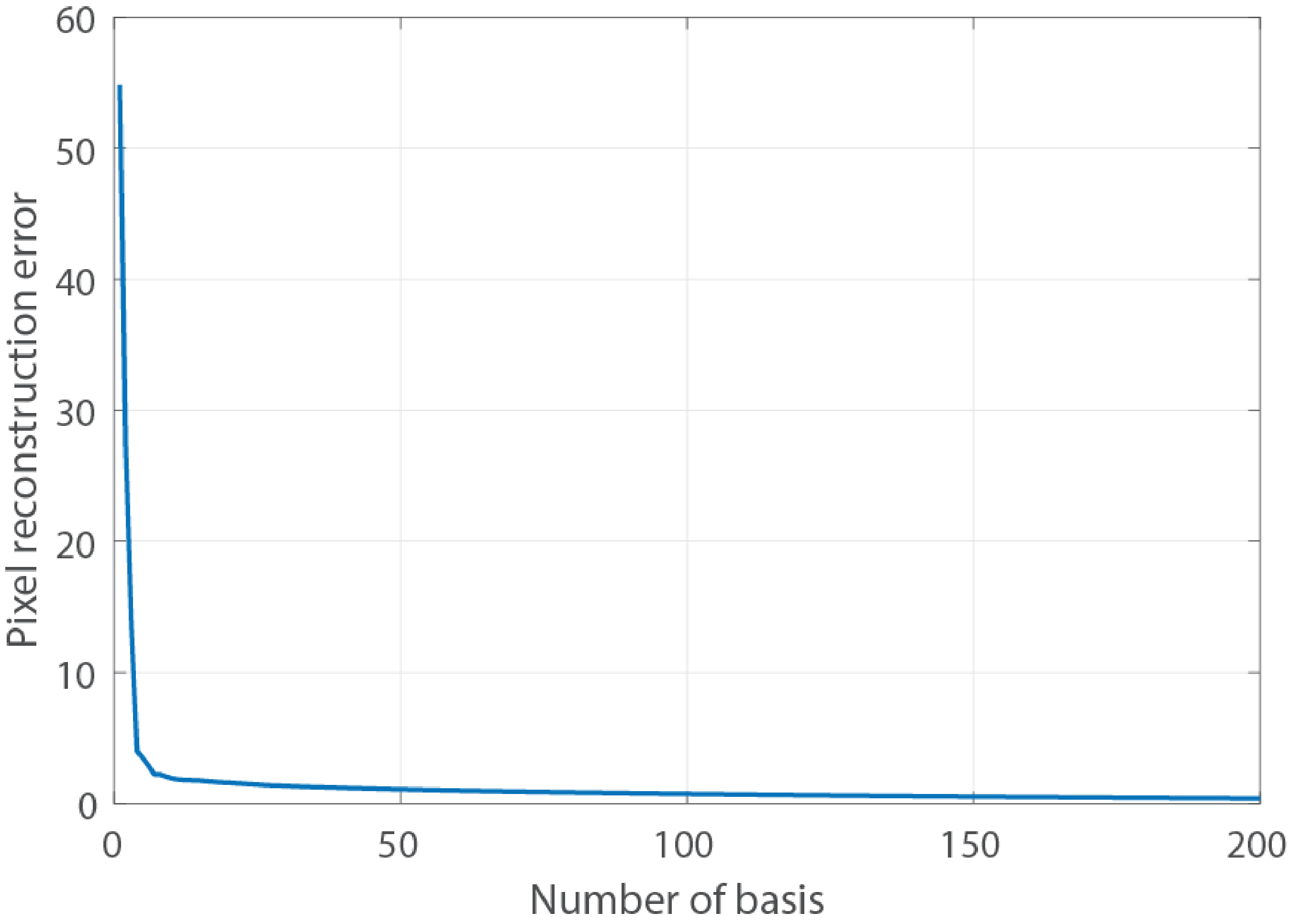


Ground truth

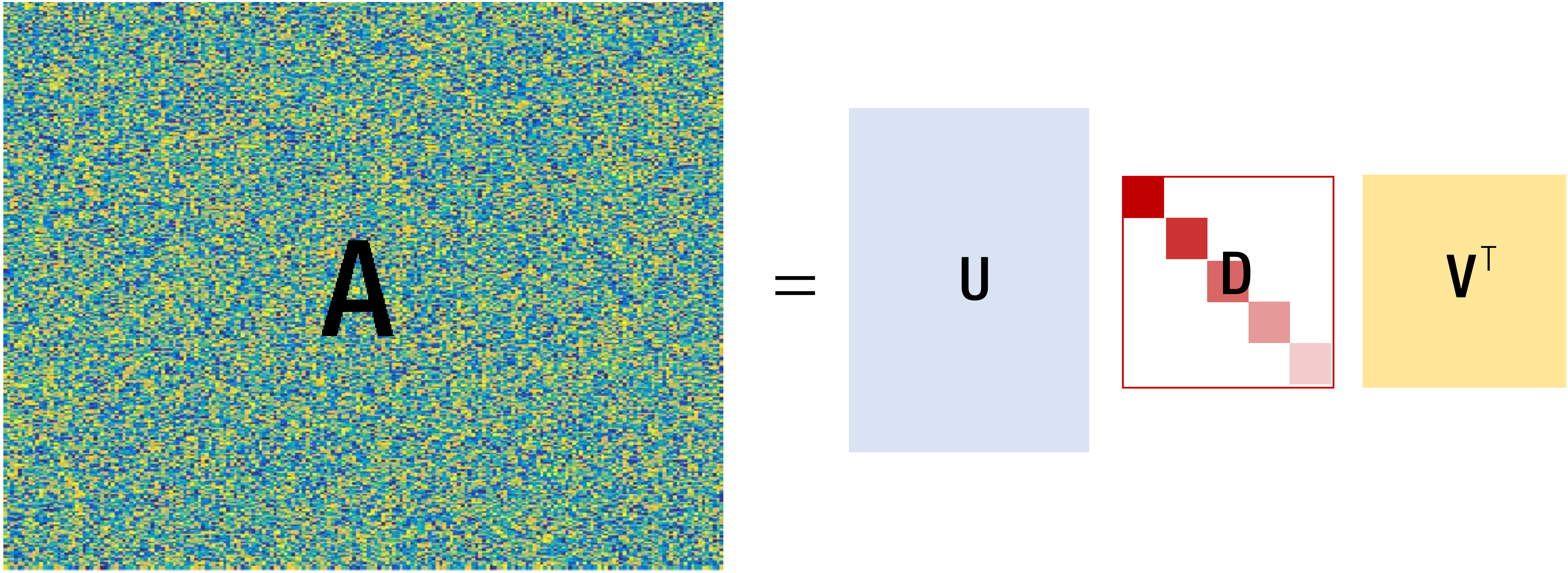
Number of basis: 7

A

MondrianSVD.m

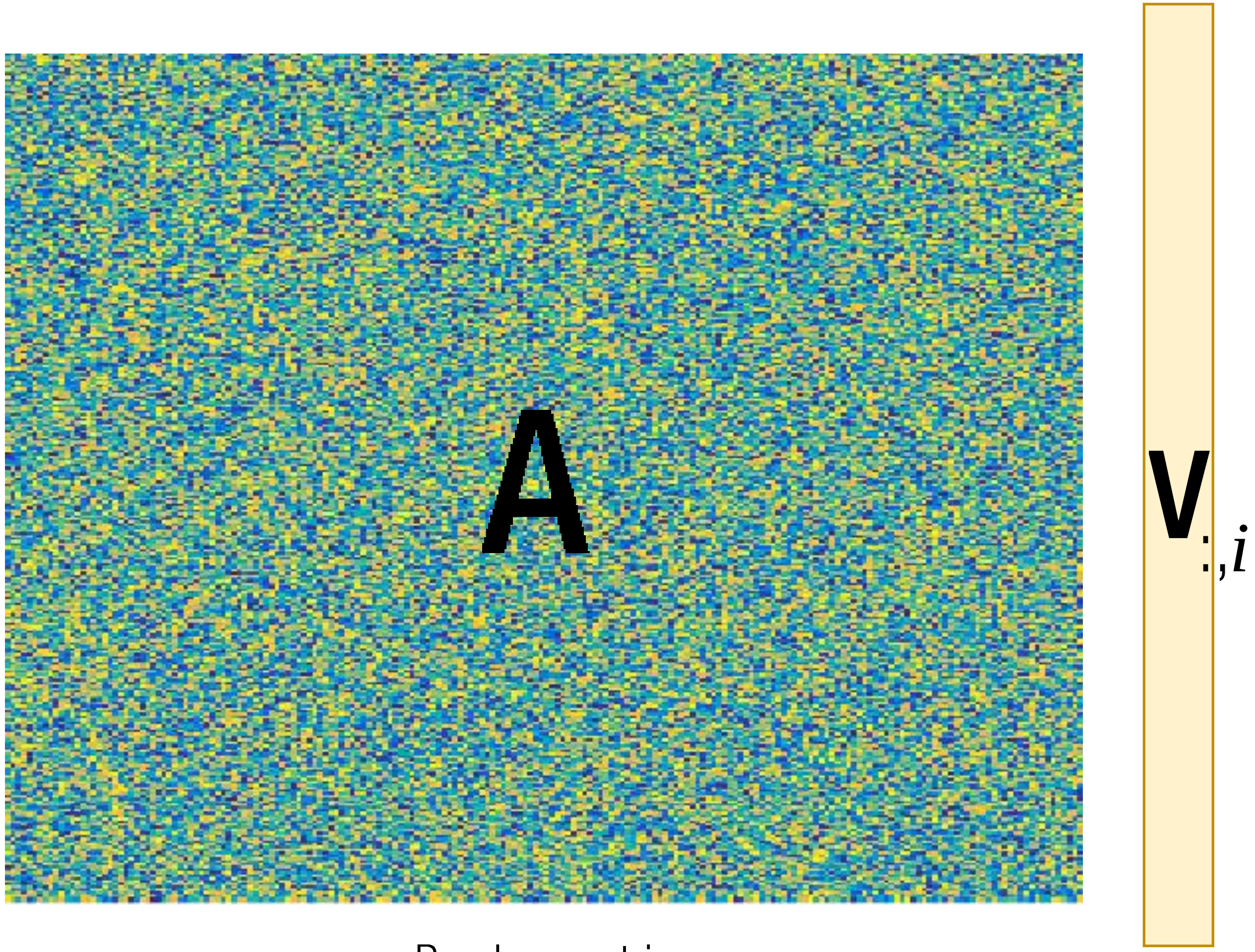


Random Matrix SVD



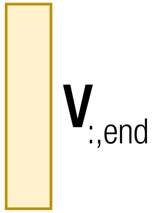
Random matrix

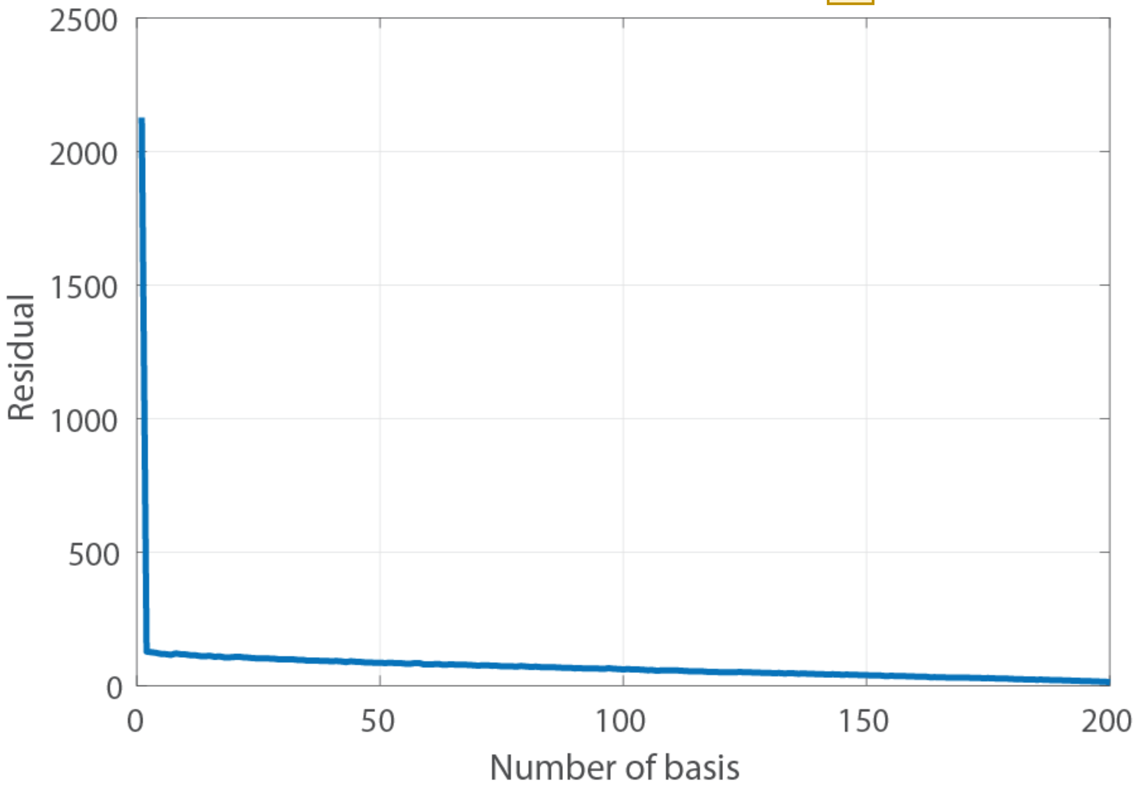
Residual (Nullspace Approximation)



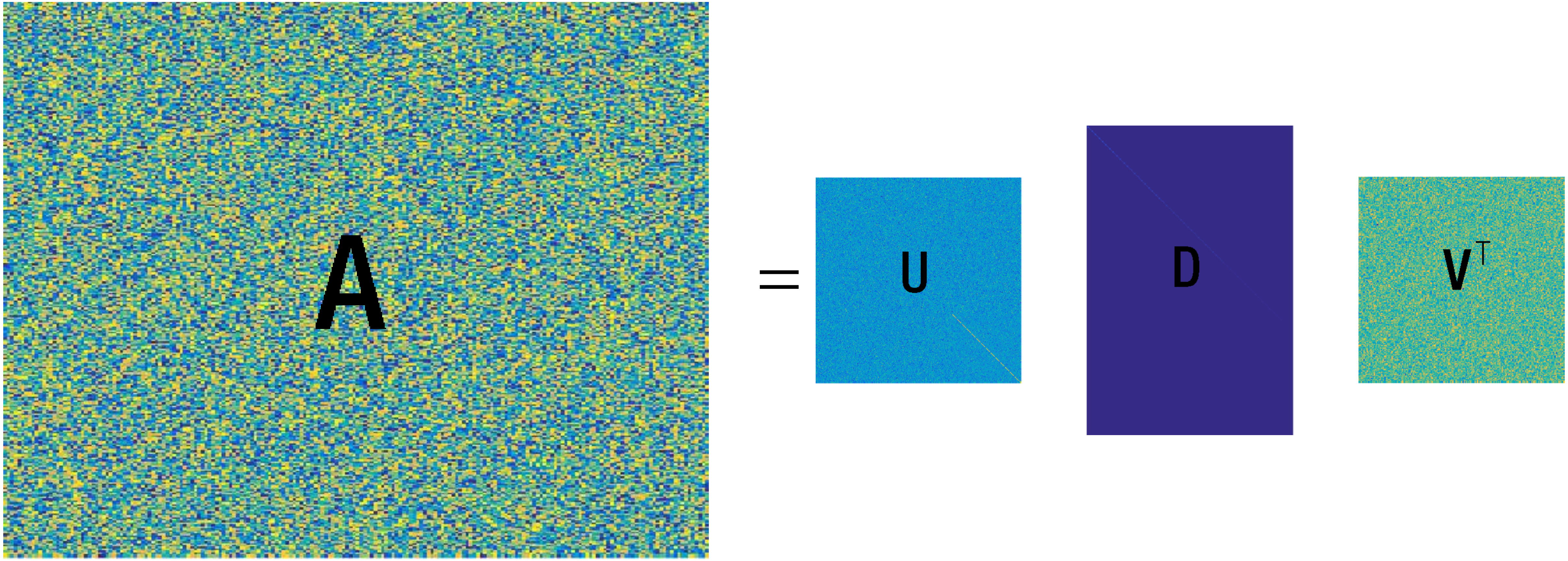
Random matrix

Approximated nullspace of **A**:



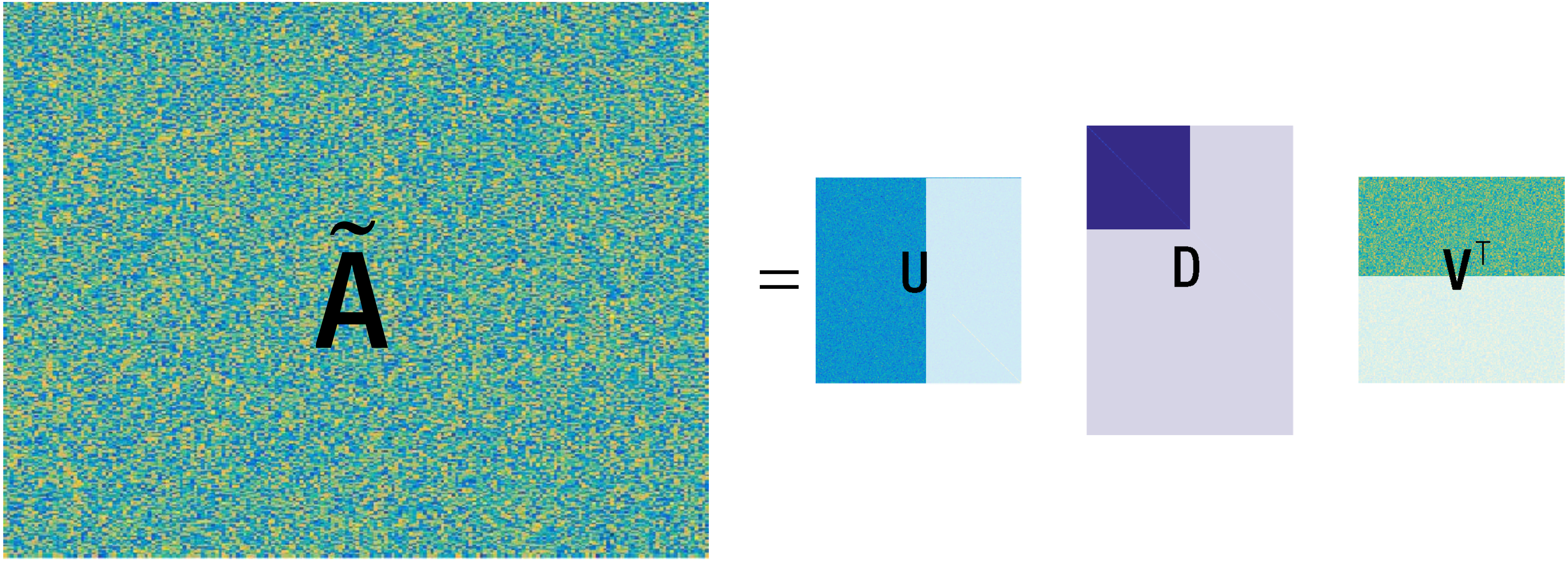


SVD Matrix Approximation



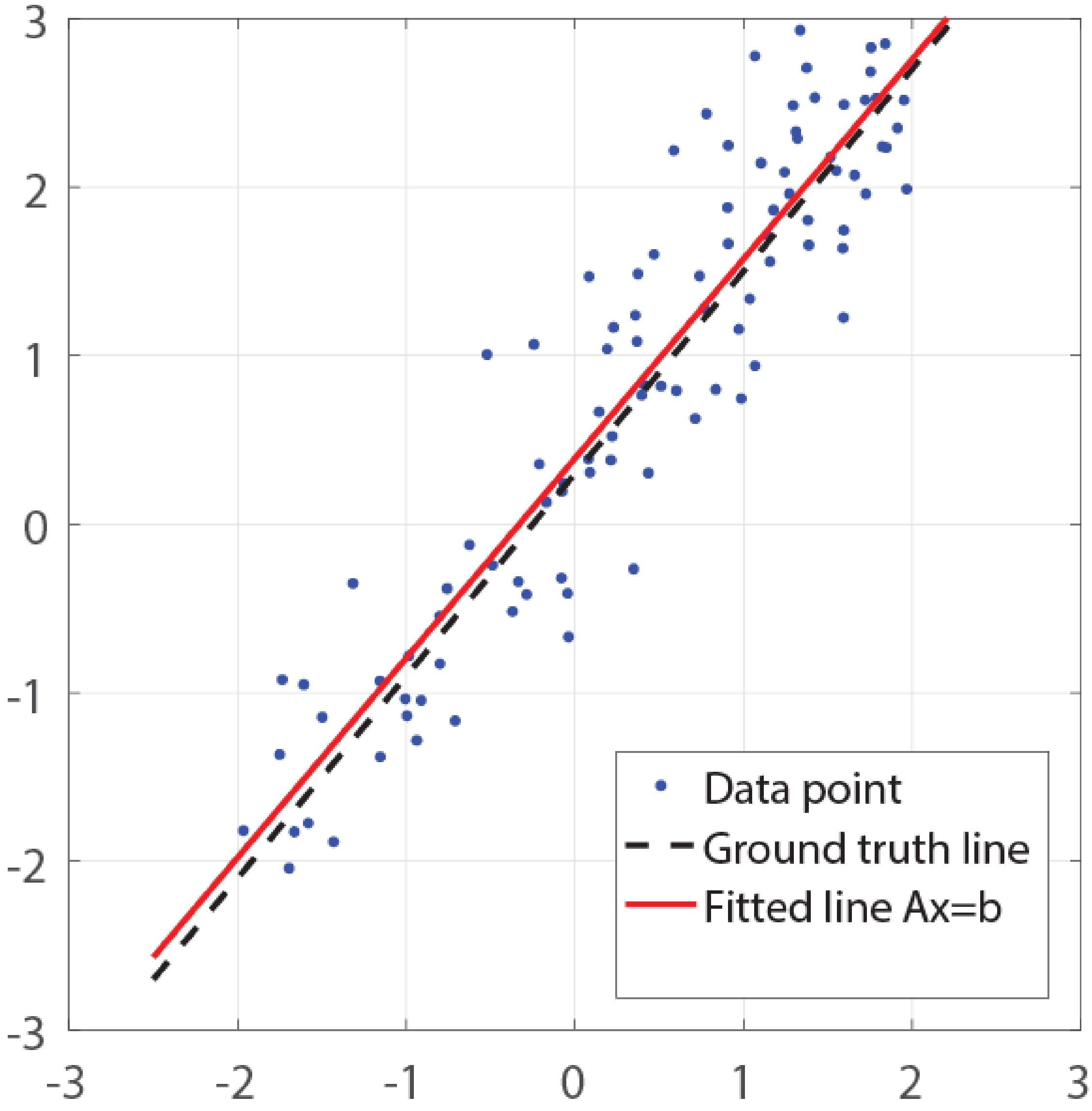
Random matrix

SVD Matrix Approximation



Reconstructed matrix

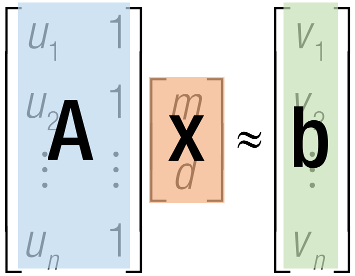
Line Fitting (Ax=b)



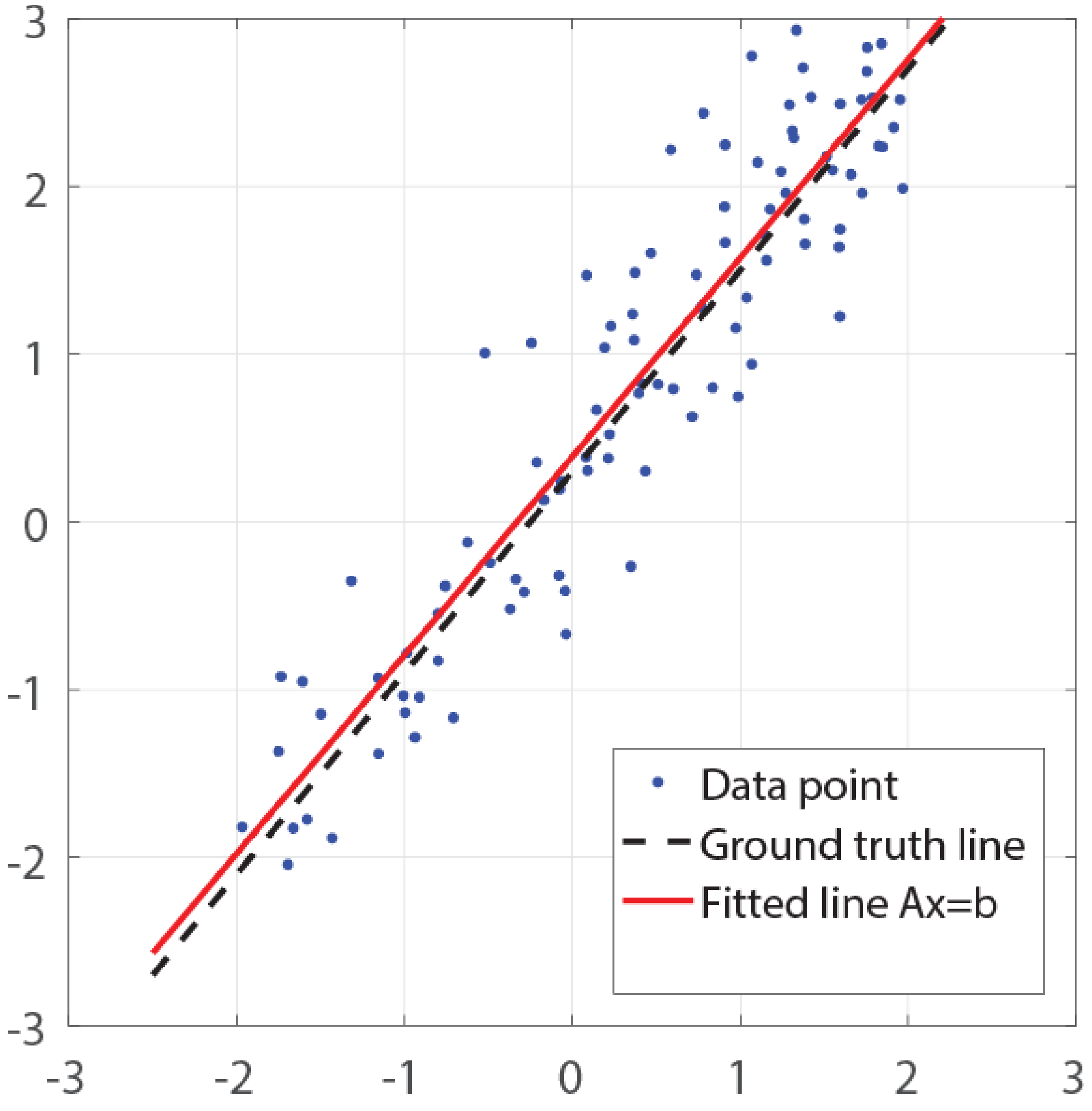
Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (m,d)

$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$\vdots$$
$$v_n \approx mu_n + d$$



Line Fitting (Ax=b)



Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

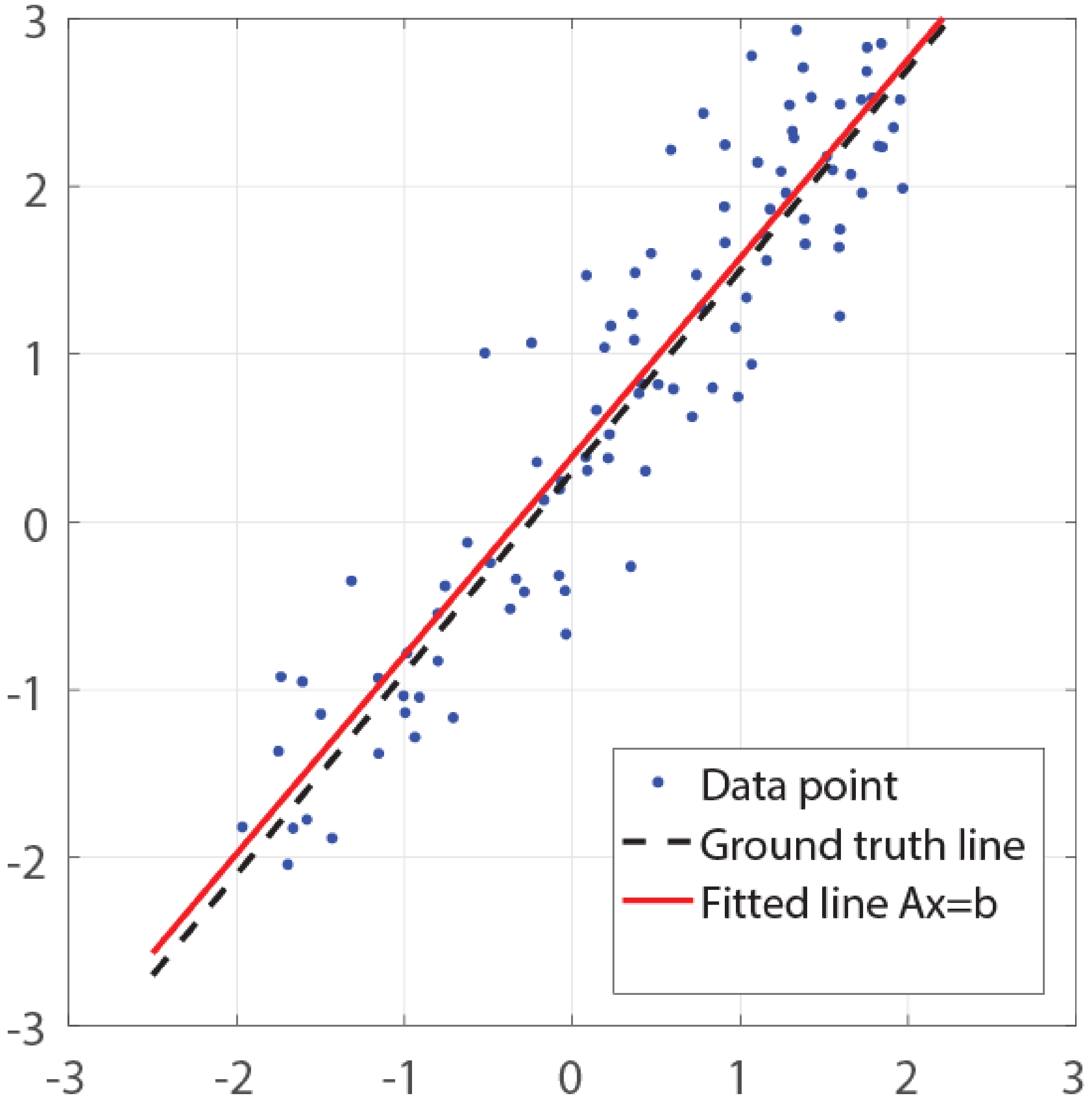
Find the best line: (a, b, c) (m, d)

$$au_1 + bv_1 + c \approx 0$$
$$au_2 + bv_2 + c \approx 0$$
$$\vdots$$
$$au_n + bv_n + c \approx 0$$

$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$\vdots$$
$$v_n \approx mu_n + d$$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

Line Fitting (Ax=b)



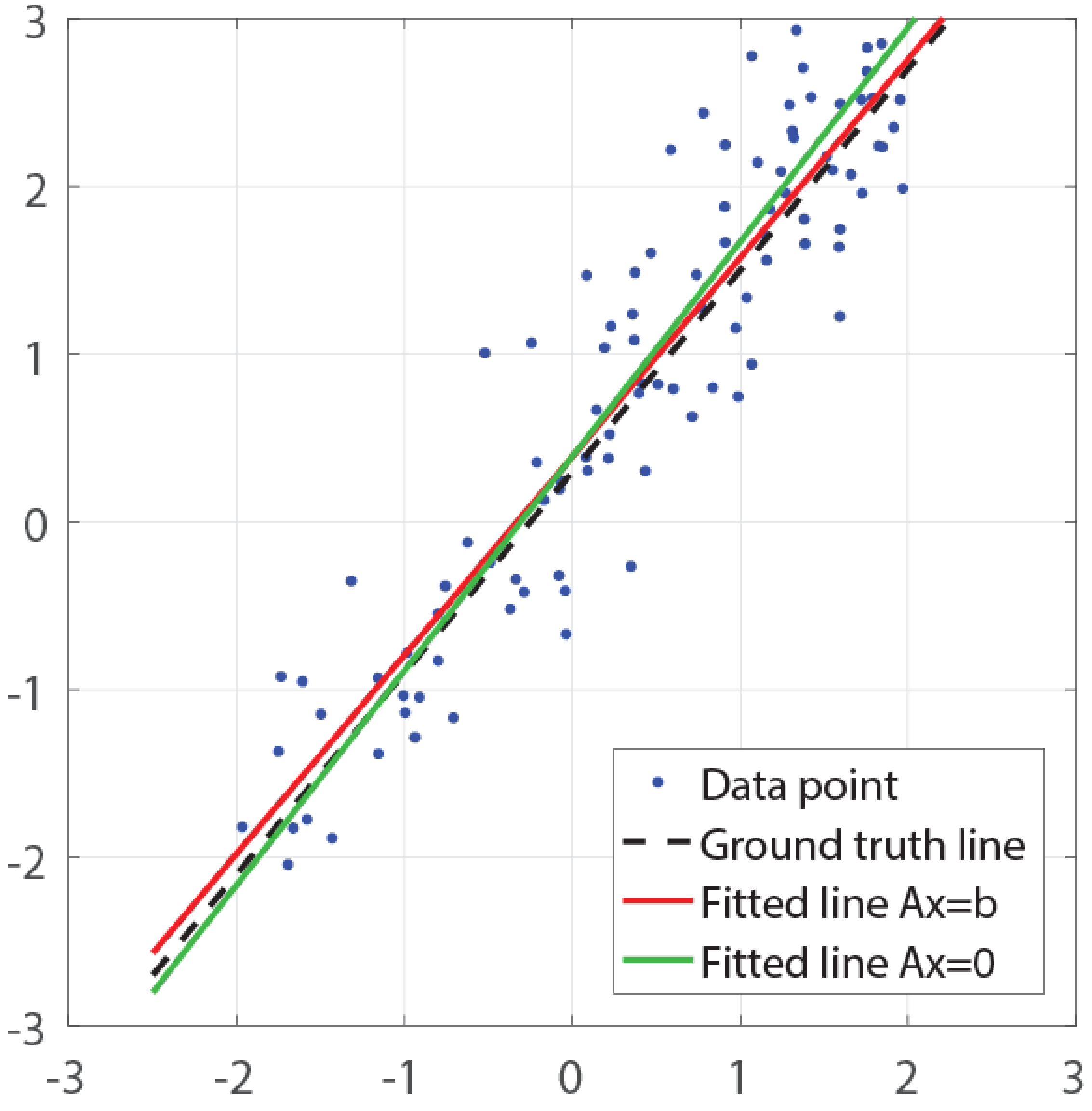
Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c) (m,d)

$$
\begin{aligned}
au_1 + bv_1 + c &\approx 0\\
au_2 + bv_2 + c &\approx 0\\
&\vdots\\
au_n + bv_n + c &\approx 0
\end{aligned}
\qquad\qquad
\begin{aligned}
v_1 &\approx mu_1 + d\\
v_2 &\approx mu_2 + d\\
&\vdots\\
v_n &\approx mu_n + d
\end{aligned}
$$

$$
\begin{bmatrix} u_1 & v_1 & 1\\ u_2 & v_2 & 1\\ \vdots & \vdots & 1\\ u_n & v_n & 1 \end{bmatrix}
\begin{bmatrix} a\\ b\\ c \end{bmatrix}
\approx
\begin{bmatrix} 0\\ 0\\ \vdots\\ 0 \end{bmatrix}
\qquad\qquad
\underbrace{\begin{bmatrix} u_1 & 1\\ u_2 & 1\\ \vdots & \vdots\\ u_n & 1 \end{bmatrix}}_{\mathbf{A}}
\underbrace{\begin{bmatrix} m\\ d \end{bmatrix}}_{\mathbf{x}}
\approx
\underbrace{\begin{bmatrix} v_1\\ v_2\\ \vdots\\ v_n \end{bmatrix}}_{\mathbf{b}}
$$

Line Fitting (Ax=b)



Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: (a,b,c)

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$$\underbrace{\begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ \vdots & \vdots & 1 \\ u_n & v_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{0}} \qquad \underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

How to compute homography?

Homography Computation

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\rightarrow \quad \begin{aligned} h_{11}u_x + h_{12}u_y + h_{13} + h_{31}u_x v_x + h_{32}u_y v_x + h_{33}v_x = 0 \\ h_{21}u_x + h_{22}u_y + h_{23} + h_{31}u_x v_y + h_{32}u_y v_y + h_{33}v_y = 0 \end{aligned}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Unknowns: $h_{11}, \cdots, h_{33}$

Equations: 2 per correspondence

Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\rightarrow \begin{aligned} h_{11}u_x + h_{12}u_y + h_{13} - h_{31}u_xv_x - h_{32}u_yv_x - h_{33}v_x &= 0 \\ h_{21}u_x + h_{22}u_y + h_{23} - h_{31}u_xv_y - h_{32}u_yv_y - h_{33}v_y &= 0 \end{aligned}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} u_x & u_y & 1 & & & & -u_xv_x & -u_yv_x & -v_x \\ & & & u_x & u_y & 1 & -u_xv_y & -u_yv_y & -v_y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\rightarrow \quad h_{11}u_x + h_{12}u_y + h_{13} - h_{31}u_x v_x - h_{32}u_y v_x - h_{33}v_x = 0$$

$$h_{21}u_x + h_{22}u_y + h_{23} - h_{31}u_x v_y - h_{32}u_y v_y - h_{33}v_y = 0$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & & -u_x v_y & -u_y v_y & -v_y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ \vdots \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

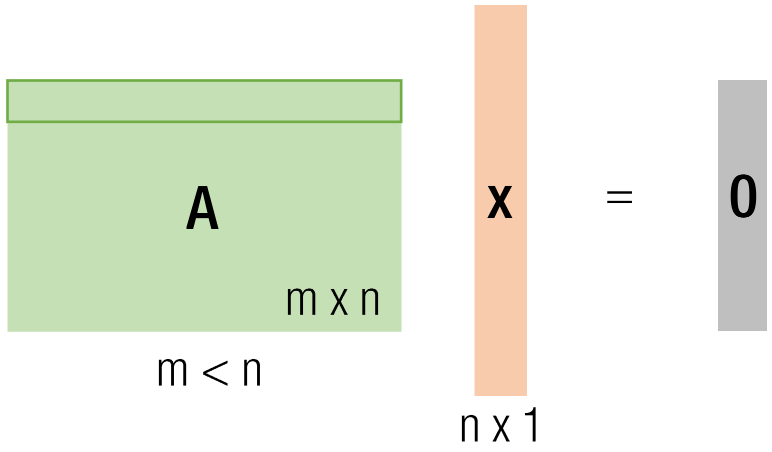
$$\mathbf{A}\,\mathbf{x} = \mathbf{0}$$

2x9

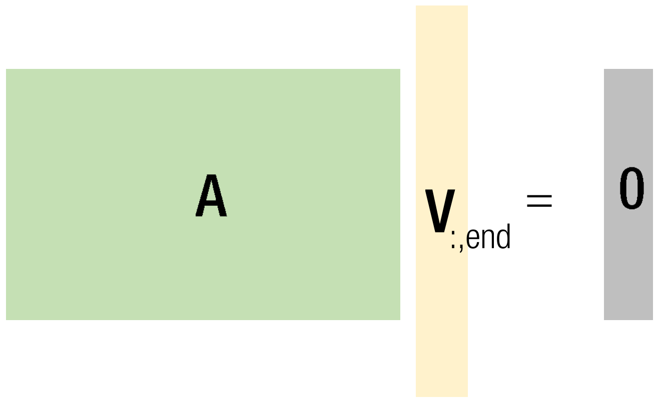
Recall: Singular Value Decomposition (SVD)

\# eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.



$\longrightarrow \; \mathbf{x} = \mathrm{null}(\mathbf{A})$



$\mathbf{V}_{:end} = \mathrm{null}(\mathbf{A})$

For a unique solution, **A** should be m x (m+1)

Homography Computation

How many correspondences are needed?

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

2x9

Homography Computation

How many correspondences are needed? 4

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\mathbf{A}\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

8x9

$$\mathbf{x} = \mathbf{V}_{:end} = \text{null}(\mathbf{A})$$

Homography Computation

How many correspondences are needed? 4

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

ComputeHomography.m

```
function H = ComputeHomography(u, X)

A = [];
for i = 1 : size(u,1)
   A = [A; X(i,:) zeros(1,3) -u(i,1)*X(i,:)];
   A = [A; zeros(1,3) X(i,:) -u(i,2)*X(i,:)];
end
```

Constructing **A**

Homography Computation

How many correspondences are needed? 4

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

ComputeHomography.m

```
function H = ComputeHomography(u, X)

A = [];
for i = 1 : size(u,1)
   A = [A; X(i,:) zeros(1,3) -u(i,1)*X(i,:)];
   A = [A; zeros(1,3) X(i,:) -u(i,2)*X(i,:)];
end

[u, d, v] = svd(A);
h = v(:,end);
H = [h(1:3)'; h(4:6)'; h(7:9)'];
H = H/norm(H);
```

Constructing **A**

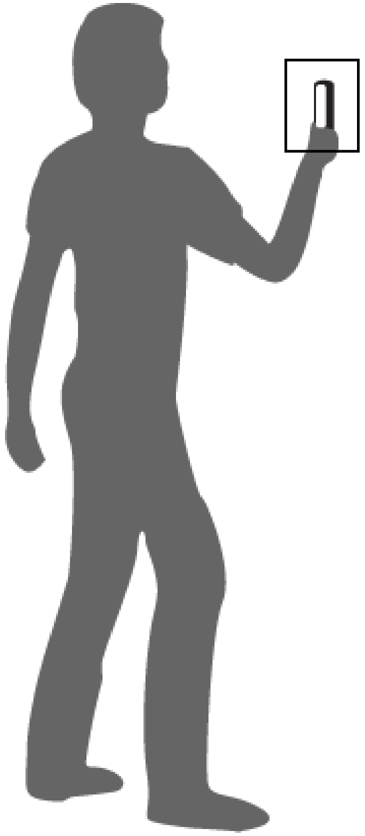
Solving **Ax**=**0**

KellerEntranceHomography.m

```
im1 = imread('keller_left.png');
im2 = imread('keller_right.png');

im_warped = zeros(2000,4000,3);

u1 = [2806 1004;     2456 753;
     1677 1234;     2325 1474];

u2 = [1483 1541;     1948 997;
     860 843;     587 1316];

u1 = [u1 ones(4,1)];
u2 = [u2 ones(4,1)];

H1 = ComputeHomography(u2, u1);
H2 = ComputeHomography(u1, u2);

im_warped1 = ImageWarping(im1, H1);
im_warped2 = ImageWarping(im2, H2);

im_1 = 0.5*im_warped1 + 0.5*im2;
im_2 = 0.5*im_warped2 + 0.5*im1;
```

Camera Calibration

Camera Intrinsic Parameter

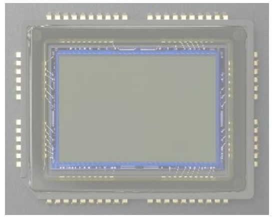


Pixel space

Metric space

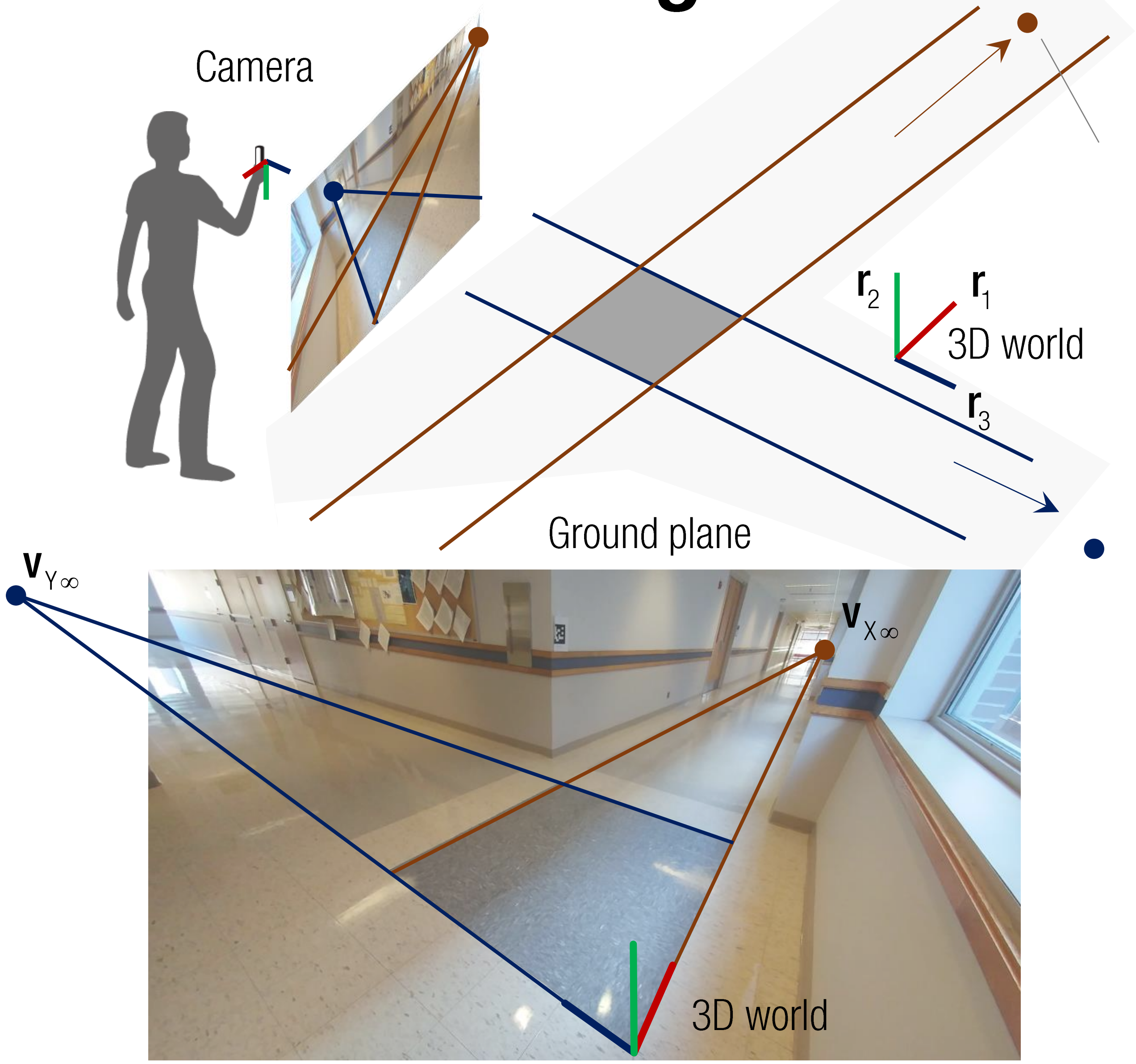
$$\lambda \begin{bmatrix} u_{\text{img}} \\ v_{\text{img}} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f_x & & p_x \\ & f_y & p_y \\ & & 1 \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

+



Camera intrinsic parameter
: metric space to pixel space

Two Vanishing Points



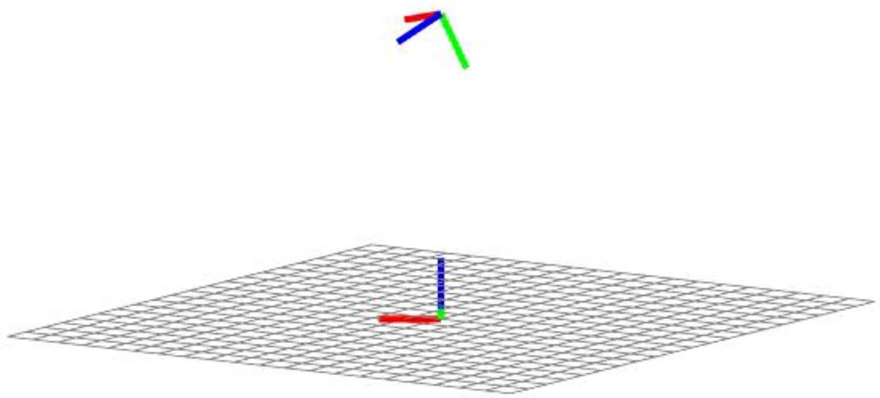
ComputeCameraUsingTwoVanishingPoints.m

```
f = 4000;
K = [f 0 size(im,2)/2;
    0 f size(im,1)/2;
    0 0 1];

l11 = GetLineFromTwoPoints(m11,m12);
l12 = GetLineFromTwoPoints(m13,m14);

l21 = GetLineFromTwoPoints(m21,m22);
l22 = GetLineFromTwoPoints(m23,m24);

v1 = GetPointFromTwoLines(l11,l12);
v2 = GetPointFromTwoLines(l21,l22);

r1 = inv(K)*v1/norm(inv(K)*v1);
r2 = inv(K)*v2/norm(inv(K)*v2);

r3 = Vec2Skew(r1)*r2;
```

Not orthogonal matrix!

```
R =
0.2448   -0.5178    0.0424
  -0.1737   -0.1960   -0.6978
   0.9539    0.8327   -0.1379

det(R) =
                0.5077

R'*R =
   0.3299    0.0294   -0.2036
   0.0294    0.5555   -0.2327
  -0.2036   -0.2327    1.6224
```

Two Vanishing Points



ComputeCameraUsingTwoVanishingPoints.m

```
f = 1224;
K = [f 0 size(im,2)/2;
    0 f size(im,1)/2;
    0 0 1];

l11 = GetLineFromTwoPoints(m11,m12);
l12 = GetLineFromTwoPoints(m13,m14);

l21 = GetLineFromTwoPoints(m21,m22);
l22 = GetLineFromTwoPoints(m23,m24);

v1 = GetPointFromTwoLines(l11,l12);
v2 = GetPointFromTwoLines(l21,l22);

r1 = inv(K)*v1/norm(inv(K)*v1);
r2 = inv(K)*v2/norm(inv(K)*v2);

r3 = Vec2Skew(r1)*r2;
```

Change focal length

Orthogonal matrix!

```
R =
   0.5846   -0.8496    0.0508
  -0.4149   -0.3216   -0.8367
   0.6972    0.4180   -0.5405

det(R) =
   0.9948

R'*R =
   1.0662   -0.0118    0.0250
  -0.0118    0.9757    0.0285
   0.0250    0.0285    0.9530
```

How to automatically and precisely calibrate camera intrinsic parameter?

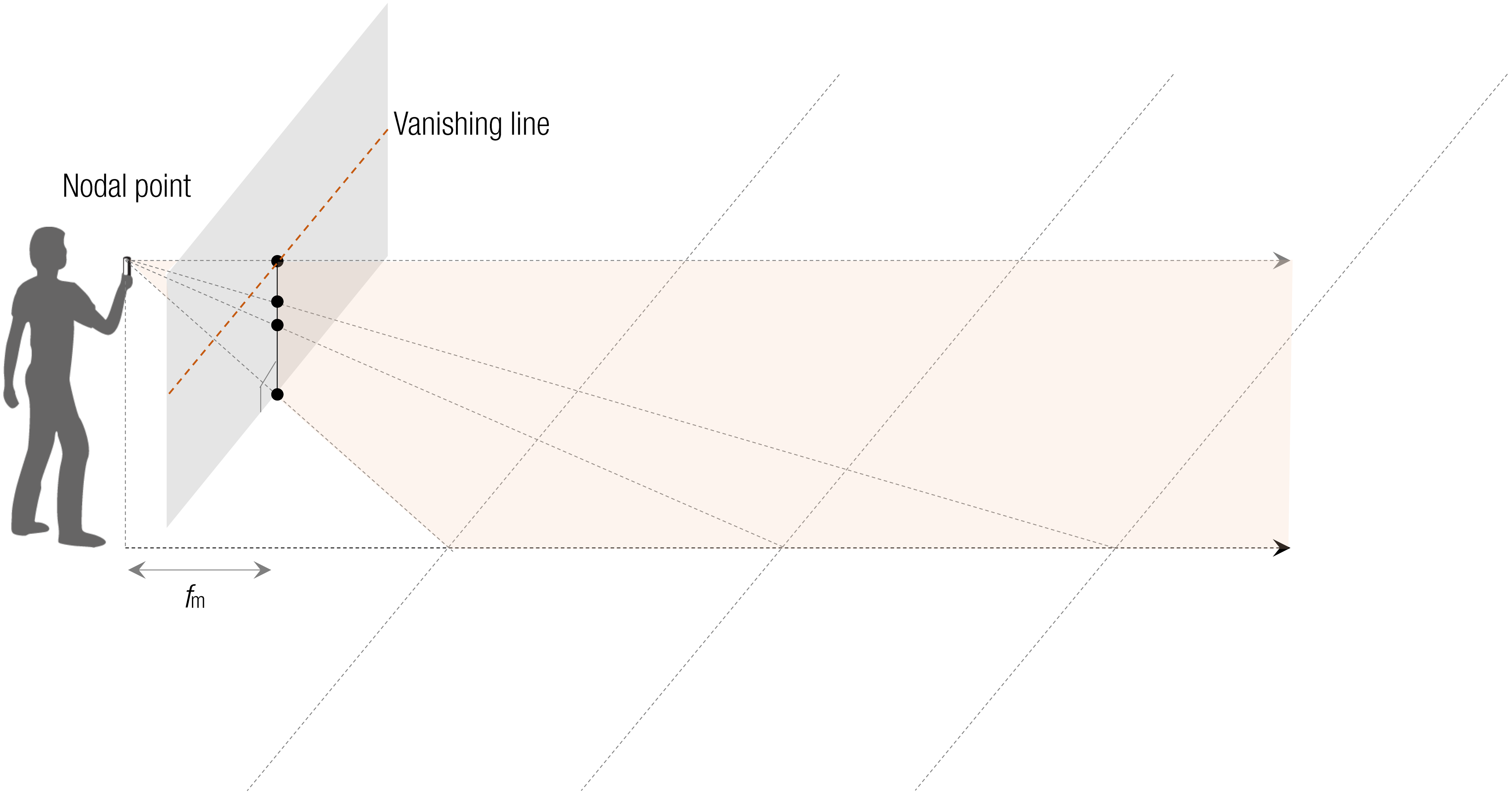
Physical Focal Point

Physical Focal Point

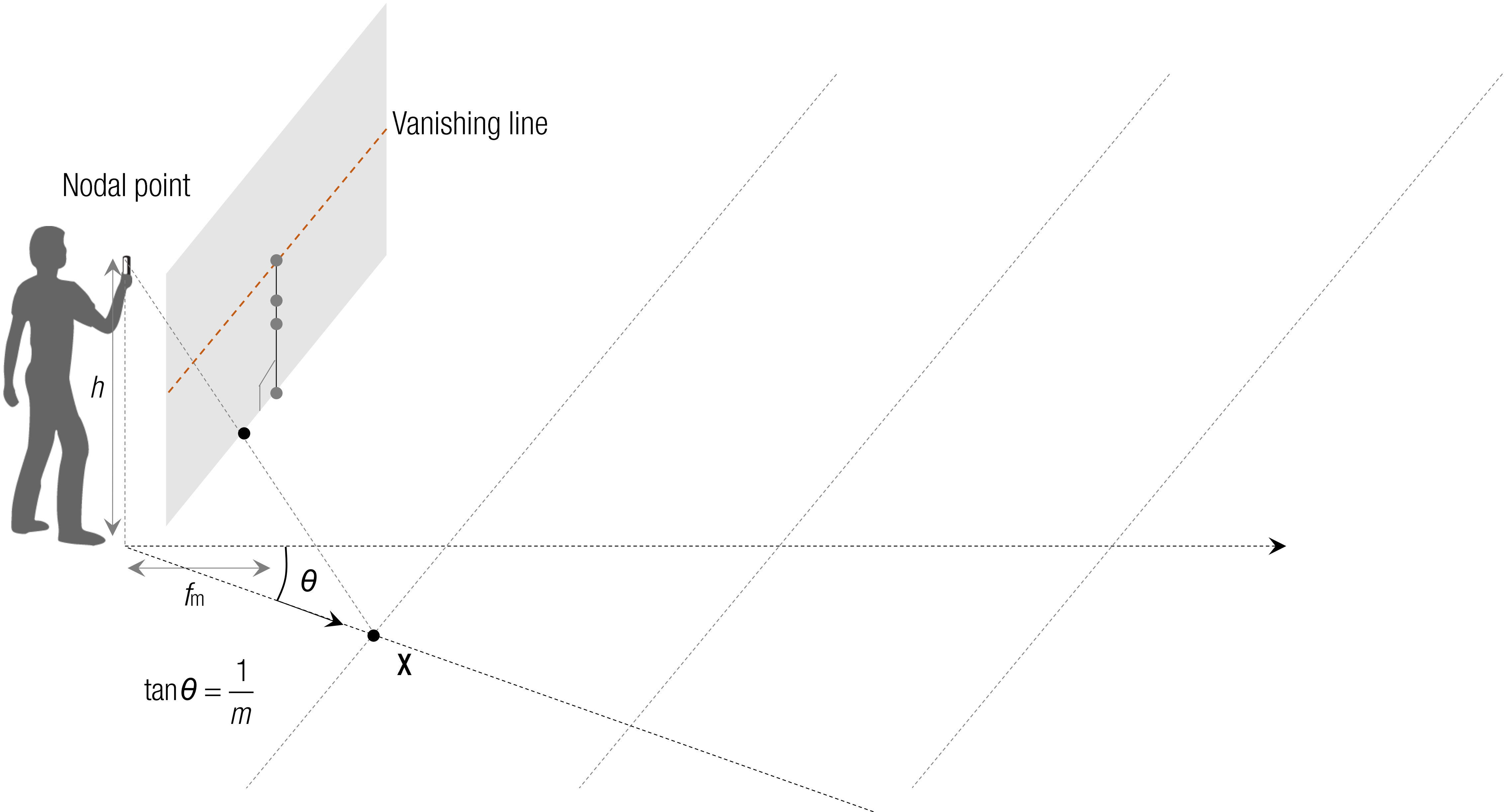
Physical Focal Point

Physical Focal Point

Physical Focal Point

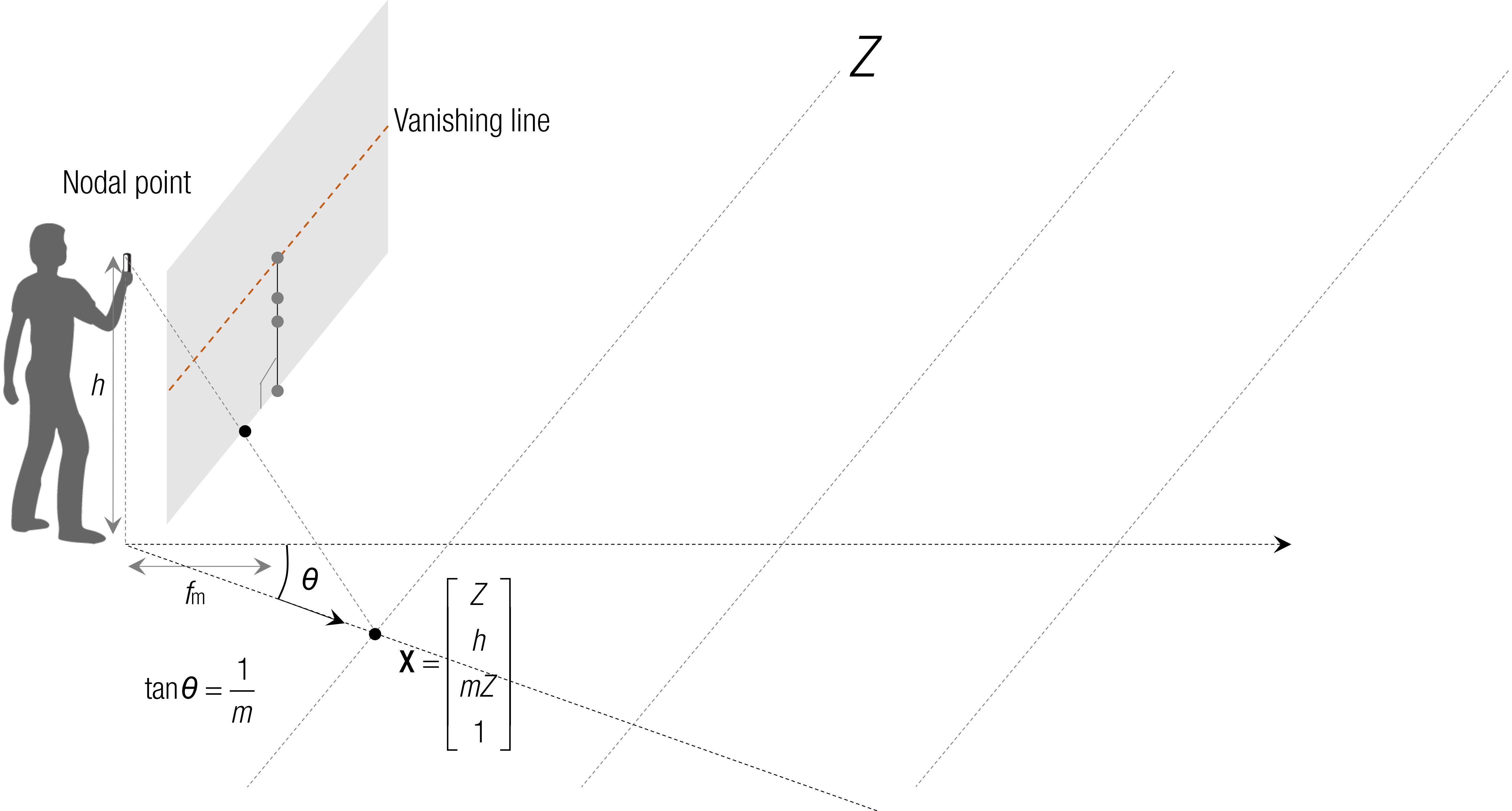


Physical Focal Point

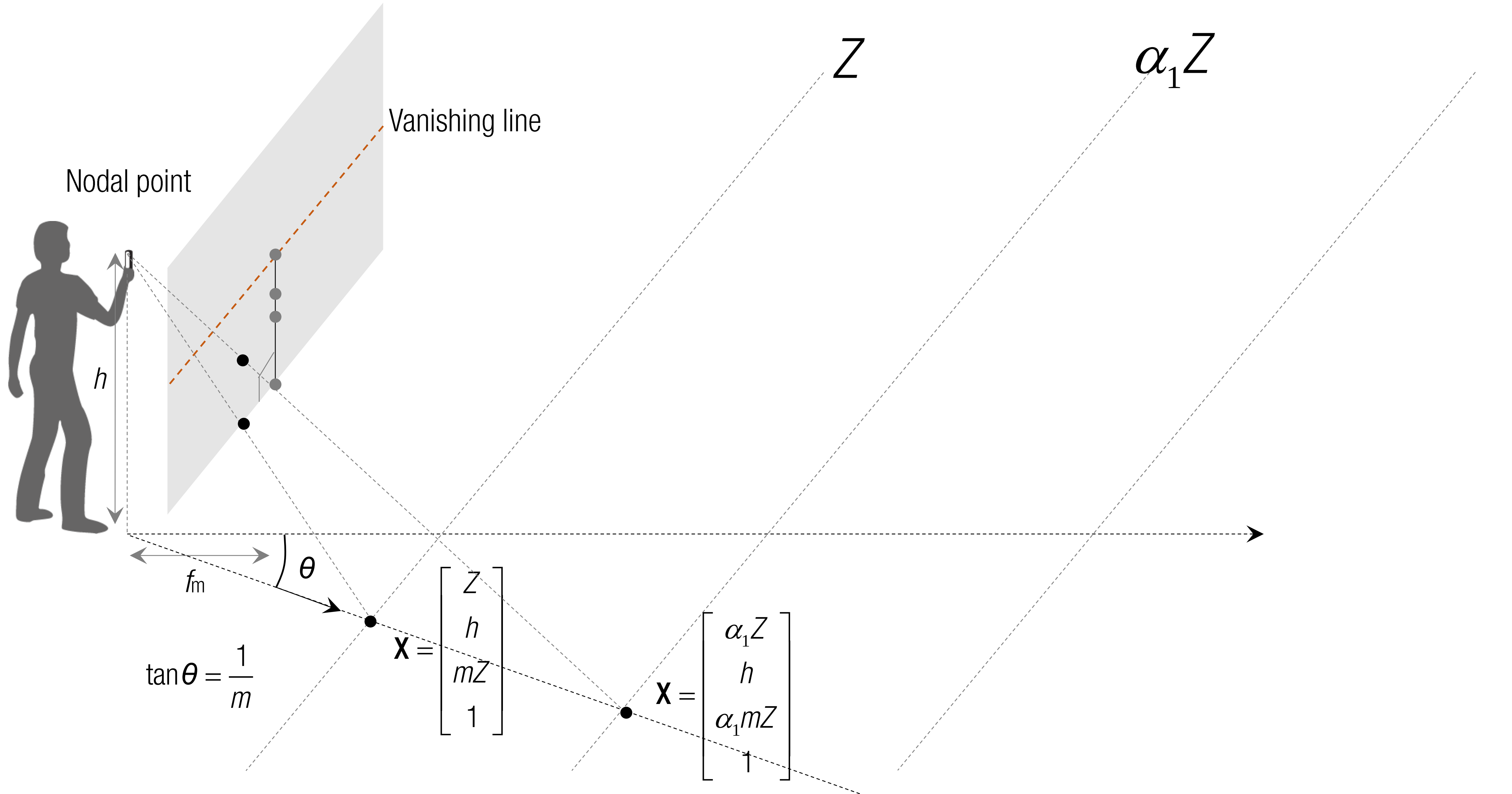


$$\tan\theta = \frac{1}{m}$$

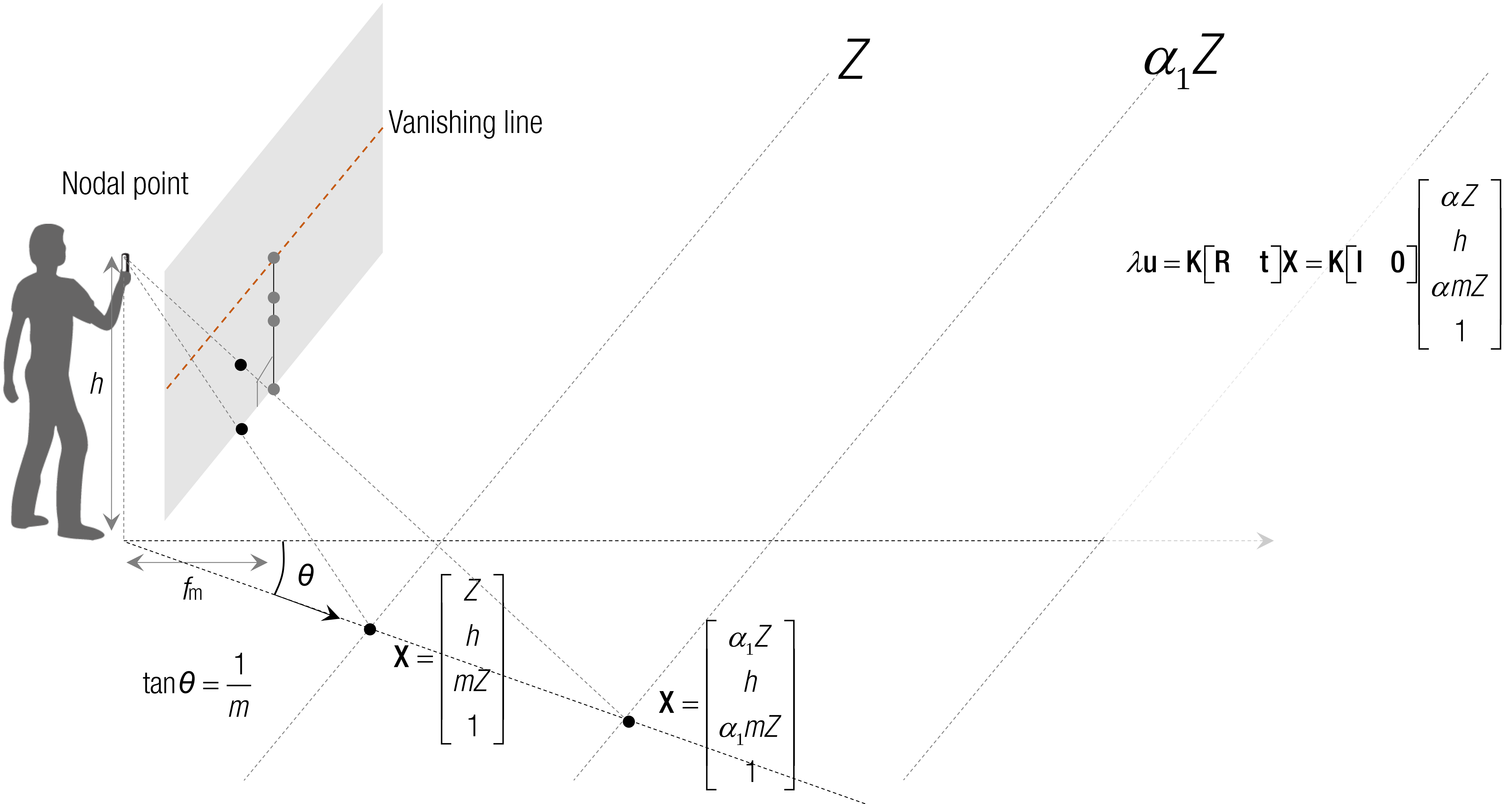
Physical Focal Point



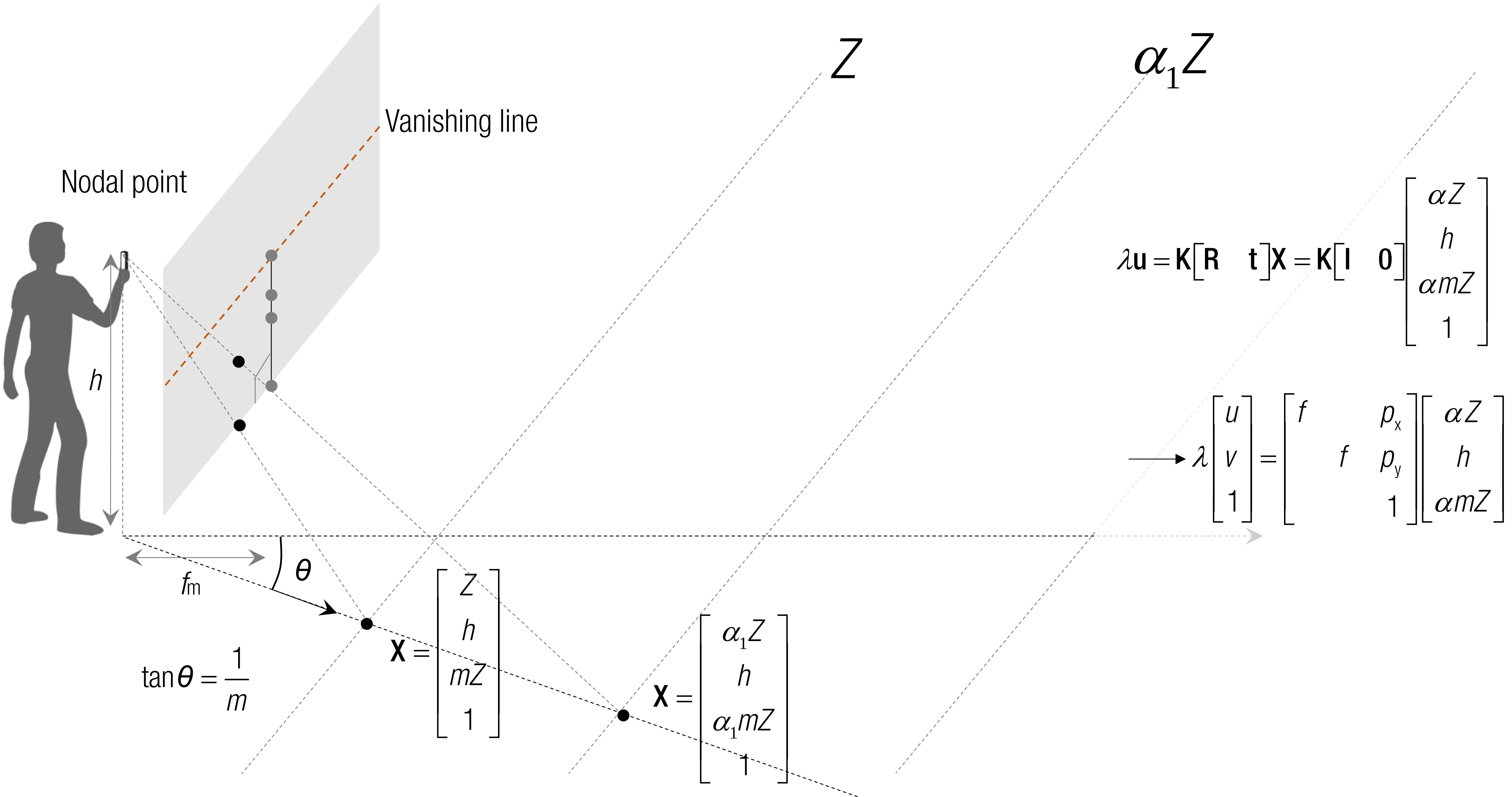
Physical Focal Point



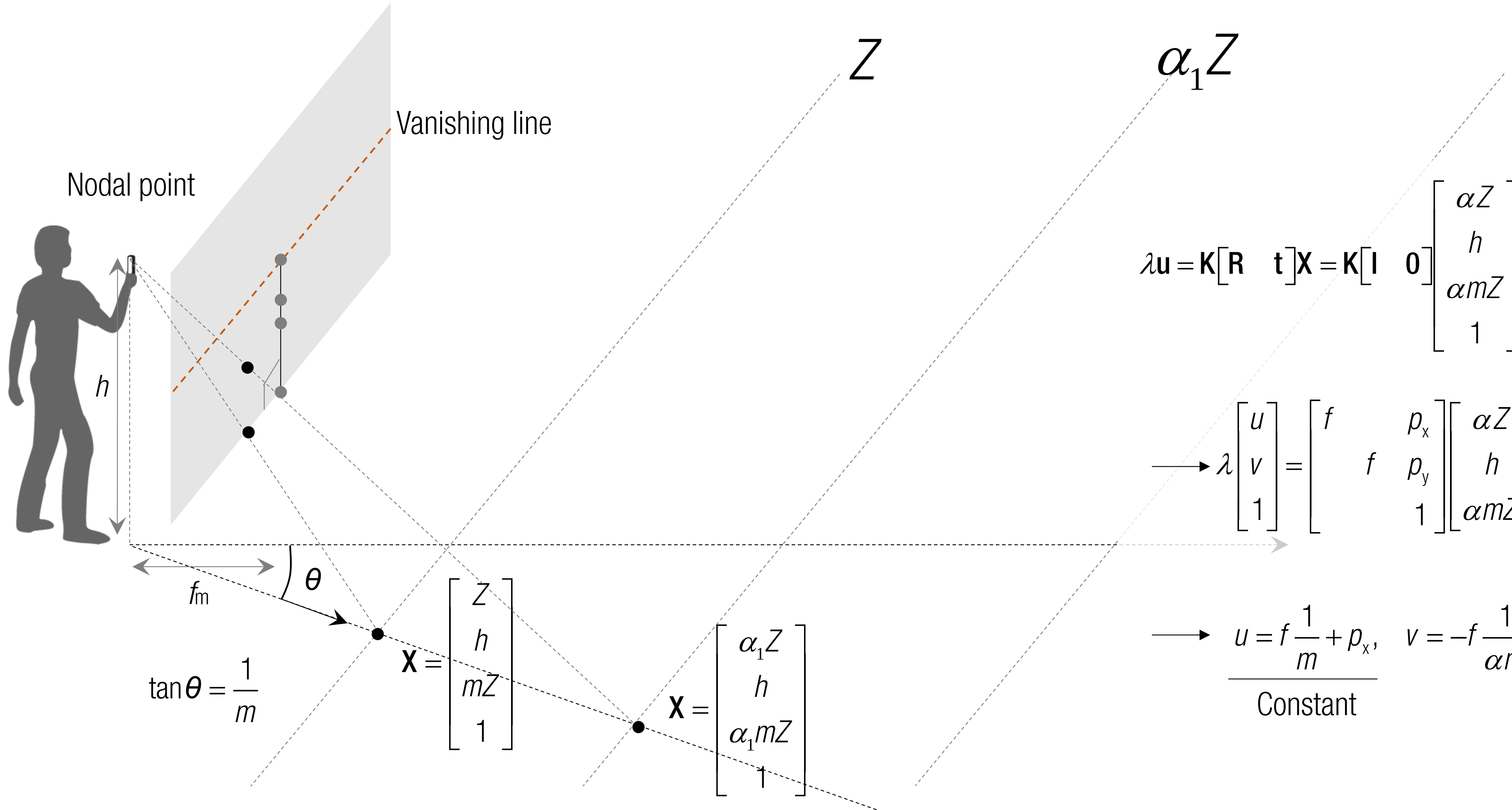
Physical Focal Point



Physical Focal Point



Physical Focal Point

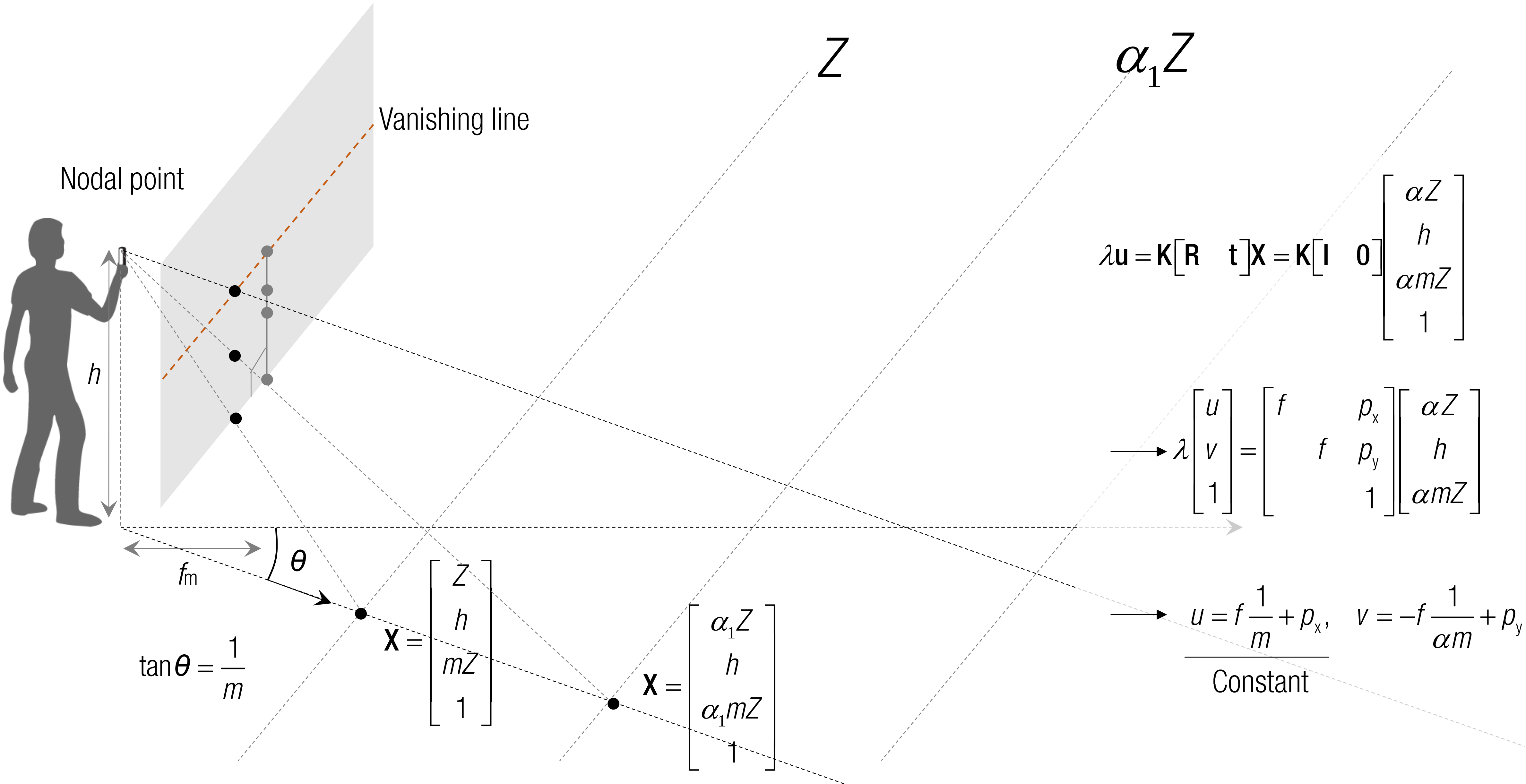


$$\lambda \mathbf{u} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}\mathbf{X} = \mathbf{K}\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \alpha Z \\ h \\ \alpha mZ \\ 1 \end{bmatrix}$$

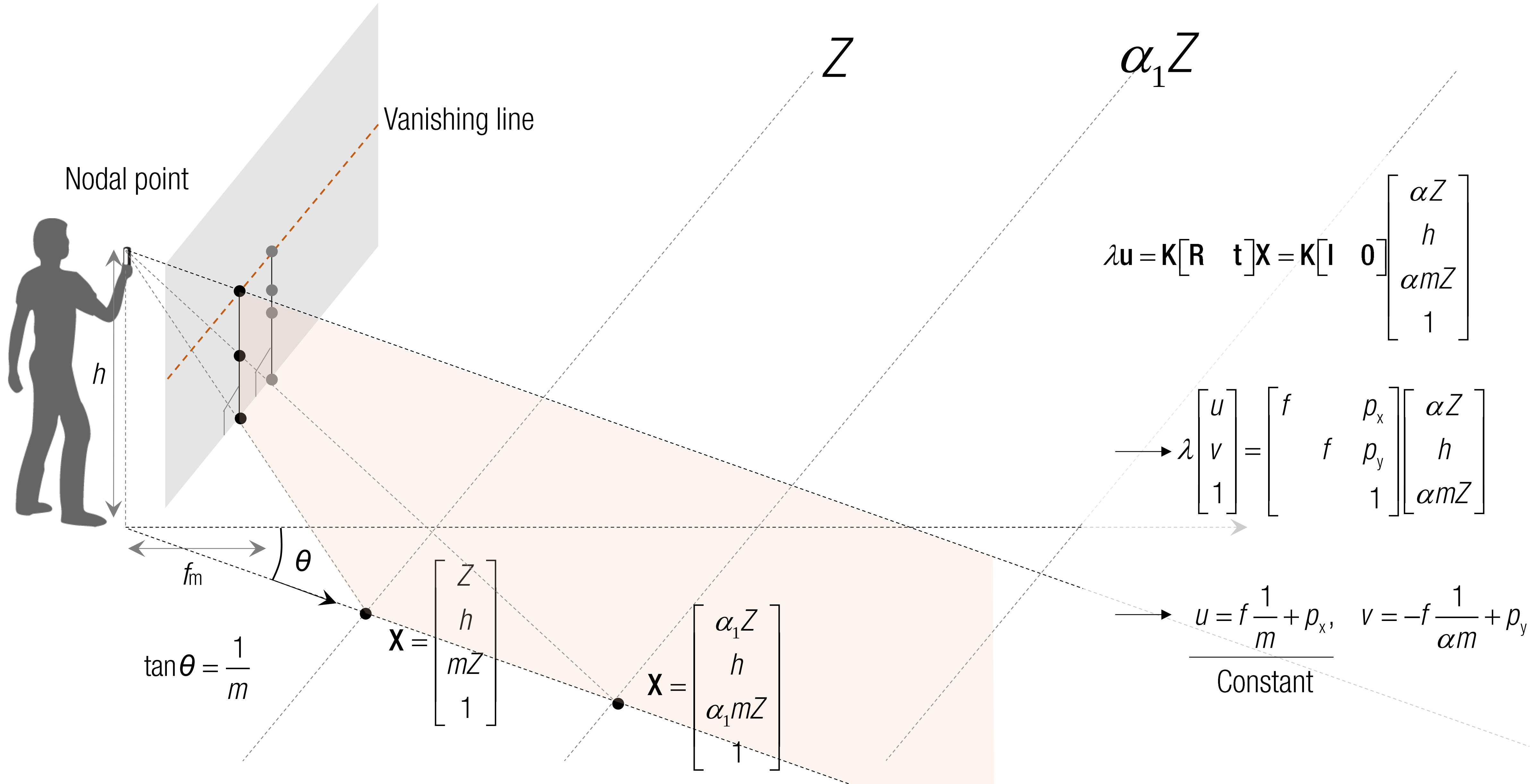
$$\rightarrow \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}\begin{bmatrix} \alpha Z \\ h \\ \alpha mZ \end{bmatrix}$$

$$\rightarrow \underbrace{u = f\frac{1}{m} + p_x}_{\text{Constant}}, \quad v = -f\frac{1}{\alpha m} + p_y$$

Physical Focal Point



Physical Focal Point



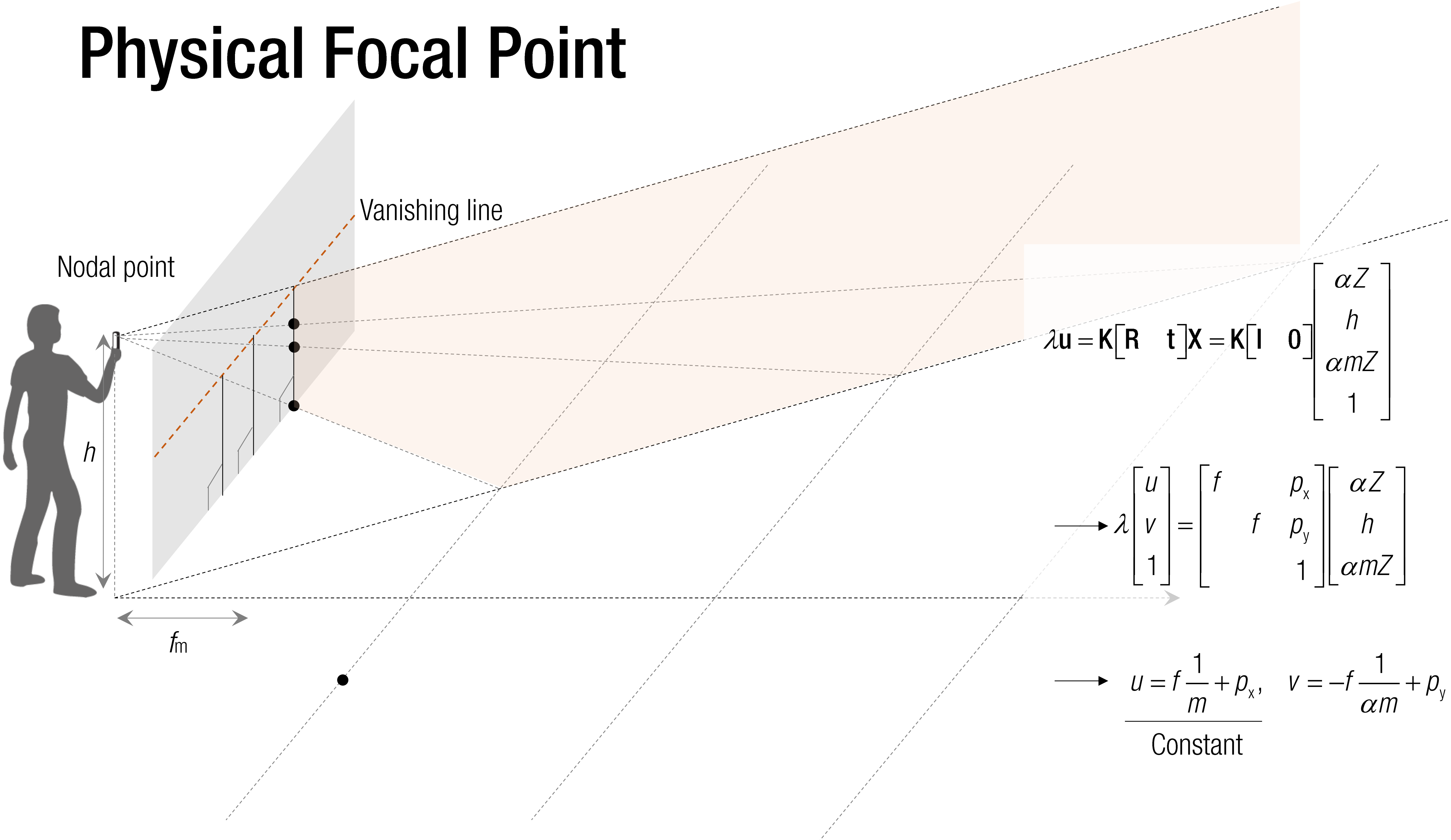
$$\lambda \mathbf{u} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}\mathbf{X} = \mathbf{K}\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \alpha Z \\ h \\ \alpha mZ \\ 1 \end{bmatrix}$$

$$\rightarrow \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}\begin{bmatrix} \alpha Z \\ h \\ \alpha mZ \end{bmatrix}$$

$$\rightarrow u = f\frac{1}{m} + p_x, \quad v = -f\frac{1}{\alpha m} + p_y$$

Constant

Physical Focal Point

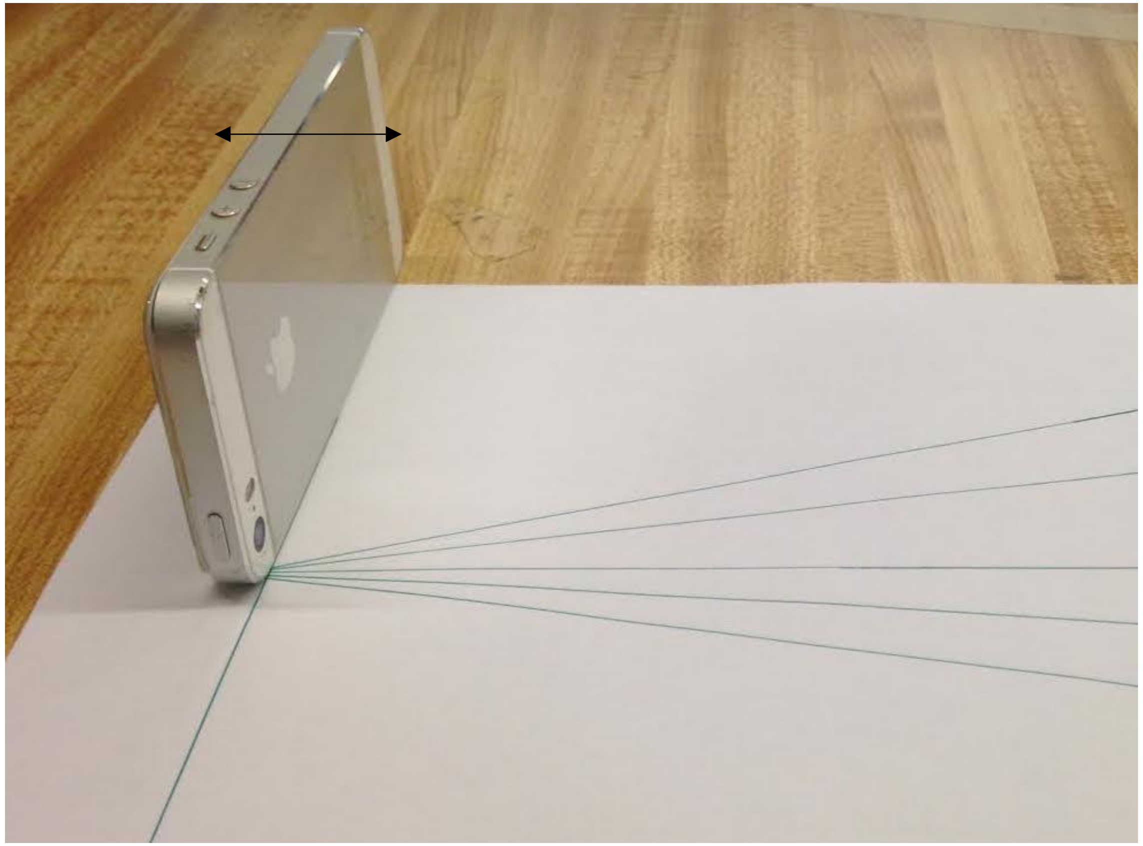


$$\lambda \mathbf{u} = \mathbf{K}[\mathbf{R} \quad \mathbf{t}]\mathbf{X} = \mathbf{K}[\mathbf{I} \quad \mathbf{0}]\begin{bmatrix}\alpha Z\\ h\\ \alpha m Z\\ 1\end{bmatrix}$$

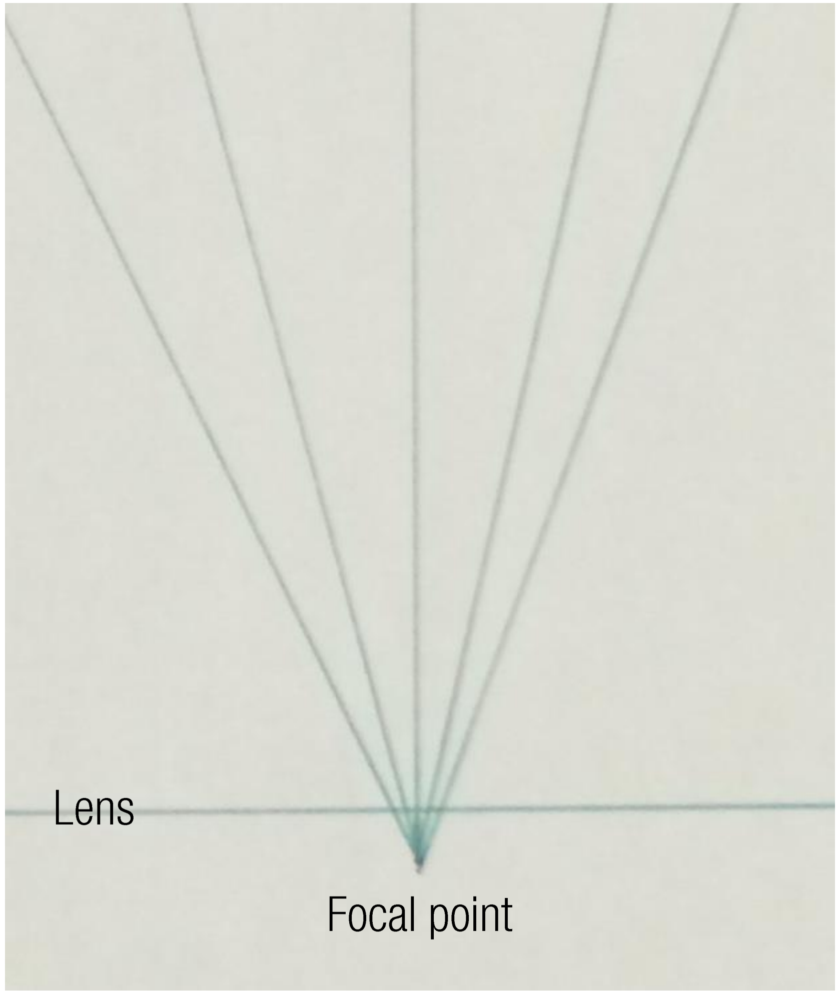
$$\rightarrow \lambda\begin{bmatrix}u\\ v\\ 1\end{bmatrix} = \begin{bmatrix} f & & p_x\\ & f & p_y\\ & & 1\end{bmatrix}\begin{bmatrix}\alpha Z\\ h\\ \alpha m Z\end{bmatrix}$$

$$\rightarrow \underbrace{u = f\frac{1}{m}+p_x}_{\text{Constant}}, \quad v = -f\frac{1}{\alpha m}+p_y$$

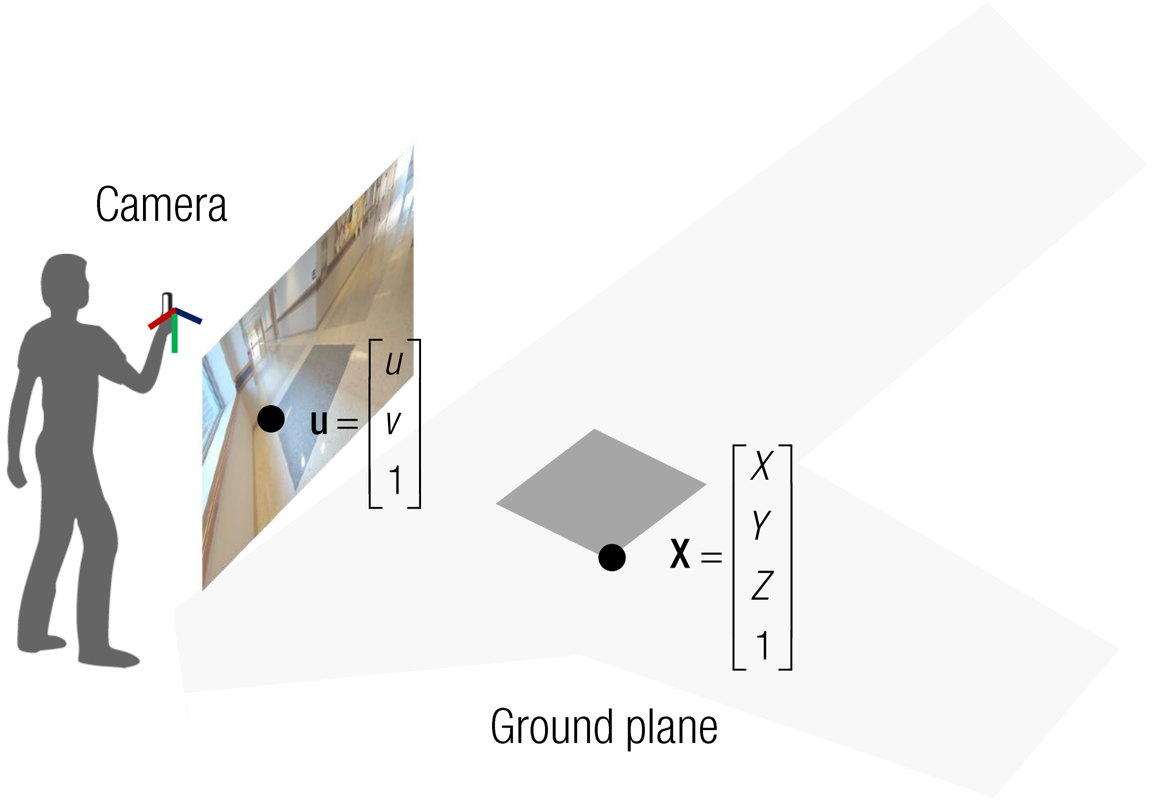
Where am I? (Focal Point)



Where am I? (Focal Point)



Camera Calibration in Pixel Space

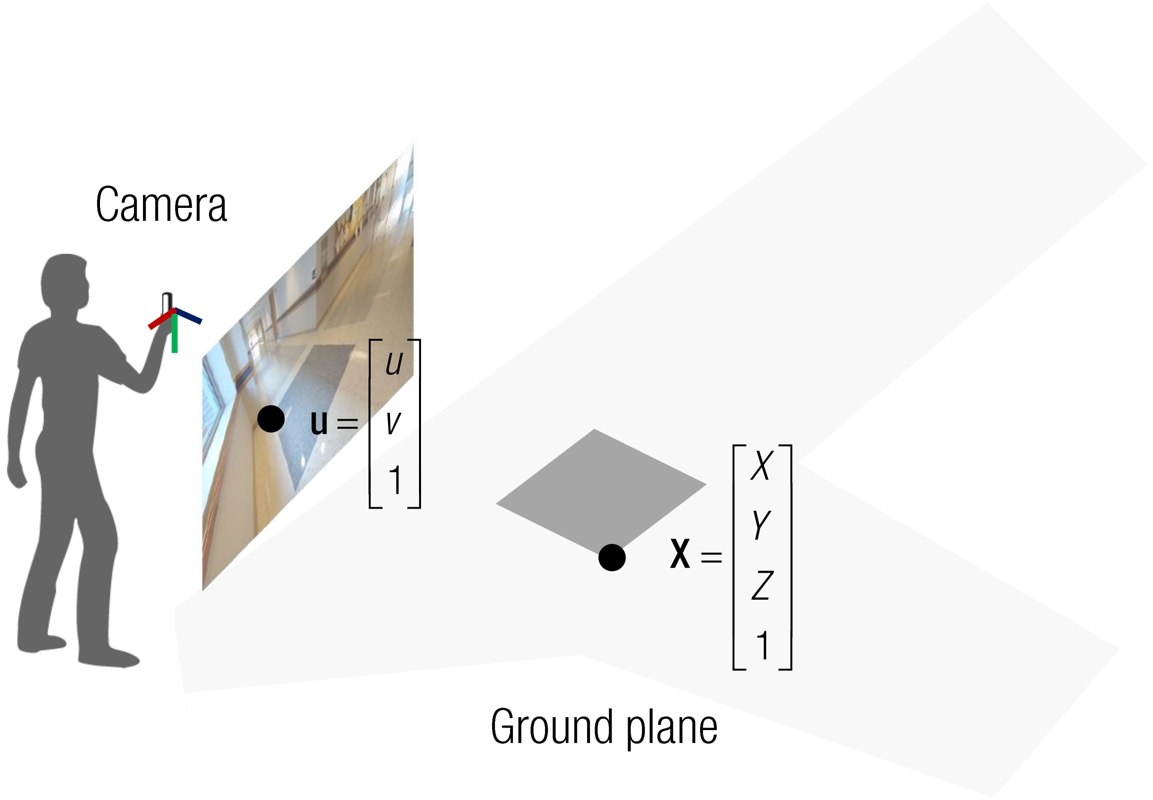


$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

# of unknowns:

# of equations:

Camera Calibration in Pixel Space

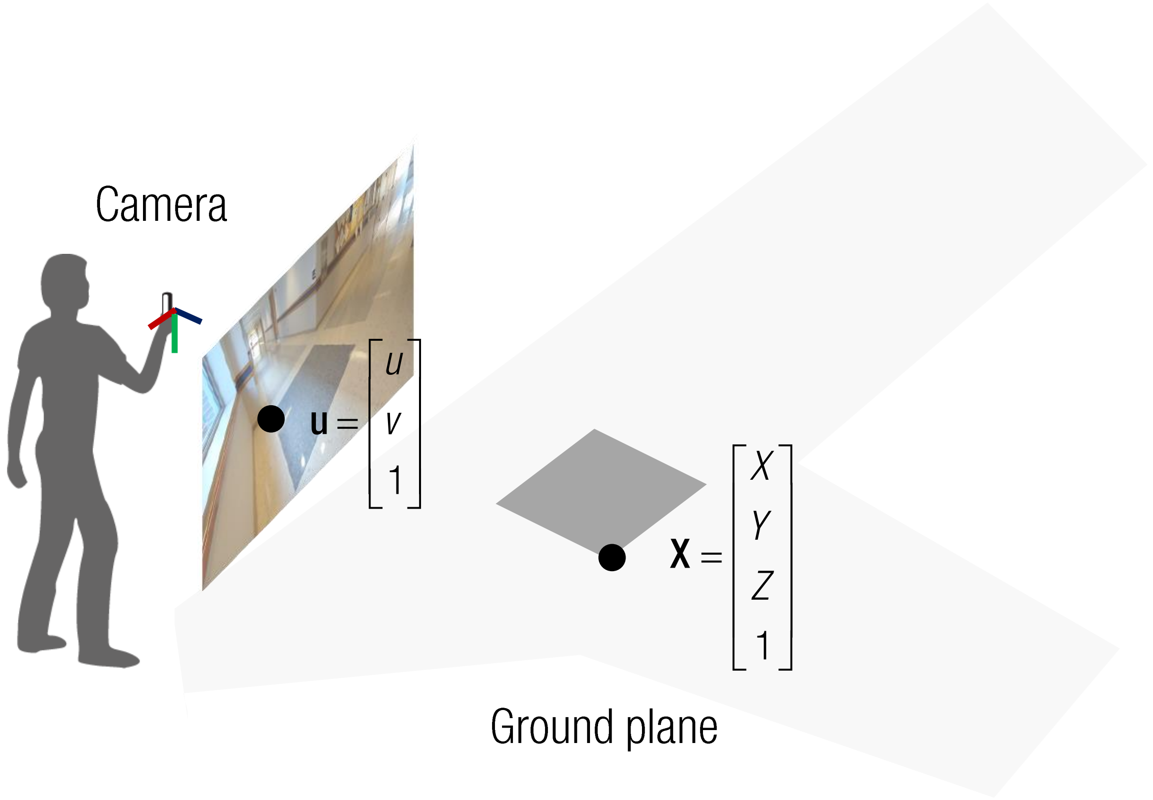


$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

# of unknowns: 3 (**K**) + 6 (**R** and **t**) + 3 (**X**)

# of equations:

Camera Calibration in Pixel Space

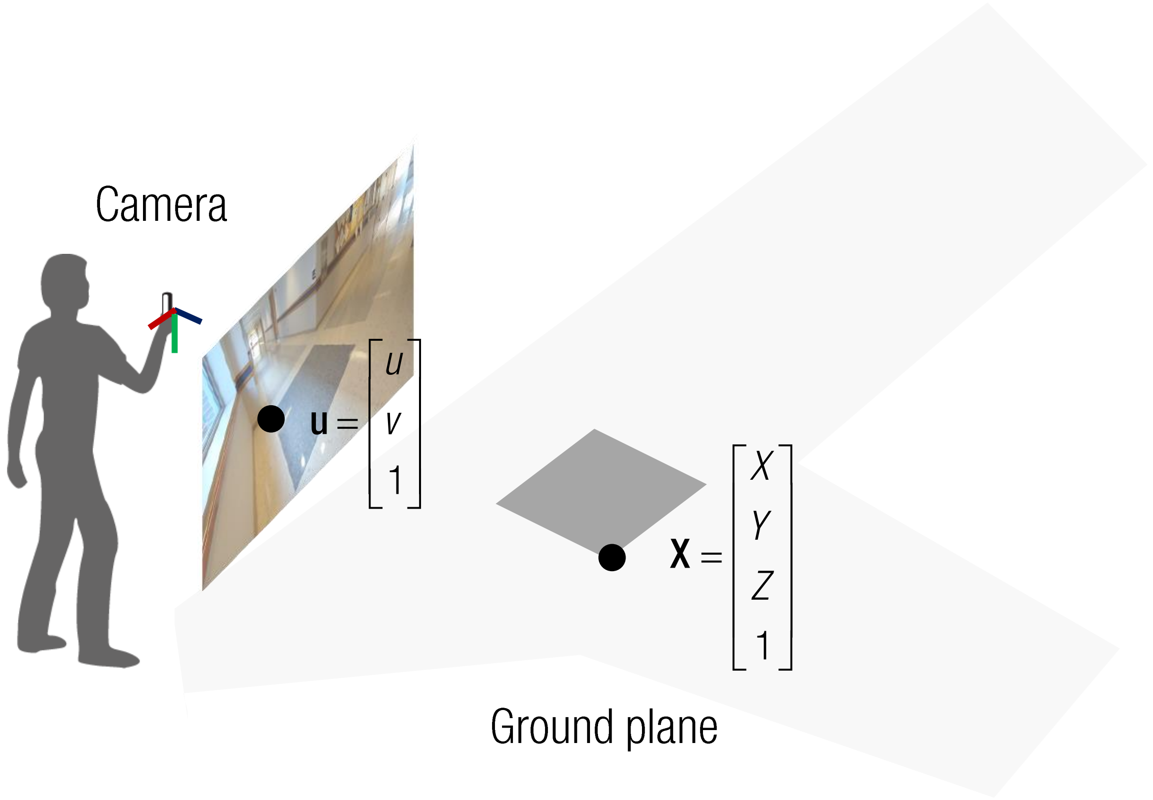


$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

# of unknowns: 3 (**K**) + 6 (**R** and **t**) + 3 (**X**)

# of equations: 2

Camera Calibration in Pixel Space



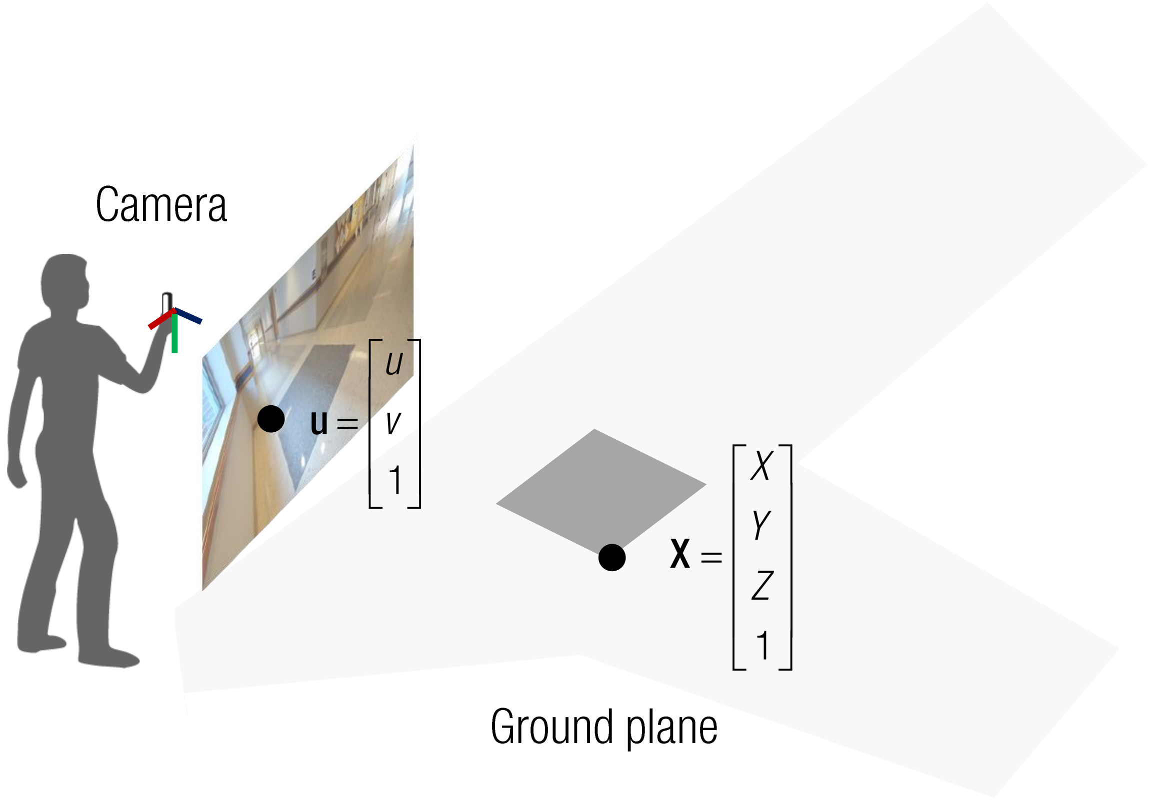
$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

# of unknowns: 3 (**K**) + 6F (**R** and **t**) + 3P (**X**)

# of equations: 2P

where F is # of images and P is # of points.

Camera Calibration in Pixel Space



$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

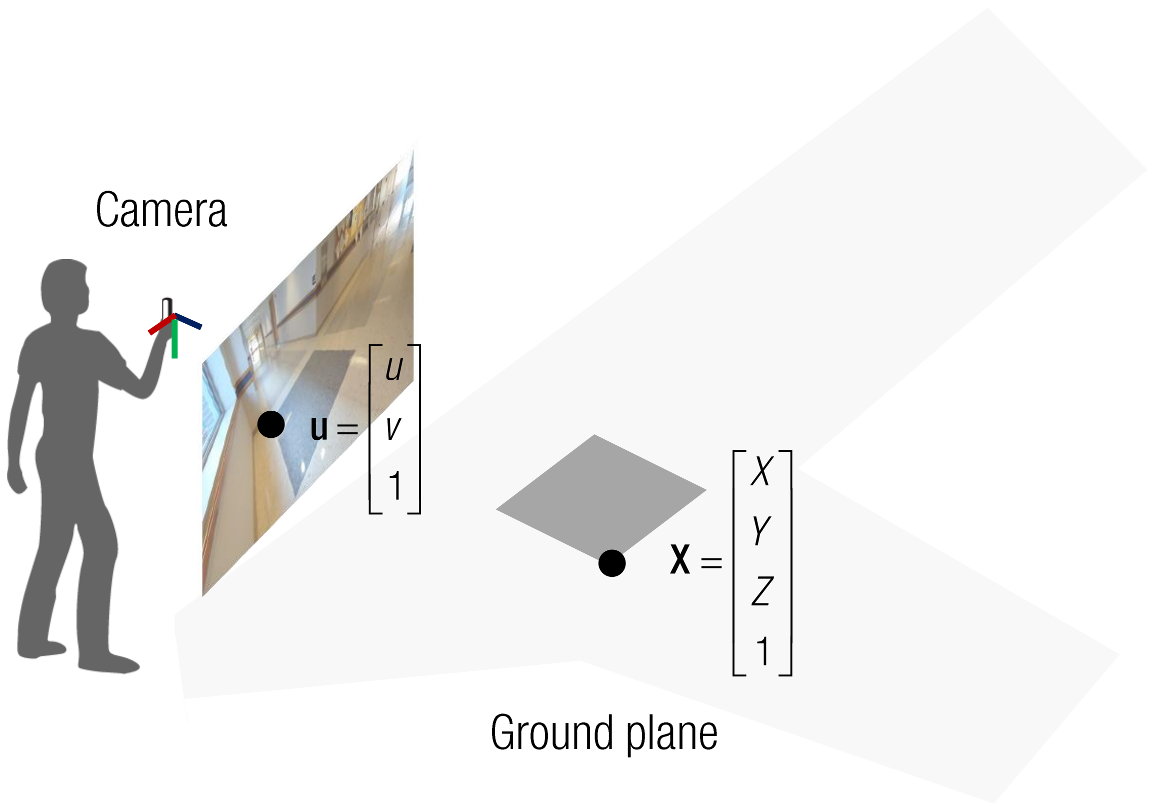
# of unknowns: 3 (**K**) + 6F (**R** and **t**) + 3P (**X**)

# of equations: 2P

where F is # of images and P is # of points.

# of unknowns > # of equations

Camera Calibration in Pixel Space



$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

# of unknowns: 3 (**K**) + 6F (**R** and **t**) + 3P (**X**)

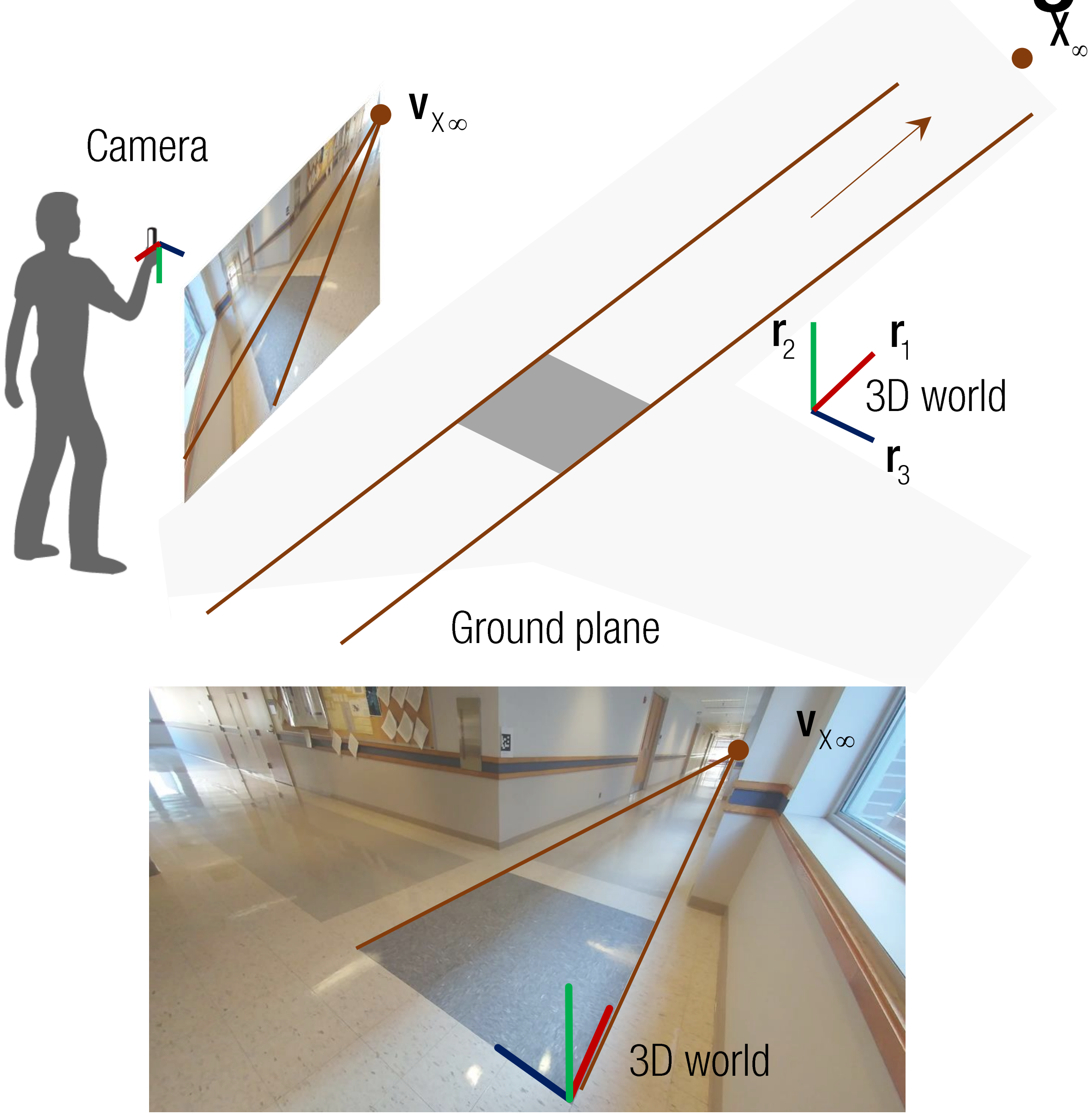
# of equations: 2P

where F is # of images and P is # of points.

# of unknowns > # of equations

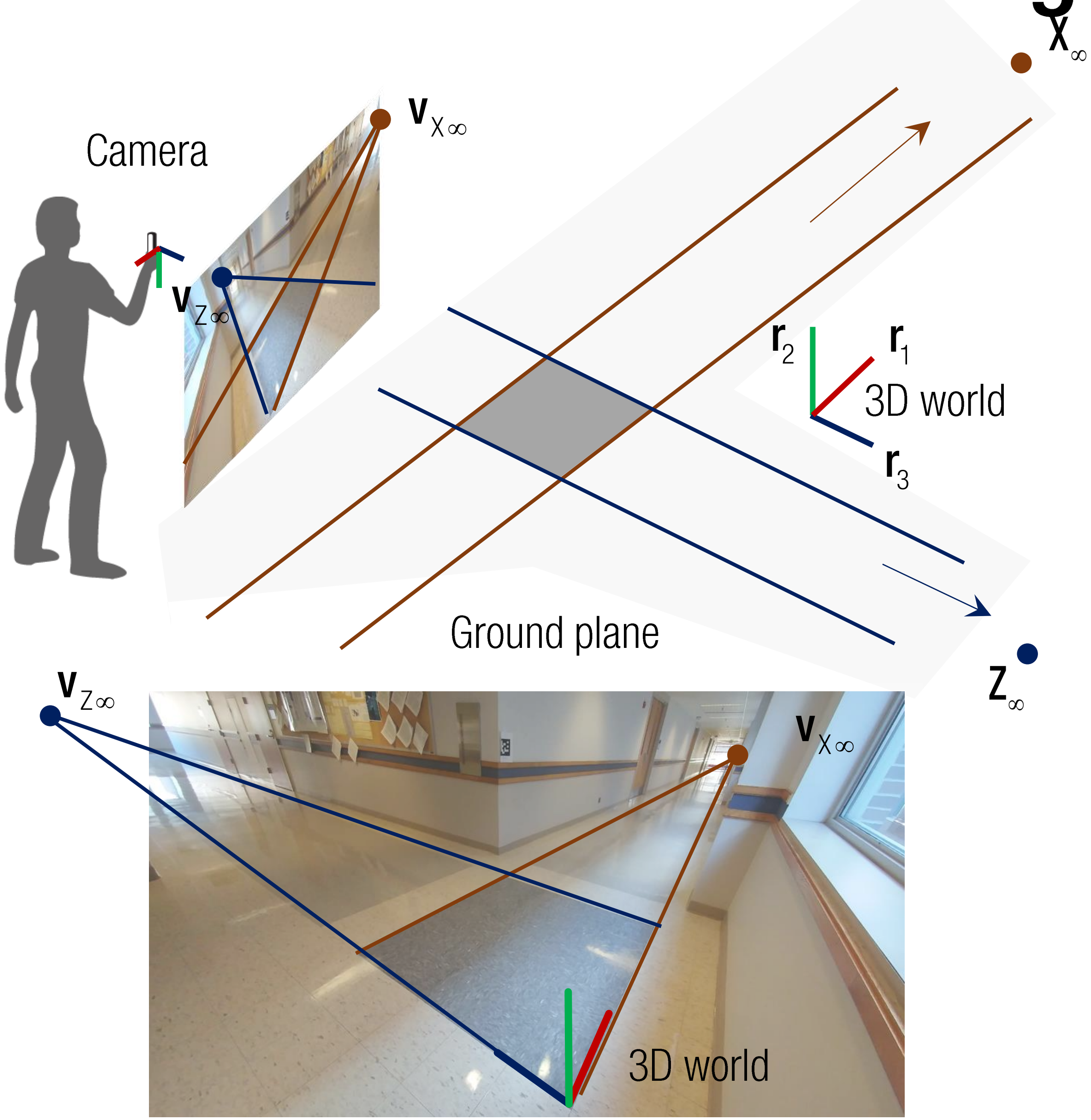
What do we know about the scene?

Camera Calibration using Vanishing Points



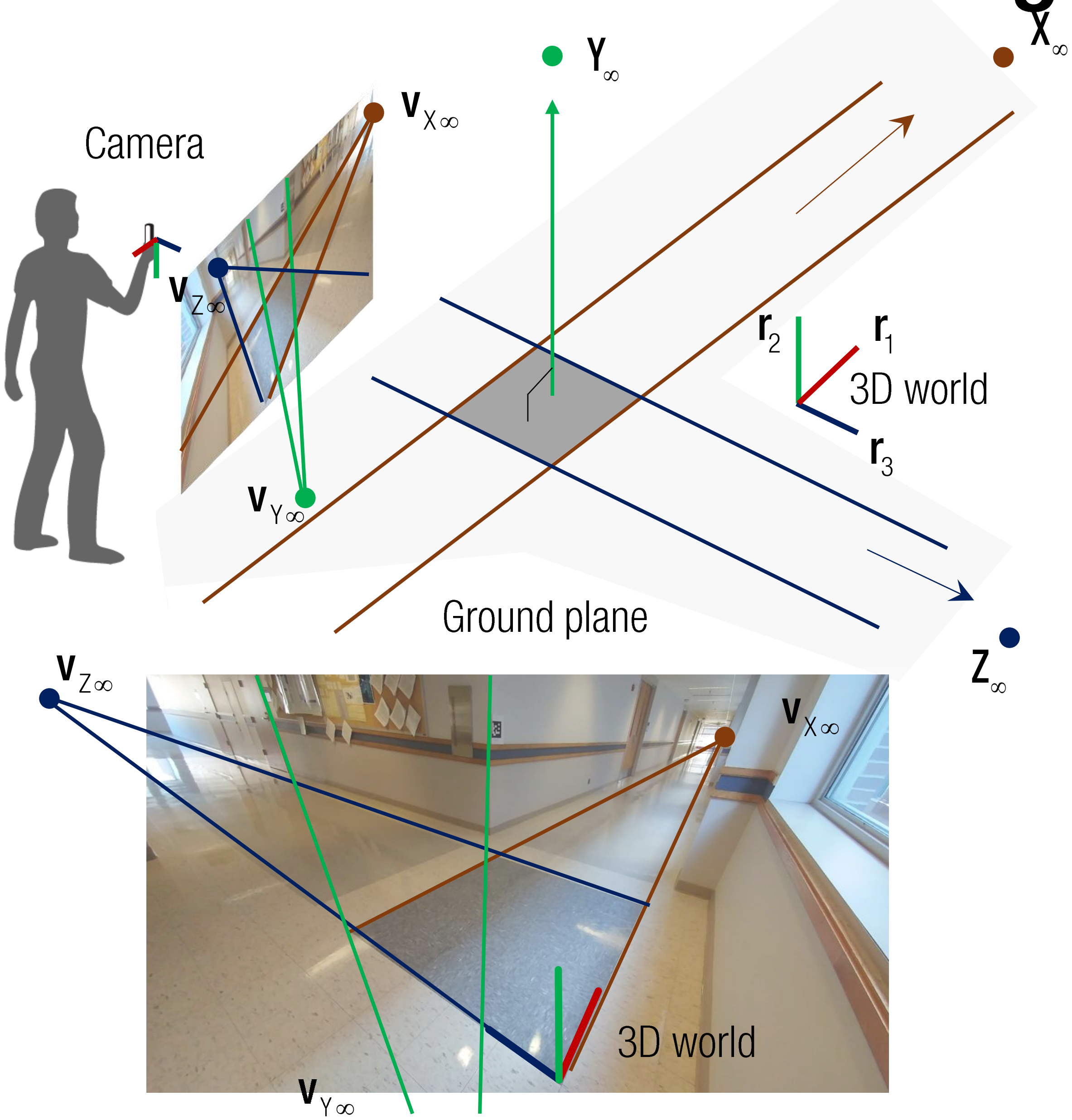
$$\lambda \mathbf{v}_{X\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{X}_\infty$$

Camera Calibration using Vanishing Points



$$\lambda \mathbf{v}_{X\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{X}_\infty \qquad \lambda \mathbf{v}_{Z\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{Z}_\infty$$

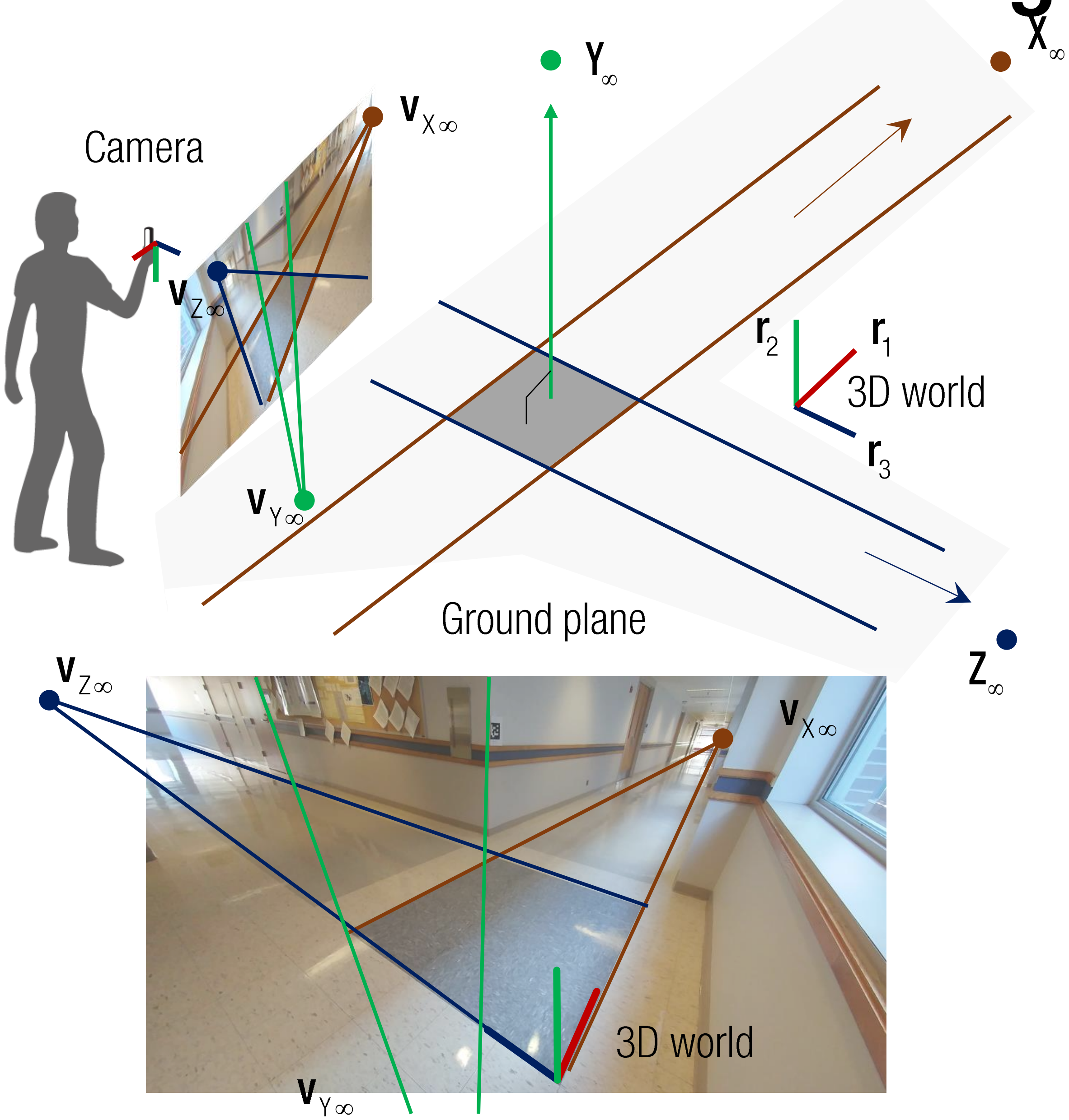
Camera Calibration using Vanishing Points



$$\lambda \mathbf{v}_{X\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{X}_\infty \quad \lambda \mathbf{v}_{Z\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{Z}_\infty \quad \lambda \mathbf{v}_{Y\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{Y}_\infty$$

Note that the camera extrinsic is still unknown (**R** and **t**).

Camera Calibration using Vanishing Points



$$\lambda \mathbf{v}_{X\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{X}_\infty \quad \lambda \mathbf{v}_{Z\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{Z}_\infty \quad \lambda \mathbf{v}_{Y\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{Y}_\infty$$

Note that the camera extrinsic is still unknown (**R** and **t**).

Known property of points at infinity:

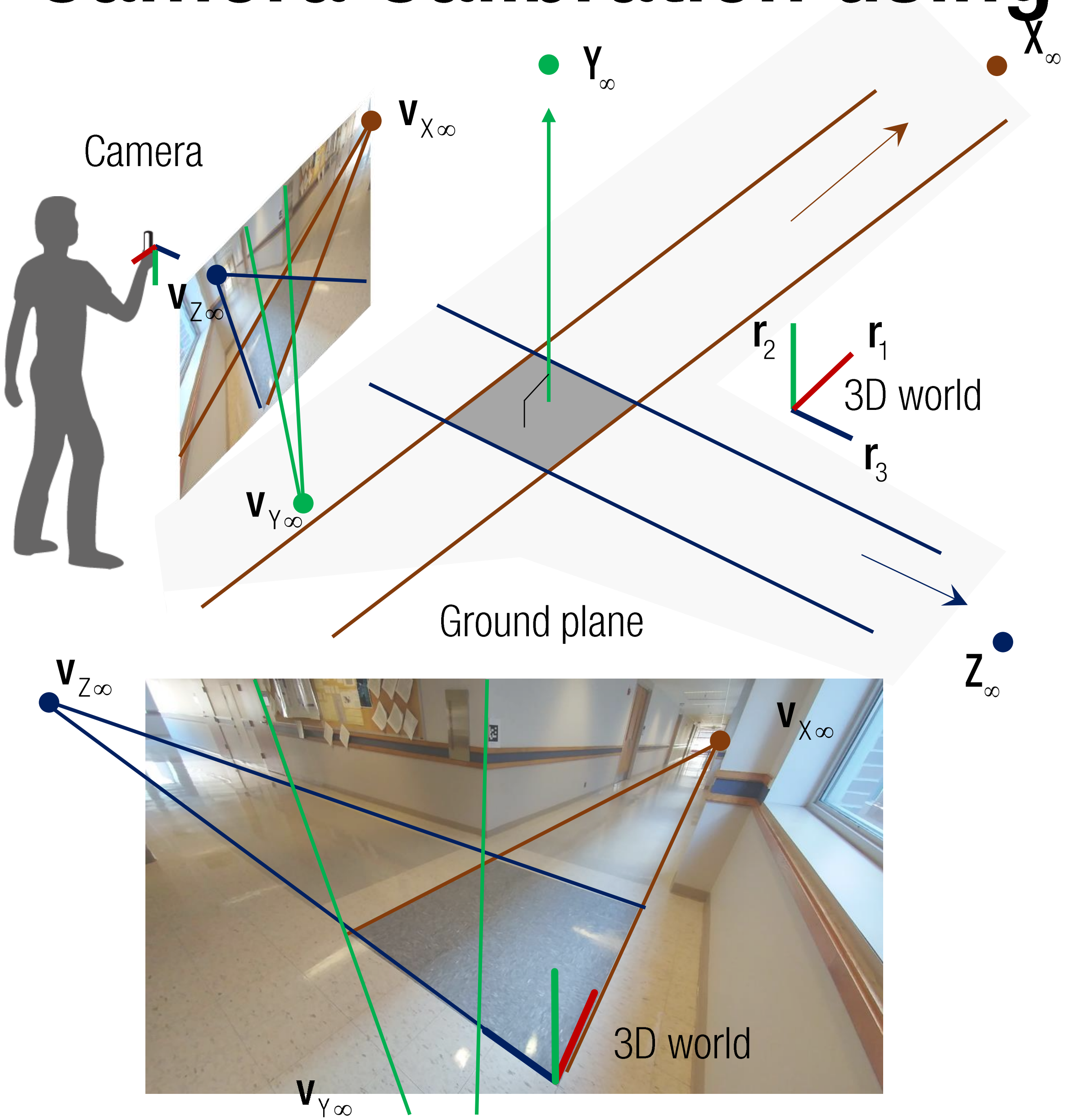
$$\left(\mathbf{X}_\infty\right)^{\mathsf{T}}\left(\mathbf{Y}_\infty\right) = 0$$

$$\left(\mathbf{Y}_\infty\right)^{\mathsf{T}}\left(\mathbf{Z}_\infty\right) = 0$$

$$\left(\mathbf{Z}_\infty\right)^{\mathsf{T}}\left(\mathbf{X}_\infty\right) = 0$$

These axes are perpendicular to each other.

Camera Calibration using Vanishing Points



$$\lambda \mathbf{v}_{X\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{X}_\infty \quad \lambda \mathbf{v}_{Z\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{Z}_\infty \quad \lambda \mathbf{v}_{Y\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{Y}_\infty$$

Note that the camera extrinsic is still unknown (**R** and **t**).

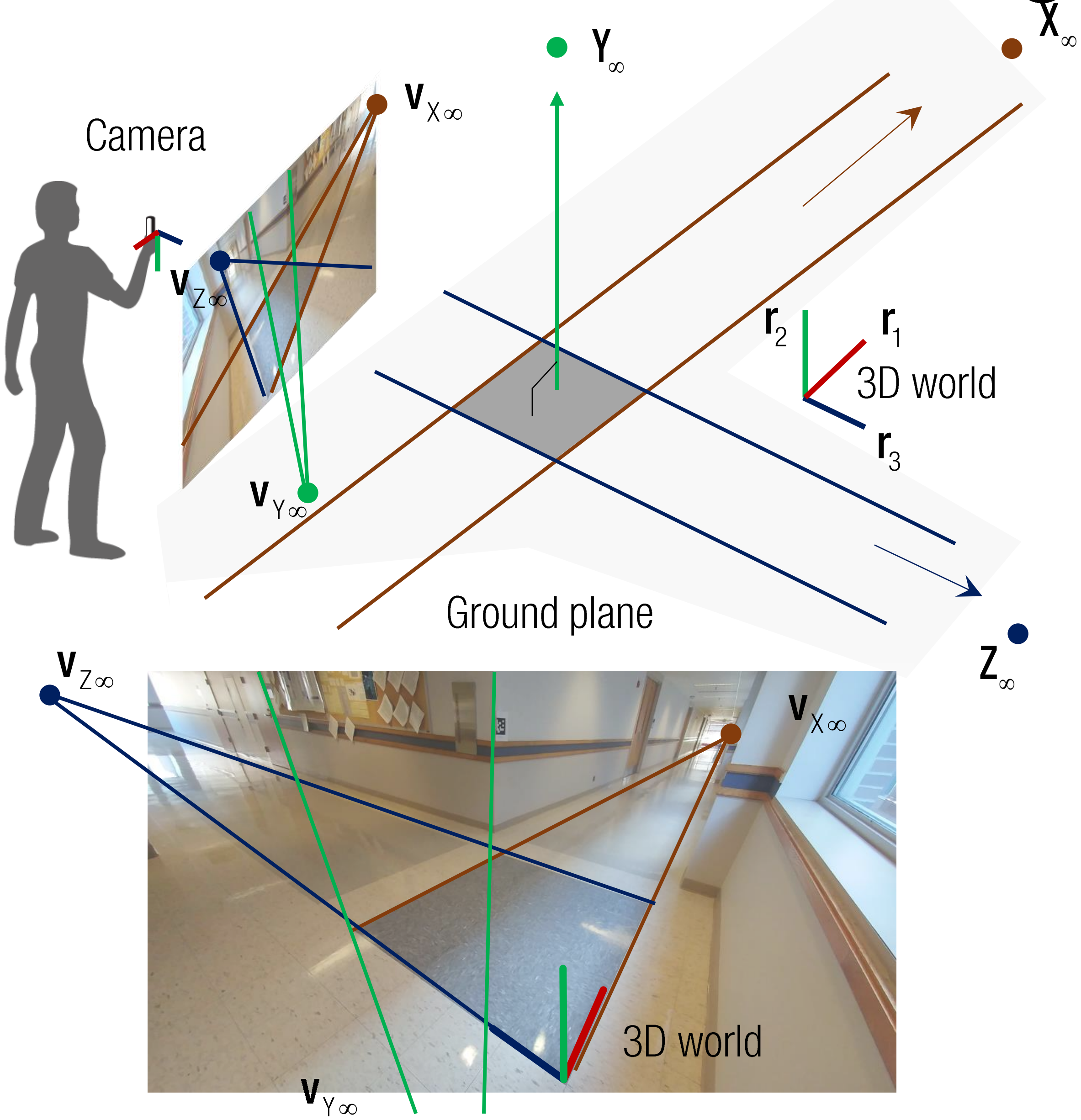
Known property of points at infinity:

$$(\mathbf{X}_\infty)^\mathsf{T}(\mathbf{Y}_\infty) = 0 \qquad (\mathbf{R}\mathbf{X}_\infty)^\mathsf{T}(\mathbf{R}\mathbf{Y}_\infty) = 0$$

$$(\mathbf{Y}_\infty)^\mathsf{T}(\mathbf{Z}_\infty) = 0 \quad \longleftrightarrow \quad (\mathbf{R}\mathbf{Y}_\infty)^\mathsf{T}(\mathbf{R}\mathbf{Z}_\infty) = 0$$

$$(\mathbf{Z}_\infty)^\mathsf{T}(\mathbf{X}_\infty) = 0 \qquad (\mathbf{R}\mathbf{Z}_\infty)^\mathsf{T}(\mathbf{R}\mathbf{X}_\infty) = 0$$

Camera Calibration using Vanishing Points



$$\lambda\mathbf{v}_{X\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}\mathbf{R}\mathbf{X}_\infty \quad \lambda\mathbf{v}_{Z\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}\mathbf{R}\mathbf{Z}_\infty \quad \lambda\mathbf{v}_{Y\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}\mathbf{R}\mathbf{Y}_\infty$$

$$\lambda\mathbf{K}^{-1}\mathbf{v}_{X\infty} = \mathbf{R}\mathbf{X}_\infty \quad \lambda\mathbf{K}^{-1}\mathbf{v}_{Y\infty} = \mathbf{R}\mathbf{Y}_\infty \quad \lambda\mathbf{K}^{-1}\mathbf{v}_{Z\infty} = \mathbf{R}\mathbf{Z}_\infty$$

Note that the camera extrinsic is still unknown (**R** and **t**).

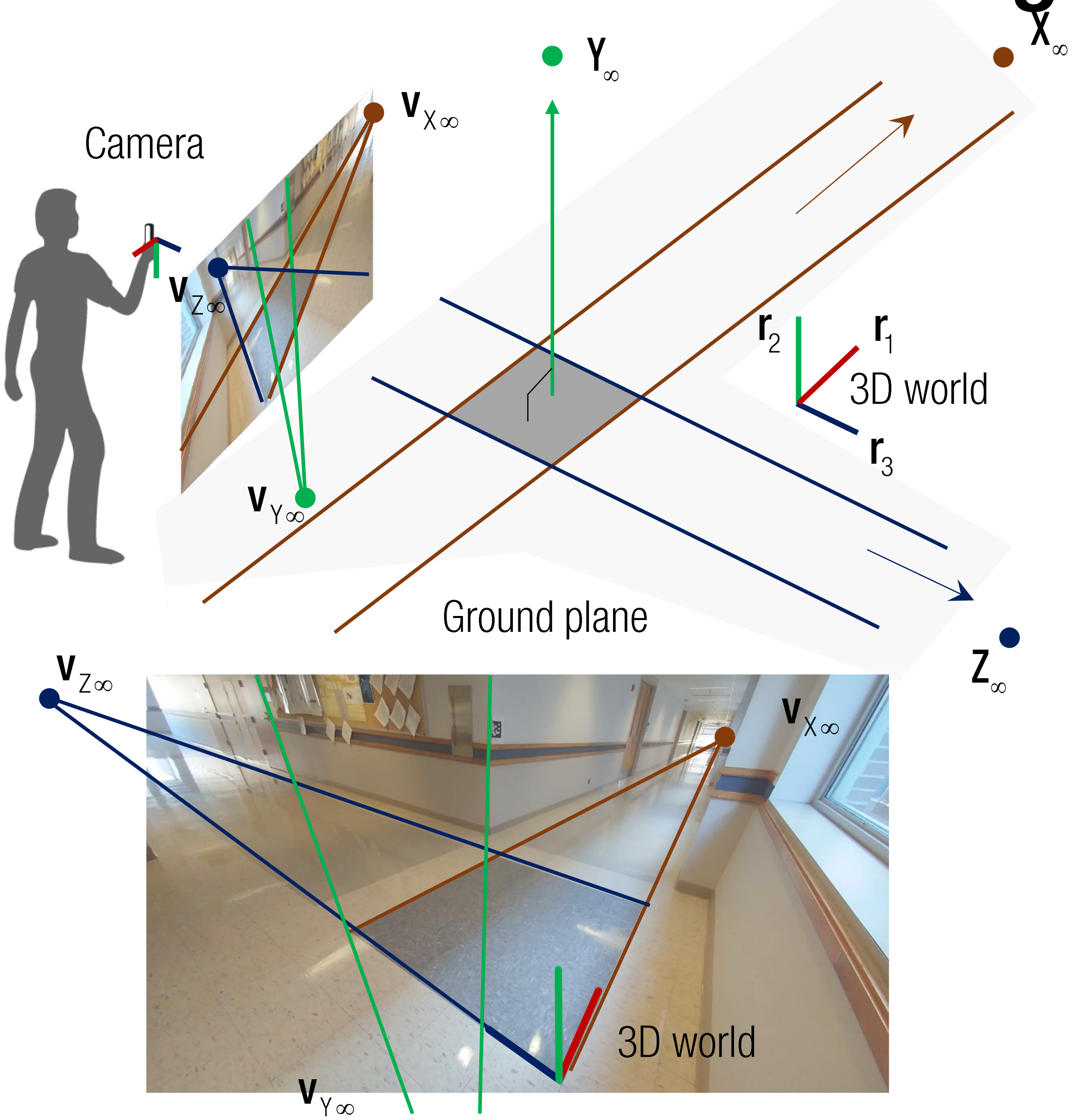
Known property of points at infinity:

$$(\mathbf{X}_\infty)^\mathsf{T}(\mathbf{Y}_\infty) = 0 \qquad (\mathbf{R}\mathbf{X}_\infty)^\mathsf{T}(\mathbf{R}\mathbf{Y}_\infty) = 0$$

$$(\mathbf{Y}_\infty)^\mathsf{T}(\mathbf{Z}_\infty) = 0 \quad \longleftrightarrow \quad (\mathbf{R}\mathbf{Y}_\infty)^\mathsf{T}(\mathbf{R}\mathbf{Z}_\infty) = 0$$

$$(\mathbf{Z}_\infty)^\mathsf{T}(\mathbf{X}_\infty) = 0 \qquad (\mathbf{R}\mathbf{Z}_\infty)^\mathsf{T}(\mathbf{R}\mathbf{X}_\infty) = 0$$

Camera Calibration using Vanishing Points



$$\lambda \mathbf{v}_{X\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{X}_\infty \quad \lambda \mathbf{v}_{Z\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{Z}_\infty \quad \lambda \mathbf{v}_{Y\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{Y}_\infty$$

$$\lambda \mathbf{K}^{-1}\mathbf{v}_{X\infty} = \mathbf{R}\mathbf{X}_\infty \qquad \lambda \mathbf{K}^{-1}\mathbf{v}_{Y\infty} = \mathbf{R}\mathbf{Y}_\infty \qquad \lambda \mathbf{K}^{-1}\mathbf{v}_{Z\infty} = \mathbf{R}\mathbf{Z}_\infty$$

Note that the camera extrinsic is still unknown (**R** and **t**).

Known property of points at infinity:

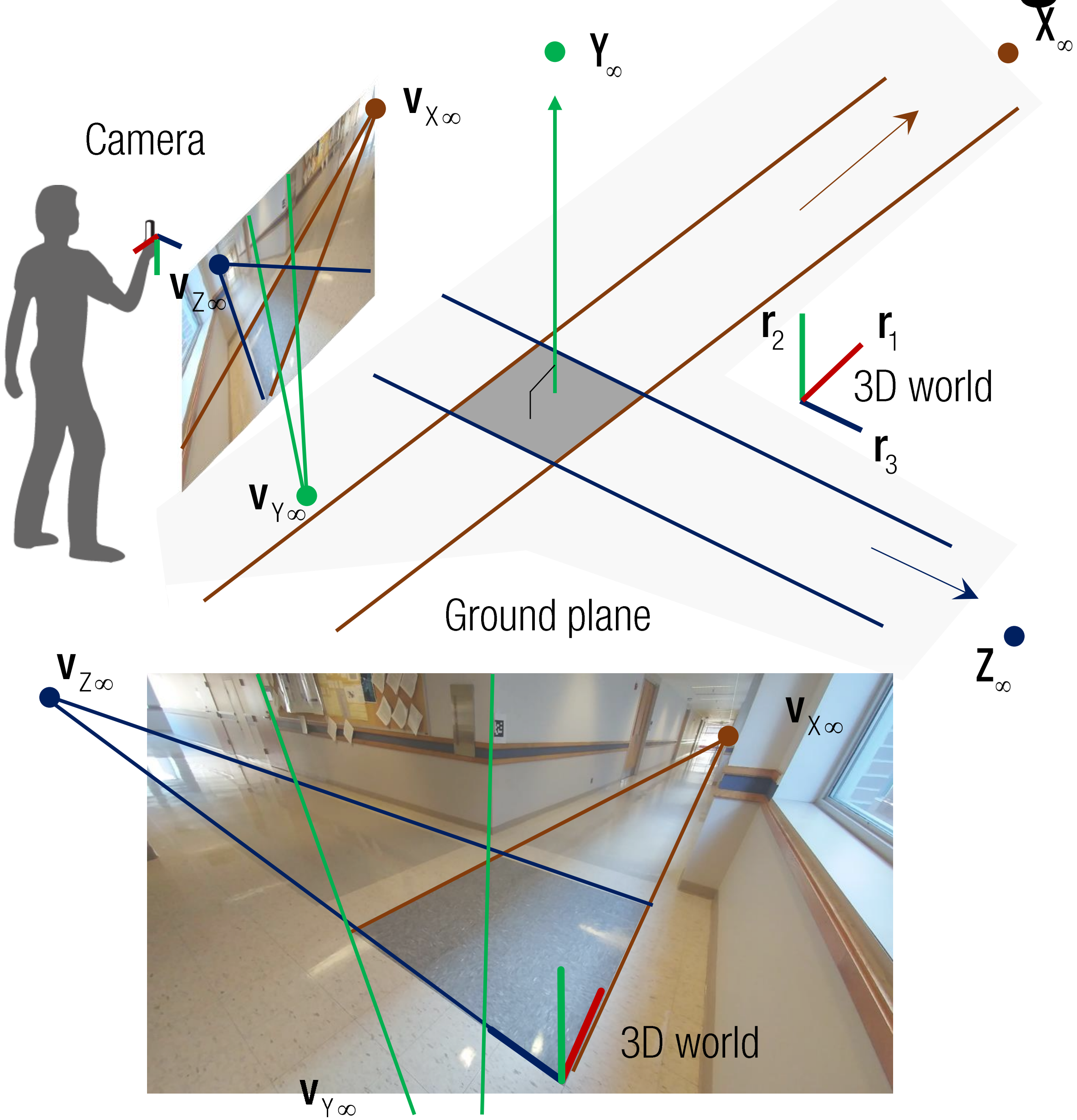
$$(\mathbf{X}_\infty)^\mathsf{T}(\mathbf{Y}_\infty) = 0 \qquad (\mathbf{R}\mathbf{X}_\infty)^\mathsf{T}(\mathbf{R}\mathbf{Y}_\infty) = 0$$

$$(\mathbf{Y}_\infty)^\mathsf{T}(\mathbf{Z}_\infty) = 0 \quad \longleftrightarrow \quad (\mathbf{R}\mathbf{Y}_\infty)^\mathsf{T}(\mathbf{R}\mathbf{Z}_\infty) = 0$$

$$(\mathbf{Z}_\infty)^\mathsf{T}(\mathbf{X}_\infty) = 0 \qquad (\mathbf{R}\mathbf{Z}_\infty)^\mathsf{T}(\mathbf{R}\mathbf{X}_\infty) = 0$$

$$(\mathbf{K}^{-1}\mathbf{v}_{X\infty})^\mathsf{T}(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}) = (\mathbf{K}^{-1}\mathbf{v}_{Y\infty})^\mathsf{T}(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}) = (\mathbf{K}^{-1}\mathbf{v}_{Z\infty})^\mathsf{T}(\mathbf{K}^{-1}\mathbf{v}_{X\infty}) = 0$$

Camera Calibration using Vanishing Points



$$\lambda \mathbf{v}_{X\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{X}_\infty \quad \lambda \mathbf{v}_{Z\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{Z}_\infty \quad \lambda \mathbf{v}_{Y\infty} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \mathbf{R}\mathbf{Y}_\infty$$

$$\lambda \mathbf{K}^{-1}\mathbf{v}_{X\infty} = \mathbf{R}\mathbf{X}_\infty \qquad \lambda \mathbf{K}^{-1}\mathbf{v}_{Y\infty} = \mathbf{R}\mathbf{Y}_\infty \qquad \lambda \mathbf{K}^{-1}\mathbf{v}_{Z\infty} = \mathbf{R}\mathbf{Z}_\infty$$

Note that the camera extrinsic is still unknown (**R** and **t**).

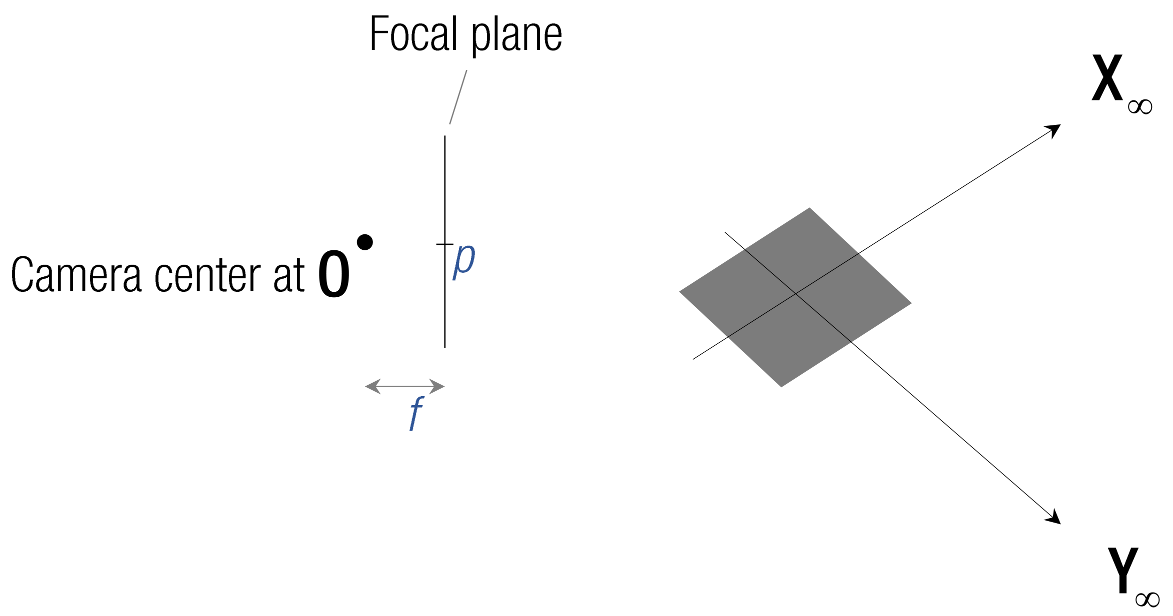
Known property of points at infinity:

$$(\mathbf{X}_\infty)^T(\mathbf{Y}_\infty) = 0 \qquad (\mathbf{R}\mathbf{X}_\infty)^T(\mathbf{R}\mathbf{Y}_\infty) = 0$$
$$(\mathbf{Y}_\infty)^T(\mathbf{Z}_\infty) = 0 \quad \longleftrightarrow \quad (\mathbf{R}\mathbf{Y}_\infty)^T(\mathbf{R}\mathbf{Z}_\infty) = 0$$
$$(\mathbf{Z}_\infty)^T(\mathbf{X}_\infty) = 0 \qquad (\mathbf{R}\mathbf{Z}_\infty)^T(\mathbf{R}\mathbf{X}_\infty) = 0$$

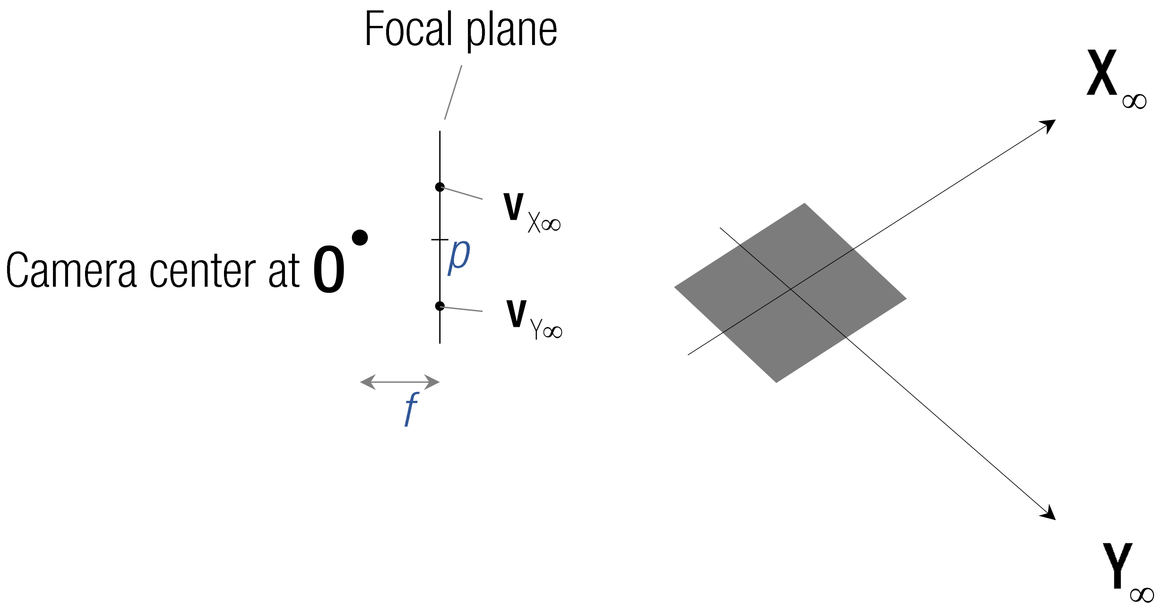
$$(\mathbf{K}^{-1}\mathbf{v}_{X\infty})^T(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}) = (\mathbf{K}^{-1}\mathbf{v}_{Y\infty})^T(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}) = (\mathbf{K}^{-1}\mathbf{v}_{Z\infty})^T(\mathbf{K}^{-1}\mathbf{v}_{X\infty}) = 0$$

: 3 unknowns and 3 equations

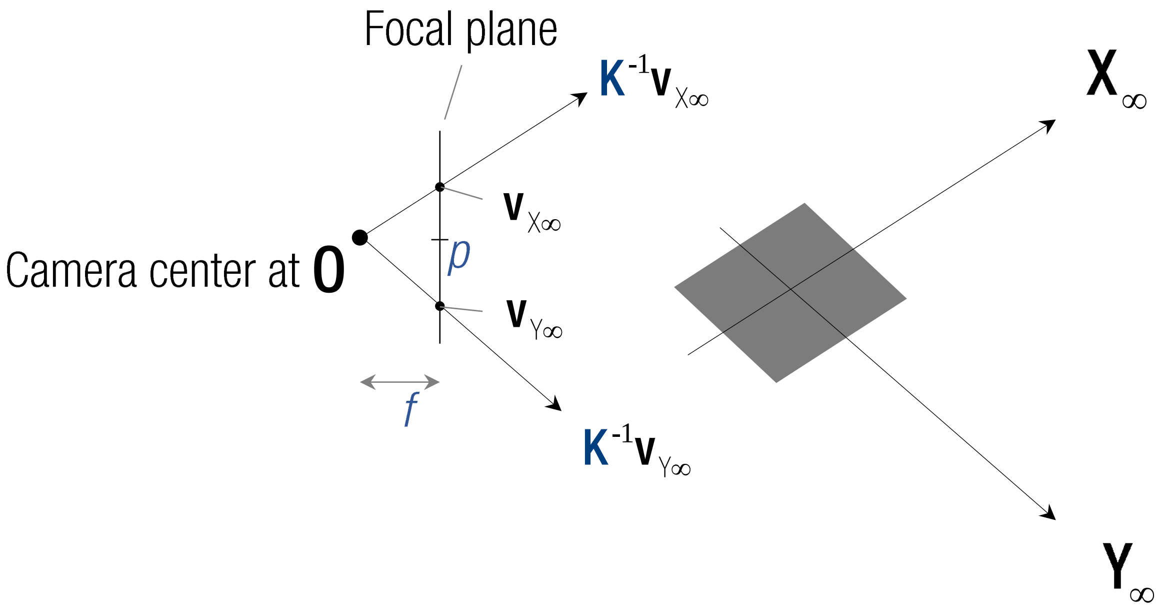
Geometric Interpretation with 1D Camera



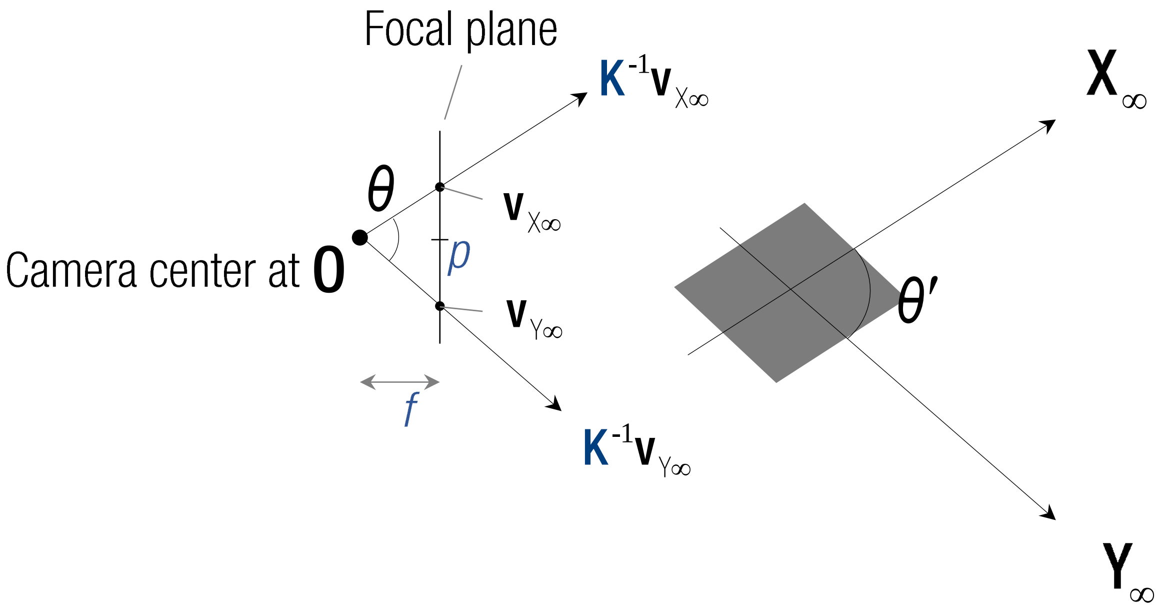
Geometric Interpretation with 1D Camera



Geometric Interpretation with 1D Camera

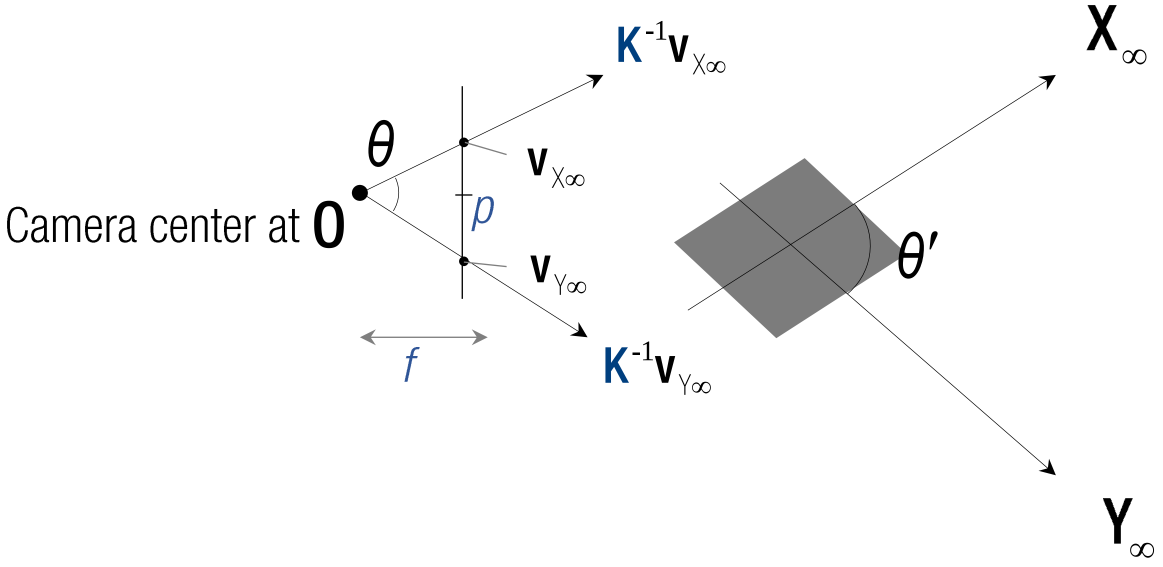


Geometric Interpretation with 1D Camera



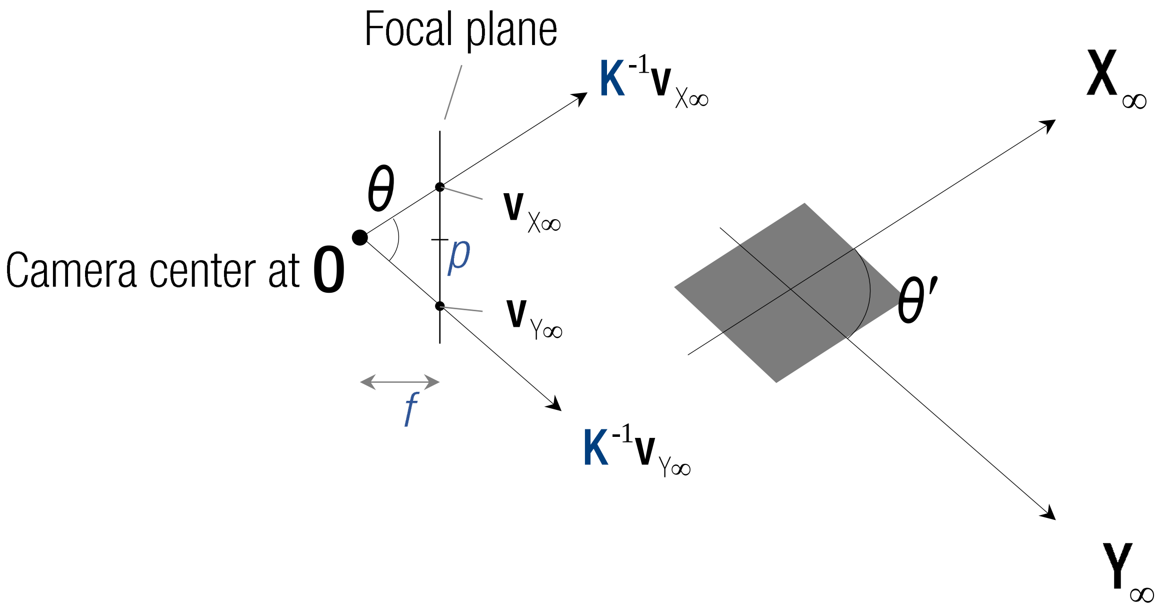
$\theta = \theta'$

Geometric Interpretation with 1D Camera



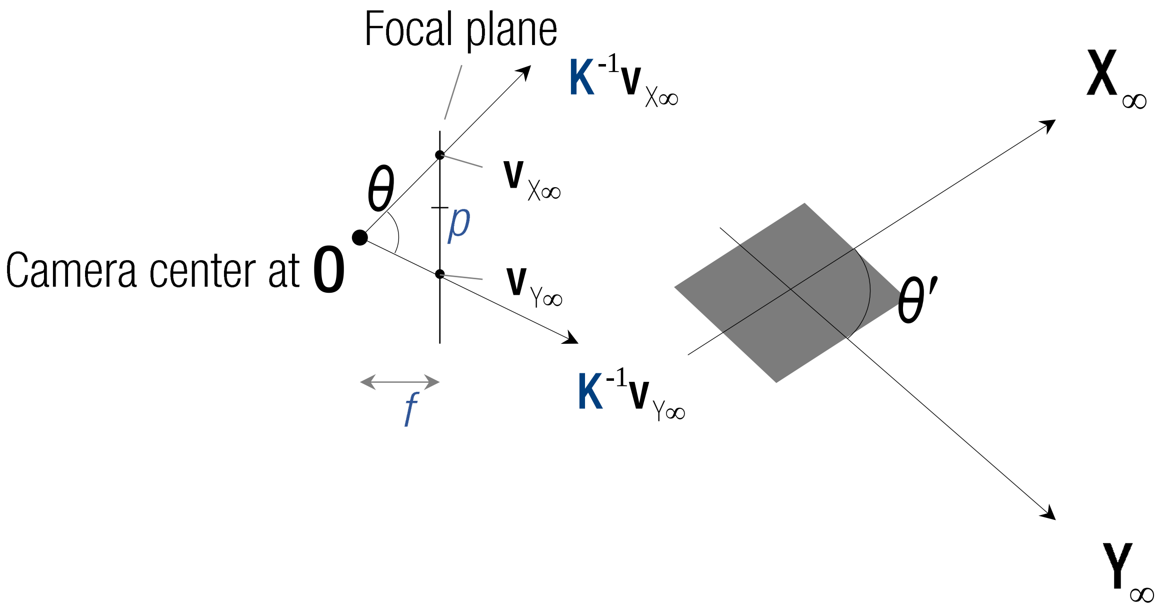
$$\theta \neq \theta'$$

Geometric Interpretation with 1D Camera



$\theta = \theta'$

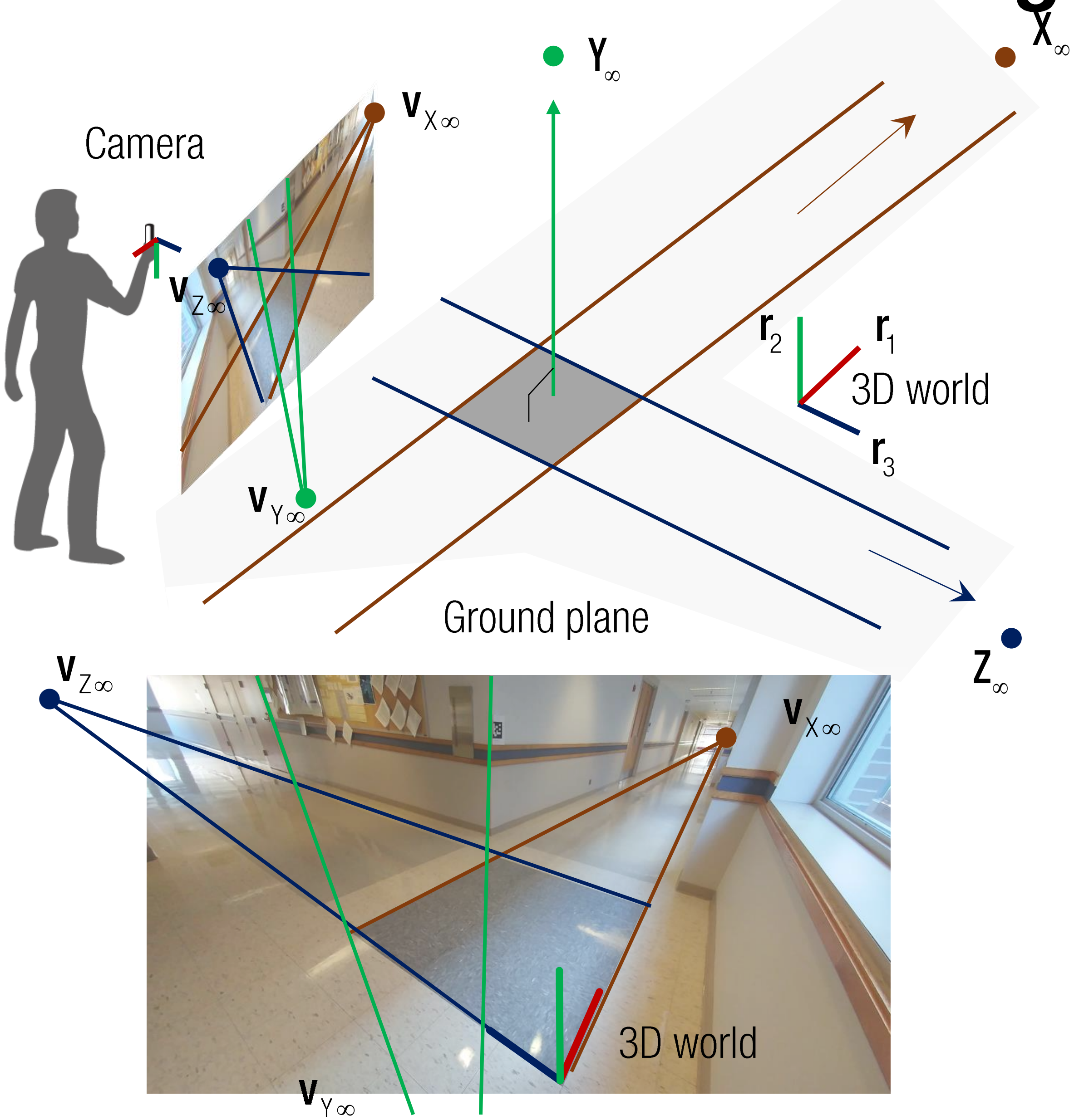
Geometric Interpretation with 1D Camera



$$\theta \neq \theta'$$

Given two vanishing points, the focal length and principal point are uniquely defined.
For the 2D camera case, another vanishing point is needed to uniquely define f, p_x, and p_y.

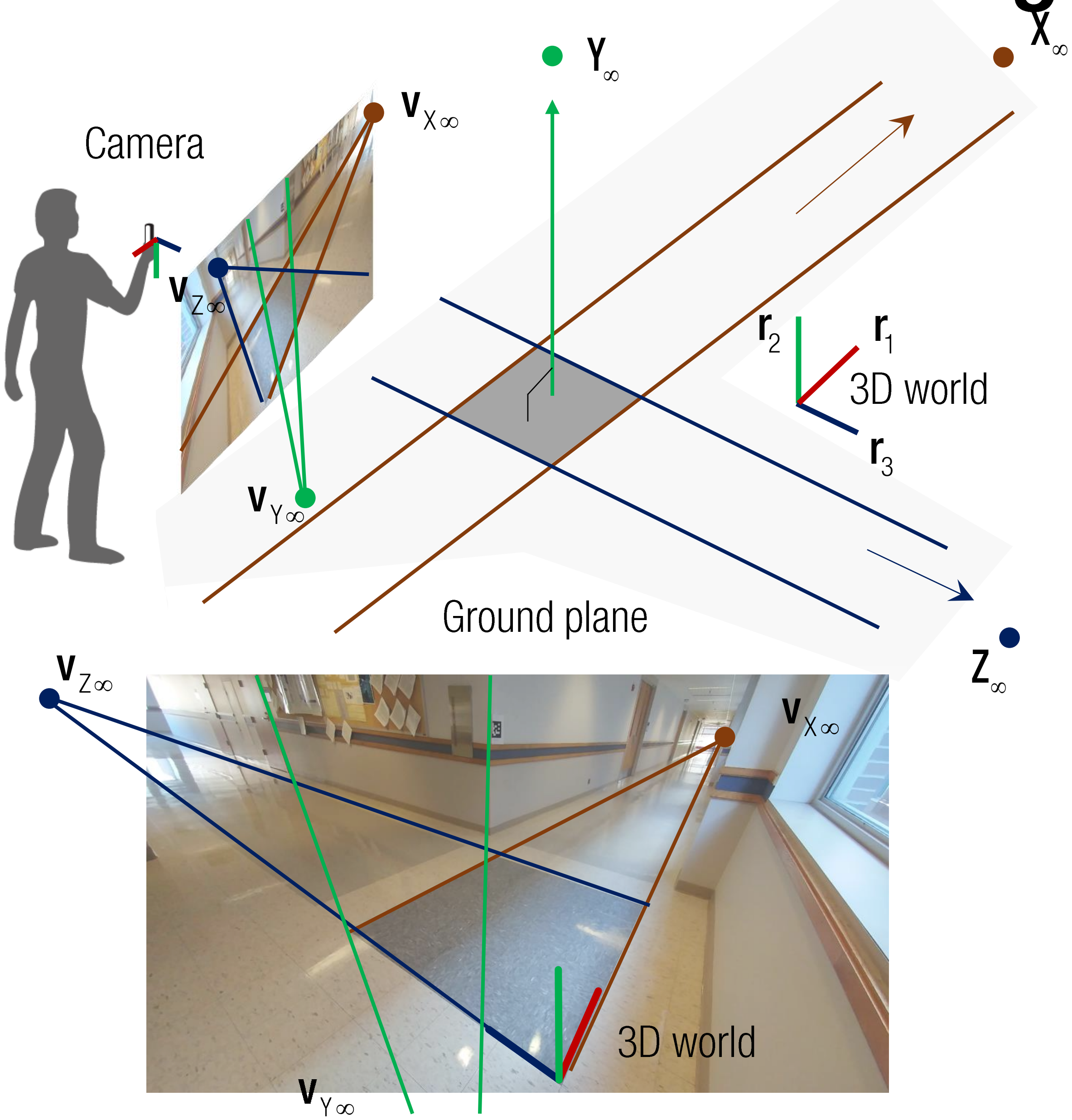
Camera Calibration using Vanishing Points



$$\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)=0$$

: 3 unknowns and 3 equations

Camera Calibration using Vanishing Points

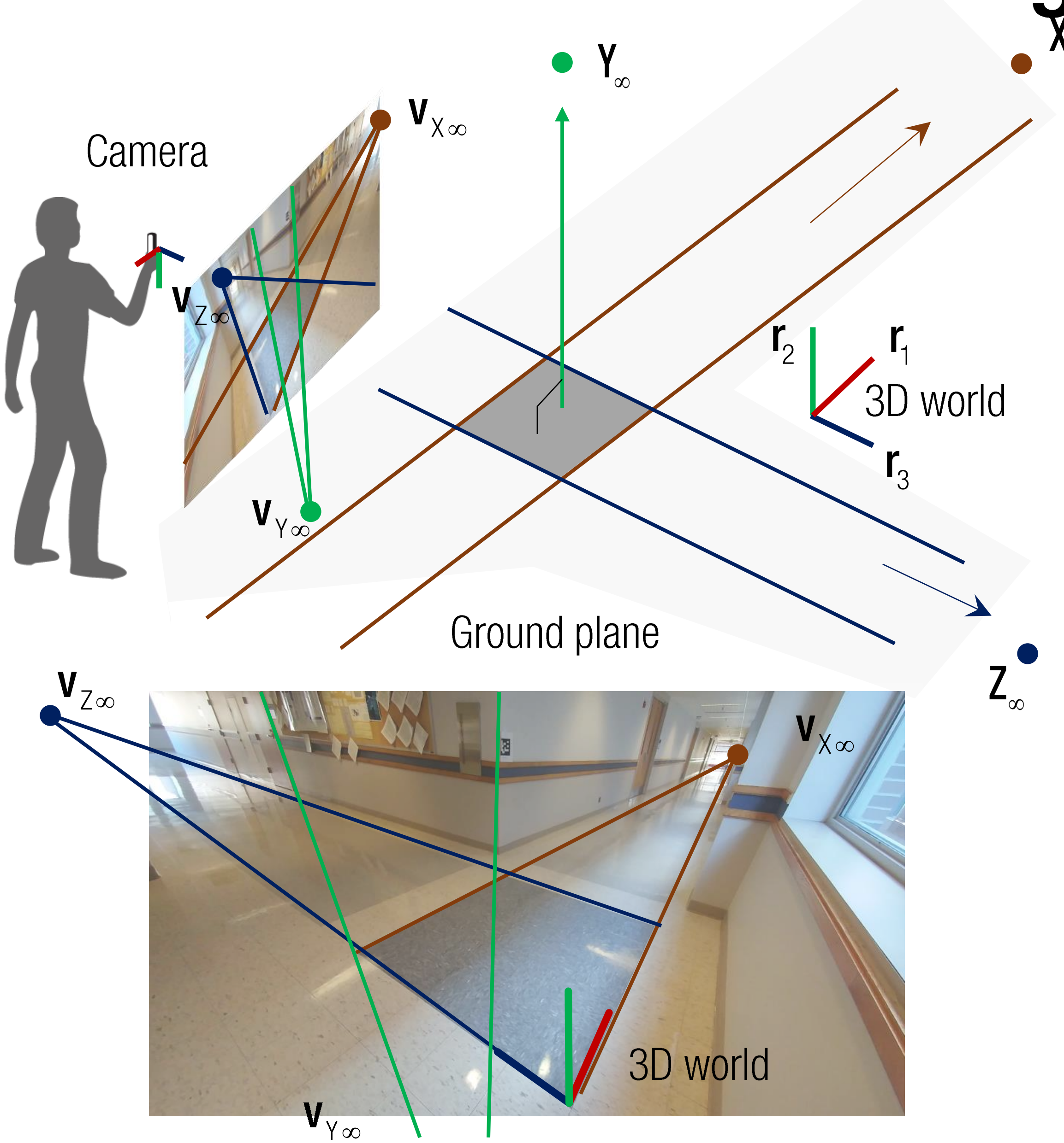


$$\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)=0$$

: 3 unknowns and 3 equations

$$\longrightarrow \quad \left(\mathbf{K}^{-1}\mathbf{v}_{i}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{j}\right)=\mathbf{v}_{i}^{T}\mathbf{K}^{-T}\mathbf{K}^{-1}\mathbf{v}_{j} \qquad i,j=X,Y,Z, \text{ and } i\neq j$$

Camera Calibration using Vanishing Points



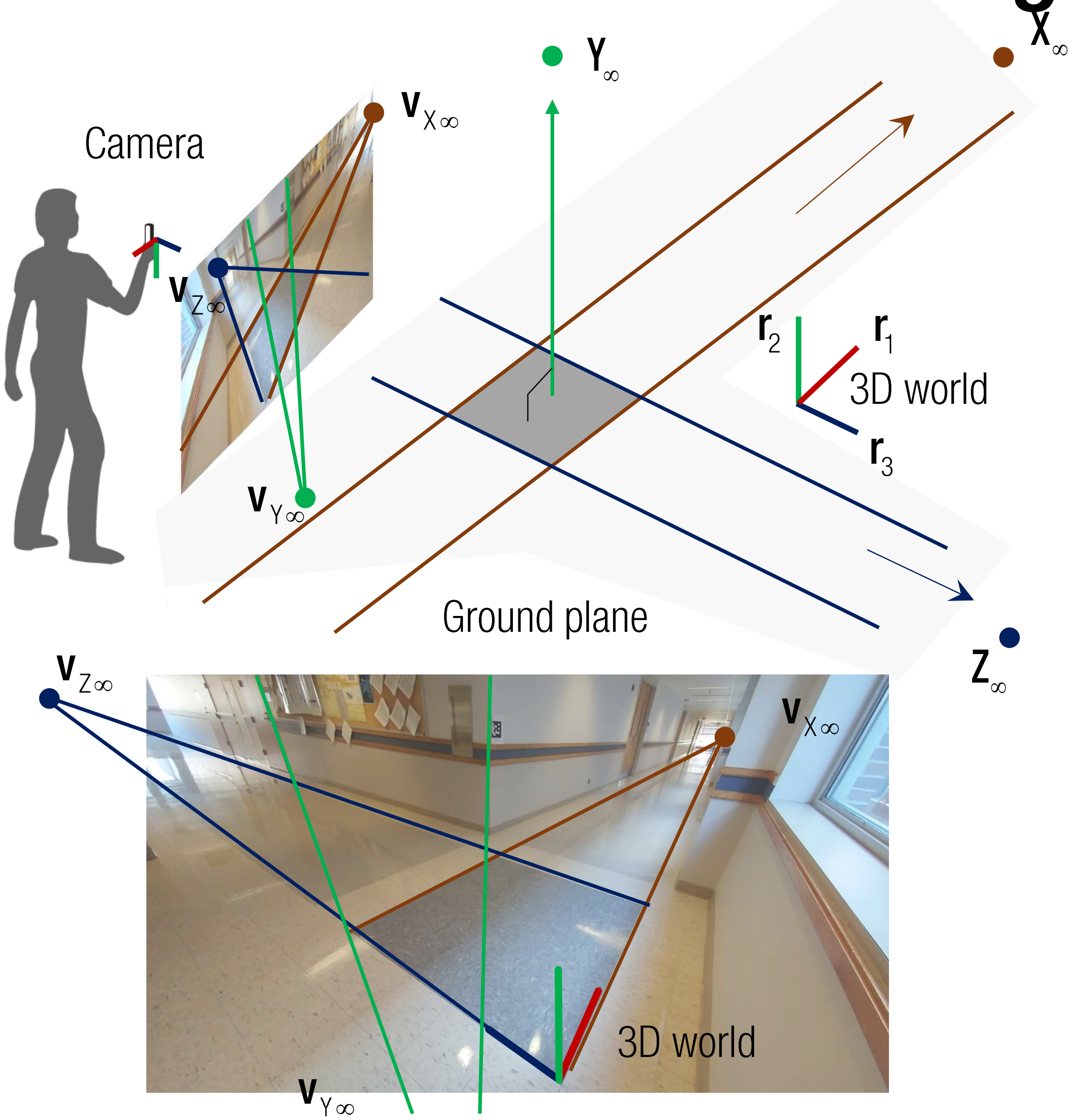
$$\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)=0$$

: 3 unknowns and 3 equations

$$\longrightarrow \quad \left(\mathbf{K}^{-1}\mathbf{v}_{i}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{j}\right)=\mathbf{v}_{i}^{T}\mathbf{K}^{-T}\mathbf{K}^{-1}\mathbf{v}_{j} \qquad i,j=X,Y,Z, \text{ and } i\neq j$$

$$\mathbf{K}^{-T}\mathbf{K}^{-1}=\begin{bmatrix}1/f & & \\ & 1/f & \\ -p_x/f & -p_y/f & 1\end{bmatrix}\begin{bmatrix}1/f & & -p_x/f \\ & 1/f & -p_y/f \\ & & 1\end{bmatrix}$$

Camera Calibration using Vanishing Points



$$\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)^T\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right) = \left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)^T\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right) = \left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)^T\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right) = 0$$

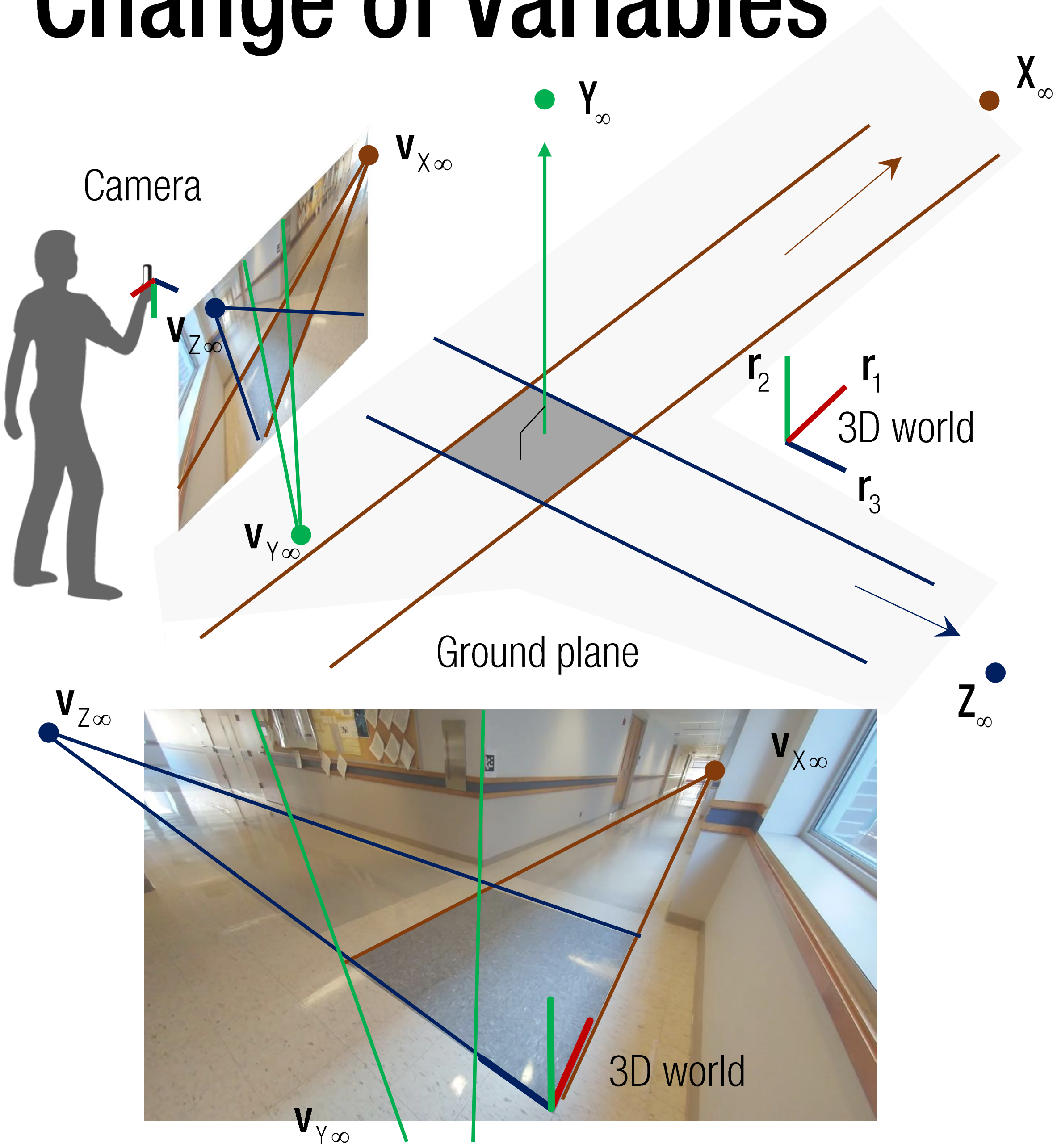
: 3 unknowns and 3 equations

$$\longrightarrow \quad \left(\mathbf{K}^{-1}\mathbf{v}_i\right)^T\left(\mathbf{K}^{-1}\mathbf{v}_j\right) = \mathbf{v}_i^T\mathbf{K}^{-T}\mathbf{K}^{-1}\mathbf{v}_j \qquad i, j = X, Y, Z, \text{ and } i \neq j$$

$$\mathbf{K}^{-T}\mathbf{K}^{-1} = \begin{bmatrix} 1/f & & \\ & 1/f & \\ -p_x/f & -p_y/f & 1 \end{bmatrix}\begin{bmatrix} 1/f & & -p_x/f \\ & 1/f & -p_y/f \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{f^2} & & -\frac{p_x}{f^2} \\ & \frac{1}{f^2} & -\frac{p_y}{f^2} \\ -\frac{p_x}{f^2} & -\frac{p_y}{f^2} & \frac{p_x^2}{f^2} + \frac{p_y^2}{f^2} + 1 \end{bmatrix}$$

Change of Variables



$$(\mathbf{K}^{-1}\mathbf{v}_{X\infty})^{T}(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}) = (\mathbf{K}^{-1}\mathbf{v}_{Y\infty})^{T}(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}) = (\mathbf{K}^{-1}\mathbf{v}_{Z\infty})^{T}(\mathbf{K}^{-1}\mathbf{v}_{X\infty}) = 0$$

: 3 unknowns and 3 equations

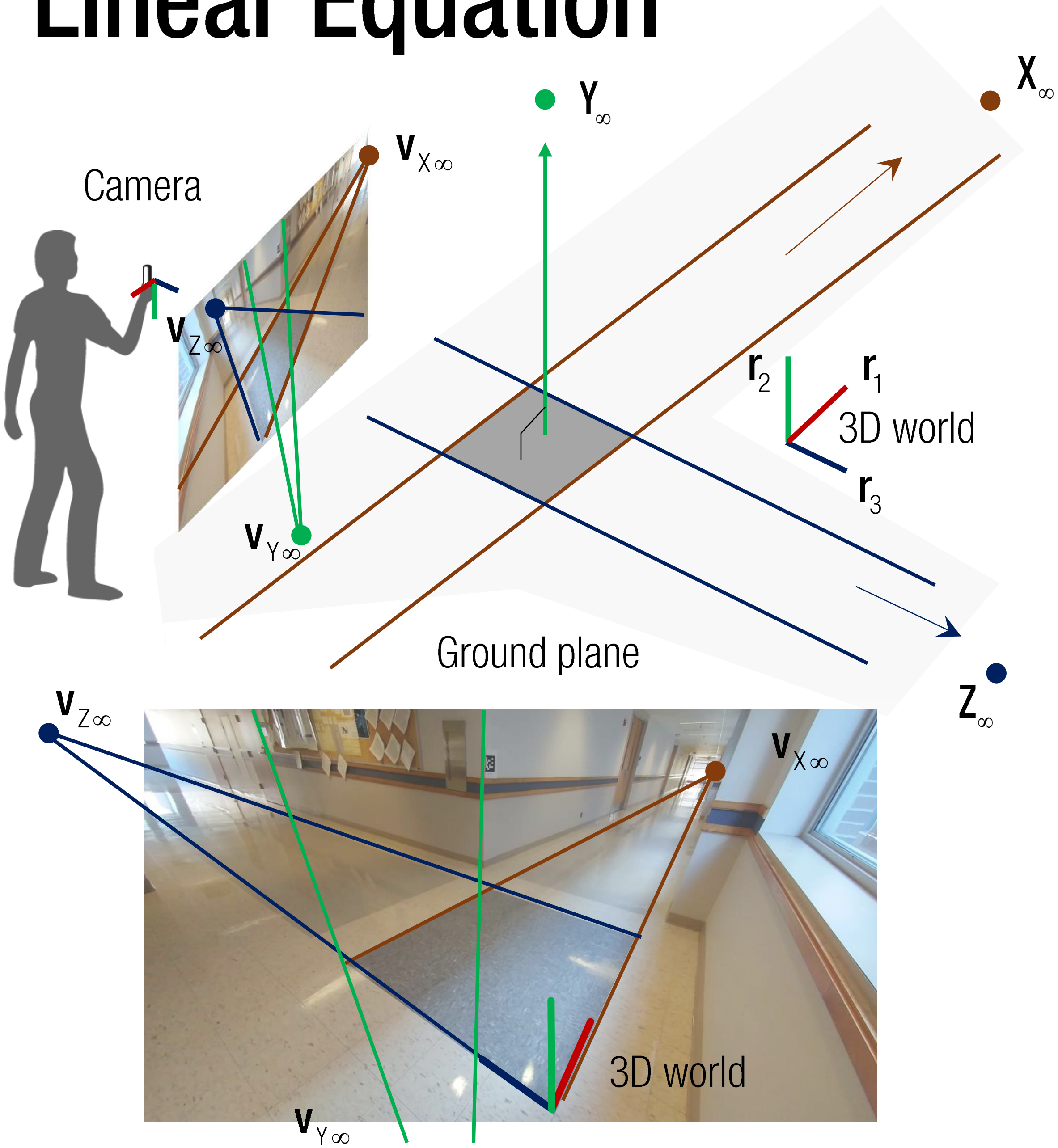
$$\longrightarrow \quad (\mathbf{K}^{-1}\mathbf{v}_{i})^{T}(\mathbf{K}^{-1}\mathbf{v}_{j}) = \mathbf{v}_i^{T}\mathbf{K}^{-T}\mathbf{K}^{-1}\mathbf{v}_j \qquad i,j = X,Y,Z, \text{ and } i \neq j$$

$$\mathbf{K}^{-T}\mathbf{K}^{-1} = \begin{bmatrix} 1/f & & \\ & 1/f & \\ -p_x/f & -p_y/f & 1 \end{bmatrix}\begin{bmatrix} 1/f & & -p_x/f \\ & 1/f & -p_y/f \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{f^2} & & -\frac{p_x}{f^2} \\ & \frac{1}{f^2} & -\frac{p_y}{f^2} \\ -\frac{p_x}{f^2} & -\frac{p_y}{f^2} & \frac{p_x^2}{f^2}+\frac{p_y^2}{f^2}+1 \end{bmatrix} = \begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix}$$

where $b_1 = \frac{1}{f^2}, \quad b_2 = -\frac{p_x}{f^2}, \quad b_3 = -\frac{p_y}{f^2}, \quad b_4 = \frac{p_x^2}{f^2} + \frac{p_y^2}{f^2} + 1$

Linear Equation



$$\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)=0$$

: 3 unknowns and 3 equations

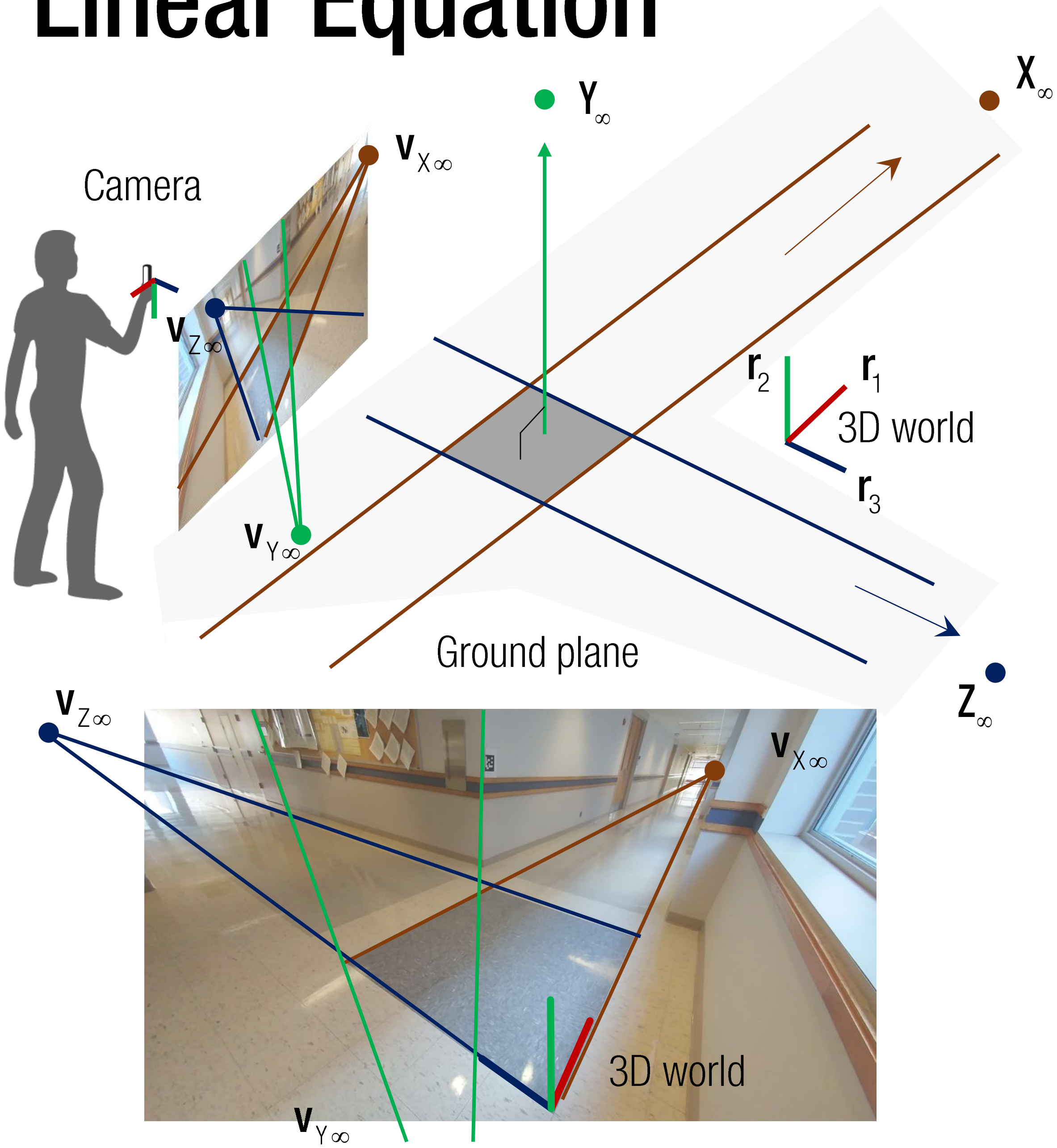
$$\rightarrow \left(\mathbf{K}^{-1}\mathbf{v}_i\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_j\right)=\mathbf{v}_i^{T}\mathbf{K}^{-T}\mathbf{K}^{-1}\mathbf{v}_j \qquad i,j = X,Y,Z, \text{ and } i \neq j$$

$$\rightarrow \mathbf{v}_i^{T}\begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix}\mathbf{v}_j = \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix}^{T}\begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix}\begin{bmatrix} u_j \\ v_j \\ 1 \end{bmatrix}=0$$

Linear in b

$$\rightarrow \begin{bmatrix} u_i u_j + v_i v_j & u_i + u_j & v_i + v_j & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}=0$$

Linear Equation



$$\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)=0$$

: 3 unknowns and 3 equations

$\rightarrow$ $\left(\mathbf{K}^{-1}\mathbf{v}_i\right)^{T}\left(\mathbf{K}^{-1}\mathbf{v}_j\right)=\mathbf{v}_i^{T}\mathbf{K}^{-T}\mathbf{K}^{-1}\mathbf{v}_j$ $\quad i,j=X,Y,Z,\ \text{and}\ i\neq j$

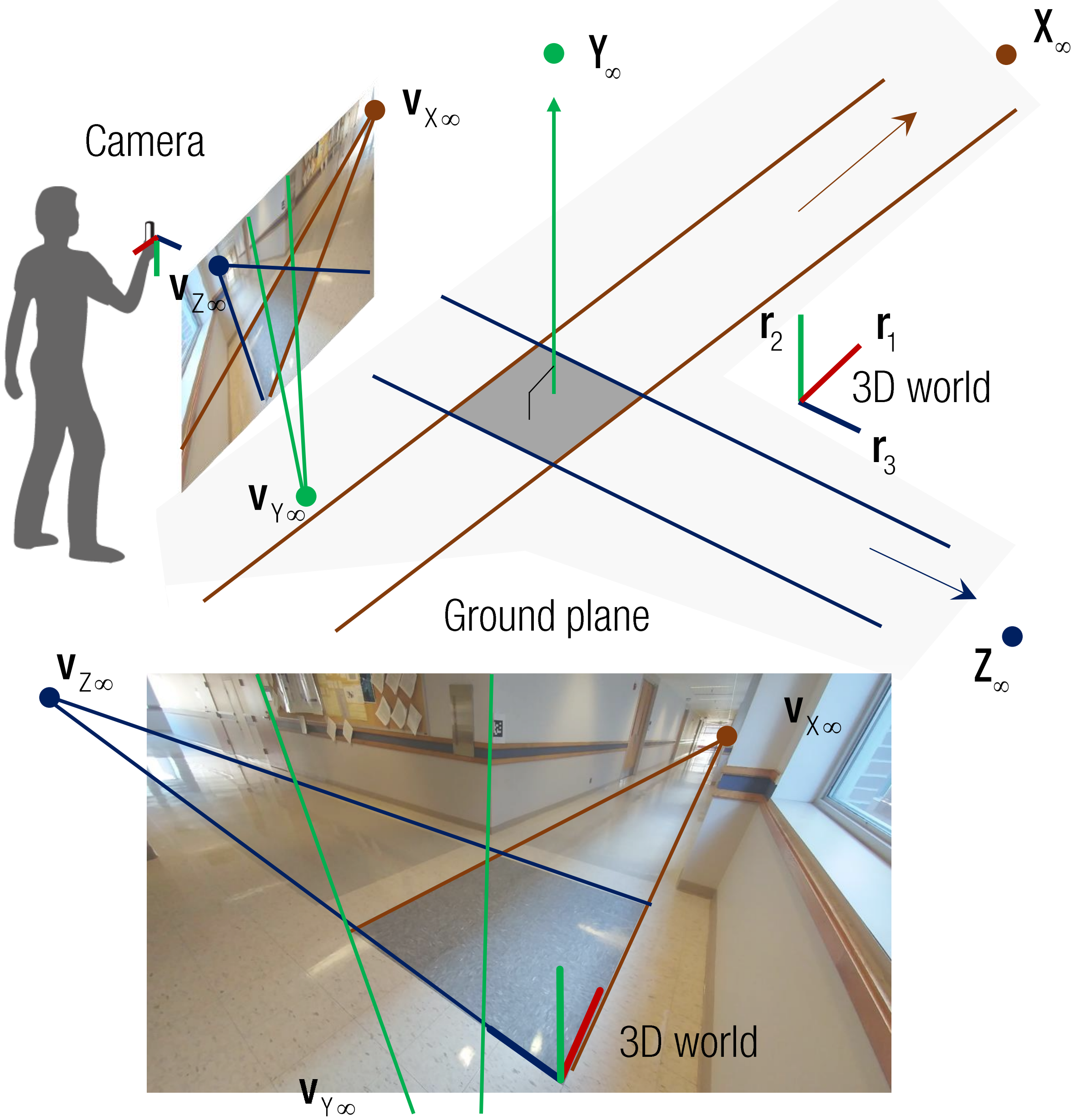
$$\rightarrow\quad \mathbf{v}_i^{T}\begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix}\mathbf{v}_j=\begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix}^{T}\begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix}\begin{bmatrix} u_j \\ v_j \\ 1 \end{bmatrix}=0$$

Linear in b

$$\rightarrow\quad \begin{bmatrix} u_1u_2+v_1v_2 & u_1+u_2 & v_1+v_2 & 1 \\ u_3u_2+v_3v_2 & u_3+u_2 & v_3+v_2 & 1 \\ u_1u_3+v_1v_3 & u_1+u_3 & v_1+v_3 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}=0$$

3x4

Linear Solve



$$\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)^T\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)^T\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)^T\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)=0$$

: 3 unknowns and 3 equations

$$\rightarrow \left(\mathbf{K}^{-1}\mathbf{v}_i\right)^T\left(\mathbf{K}^{-1}\mathbf{v}_j\right)=\mathbf{v}_i^T\mathbf{K}^{-T}\mathbf{K}^{-1}\mathbf{v}_j \qquad i,j=X,Y,Z, \text{ and } i\neq j$$

$$\rightarrow \mathbf{v}_i^T\begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix}\mathbf{v}_j=\begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix}^T\begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix}\begin{bmatrix} u_j \\ v_j \\ 1 \end{bmatrix}=0$$

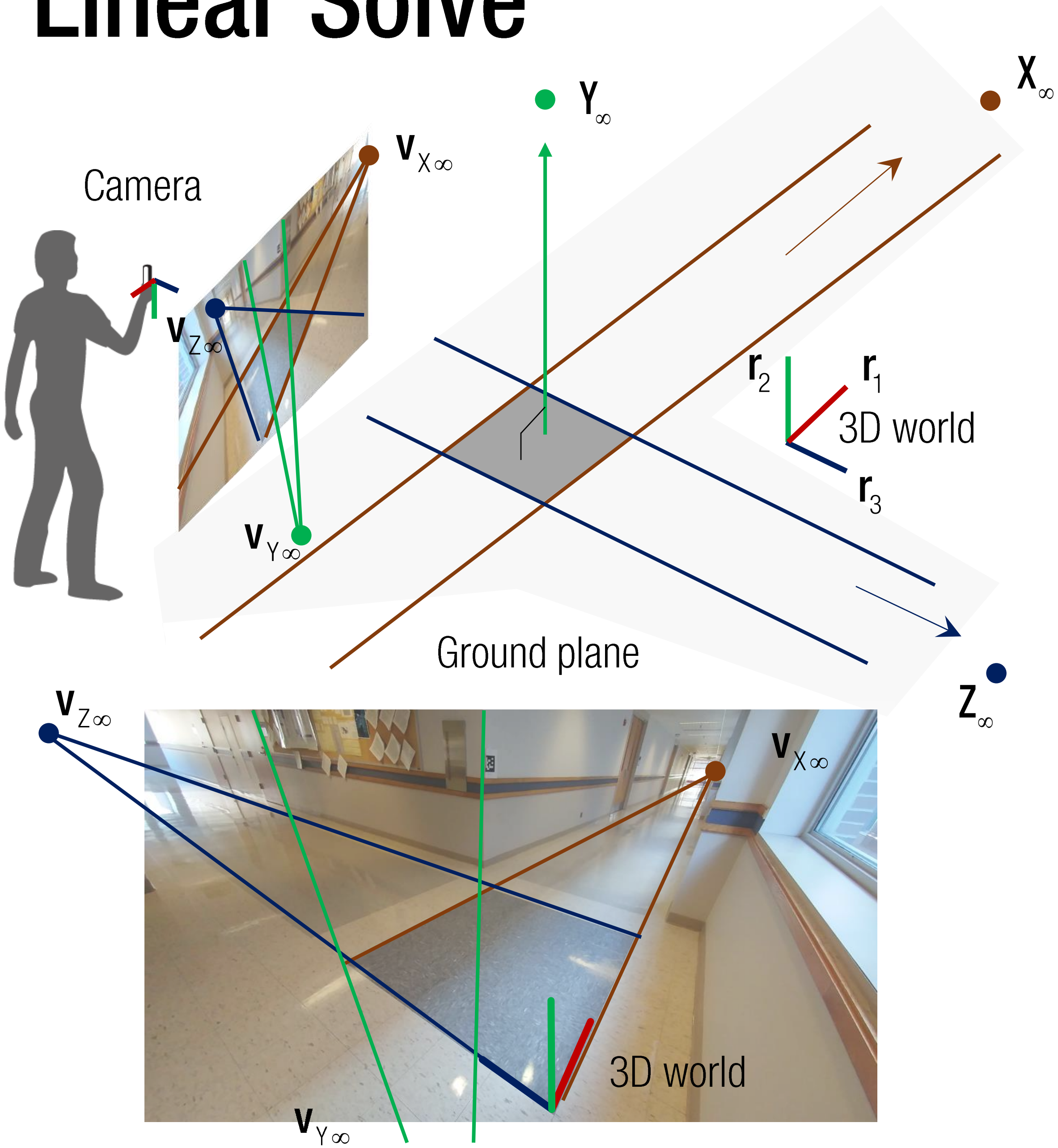
Linear in b

$$\rightarrow \begin{bmatrix} u_1u_2+v_1v_2 & u_1+u_2 & v_1+v_2 & 1 \\ u_3u_2+v_3v_2 & u_3+u_2 & v_3+v_2 & 1 \\ u_1u_3+v_1v_3 & u_1+u_3 & v_1+v_3 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}=\mathbf{0}$$

A **x** = **0**

3x4

Linear Solve



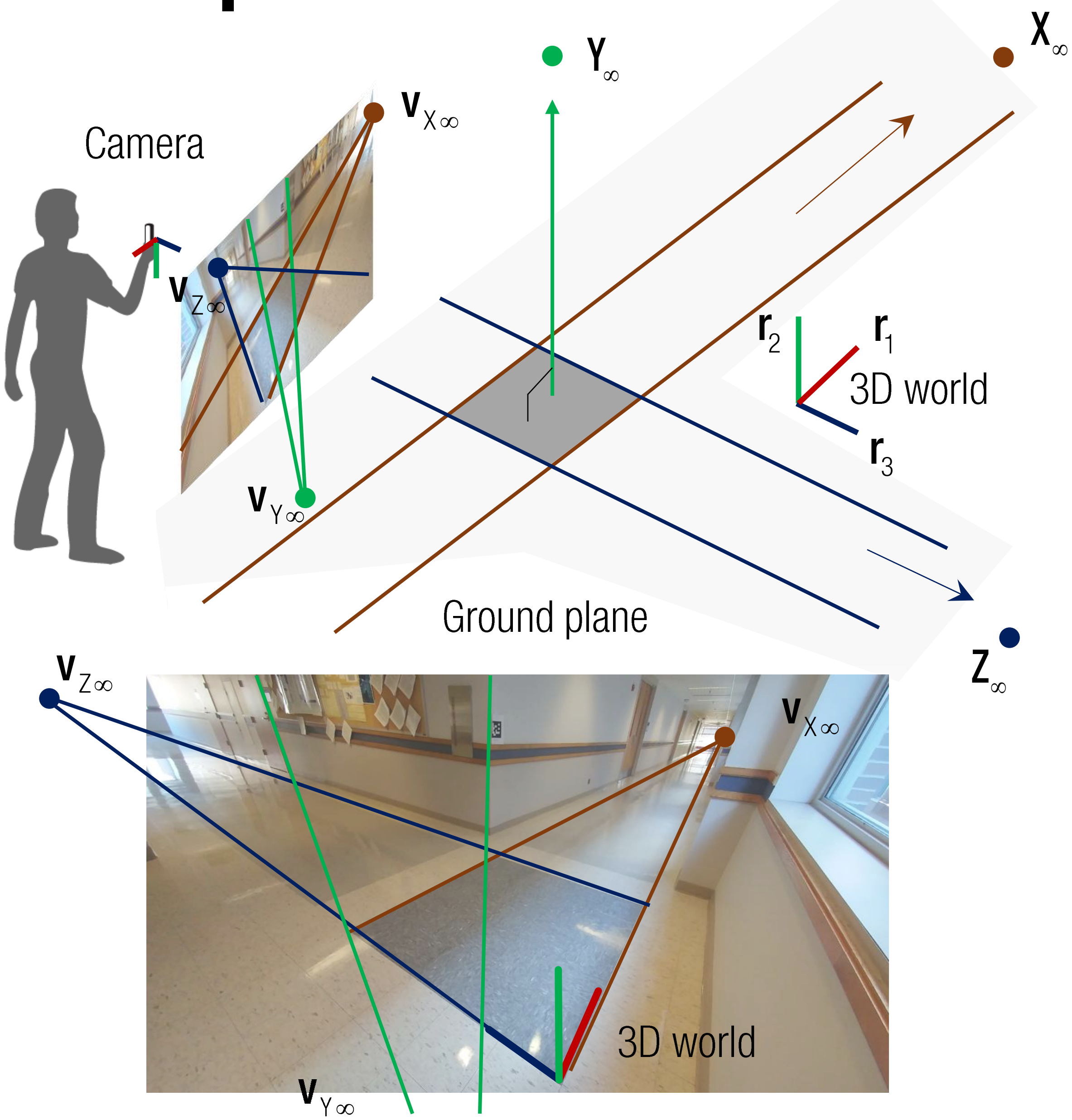
$$\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)^T\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)^T\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)^T\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)=0$$

: 3 unknowns and 3 equations

$$\rightarrow \quad \mathbf{x}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

where $b_1=\frac{1}{f^2},\quad b_2=-\frac{p_x}{f^2},\quad b_3=-\frac{p_y}{f^2},\quad b_4=\frac{p_x^{\ 2}}{f^2}+\frac{p_y^2}{f^2}+1$

Compute Extrinsic



$$\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)^{\mathsf{T}}\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Y\infty}\right)^{\mathsf{T}}\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)=\left(\mathbf{K}^{-1}\mathbf{v}_{Z\infty}\right)^{\mathsf{T}}\left(\mathbf{K}^{-1}\mathbf{v}_{X\infty}\right)=0$$

: 3 unknowns and 3 equations

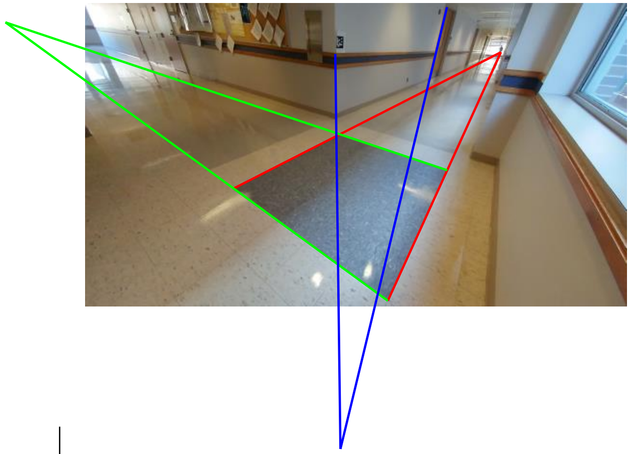
$$\rightarrow \quad \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

where $b_1 = \frac{1}{f^2}, \quad b_2 = -\frac{p_x}{f^2}, \quad b_3 = -\frac{p_y}{f^2}, \quad b_4 = \frac{p_x^{\,2}}{f^2} + \frac{p_y^2}{f^2} + 1$

$$\rightarrow \quad p_x = -\frac{b_2}{b_1}, \quad p_y = -\frac{b_3}{b_1}, \quad f = \sqrt{\frac{b_4}{b_1} - (p_x^2 + p_y^2)}$$

Camera Calibration

```
function CameraCalibration

m11 = [2145;2120;1];m12 = [2566;1191;1];
m13 = [1804;935;1];m14 = [1050;1320;1];

z11 = [1772; 364; 1];z12 = [1778; 823; 1];
z21 = [2564; 31; 1];z22 = [2439; 551; 1];

m21 = m11;m22 = m14;m23 = m12;m24 = m13;

l11 = GetLineFromTwoPoints(m11,m12);
l12 = GetLineFromTwoPoints(m13,m14);

l21 = GetLineFromTwoPoints(m21,m22);
l22 = GetLineFromTwoPoints(m23,m24);

l31 = GetLineFromTwoPoints(z11,z12);
l32 = GetLineFromTwoPoints(z21,z22);

x = GetPointFromTwoLines(l11,l12);
y = GetPointFromTwoLines(l21,l22);
z = GetPointFromTwoLines(l31,l32);
```



Vanishing points

CameraCalibration.m

```
A = [x(1)*y(1)+x(2)*y(2) x(1)+y(1) x(2)+y(2) 1;
    z(1)*y(1)+z(2)*y(2) z(1)+y(1) z(2)+y(2) 1;
    x(1)*z(1)+x(2)*z(2) x(1)+z(1) x(2)+z(2) 1];

[u d v] = svd(A);
x = v(:,end);

px = -x(2)/x(1);
py = -x(3)/x(1);
f = sqrt(x(4)/x(1)-px^2-py^2);

K = [f 0 px;
    0 f py;
    0 0 1]
```

Linear solve using SVD

```
K =

   1317.2        0   1931.8
        0   1317.2   1146.1
        0        0        1
```

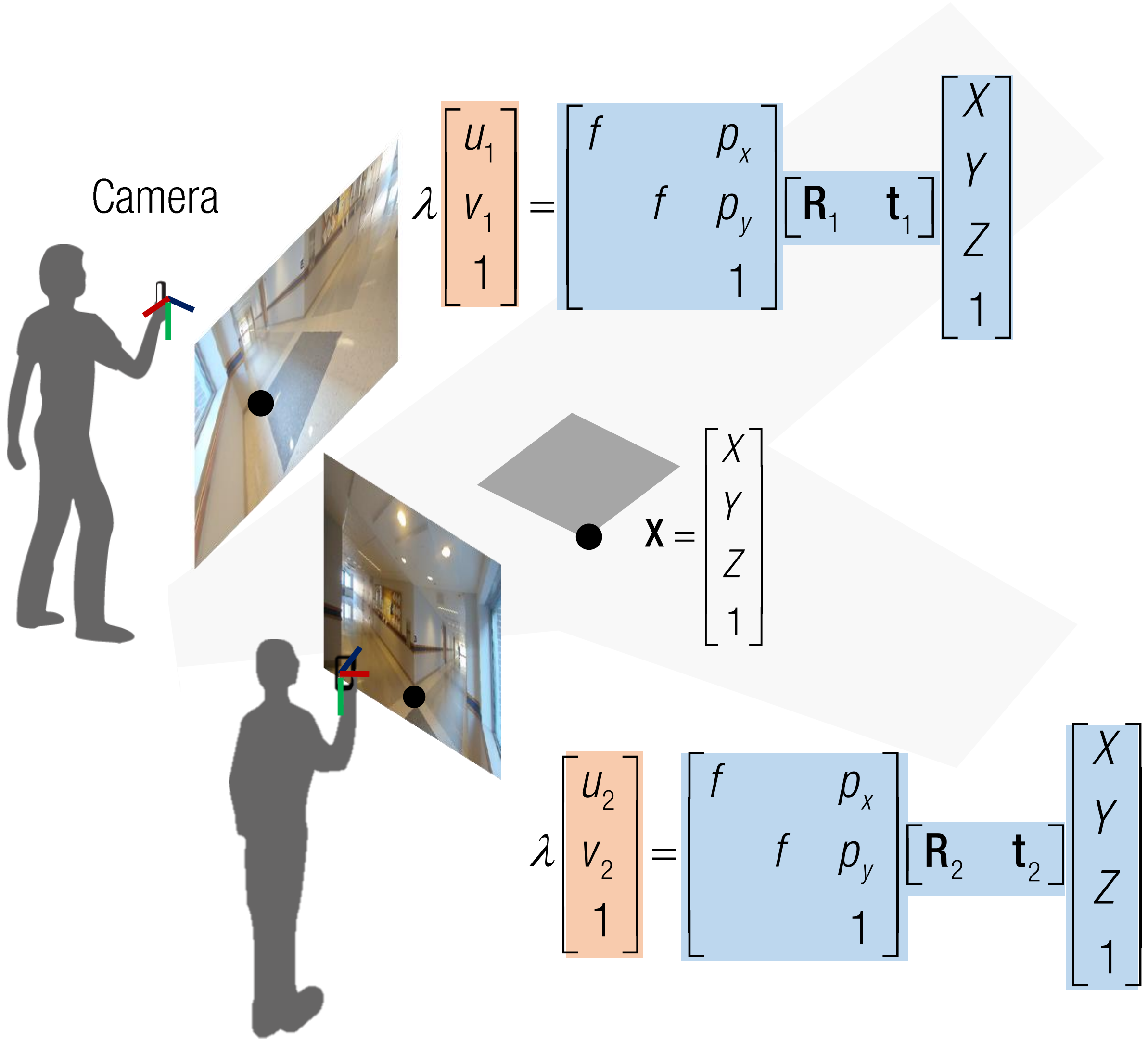
$f = 1500$

$p_x = \text{size(im,2)}/2 = 1920$

$p_y = \text{size(im,1)}/2 = 1080$

Previous manual estimate

Robust Camera Calibration



# of unknowns: 3 (**K**) + 6n (**R** and **t**) + 3 (X)

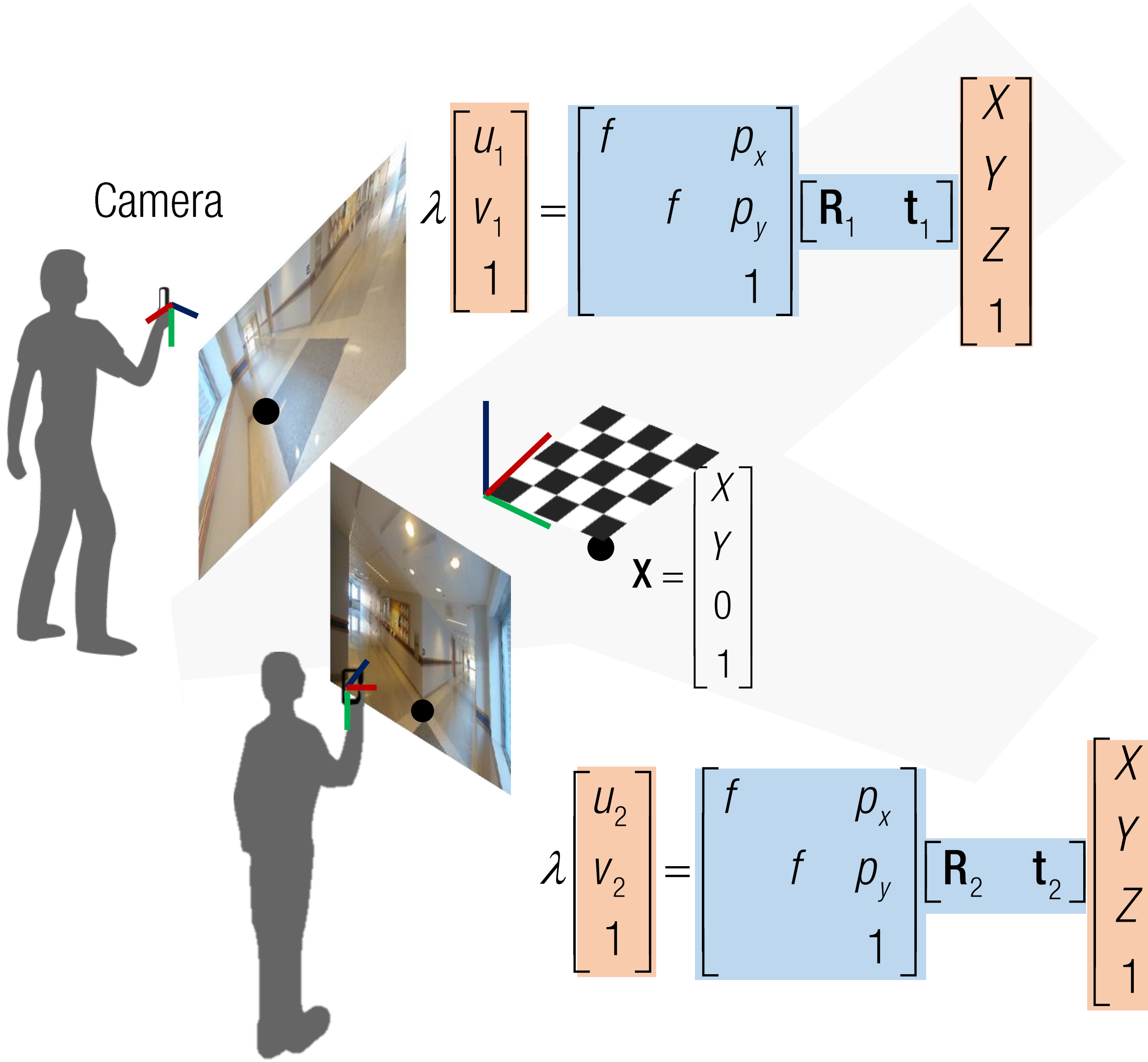
n: the number of images

What if we know 3D points common to all images?

: Knowns

: Unknowns

Insight: Known Common 3D Points in a Plane



$$\lambda \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 & \mathbf{t}_1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda \begin{bmatrix} u_2 \\ v_2 \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_2 & \mathbf{t}_2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

# of unknowns: 3 (**K**) + 6n (**R** and **t**)

n: the number of images

# of equations: 2nm (**X**)

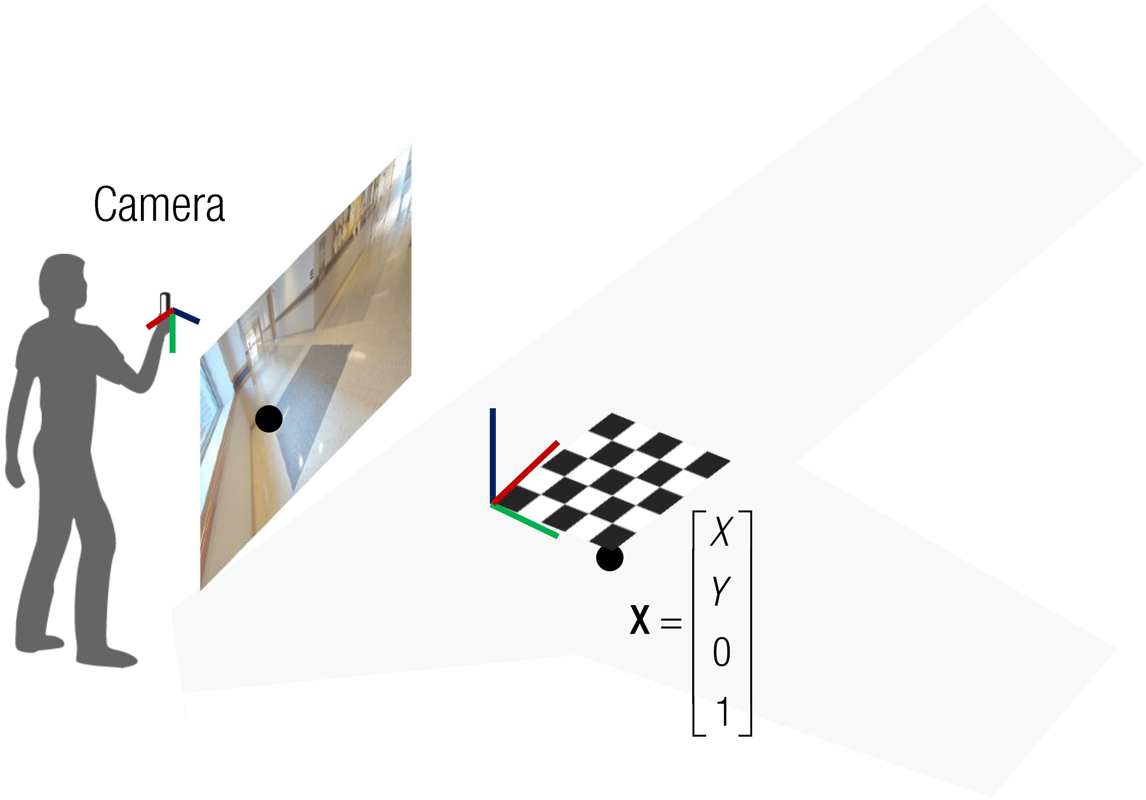
m: the number of known 3D points

We can solve for **K**, **R**, **t** if $3 + 6n < 2\,nm$

: Knowns

: Unknowns

Homography Mapping



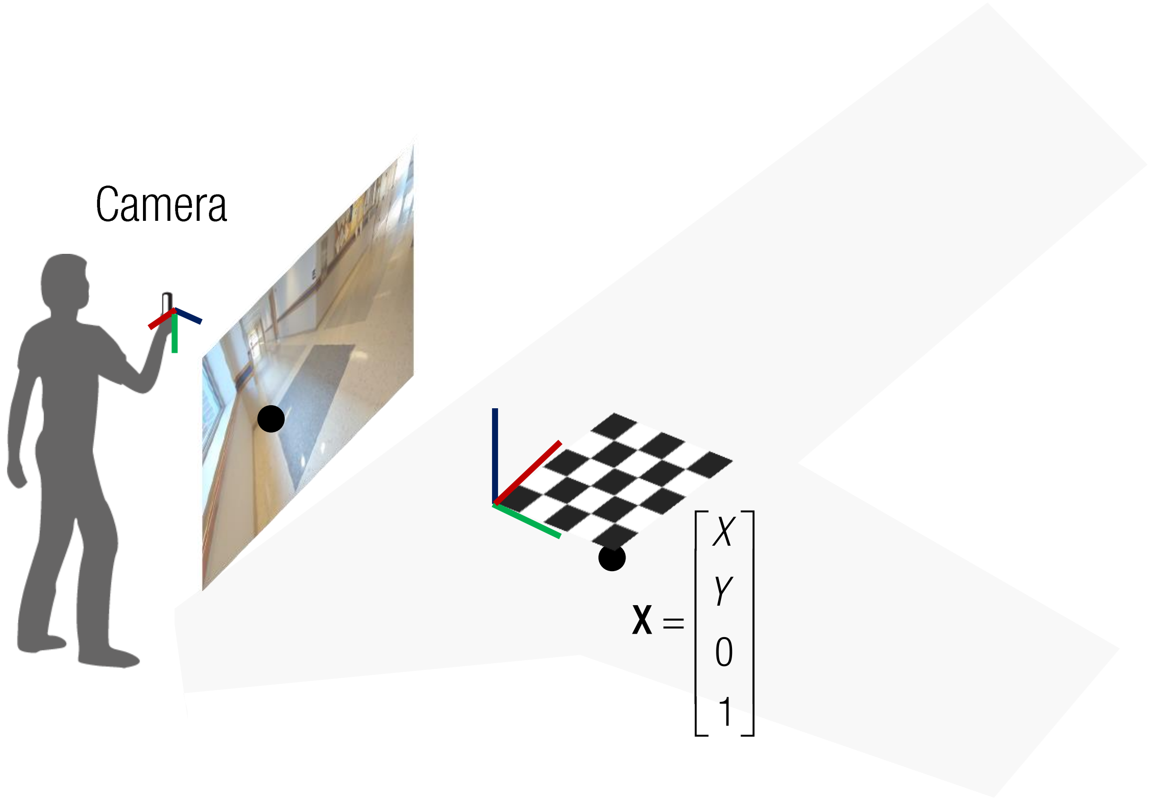
Points in 2D plane are mapped to an image with homography:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t} \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}$$

: Knowns

: Unknowns

Homography Mapping



Points in 2D plane are mapped to an image with homography:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t} \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}$$

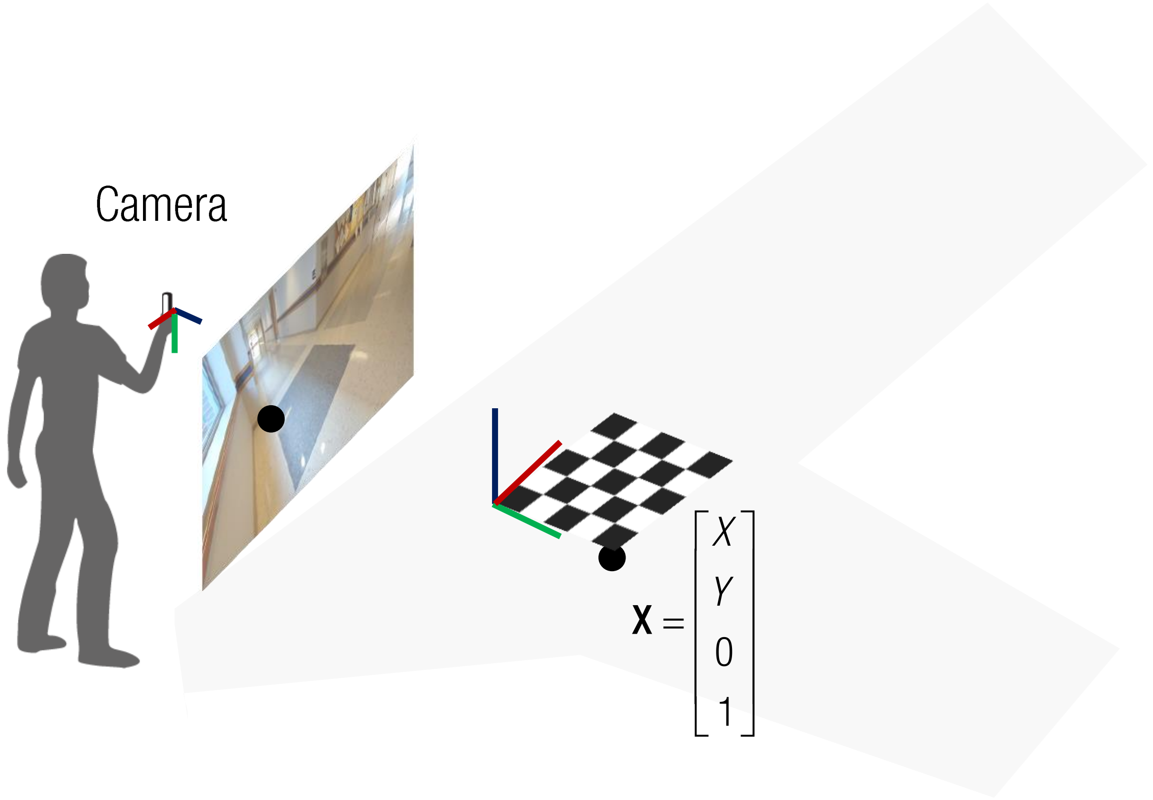
$$= \underbrace{\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t} \end{bmatrix}}_{\mathbf{Q}} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \underbrace{\mathbf{H}}_{3x3} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

: Knowns

: Unknowns

Homography can be directly computed by 2D-2D mapping and then, **H** can be factorized to **K Q**.

Homography Factorization



: Knowns

: Unknowns

Homography factorization:

$$\mathbf{H} = \mathbf{KQ} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t} \end{bmatrix}$$

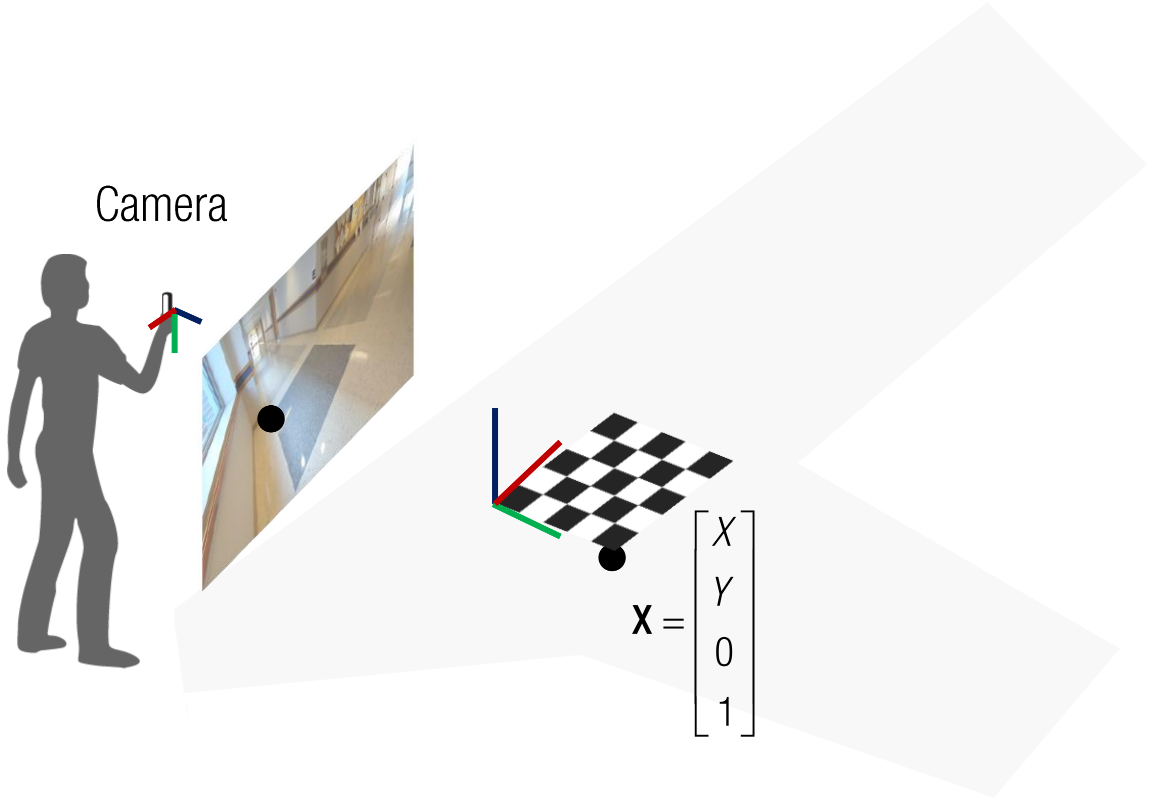
Orthogonality of rotation matrix property:

$$\mathbf{r}_1^{\mathsf{T}} \mathbf{r}_2 = 0 \qquad \|\mathbf{r}_1\| = 1 \qquad \|\mathbf{r}_2\| = 1$$

where $\mathbf{r}_1 = \mathbf{K}^{-1}\mathbf{h}_1 \qquad \mathbf{r}_2 = \mathbf{K}^{-1}\mathbf{h}_2 \qquad \mathbf{r}_3 = \mathbf{K}^{-1}\mathbf{h}_3$

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 \end{bmatrix}$$

Homography Factorization



Orthogonality of rotation matrix property:

$$\mathbf{r}_1^{\mathrm{T}}\mathbf{r}_2 = 0 \qquad \left\|\mathbf{r}_1\right\| = 1 \qquad \left\|\mathbf{r}_2\right\| = 1$$

where $\mathbf{r}_1 = \lambda\mathbf{K}^{-1}\mathbf{h}_1 \qquad \mathbf{r}_2 = \lambda\mathbf{K}^{-1}\mathbf{h}_2 \qquad \mathbf{r}_3 = \lambda\mathbf{K}^{-1}\mathbf{h}_3$

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 \end{bmatrix}$$

$$\rightarrow \left(\mathbf{K}^{-1}\mathbf{h}_1\right)^{\mathrm{T}}\left(\mathbf{K}^{-1}\mathbf{h}_2\right) = \mathbf{h}_1^{\mathrm{T}}\mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1}\mathbf{h}_2 = 0 \qquad (1)$$

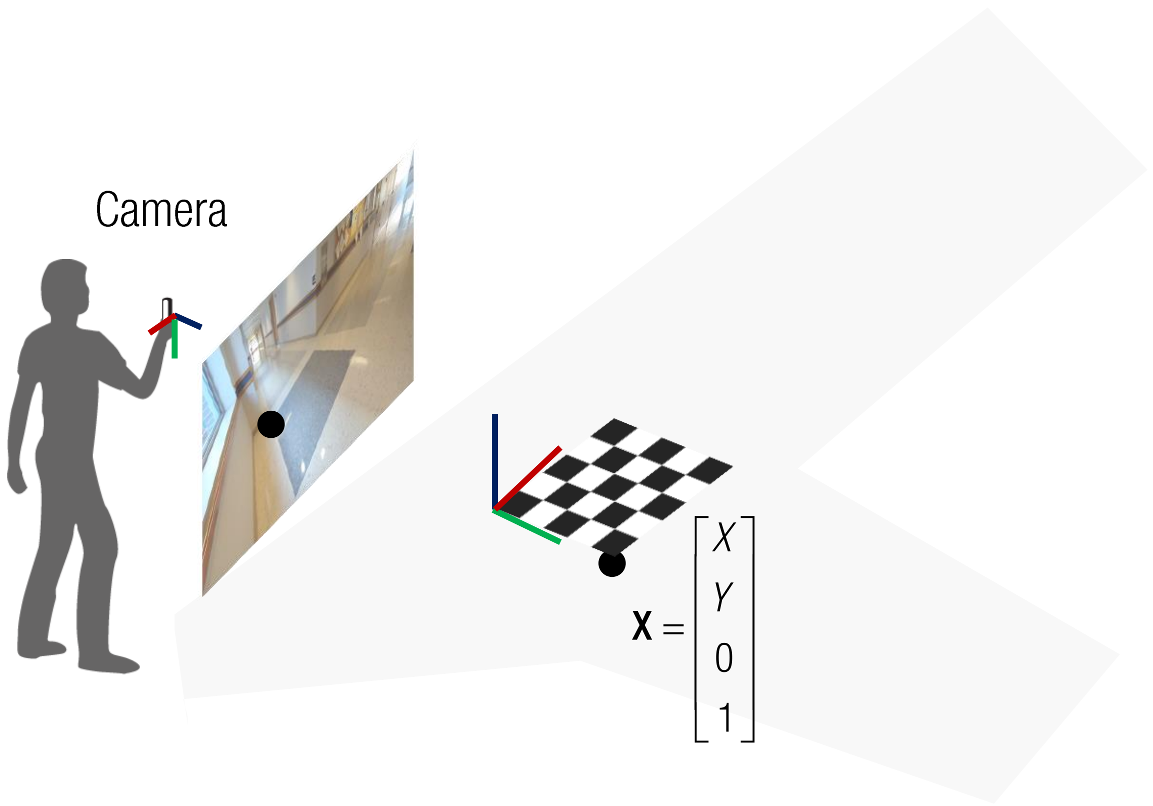
$$\left\|\mathbf{K}^{-1}\mathbf{h}_1\right\| = \left\|\mathbf{K}^{-1}\mathbf{h}_2\right\| \quad \text{or,} \quad \mathbf{h}_1^{\mathrm{T}}\mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1}\mathbf{h}_1 = \mathbf{h}_2^{\mathrm{T}}\mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1}\mathbf{h}_2 \qquad (2)$$

$$\mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1} = \begin{bmatrix} 1/f & & \\ & 1/f & \\ -p_x/f & -p_y/f & 1 \end{bmatrix}\begin{bmatrix} 1/f & & -p_x/f \\ & 1/f & -p_y/f \\ & & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix}}_{\mathbf{B}}$$

where $b_1 = \frac{1}{f^2}, \quad b_2 = -\frac{p_x}{f^2}, \quad b_3 = -\frac{p_y}{f^2}, \quad b_4 = \frac{p_x^2}{f^2} + \frac{p_y^2}{f^2} + 1$

B is again linear in Eq (1) and (2).

Linear Solve



$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{h}_1^{\mathrm{T}}\mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1}\mathbf{h}_2 = \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} h_{12} \\ h_{22} \\ h_{32} \end{bmatrix} = 0$$

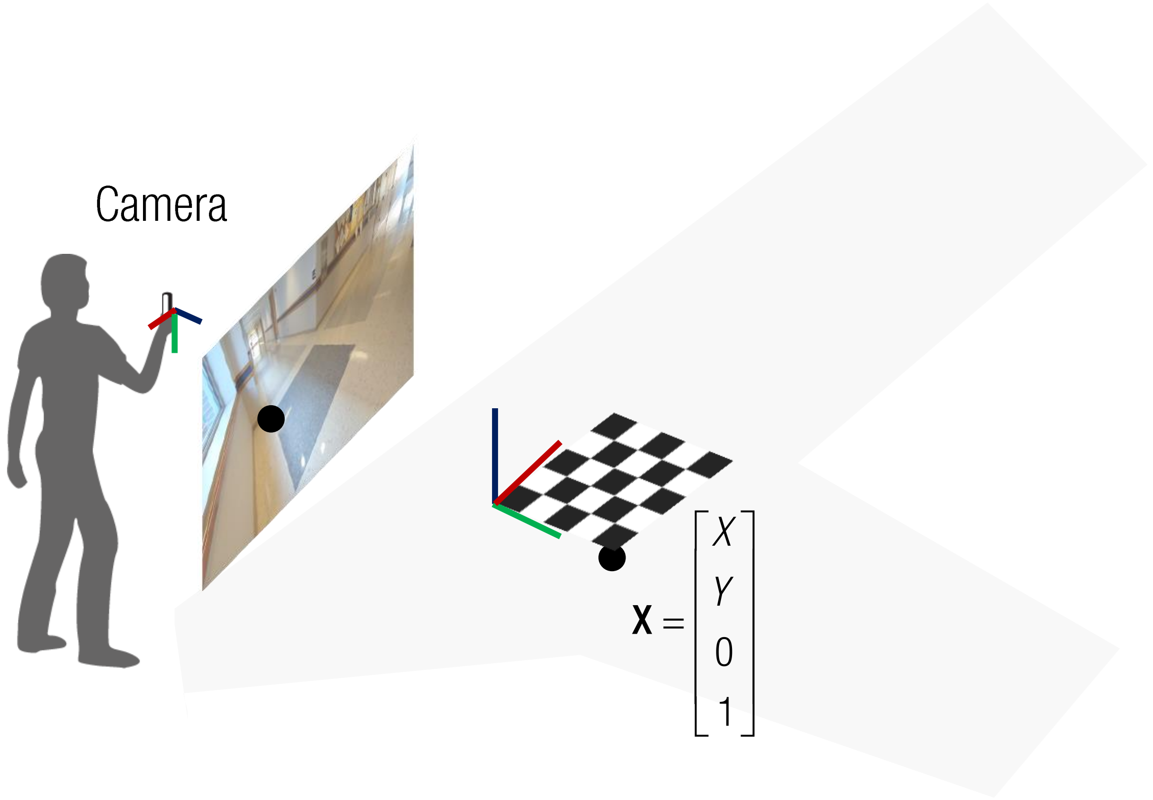
$$\mathbf{h}_1^{\mathrm{T}}\mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1}\mathbf{h}_1 = \mathbf{h}_2^{\mathrm{T}}\mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1}\mathbf{h}_2$$

$$\begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix} = \begin{bmatrix} h_{12} \\ h_{22} \\ h_{32} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} h_{12} \\ h_{22} \\ h_{32} \end{bmatrix}$$

$$\begin{bmatrix} h_{11}h_{12} + h_{21}h_{22} & h_{11} + h_{12} & h_{21} + h_{22} & 1 \\ h_{11}^2 - h_{12}^2 + h_{21}^2 - h_{22}^2 & 2(h_{11} - h_{12}) & 2(h_{21} - h_{22}) & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{0}$$

2x4

Linear Solve



$$\mathbf{h}_1^{\mathsf{T}}\mathbf{K}^{-\mathsf{T}}\mathbf{K}^{-1}\mathbf{h}_2 = \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} h_{12} \\ h_{22} \\ h_{32} \end{bmatrix} = 0$$

$$\mathbf{h}_1^{\mathsf{T}}\mathbf{K}^{-\mathsf{T}}\mathbf{K}^{-1}\mathbf{h}_1 = \mathbf{h}_2^{\mathsf{T}}\mathbf{K}^{-\mathsf{T}}\mathbf{K}^{-1}\mathbf{h}_2$$

$$\begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix} = \begin{bmatrix} h_{12} \\ h_{22} \\ h_{32} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} h_{12} \\ h_{22} \\ h_{32} \end{bmatrix}$$

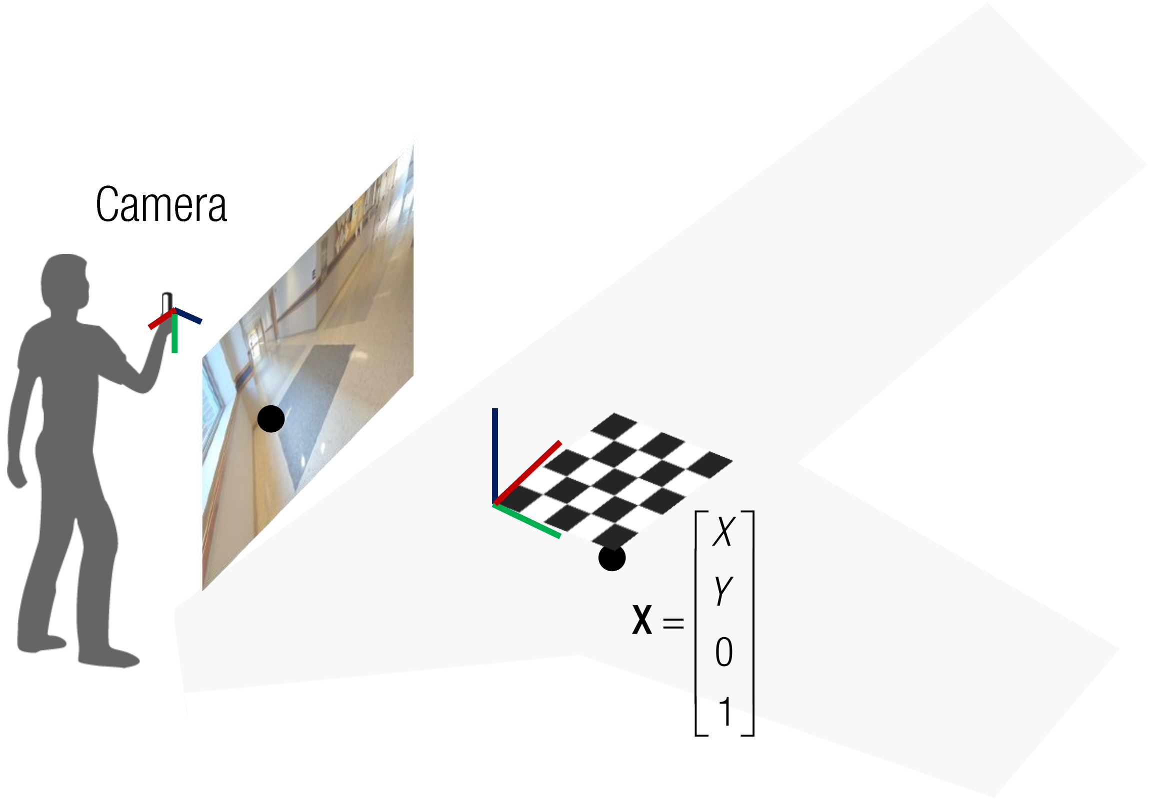
$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} h_{11}h_{12} + h_{21}h_{22} & h_{11} + h_{12} & h_{21} + h_{22} & 1 \\ h_{11}^2 - h_{12}^2 + h_{21}^2 - h_{22}^2 & 2(h_{11} - h_{12}) & 2(h_{21} - h_{22}) & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{\mathbf{x}} = \mathbf{0}$$

2x4

Each image produces 2 equations and therefore, **x** can be computed with minimum 2 images.

Linear Solve



$$\mathbf{h}_1^{\mathsf{T}}\mathbf{K}^{-\mathsf{T}}\mathbf{K}^{-1}\mathbf{h}_2 = \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} h_{12} \\ h_{22} \\ h_{32} \end{bmatrix} = 0$$

$$\mathbf{h}_1^{\mathsf{T}}\mathbf{K}^{-\mathsf{T}}\mathbf{K}^{-1}\mathbf{h}_1 = \mathbf{h}_2^{\mathsf{T}}\mathbf{K}^{-\mathsf{T}}\mathbf{K}^{-1}\mathbf{h}_2$$

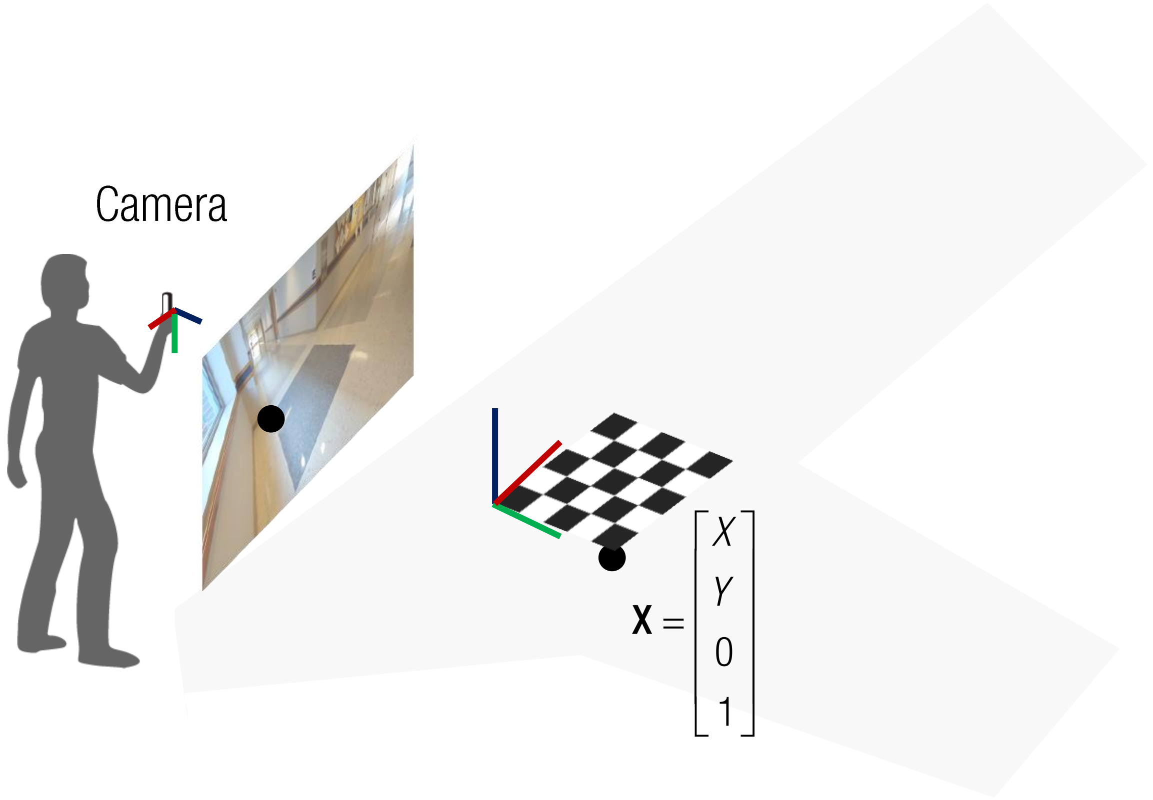
$$\begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix} = \begin{bmatrix} h_{12} \\ h_{22} \\ h_{32} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} b_1 & & b_2 \\ & b_1 & b_3 \\ b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} h_{12} \\ h_{22} \\ h_{32} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} h_{11}h_{12} + h_{21}h_{22} & h_{11} + h_{12} & h_{21} + h_{22} & 1 \\ h_{11}^2 - h_{12}^2 + h_{21}^2 - h_{22}^2 & 2(h_{11} - h_{12}) & 2(h_{21} - h_{22}) & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{\mathbf{x}} = \mathbf{0}$$

2x4

$$p_x = -\frac{b_2}{b_1}, \quad p_y = -\frac{b_3}{b_1}, \quad f = \sqrt{\frac{b_4}{b_1} - (p_x^2 + p_y^2)}$$

Extrinsic Parameter



Homography factorization:

: Knowns

: Unknowns

$$\mathbf{H} = \mathbf{KQ} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t} \end{bmatrix}$$

$$\mathbf{r}_1 = \frac{\mathbf{K}^{-1}\mathbf{h}_1}{\left\|\mathbf{K}^{-1}\mathbf{h}_1\right\|}, \quad \mathbf{r}_2 = \frac{\mathbf{K}^{-1}\mathbf{h}_2}{\left\|\mathbf{K}^{-1}\mathbf{h}_1\right\|}, \quad \mathbf{t} = \frac{\mathbf{K}^{-1}\mathbf{h}_3}{\left\|\mathbf{K}^{-1}\mathbf{h}_1\right\|}, \quad \mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$$

Divided by constant factor

MATLAB Calibration Toolbox Demo