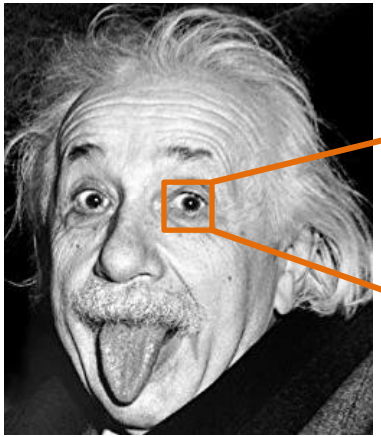
Filtering/Convolution/Gradient

IMAGE SPATIAL FILTERING ~ CORRELATION

$$I_f(i,j) = \sum_{k,l} I(i+k, j+l) z(k,l)$$



2	1	4	4	7
1	2	2	3	6
3	3	5	8	9
5	2	2	6	7
8	3	2	1	3

I

Image

$\otimes$

a	b	c
d	e	f
g	h	i

z

Filter

$=$

	y			

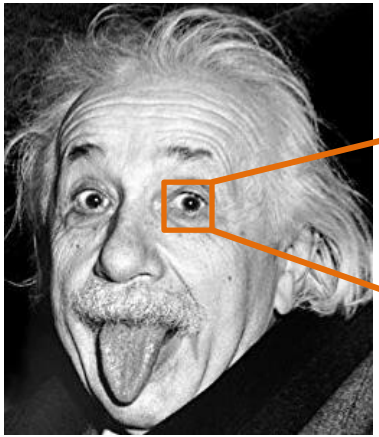
I_f

Filtered image

$$y = 2a + b + 4c + d + 2e + 2f + 3g + 3h + 5i$$

BOUNDARY

$$I_f(i,j) = \sum_{k,l} I(i+k, j+l) z(k,l)$$



0	0	0	0	0	0	0
0	2	1	4	4	7	0
0	1	2	2	3	6	0
0	3	3	5	8	9	0
0	5	2	2	6	7	0
0	8	3	2	1	3	0
0	0	0	0	0	0	0

I

Image

Zero padding

$\otimes$

a	b	c
d	e	f
g	h	i

z

Filter

$=$

y				

I_f

Filtered image

$$y = 2e + f + h + 2i$$

Correlation vs. Convolution

Image correlation:

$$I \otimes z = I_f$$

$$\sum_{k,l} I(i+k, j+l) z(k,l) = I_f(i,j)$$

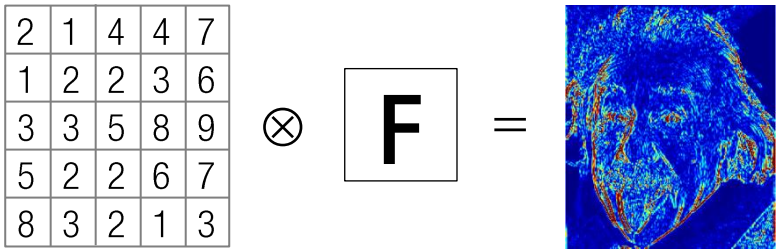
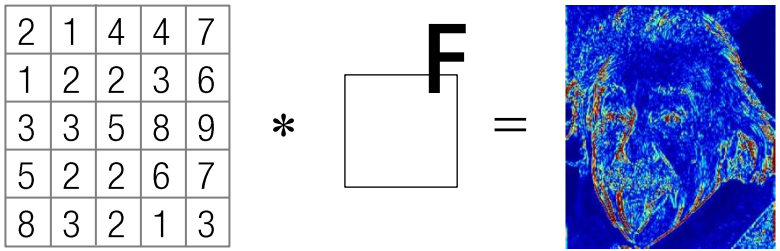


Image convolution:

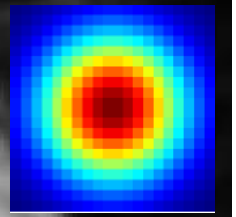
$$I * z = J$$

$$\sum_{k,l} I(i-k, j-l) z(k,l) = J(i,j)$$



Flip the filter in both dimension (bottom to top, right to left)

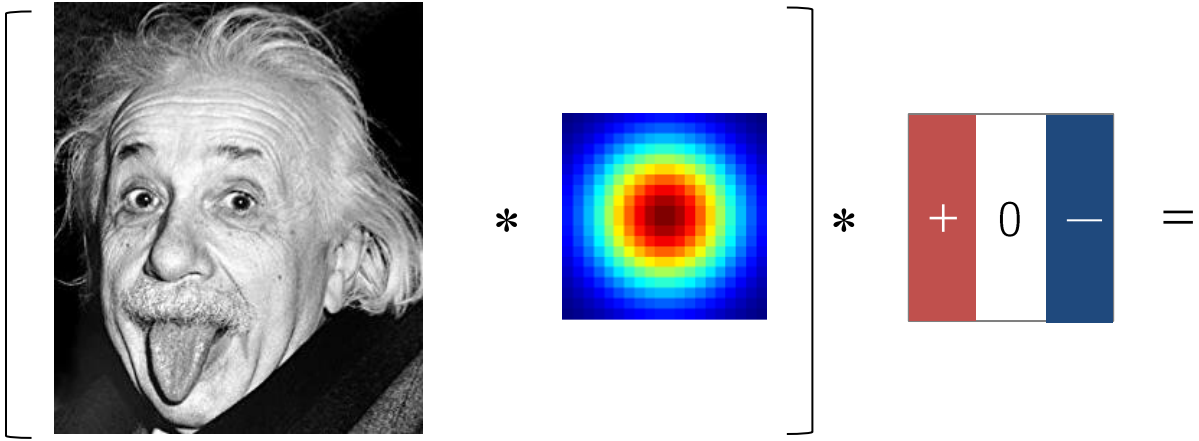
Recall: Gaussian Blurring~Denoising



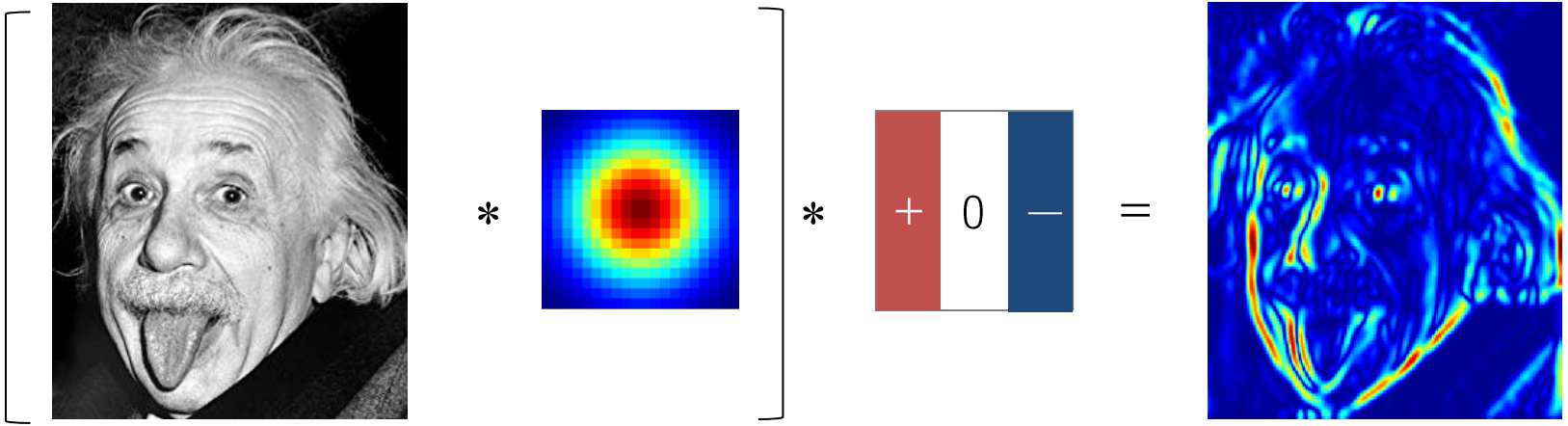
Noisy image

Denoised image

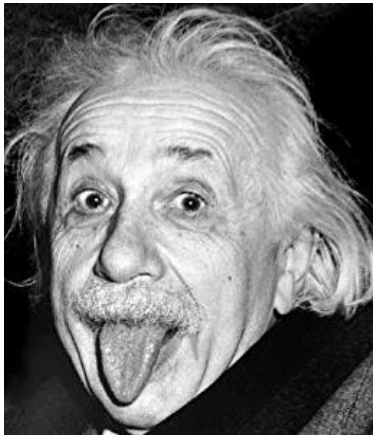
Strategy: Denoise and Differentiate



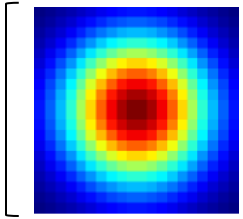
Strategy: Denoise and Differentiate



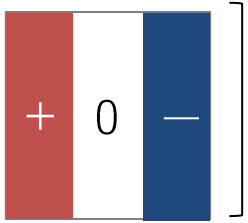
ASSOCIATIVITY



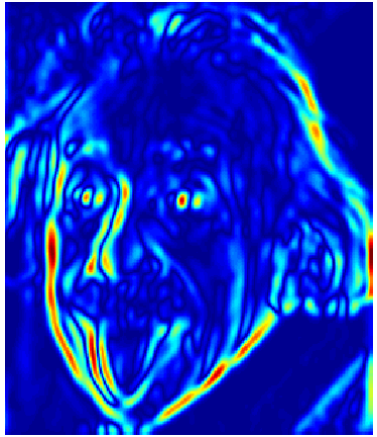
$*$



$*$

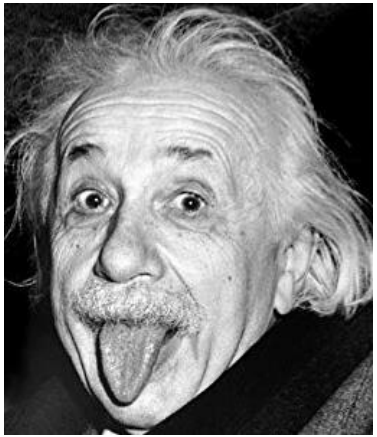


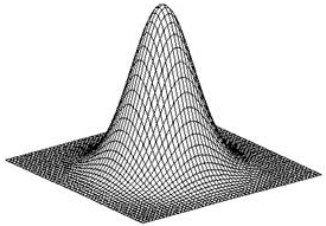
$=$



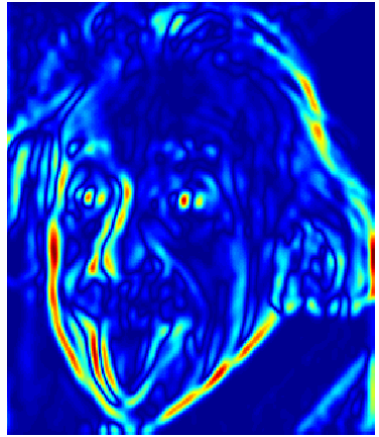
$$\frac{1}{2\pi\sigma^2} e^{-\frac{k^2+l^2}{2\sigma^2}} \qquad \frac{\partial}{\partial u}$$

COMMUTATIVITY



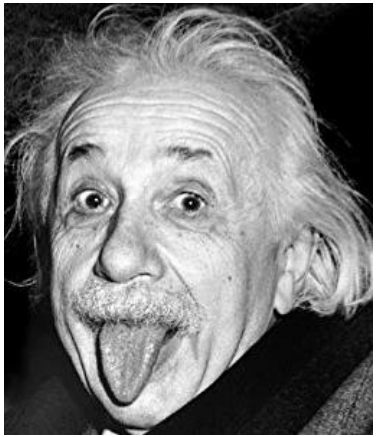


$$* \left[\begin{array}{|c|c|c|} \hline + & 0 & - \\ \hline \end{array} \right] * \left[\begin{array}{|c|} \hline \\ \hline \end{array} \right] =$$



$$\frac{\partial}{\partial u} \qquad \frac{1}{2\pi\sigma^2} e^{-\frac{k^2+l^2}{2\sigma^2}}$$

SOBEL FILTER



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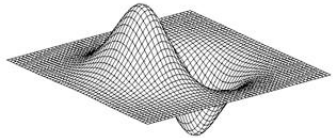
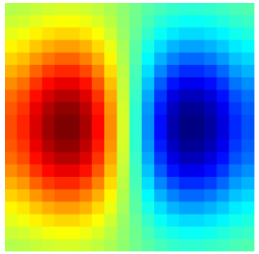
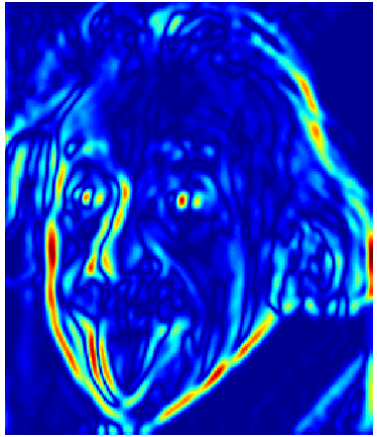




$$\frac{\partial}{\partial u}\frac{1}{2\pi\sigma^2}e^{-\frac{k^2+l^2}{2\sigma^2}}$$

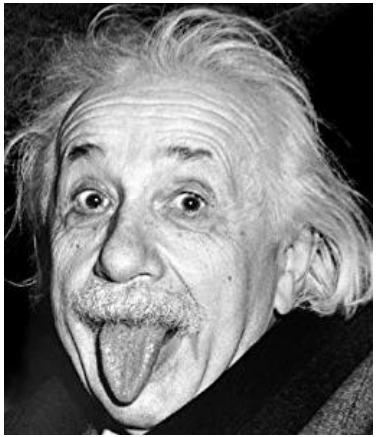
=



Sobel filter: derivative of Gaussian filter, e.g.,

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

SOBEL FILTER



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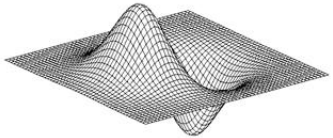
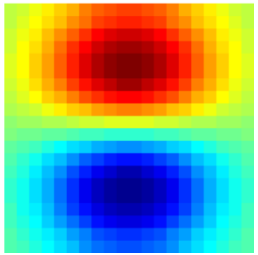
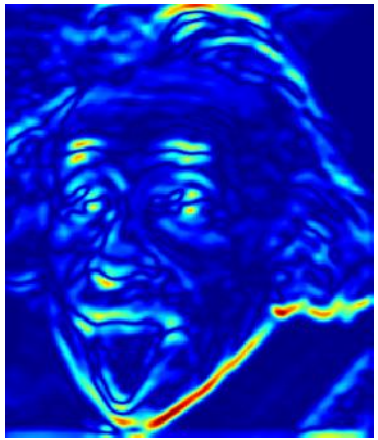




$$\frac{\partial}{\partial v}\frac{1}{2\pi\sigma^2}e^{-\frac{k^2+l^2}{2\sigma^2}}$$

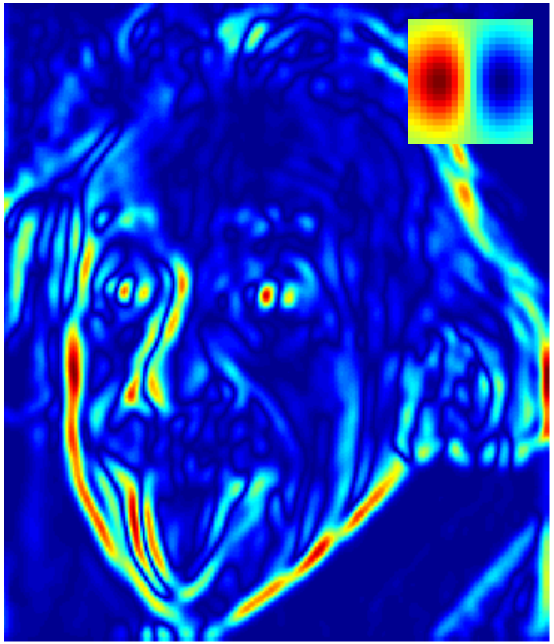
=

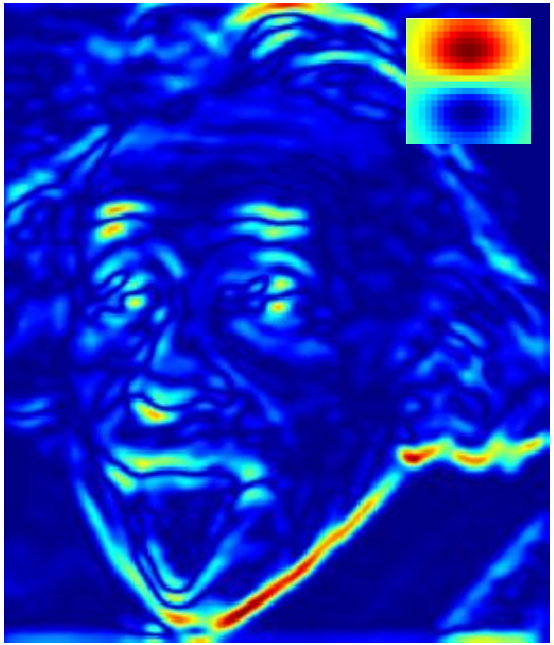


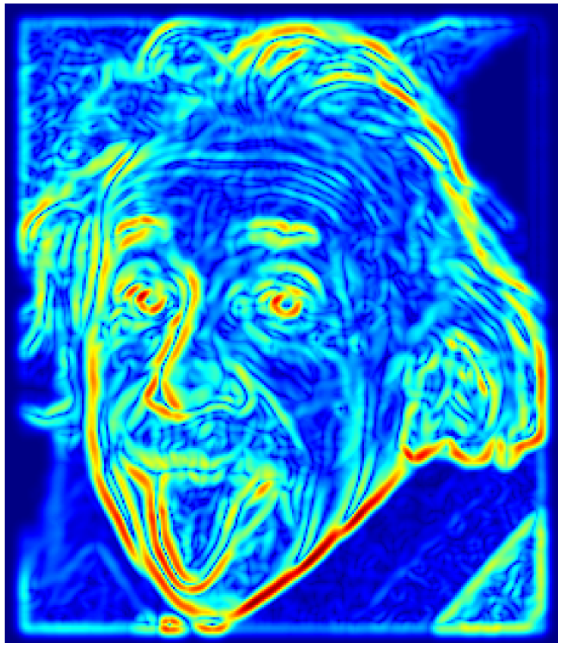
Sobel filter: derivative of Gaussian filter, e.g.,

$$\begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Image Gradient Magnitude





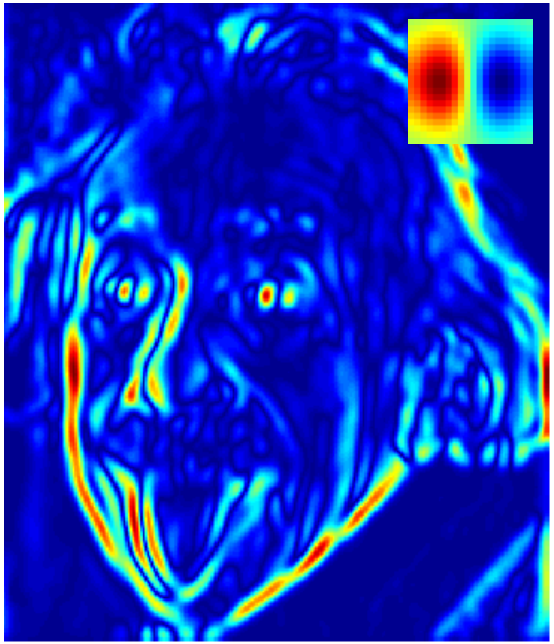


$$\frac{\partial I}{\partial u}$$

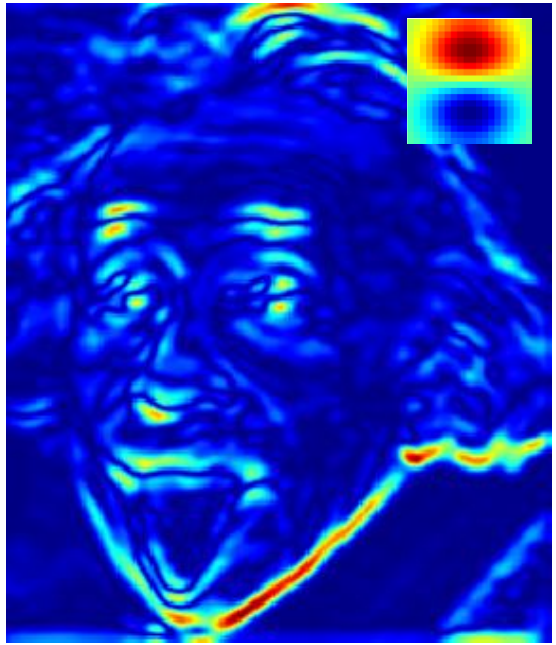
$$\frac{\partial I}{\partial v}$$

$$\|\nabla I\| = \sqrt{\left(\frac{\partial I}{\partial u}\right)^2 + \left(\frac{\partial I}{\partial v}\right)^2}$$

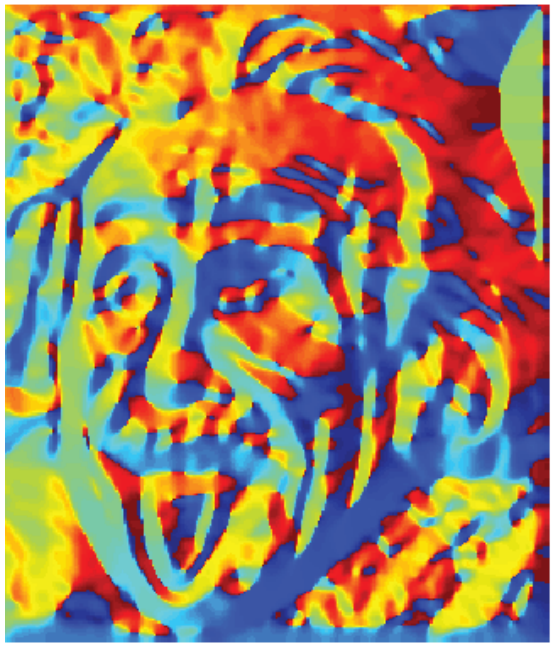
IMAGE GRADIENT DIRECTION



$$\frac{\partial I}{\partial u}$$

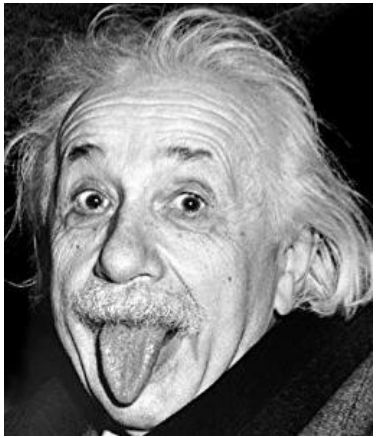


$$\frac{\partial I}{\partial v}$$



$$\measuredangle \nabla I = \tan^{-1}\left(\frac{\partial I}{\partial v} / \frac{\partial I}{\partial u}\right)$$

LAPLACIAN



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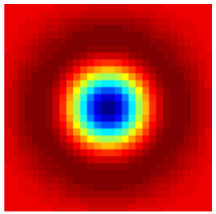

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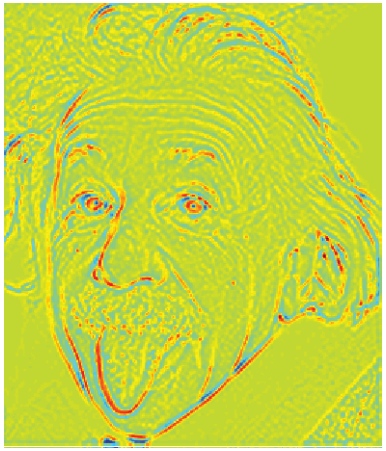
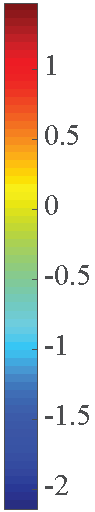
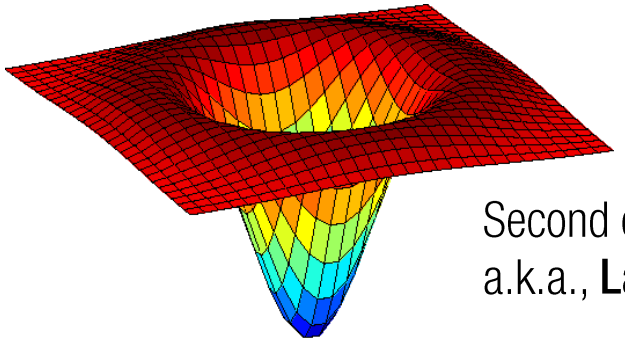


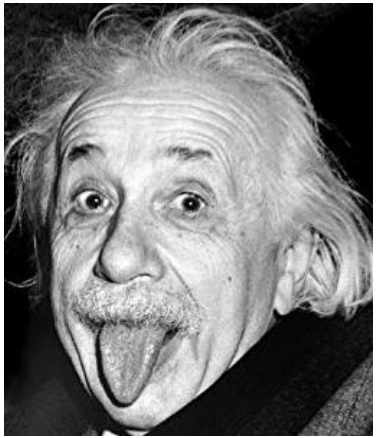


$$\nabla \cdot \nabla G = \nabla \left(\frac{\partial G}{\partial u}\mathbf{i} + \frac{\partial G}{\partial v}\mathbf{j} \right) = \frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2}$$



Second order derivative of Gaussian,
a.k.a., **Laplacian of Gaussian**

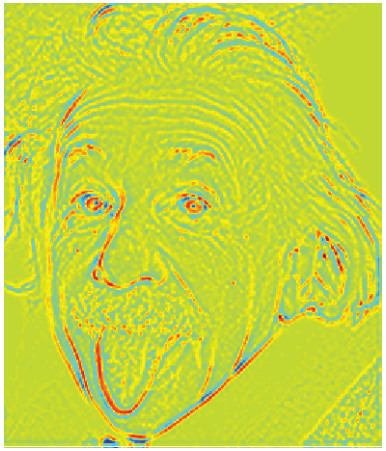
LAPLACIAN

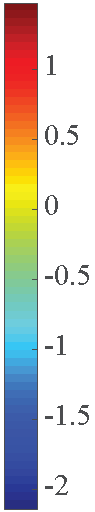


*

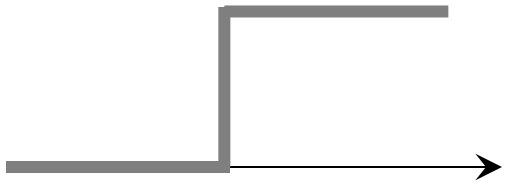
0	1	0
1	-4	1
0	1	0

=



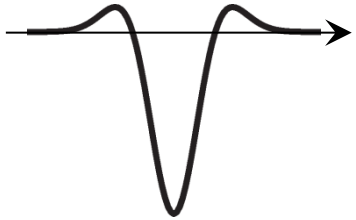


$$\nabla \cdot \nabla G = \nabla\left(\frac{\partial G}{\partial u}\mathbf{i} + \frac{\partial G}{\partial v}\mathbf{j}\right) = \frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2}$$



Edge

*



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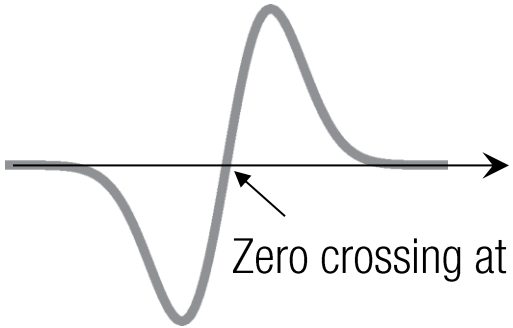
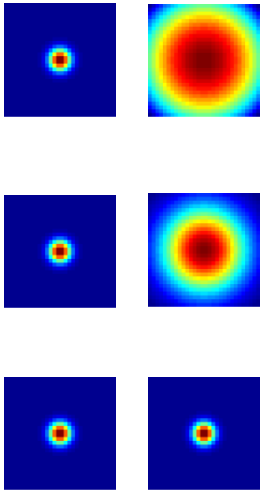


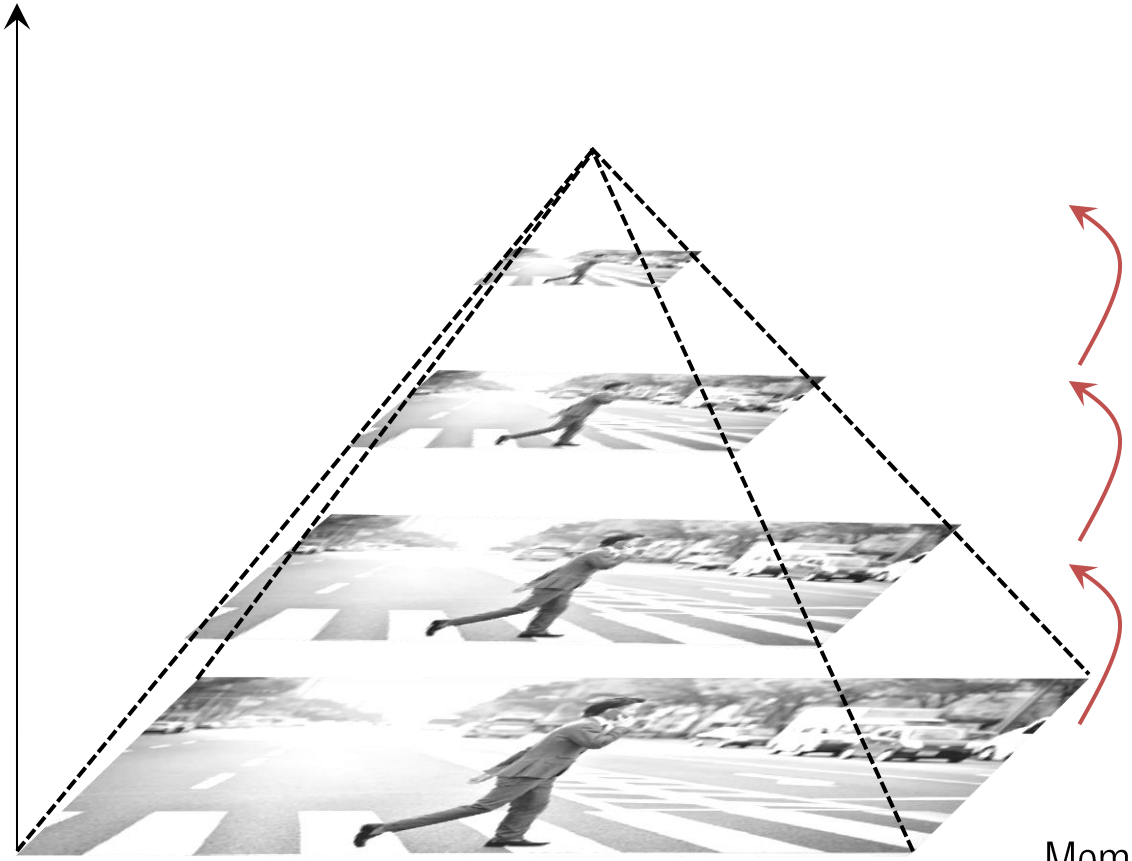
Zero crossing at edge

Pyramid

Image Reconstruction: Upsampling and Gaussian Blurring

CF) NAÏVE IMAGE SUBSAMPLING AND UPSAMPLING





½ subsampling

½ subsampling

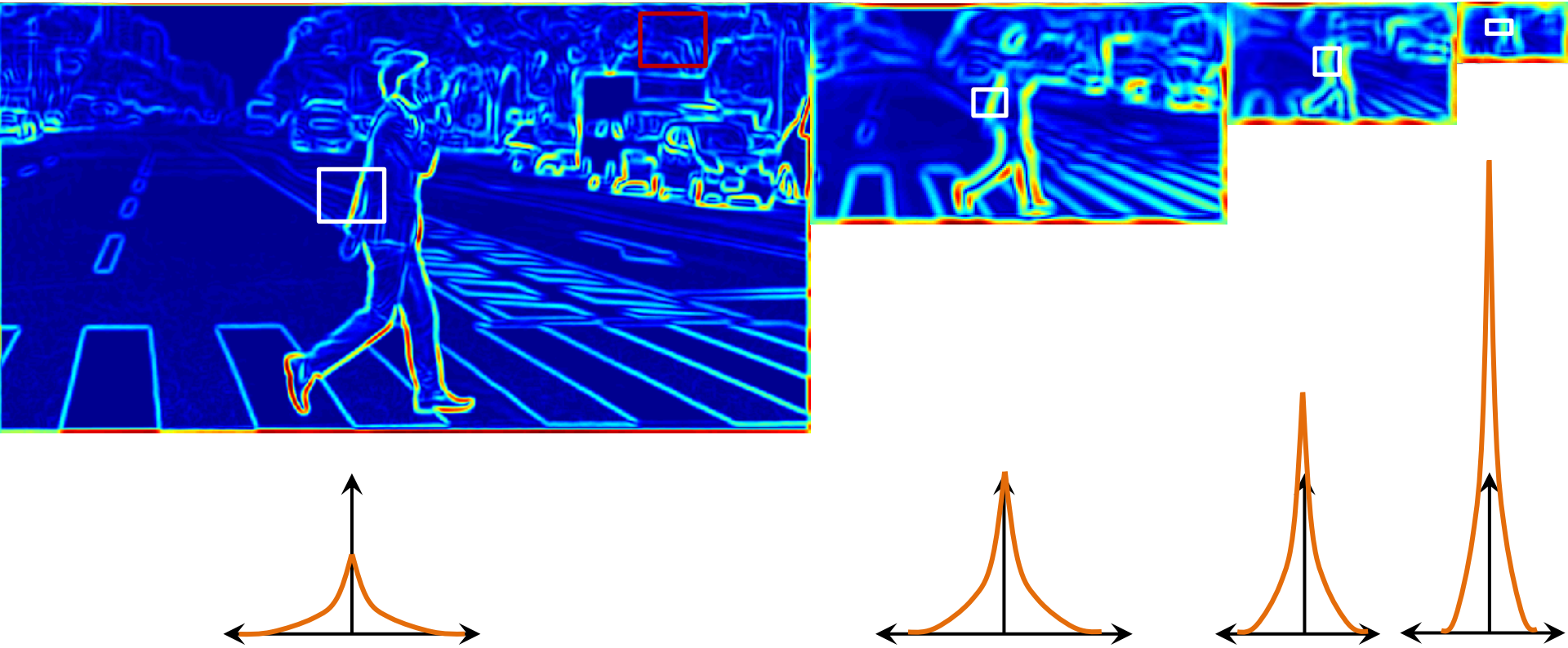
½ subsampling

Memory consumption

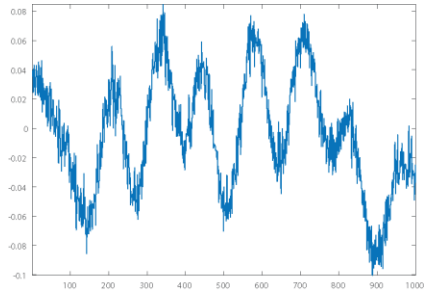
$$|I|(1+\frac{1}{4}+\frac{1}{16}+\cdots)=\frac{4}{3}|I|$$

GAUSSIAN IMAGE PYRAMID

Redundant Representation of Gaussian Pyramid



LAPLACIAN OF GAUSSIAN (LOG) ~ DOG



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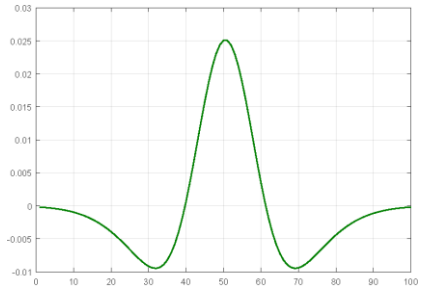
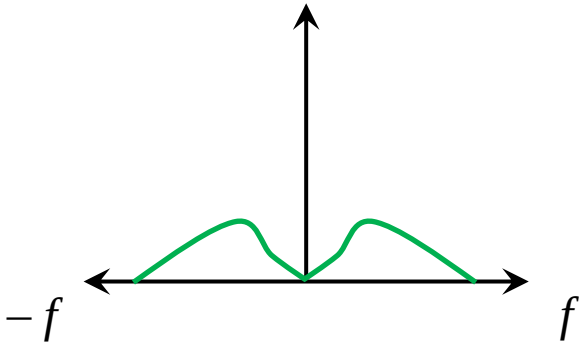


FT

Inverse FT



$x(t)$ * $g(t;\sigma_1) - g(t;\sigma_2)$

$\approx \nabla \cdot \nabla g$

Laplacian of Gaussian

$X(f)\left(G(f;\sigma_1) - G(f;\sigma_2)\right)$

Laplacian of Gaussian (LoG) ~ DoG

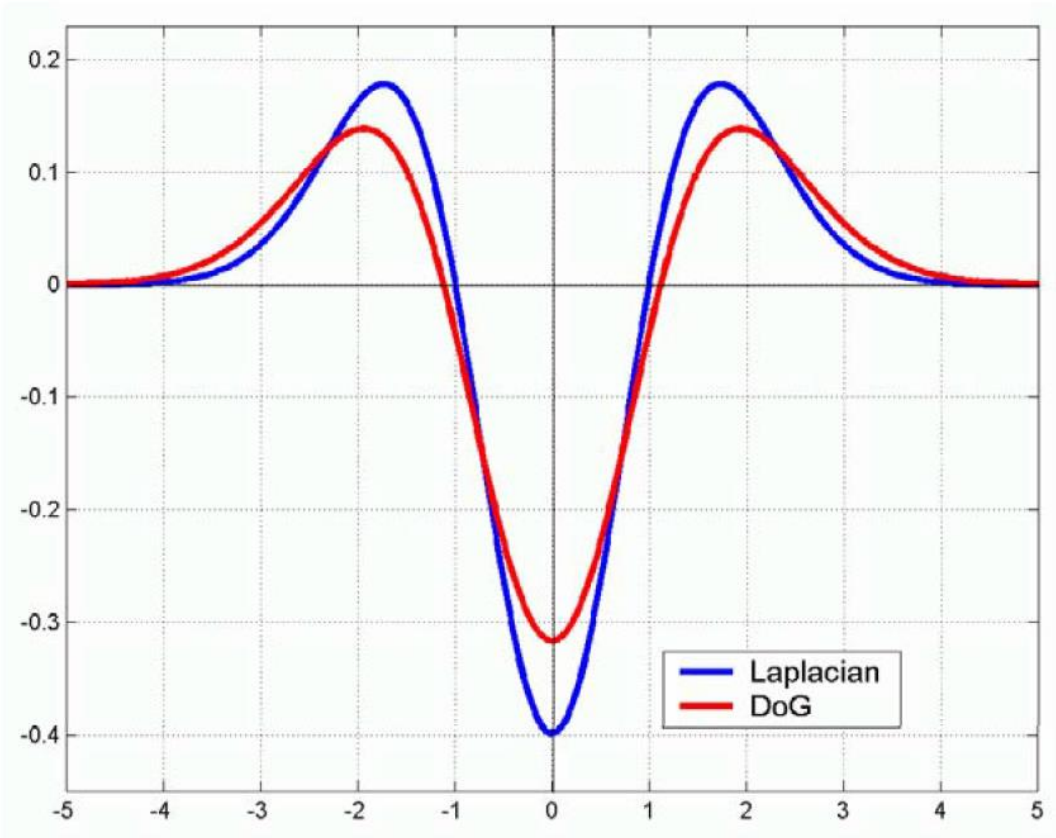
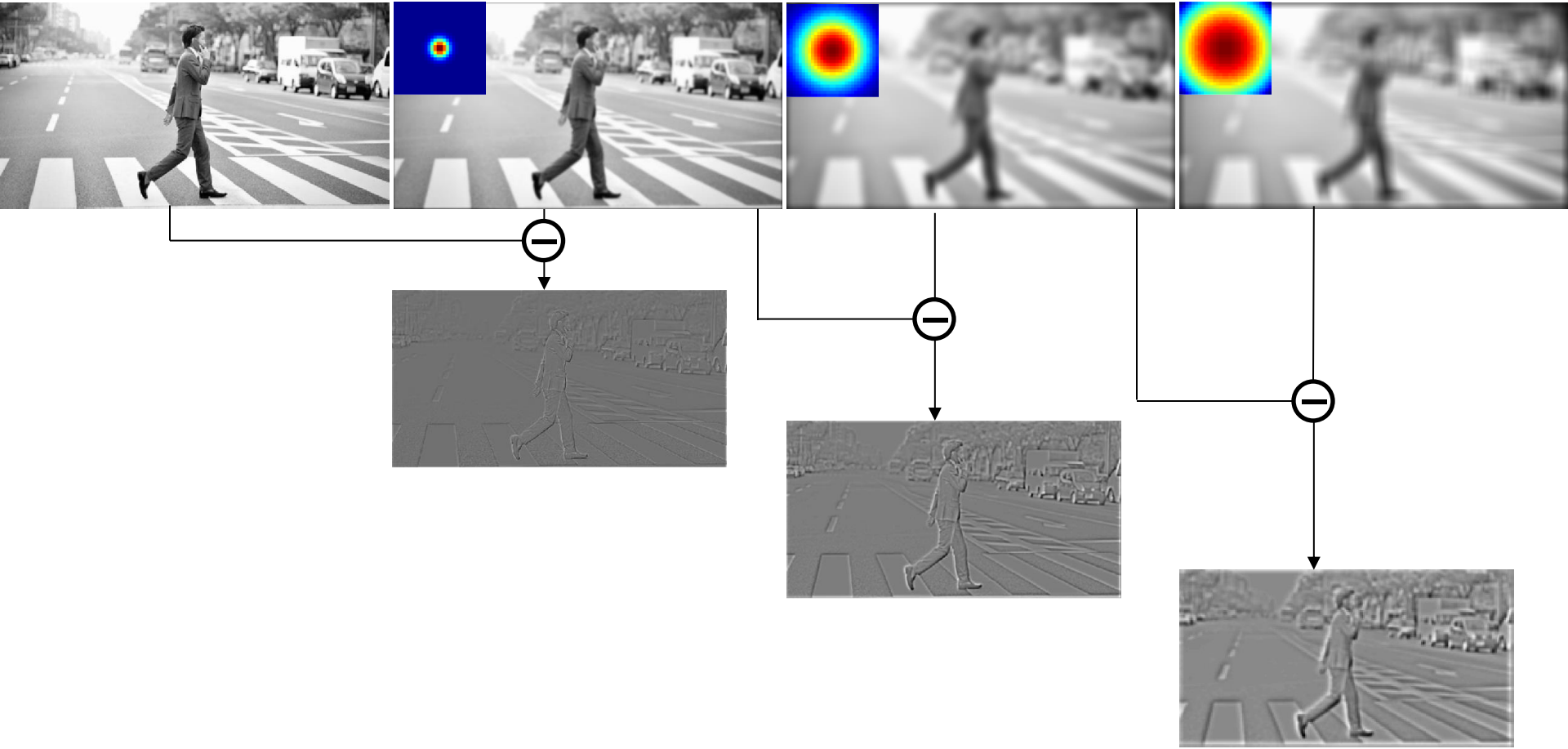
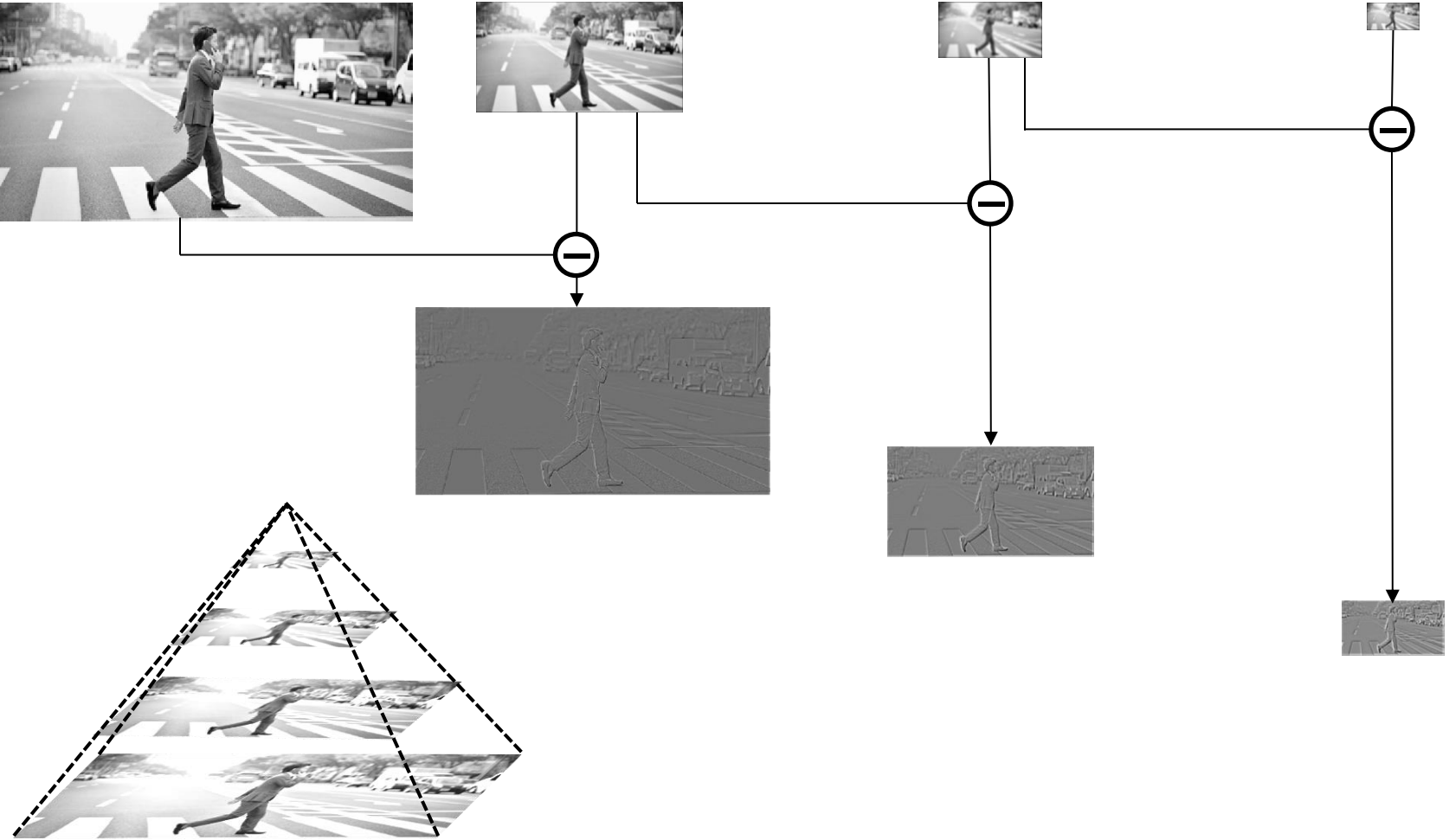
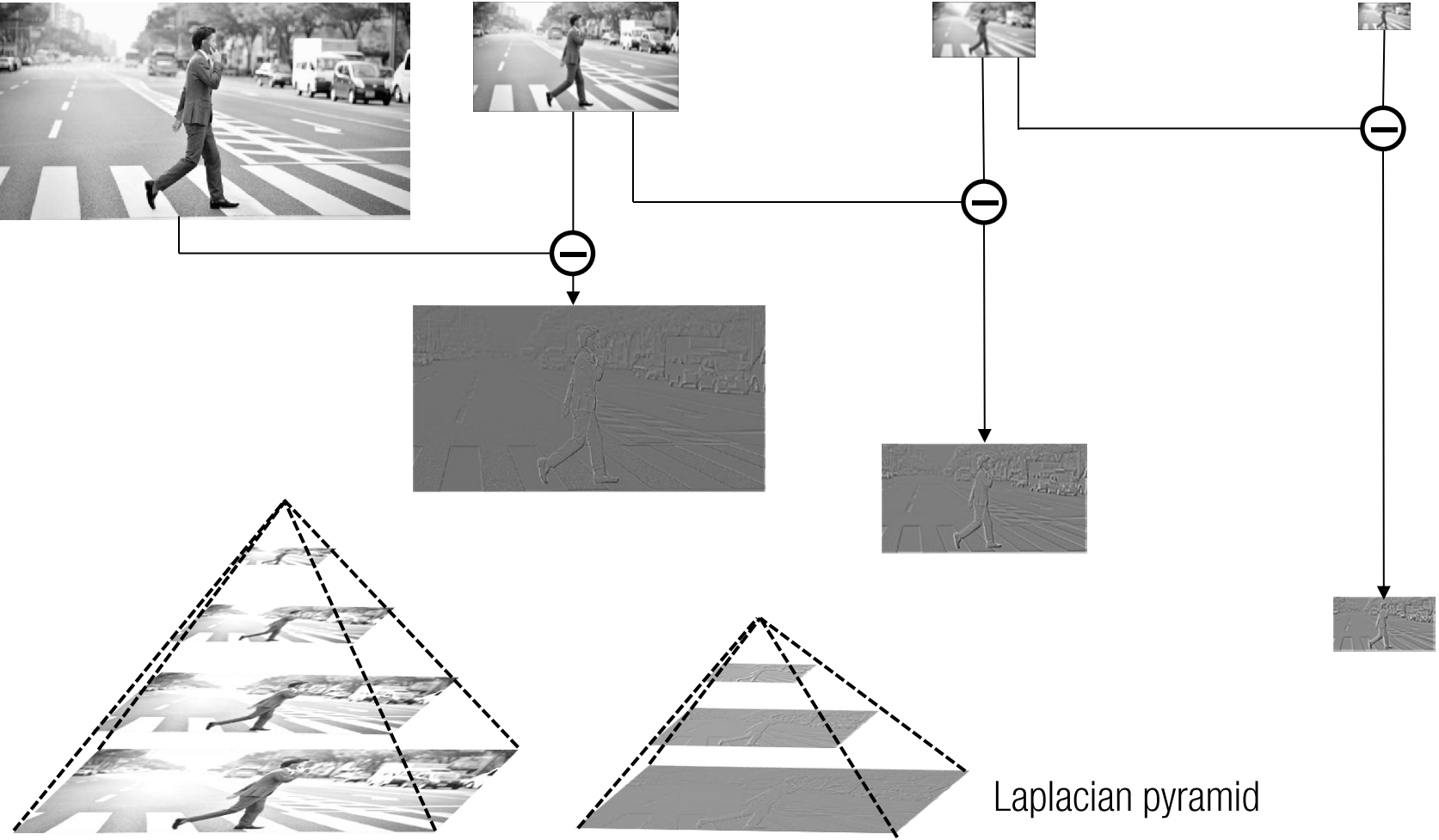


IMAGE LAPLACIAN

IMAGE LAPLACIAN





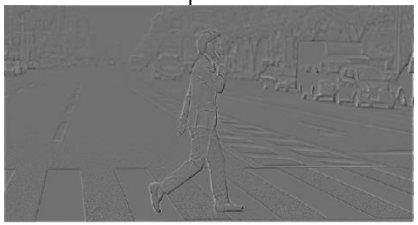


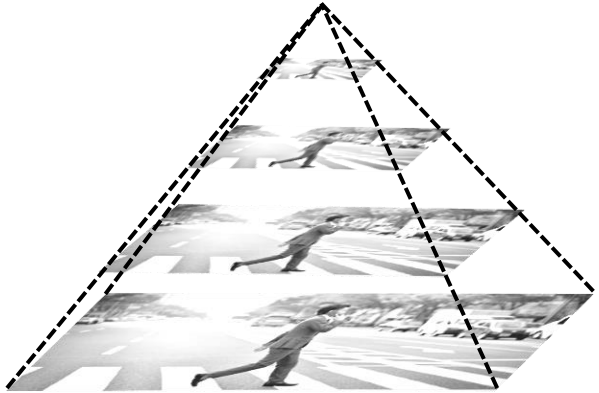
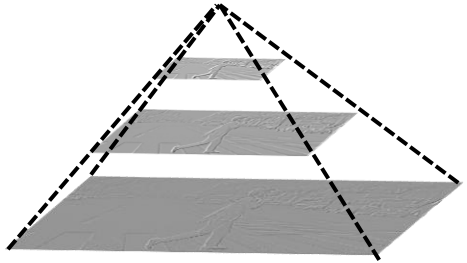
Laplacian pyramid









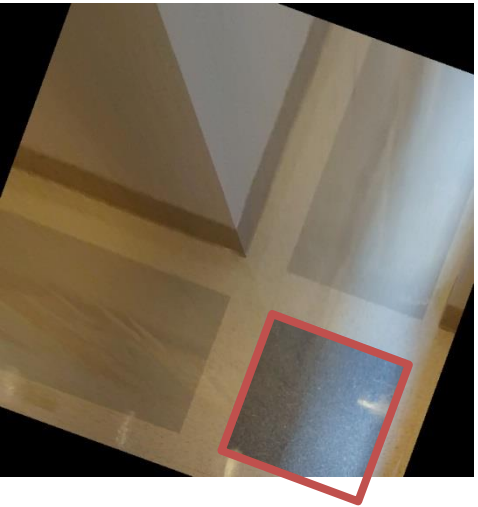




Upsample and blur

Laplacian pyramid

Image Transformation

HIERACHY OF TRANSFORMATIONS



Euclidean (3 dof)

- Length
- Angle
- Area

$$\begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ & & 1 \end{bmatrix}$$

Similarity (4 dof)

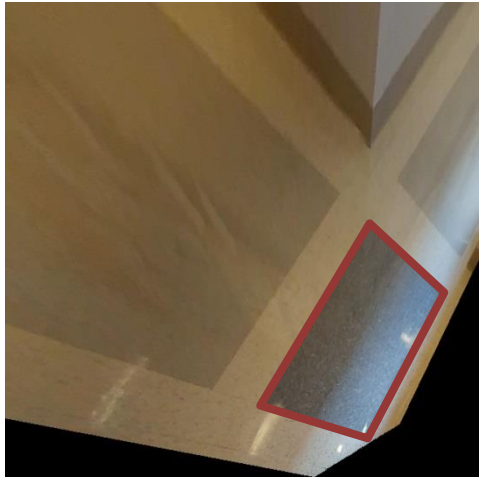
- Length ratio
- Angle

$$\begin{bmatrix} \alpha\cos\theta & -\alpha\sin\theta & t_x \\ \alpha\sin\theta & \alpha\cos\theta & t_y \\ & & 1 \end{bmatrix}$$

Affine (6 dof)

- Parallelism
- Ratio of area
- Ratio of length

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

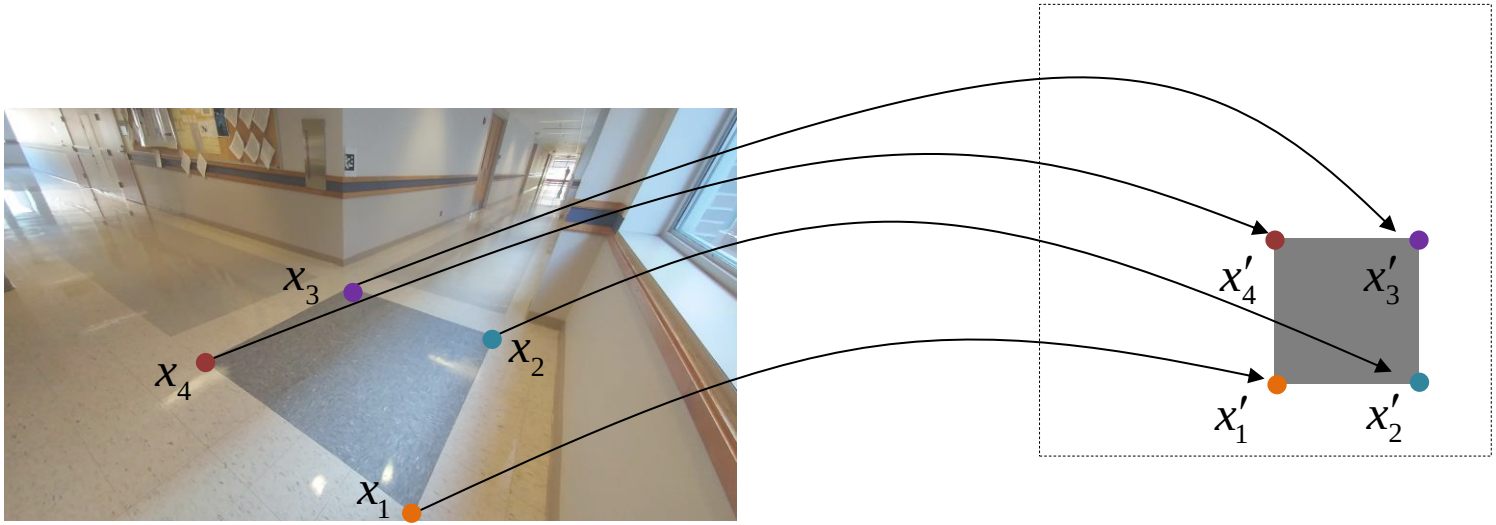


Projective (8 dof)

- Cross ratio
- Concurrency
- Colinearity

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix}$$

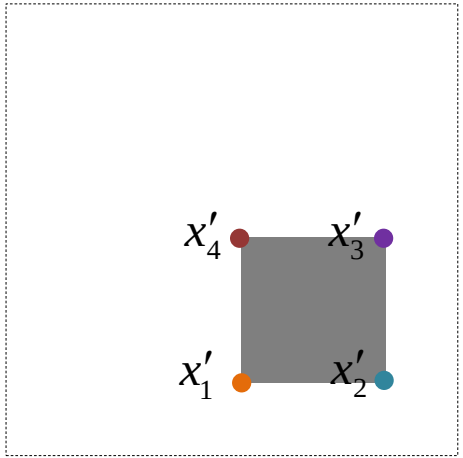
HOMOGRAPHY COMPUTATION



$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix}$$

The image can be rectified as if it is seen from top view.

HOMOGRAPHY COMPUTATION



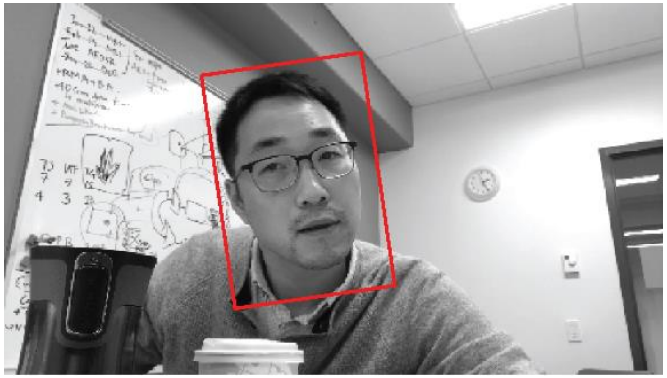
$$\begin{bmatrix} u_1 & v_1 & 1 & 0 & 0 & 0 & -u_1u'_1 & -v_1u'_1 \\ 0 & 0 & 0 & u_1 & v_1 & 1 & -u_1v'_1 & -v_1v'_1 \\ & & & & \vdots & & & \\ u_4 & v_4 & 1 & 0 & 0 & 0 & -u_4u'_4 & -v_4u'_4 \\ 0 & 0 & 0 & u_4 & v_4 & 1 & -u_4v'_4 & -v_4v'_4 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ \vdots \\ h_{32} \end{bmatrix} = \begin{bmatrix} u'_1 \\ v'_1 \\ \vdots \\ u'_4 \\ v'_4 \end{bmatrix}$$

$$Ax = b \longrightarrow x = \left(A^T A\right)^{-1} A^T b$$

Optical Flow

IMAGE ALIGNMENT

Template
$T(x)$



Target image
$I(x)$

Brightness constancy

$$I(W(x;p)) \approx T(x)$$

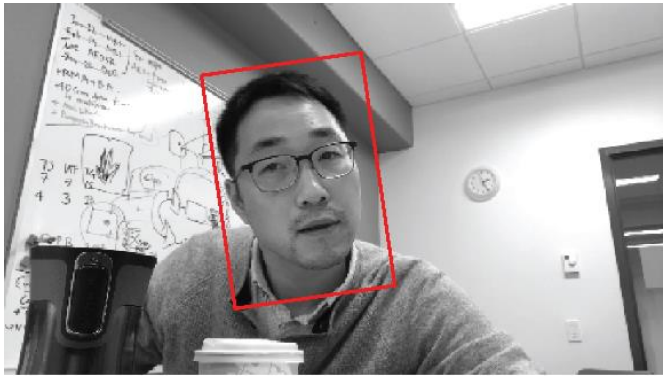
ex) affine transform

$$W(x;p) = \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} p_1 u + p_2 v + p_3 \\ p_4 u + p_5 v + p_6 \end{bmatrix}$$

$$= \begin{bmatrix} (p_1 + 1)u + p_2 v + p_3 \\ p_4 u + (p_5 + 1)v + p_6 \end{bmatrix}$$

IMAGE ALIGNMENT

Template
$T(x)$



Target image
$I(x)$

Brightness constancy

$$I(W(x;p)) \approx T(x)$$

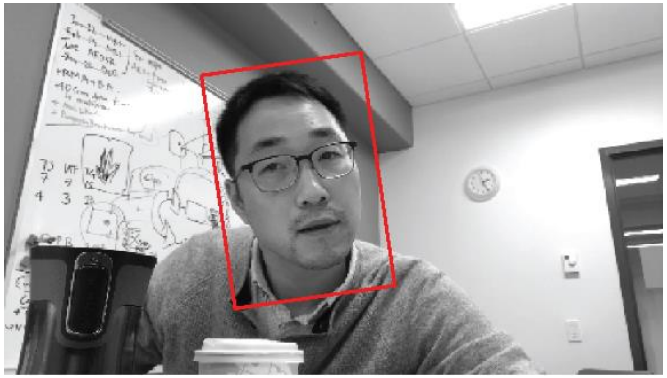
Objective: to find the optimal warping parameter p that minimizes warping error.

$$p* = \underset{p}{\text{minimize}} \sum_{x} \Big(I(W(x;p)) - \overline{T(x)} \Big)^2$$

Template

IMAGE ALIGNMENT

Template
$T(x)$



Target image
$I(x)$

Brightness constancy

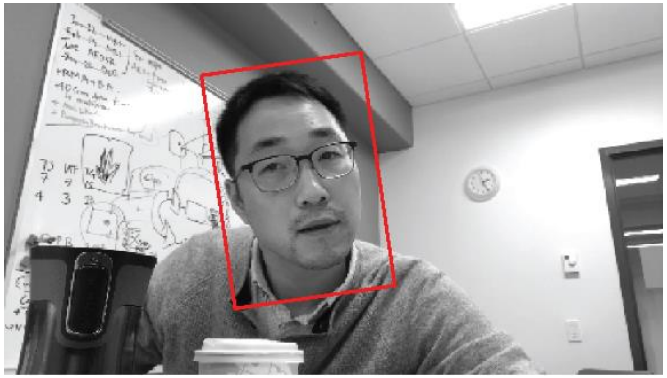
$$I(W(x;p)) \approx T(x)$$

Objective: to find the optimal warping parameter p that minimizes warping error.

$$p* = \underset{p}{\text{minimize}} \sum_{x} \Big(\underbrace{I(W(x;p))}_{\text{Warped image}} - \underbrace{T(x)}_{\text{Template}} \Big)^2$$

IMAGE ALIGNMENT

Template
$T(x)$



Target image
$I(x)$

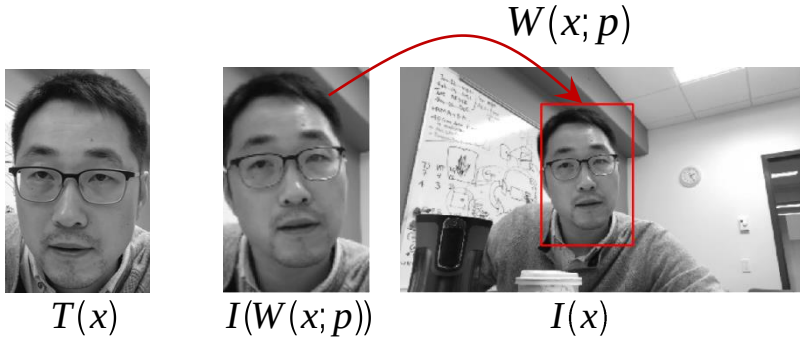
Brightness constancy

$$I(W(x;p)) \approx T(x)$$

Objective: to find the optimal warping parameter p that minimizes warping error.

$$p* = \underset{p}{\text{minimize}} \sum_{x} \underbrace{\Big(I(W(x;p)) - T(x) \Big)^2}_{\text{Error image}}$$

IMAGE ALIGNMENT OBJECTIVE



$$p* = \underset{p}{\text{minimize}} \sum_x \left(I(W(x;p) - T(x) \right)^2$$

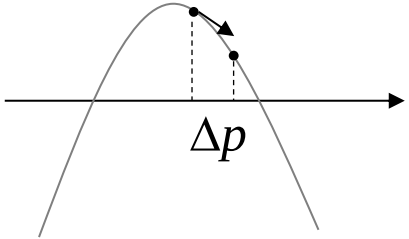
Guass-Newton's method

1.Linearize the obj. function at p

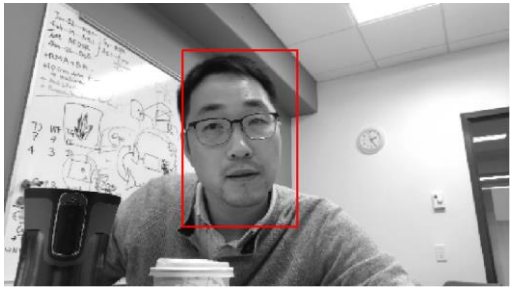
$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I \frac{\partial W}{\partial p} \Delta p$$

2. Find Δp that minimizes the obj. function at p

$$\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T (T(x) - I(W(x;p)))$$

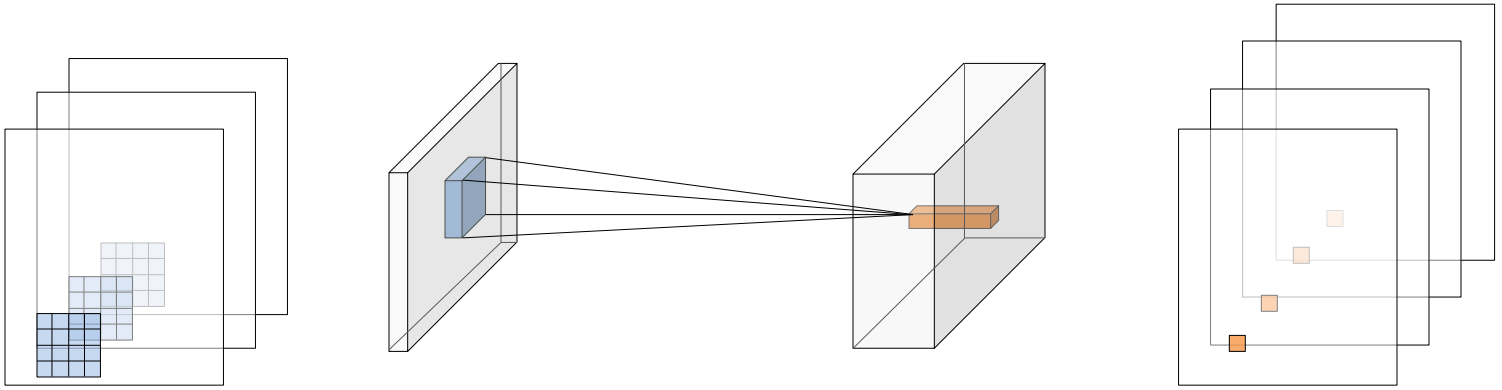


3. Update $p \leftarrow p + \Delta p$



CNN

Convolutional Layer



$H \times W \times C_i$

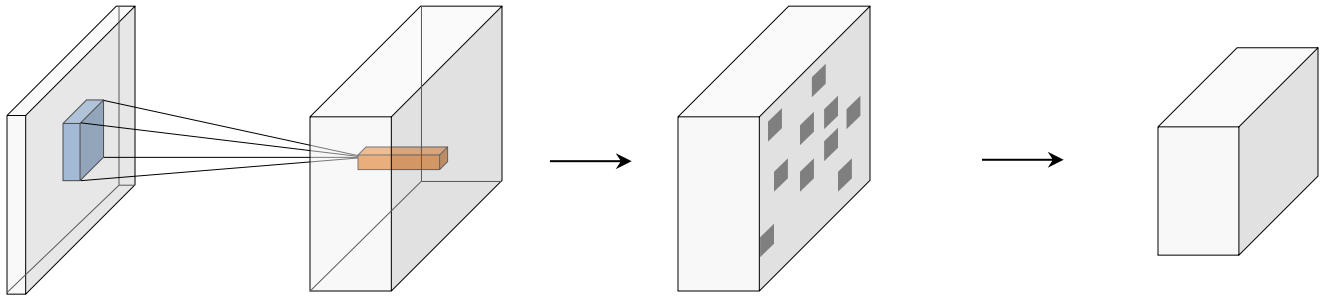
$H \times W \times C_o$

Multi-channel input

Multi-channel output

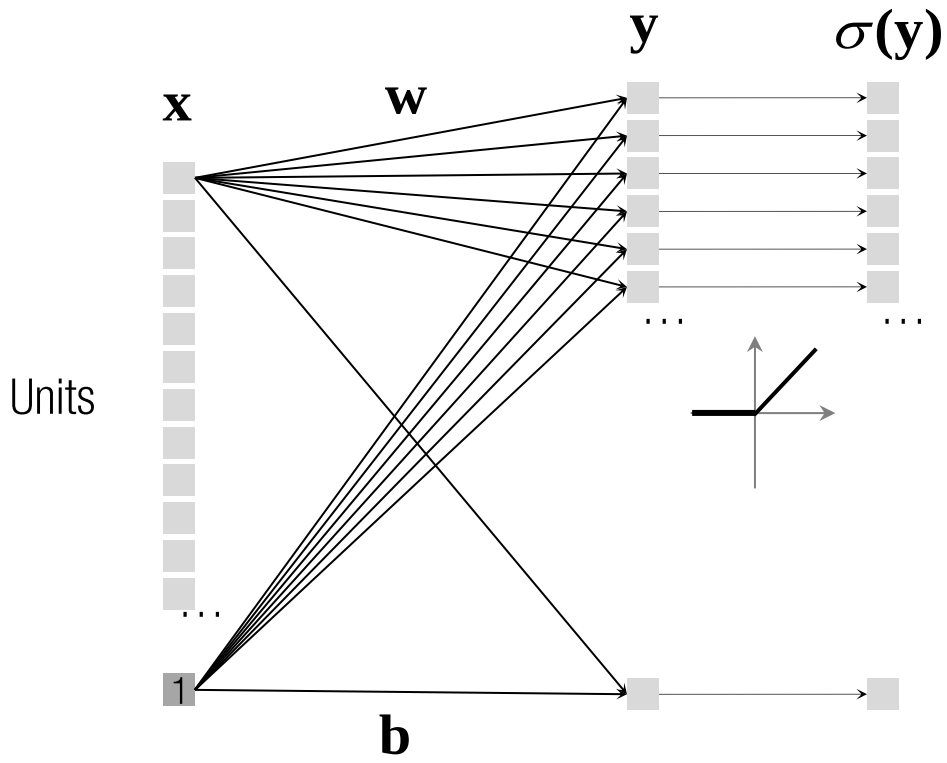
Feature map

CONV+RELU+POOL

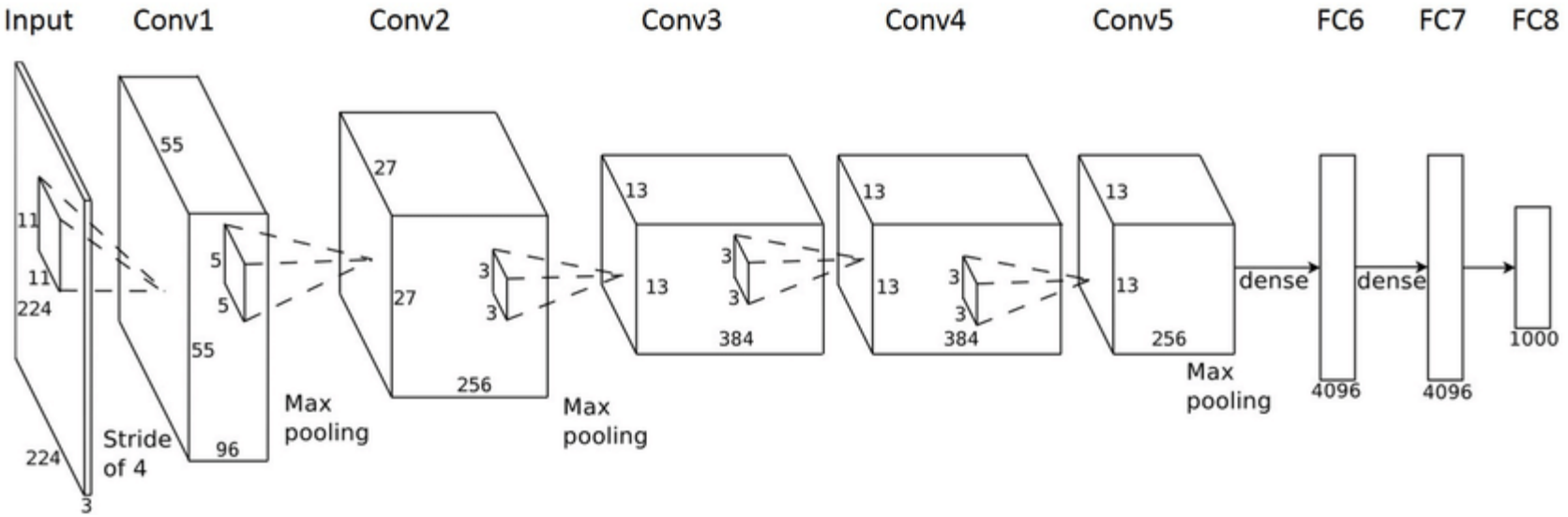


Operations:	Conv	ReLu	Max-pool	
# of units:	$H \times W \times C_1$	$H \times W \times C_2$	$H \times W \times C_2$	$H_1 \times W_1 \times C_2$
# of weights:	$F_1 \times F_2 \times C_1 \times C_2$	0	0	
# of biases:	$1 \times C_2$	0	0	

FC+ReLu



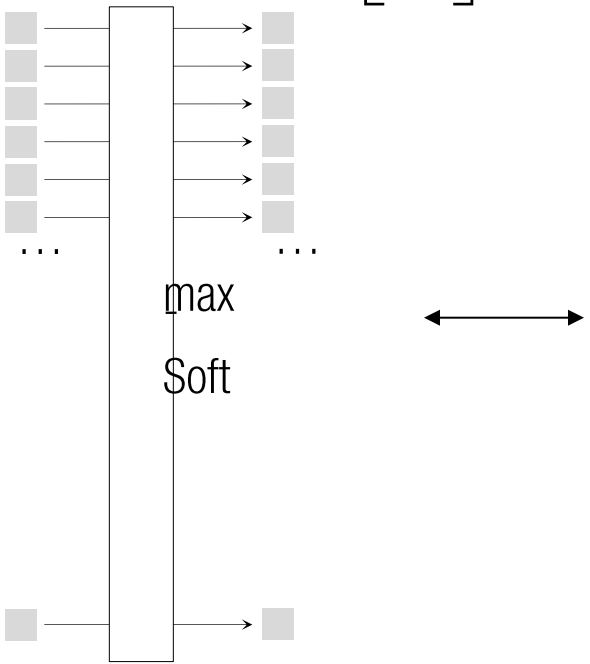
ALEX NET


Input
Conv1
Conv2
Conv3
Conv4
Conv5
FC6
FC7
FC8
224
224
3
11
11
55
55
96
5
5
Stride of 4
27
27
256
3
3
Max pooling
13
13
384
3
3
Max pooling
13
13
384
3
3
13
13
256
Max pooling
dense
4096
dense
4096
1000

ERROR MEASURE (LOSS)

$\mathbf{x} \in \mathbb{R}^L$ $\tilde{\mathbf{y}} \in [0,1]^L$ $\tilde{\mathbf{y}} \in \{0,1\}^L$

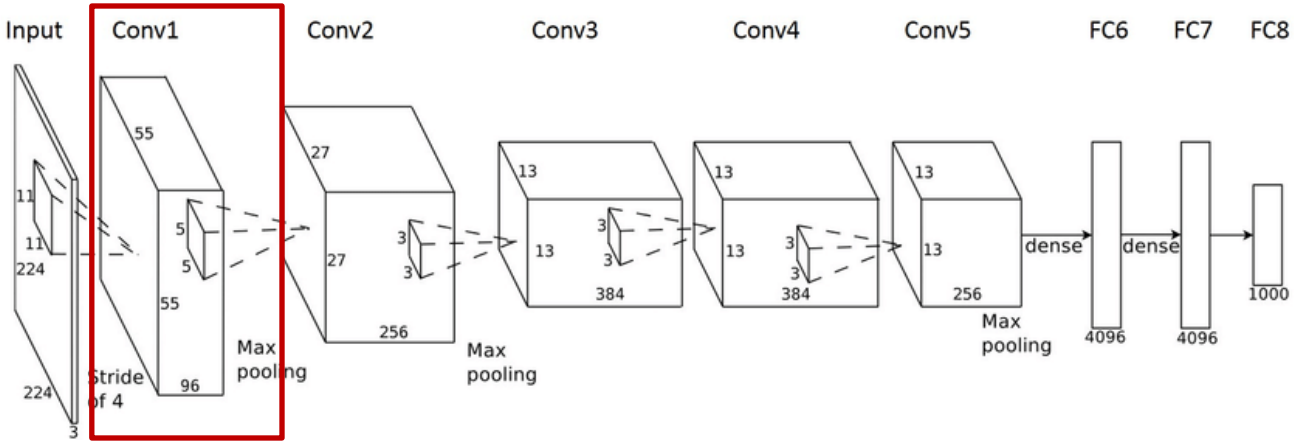
L : # of class labels



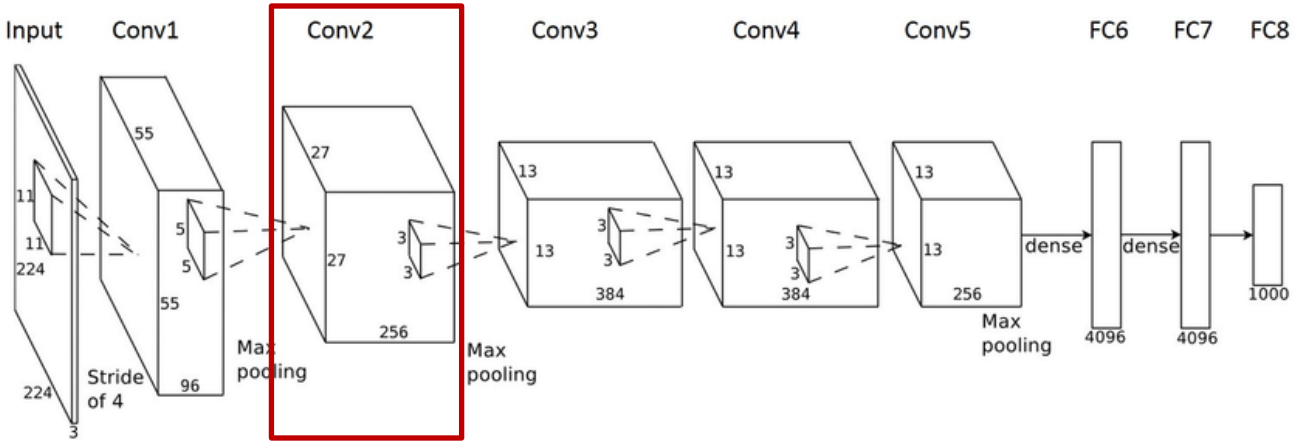
Prediction Ground truth

Cross-entropy loss (matching prob. distribution)

$$L = \sum_i \mathbf{y}_i \log \tilde{\mathbf{y}}_i$$


Input
Conv1
Conv2
Conv3
Conv4
Conv5
FC6
FC7
FC8
224
224
3
11
11
55
55
96
5
5
Stride of 4
Max pooling
27
27
256
3
3
Max pooling
13
13
384
3
3
13
13
384
3
3
13
13
256
Max pooling
dense
4096
dense
4096
1000

Layer 1


Input
Conv1
Conv2
Conv3
Conv4
Conv5
FC6
FC7
FC8
224
224
3
11
11
55
55
96
5
5
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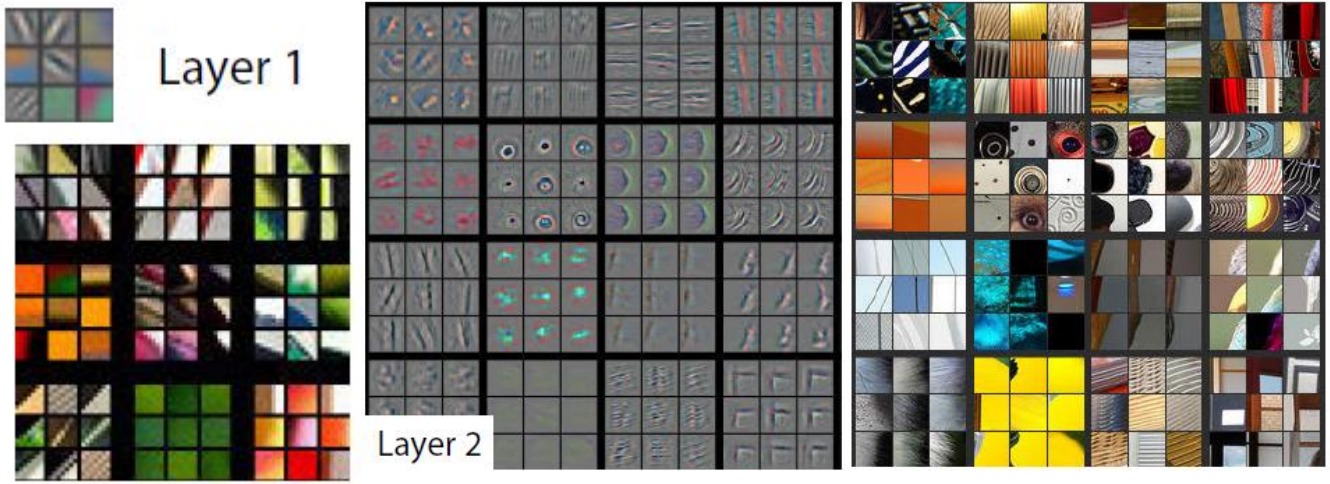
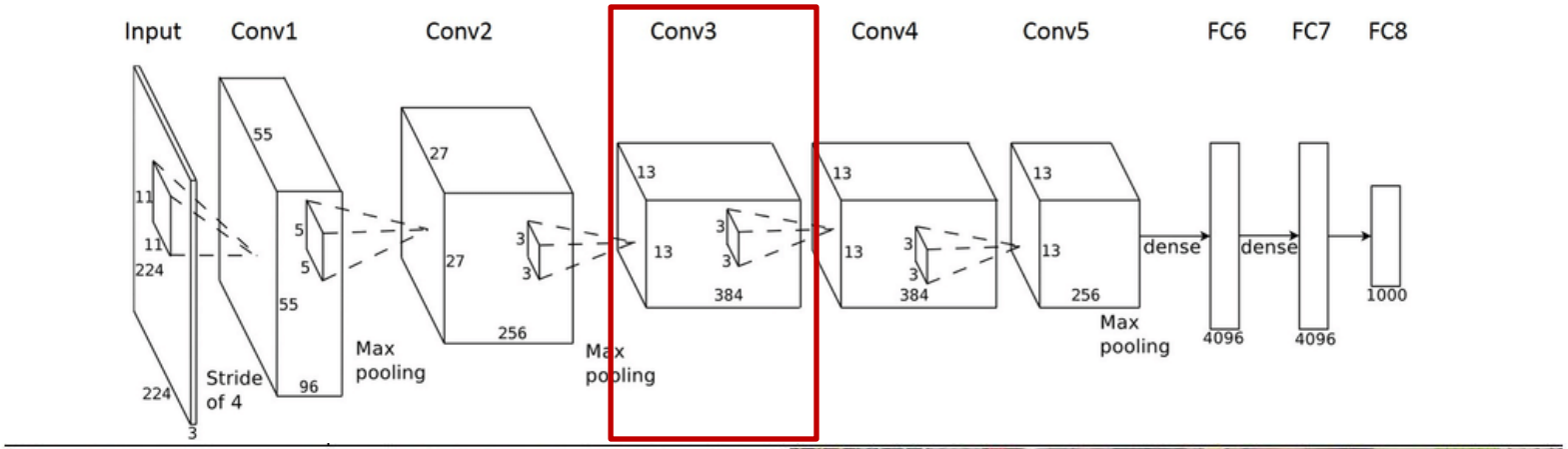
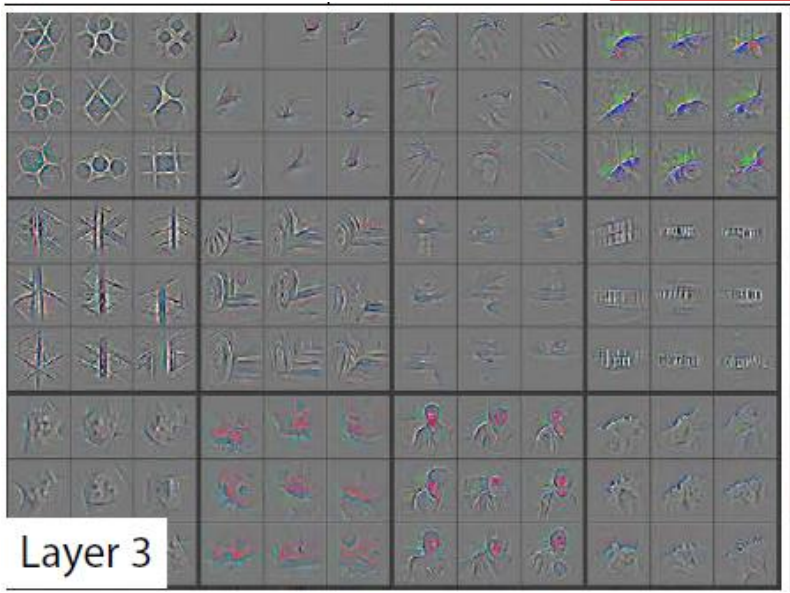
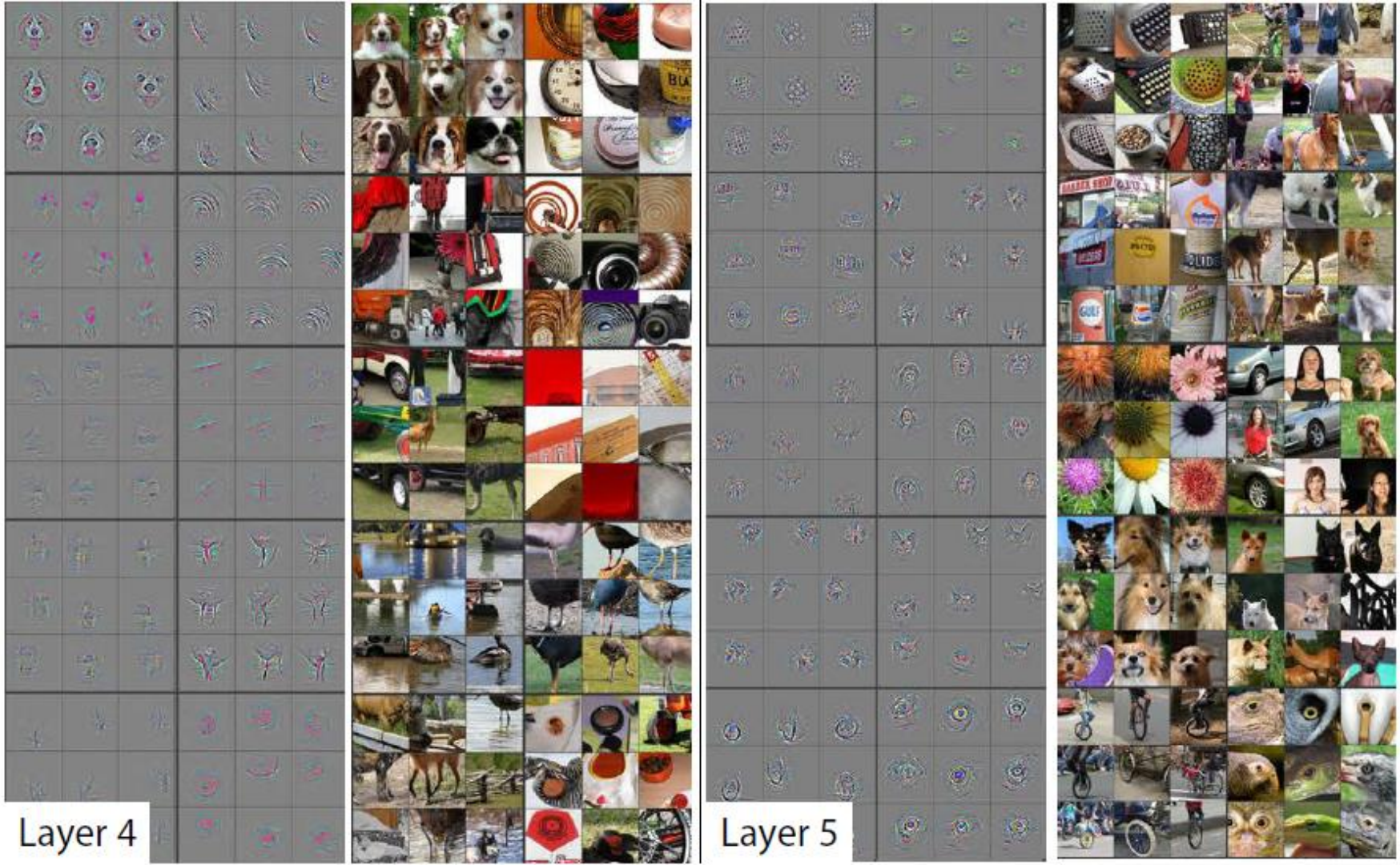
27
27
256
3
3
Max pooling
13
13
384
Max pooling
13
13
384
13
13
256
Max pooling
dense
4096
dense
4096
1000

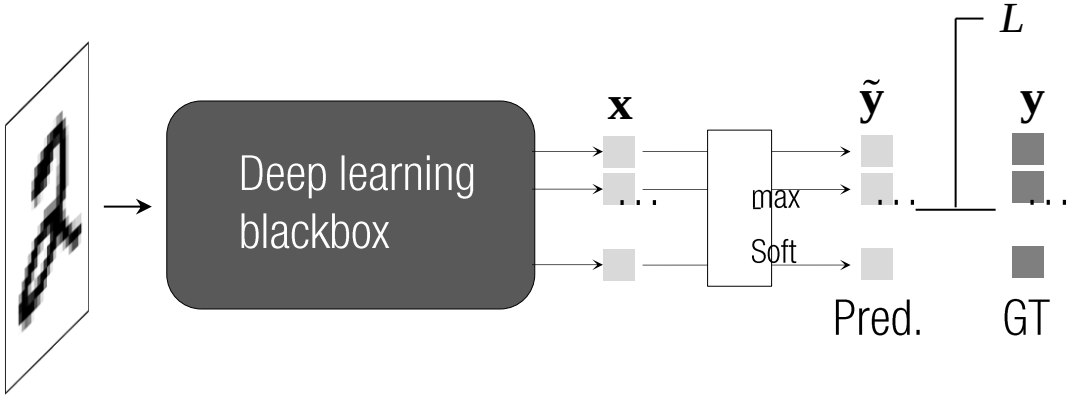

Layer 1
Layer 2


Input
Conv1
Conv2
Conv3
Conv4
Conv5
FC6
FC7
FC8
11
11
224
224
3
55
55
5
5
96
Stride
of 4
27
27
5
256
Max
pooling
13
13
3
3
384
Max
pooling
13
13
3
3
384
13
13
3
3
256
Max
pooling
dense
4096
dense
4096
1000

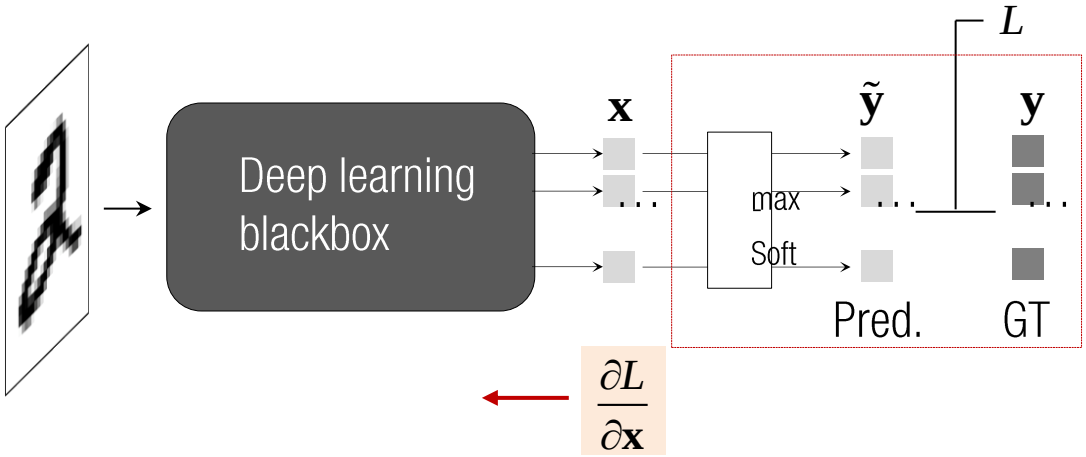

Layer 3


Layer 4
Layer 5

ENTROPY LOSS DERIVATIVE



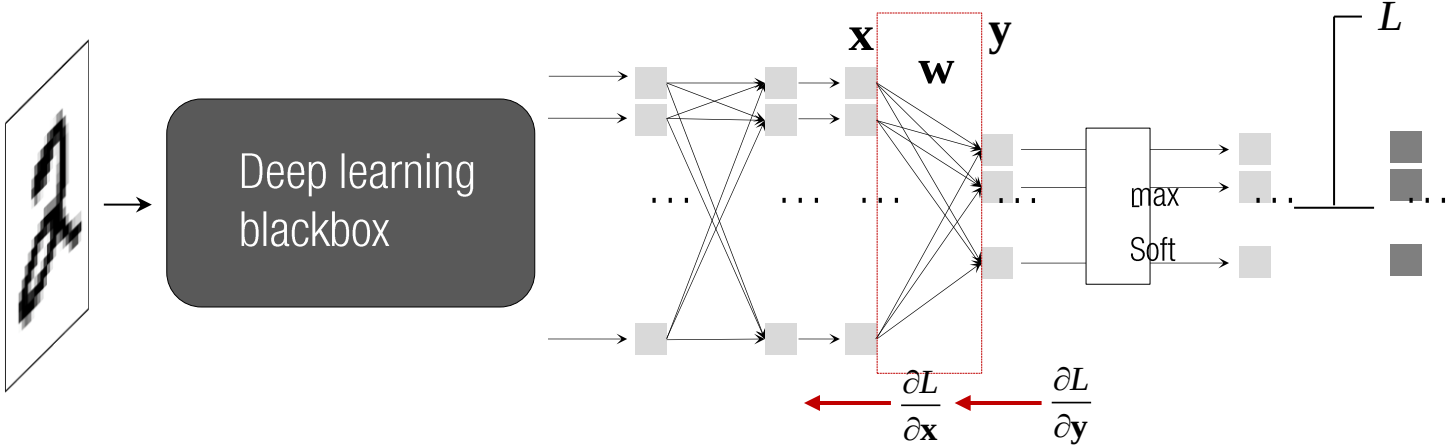
ENTROPY LOSS DERIVATIVE



function [L, dLdx] = Loss(x, y)

Input: $\mathbf{x}$	$\frac{\partial L}{\partial \mathbf{x}_i} = \tilde{\mathbf{y}}_i - \mathbf{y}_i$	1 x n
Trainable var.: None	None	
Output: $L = \sum_i \mathbf{y}_i \log \tilde{\mathbf{y}}_i$ where $\tilde{\mathbf{y}}_i = \frac{e^{\mathbf{x}_i}}{\sum_i e^{\mathbf{x}_i}}$		

FULLY CONNECTED LAYER



Input: $\mathbf{x} \in \mathbb{R}^n$

Trainable var.: $\mathbf{w} \in \mathbb{R}^{m \times n}$

Output: $\mathbf{y} = \mathbf{w}\mathbf{x}$

$\mathbf{y} \in \mathbb{R}^m$

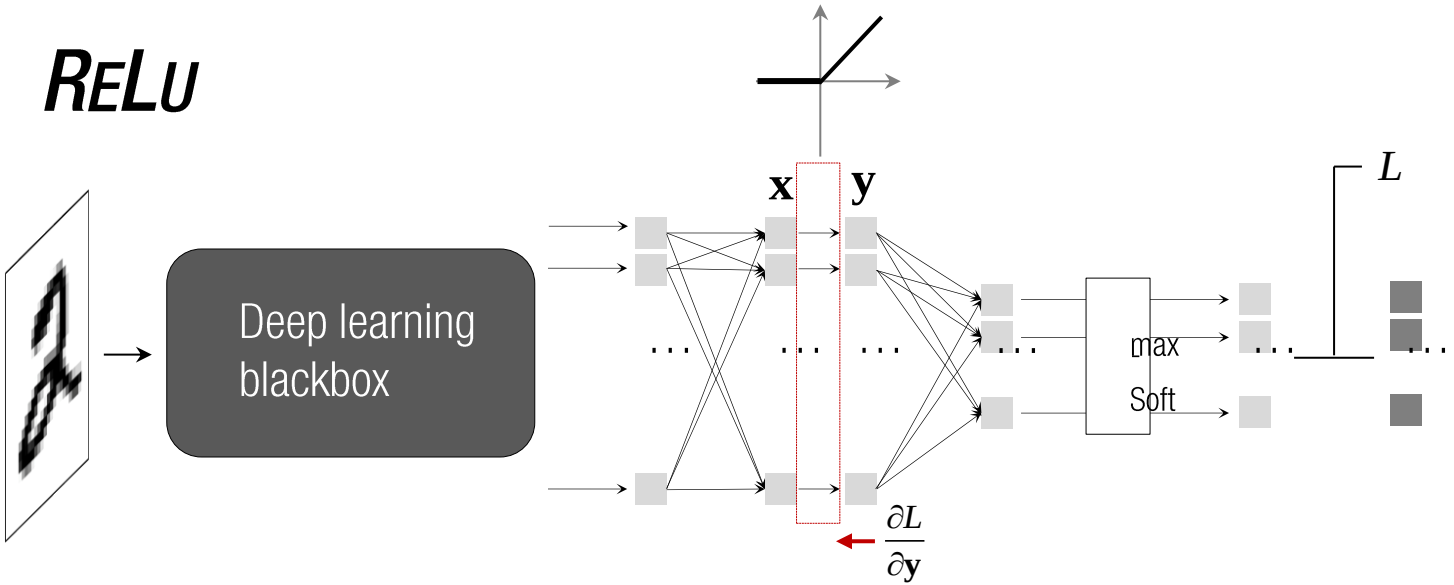
$$\frac{\partial L}{\partial \mathbf{x}_i} = \sum_j \frac{\partial L}{\partial \mathbf{y}_j} \frac{\partial \mathbf{y}_j}{\partial \mathbf{x}_i} = \sum_j \frac{\partial L}{\partial \mathbf{y}_j} \mathbf{w}_{ji} \qquad 1\text{xn}$$

$$\frac{\partial L}{\partial \mathbf{w}_{ij}} = \sum_i \frac{\partial L}{\partial \mathbf{y}_i} \frac{\partial \mathbf{y}_i}{\partial \mathbf{w}_{ij}} = \frac{\partial L}{\partial \mathbf{y}_i} \mathbf{x}_j \qquad 1\text{x(nm)}$$

function [y] = FC(x, w)

function [dLdx, dLdw, dLdb] = FC_back(dLdy, x, w, y)

ReLu



Input: $\mathbf{x} \in \mathbb{R}^n$

$$\frac{\partial L}{\partial \mathbf{x}_i} = \sum_j \frac{\partial L}{\partial \mathbf{y}_j} \frac{\partial}{\partial \mathbf{x}_i} \max(0, \mathbf{x}_j) = \frac{\partial L}{\partial \mathbf{y}_i} \frac{\partial}{\partial \mathbf{x}_i} \max(0, \mathbf{x}_i) = \begin{cases} \frac{\partial L}{\partial \mathbf{y}_i} & \text{if } \mathbf{y}_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

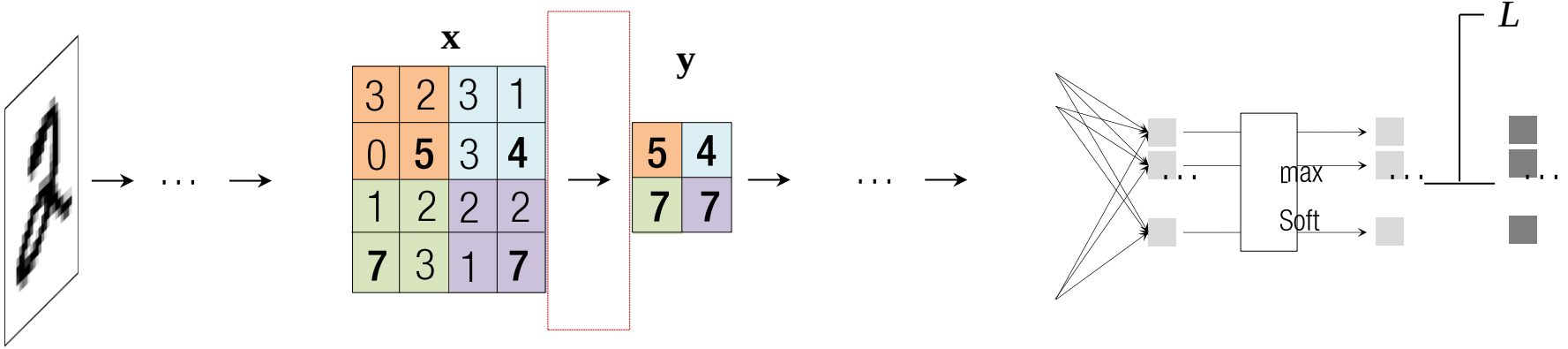
Trainable var.: None

Output: $\mathbf{y}_i = \max(0, \mathbf{x}_i)$

$\mathbf{y} \in \mathbb{R}^n$

function [y] = Relu(x)
function [dLdx] = Relu_back(dLdy, x, w, y)

MAX-POOL



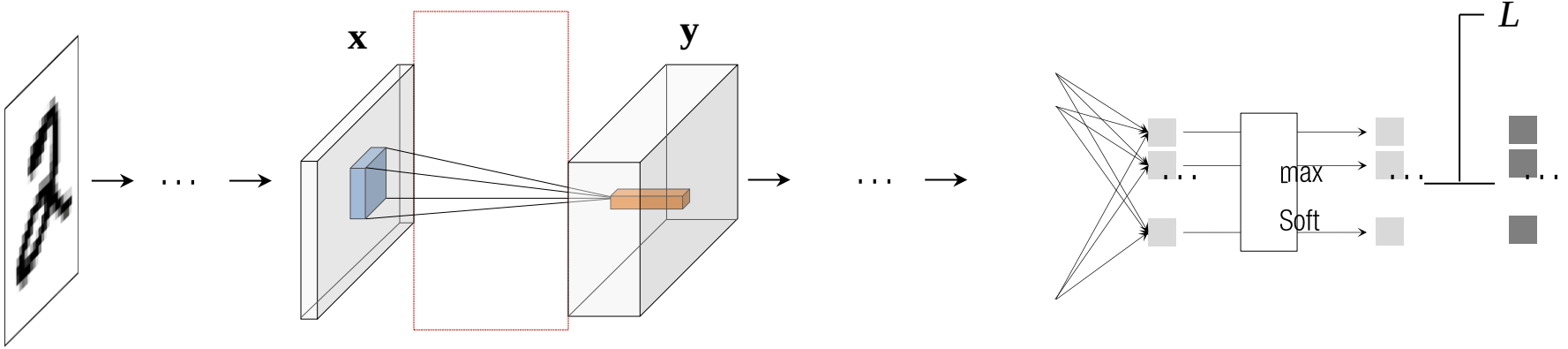
Input: $\mathbf{x} \in \mathbb{R}^{H \times W \times C}$

Trainable var.: None

Output: $\mathbf{y} \in \mathbb{R}^{\frac{H}{2} \times \frac{W}{2} \times C}$

```
function [y] = Maxpool(x, size, stride)
function [dLdx] = Maxpool_back(dLdy, x, size, stride, y)
```

CONVOLUTIONAL OPERATION



Input: $\mathbf{x} \in \mathbb{R}^{H \times W \times C_1}$

Trainable var.: $\mathbf{w} \in \mathbb{R}^{F \times F \times C_1 \times C_2}$

Output: $\mathbf{y} = \mathbf{x} * \mathbf{w}$

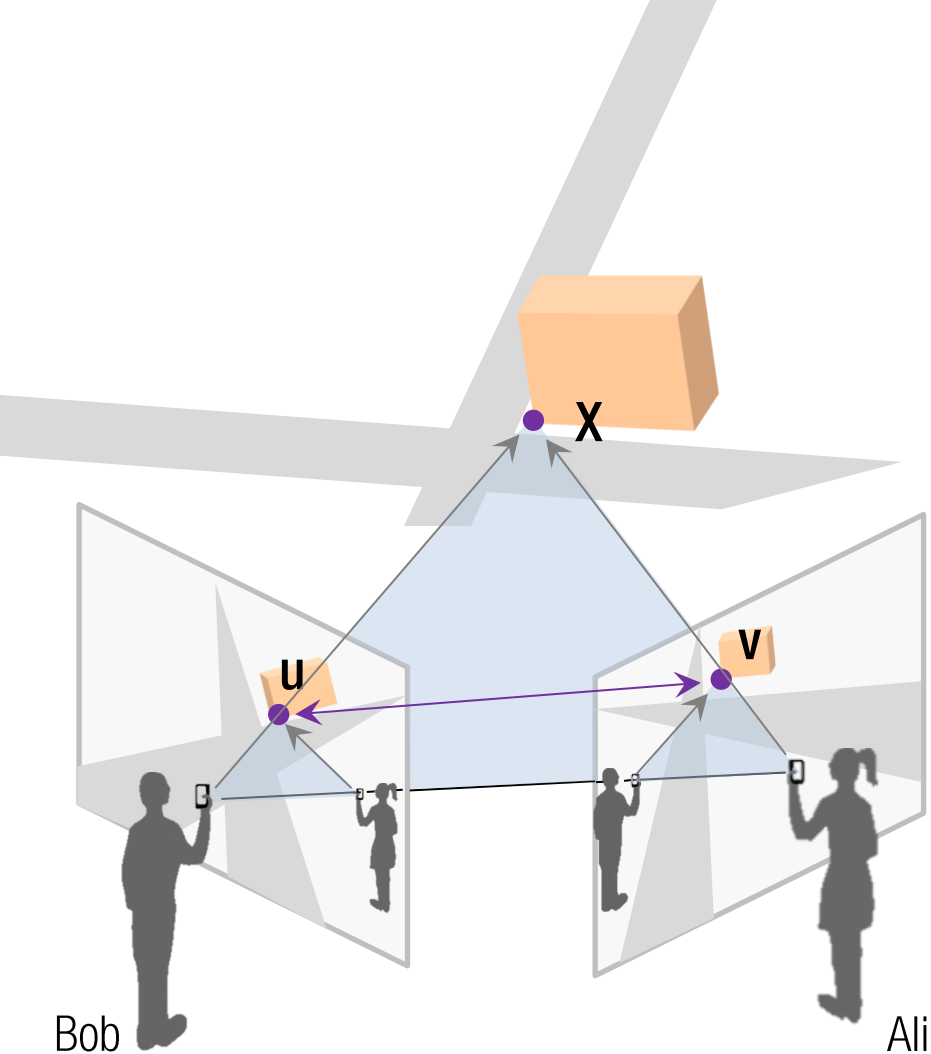
$\mathbf{y} \in \mathbb{R}^{H \times W \times C_2}$

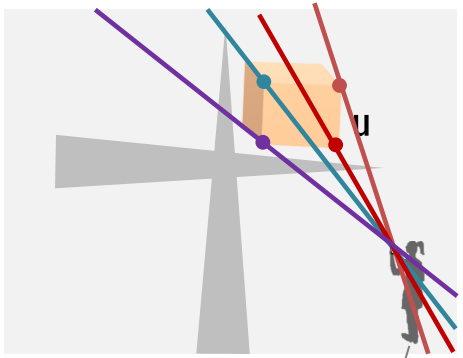
$$\frac{\partial L}{\partial \mathbf{x}_{ijk}} = \sum_m \sum_n \sum_l L_{mnl} \mathbf{w}_{m-i,n-j,k,l}$$

$$\frac{\partial L}{\partial \mathbf{w}_{ijcd}} = \sum_k \sum_l L_{kld} \mathbf{x}_{k+i,l+j,c}$$

function [y] = Conv(x, w, b)
function [dLdx dLdw dLdb] = Conv_back(dLdy, x, w, b, y)

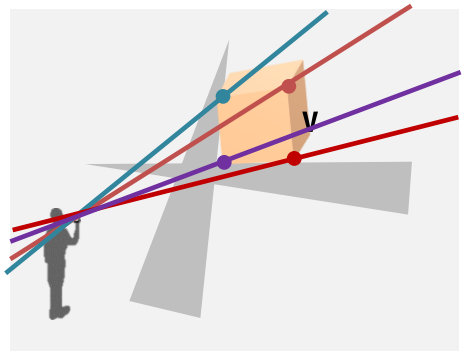
Epipolar Geometry





Bob's image

Alice from Bob's view

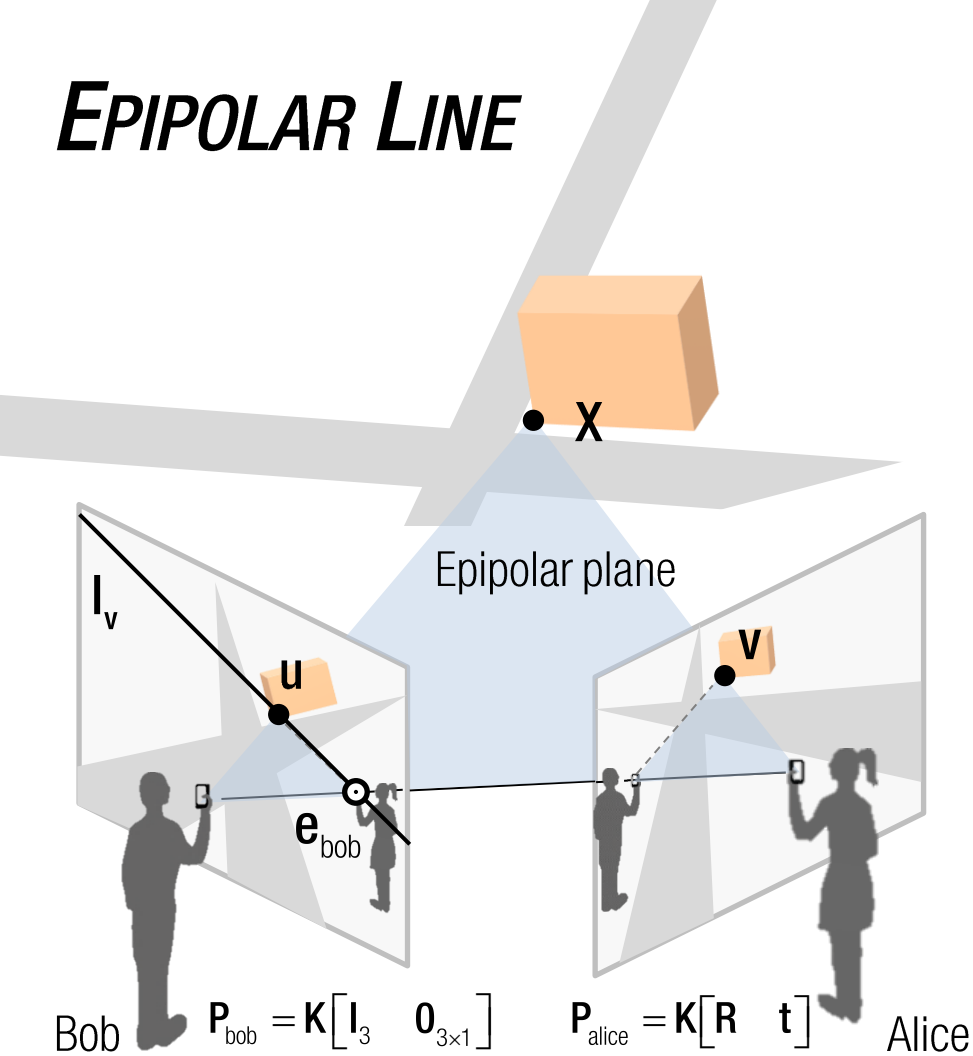


Alice's image

Epipolar constraint between two images:

1. A point, **u**, in Bob's image corresponds to an epipolar line $\mathbf{l}_u$ in Alice's image.
2. The epipolar line passes the corresponding point in Alice's image, **v**: $\mathbf{v}^T\mathbf{l}_u = 0$
3. Any point along the epipolar line can be a candidate of correspondences.
4. Epiploar lines meet at the epipole: $\mathbf{e}_{bob}^T\mathbf{l}_u = 0$ $\mathbf{e}_{alice}^T\mathbf{l}_v = 0$

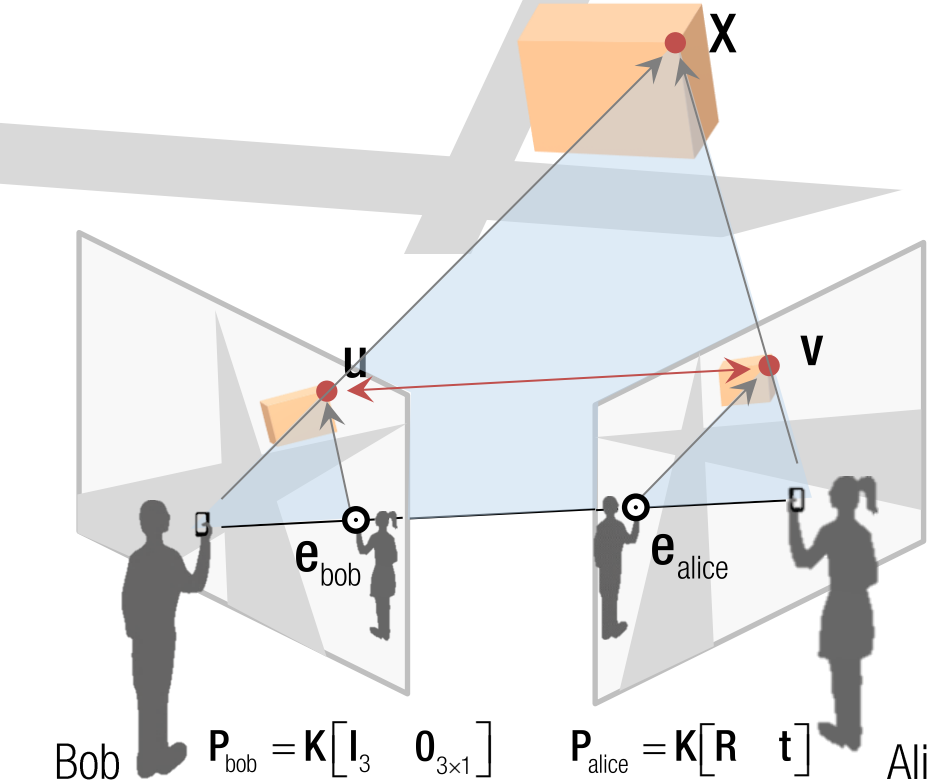
EPIPOLAR LINE

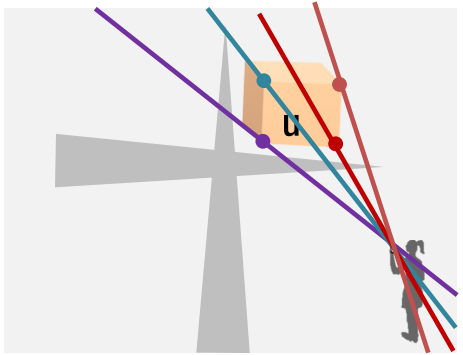


$$l_v = Fv$$

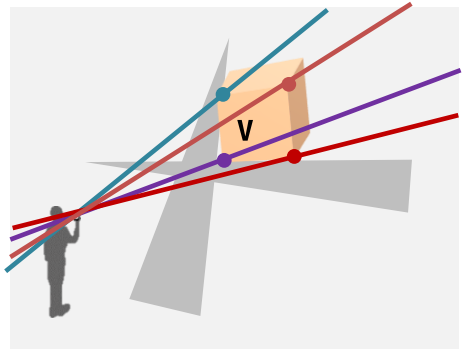
Fundamental matrix

FUNDAMENTAL MATRIX





Bob's image

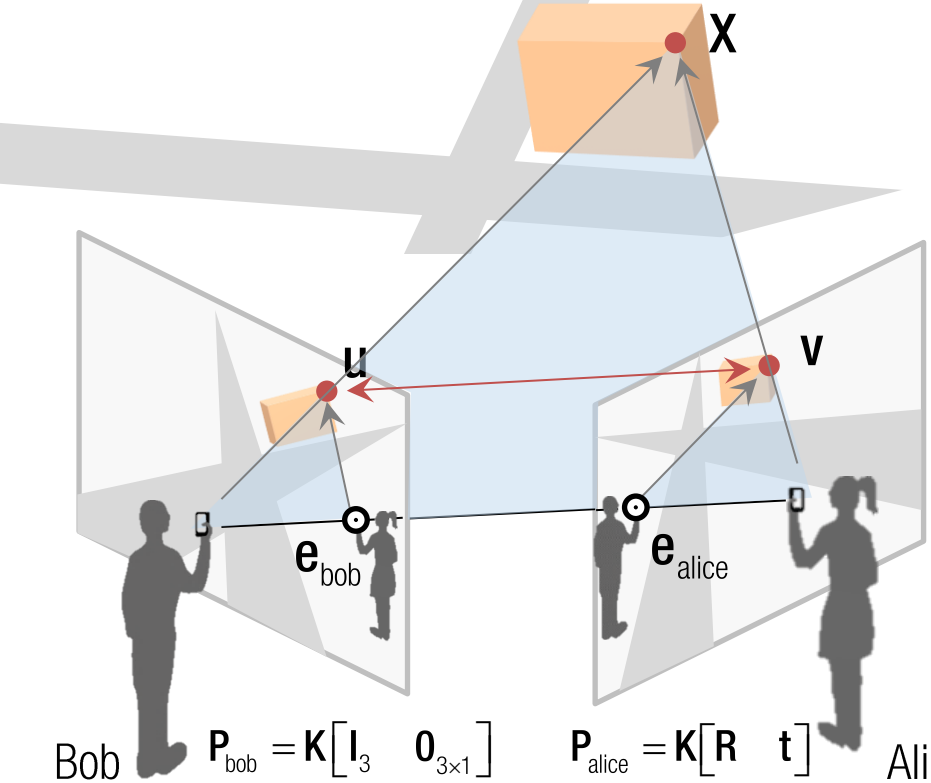


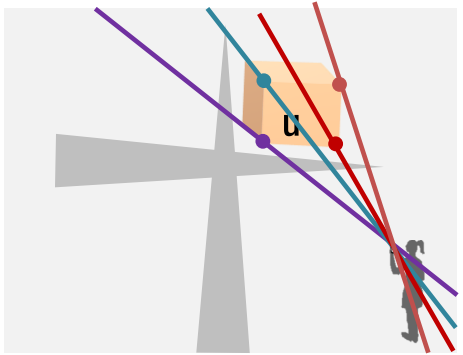
Alice's image

Properties of Fundamental Matrix

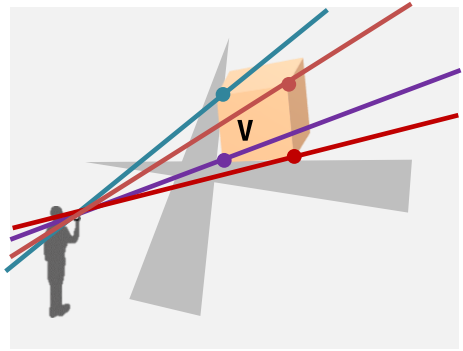
- Transpose: if **F** is for $\mathbf{P}_{bob}$, $\mathbf{P}_{alice}$, then $\mathbf{F}^T$ is for $\mathbf{P}_{alice}$, $\mathbf{P}_{bob}$.

FUNDAMENTAL MATRIX





Bob's image

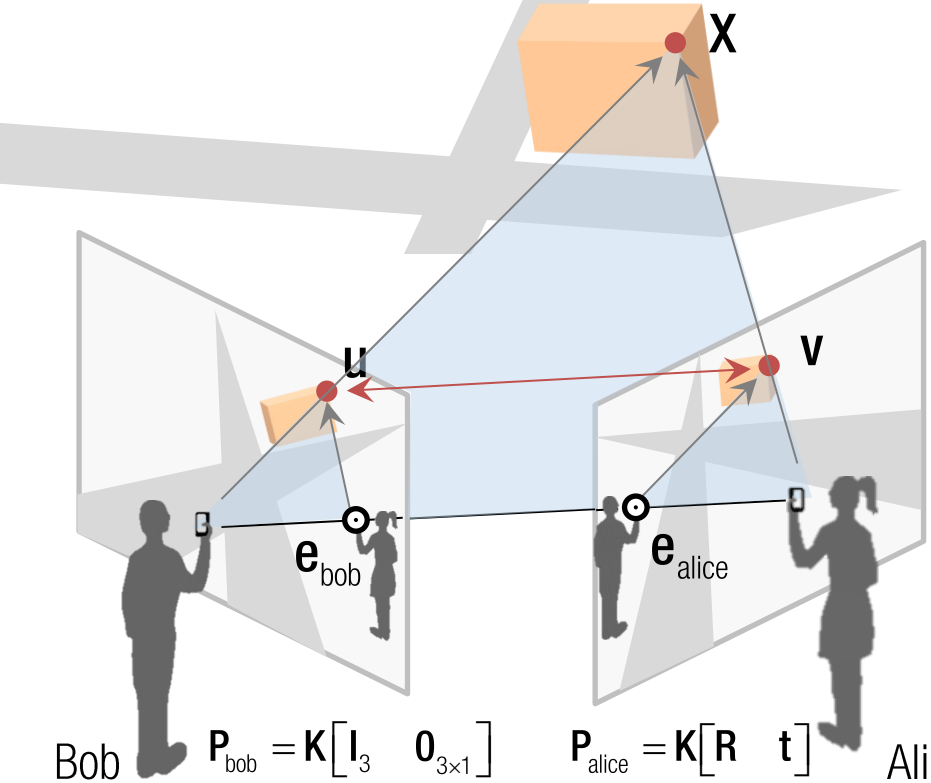


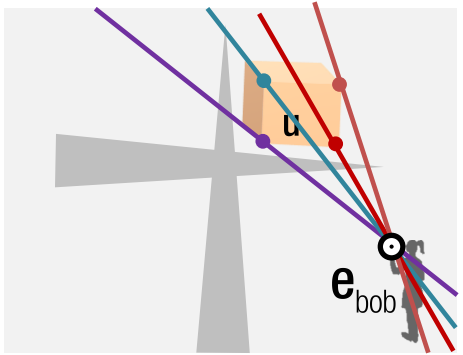
Alice's image

Properties of Fundamental Matrix

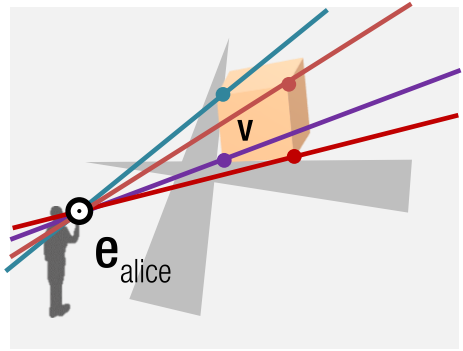
- Transpose: if $\mathbf{F}$ is for $\mathbf{P}_{bob}$, $\mathbf{P}_{alice}$, then $\mathbf{F}^T$ is for $\mathbf{P}_{alice}$, $\mathbf{P}_{bob}$.
- Epipolar line: $l_u = Fu \quad l_v = F^T v$

FUNDAMENTAL MATRIX





Bob's image



Alice's image

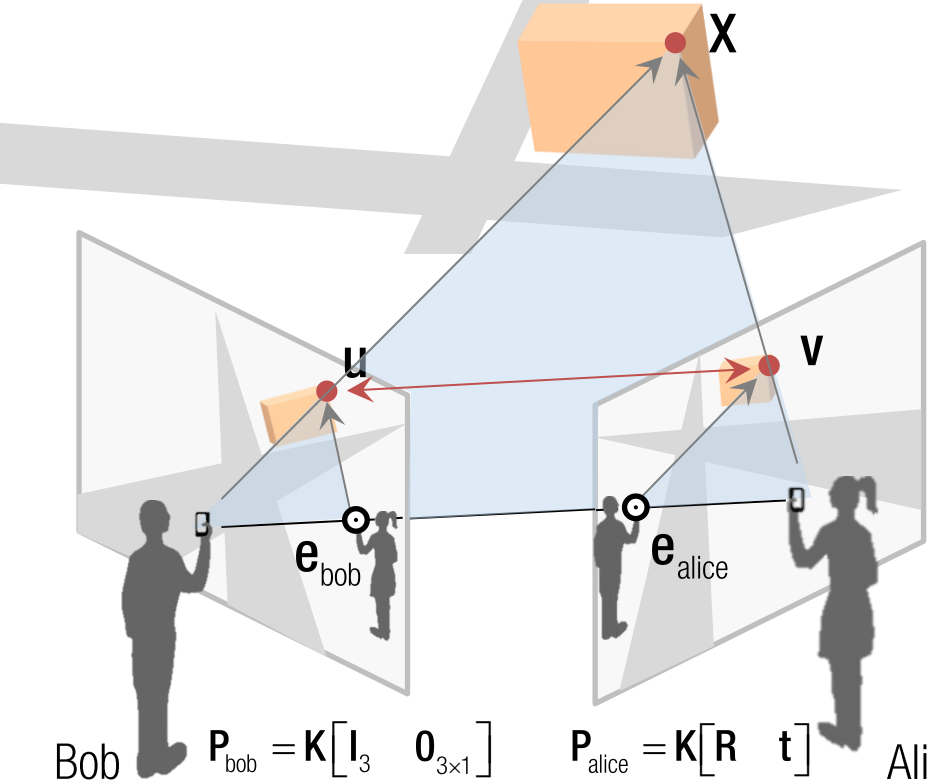
Properties of Fundamental Matrix

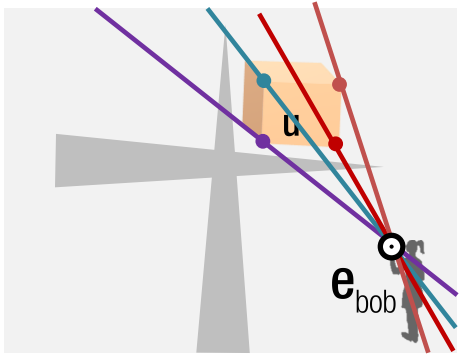
- Transpose: if $\mathbf{F}$ is for $\mathbf{P}_{bob}$, $\mathbf{P}_{alice}$, then $\mathbf{F}^T$ is for $\mathbf{P}_{alice}$, $\mathbf{P}_{bob}$.
- Epipolar line: $\mathbf{l}_u = \mathbf{F}\mathbf{u} \quad \mathbf{l}_v = \mathbf{F}^T\mathbf{v}$

  $\mathbf{F}\mathbf{e}_{bob} = \mathbf{0} \quad \mathbf{F}^T\mathbf{e}_{alice} = \mathbf{0}$
- Epipole: $\because \mathbf{v}_i^T\mathbf{F}\mathbf{e}_{bob} = 0, \quad \mathbf{u}_i^T\mathbf{F}^T\mathbf{e}_{alice} = 0, \quad \forall i$

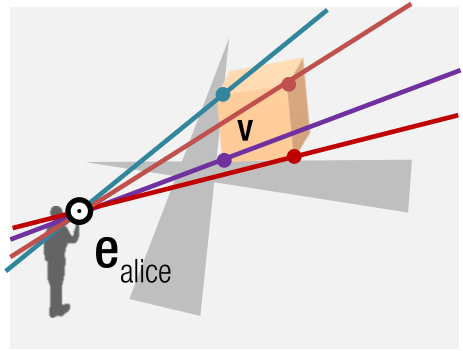
  $\longrightarrow \quad \mathbf{e}_{bob} = \text{null}(\mathbf{F}), \quad \mathbf{e}_{alice} = \text{null}(\mathbf{F}^T)$

FUNDAMENTAL MATRIX





Bob's image



Alice's image

Properties of Fundamental Matrix

- Transpose: if $\mathbf{F}$ is for $\mathbf{P}_{bob}$, $\mathbf{P}_{alice}$, then $\mathbf{F}^T$ is for $\mathbf{P}_{alice}$, $\mathbf{P}_{bob}$.
- Epipolar line: $\mathbf{l}_u = \mathbf{F}\mathbf{u} \quad \mathbf{l}_v = \mathbf{F}^T\mathbf{v}$
  $\mathbf{F}\mathbf{e}_{bob} = \mathbf{0} \quad \mathbf{F}^T\mathbf{e}_{alice} = \mathbf{0}$
- Epipole:
- rank($\mathbf{F}$)=2:
  DoF 9 (3x3 matrix)-1 (scale)-1 (rank)=7