Dense Optical Flow

Hyun Soo Park

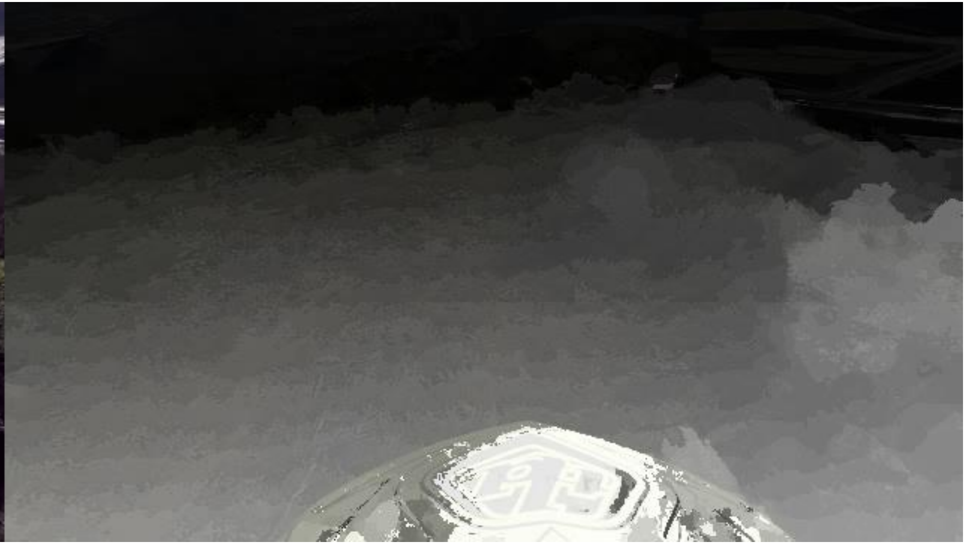
Good features to track

Weak feature to track

Weak feature to track

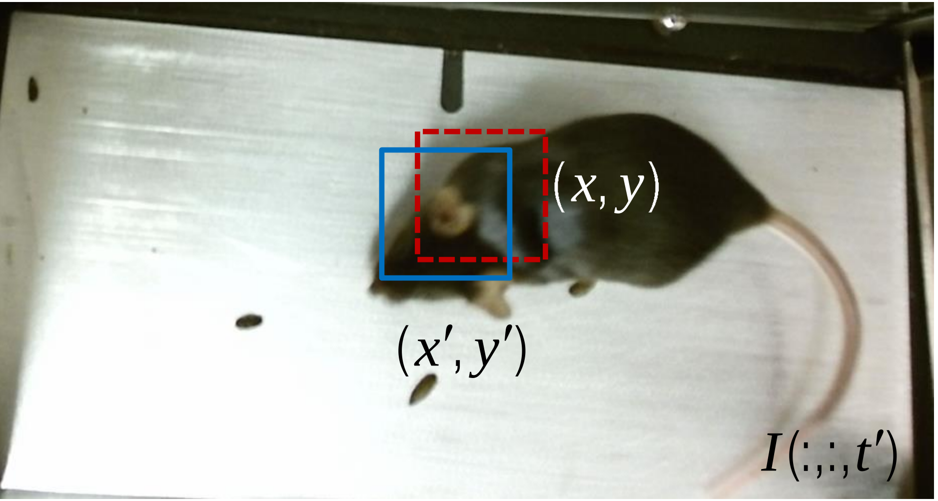
Spatial Smoothness of Optical Flow

The brighter, the bigger flow magnitude.



The brighter, the bigger flow magnitude.

Recall: Local Patch Tracking



Neighboring pixels move similarly.

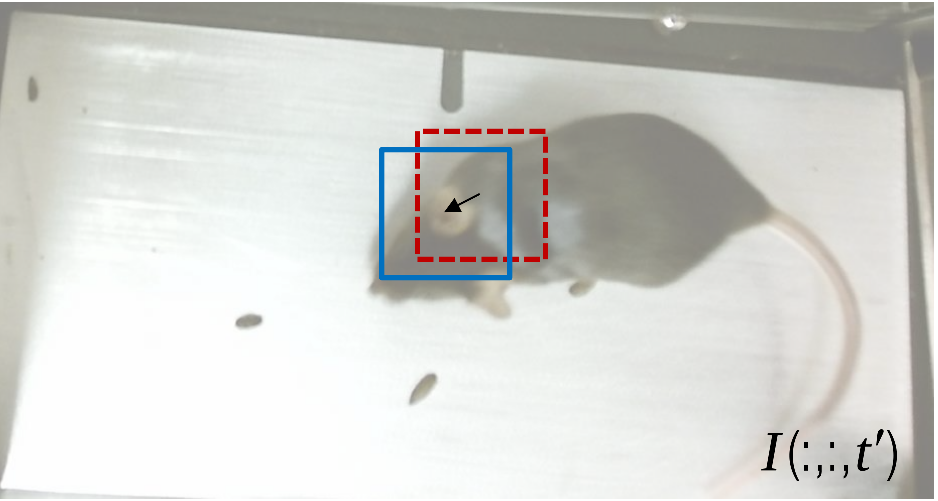
$\left\{ I(x_i, y_i, t) \right\}_{(x_i, y_i) \in N(x,y)}$ $\quad$ $\left\{ I(x_i', y_i', t) \right\}_{(x_i', y_i') \in N(x',y')}$

$$\left.\begin{array}{c} \left.\dfrac{\partial I}{\partial x}\right|_1 u + \left.\dfrac{\partial I}{\partial y}\right|_1 v = -\left.\dfrac{\partial I}{\partial t}\right|_1 \\ \vdots \\ \left.\dfrac{\partial I}{\partial x}\right|_n u + \left.\dfrac{\partial I}{\partial y}\right|_n v = -\left.\dfrac{\partial I}{\partial t}\right|_n \end{array}\right\}$$

# of unknowns: 2

# of equations: # of pixels in the local patch

Assumption: Constant Motion



$I(:,:,t')$

Neighboring pixels move similarly.

$\left\{I(x_i, y_i, t)\right\}_{(x_i, y_i)\in N(x,y)}$

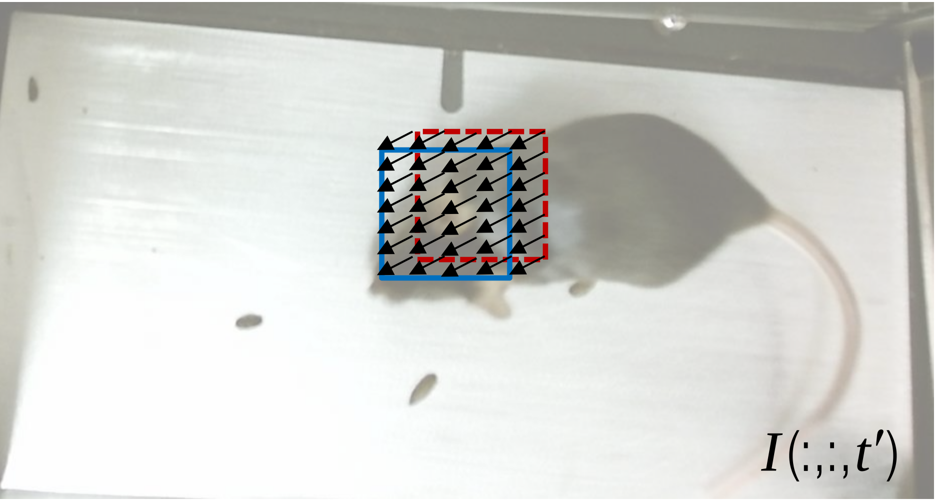
$\left\{I(x'_i, y'_i, t)\right\}_{(x'_i, y'_i)\in N(x',y')}$

$$\left.\begin{array}{c} \left.\dfrac{\partial I}{\partial x}\right|_1 u + \left.\dfrac{\partial I}{\partial y}\right|_1 v = -\left.\dfrac{\partial I}{\partial t}\right|_1 \\ \vdots \\ \left.\dfrac{\partial I}{\partial x}\right|_n u + \left.\dfrac{\partial I}{\partial y}\right|_n v = -\left.\dfrac{\partial I}{\partial t}\right|_n \end{array}\right\}$$

\# of unknowns: 2

\# of equations: # of pixels in the local patch

Assumption: Constant Motion



Neighboring pixels move similarly.

$\{I(x_i, y_i, t)\}_{(x_i, y_i) \in N(x, y)}$

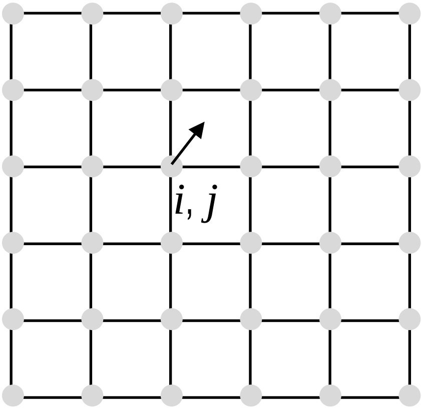
$\{I(x_i', y_i', t)\}_{(x_i', y_i') \in N(x', y')}$

$$\left.\begin{array}{c} \left.\frac{\partial I}{\partial x}\right|_1 u + \left.\frac{\partial I}{\partial y}\right|_1 v = -\left.\frac{\partial I}{\partial t}\right|_1 \\ \vdots \\ \left.\frac{\partial I}{\partial x}\right|_n u + \left.\frac{\partial I}{\partial y}\right|_n v = -\left.\frac{\partial I}{\partial t}\right|_n \end{array}\right\}$$

# of unknowns: 2

# of equations: # of pixels in the local patch

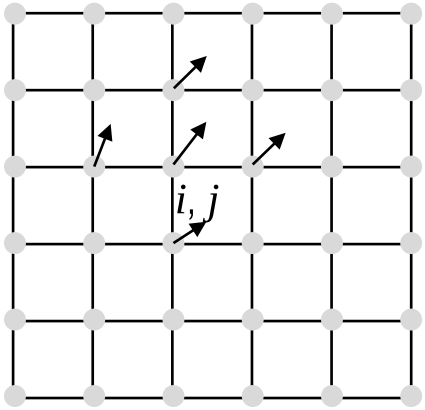
DENSE OPTICAL FLOW



Optical flow equation:

$$I_x\big|_{i,j}\, u_{i,j} + I_y\big|_{i,j}\, v_{i,j} = -I_t\big|_{i,j}$$

DENSE OPTICAL FLOW



Optical flow equation:

$$I_x\big|_{i,j} u_{i,j} + I_y\big|_{i,j} v_{i,j} = -I_t\big|_{i,j}$$

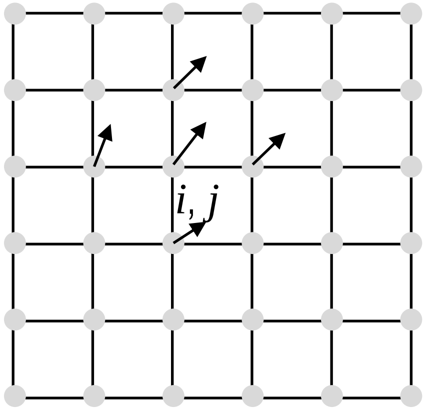
$$\vdots$$

Spatial smoothness constraint:

$$\left(u_{i,j} - u_{i-1,j}\right)^2 + \left(u_{i,j} - u_{i+1,j}\right)^2 + \left(u_{i,j} - u_{i,j-1}\right)^2 + \left(u_{i,j} - u_{i,j+1}\right)^2$$

$$\left(v_{i,j} - v_{i-1,j}\right)^2 + \left(v_{i,j} - v_{i+1,j}\right)^2 + \left(v_{i,j} - v_{i,j-1}\right)^2 + \left(v_{i,j} - v_{i,j+1}\right)^2$$

DENSE OPTICAL FLOW



Optical flow equation:

$$I_x\big|_{i,j}\, u_{i,j} + I_y\big|_{i,j}\, v_{i,j} = -I_t\big|_{i,j}$$

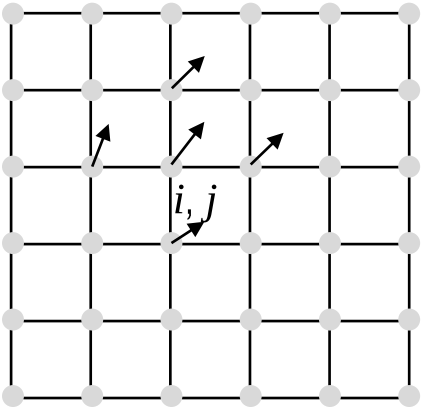
$$\vdots$$

Spatial smoothness constraint:

$$\|\nabla u\|^2$$

$$\|\nabla v\|^2$$

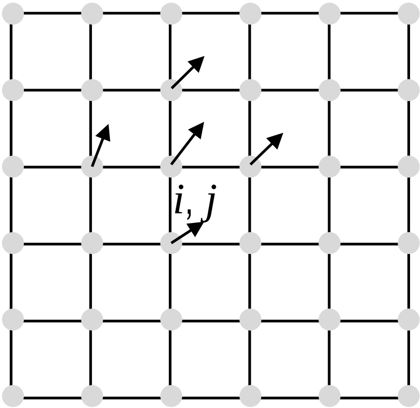
Dense Optical Flow



Objective:

$$\underset{\{u_{i,j},v_{i,j}\}}{\text{minimize}} \sum_i \sum_j \underbrace{\left(I_x u_{i,j} + I_y v_{i,j} + I_t\right)^2}_{\text{Optical flow eq.}} + \underbrace{\lambda \left\|\nabla u_{i,j}\right\|^2 + \lambda \left\|\nabla v_{i,j}\right\|^2}_{\text{Spatial smoothness}}$$

Dense Optical Flow



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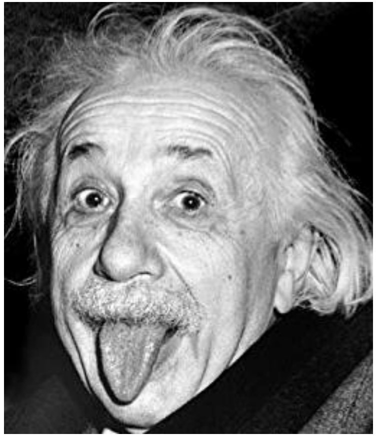
Taking derivative w.r.t. $u_{i,j}, v_{i,j}$

$$I_x\left(I_x u_{i,j} + I_y v_{i,j} + I_t\right) - \lambda \nabla^2 u_{i,j} = 0$$

$$I_y\left(I_x u_{i,j} + I_y v_{i,j} + I_t\right) - \lambda \nabla^2 v_{i,j} = 0$$

Laplace operator: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

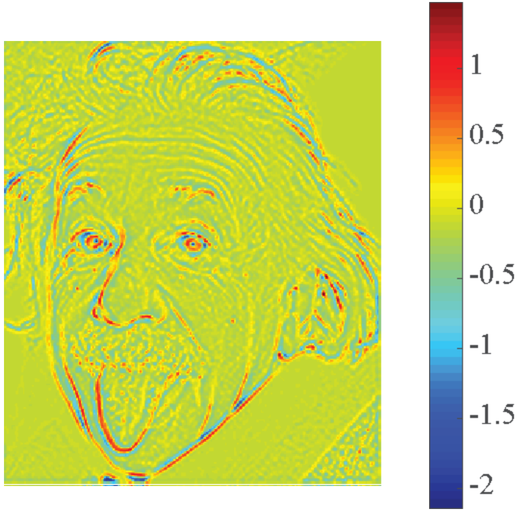
RECALL: LAPLACE OPERATOR



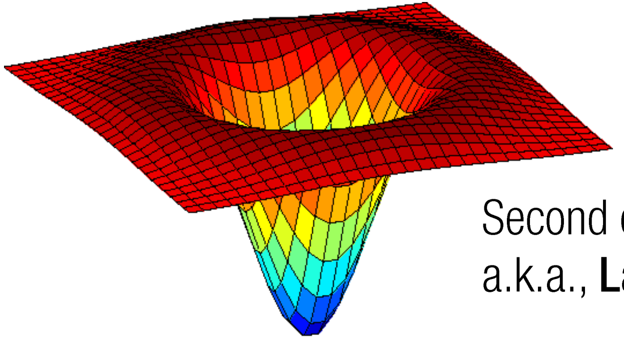
$*$

0	1	0
1	-4	1
0	1	0

$=$

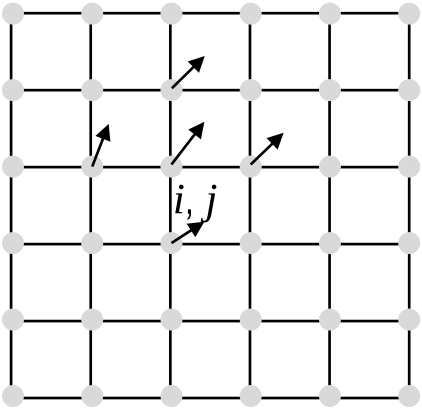


$$\nabla \cdot \nabla G = \nabla\left(\frac{\partial G}{\partial u}\mathbf{i} + \frac{\partial G}{\partial v}\mathbf{j}\right) = \frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2}$$



Second order derivative of Gaussian,
a.k.a., **Laplacian of Gaussian**

DENSE OPTICAL FLOW



Objective:

$$\underset{\{u_{i,j}, v_{i,j}\}}{\text{minimize}} \sum_i \sum_j \underbrace{\left(I_x u_{i,j} + I_y v_{i,j} + I_t \right)^2}_{\text{Optical flow eq.}} + \underbrace{\lambda \left\| \nabla u_{i,j} \right\|^2 + \lambda \left\| \nabla v_{i,j} \right\|^2}_{\text{Spatial smoothness}}$$

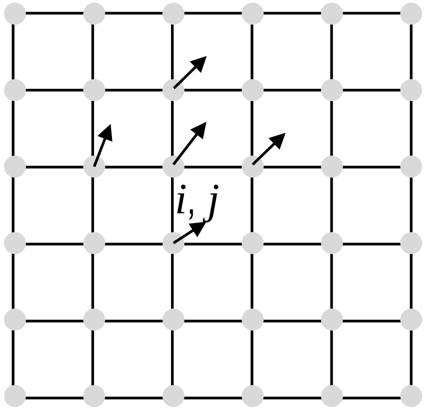
Taking derivative w.r.t. $u_{i,j}, v_{i,j}$

$$I_x \left(I_x u_{i,j} + I_y v_{i,j} + I_t \right) - \lambda \nabla^2 u_{i,j} = 0$$

$$I_y \left(I_x u_{i,j} + I_y v_{i,j} + I_t \right) - \lambda \nabla^2 v_{i,j} = 0$$

Laplace operator: $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \underbrace{\frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} \right) - u_{i,j}}_{\text{Weighted average of neighbors}}$

Dense Optical Flow



Objective:

$$\underset{\{u_{i,j},v_{i,j}\}}{\text{minimize}} \sum_i \sum_j \underbrace{\left(I_x u_{i,j} + I_y v_{i,j} + I_t\right)^2}_{\text{Optical flow eq.}} + \underbrace{\lambda \left\|\nabla u_{i,j}\right\|^2 + \lambda \left\|\nabla v_{i,j}\right\|^2}_{\text{Spatial smoothness}}$$

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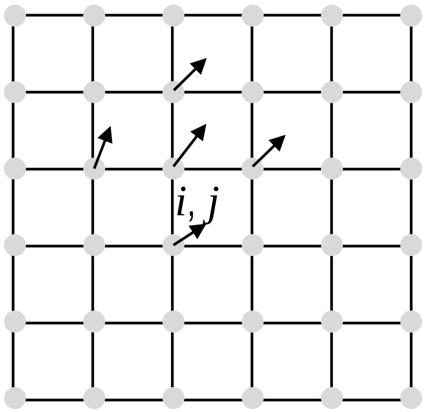
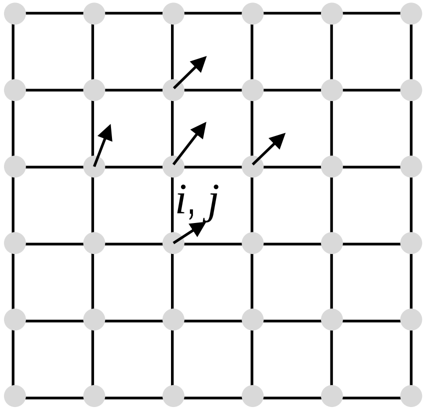
DENSE OPTICAL FLOW



$$I_x\left(I_x u_{i,j} + I_y v_{i,j} + I_t\right) - \lambda(\bar{u}_{i,j} - u_{i,j}) = 0$$

$$I_y\left(I_x u_{i,j} + I_y v_{i,j} + I_t\right) - \lambda(\bar{v}_{i,j} - v_{i,j}) = 0$$

DENSE OPTICAL FLOW



$$I_x\left(I_x u_{i,j} + I_y v_{i,j} + I_t\right) - \lambda(\bar{u}_{i,j} - u_{i,j}) = 0$$

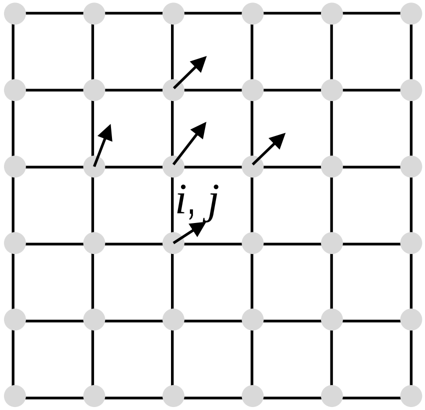
$$I_y\left(I_x u_{i,j} + I_y v_{i,j} + I_t\right) - \lambda(\bar{v}_{i,j} - v_{i,j}) = 0$$

$$\longrightarrow \begin{aligned} \left(I_x^2 + \lambda\right)u_{i,j} + I_x I_y v_{i,j} &= \lambda\bar{u}_{i,j} - I_x I_t \\ I_x I_y u_{i,j} + \left(I_y^2 + \lambda\right)v_{i,j} &= \lambda\bar{v}_{i,j} - I_y I_t \end{aligned}$$

# of unknowns: ?

# of equations: ?

Dense Optical Flow



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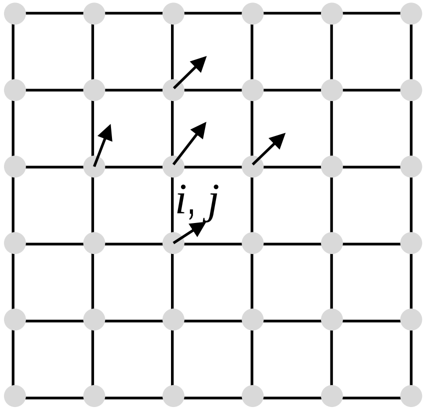
\# of unknowns: ?
\# of equations: ?

Are we done?

Dense Optical Flow



$$I_x\left(I_x u_{i,j} + I_y v_{i,j} + I_t\right) - \lambda(\bar{u}_{i,j} - u_{i,j}) = 0$$

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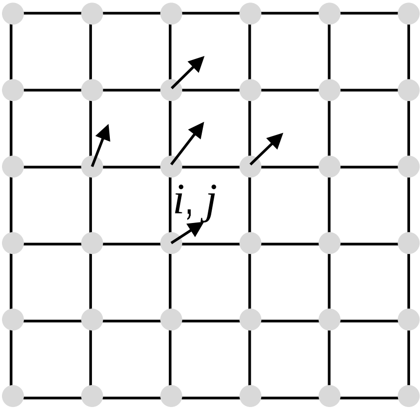
$$\longrightarrow \begin{aligned} \left(I_x^2 + \lambda\right)u_{i,j} + I_x I_y v_{i,j} &= \lambda\boxed{\bar{u}_{i,j}} - I_x I_t \\ I_x I_y u_{i,j} + \left(I_y^2 + \lambda\right)v_{i,j} &= \lambda\boxed{\bar{v}_{i,j}} - I_y I_t \end{aligned}$$

# of unknowns: ?
# of equations: ?

Are we done? No, because the solution $u_{i,j}, v_{i,j}$ depends on neighboring pixels.

$\longrightarrow$ The neighboring pixels need to be updated.

Dense Optical Flow



$$I_x\left(I_x u_{i,j} + I_y v_{i,j} + I_t\right) - \lambda(\bar{u}_{i,j} - u_{i,j}) = 0$$

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$$\longrightarrow \quad \begin{aligned} \left(I_x^2 + \lambda\right)u_{i,j} + I_x I_y v_{i,j} &= \lambda \bar{u}_{i,j} - I_x I_t \\ I_x I_y u_{i,j} + \left(I_y^2 + \lambda\right)v_{i,j} &= \lambda \bar{v}_{i,j} - I_y I_t \end{aligned}$$

$$u_{i,j}^{k+1} = \bar{u}_{i,j}^{k} - \frac{I_x\left(I_x \bar{u}_{i,j}^{k} + I_y \bar{v}_{i,j}^{k} + I_t\right)}{\lambda + I_x^2 + I_y^2}$$

$$\underbrace{v_{i,j}^{k+1}}_{\text{new}} = \underbrace{\bar{v}_{i,j}^{k}}_{\text{old}} - \frac{I_y\left(I_x \bar{u}_{i,j}^{k} + I_y \bar{v}_{i,j}^{k} + I_t\right)}{\lambda + I_x^2 + I_y^2}$$

Update all pixel simultaneously until converges.

https://www.youtube.com/watch?v=sslNeWRb58M

https://www.youtube.com/watch?v=JSzUdVBmQP4

Faculty Candidate Talk (TBA):

Deqing Sun, a.k.a., Master of Optical Flow

https://www.youtube.com/watch?v=0guj3TIdWZk