Compositional Warping

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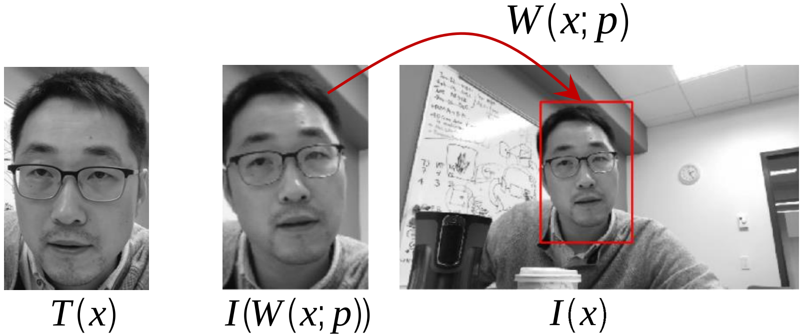
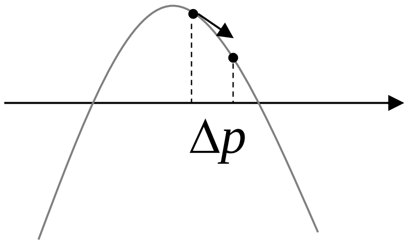
RECALL: IMAGE ALIGNMENT OBJECTIVE





$$p* = \underset{p}{\text{minimize}} \sum_{x} \left(I(W(x;p) - T(x) \right)^2$$

Guass-Newton's method

1.Linearize the obj. function at p

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I \frac{\partial W}{\partial p} \Delta p$$

2. Find Δp that minimizes the obj. function at p

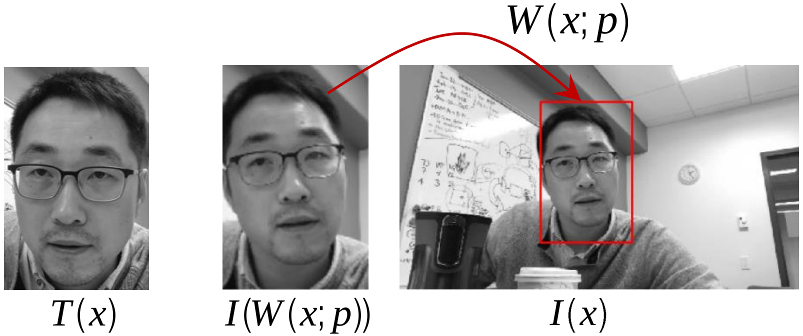
$$I(W(x;p)) + \nabla I \frac{\partial W}{\partial p} \Delta p - T(x) = 0 \longrightarrow \nabla I \frac{\partial W}{\partial p} \Delta p = T(x) - I(W(x;p))$$

$$\Delta p = H^{-1} \sum_{x} \left(\nabla I \frac{\partial W}{\partial p} \right)^T (T(x) - I(W(x;p)))$$

where $H = \sum_{x} \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(\nabla I \frac{\partial W}{\partial p} \right)$

3. Update $p \leftarrow p + \Delta p$

COMPUTATIONAL ASPECT



$$p* = \underset{p}{\text{minimize}} \sum_{x} \left(I(W(x;p) - T(x) \right)^2$$

Guass-Newton's method

1.Linearize the obj. function at p

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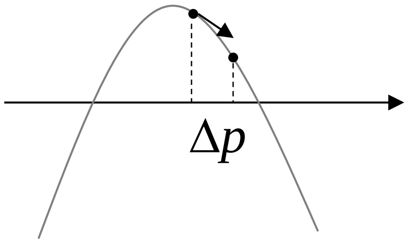


$$I(W(x;p)) + \nabla I \frac{\partial W}{\partial p} \Delta p - T(x) = 0 \longrightarrow \nabla I \frac{\partial W}{\partial p} \Delta p = T(x) - I(W(x;p))$$

$$\boxed{\begin{aligned} &\Delta p = H^{-1} \sum_{x} \left(\nabla I \frac{\partial W}{\partial p} \right)^{T} (T(x) - I(W(x;p))) \\ &\text{where } H = \sum_{x} \left(\nabla I \frac{\partial W}{\partial p} \right)^{T} \left(\nabla I \frac{\partial W}{\partial p} \right) \end{aligned}}$$

3. Update $\boxed{p \leftarrow p + \Delta p}$

LUCAS-KANADE ALGORITHM

1. Warp the target image $I(W(x;p))$

$I(x)$

$I(W(x;p))$

$$\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(T(x) - I(W(x;p)) \right)$$

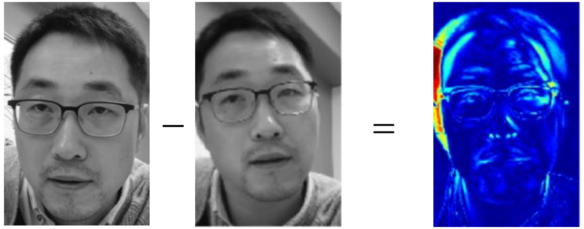
where $H = \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(\nabla I \frac{\partial W}{\partial p} \right)$

$$p \leftarrow p + \Delta p$$

LUCAS-KANADE ALGORITHM

1. Warp the target image $I(W(x;p))$

2. Compute the error image $T(x) - I(W(x;p))$



$T(x)$ $I(W(x;p))$

$$\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T (T(x) - I(W(x;p)))$$

where $H = \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(\nabla I \frac{\partial W}{\partial p} \right)$

$$p \leftarrow p + \Delta p$$

Lucas-Kanade Algorithm

$$\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(T(x) - I(W(x;p)) \right)$$

where $H = \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(\nabla I \frac{\partial W}{\partial p} \right)$

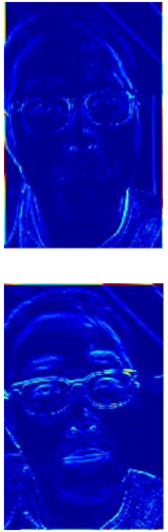
$$p \leftarrow p + \Delta p$$

1. Warp the target image $I(W(x;p))$

2. Compute the error image $T(x) - I(W(x;p))$

3. Warp the gradient image

$\nabla I(x)$



$\nabla I(W(x;p))$

LUCAS-KANADE ALGORITHM

$$\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(T(x) - I(W(x;p)) \right)$$

where $H = \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(\nabla I \frac{\partial W}{\partial p} \right)$

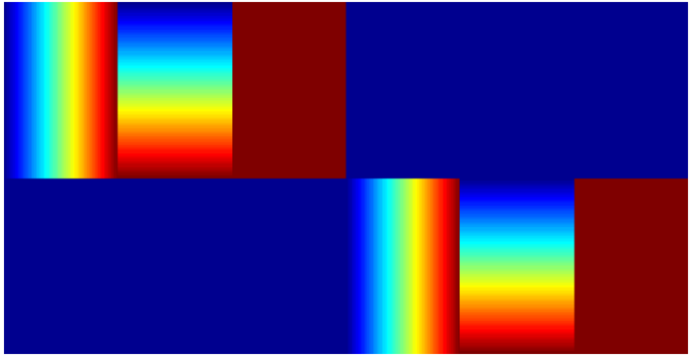
$p \leftarrow p + \Delta p$

1. Warp the target image $I(W(x;p))$

2. Compute the error image $T(x) - I(W(x;p))$

3. Warp the gradient image $\nabla I(W(x;p))$

4. Compute Jacobian $\frac{\partial W}{\partial p}$



Lucas-Kanade Algorithm

$$\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(T(x) - I(W(x;p)) \right)$$

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$$p \leftarrow p + \Delta p$$

1. Warp the target image $I(W(x;p))$
2. Compute the error image $T(x) - I(W(x;p))$
3. Warp the gradient image $\nabla I(W(x;p))$
4. Compute Jacobian $\frac{\partial W}{\partial p}$
5. Compute steepest descent images $\nabla I \frac{\partial W}{\partial p}$

LUCAS-KANADE ALGORITHM

$$\Delta p = H^{-1}\sum_x \left(\nabla I \frac{\partial W}{\partial p}\right)^T (T(x) - I(W(x;p)))$$

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$$p \leftarrow p + \Delta p$$

1. Warp the target image $I(W(x;p))$

2. Compute the error image $T(x) - I(W(x;p))$

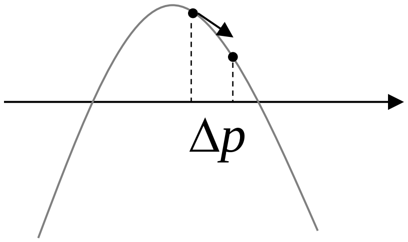
3. Warp the gradient image $\nabla I(W(x;p))$

4. Compute Jacobian $\frac{\partial W}{\partial p}$

5. Compute steepest descent images $\nabla I \frac{\partial W}{\partial p}$

6. Compute Hessian $H = \sum_x \left(\nabla I \frac{\partial W}{\partial p}\right)^T \left(\nabla I \frac{\partial W}{\partial p}\right)$

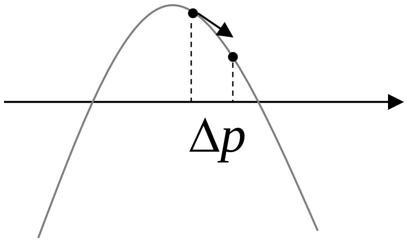
7. Compute $\Delta p = H^{-1}\sum_x \left(\nabla I \frac{\partial W}{\partial p}\right)^T (T(x) - I(W(x;p)))$



LUCAS-KANADE ALGORITHM

1. Warp the target image $I(W(x;p))$
2. Compute the error image $T(x) - I(W(x;p))$
3. Warp the gradient image $\nabla I(W(x;p))$
4. Compute Jacobian $\frac{\partial W}{\partial p}$
5. Compute steepest descent images $\nabla I \frac{\partial W}{\partial p}$
6. Compute Hessian $H = \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(\nabla I \frac{\partial W}{\partial p} \right)$
7. Compute $\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T (T(x) - I(W(x;p)))$
8. Update $p \leftarrow p + \Delta p$
9. Goto 1 unless $\|\Delta p\| < \varepsilon$

$$\boxed{\begin{array}{l} \Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T (T(x) - I(W(x;p))) \\ \text{where } H = \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(\nabla I \frac{\partial W}{\partial p} \right) \\ p \leftarrow p + \Delta p \end{array}}$$



COMPUTATION BOTTLENECK

1. Warp the target image $I(W(x;p))$

2. Compute the error image $T(x) - I(W(x;p))$

3. Warp the gradient image $\nabla I(W(x;p))$

4. Compute Jacobian $\frac{\partial W}{\partial p}$

5. Compute steepest descent images $\nabla I \frac{\partial W}{\partial p}$

6. Compute Hessian $H = \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(\nabla I \frac{\partial W}{\partial p} \right)$

7. Compute $\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T (T(x) - I(W(x;p)))$

8. Update $p \leftarrow p + \Delta p$

9. Goto 1 unless $\|\Delta p\| < \varepsilon$

Additive vs. Compositional

Additive mapping

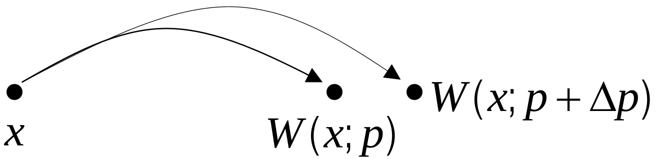
$$I(W(x;p+\Delta p)) \approx I(W(x;p))$$

$$\begin{tikzcd} x \arrow[r, bend left] & W(x;p) \end{tikzcd}$$

ADDITIVE VS. COMPOSITIONAL

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\Big|_p \frac{\partial W}{\partial p}\Big|_p \Delta p$$



Additive vs. Compositional

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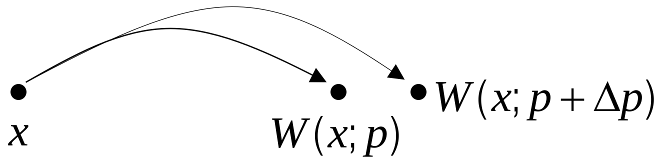


Compositional mapping I

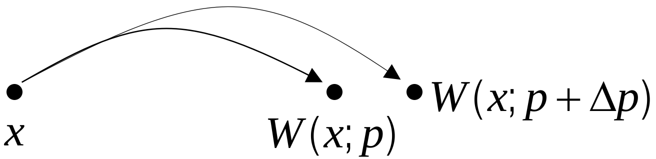
$$I(W(W(x;p);\Delta p))$$

Note) $\left(f \circ g\right)' = \left(f' \circ g\right) g'$

ADDITIVE VS. COMPOSITIONAL

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\Big|_p \frac{\partial W}{\partial p}\bigg|_p \Delta p$$



Compositional mapping I

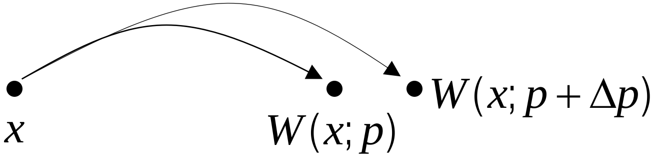
$$I(W(W(x;p);\Delta p)) \approx I(W(W(x;p);0)) + \nabla I(W)\frac{\partial W}{\partial p}\bigg|_p \Delta p$$

Note) $(f \circ g)' = \left(f' \circ g\right)g'$

Additive vs. Compositional

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\Big|_p \frac{\partial W}{\partial p}\Big|_p \Delta p$$



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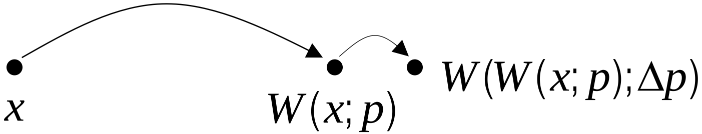
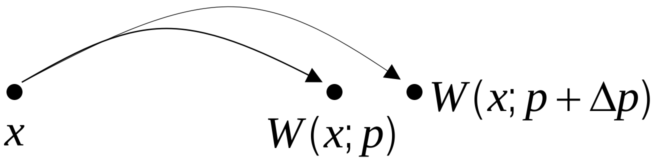


Note) $(f \circ g)' = \left(f' \circ g\right)g'$

ADDITIVE VS. COMPOSITIONAL

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{p} \Delta p$$



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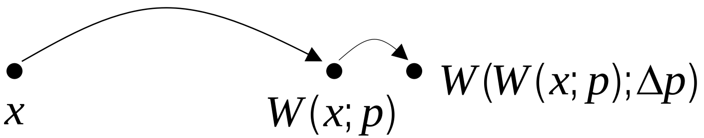
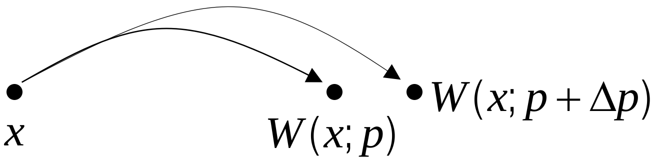


Note) $(f \circ g)' = \left(f' \circ g\right) g'$ and $W(x;0) = x$

ADDITIVE VS. COMPOSITIONAL

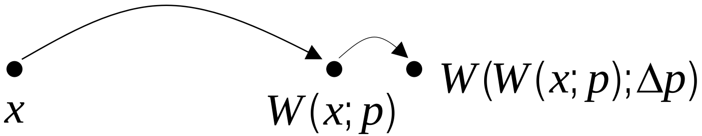
Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{p} \Delta p$$



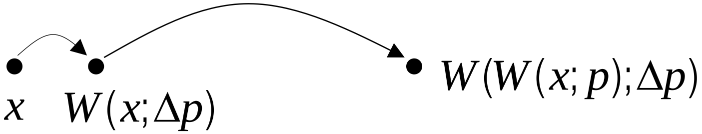
Compositional mapping I

$$I(W(W(x;p);\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{p} \Delta p$$



Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(W(x;0);p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{0} \Delta p$$

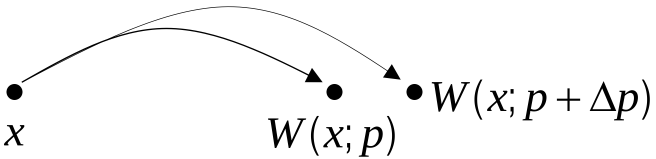


Note) $\left(f \circ g\right)' = \left(f' \circ g\right)g'$ and $W(x;0) = x$

ADDITIVE VS. COMPOSITIONAL

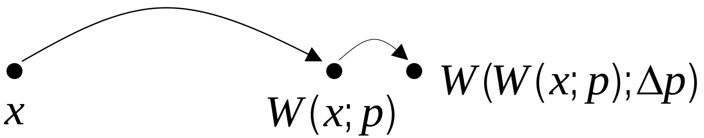
Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{p} \Delta p$$



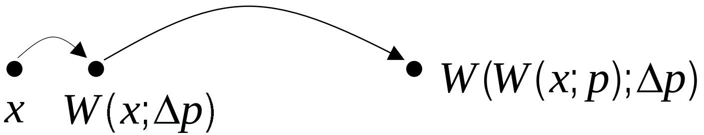
Compositional mapping I

$$I(W(W(x;p);\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{p} \Delta p$$



Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{0} \Delta p$$



Note) $\left(f \circ g\right)' = \left(f' \circ g\right)g'$ and $W(x;0) = x$

Additive vs. Compositional

Location of linearization

Additive mapping

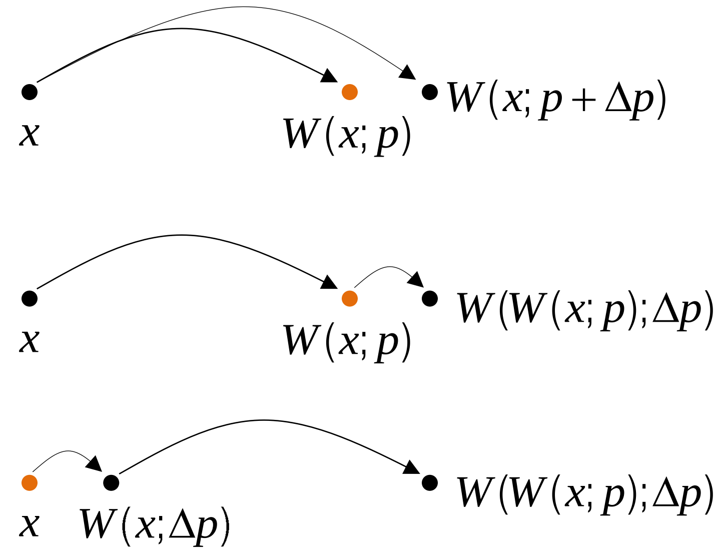
$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{p} \Delta p$$

Compositional mapping I

$$I(W(W(x;p);\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{p} \Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\underbrace{\boxed{\left.\frac{\partial W}{\partial p}\right|_{0}}}_{\text{constant}} \Delta p$$



Note) $(f \circ g)' = (f' \circ g)g'$ and $W(x;0) = x$

$\left.\frac{\partial W}{\partial p}\right|_{0}$ is constant

Additive vs. Compositional

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_p \Delta p$$

Compositional mapping I

$$I(W(W(x;p);\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_p \Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\boxed{\left.\frac{\partial W}{\partial p}\right|_0} \Delta p$$

constant

Note) $(f \circ g)' = (f' \circ g)g'$ and $W(x;0) = x$

1. Warp the target image $I(W(x;p))$
2. Compute the error image $T(x) - I(W(x;p))$
3. Warp the gradient image $\nabla I(W(x;p))$
4. Compute Jacobian $\frac{\partial W}{\partial p}$
5. Compute steepest descent images $\nabla I \frac{\partial W}{\partial p}$
6. Compute Hessian $H = \sum_x \left(\nabla I \frac{\partial W}{\partial p}\right)^T \left(\nabla I \frac{\partial W}{\partial p}\right)$
7. Compute $\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p}\right)^T (T(x) - I(W(x;p)))$
8. Update $p \leftarrow p + \Delta p$
9. Goto 1 unless $\|\Delta p\| < \varepsilon$

Additive vs. Compositional

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W) \left.\frac{\partial W}{\partial p}\right|_p \Delta p$$

Compositional mapping I

$$I(W(W(x;p);\Delta p)) \approx I(W(x;p)) + \nabla I(W) \left.\frac{\partial W}{\partial p}\right|_p \Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W) \boxed{\left.\frac{\partial W}{\partial p}\right|_0} \Delta p$$

constant

Note) $(f \circ g)' = (f' \circ g) g'$ and $W(x;0) = x$

1. Warp the target image $I(W(x;p))$
2. Compute the error image $T(x) - I(W(x;p))$
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8. Update $p \leftarrow p + \Delta p$
9. Goto 1 unless $\|\Delta p\| < \varepsilon$

Additive vs. Compositional

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{p}\Delta p \qquad W(x;p+\Delta p) = \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} p_1+\Delta p_1 & p_2+\Delta p_2 & p_3+\Delta p_3 \\ p_4+\Delta p_4 & p_5+\Delta p_5 & p_6+\Delta p_6 \end{bmatrix}\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Compositional mapping I

$$I(W(W(x;p);\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{p}\Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\boxed{\left.\frac{\partial W}{\partial p}\right|_{0}}\Delta p$$

constant

Note) $(f \circ g)' = \left(f' \circ g\right)g'$ and $W(x;0) = x$

ADDITIVE VS. COMPOSITIONAL

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{p} \Delta p$$

$$W(x;p+\Delta p) = \begin{bmatrix} 1+p_1+\Delta p_1 & p_2+\Delta p_2 & p_3+\Delta p_3 \\ p_4+\Delta p_4 & 1+p_5+\Delta p_5 & p_6+\Delta p_6 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Compositional mapping I

$$I(W(W(x;p);\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_{p} \Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\underbrace{\boxed{\left.\frac{\partial W}{\partial p}\right|_{0}}}_{\text{constant}} \Delta p$$

Note) $(f \circ g)' = (f' \circ g)g'$ and $W(x;0) = x$

Additive vs. Compositional

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_p \Delta p$$

Compositional mapping I

$$I(W(W(x;p);\Delta p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_p \Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\boxed{\left.\frac{\partial W}{\partial p}\right|_0} \Delta p$$

constant

Note) $(f \circ g)' = \left(f' \circ g\right)g'$ and $W(x;0) = x$

Abuse of notation

$$W(x;p+\Delta p) = \begin{bmatrix} 1+p_1+\Delta p_1 & p_2+\Delta p_2 & p_3+\Delta p_3 \\ p_4+\Delta p_4 & 1+p_5+\Delta p_5 & p_6+\Delta p_6 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Additive vs. Compositional

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W) \left.\frac{\partial W}{\partial p}\right|_p \Delta p$$

Abuse of notation

$$W(x;p+\Delta p) = \quad A(p+\Delta p)\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Compositional mapping I

$$I(W(W(x;p);\Delta p)) \approx I(W(x;p)) + \nabla I(W) \left.\frac{\partial W}{\partial p}\right|_p \Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W) \boxed{\left.\frac{\partial W}{\partial p}\right|_0} \Delta p$$

constant

$$W(W(x;\Delta p);p) = A(p)A(\Delta p)\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Note) $\left(f \circ g\right)' = \left(f' \circ g\right) g'$ and $W(x;0) = x$

Lucas-Kanade

1. Warp the target image $I(W(x;p))$
2. Compute the error image $T(x) - I(W(x;p))$
3. Warp the gradient image $\nabla I(W(x;p))$
4. Compute Jacobian $\frac{\partial W}{\partial p}$
5. Compute steepest descent images $\nabla I \frac{\partial W}{\partial p}$
6. Compute Hessian $H = \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(\nabla I \frac{\partial W}{\partial p} \right)$
7. Compute $\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T (T(x) - I(W(x;p)))$
8. Update $p \leftarrow p + \Delta p$
9. Goto 1 unless $\|\Delta p\| < \varepsilon$

Compositional Alignment

1. Compute Jacobian $\frac{\partial W}{\partial p}$

---

2. Warp the target image $I(W(x;p))$
3. Compute the error image $T(x) - I(W(x;p))$
4. Warp the gradient image $\nabla I(W(x;p))$
5. Compute steepest descent images $\nabla I \frac{\partial W}{\partial p}$
6. Compute Hessian $H = \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(\nabla I \frac{\partial W}{\partial p} \right)$
7. Compute $\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T (T(x) - I(W(x;p)))$
8. Update $W(x;p) \leftarrow W(x;p) \circ W(x;\Delta p)$
9. Goto 2 unless $\|\Delta p\| < \varepsilon$

DUALITY

$$p* = \underset{p}{\text{minimize}} \sum_{x} \left(I(W(x;p) - T(x) \right)^{2}$$

$T(x)$

$I(W(x;p))$

$$p* = \underset{p}{\text{minimize}} \sum_{x} \left(I(x) - T(W^{-1}(x;p)) \right)^{2}$$

$T(W^{-1}(x;p))$

$I(x)$

Inverse Compositional

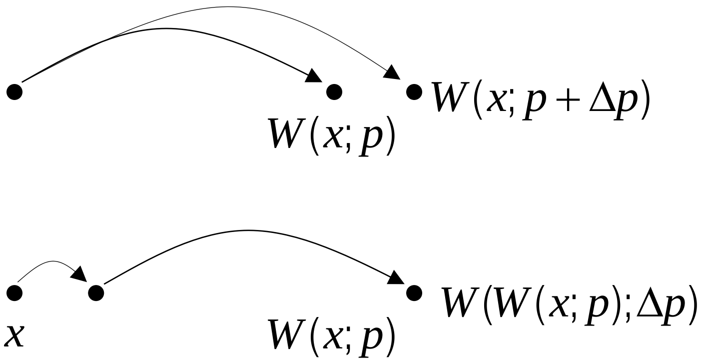
- Location of linearization

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\frac{\partial W}{\partial p}\Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p$$



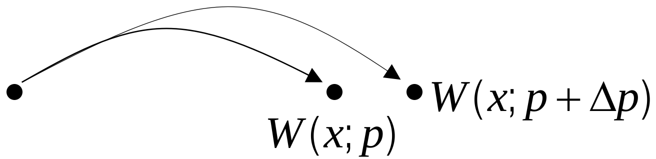
Note) $(f \circ g)' = (f' \circ g)g'$ and $W(x;0) = x$

INVERSE COMPOSITIONAL

- Location of linearization

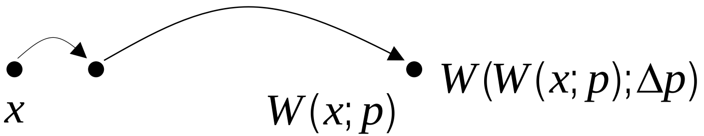
Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\frac{\partial W}{\partial p}\Delta p$$



Compositional mapping II

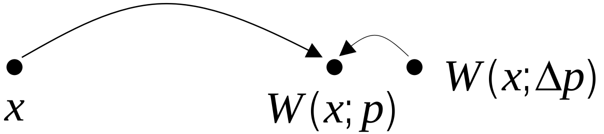
$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p$$



Inverse compositional mapping

$$I(W(x;p))$$

$$T(W(x;\Delta p)) \approx T(W(x;0)) + \nabla T(x)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p$$



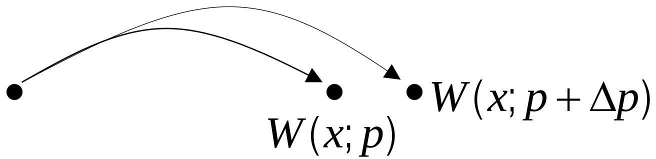
Note) $(f\circ g)' = \left(f'\circ g\right)g'$ and $W(x;0)=x$

Inverse Compositional

● Location of linearization

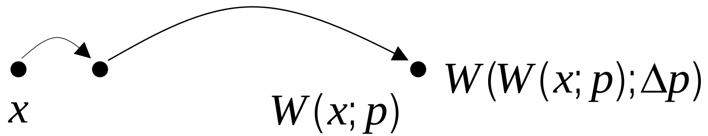
Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\frac{\partial W}{\partial p}\Delta p$$



Compositional mapping II

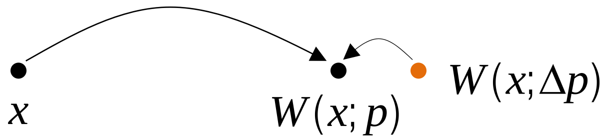
$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p$$



Inverse compositional mapping

$$I(W(x;p))$$

$$T(W(x;\Delta p)) \approx T(x) + \nabla T(x)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p$$



Note) $(f \circ g)' = \left(f' \circ g\right)g'$ and $W(x;0) = x$

DUALITY

$$p^* = \underset{p}{\text{minimize}} \sum_x \left(I(W(x;p) - T(x) \right)^2$$

$T(x)$

$I(W(x;p))$

$$p^* = \underset{p}{\text{minimize}} \sum_x \left(I(W(x;p)) - T(W(x;\Delta p)) \right)^2$$

$T(W(x;\Delta p))$

$I(W(x;p))$

Inverse Compositional

Location of linearization

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\frac{\partial W}{\partial p}\Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p$$

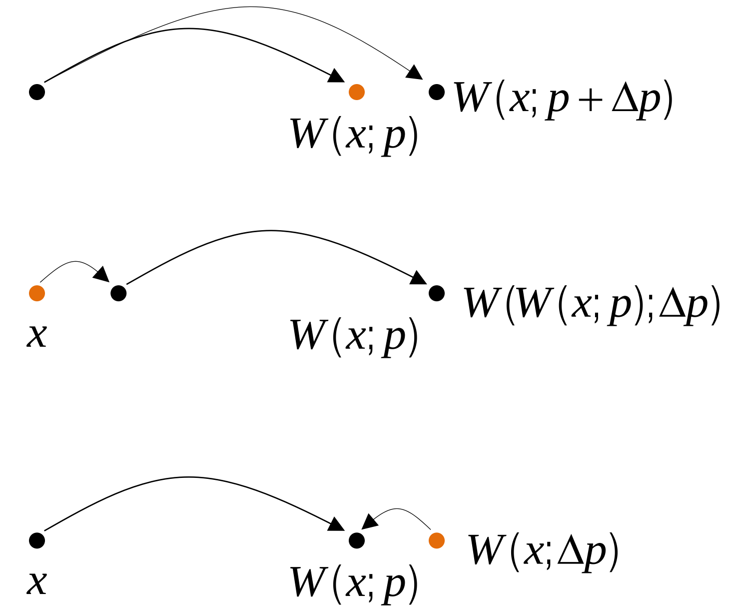
Inverse compositional mapping

$$I(W(x;p))$$

constant

$$T(W(x;\Delta p)) \approx T(x) + \nabla T(x)\boxed{\left.\frac{\partial W}{\partial p}\right|_0}\Delta p$$

Note) $(f \circ g)' = (f' \circ g)g'$ and $W(x;0) = x$



Inverse Compositional

● Location of linearization

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\frac{\partial W}{\partial p}\Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p$$

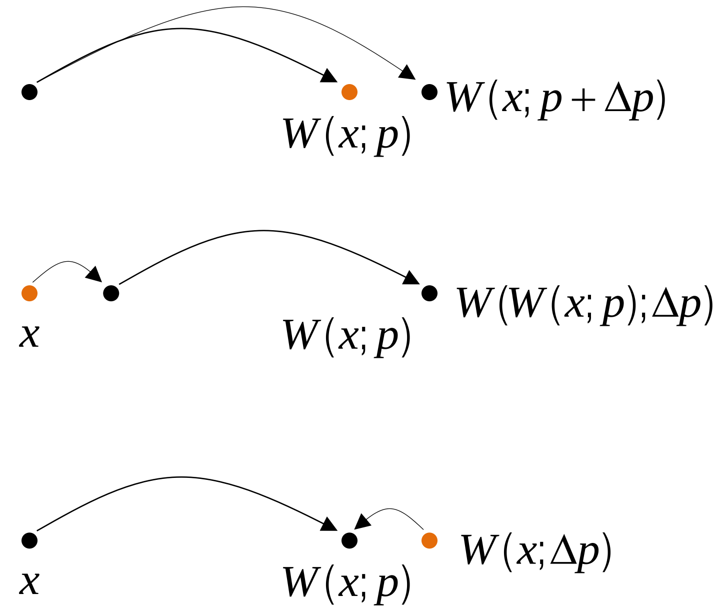
Inverse compositional mapping

$$I(W(x;p))$$

constant

$$T(W(x;\Delta p)) \approx \boxed{T(x) + \nabla T(x)\left.\frac{\partial W}{\partial p}\right|_0}\Delta p$$

Note) $(f \circ g)' = (f' \circ g)g'$ and $W(x;0) = x$



Inverse Compositional

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\frac{\partial W}{\partial p}\Delta p \qquad W(x;p+\Delta p) = A(p+\Delta p)\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p \qquad W(x;p)\circ W(x;\Delta p) = A(p)A(\Delta p)\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Inverse compositional mapping

$$I(W(x;p))$$

$$T(W(x;\Delta p)) \approx \underbrace{\boxed{T(x) + \nabla T(x)\left.\frac{\partial W}{\partial p}\right|_0}}_{\text{constant}} \Delta p$$

Note) $(f \circ g)' = \left(f' \circ g\right) g'$ and $W(x;0) = x$

Inverse Compositional

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\frac{\partial W}{\partial p}\Delta p \qquad W(x;p+\Delta p) = A(p+\Delta p)\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p \qquad W(x;p)\circ W(x;\Delta p) = A(p)A(\Delta p)\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Inverse compositional mapping

$$I(W(x;p))$$

constant

$$T(W(x;\Delta p)) \approx \boxed{T(x) + \nabla T(x)\left.\frac{\partial W}{\partial p}\right|_0} \Delta p \qquad W(x;p)\circ W^{-1}(x;\Delta p) = A(p)A^{-1}(\Delta p)\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Note) $(f\circ g)' = (f'\circ g)g'$ and $W(x;0) = x$

INVERSE COMPOSITIONAL

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\frac{\partial W}{\partial p}\Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p$$

Inverse compositional mapping

$$I(W(x;p))$$

constant

$$T(W(x;\Delta p)) \approx \boxed{T(x) + \nabla T(x)\left.\frac{\partial W}{\partial p}\right|_0}\,\Delta p$$

Note) $(f \circ g)' = (f' \circ g)g'$ and $W(x;0) = x$

1. Compute Jacobian $\frac{\partial W}{\partial p}$

---

2. Warp the target image $I(W(x;p))$
3. Compute the error image $T(x) - I(W(x;p))$
4. Warp the gradient image $\nabla I(W(x;p))$
5. Compute steepest descent images $\nabla I\frac{\partial W}{\partial p}$
6. Compute Hessian $H = \sum_x \left(\nabla I\frac{\partial W}{\partial p}\right)^T \left(\nabla I\frac{\partial W}{\partial p}\right)$
7. Compute $\Delta p = H^{-1}\sum_x \left(\nabla I\frac{\partial W}{\partial p}\right)^T (T(x) - I(W(x;p)))$
8. Update $W(x;p) \leftarrow W(x;p) \circ W(x;\Delta p)$
9. Goto 2 unless $\|\Delta p\| < \varepsilon$

INVERSE COMPOSITIONAL

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\frac{\partial W}{\partial p}\Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p$$

Inverse compositional mapping

$$I(W(x;p))$$

constant

$$T(W(x;\Delta p)) \approx \boxed{T(x) + \nabla T(x)\left.\frac{\partial W}{\partial p}\right|_0} \Delta p$$

Note) $(f \circ g)' = \left(f' \circ g\right) g'$ and $W(x;0) = x$

1. Compute Jacobian $\frac{\partial W}{\partial p}$

---

2. Warp the target image $I(W(x;p))$
3. Compute the error image $T(x) - I(W(x;p))$
4. The gradient image $\nabla T(x)$
5. Compute steepest descent images $\nabla T\frac{\partial W}{\partial p}$
6. Compute Hessian $H = \sum_x \left(\nabla T\frac{\partial W}{\partial p}\right)^T \left(\nabla T\frac{\partial W}{\partial p}\right)$
7. Compute $\Delta p = H^{-1}\sum_x \left(\nabla T\frac{\partial W}{\partial p}\right)^T (T(x) - I(W(x;p)))$
8. Update $W(x;p) \leftarrow W(x;p) \circ W^{-1}(x;\Delta p)$
9. Goto 2 unless $\|\Delta p\| < \varepsilon$

Inverse Compositional

Additive mapping

$$I(W(x;p+\Delta p)) \approx I(W(x;p)) + \nabla I(W)\frac{\partial W}{\partial p}\Delta p$$

Compositional mapping II

$$I(W(W(x;\Delta p);p)) \approx I(W(x;p)) + \nabla I(W)\left.\frac{\partial W}{\partial p}\right|_0 \Delta p$$

Inverse compositional mapping

$$I(W(x;p))$$

constant

$$T(W(x;\Delta p)) \approx \boxed{T(x) + \nabla T(x)\left.\frac{\partial W}{\partial p}\right|_0}\Delta p$$

Note) $(f \circ g)' = (f' \circ g)g'$ and $W(x;0) = x$

1. Compute Jacobian $\frac{\partial W}{\partial p}$

---

2. Warp the target image $I(W(x;p))$
3. Compute the error image $T(x) - I(W(x;p))$
4. The gradient image $\nabla T(x)$
5. Compute steepest descent images $\nabla T \frac{\partial W}{\partial p}$
6. Compute Hessian $H = \sum_x \left(\nabla T \frac{\partial W}{\partial p}\right)^T \left(\nabla T \frac{\partial W}{\partial p}\right)$
7. Compute $\Delta p = H^{-1} \sum_x \left(\nabla T \frac{\partial W}{\partial p}\right)^T (T(x) - I(W(x;p)))$
8. Update $W(x;p) \leftarrow W(x;p) \circ W^{-1}(x;\Delta p)$
9. Goto 2 unless $\|\Delta p\| < \varepsilon$

Compositional Alignment

1. Compute Jacobian $\frac{\partial W}{\partial p}$

---

2. Warp the target image $I(W(x;p))$

3. Compute the error image $T(x) - I(W(x;p))$

4. Warp the gradient image $\nabla I(W(x;p))$

5. Compute steepest descent images $\nabla I \frac{\partial W}{\partial p}$

6. Compute Hessian $H = \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T \left(\nabla I \frac{\partial W}{\partial p} \right)$

7. Compute $\Delta p = H^{-1} \sum_x \left(\nabla I \frac{\partial W}{\partial p} \right)^T (T(x) - I(W(x;p)))$

8. Update $W(x;p) \leftarrow W(x;p) \circ W(x;\Delta p)$

9. Goto 2 unless $\|\Delta p\| < \varepsilon$

Inv. Compositional Alignment

1. Compute Jacobian $\frac{\partial W}{\partial p}$

4. Warp the gradient image $\nabla T(x)$

5. Compute steepest descent images $\nabla T \frac{\partial W}{\partial p}$

6. Compute Hessian $H = \sum_x \left(\nabla T \frac{\partial W}{\partial p} \right)^T \left(\nabla T \frac{\partial W}{\partial p} \right)$

---

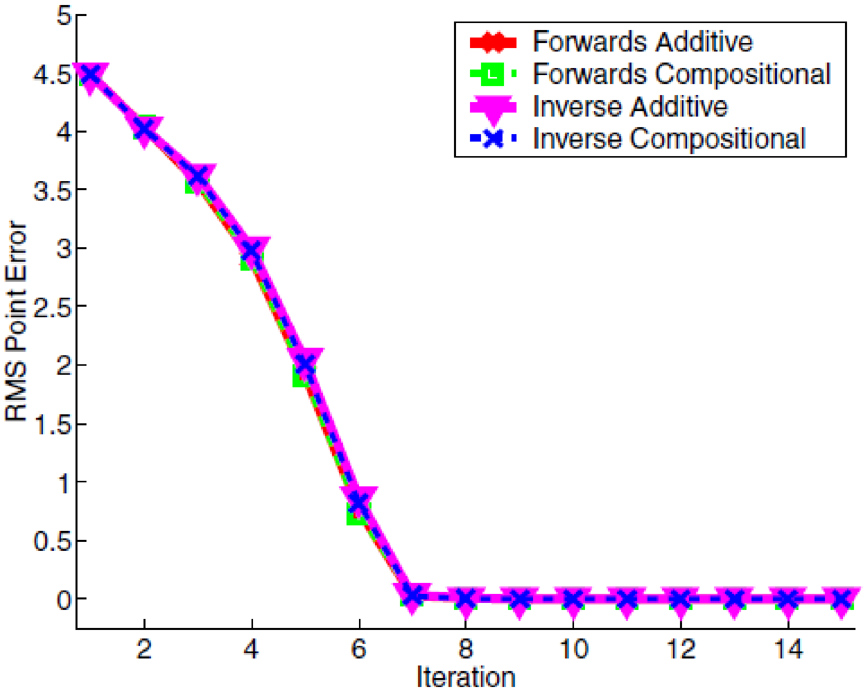
2. Warp the target image $I(W(x;p))$

3. Compute the error image $T(x) - I(W(x;p))$

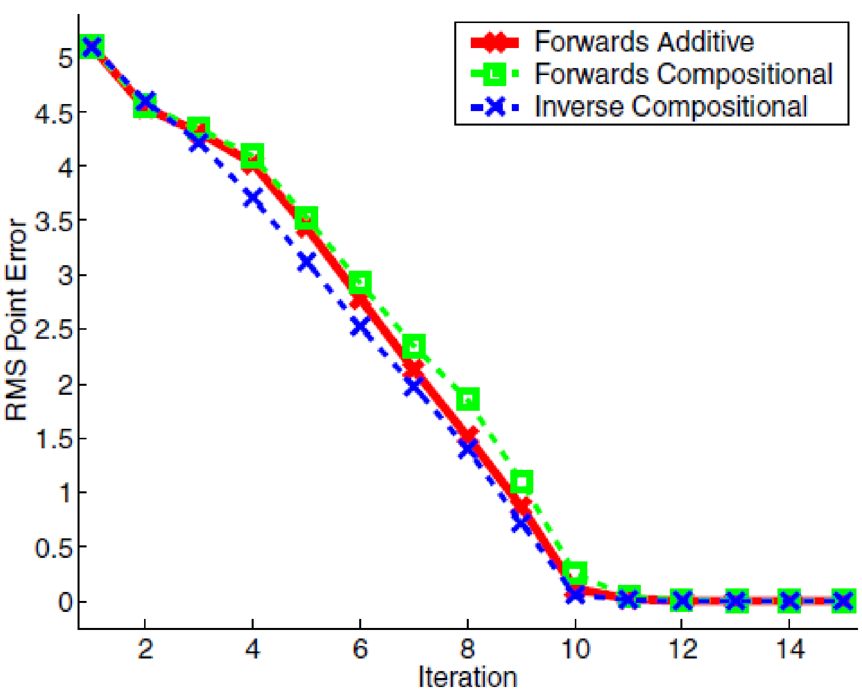
7. Compute $\Delta p = H^{-1} \sum_x \left(\nabla T \frac{\partial W}{\partial p} \right)^T (T(x) - I(W(x;p)))$

8. Update $W(x;p) \leftarrow W(x;p) \circ W^{-1}(x;\Delta p)$

9. Goto 2 unless $\|\Delta p\| < \varepsilon$



(e) Example Convergence for an Affine Warp



(f) Example Convergence for a Homography

Table 5. Timing results for our Matlab implementation of the four algorithms in milliseconds. These results are for the 6-parameter affine warp using a 100×100 pixel template on a 933 MHz Pentium-IV.

	Step 3	Step 4	Step 5	Step 6	Total
Pre-computation:					
Forwards Additive (FA)	–	–	–	–	0.0
Forwards Compositional (FC)	–	17.4	–	–	17.4
Inverse Additive (IA)	8.30	17.1	27.5	37.0	89.9
Inverse Compositional (IC)	8.31	17.1	27.5	37.0	90.0

	Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	Step 8	Step 9	Total
Per iteration:										
FA	1.88	0.740	36.1	17.4	27.7	37.2	6.32	0.111	0.108	127
FC	1.88	0.736	8.17	–	27.6	37.0	6.03	0.106	0.253	81.7
IA	1.79	0.688	–	–	–	–	6.22	0.106	0.624	9.43
IC	1.79	0.687	–	–	–	–	6.22	0.106	0.409	9.21

$I(x)$

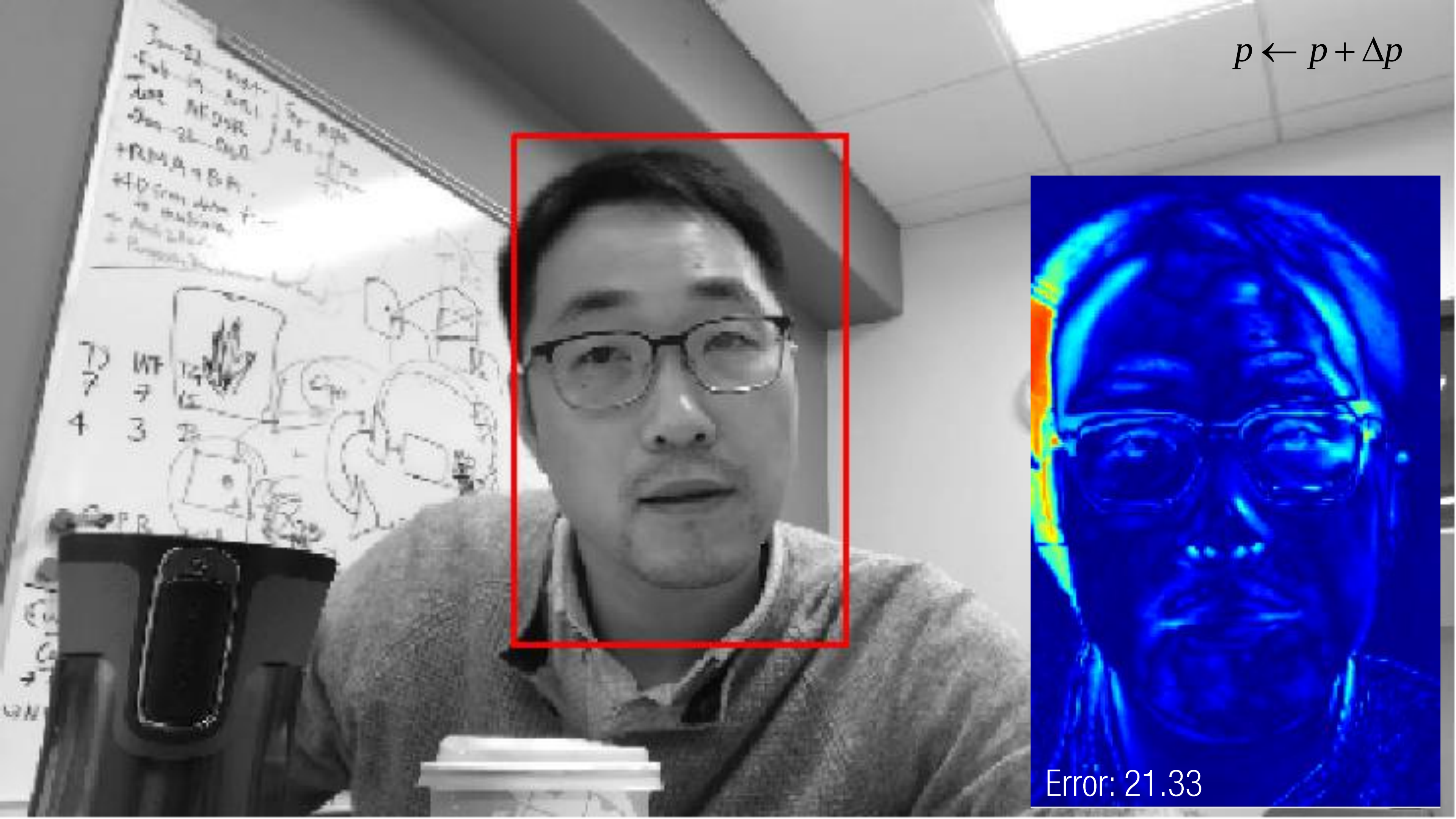
$T(x)$

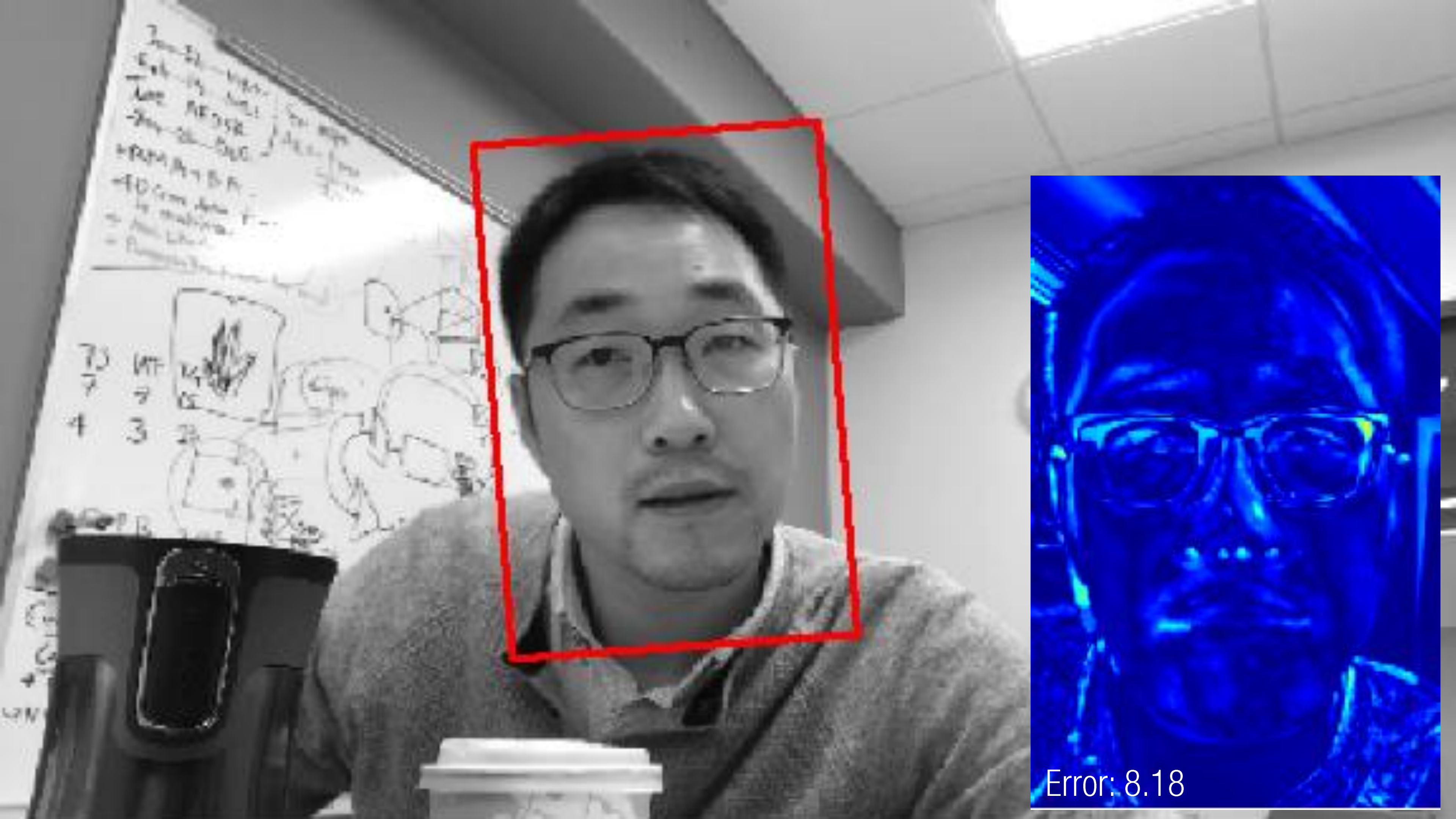
$I(W(x;p))$

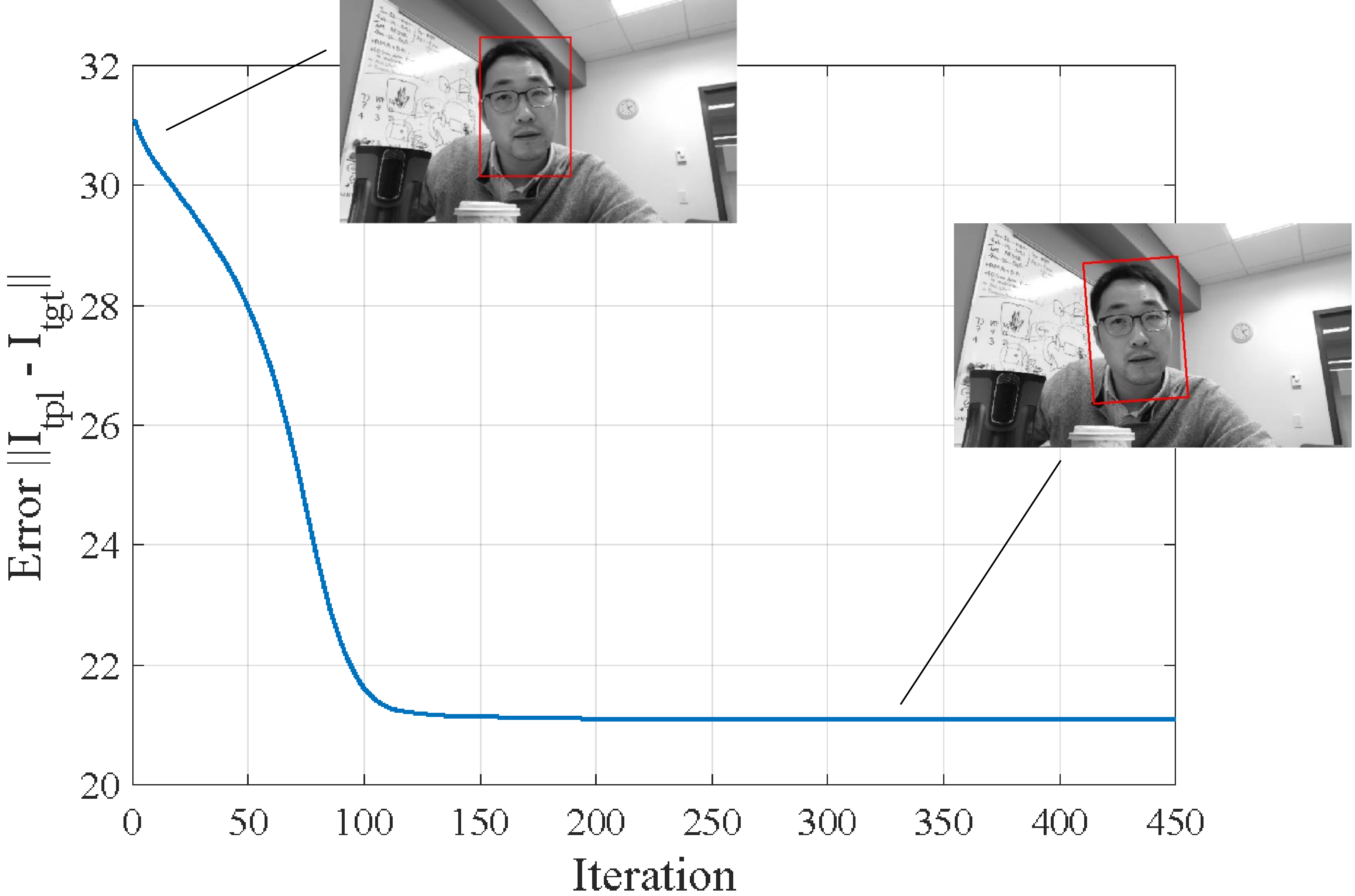
$0.5T(x) + 0.5I(W(x;p))$

$\left|T(x) - I(W(x;p))\right|$

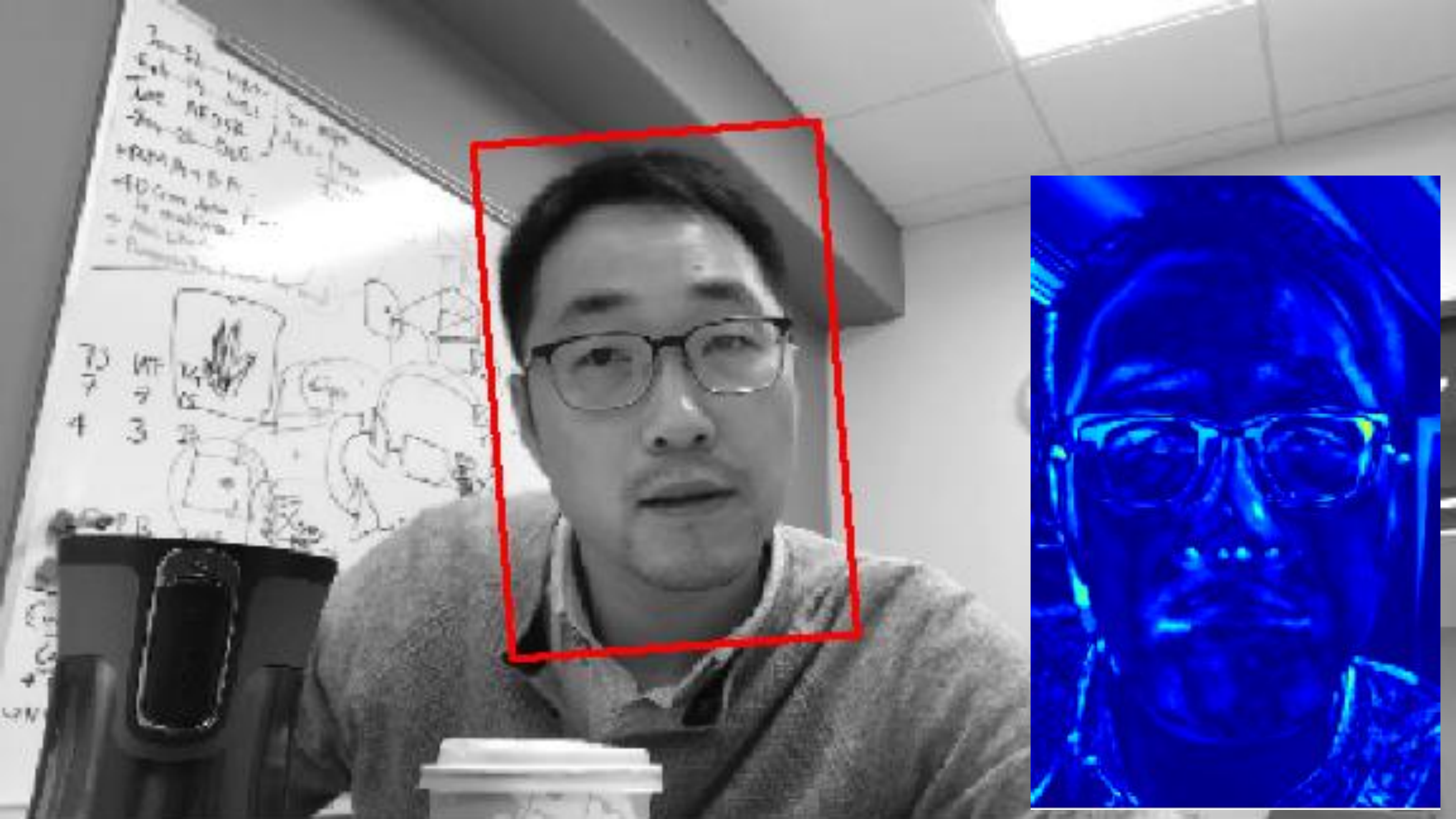
$p \leftarrow p + \Delta p$

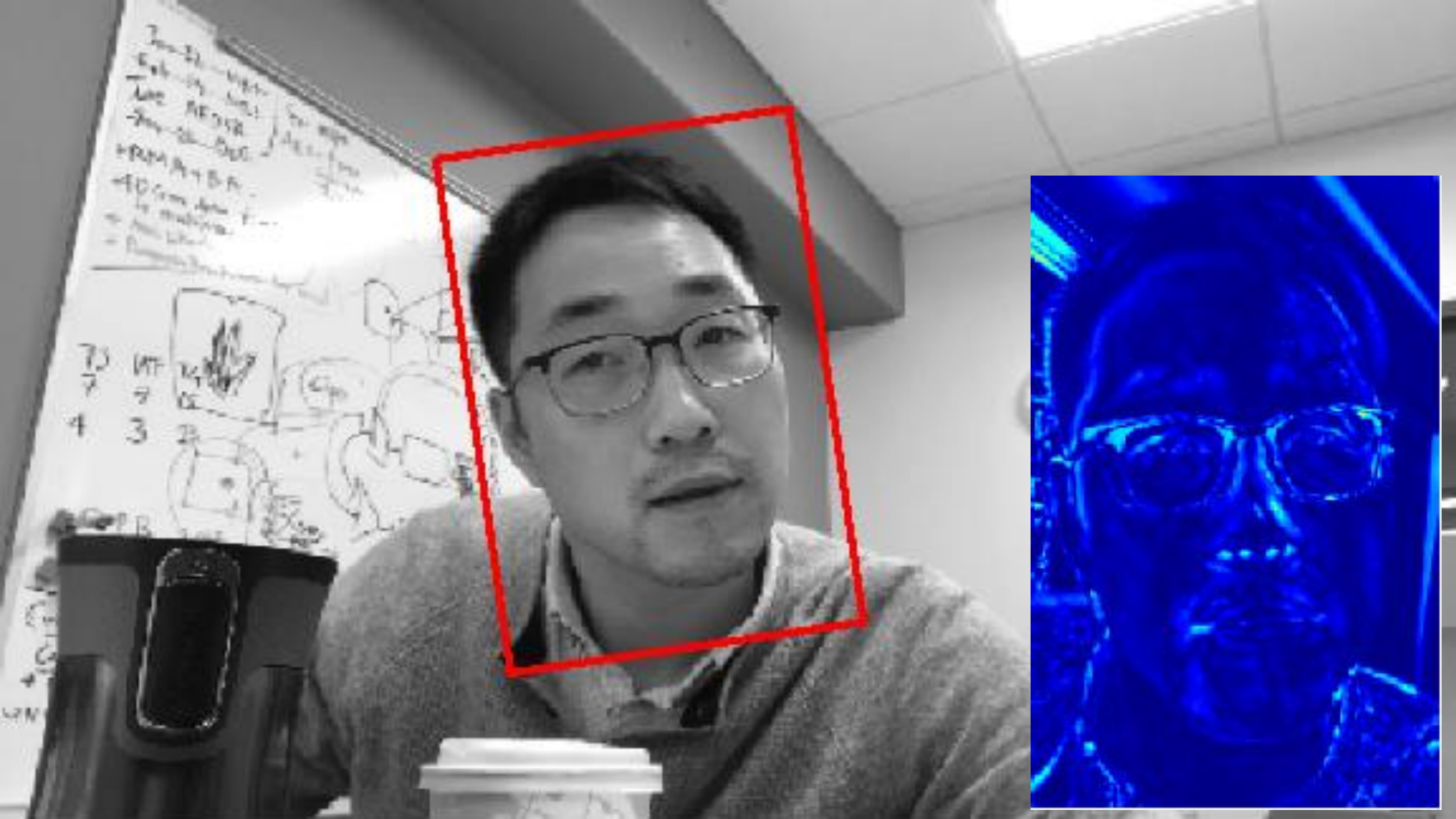


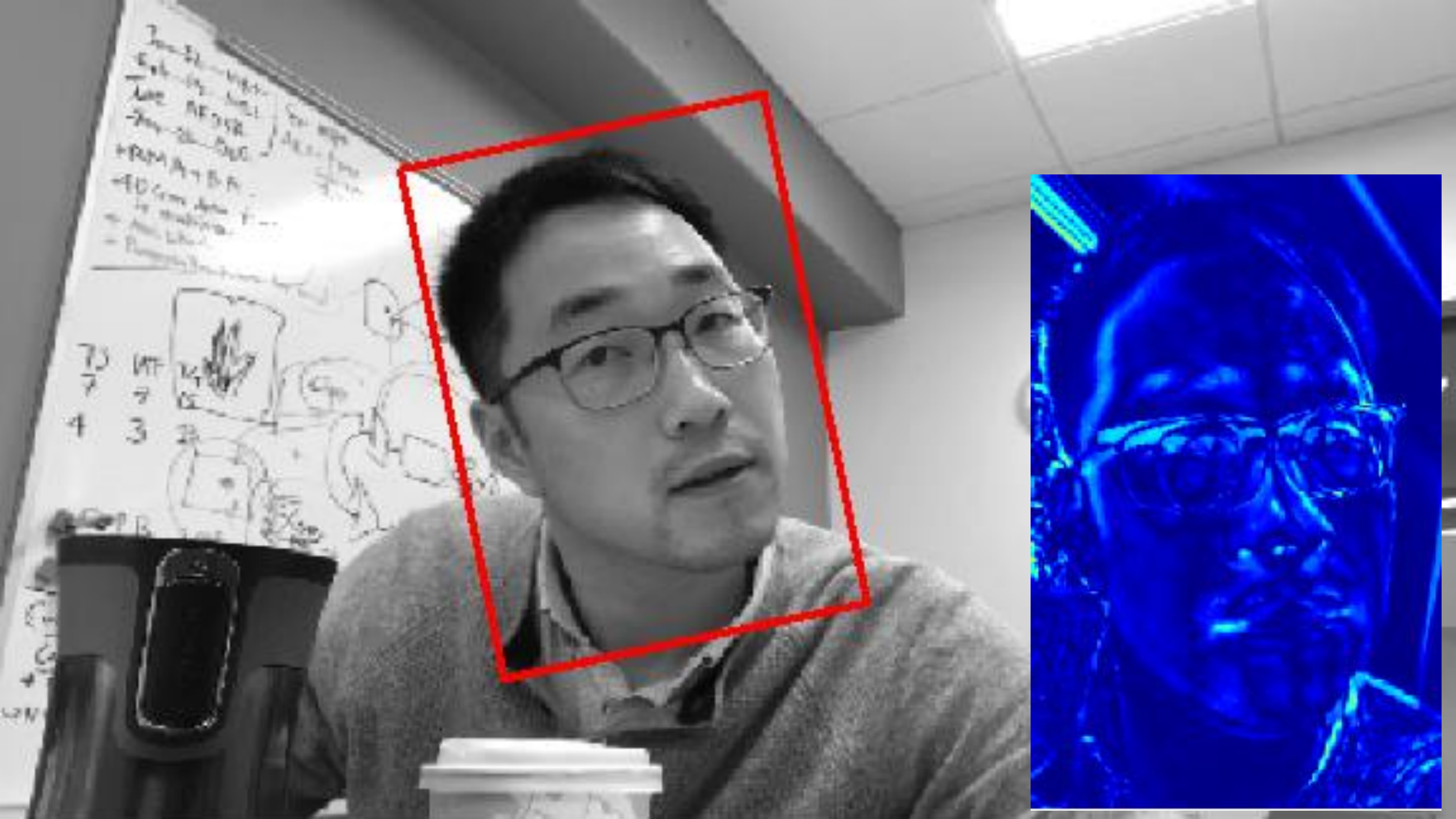

Error: 8.18



32
30
28
26
24
22
20
0
50
100
150
200
250
300
350
400
450
Iteration
Error $||I_{tpl} - I_{tgt}||$







https://www.youtube.com/watch?v=qeZ7H40BQv4