

# *DENSE OPTICAL FLOW*

HYUN SOO PARK







Good features to track



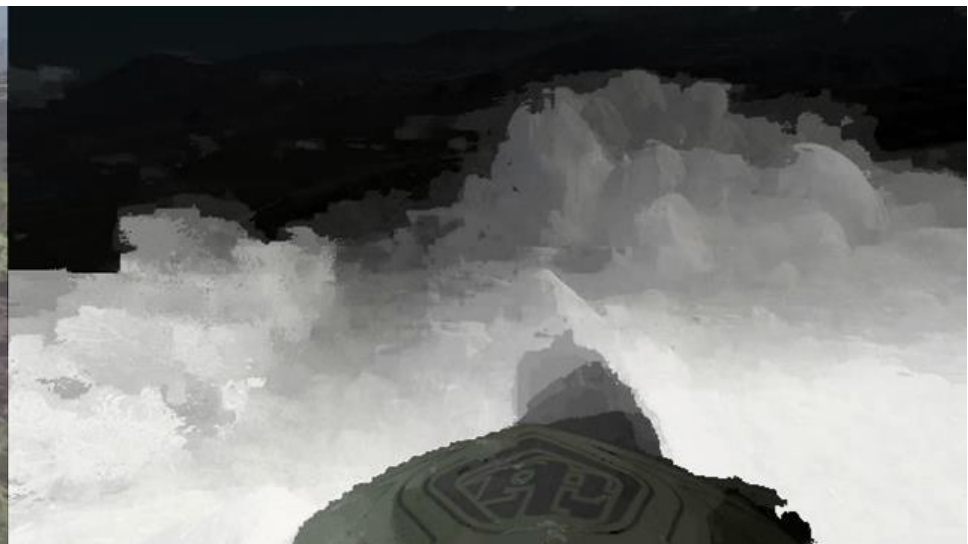
Weak feature to track



Weak feature to track



# ***SPATIAL SMOOTHNESS OF OPTICAL FLOW***

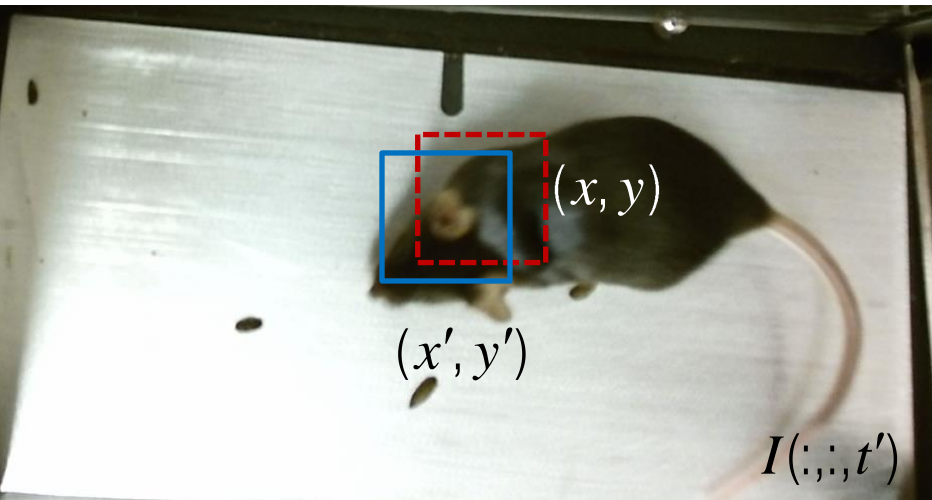


The brighter, the bigger flow magnitude.

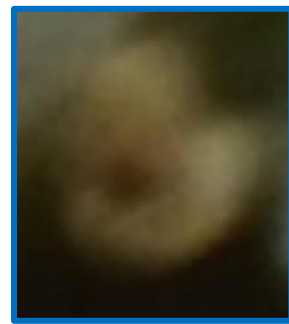


The brighter, the bigger flow magnitude.

# RECALL: LOCAL PATCH TRACKING



Neighboring pixels move similarly.



$$\{I(x_i, y_i, t)\}_{(x_i, y_i) \in N(x, y)}$$

$$\{I(x'_i, y'_i, t)\}_{(x'_i, y'_i) \in N(x', y')}$$

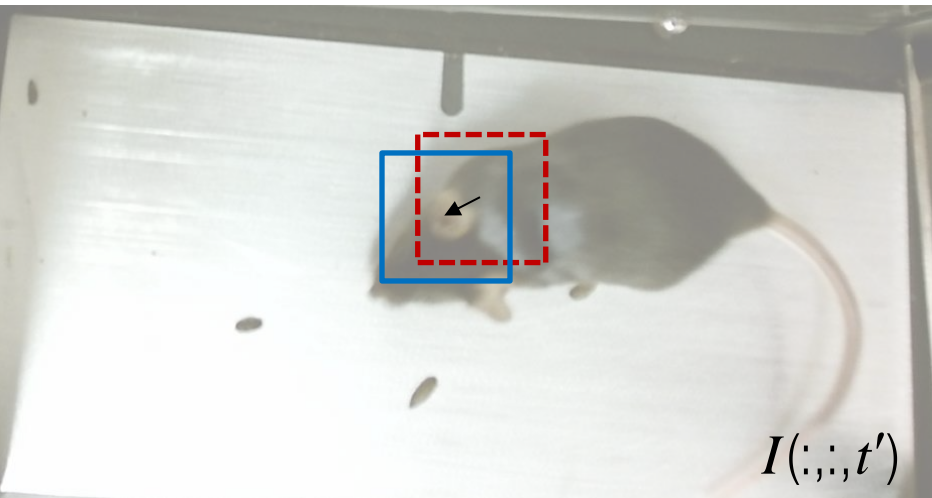
$$\left. \begin{aligned} \frac{\partial I}{\partial x} \Big|_1 u + \frac{\partial I}{\partial y} \Big|_1 v = - \frac{\partial I}{\partial t} \Big|_1 \\ \vdots \\ \frac{\partial I}{\partial x} \Big|_n u + \frac{\partial I}{\partial y} \Big|_n v = - \frac{\partial I}{\partial t} \Big|_n \end{aligned} \right\}$$

# of unknowns: 2

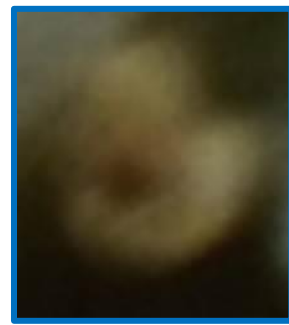
# of equations: # of pixels in the local patch



# ASSUMPTION: CONSTANT MOTION



Neighboring pixels move similarly.



$$\{I(x_i, y_i, t)\}_{(x_i, y_i) \in N(x, y)}$$

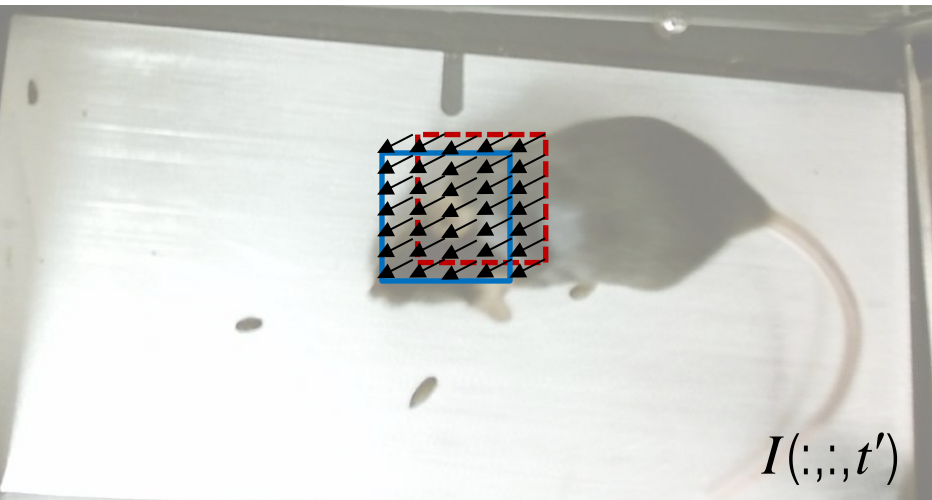
$$\{I(x'_i, y'_i, t)\}_{(x'_i, y'_i) \in N(x', y')}$$

$$\left. \begin{aligned} \frac{\partial I}{\partial x} \Big|_1 u + \frac{\partial I}{\partial y} \Big|_1 v = - \frac{\partial I}{\partial t} \Big|_1 \\ \vdots \\ \frac{\partial I}{\partial x} \Big|_n u + \frac{\partial I}{\partial y} \Big|_n v = - \frac{\partial I}{\partial t} \Big|_n \end{aligned} \right\}$$

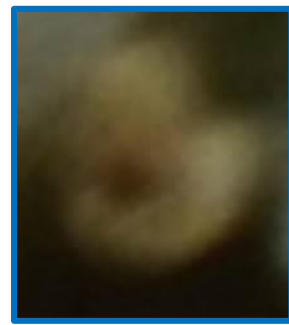
# of unknowns: 2

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$$\{I(x_i, y_i, t)\}_{(x_i, y_i) \in N(x, y)}$$

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$$\left. \begin{aligned} \frac{\partial I}{\partial x} \Big|_1 u + \frac{\partial I}{\partial y} \Big|_1 v &= - \frac{\partial I}{\partial t} \Big|_1 \\ &\vdots \\ \frac{\partial I}{\partial x} \Big|_n u + \frac{\partial I}{\partial y} \Big|_n v &= - \frac{\partial I}{\partial t} \Big|_n \end{aligned} \right\}$$

# of unknowns: 2

# of equations: # of pixels in the local patch



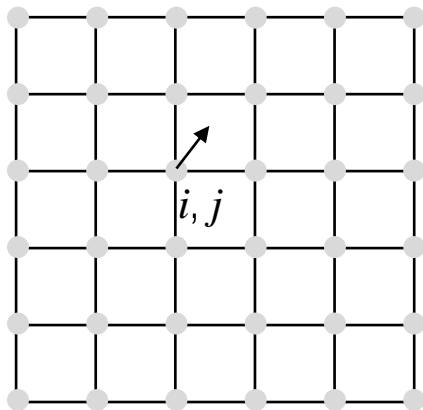
Weak feature to track

$$\frac{\partial I}{\partial x} \Big|_1 u_1 + \frac{\partial I}{\partial y} \Big|_1 v_1 = - \frac{\partial I}{\partial t} \Big|_1$$

⋮

$$\frac{\partial I}{\partial x} \Big|_n u_n + \frac{\partial I}{\partial y} \Big|_n v_n = - \frac{\partial I}{\partial t} \Big|_n$$

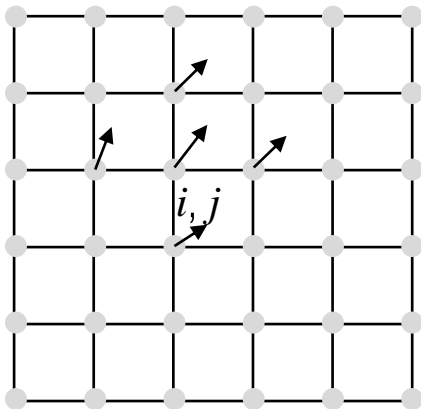
# *DENSE OPTICAL FLOW*



Optical flow equation:

$$I_x|_{i,j} u_{i,j} + I_y|_{i,j} v_{i,j} = -I_t|_{i,j}$$

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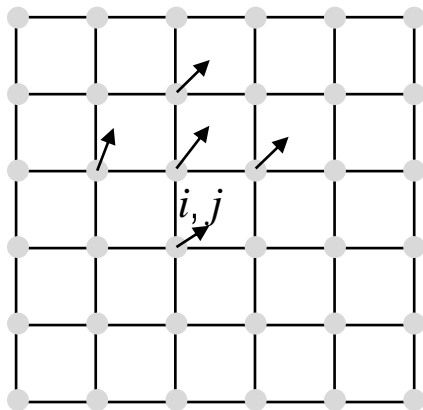
⋮

Spatial smoothness constraint:

$$\begin{aligned} & (u_{i,j} - u_{i-1,j})^2 + (u_{i,j} - u_{i+1,j})^2 + (u_{i,j} - u_{i,j-1})^2 + (u_{i,j} - u_{i,j+1})^2 \\ & (v_{i,j} - v_{i-1,j})^2 + (v_{i,j} - v_{i+1,j})^2 + (v_{i,j} - v_{i,j-1})^2 + (v_{i,j} - v_{i,j+1})^2 \end{aligned}$$



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Optical flow equation:

$$I_x|_{i,j} u_{i,j} + I_y|_{i,j} v_{i,j} = -I_t|_{i,j}$$

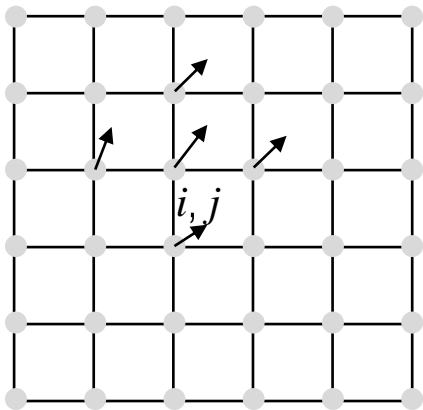
⋮

Spatial smoothness constraint:

$$\|\nabla u\|^2$$

$$\|\nabla v\|^2$$

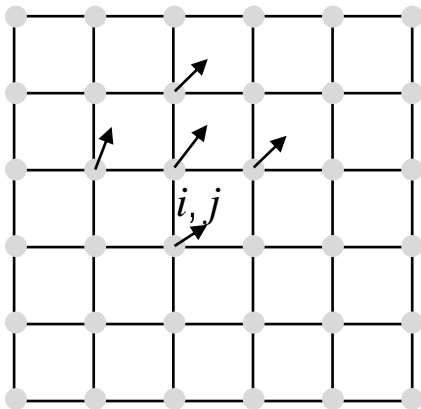
# DENSE OPTICAL FLOW



Objective:

$$\underset{\{u_{i,j}, v_{i,j}\}}{\text{minimize}} \sum_i \sum_j \underbrace{(I_x u_{i,j} + I_y v_{i,j} + I_t)^2}_{\text{Optical flow eq.}} + \lambda \underbrace{\|\nabla u_{i,j}\|^2 + \|\nabla v_{i,j}\|^2}_{\text{Spatial smoothness}}$$

# DENSE OPTICAL FLOW

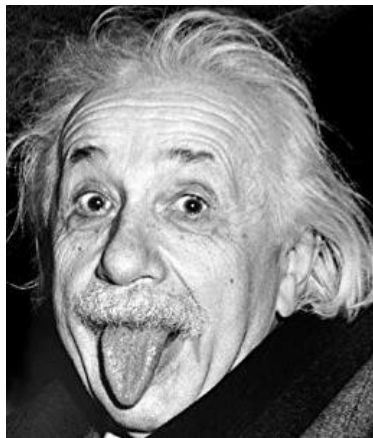


Objective:

$$\underset{\{u_{i,j}, v_{i,j}\}}{\text{minimize}} \sum_i \sum_j \underbrace{(I_x u_{i,j} + I_y v_{i,j} + I_t)^2}_{\text{Optical flow eq.}} + \lambda \underbrace{\|\nabla u_{i,j}\|^2 + \|\nabla v_{i,j}\|^2}_{\text{Spatial smoothness}}$$

Taking derivative w.r.t.  $u_{i,j}, v_{i,j}$

# RECALL: LAPLACE OPERATOR

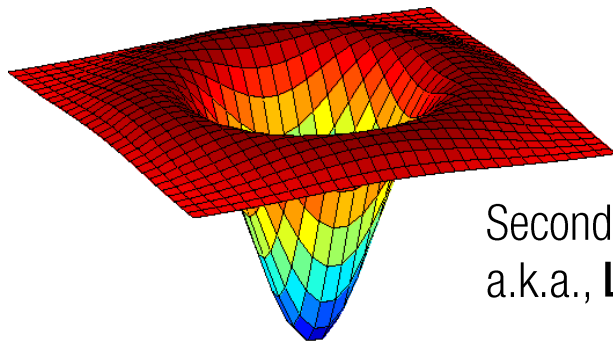
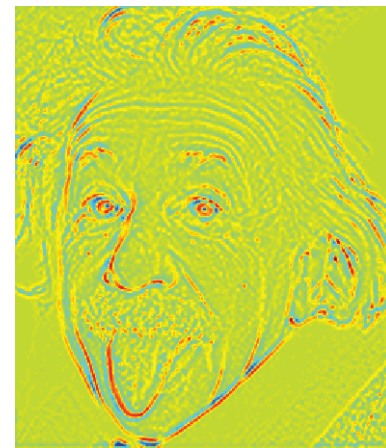


\*

0	1	0
1	-4	1
0	1	0

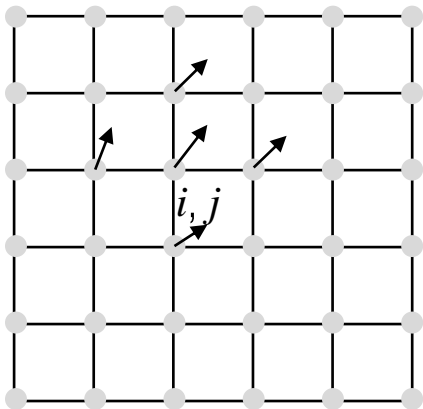
=

$$\nabla \cdot \nabla G = \nabla \left( \frac{\partial G}{\partial u} \mathbf{i} + \frac{\partial G}{\partial v} \mathbf{j} \right) = \frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2}$$



Second order derivative of Gaussian,  
a.k.a., **Laplacian of Gaussian**

# DENSE OPTICAL FLOW



Objective:

$$\underset{\{u_{i,j}, v_{i,j}\}}{\text{minimize}} \sum_i \sum_j \underbrace{\left( I_x u_{i,j} + I_y v_{i,j} + I_t \right)^2}_{\text{Optical flow eq.}} + \lambda \underbrace{\left\| \nabla u_{i,j} \right\|^2 + \lambda \left\| \nabla v_{i,j} \right\|^2}_{\text{Spatial smoothness}}$$

Taking derivative w.r.t.  $u_{i,j}, v_{i,j}$

$$I_x \left( I_x u_{i,j} + I_y v_{i,j} + I_t \right) - \lambda \nabla^2 u_{i,j} = 0$$

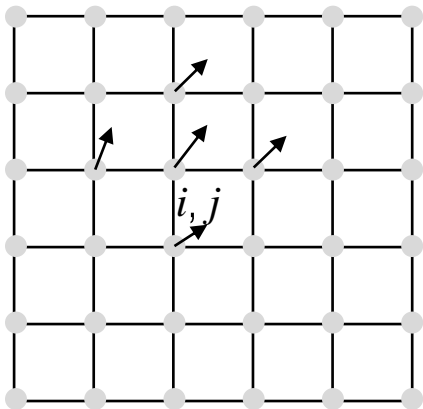
$$I_y \left( I_x u_{i,j} + I_y v_{i,j} + I_t \right) - \lambda \nabla^2 v_{i,j} = 0$$

$$\text{Laplace operator: } \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \frac{1}{4} \left( u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} \right) - u_{i,j}$$

Weighted average of neighbors



# DENSE OPTICAL FLOW



Objective:

$$\underset{\{u_{i,j}, v_{i,j}\}}{\text{minimize}} \sum_i \sum_j \underbrace{\left( I_x u_{i,j} + I_y v_{i,j} + I_t \right)^2}_{\text{Optical flow eq.}} + \lambda \underbrace{\left\| \nabla u_{i,j} \right\|^2 + \lambda \left\| \nabla v_{i,j} \right\|^2}_{\text{Spatial smoothness}}$$

Taking derivative w.r.t.  $u_{i,j}, v_{i,j}$

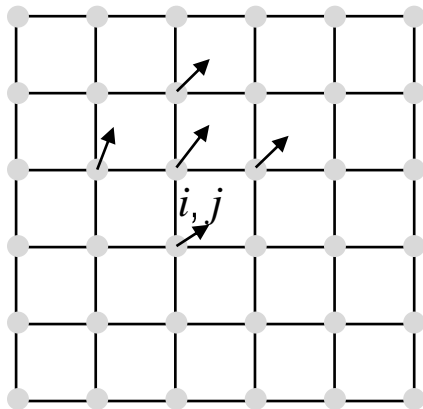
$$I_x \left( I_x u_{i,j} + I_y v_{i,j} + I_t \right) - \lambda \nabla^2 u_{i,j} = 0$$

$$I_y \left( I_x u_{i,j} + I_y v_{i,j} + I_t \right) - \lambda \nabla^2 v_{i,j} = 0$$

$$\text{Laplace operator: } \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \bar{u}_{i,j} - u_{i,j}$$

Weighted average of neighbors

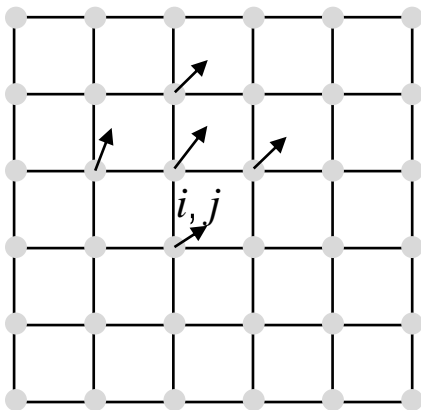
# DENSE OPTICAL FLOW



$$I_x (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{u}_{i,j} - u_{i,j}) = 0$$

$$I_y (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{v}_{i,j} - v_{i,j}) = 0$$

# DENSE OPTICAL FLOW



$$I_x (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{u}_{i,j} - u_{i,j}) = 0$$

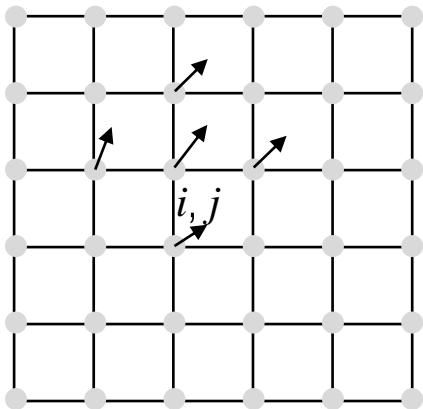
$$I_y (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{v}_{i,j} - v_{i,j}) = 0$$

$$\begin{aligned} \rightarrow & (I_x^2 + \lambda) u_{i,j} + I_x I_y v_{i,j} = \lambda \bar{u}_{i,j} - I_x I_t \\ & I_x I_y u_{i,j} + (I_y^2 + \lambda) v_{i,j} = \lambda \bar{v}_{i,j} - I_y I_t \end{aligned}$$

# of unknowns: ?

# of equations: ?

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$$I_x (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{u}_{i,j} - u_{i,j}) = 0$$

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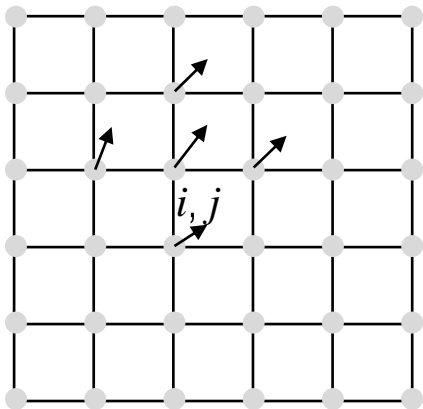
$$\rightarrow \begin{aligned} (I_x^2 + \lambda) u_{i,j} + I_x I_y v_{i,j} &= \lambda \bar{u}_{i,j} - I_x I_t \\ I_x I_y u_{i,j} + (I_y^2 + \lambda) v_{i,j} &= \lambda \bar{v}_{i,j} - I_y I_t \end{aligned}$$

# of unknowns: ?

# of equations: ?

Are we done?

# DENSE OPTICAL FLOW



$$I_x (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{u}_{i,j} - u_{i,j}) = 0$$

$$I_y (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{v}_{i,j} - v_{i,j}) = 0$$

$$\begin{aligned} \longrightarrow & (I_x^2 + \lambda) u_{i,j} + I_x I_y v_{i,j} = \lambda \bar{u}_{i,j} - I_x I_t \\ & I_x I_y u_{i,j} + (I_y^2 + \lambda) v_{i,j} = \lambda \bar{v}_{i,j} - I_y I_t \end{aligned}$$

# of unknowns: ?

# of equations: ?

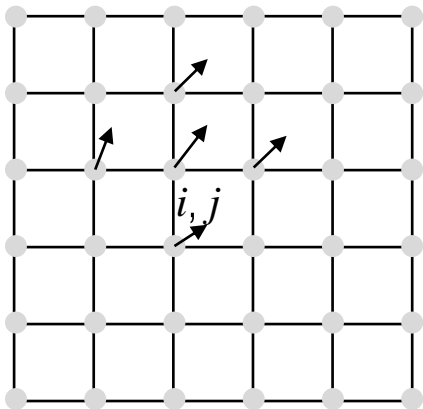
Are we done?

No, because the solution  $u_{i,j}, v_{i,j}$  depends on neighboring pixels.

→ The neighboring pixels need to be updated.



# DENSE OPTICAL FLOW



$$I_x (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{u}_{i,j} - u_{i,j}) = 0$$

$$I_y (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{v}_{i,j} - v_{i,j}) = 0$$

$$\begin{aligned} \rightarrow (I_x^2 + \lambda) u_{i,j} + I_x I_y v_{i,j} &= \lambda \bar{u}_{i,j} - I_x I_t \\ I_x I_y u_{i,j} + (I_y^2 + \lambda) v_{i,j} &= \lambda \bar{v}_{i,j} - I_y I_t \end{aligned}$$

$$u_{i,j}^{k+1} = \bar{u}_{i,j}^k - \frac{I_x (I_x \bar{u}_{i,j}^k + I_y \bar{v}_{i,j}^k + I_t)}{\lambda + I_x^2 + I_y^2}$$

$$v_{i,j}^{k+1} = \bar{v}_{i,j}^k - \frac{I_y (I_x \bar{u}_{i,j}^k + I_y \bar{v}_{i,j}^k + I_t)}{\lambda + I_x^2 + I_y^2}$$

new old

Update all pixel simultaneously until converges.

<https://www.youtube.com/watch?v=ssINeWRb58M>

<https://www.youtube.com/watch?v=JSzUdVBmQP4>