

DENSE OPTICAL FLOW

HYUN SOO PARK







Good features to track



Weak feature to track



Weak feature to track

SPATIAL SMOOTHNESS OF OPTICAL FLOW

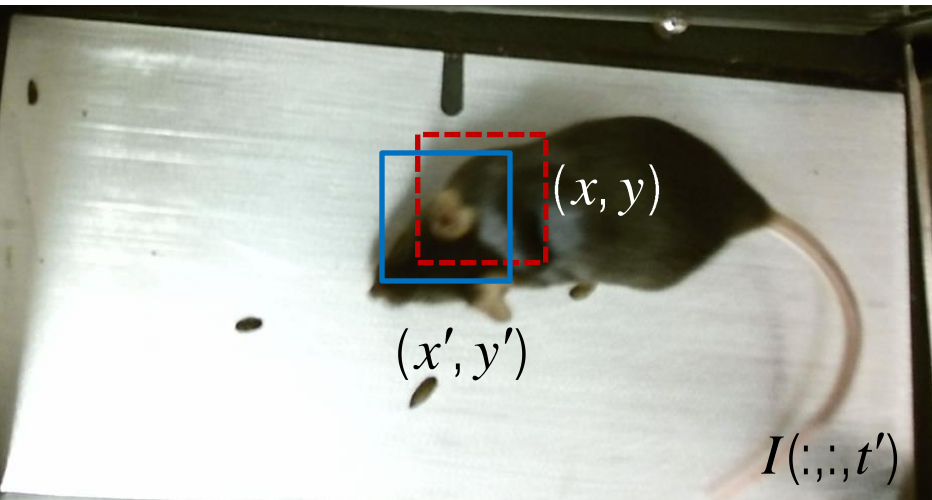


The brighter, the bigger flow magnitude.

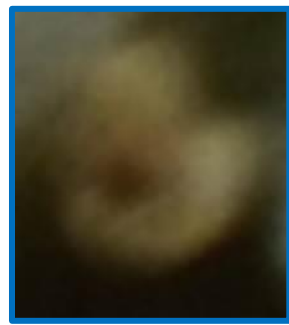


The brighter, the bigger flow magnitude.

RECALL: LOCAL PATCH TRACKING



Neighboring pixels move similarly.



$$\{I(x_i, y_i, t)\}_{(x_i, y_i) \in N(x, y)}$$

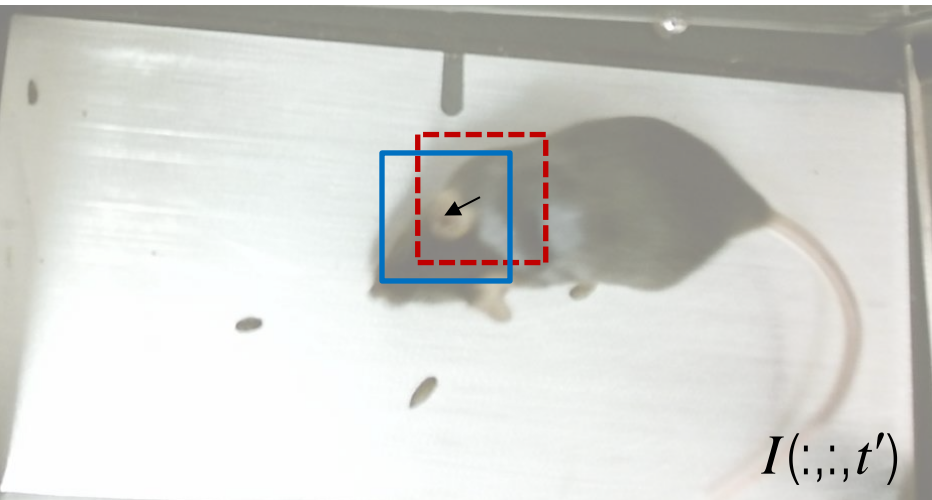
$$\{I(x'_i, y'_i, t)\}_{(x'_i, y'_i) \in N(x', y')}$$

$$\left. \begin{aligned} \frac{\partial I}{\partial x} \Big|_1 u + \frac{\partial I}{\partial y} \Big|_1 v = - \frac{\partial I}{\partial t} \Big|_1 \\ \vdots \\ \frac{\partial I}{\partial x} \Big|_n u + \frac{\partial I}{\partial y} \Big|_n v = - \frac{\partial I}{\partial t} \Big|_n \end{aligned} \right\}$$

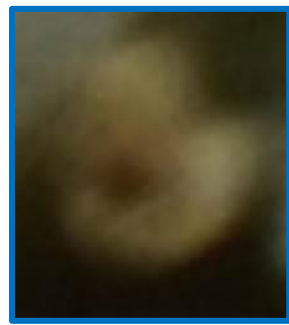
of unknowns: 2

of equations: # of pixels in the local patch

ASSUMPTION: CONSTANT MOTION



Neighboring pixels move similarly.



$$\{I(x_i, y_i, t)\}_{(x_i, y_i) \in N(x, y)}$$

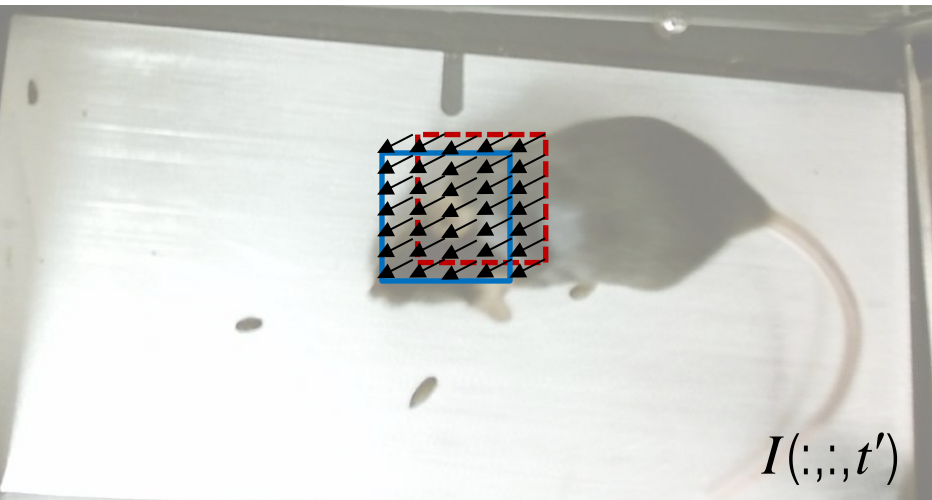
$$\{I(x'_i, y'_i, t)\}_{(x'_i, y'_i) \in N(x', y')}$$

$$\left. \begin{aligned} \frac{\partial I}{\partial x} \Big|_1 u + \frac{\partial I}{\partial y} \Big|_1 v = - \frac{\partial I}{\partial t} \Big|_1 \\ \vdots \\ \frac{\partial I}{\partial x} \Big|_n u + \frac{\partial I}{\partial y} \Big|_n v = - \frac{\partial I}{\partial t} \Big|_n \end{aligned} \right\}$$

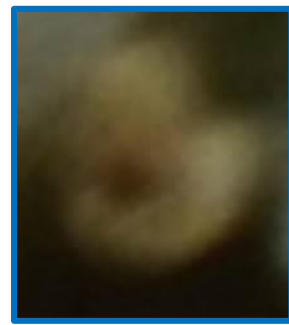
of unknowns: 2

of equations: # of pixels in the local patch

ASSUMPTION: CONSTANT MOTION



Neighboring pixels move similarly.



$$\{I(x_i, y_i, t)\}_{(x_i, y_i) \in N(x, y)}$$

$$\{I(x'_i, y'_i, t)\}_{(x'_i, y'_i) \in N(x', y')}$$

$$\left. \begin{aligned} \frac{\partial I}{\partial x} \Big|_1 u + \frac{\partial I}{\partial y} \Big|_1 v = - \frac{\partial I}{\partial t} \Big|_1 \\ \vdots \\ \frac{\partial I}{\partial x} \Big|_n u + \frac{\partial I}{\partial y} \Big|_n v = - \frac{\partial I}{\partial t} \Big|_n \end{aligned} \right\}$$

of unknowns: 2

of equations: # of pixels in the local patch



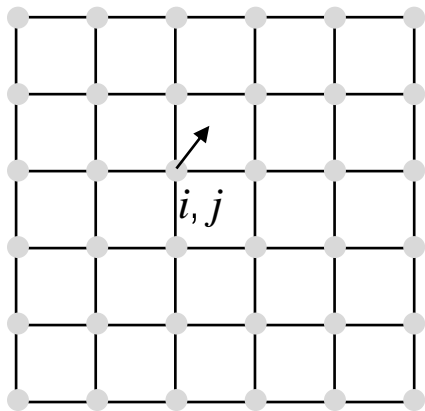
Weak feature to track

$$\frac{\partial I}{\partial x} \Big|_1 u_1 + \frac{\partial I}{\partial y} \Big|_1 v_1 = - \frac{\partial I}{\partial t} \Big|_1$$

⋮

$$\frac{\partial I}{\partial x} \Big|_n u_n + \frac{\partial I}{\partial y} \Big|_n v_n = - \frac{\partial I}{\partial t} \Big|_n$$

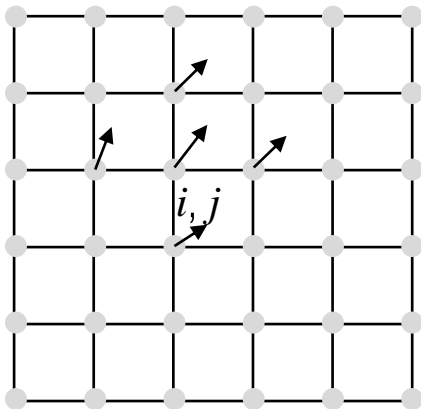
DENSE OPTICAL FLOW



Optical flow equation:

$$I_x|_{i,j} u_{i,j} + I_y|_{i,j} v_{i,j} = -I_t|_{i,j}$$

DENSE OPTICAL FLOW



Optical flow equation:

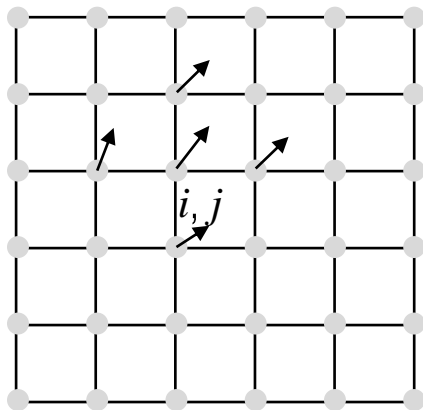
$$I_x|_{i,j} u_{i,j} + I_y|_{i,j} v_{i,j} = -I_t|_{i,j}$$

⋮

Spatial smoothness constraint:

$$\begin{aligned} & (u_{i,j} - u_{i-1,j})^2 + (u_{i,j} - u_{i+1,j})^2 + (u_{i,j} - u_{i,j-1})^2 + (u_{i,j} - u_{i,j+1})^2 \\ & (v_{i,j} - v_{i-1,j})^2 + (v_{i,j} - v_{i+1,j})^2 + (v_{i,j} - v_{i,j-1})^2 + (v_{i,j} - v_{i,j+1})^2 \end{aligned}$$

DENSE OPTICAL FLOW



Optical flow equation:

$$I_x|_{i,j} u_{i,j} + I_y|_{i,j} v_{i,j} = -I_t|_{i,j}$$

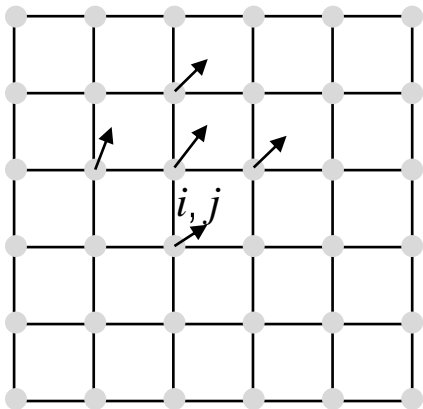
⋮

Spatial smoothness constraint:

$$\|\nabla u\|^2$$

$$\|\nabla v\|^2$$

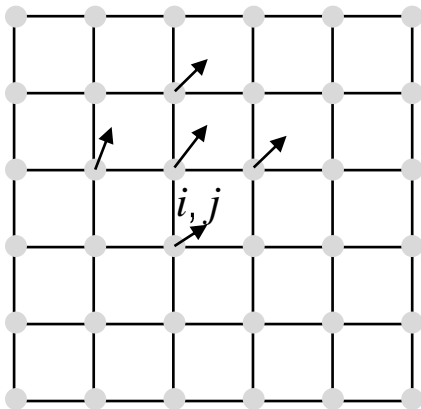
DENSE OPTICAL FLOW



Objective:

$$\underset{\{u_{i,j}, v_{i,j}\}}{\text{minimize}} \sum_i \sum_j \underbrace{(I_x u_{i,j} + I_y v_{i,j} + I_t)^2}_{\text{Optical flow eq.}} + \lambda \underbrace{\|\nabla u_{i,j}\|^2 + \|\nabla v_{i,j}\|^2}_{\text{Spatial smoothness}}$$

DENSE OPTICAL FLOW

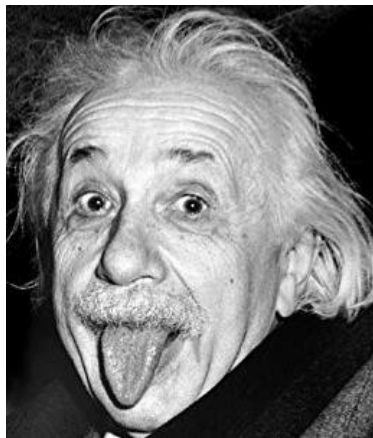


Objective:

$$\underset{\{u_{i,j}, v_{i,j}\}}{\text{minimize}} \sum_i \sum_j \underbrace{(I_x u_{i,j} + I_y v_{i,j} + I_t)^2}_{\text{Optical flow eq.}} + \lambda \underbrace{\|\nabla u_{i,j}\|^2 + \|\nabla v_{i,j}\|^2}_{\text{Spatial smoothness}}$$

Taking derivative w.r.t. $u_{i,j}, v_{i,j}$

RECALL: LAPLACE OPERATOR

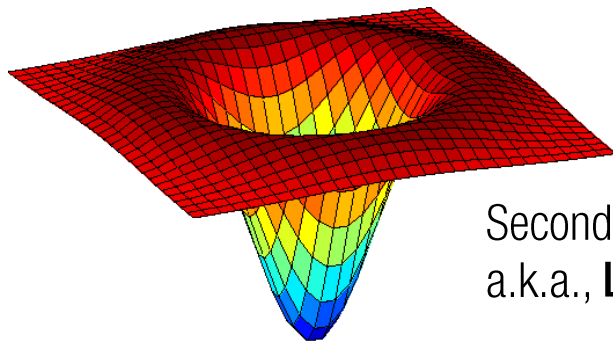
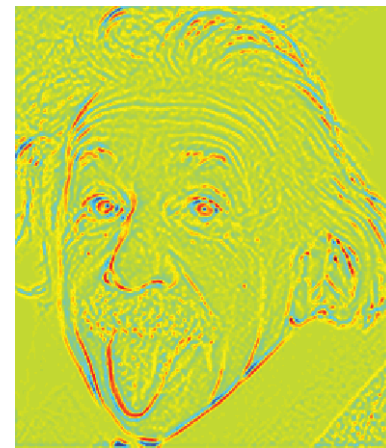


*

0	1	0
1	-4	1
0	1	0

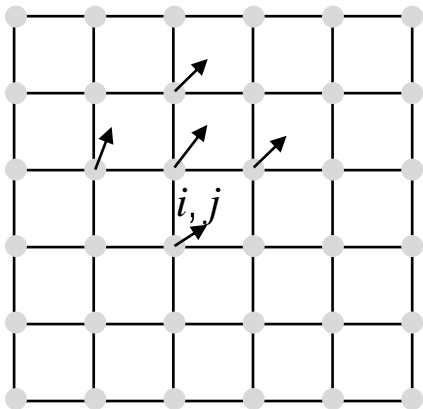
=

$$\nabla \cdot \nabla G = \nabla \left(\frac{\partial G}{\partial u} \mathbf{i} + \frac{\partial G}{\partial v} \mathbf{j} \right) = \frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2}$$



Second order derivative of Gaussian,
a.k.a., **Laplacian of Gaussian**

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Objective:

$$\underset{\{u_{i,j}, v_{i,j}\}}{\text{minimize}} \sum_i \sum_j \underbrace{\left(I_x u_{i,j} + I_y v_{i,j} + I_t \right)^2}_{\text{Optical flow eq.}} + \lambda \underbrace{\left\| \nabla u_{i,j} \right\|^2 + \lambda \left\| \nabla v_{i,j} \right\|^2}_{\text{Spatial smoothness}}$$

Taking derivative w.r.t. $u_{i,j}, v_{i,j}$

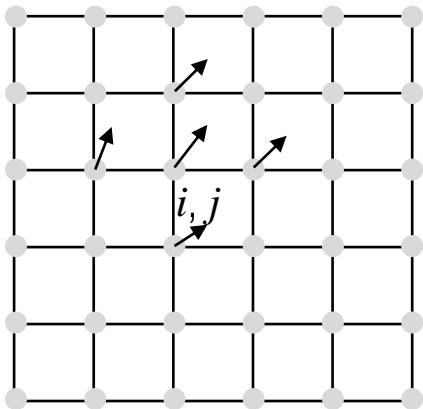
$$I_x \left(I_x u_{i,j} + I_y v_{i,j} + I_t \right) - \lambda \nabla^2 u_{i,j} = 0$$

$$I_y \left(I_x u_{i,j} + I_y v_{i,j} + I_t \right) - \lambda \nabla^2 v_{i,j} = 0$$

$$\text{Laplace operator: } \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \frac{1}{4} \left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} \right) - u_{i,j}$$

Weighted average of neighbors

DENSE OPTICAL FLOW



Objective:

$$\underset{\{u_{i,j}, v_{i,j}\}}{\text{minimize}} \sum_i \sum_j \underbrace{\left(I_x u_{i,j} + I_y v_{i,j} + I_t \right)^2}_{\text{Optical flow eq.}} + \lambda \underbrace{\left\| \nabla u_{i,j} \right\|^2 + \lambda \left\| \nabla v_{i,j} \right\|^2}_{\text{Spatial smoothness}}$$

Taking derivative w.r.t. $u_{i,j}, v_{i,j}$

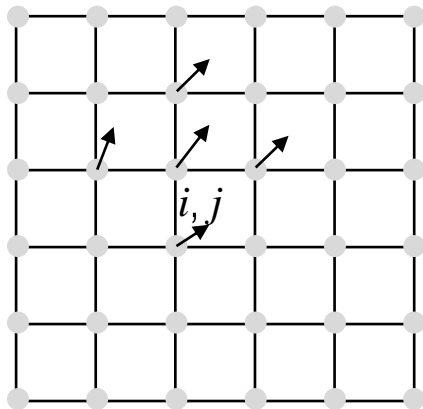
$$I_x \left(I_x u_{i,j} + I_y v_{i,j} + I_t \right) - \lambda \nabla^2 u_{i,j} = 0$$

$$I_y \left(I_x u_{i,j} + I_y v_{i,j} + I_t \right) - \lambda \nabla^2 v_{i,j} = 0$$

$$\text{Laplace operator: } \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \bar{u}_{i,j} - u_{i,j}$$

Weighted average of neighbors

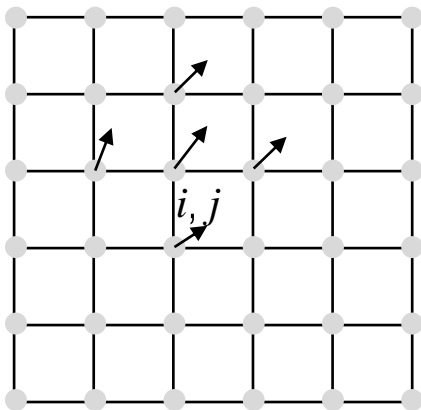
DENSE OPTICAL FLOW



$$I_x (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{u}_{i,j} - u_{i,j}) = 0$$

$$I_y (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{v}_{i,j} - v_{i,j}) = 0$$

DENSE OPTICAL FLOW



$$I_x (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{u}_{i,j} - u_{i,j}) = 0$$

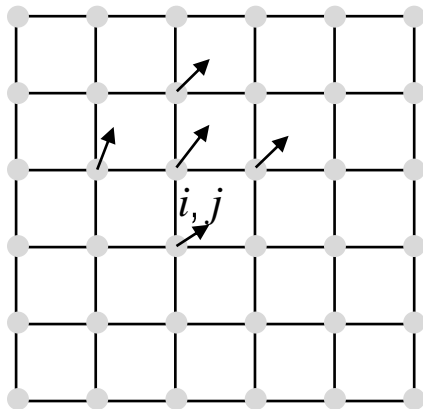
$$I_y (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{v}_{i,j} - v_{i,j}) = 0$$

$$\begin{aligned} \rightarrow & (I_x^2 + \lambda) u_{i,j} + I_x I_y v_{i,j} = \lambda \bar{u}_{i,j} - I_x I_t \\ & I_x I_y u_{i,j} + (I_y^2 + \lambda) v_{i,j} = \lambda \bar{v}_{i,j} - I_y I_t \end{aligned}$$

of unknowns: ?

of equations: ?

DENSE OPTICAL FLOW



$$I_x (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{u}_{i,j} - u_{i,j}) = 0$$

$$I_y (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{v}_{i,j} - v_{i,j}) = 0$$

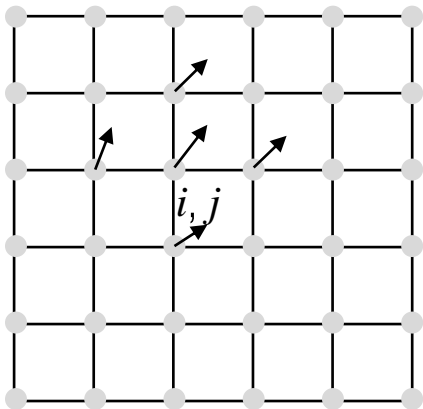
$$\rightarrow \begin{aligned} (I_x^2 + \lambda) u_{i,j} + I_x I_y v_{i,j} &= \lambda \bar{u}_{i,j} - I_x I_t \\ I_x I_y u_{i,j} + (I_y^2 + \lambda) v_{i,j} &= \lambda \bar{v}_{i,j} - I_y I_t \end{aligned}$$

of unknowns: ?

of equations: ?

Are we done?

DENSE OPTICAL FLOW



$$I_x (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{u}_{i,j} - u_{i,j}) = 0$$

$$I_y (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{v}_{i,j} - v_{i,j}) = 0$$

$$\begin{aligned} \rightarrow & (I_x^2 + \lambda) u_{i,j} + I_x I_y v_{i,j} = \lambda \bar{u}_{i,j} - I_x I_t \\ & I_x I_y u_{i,j} + (I_y^2 + \lambda) v_{i,j} = \lambda \bar{v}_{i,j} - I_y I_t \end{aligned}$$

of unknowns: ?

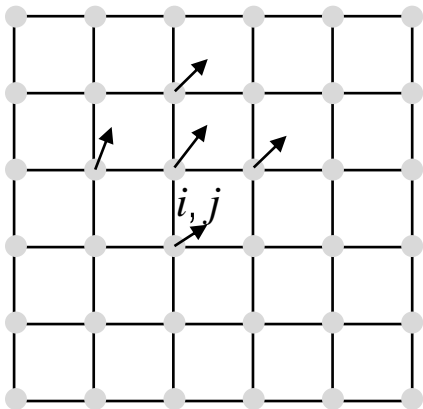
of equations: ?

Are we done?

No, because the solution $u_{i,j}, v_{i,j}$ depends on neighboring pixels.

→ The neighboring pixels need to be updated.

DENSE OPTICAL FLOW



$$I_x (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{u}_{i,j} - u_{i,j}) = 0$$

$$I_y (I_x u_{i,j} + I_y v_{i,j} + I_t) - \lambda (\bar{v}_{i,j} - v_{i,j}) = 0$$

$$\begin{aligned} \rightarrow (I_x^2 + \lambda) u_{i,j} + I_x I_y v_{i,j} &= \lambda \bar{u}_{i,j} - I_x I_t \\ I_x I_y u_{i,j} + (I_y^2 + \lambda) v_{i,j} &= \lambda \bar{v}_{i,j} - I_y I_t \end{aligned}$$

$$u_{i,j}^{k+1} = \bar{u}_{i,j}^k - \frac{I_x (I_x \bar{u}_{i,j}^k + I_y \bar{v}_{i,j}^k + I_t)}{\lambda + I_x^2 + I_y^2}$$

$$v_{i,j}^{k+1} = \bar{v}_{i,j}^k - \frac{I_y (I_x \bar{u}_{i,j}^k + I_y \bar{v}_{i,j}^k + I_t)}{\lambda + I_x^2 + I_y^2}$$

new old

Update all pixel simultaneously until converges.

<https://www.youtube.com/watch?v=ssINeWRb58M>

<https://www.youtube.com/watch?v=JSzUdVBmQP4>