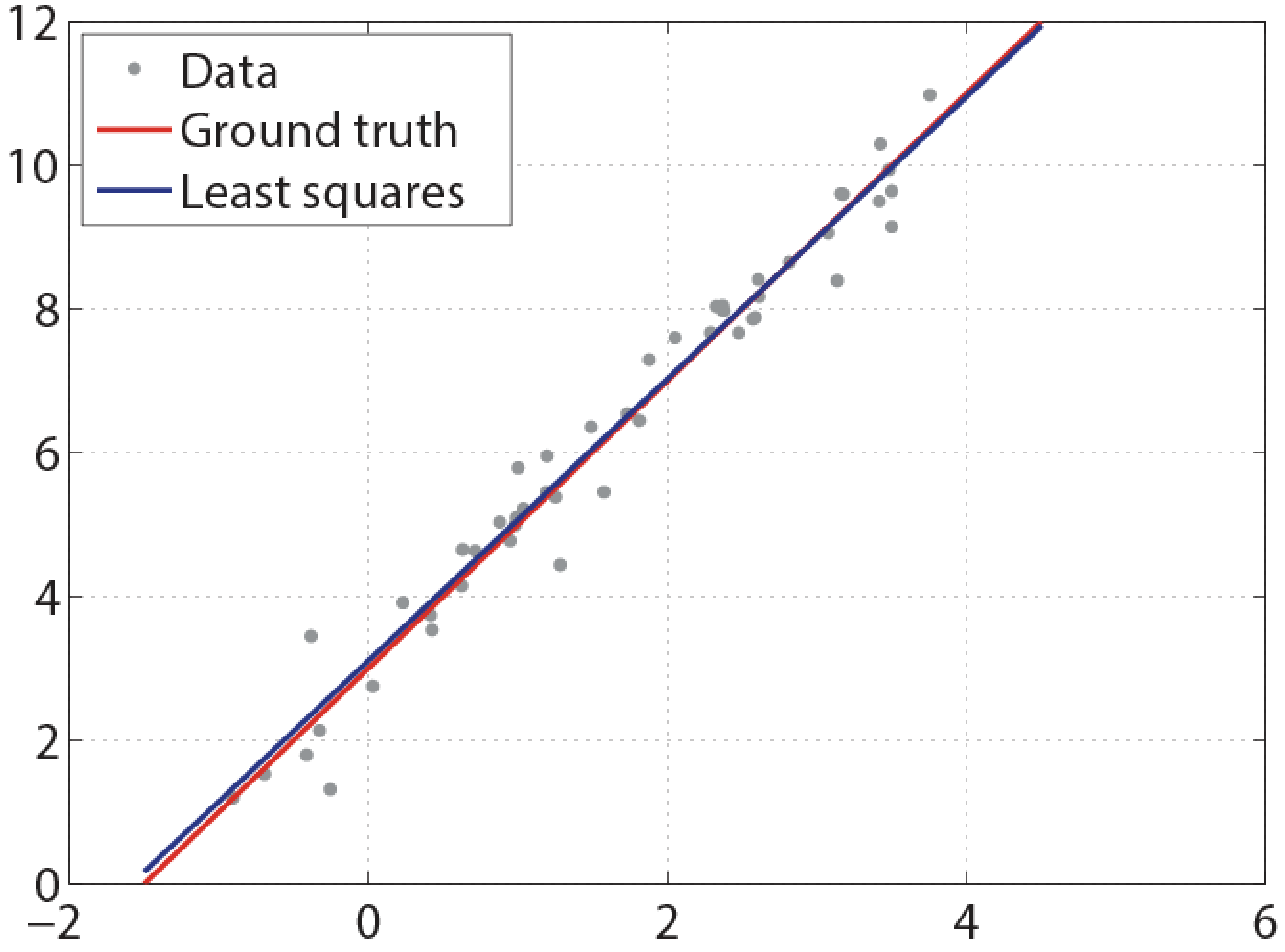
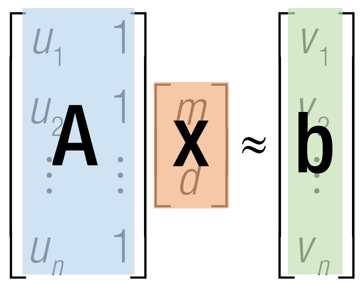
Nonlinear Estimation

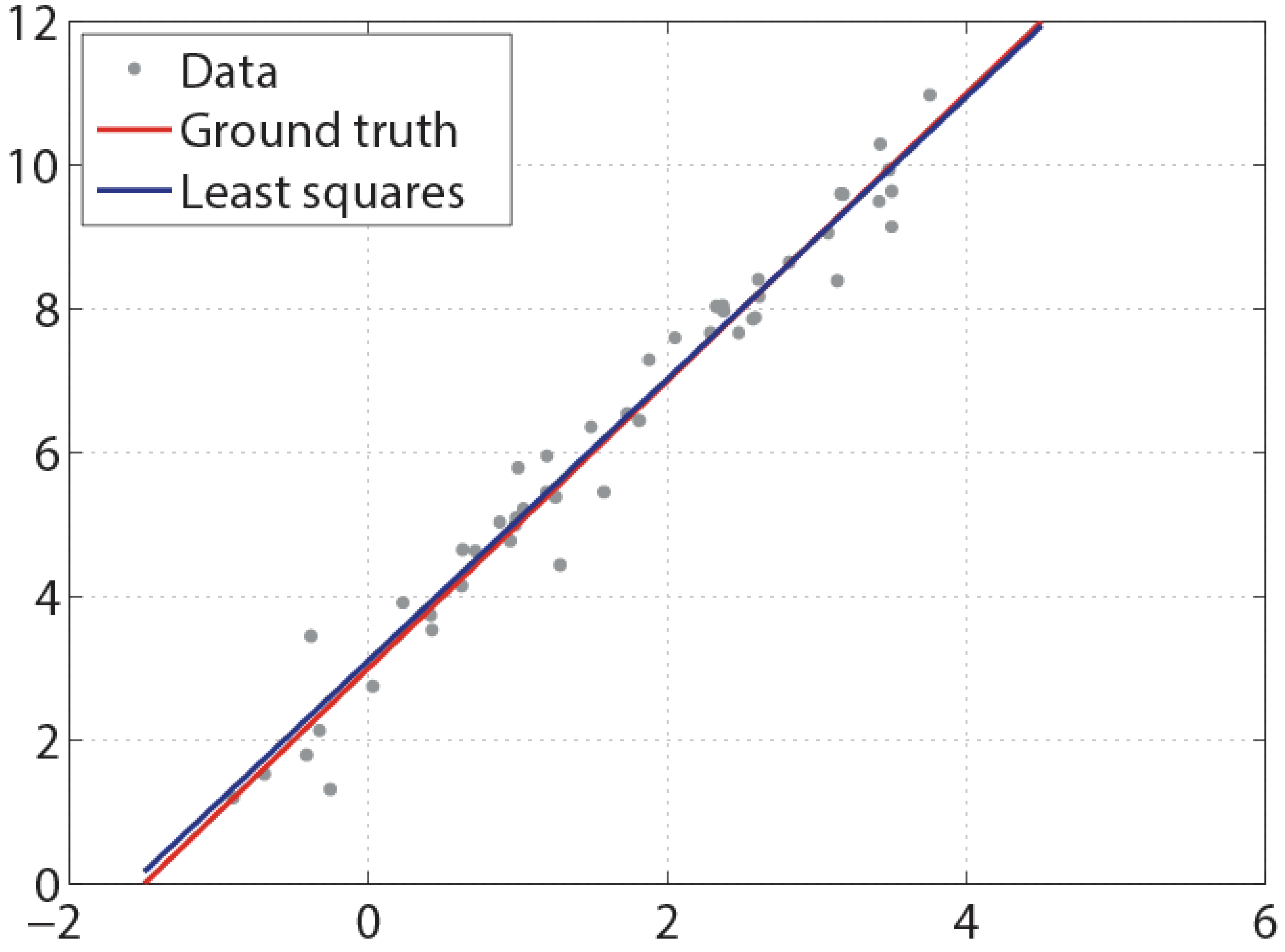
Hyun Soo Park

Recall: Line Fitting (Ax=b)





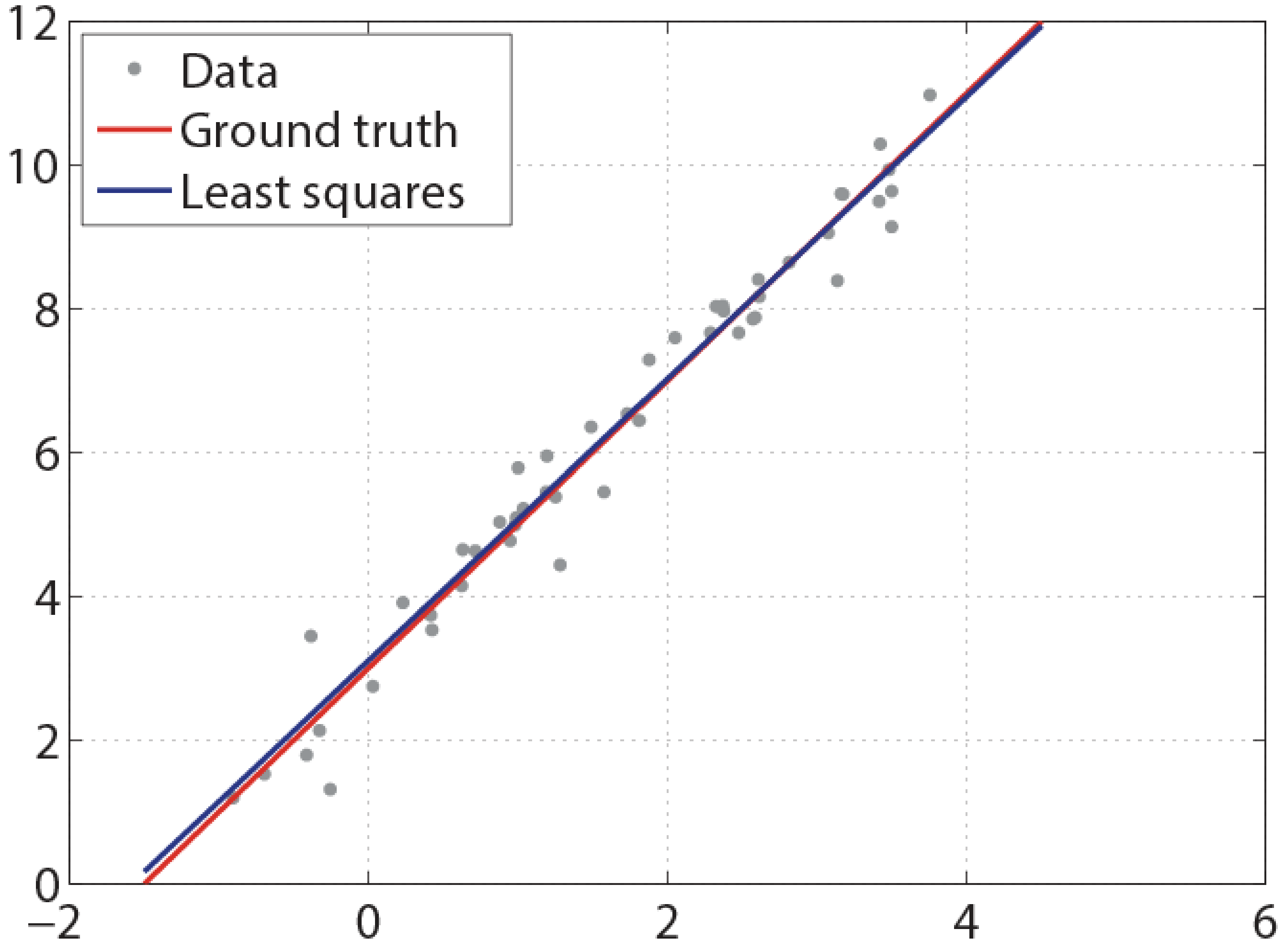
Recall: Line Fitting (Ax=b)



$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$$

$$A^T A x = A^T b$$

Recall: Line Fitting (Ax=b)

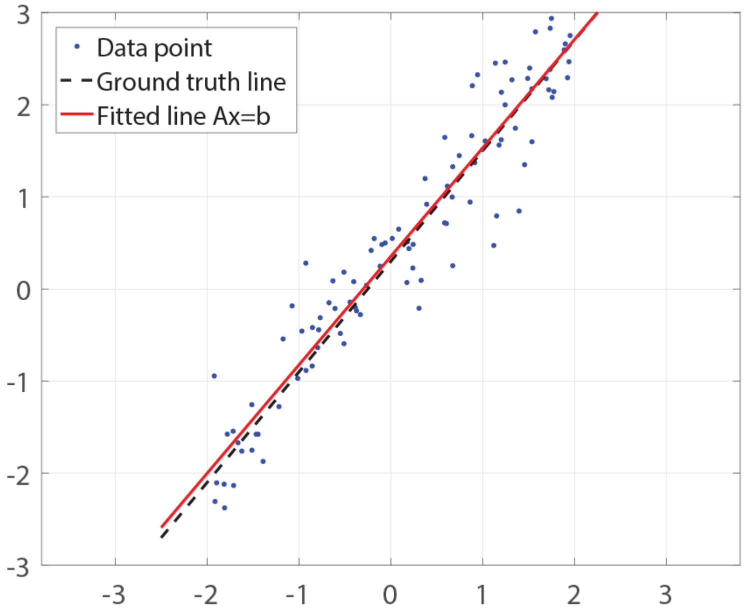


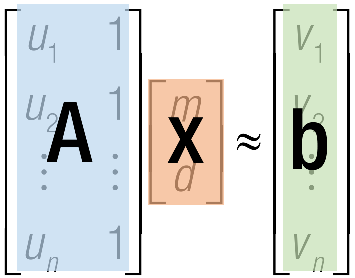
$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{b}$$

$$A^T A x = A^T b$$

$$x = \left[A^T A \right]^{-1} A^T b$$

Recall: Line Fitting (Ax=b)

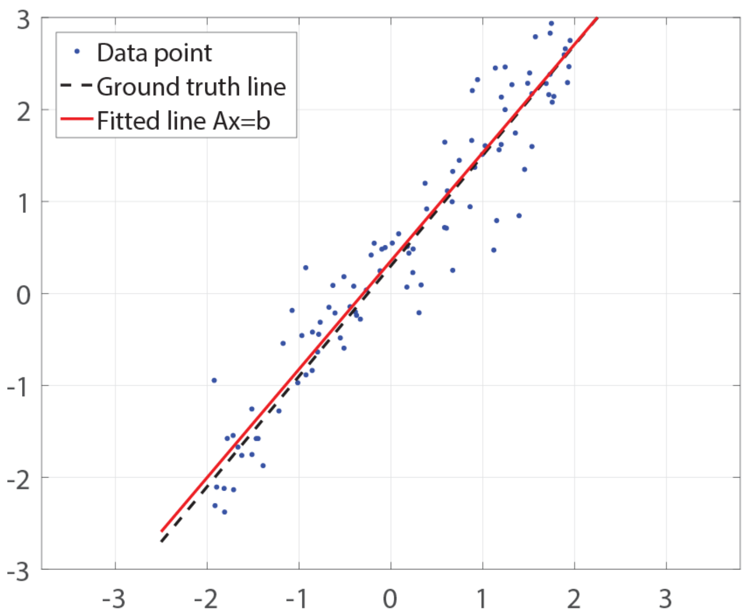


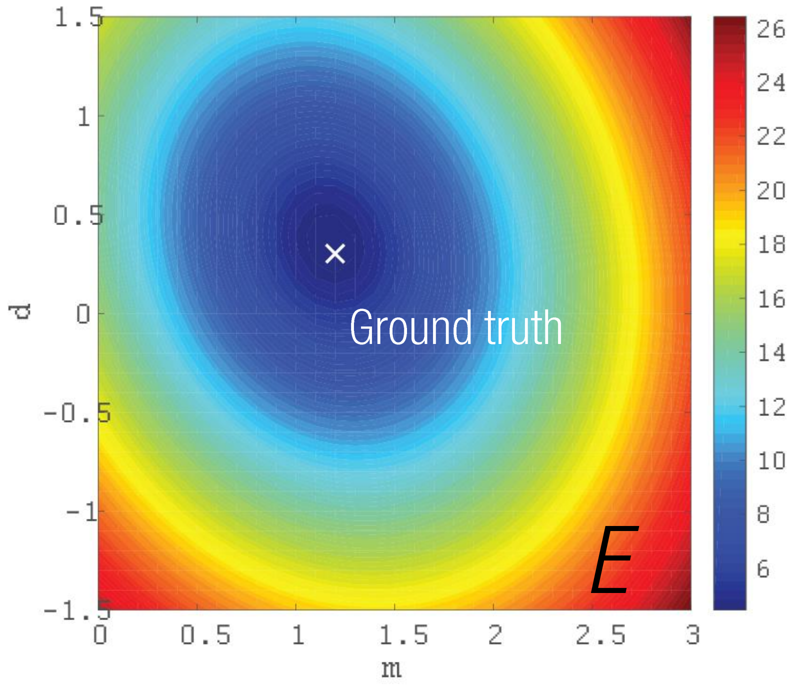


Error:

Recall: Line Fitting (Ax=b)

$$\mathbf{A}\,\mathbf{x} \approx \mathbf{b}: \quad \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$



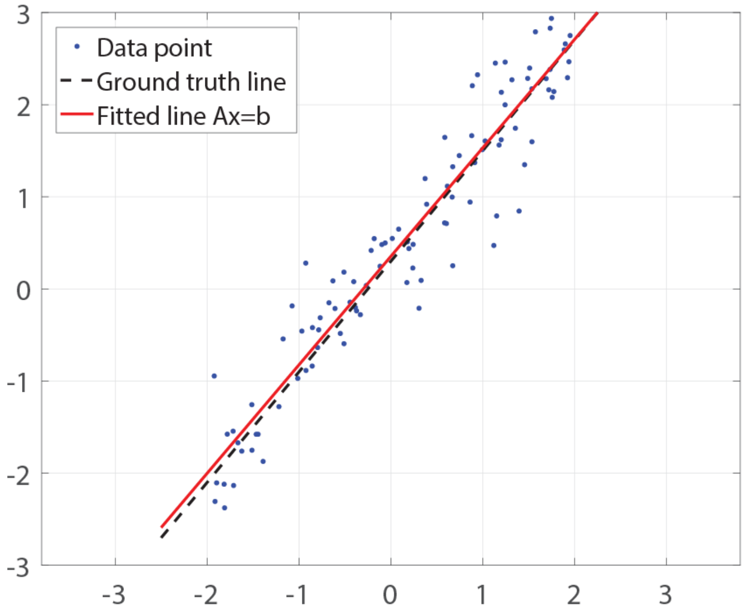


Error:

$$E = \left\| \begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|^2$$

Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$



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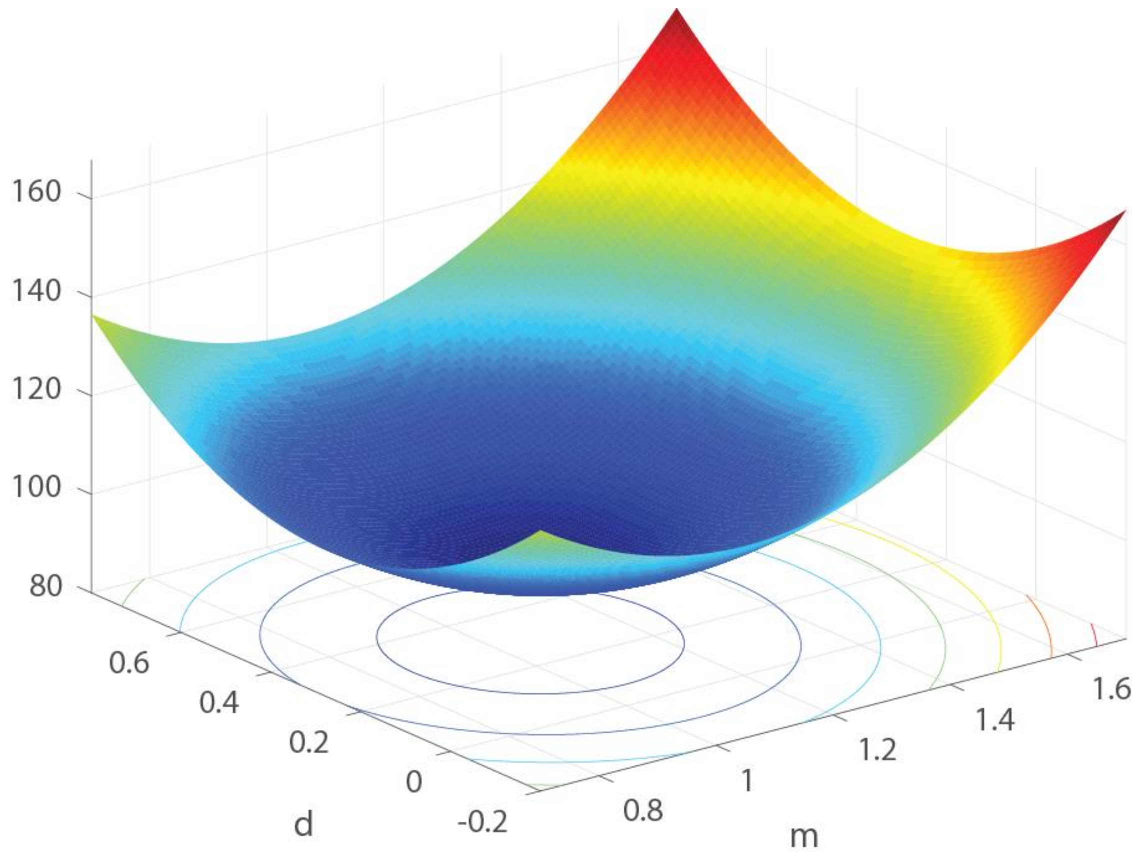
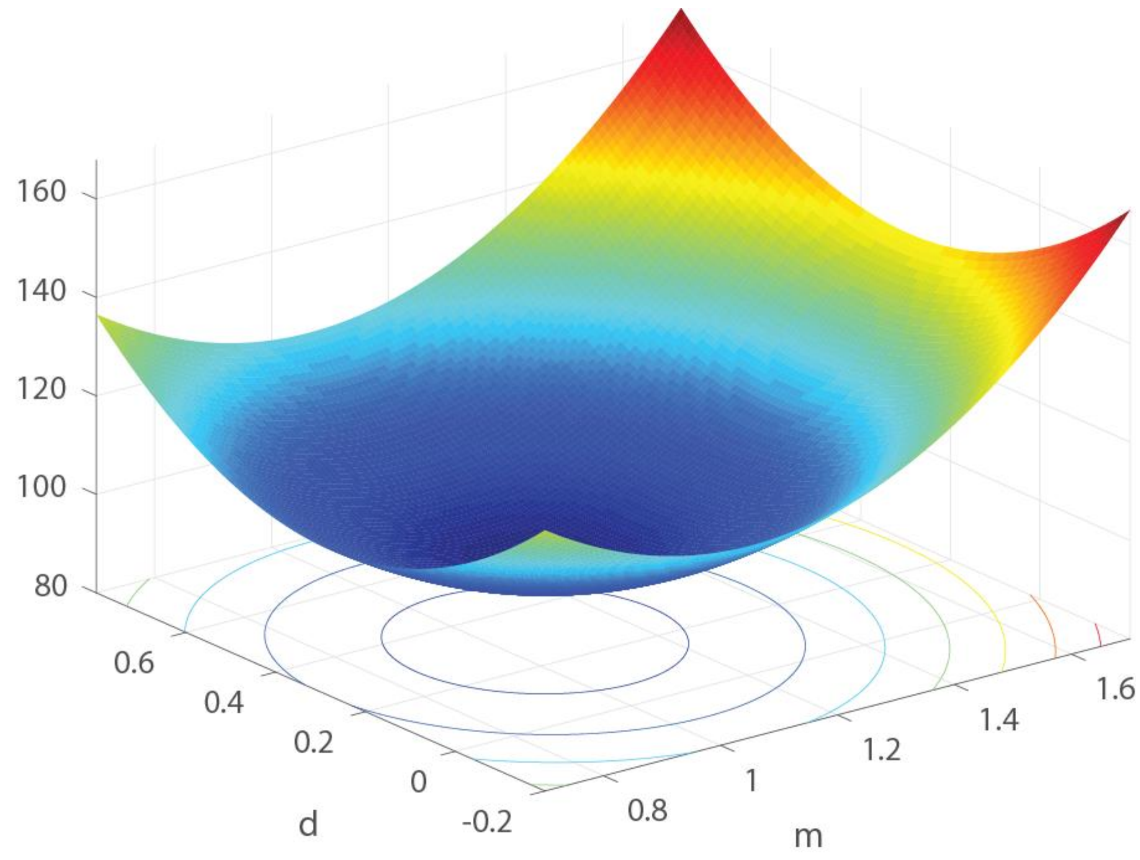
Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$



Recall: Line Fitting



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$v_1 \approx mu_1 + d$

$v_2 \approx mu_2 + d$

$v_n \approx mu_n + d$

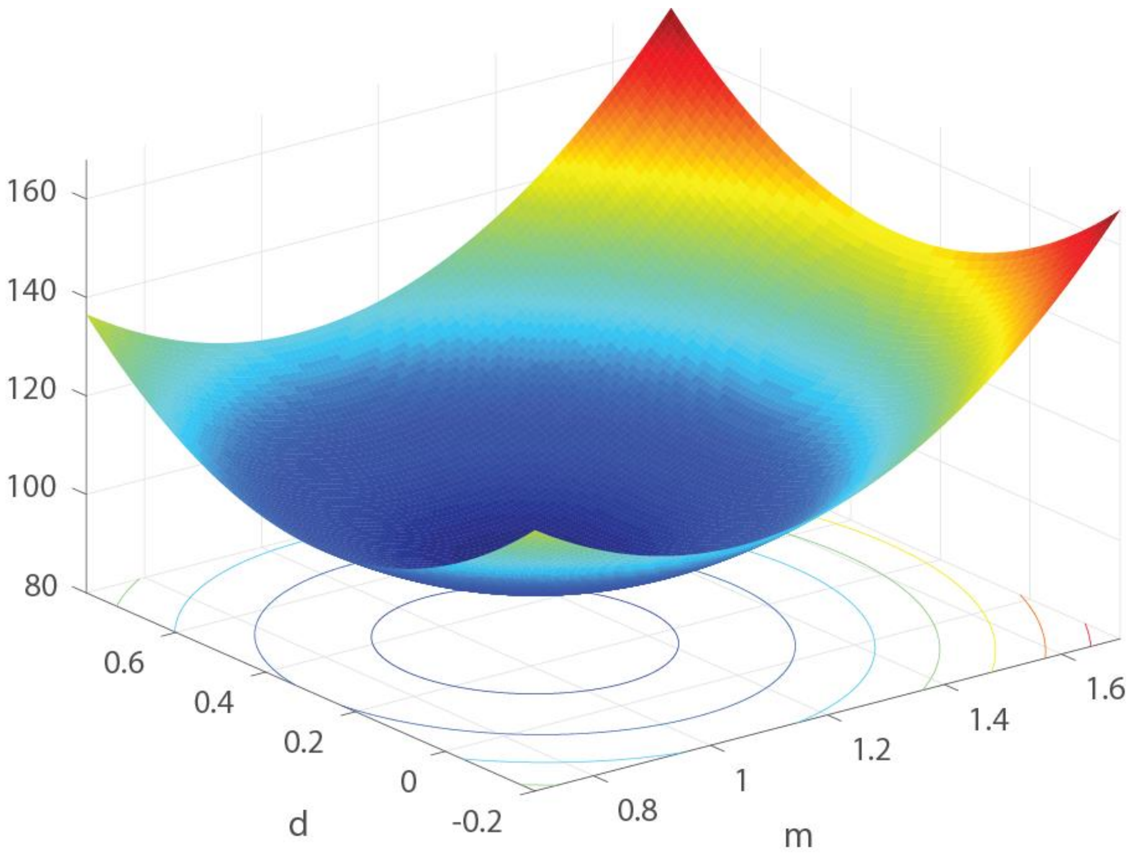
→

$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$

$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$

$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$

Recall: Line Fitting



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$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$v_n \approx mu_n + d$$

$\rightarrow$

$$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$
$$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$
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$\rightarrow$

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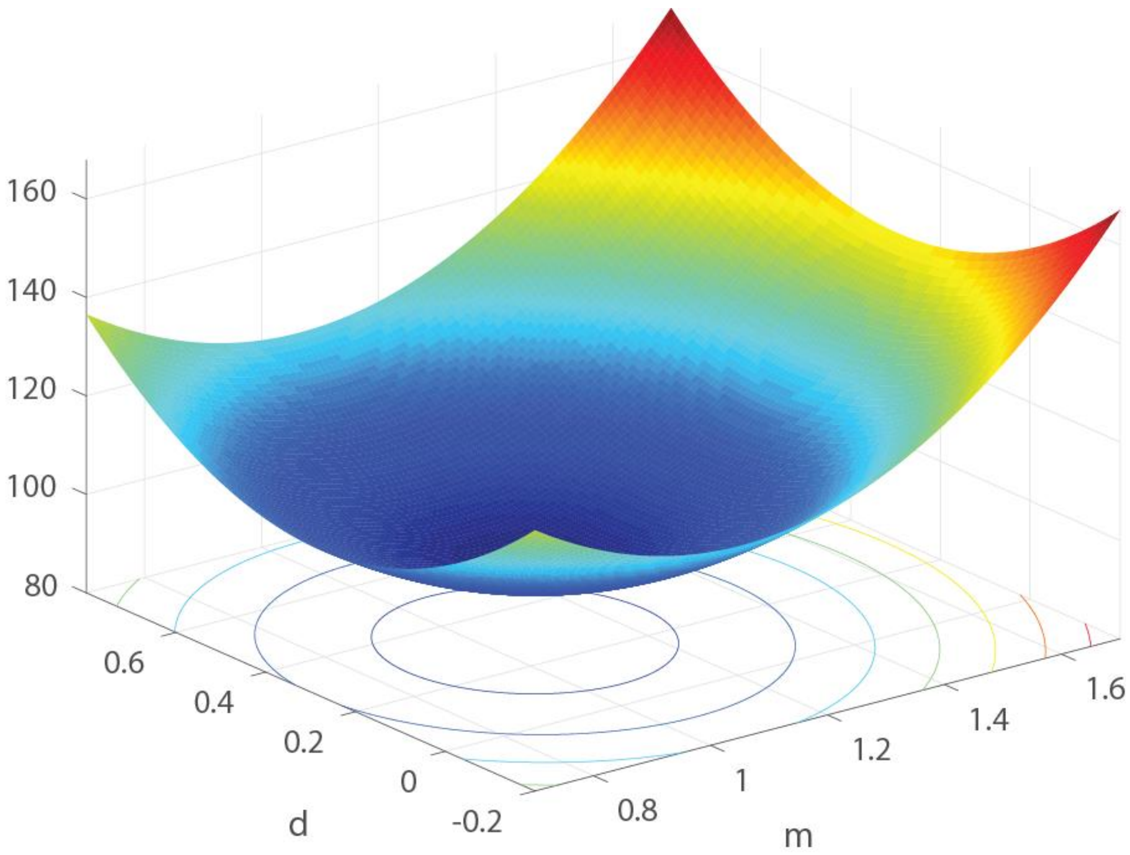
Recall: Line Fitting



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$\rightarrow$

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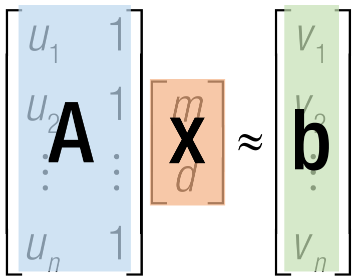
$\rightarrow$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

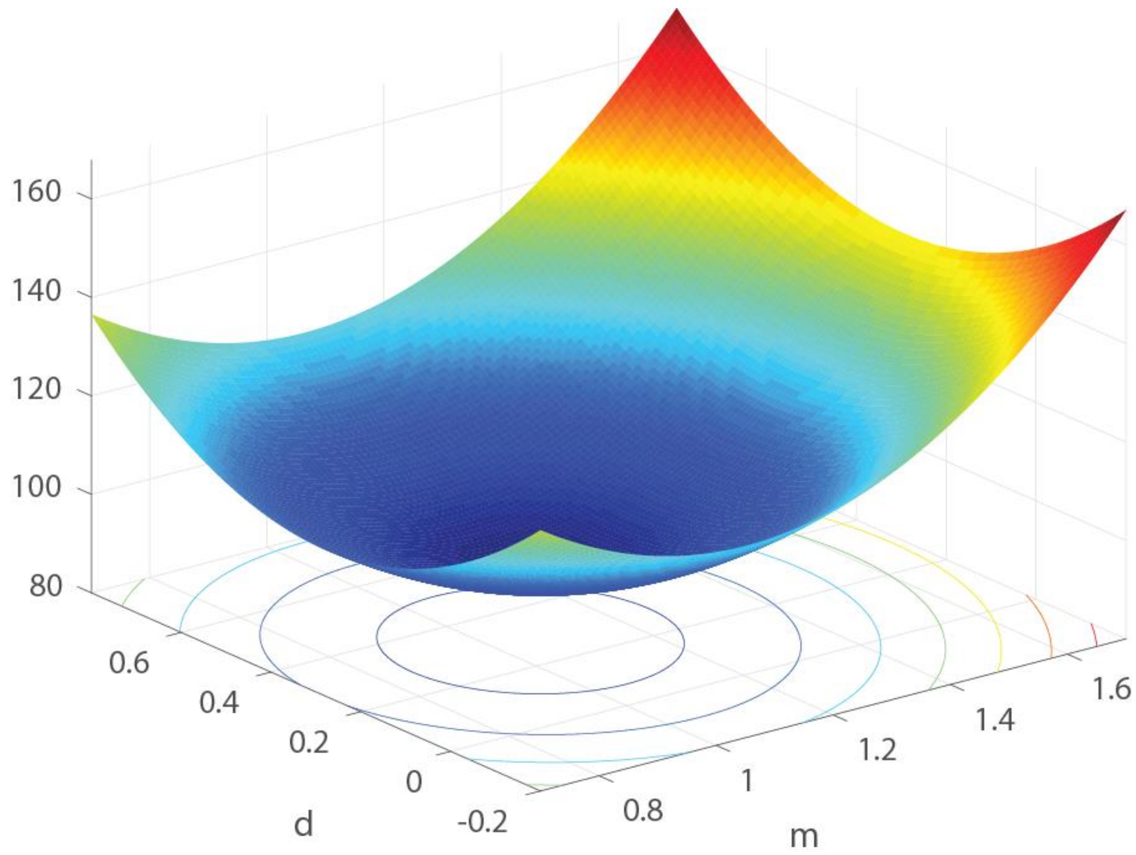
We can't invert **A**.

Recall: Line Fitting

Total error: $$E = \sum_{i=1}^{n} \left(v_i - (\textcolor{orange}{m} u_i + \textcolor{orange}{d})\right)^2$$



$$E = \left(\mathbf{Ax} - \mathbf{b}\right)^{\mathrm{T}} \left(\mathbf{Ax} - \mathbf{b}\right) = \left\|\mathbf{Ax} - \mathbf{b}\right\|^2$$



Recall: Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

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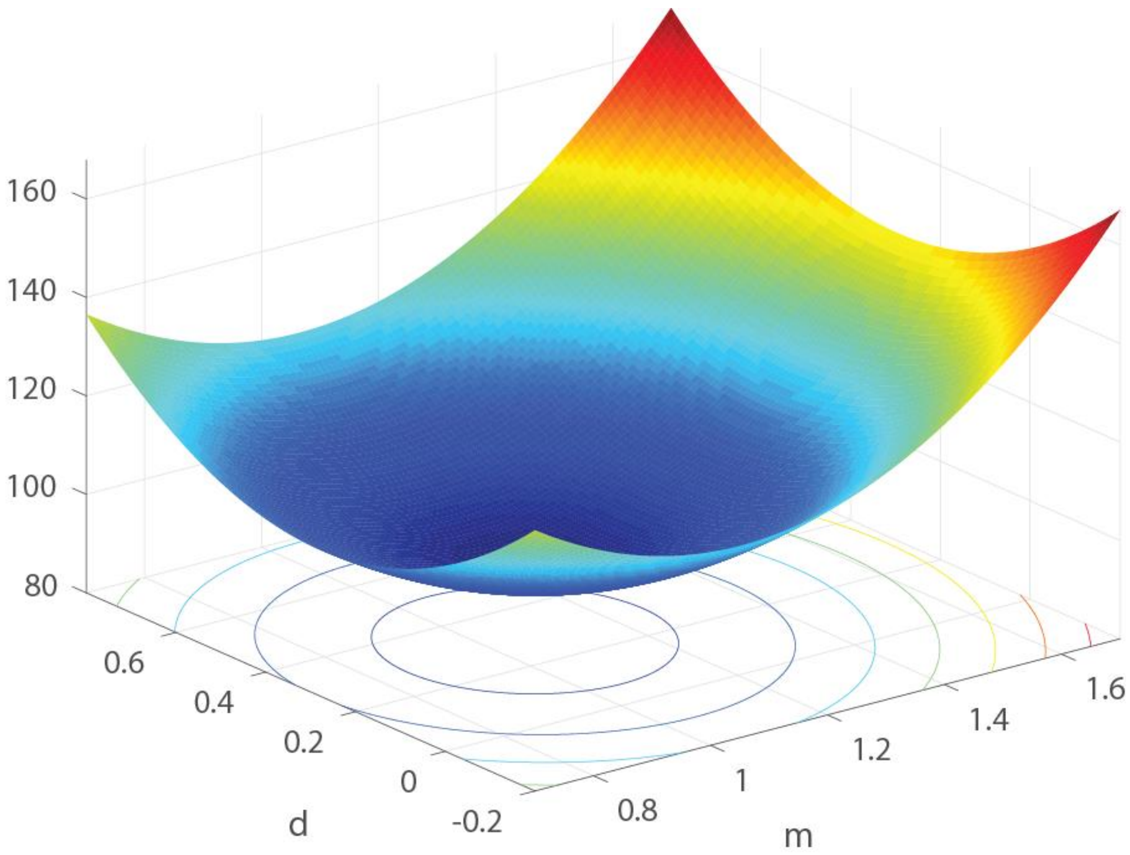
$$\mathbf{A}\mathbf{x} \approx \mathbf{b}$$



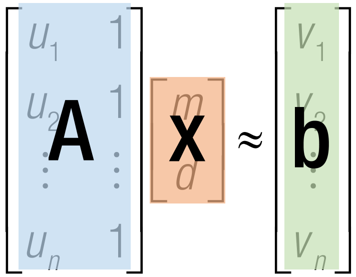
$$E = (\mathbf{Ax} - \mathbf{b})^{\mathsf{T}} (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \quad ?$$

Recall: Line Fitting

Total error: $E=\sum_{i=1}^{n}\left(v_i-(mu_i+d)\right)^2$



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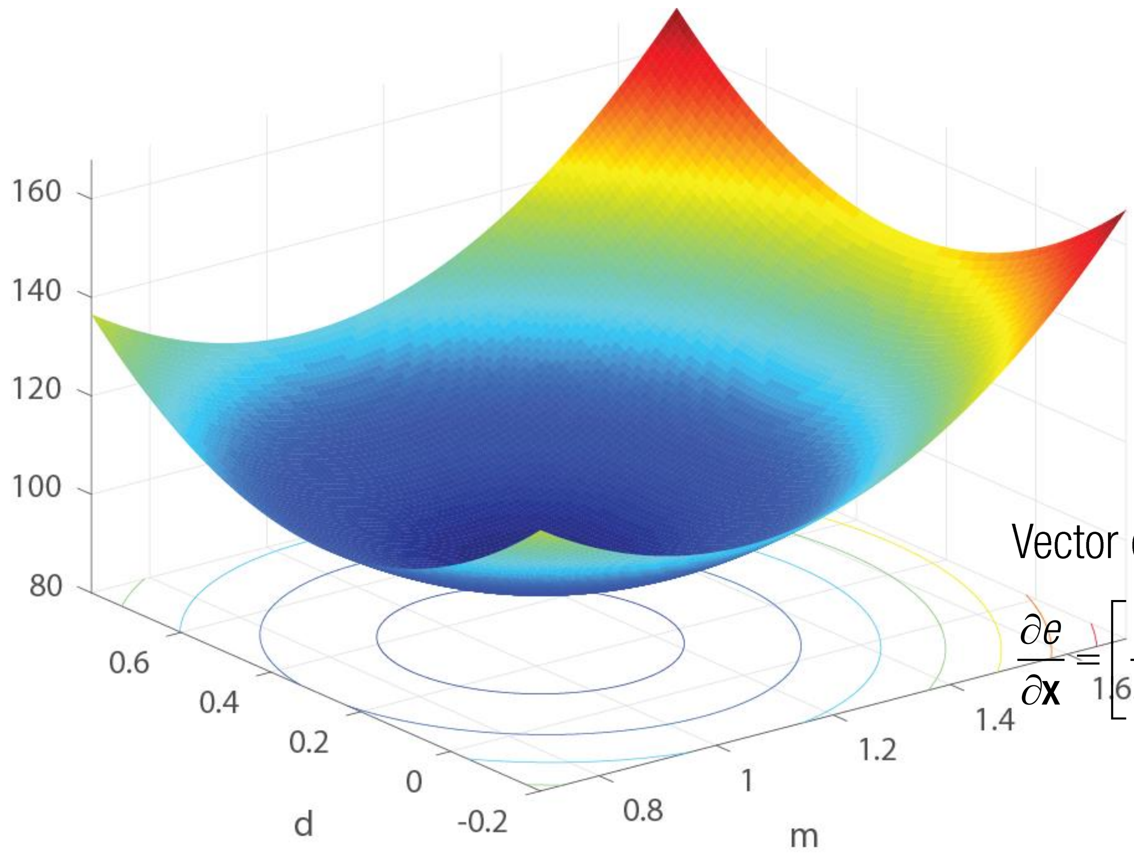

Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}}=\begin{bmatrix}\frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n}\end{bmatrix}$$

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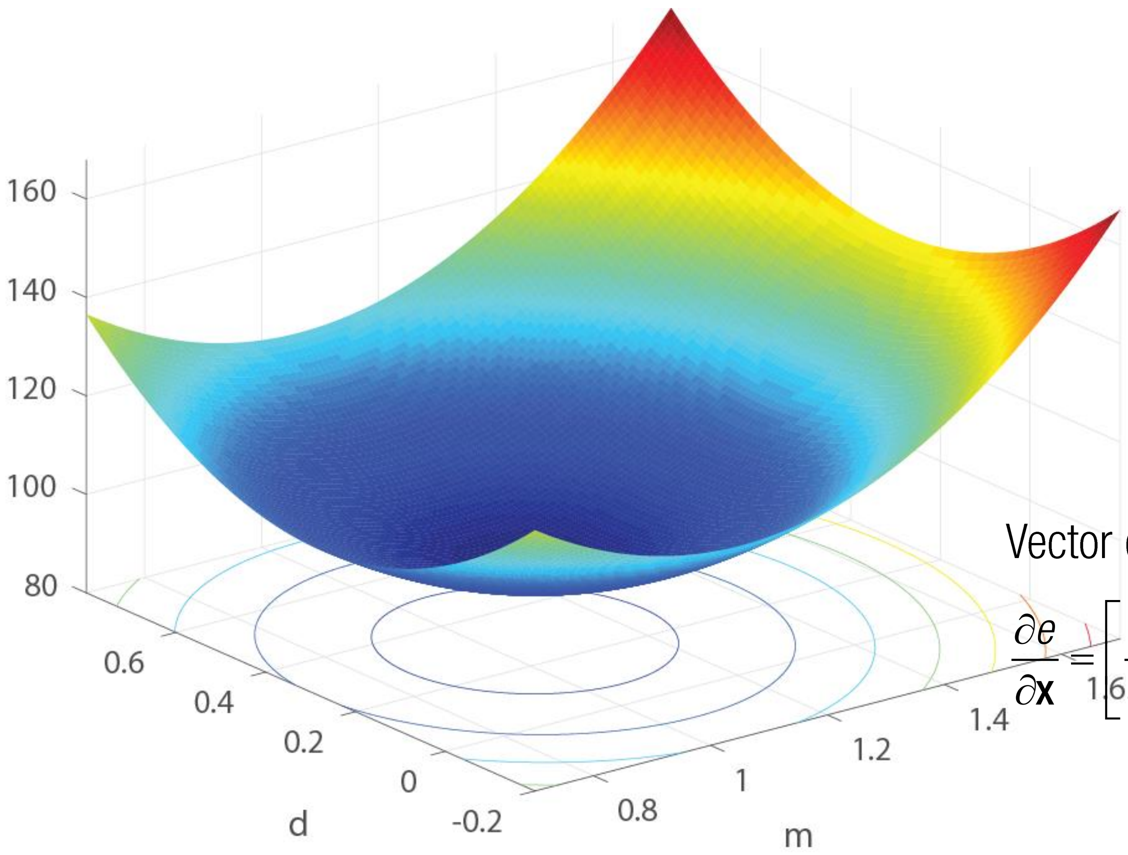

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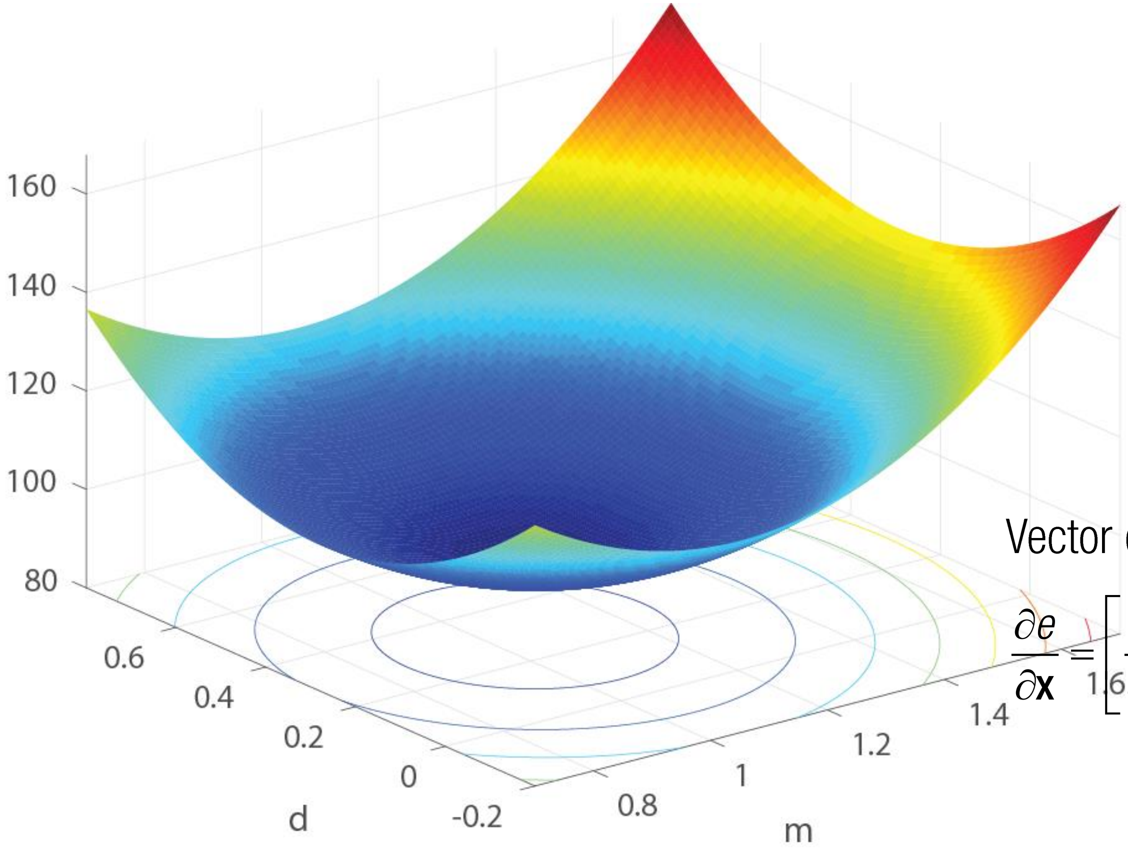
Ex) $e = \mathbf{c}^{\mathsf{T}}\mathbf{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^{\mathsf{T}}\mathbf{x}}{\partial \mathbf{x}} =$$

Recall: Line Fitting



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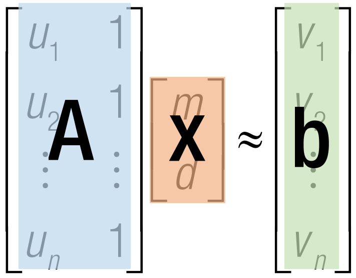

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$$\frac{\partial \mathbf{c}^{\mathsf{T}}\mathbf{x}}{\partial \mathbf{x}}=\frac{\partial}{\partial \mathbf{x}}\left(c_1x_1+\cdots+c_nx_n\right)$$

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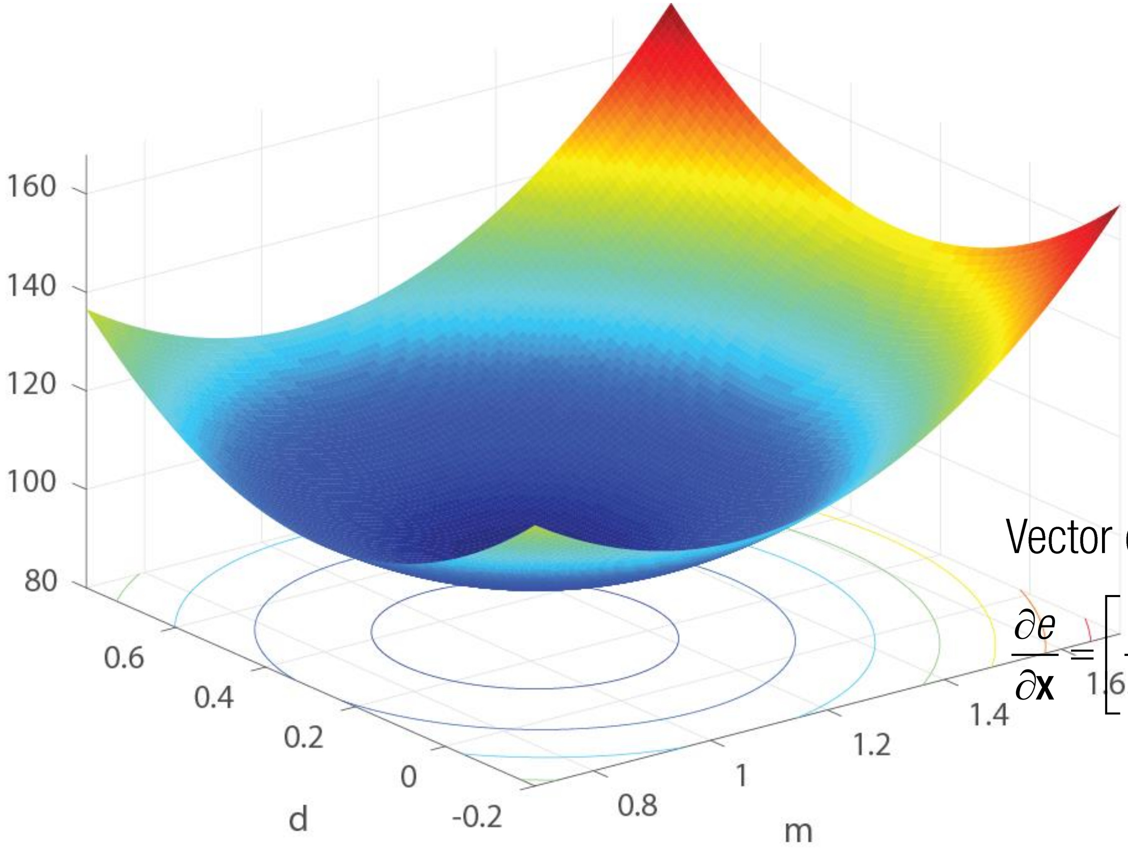

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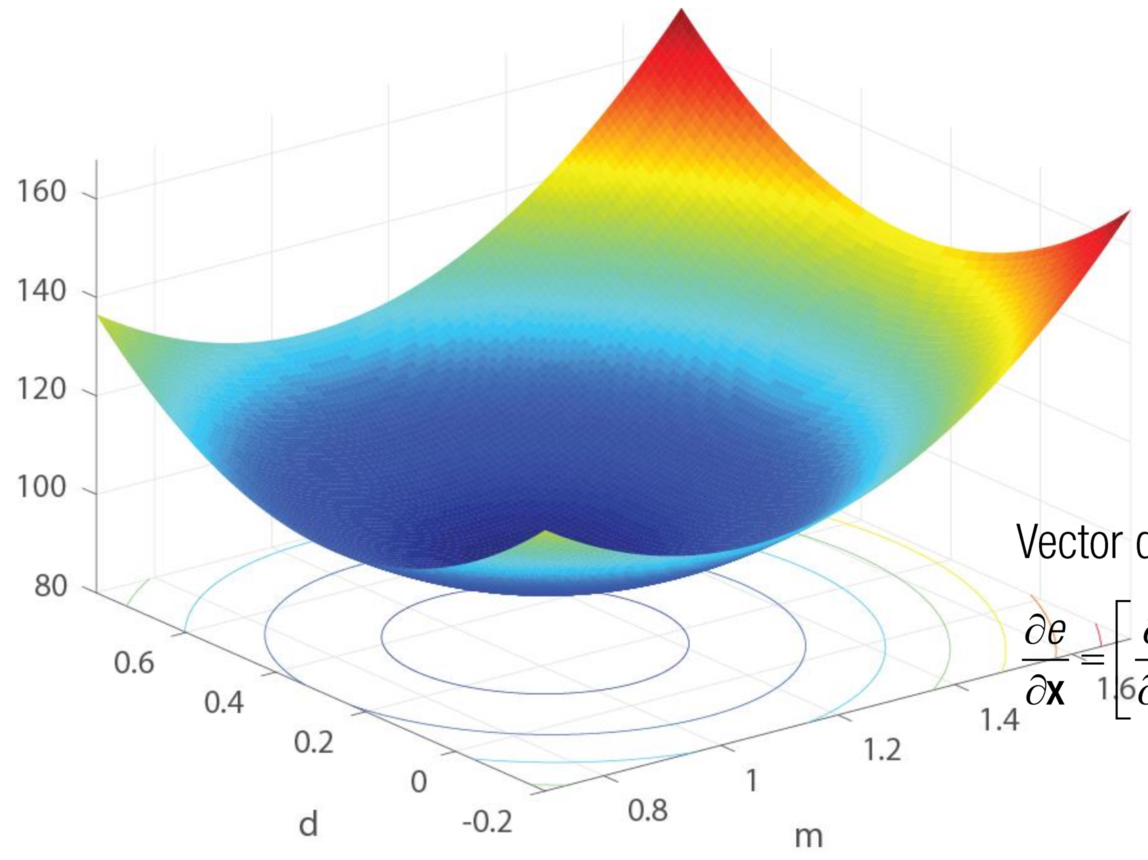

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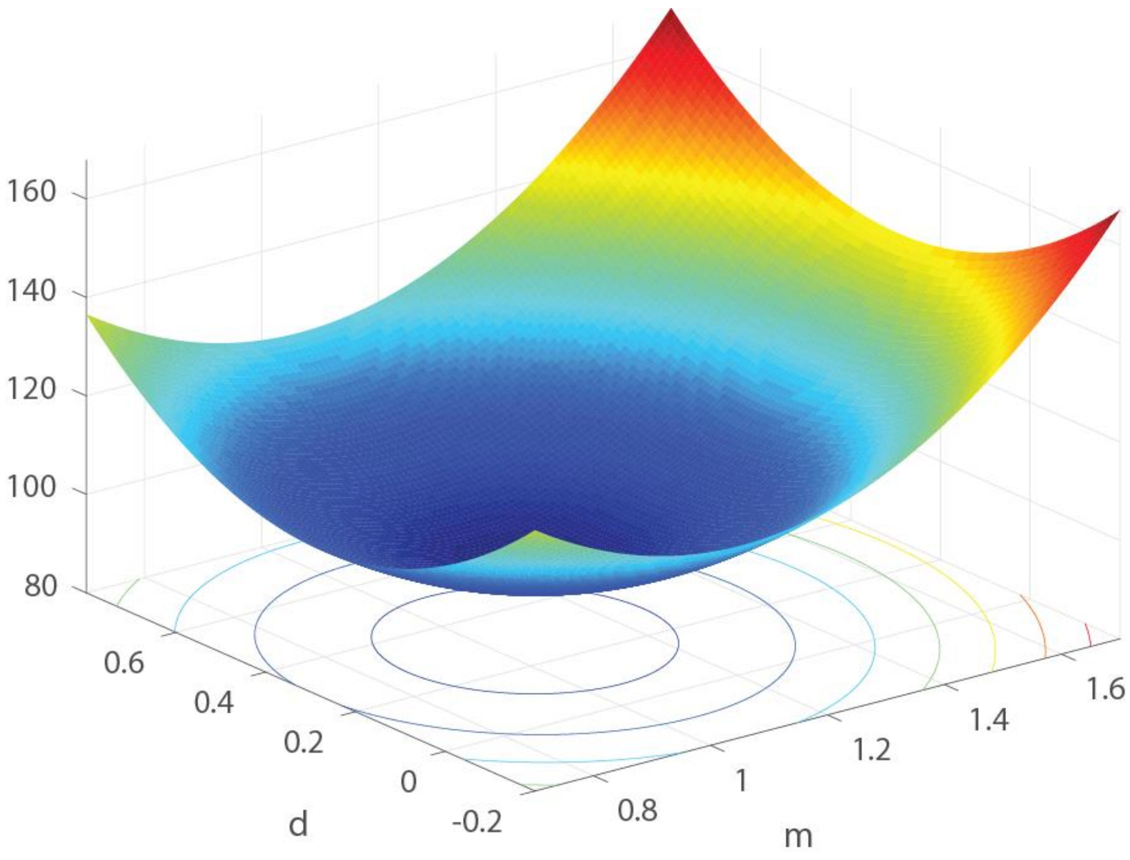

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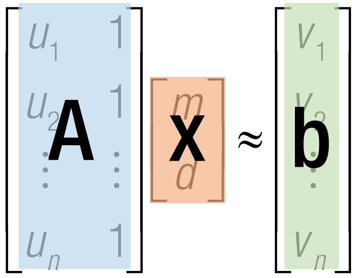
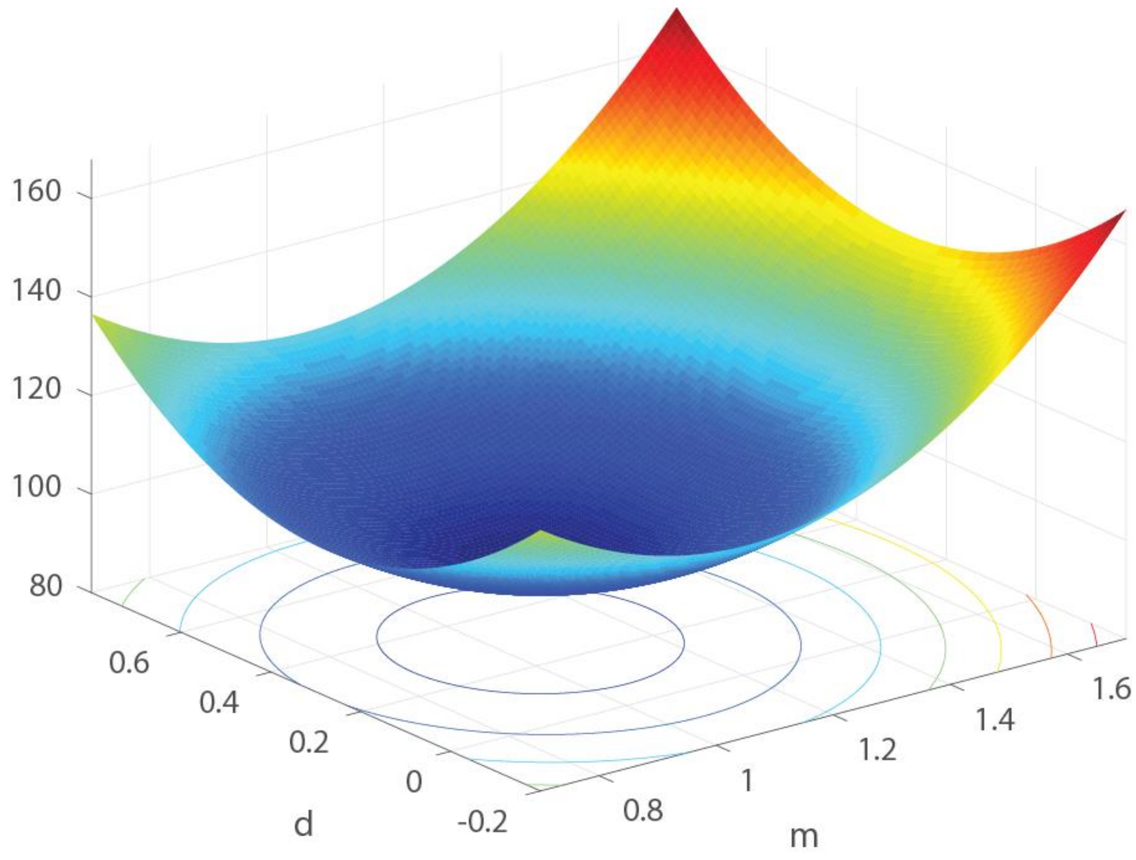
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{A}^{\mathsf{T}}\mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^{\mathsf{T}}\mathbf{Ax} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$$



Recall: Line Fitting



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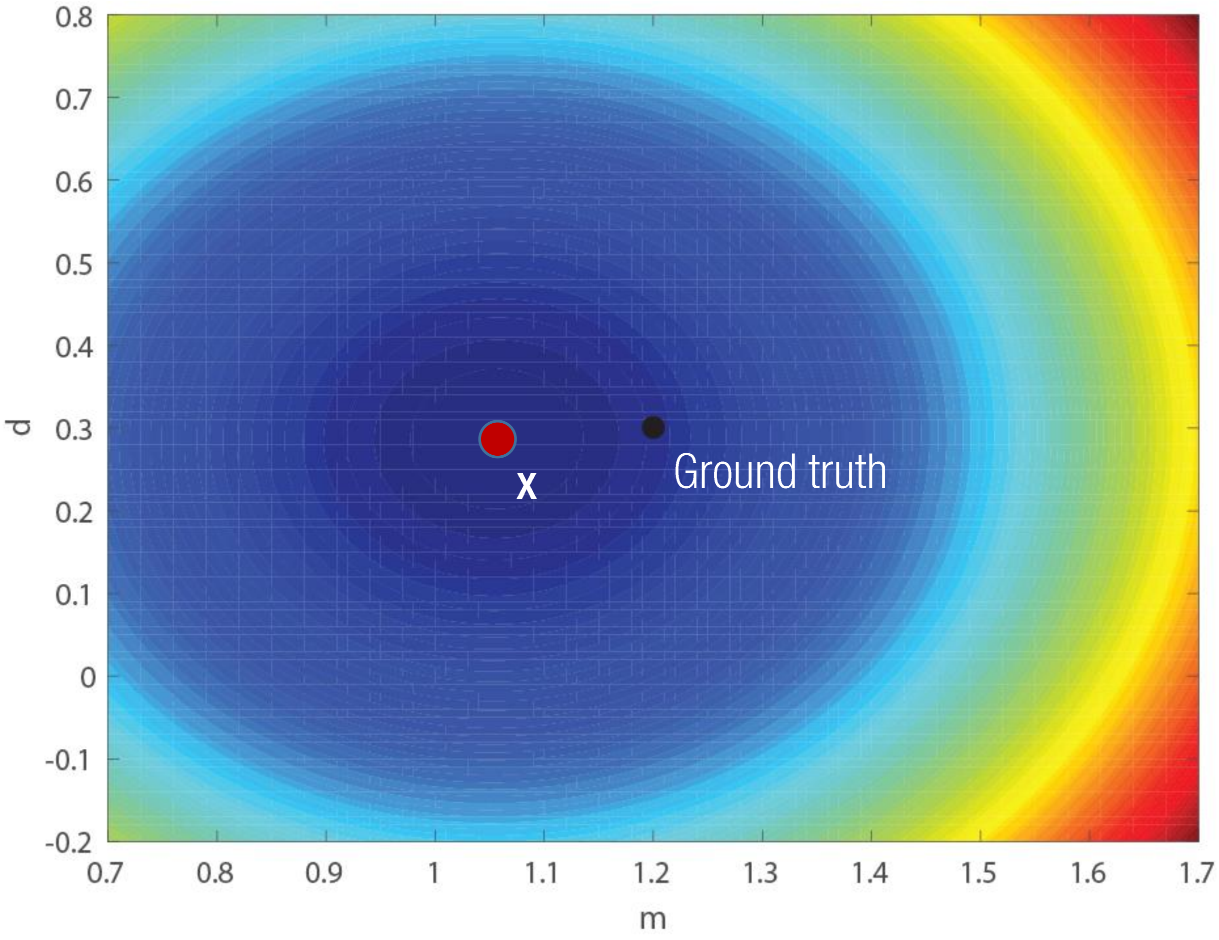
$$\longrightarrow \mathbf{A}^{\mathsf{T}}\mathbf{Ax} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$$

$$\mathbf{A}^{\mathsf{T}} \quad \mathbf{A} \quad \mathbf{x} = \mathbf{A}^{\mathsf{T}} \quad \mathbf{b}$$

Normal equation

Recall: Line Fitting



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

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$$\mathbf{A}\mathbf{x} \approx \mathbf{b}$$

$$E = (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} - 2\mathbf{x}^T\mathbf{A}^T\mathbf{b} + \mathbf{b}^T\mathbf{b}$$

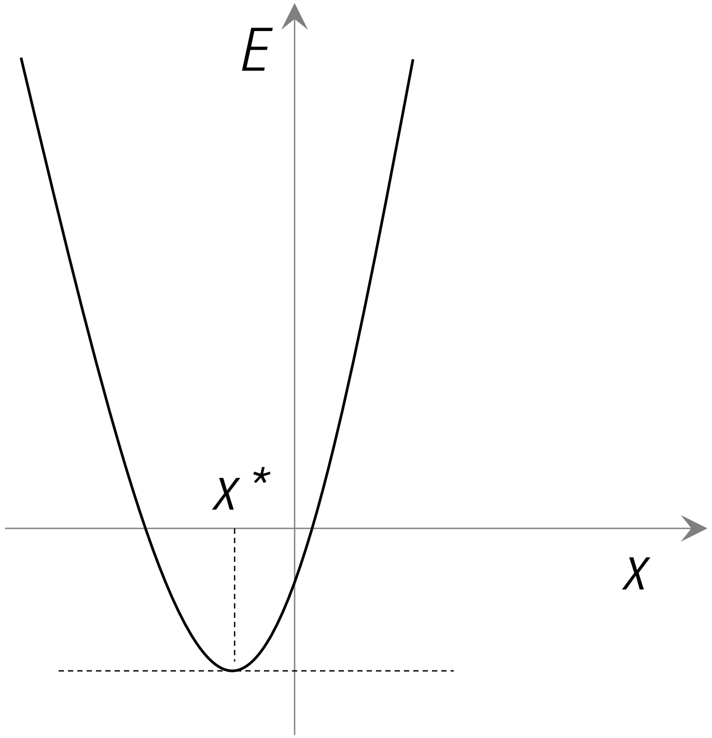
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^T\mathbf{A}\mathbf{x} - 2\mathbf{A}^T\mathbf{b} = 0$$

$$\longrightarrow \mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$$

$$\mathbf{A}^T \; \mathbf{A} \; \mathbf{x} = \mathbf{A}^T \; \mathbf{b}$$

$$\mathbf{x} = \left[\mathbf{A}^T \; \mathbf{A}\right]^{-1} \mathbf{A}^T \; \mathbf{b}$$

Linear System Recap



$$\left.\frac{dE}{dx}\right|_{x=x^*} = 0$$

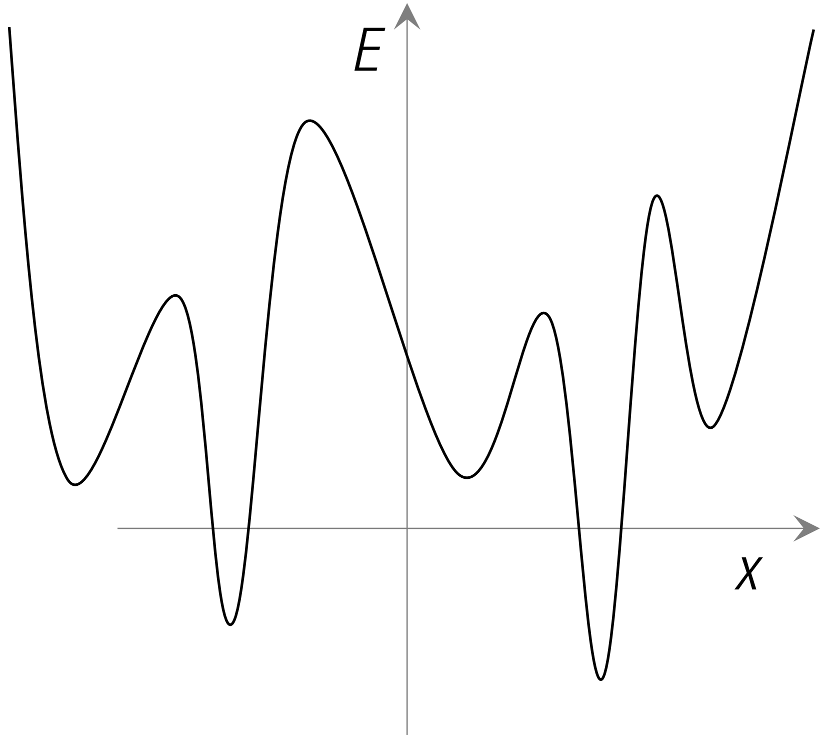
$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$\mathbf{x} = \left[\mathbf{A}^T \mathbf{A} \right]^{-1} \mathbf{A}^T \mathbf{b}$$

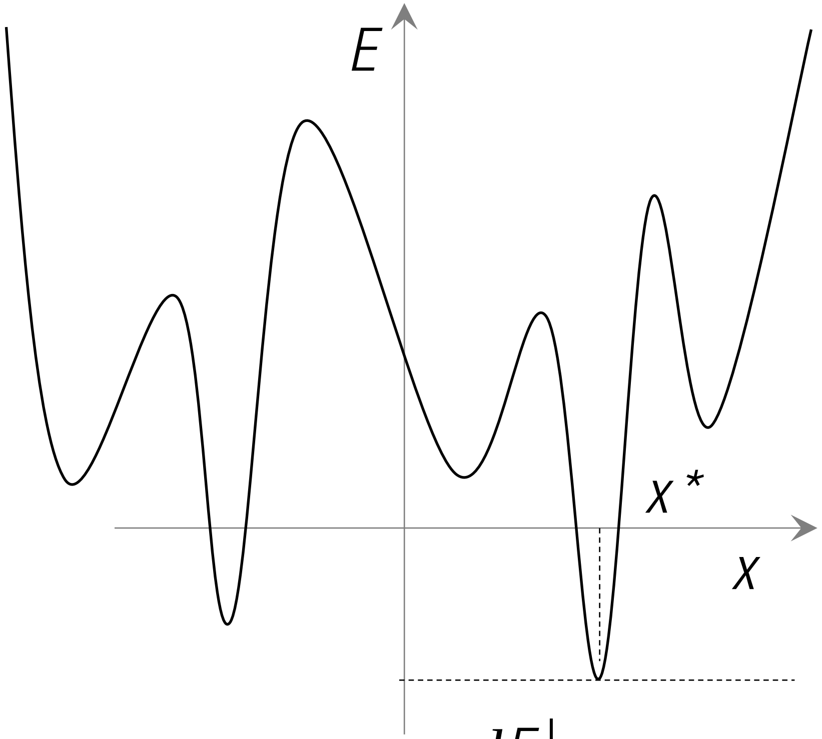
- Has the global solution
- Has the closed form solution (non-iterative solve)
- Is solved efficiently ($O(n^2)$)
- Does not require an initialization

Nonlinear System

$$f(\mathbf{x}) = \mathbf{b}$$



Nonlinear System

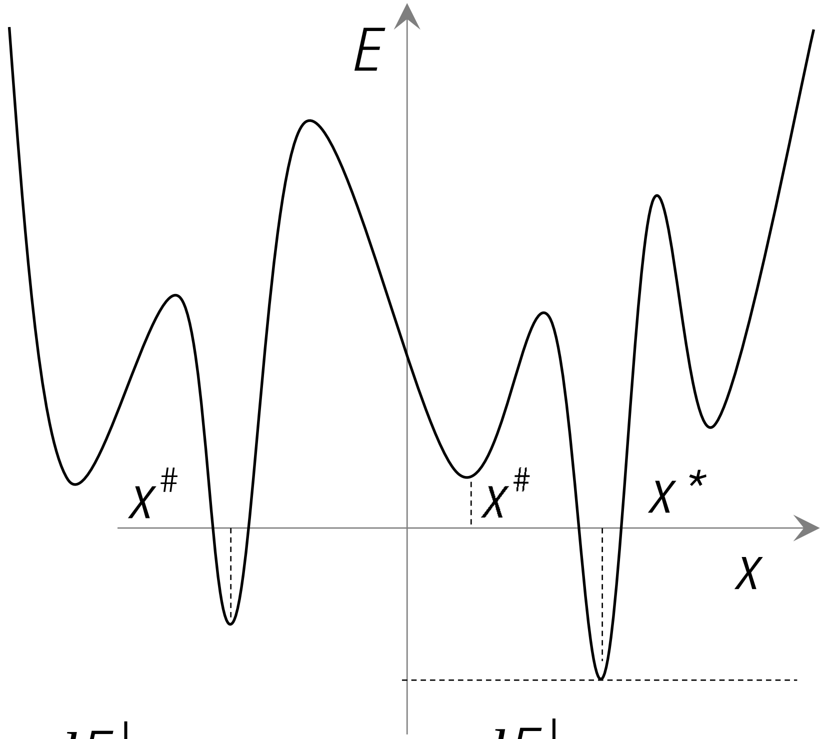


$$\left.\frac{dE}{dx}\right|_{x=x^*} = 0$$

$$f(\mathbf{x}) = \mathbf{b}$$

$$E = \left\| f(\mathbf{x}) - \mathbf{b} \right\|^2$$

Nonlinear System



$$\left.\frac{dE}{dx}\right|_{x=x^{\#}} = 0 \qquad \left.\frac{dE}{dx}\right|_{x=x^{*}} = 0$$

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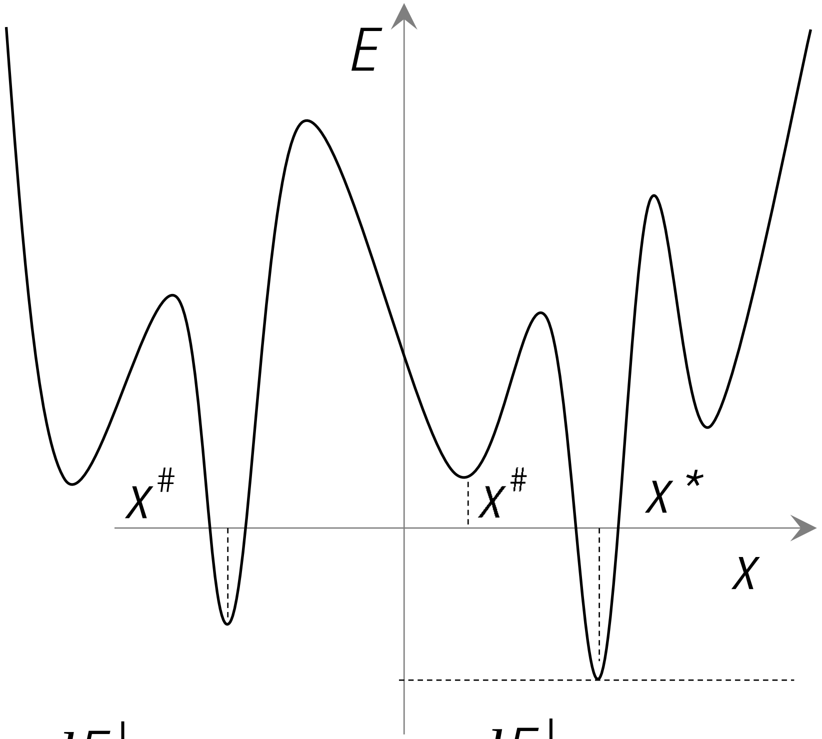

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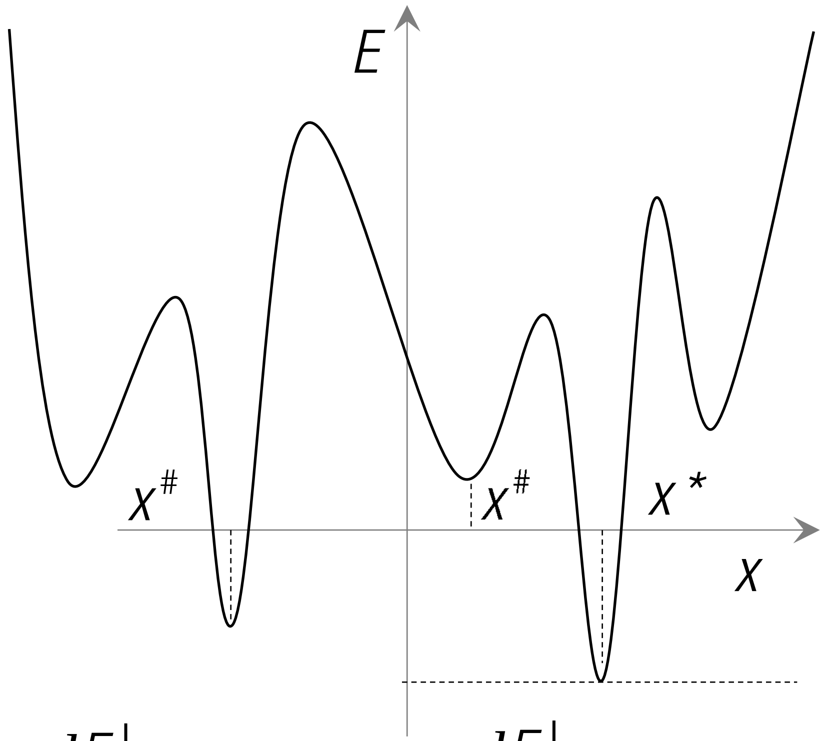
$$E=\left(f(\mathbf{x})-\mathbf{b}\right)^{\mathsf{T}}\left(f(\mathbf{x})-\mathbf{b}\right)$$
$$=f(\mathbf{x})^{\mathsf{T}}f(\mathbf{x})-2f(\mathbf{x})^{\mathsf{T}}\mathbf{b}+\mathbf{b}^{\mathsf{T}}\mathbf{b}$$

Nonlinear System



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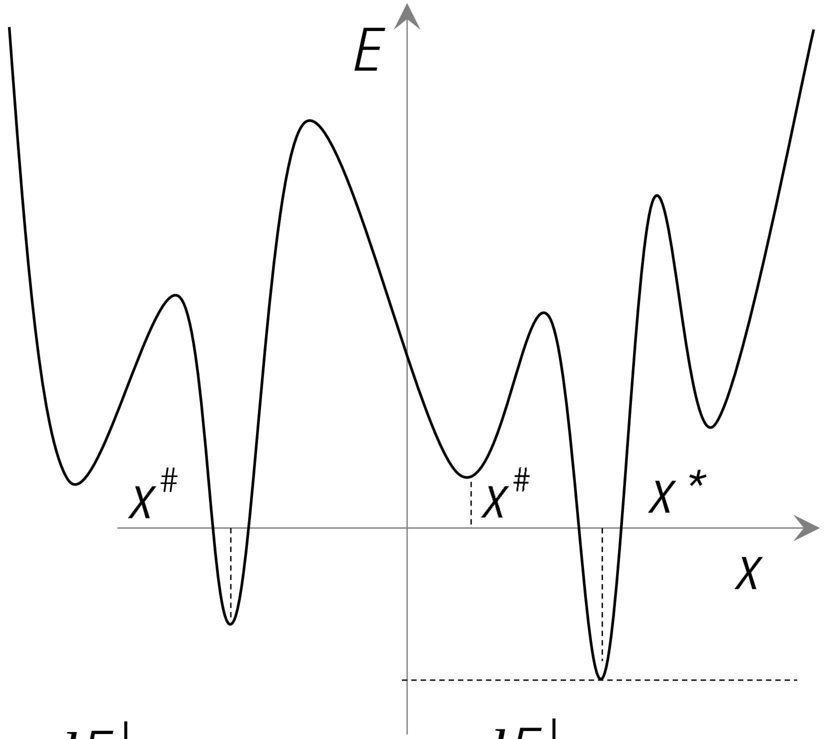
$$\frac{\partial E}{\partial \mathbf{x}}=2\frac{\partial f(\mathbf{x})^{\mathsf{T}}}{\partial \mathbf{x}}f(\mathbf{x})-2\frac{\partial f(\mathbf{x})^{\mathsf{T}}}{\partial \mathbf{x}}\mathbf{b}=\mathbf{0}$$

Nonlinear System



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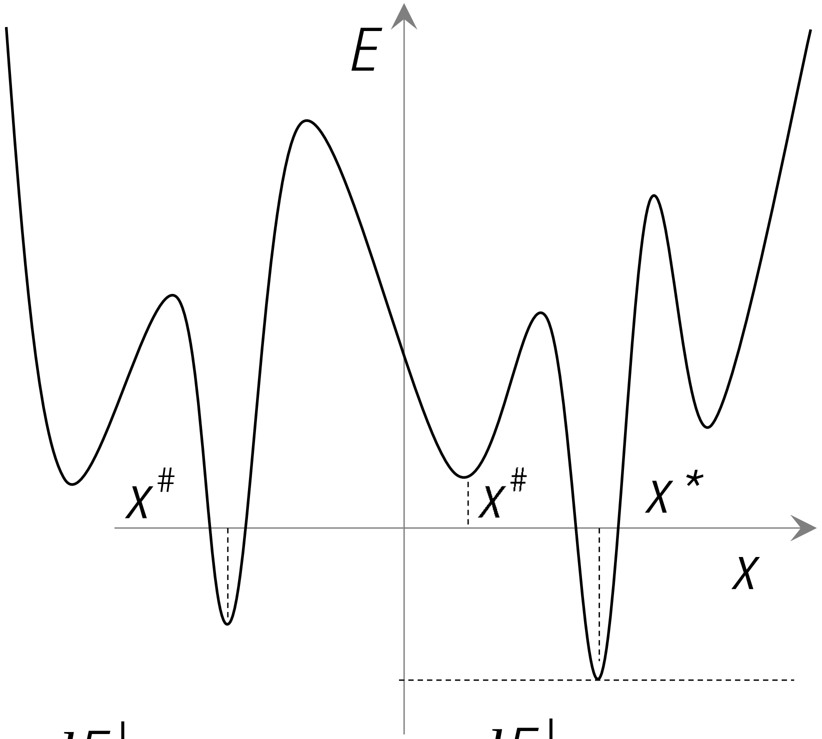
Nonlinear System



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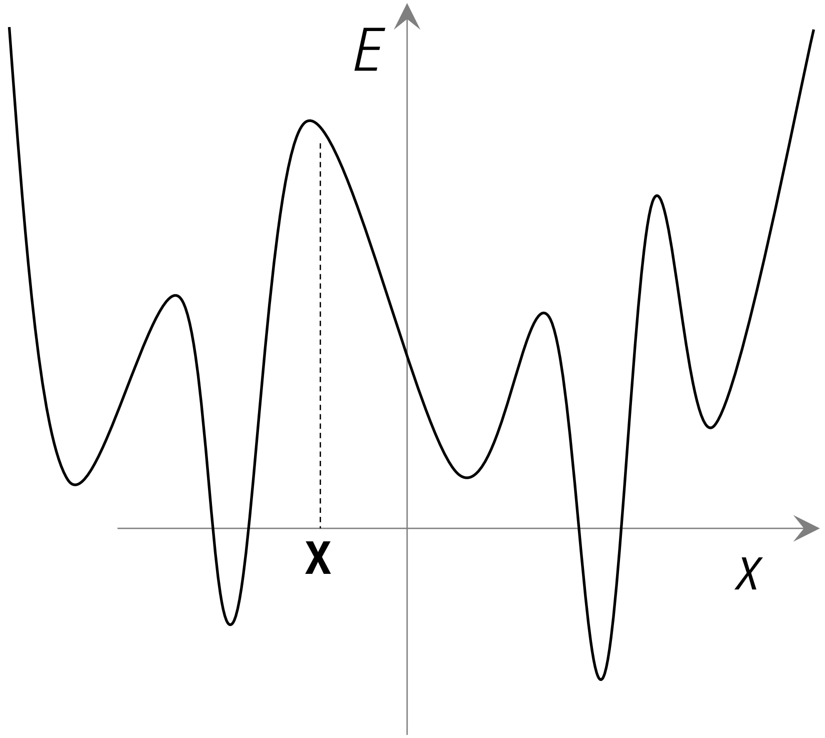
Find **x** such that the following equation is satisfied:

$$\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}}f(\mathbf{x})=\frac{\partial f(\mathbf{x})^{\mathrm{T}}}{\partial \mathbf{x}}\mathbf{b}$$

How?

Nonlinear System



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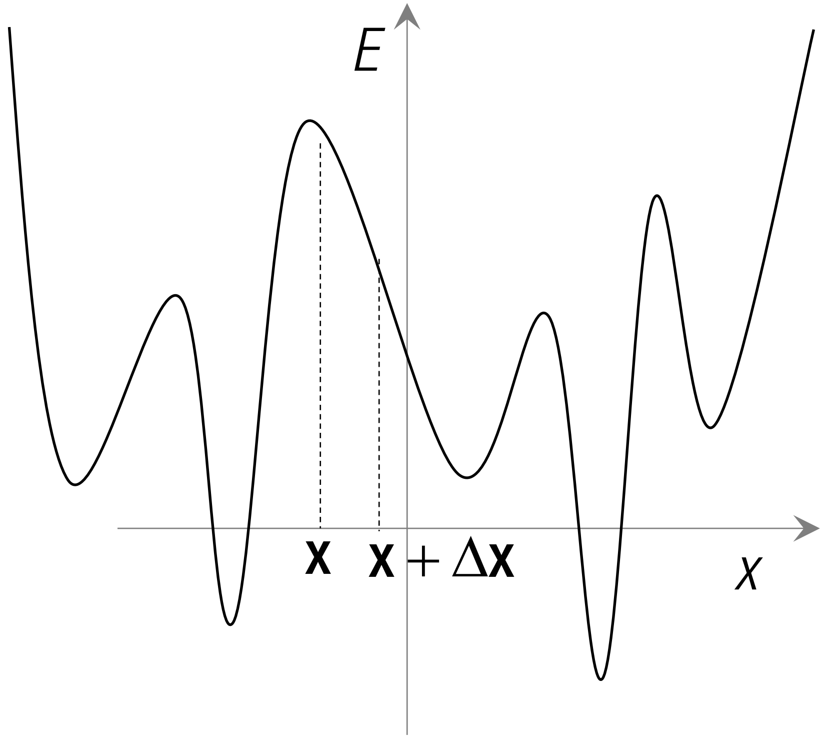
How?

Strategy: Given **x**,

Nonlinear System



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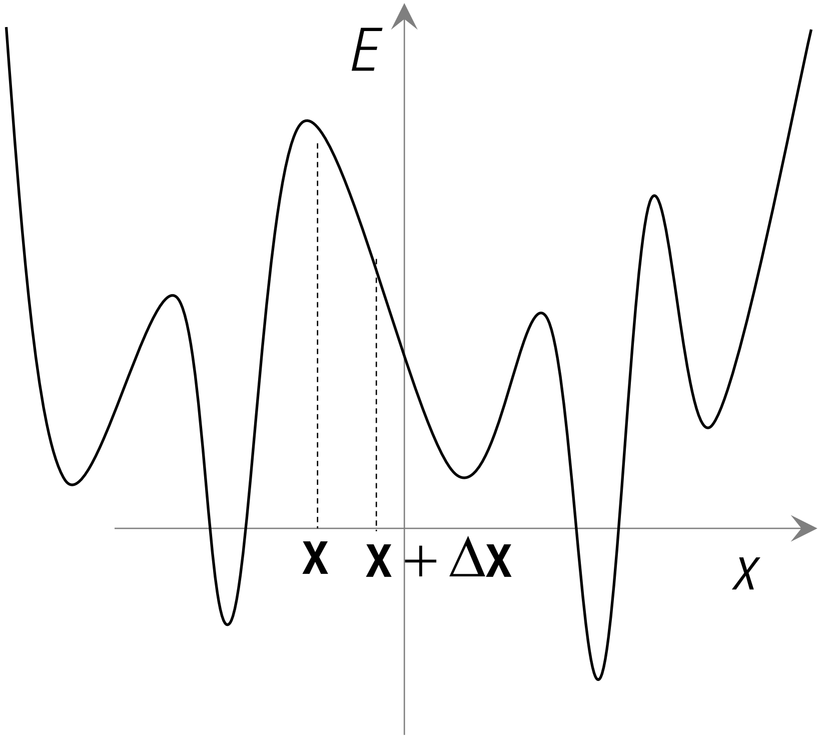
Strategy: Given **x,** move $\Delta\mathbf{x}$ such that $E(\mathbf{x}+\Delta\mathbf{x}) \leq E(\mathbf{x})$

Nonlinear System



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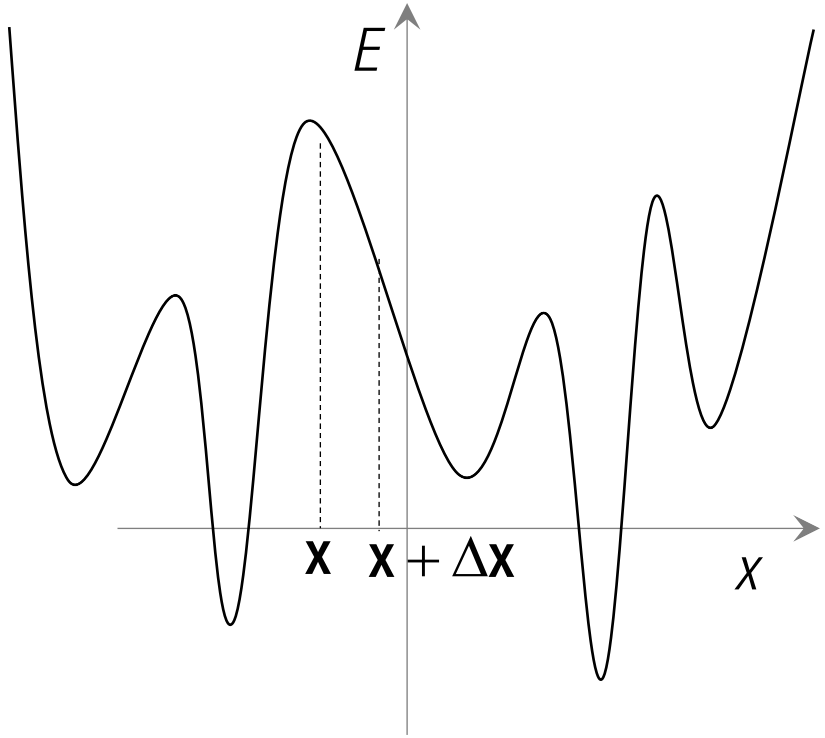
Taylor expansion:

$$f(\mathbf{x}+\Delta\mathbf{x}) =$$

Nonlinear System



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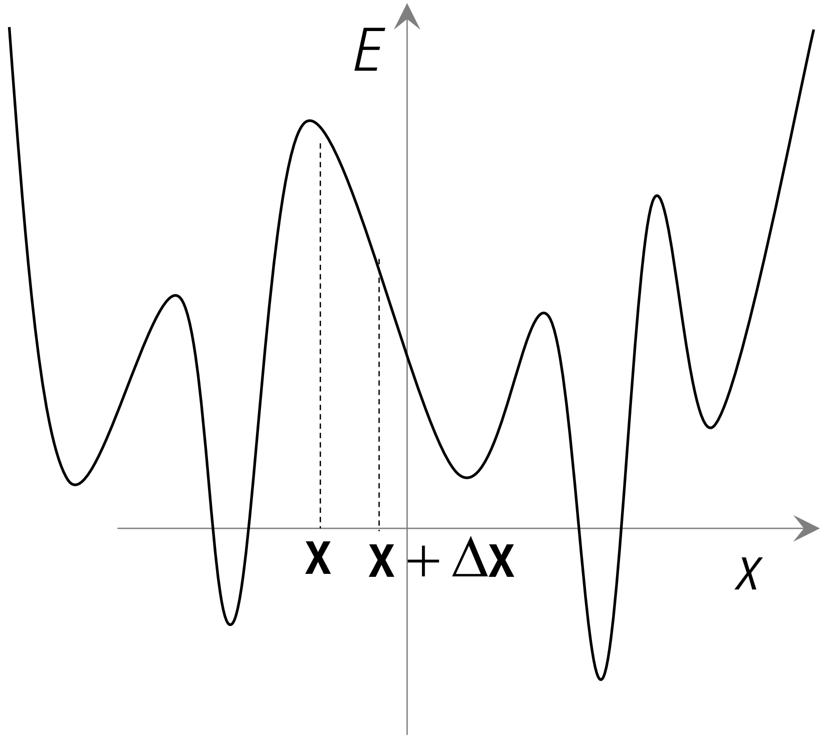
$$f(\mathbf{x}+\Delta\mathbf{x}) = f(\mathbf{x}) + \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta\mathbf{x} + \text{H.O.T.}$$

Nonlinear System



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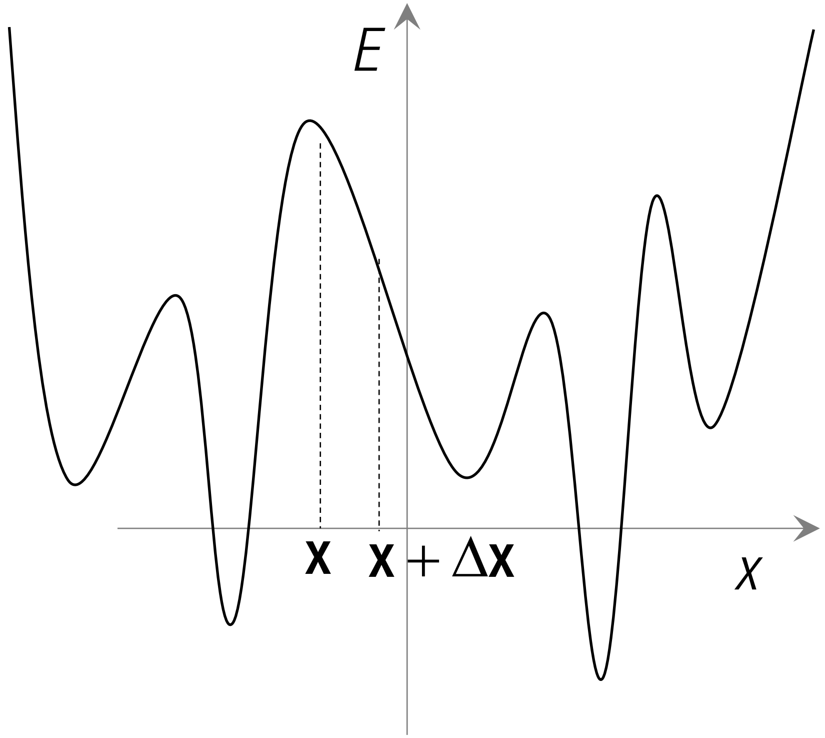
Nonlinear System



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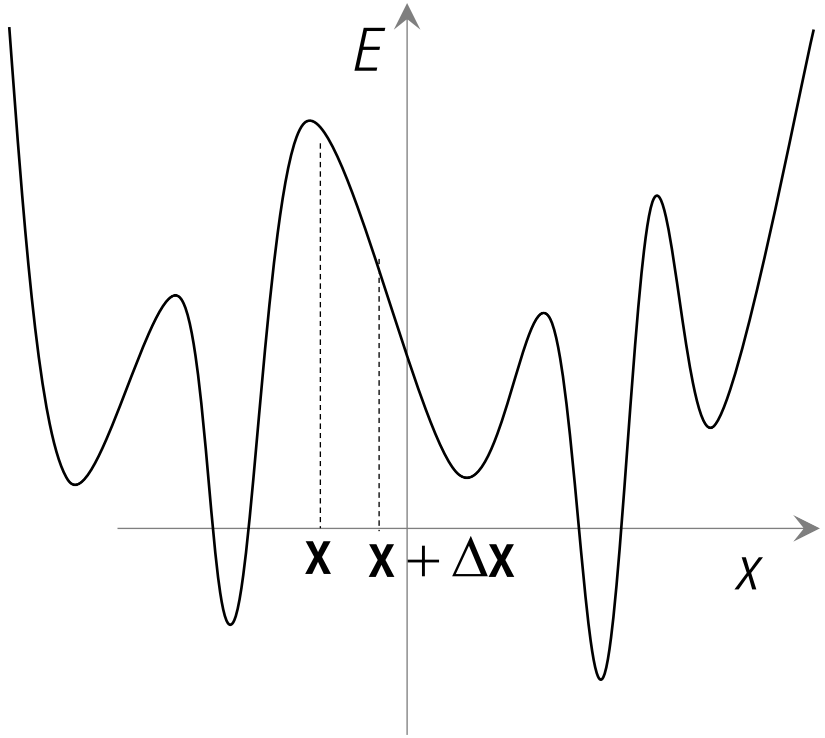
Nonlinear System



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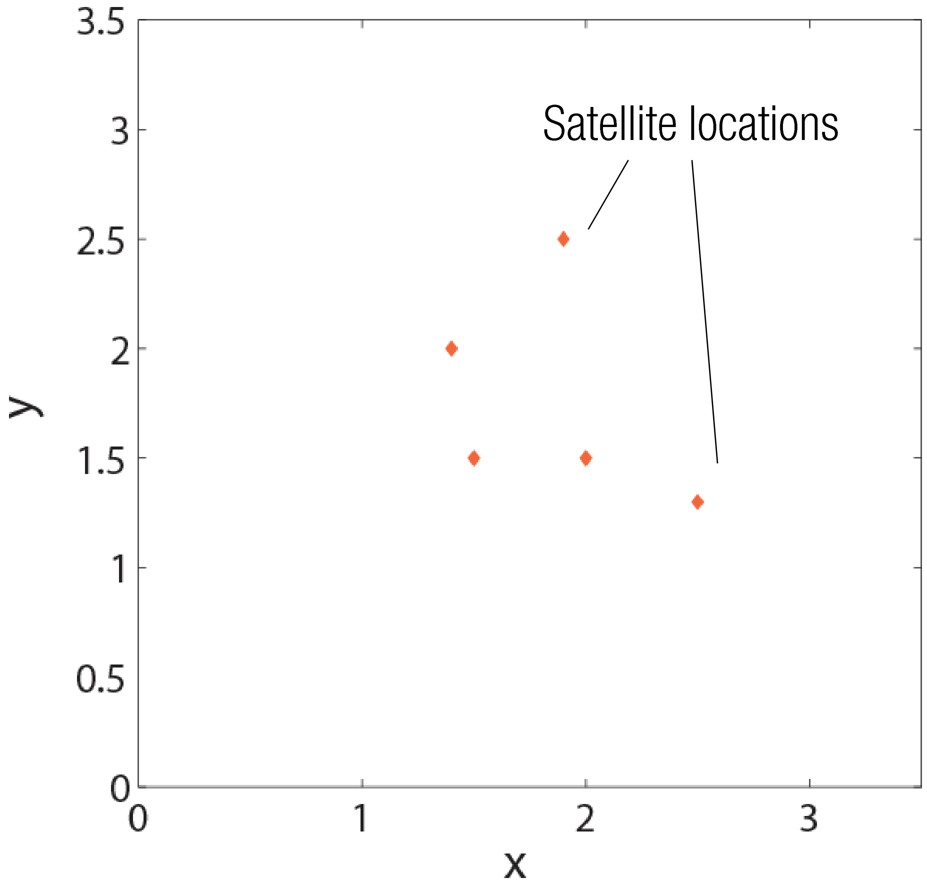
Cf.) $\mathbf{X} = \left[\mathbf{A}^{T} \mathbf{A} \right]^{-1} \mathbf{A}^{T} \mathbf{b}$

Where am I?: GPS Localization

Example:
Localization using range data from beacons



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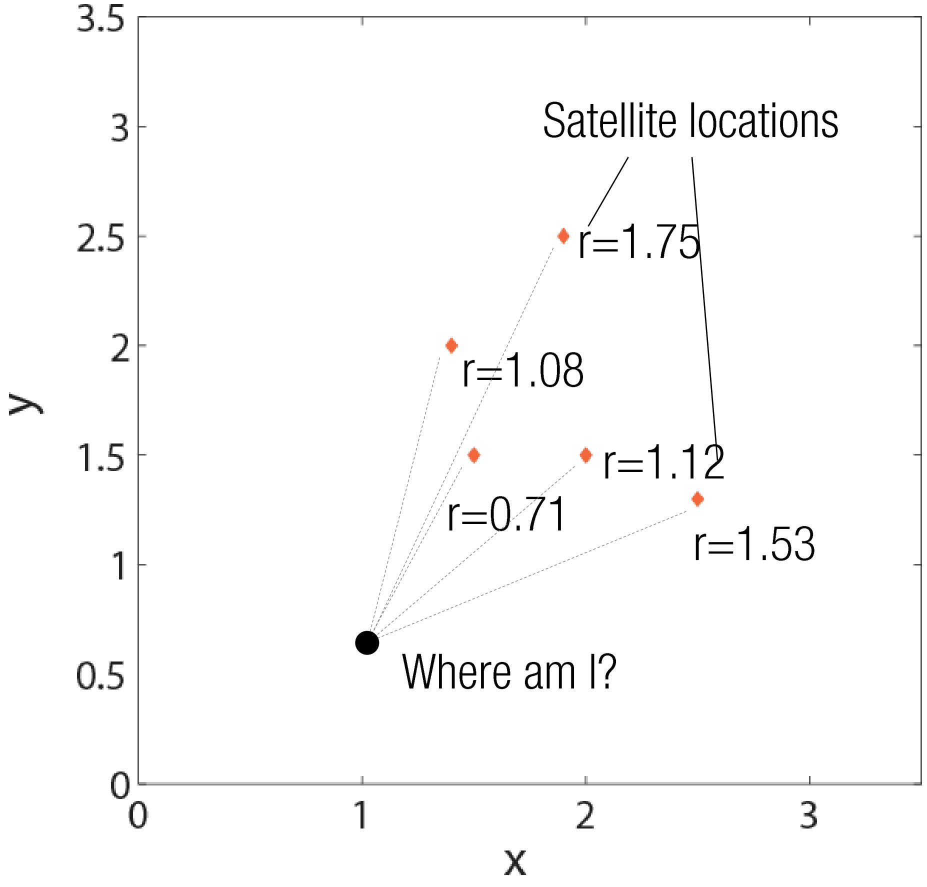

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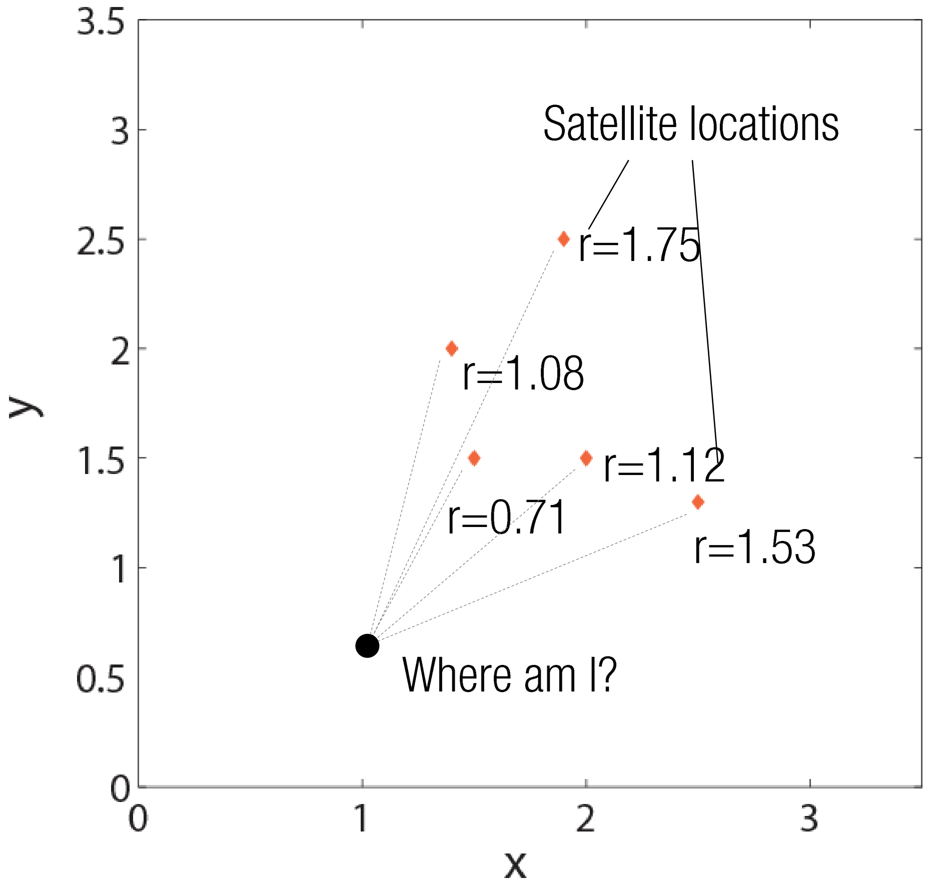

$E =$

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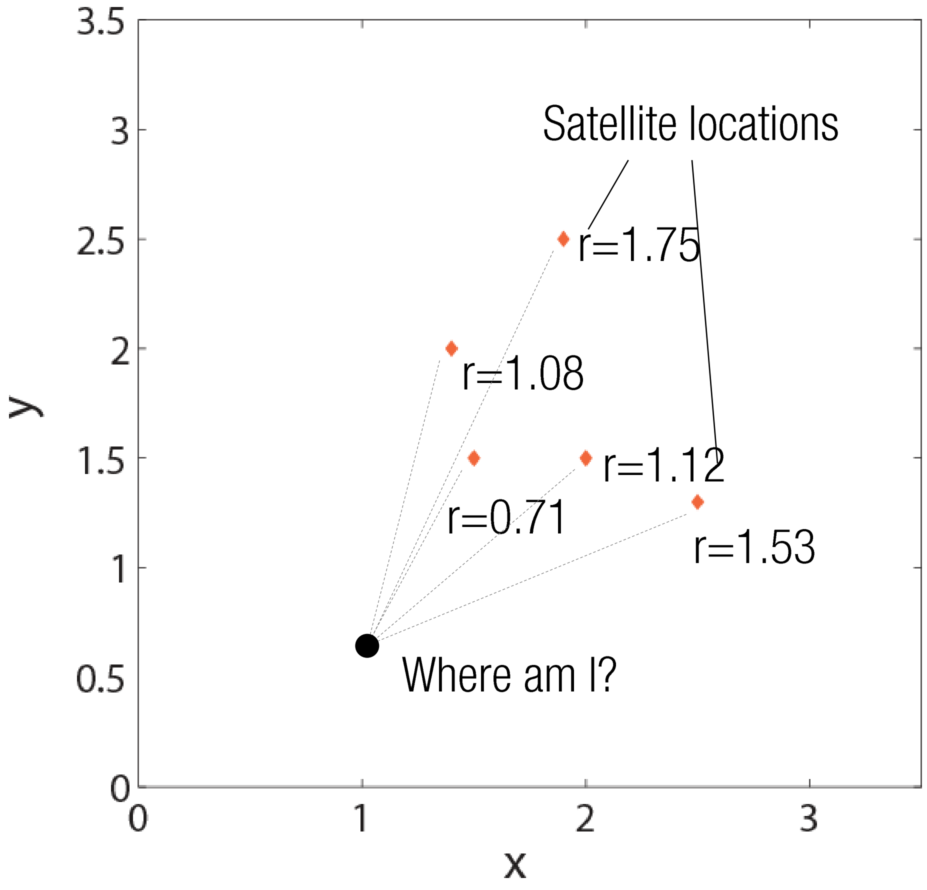
Localization using range data from beacons



$$E = \left\| \begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix} - \begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix} \right\|^2$$

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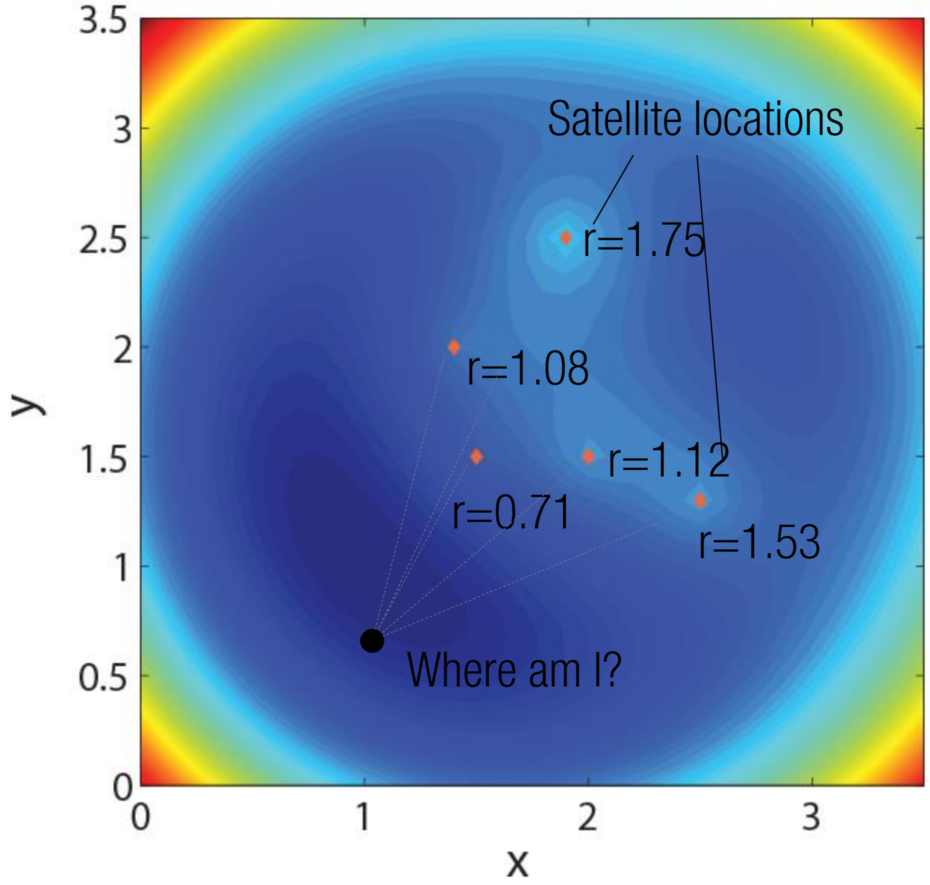
Localization using range data from beacons



$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

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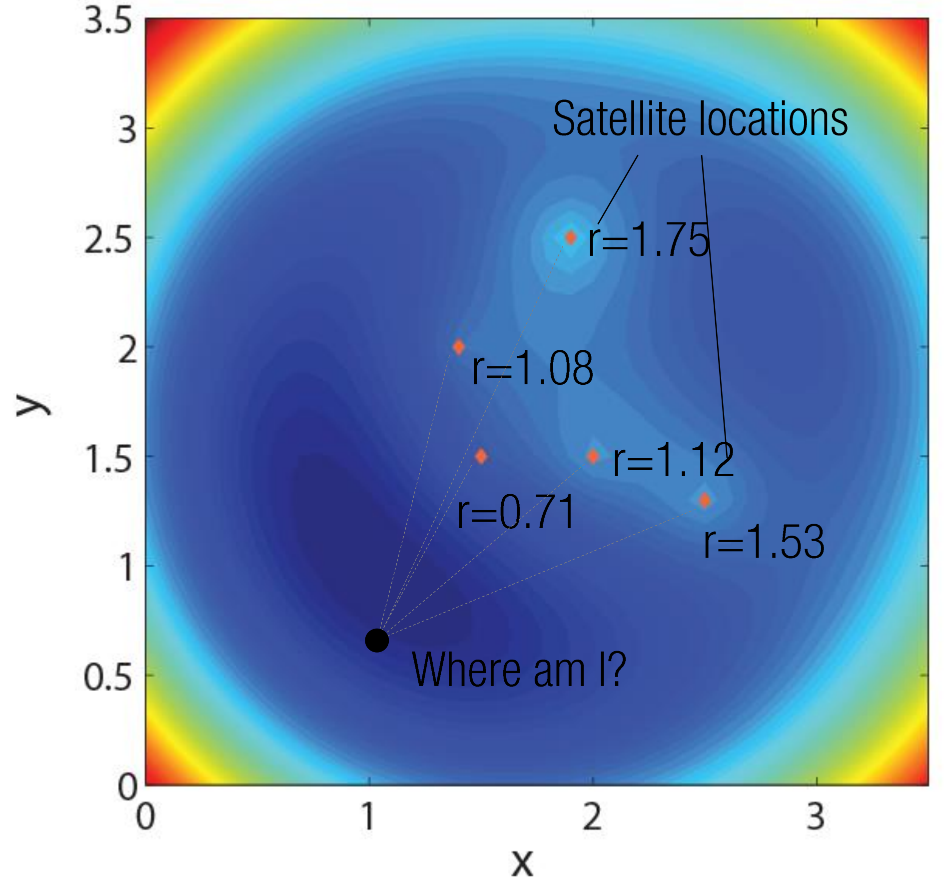

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$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathsf{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

where $\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial \mathbf{x}_1} & \cdots & \frac{\partial f_1}{\partial \mathbf{x}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial \mathbf{x}_n} & \cdots & \frac{\partial f_m}{\partial \mathbf{x}_n} \end{bmatrix}$: Jacobian

Where am I?: GPS Localization

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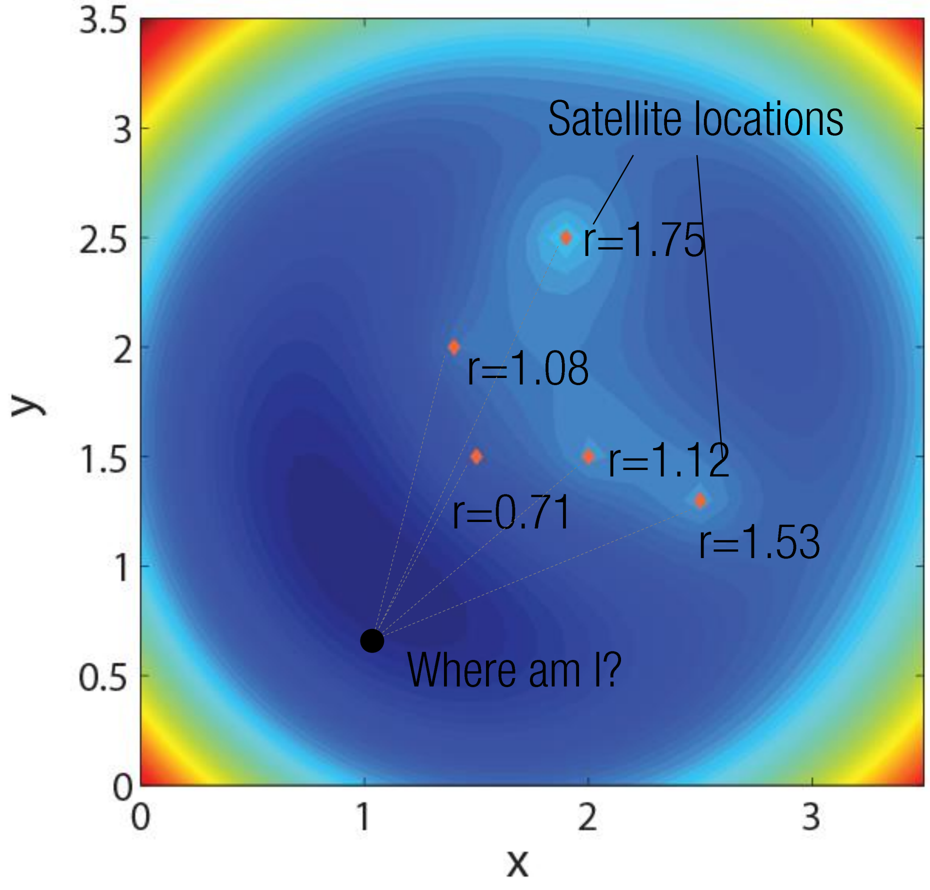
Localization using range data from beacons



$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - f(\mathbf{x}) \right)$$

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} -\dfrac{u_1 - x}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} & -\dfrac{v_1 - y}{\sqrt{(u_1 - x)^2 + (v_1 - y)^2}} \\ \vdots & \vdots \\ -\dfrac{u_5 - x}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} & -\dfrac{v_5 - y}{\sqrt{(u_5 - x)^2 + (v_5 - y)^2}} \end{bmatrix}$$

Where am I?: GPS Localization

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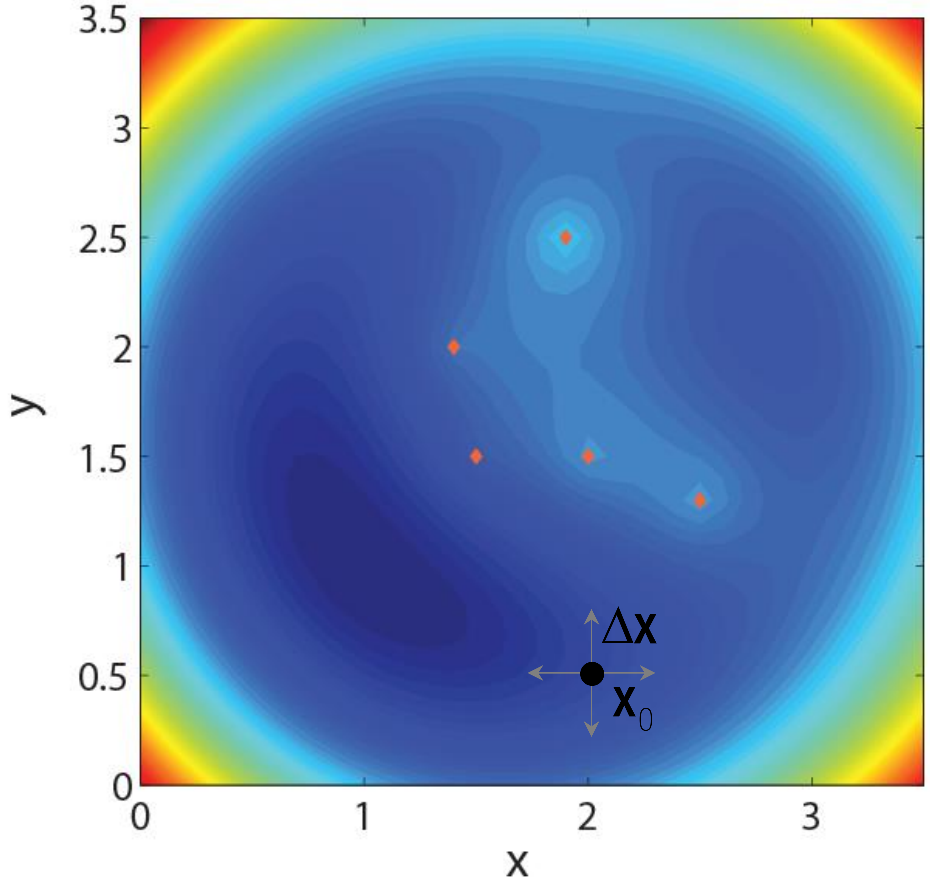

$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

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$$\Delta \mathbf{x} = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^{\mathrm{T}} \left(\mathbf{b} - f(\mathbf{x})\right)$$

Where am I?: GPS Localization

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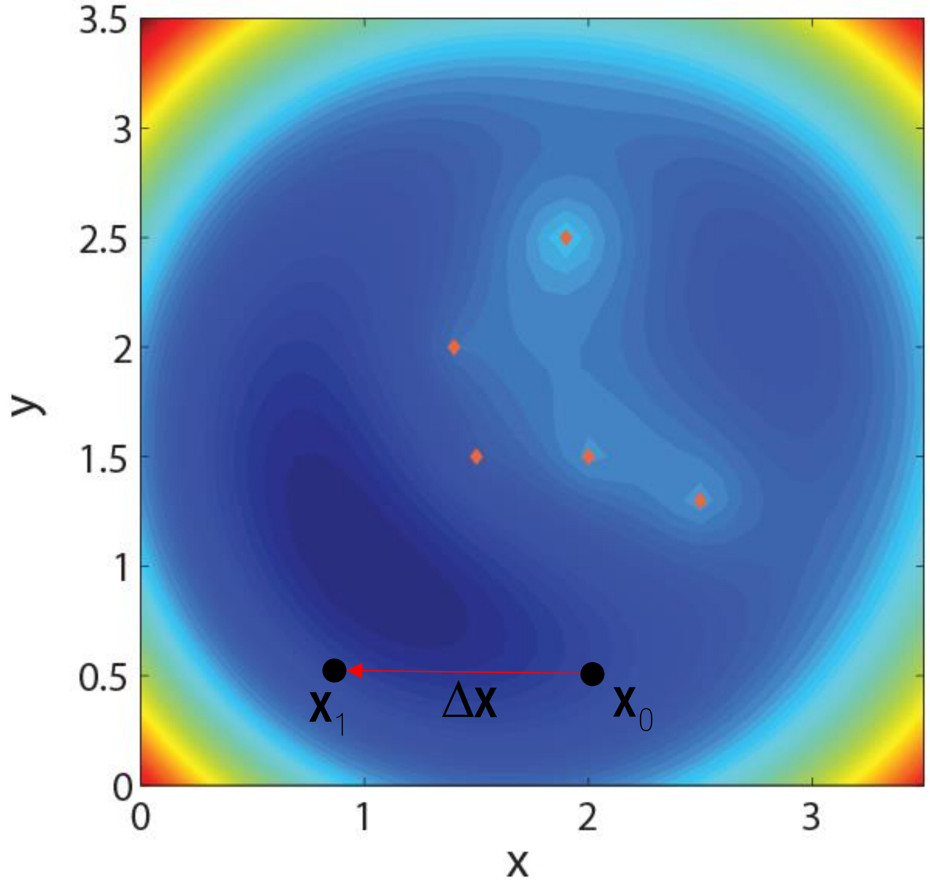
Localization using range data from beacons



$$E = \left\| \underbrace{\begin{bmatrix} \sqrt{(u_1 - x)^2 + (v_1 - y)^2} \\ \vdots \\ \sqrt{(u_5 - x)^2 + (v_5 - y)^2} \end{bmatrix}}_{f(\mathbf{x})} - \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_5 \end{bmatrix}}_{\mathbf{b}} \right\|^2$$

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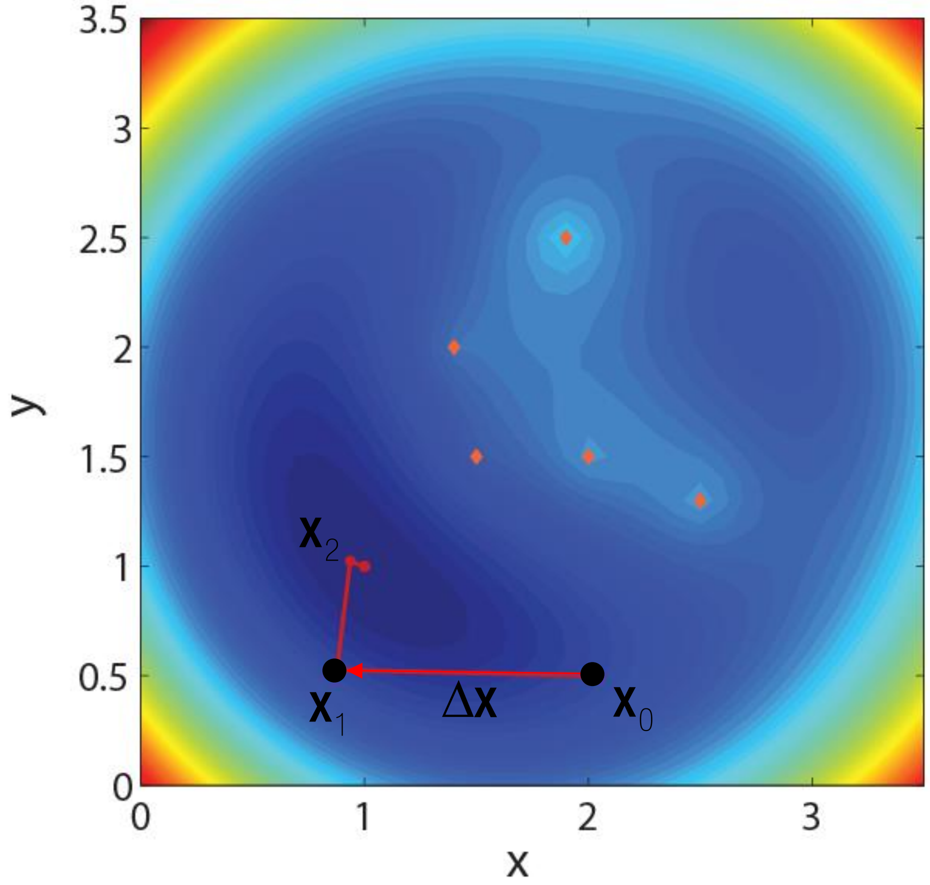

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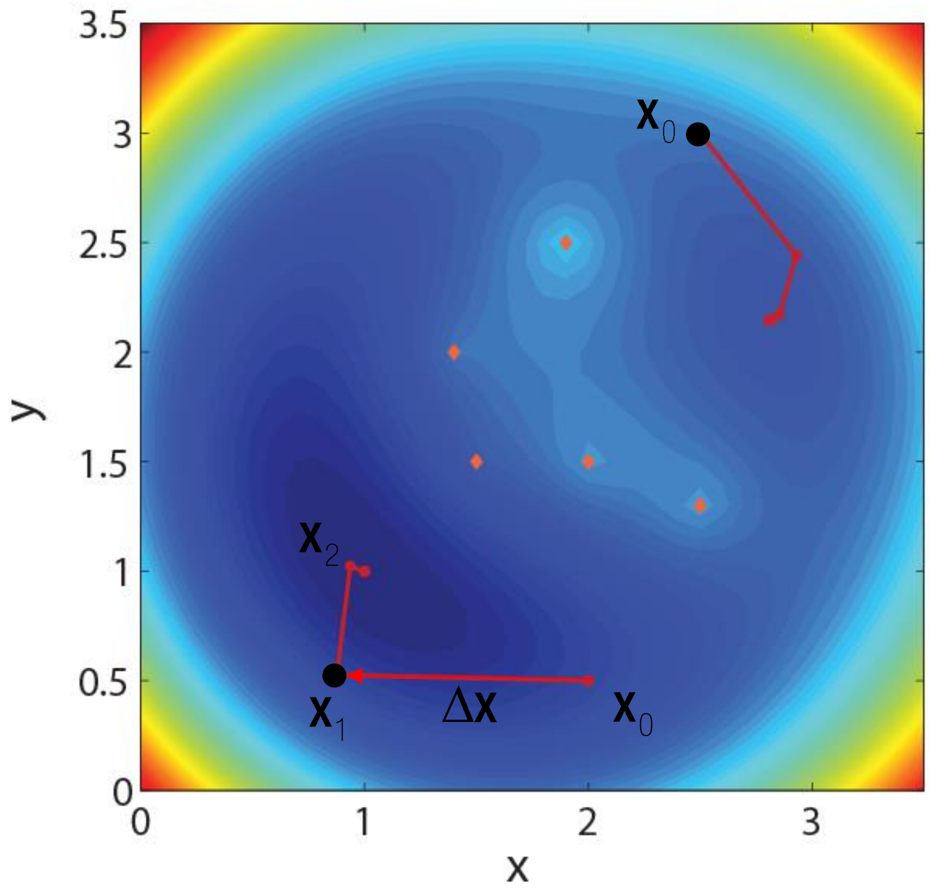
Where am I?: GPS Localization

Example:
Localization using range data from beacons



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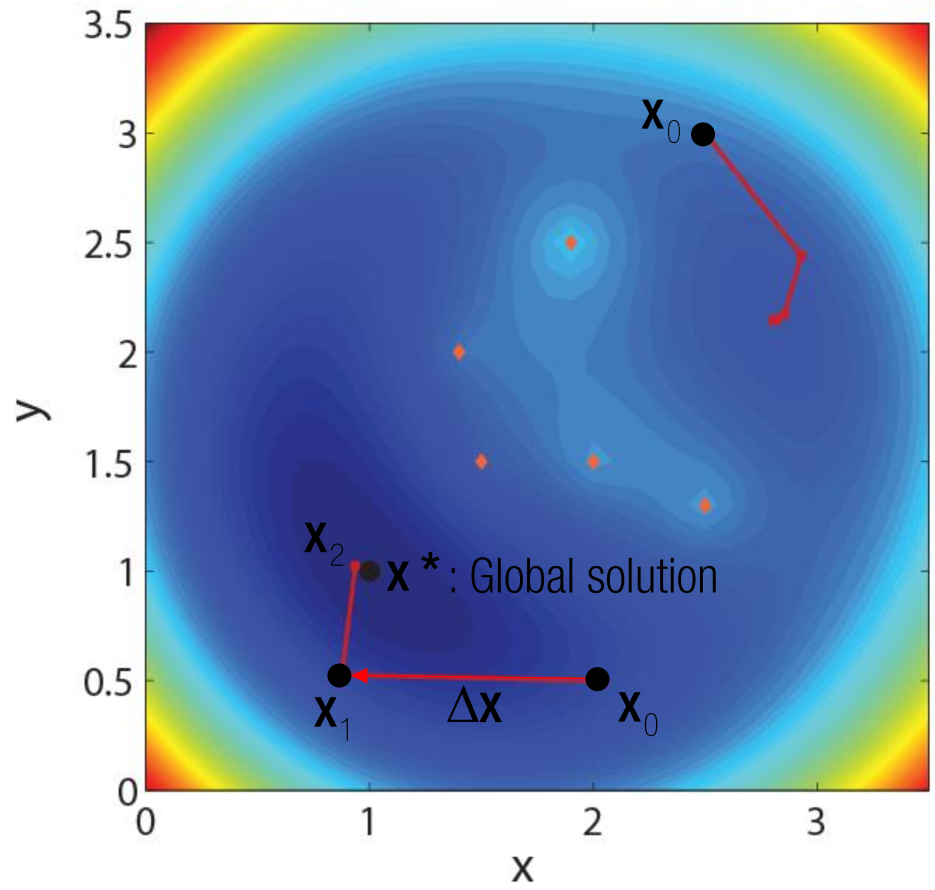

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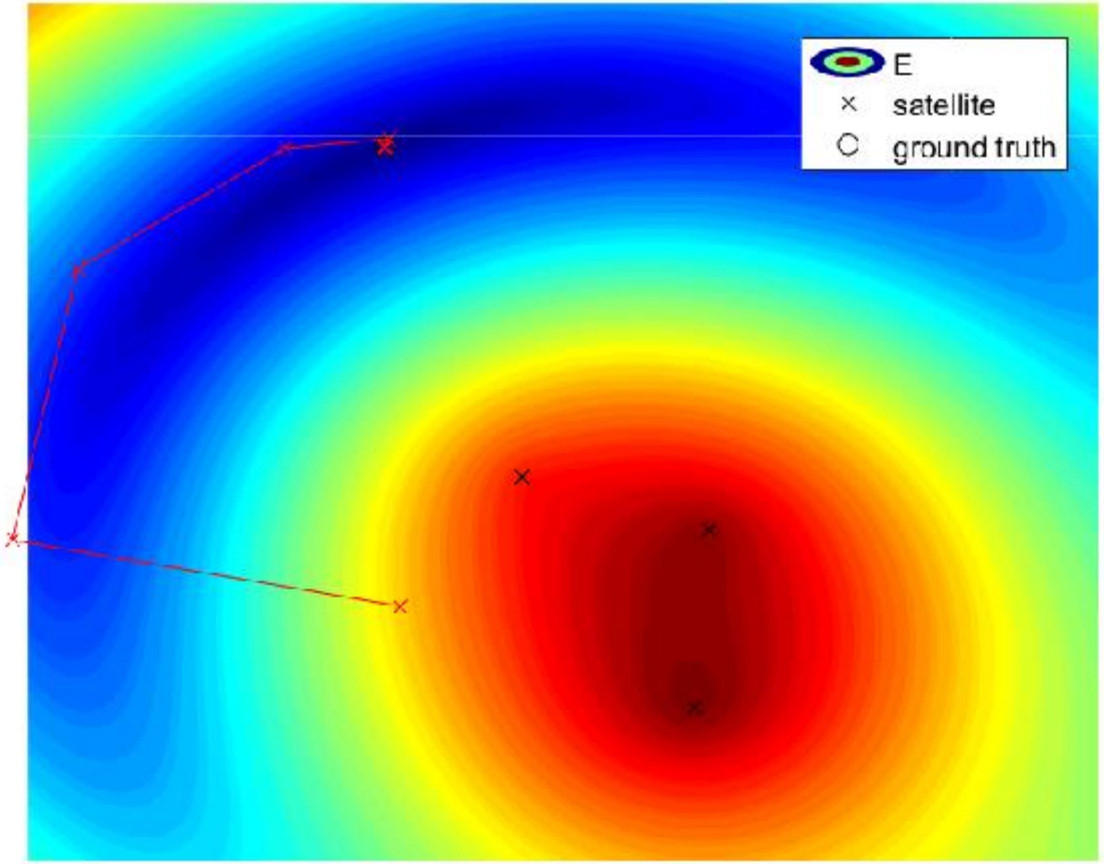

Beacon.m

```
u = 2*(rand(3,2)-0.5);
x = [-0.5 1];

for i = 1 : size(u,1)
   d(i,1) = norm(x-u(i,:));
end

[x_grid, y_grid] = meshgrid(-1.5:0.01:1.5, -1.5:0.01:1.5);

E = zeros(size(x_grid));
for i = 1 : size(u,1)
   E = E + (sqrt((x_grid-u(i,1)).^2 +(y_grid-u(i,2)).^2)-d(i)).^2 ;
end
E = sqrt(E);

x0 = 2*(rand(2,1)-0.5);
for j = 1 : 10
   J = [];
   fx = [];
   for i = 1 : size(u,1)
      denom = norm(u(i,:)-x0');
      J = [J; (u(i,:)-x0')/denom];
      fx = [fx; norm(u(i,:)-x0')];
   end
   b = d;
   delta_x = -inv(J'*J)*J'*(b-fx);
   x0 = x0 + delta_x;
end
```