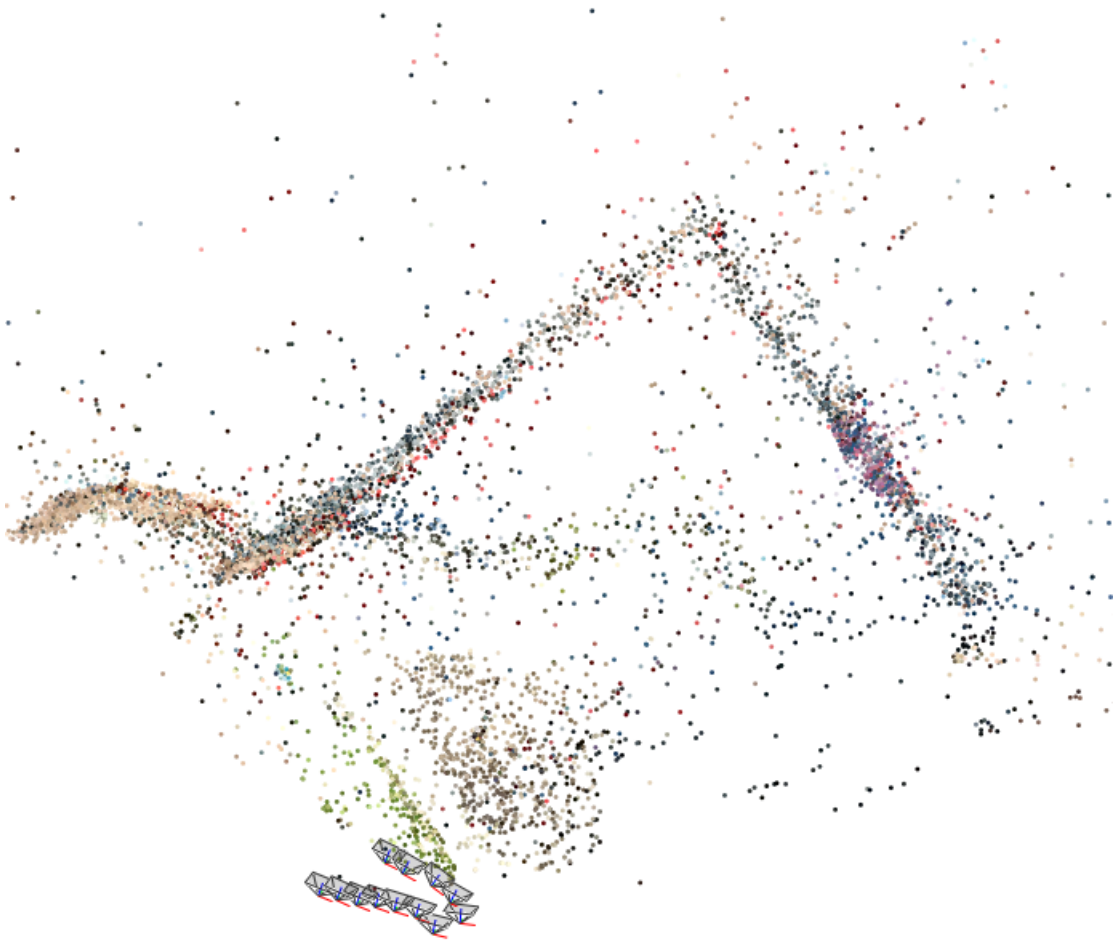


Bundle Adjustment

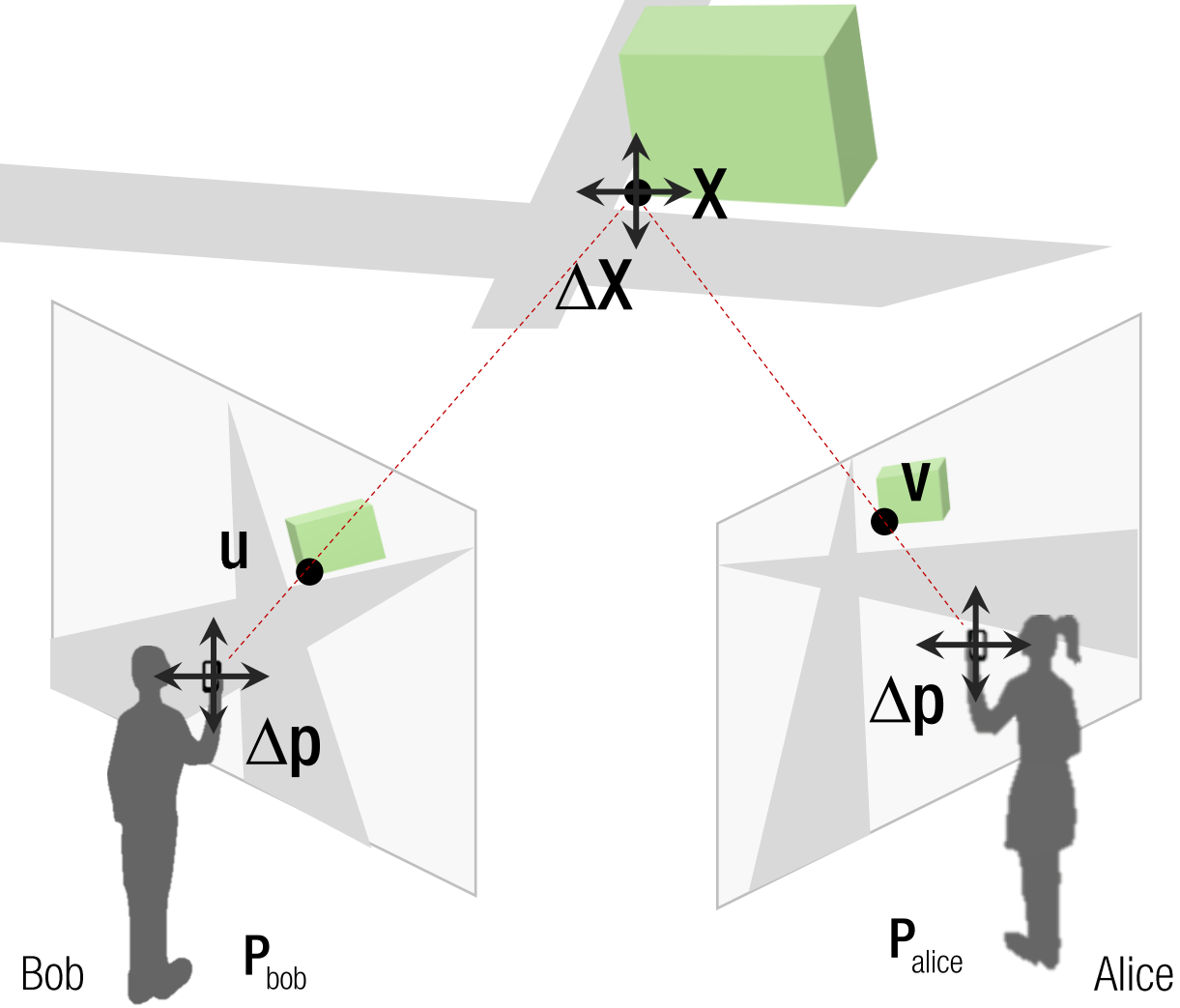
Hyun Soo Park

Black: given variables
Red: unknowns

Camera & Point Jacobian

$$E_{\text{geom}} = \left\| \hat{\mathbf{u}} - \mathbf{u} \right\|^2$$

$$= \left(\frac{\mathbf{P}_1\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - x \right)^2 + \left(\frac{\mathbf{P}_2\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - y \right)^2$$

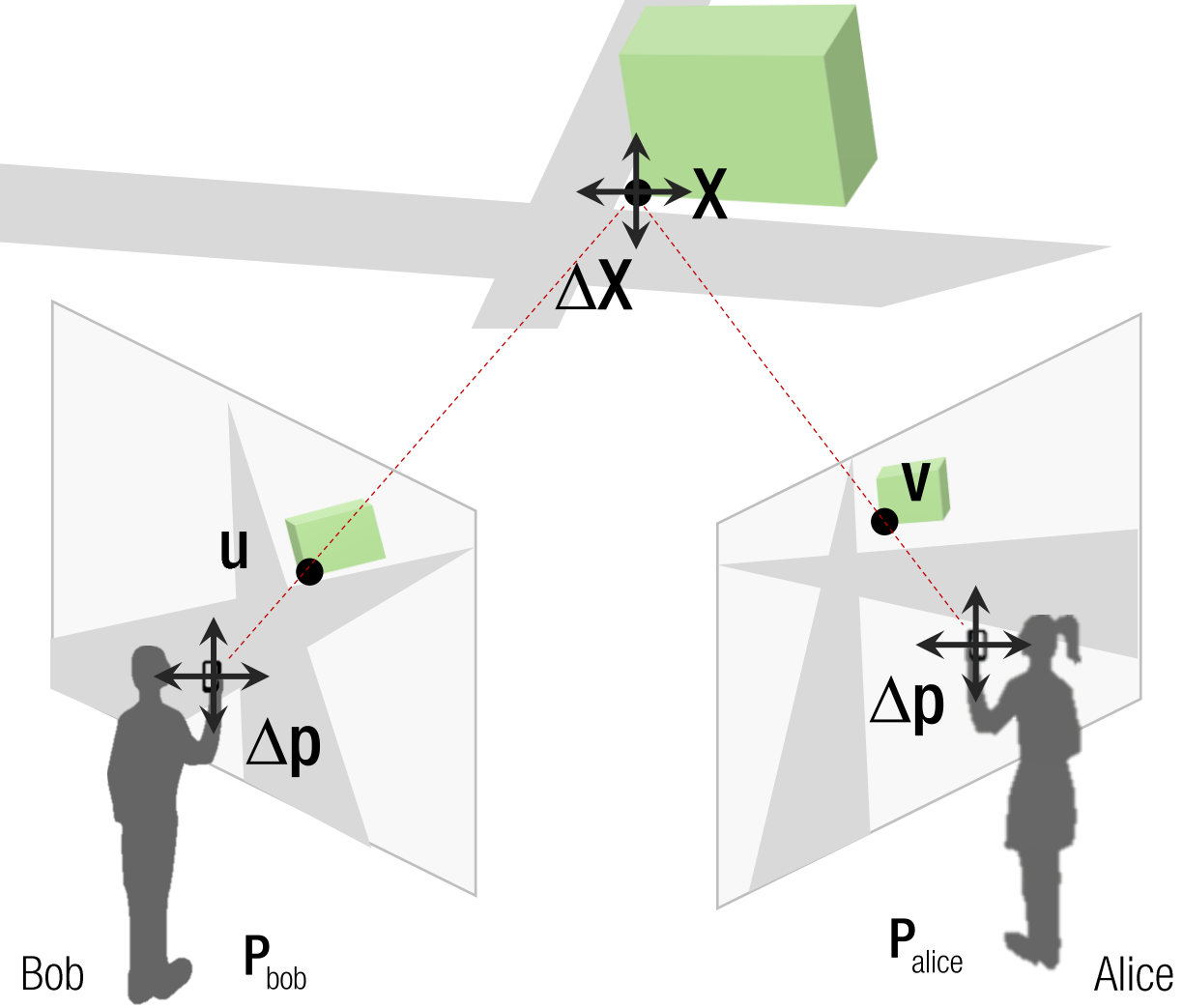


Black: given variables
Red: unknowns

Camera & Point Jacobian

$$E_{\text{geom}} = \|\hat{\mathbf{u}} - \mathbf{u}\|^2$$

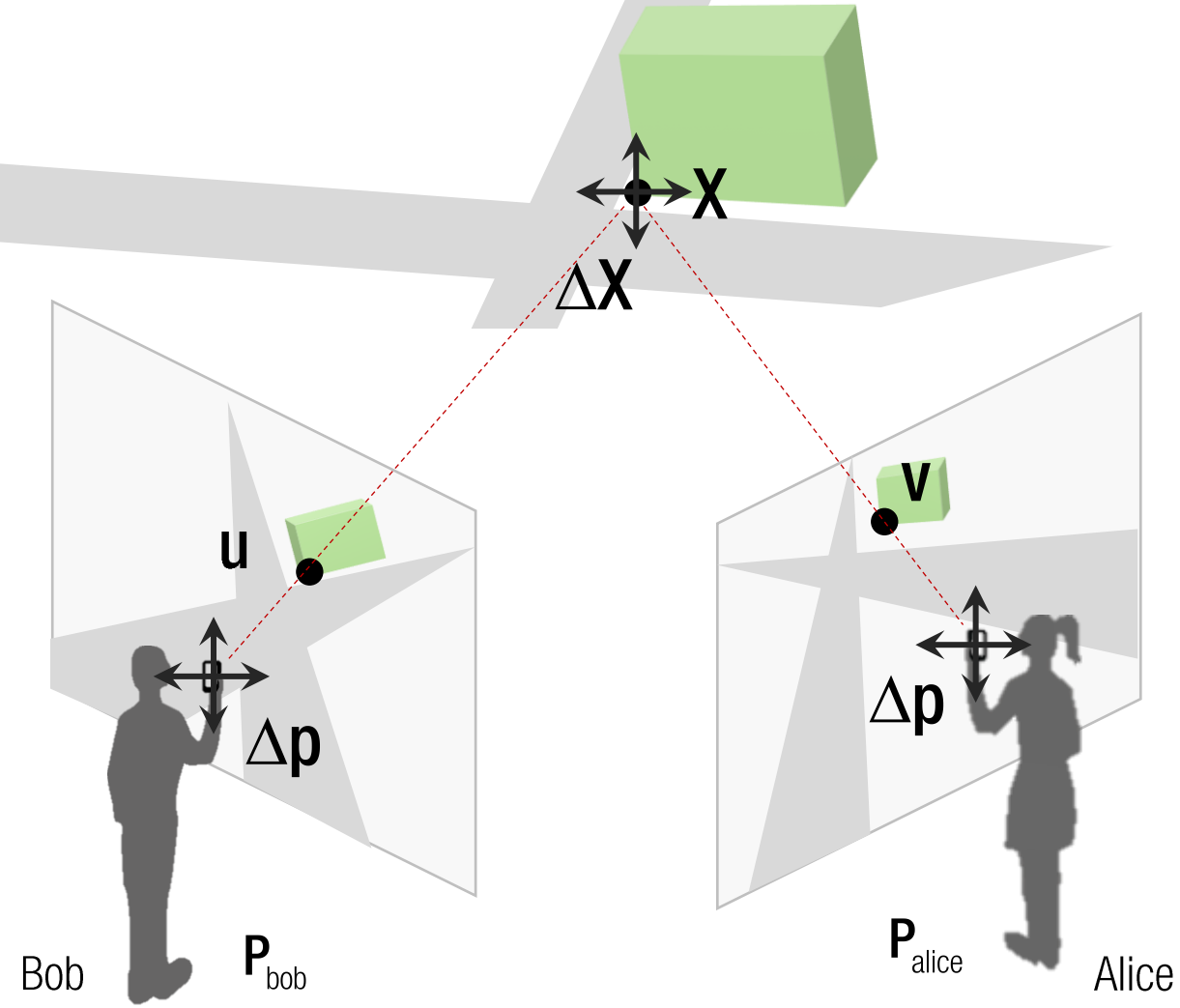
$$= \left(\frac{\mathbf{P}_1\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - x\right)^2 + \left(\frac{\mathbf{P}_2\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - y\right)^2$$



$$f(\mathbf{p},\mathbf{X}) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix}$$

Black: given variables
Red: unknowns

Camera & Point Jacobian



$$E_{\text{geom}} = \|\hat{\mathbf{u}} - \mathbf{u}\|^2$$

$$= \left(\frac{\mathbf{P}_1\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - x\right)^2 + \left(\frac{\mathbf{P}_2\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - y\right)^2$$

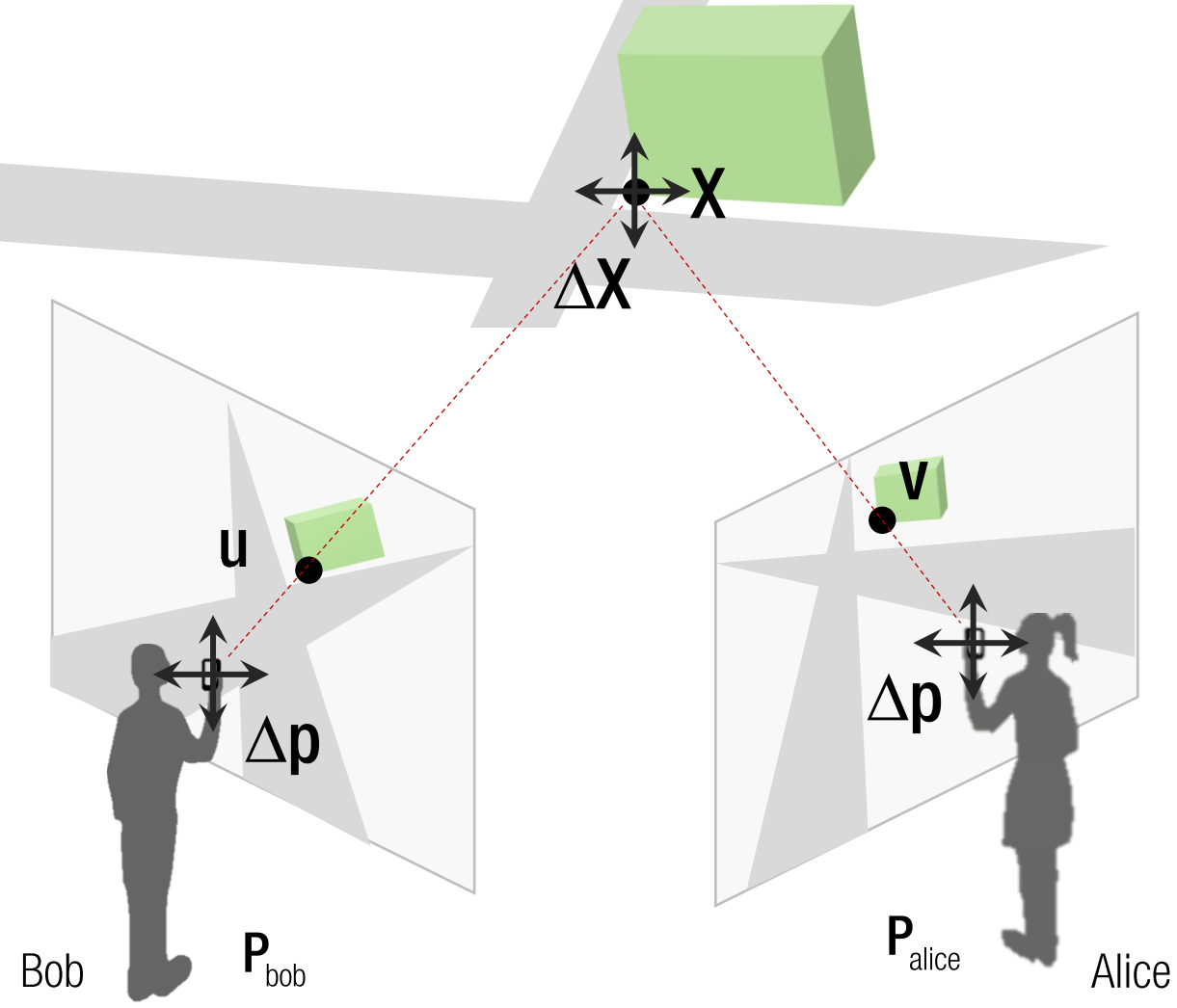
$$f(\mathbf{p},\mathbf{X}) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} \longrightarrow \frac{\partial f(\mathbf{p},\mathbf{X})}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \frac{w\frac{\partial u}{\partial \mathbf{p}} - u\frac{\partial w}{\partial \mathbf{p}}}{w^2} \\ \frac{v\frac{\partial u}{\partial \mathbf{p}} - v\frac{\partial w}{\partial \mathbf{p}}}{w^2} \end{bmatrix}$$

Black: given variables
Red: unknowns

Camera & Point Jacobian

$$E_{\text{geom}} = \|\hat{\mathbf{u}} - \mathbf{u}\|^2$$

$$= \left(\frac{\mathbf{P}_1\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - x\right)^2 + \left(\frac{\mathbf{P}_2\mathbf{X}}{\mathbf{P}_3\mathbf{X}} - y\right)^2$$



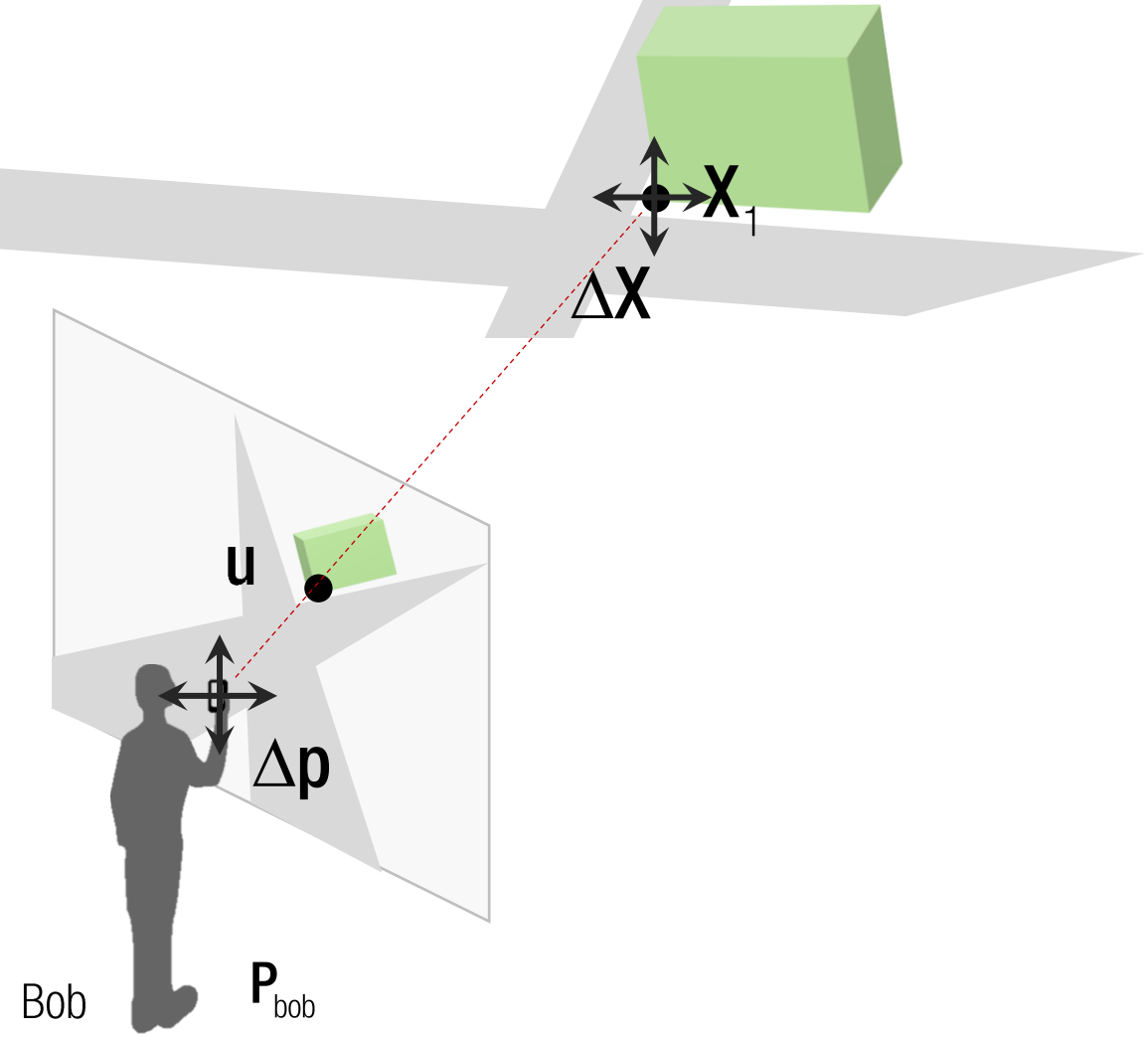
$$f(\mathbf{p},\mathbf{X}) = \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} \longrightarrow \frac{\partial f(\mathbf{p},\mathbf{X})}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \frac{w\frac{\partial u}{\partial \mathbf{p}} - u\frac{\partial w}{\partial \mathbf{p}}}{w^2} \\ \frac{v\frac{\partial u}{\partial \mathbf{p}} - v\frac{\partial w}{\partial \mathbf{p}}}{w^2} \end{bmatrix}$$

$$\longrightarrow \frac{\partial f(\mathbf{p},\mathbf{X})}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \frac{w\frac{\partial u}{\partial \mathbf{X}} - u\frac{\partial w}{\partial \mathbf{X}}}{w^2} \\ \frac{v\frac{\partial u}{\partial \mathbf{X}} - v\frac{\partial w}{\partial \mathbf{X}}}{w^2} \end{bmatrix}$$

Black: given variables
Red: unknowns

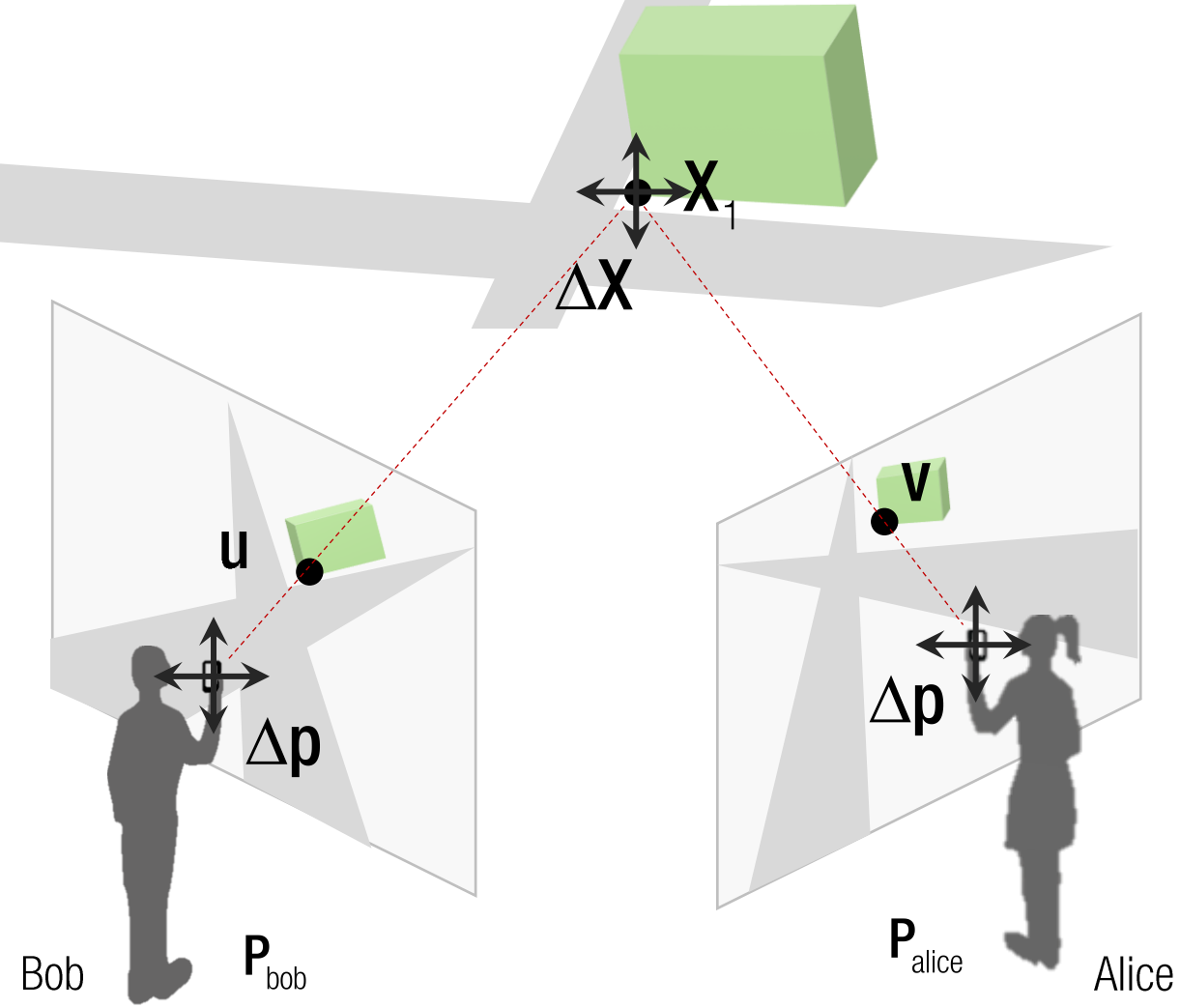
Camera & Point Jacobian

$$\mathbf{J}_{ij} = \left[\frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j} \quad \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i} \right] = \left[\mathbf{J}_{\mathbf{p}ij} \quad \mathbf{J}_{\mathbf{X}ij} \right]$$



Black: given variables
Red: unknowns

Camera & Point Jacobian

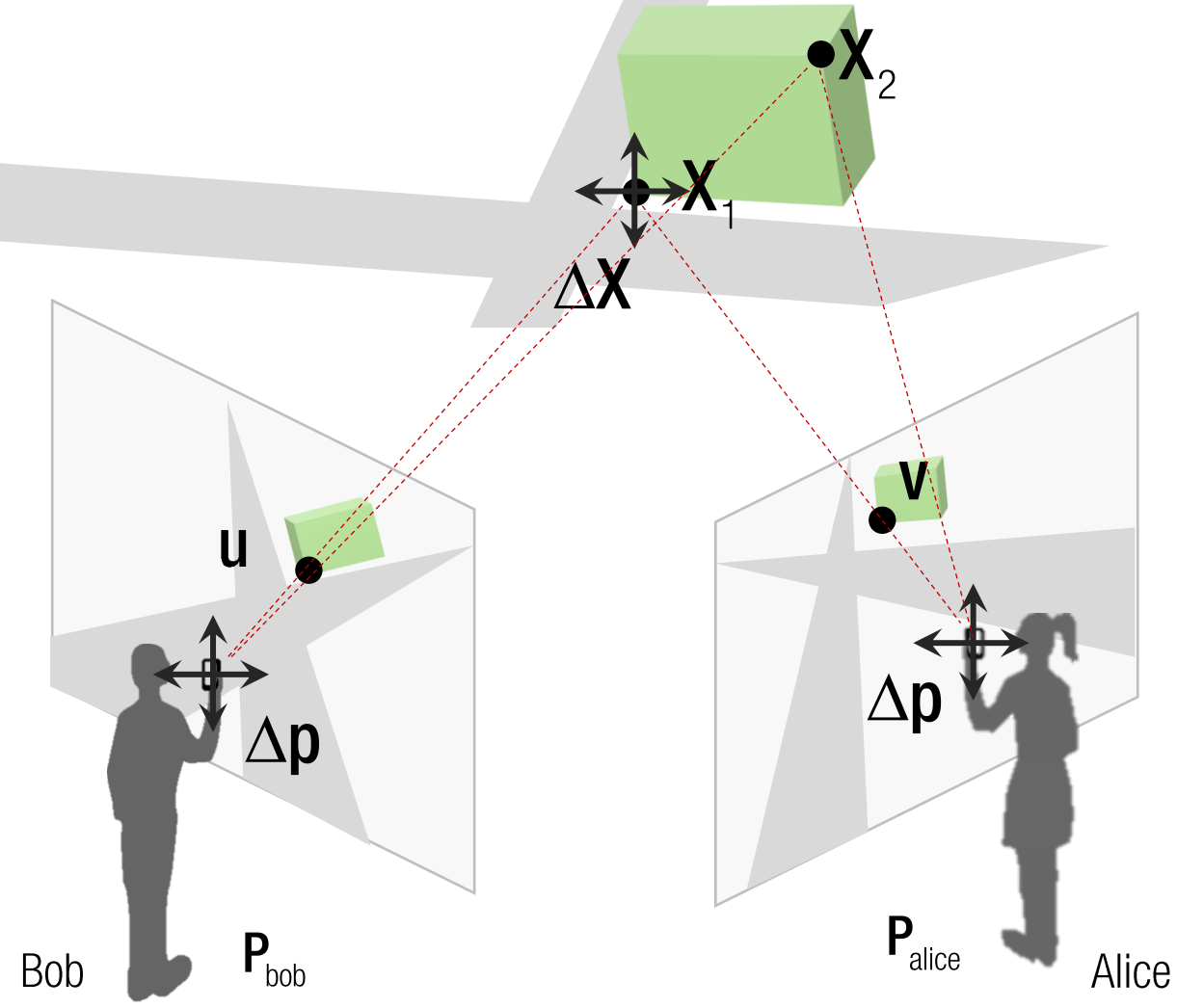


$$\mathbf{J}_{ij} = \left[\frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j} \quad \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i} \right] = \left[\mathbf{J}_{\mathbf{p}ij} \quad \mathbf{J}_{\mathbf{X}ij} \right]$$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{\mathbf{p}1,bob} & \mathbf{0}_{2\times7} & \mathbf{J}_{\mathbf{X}1,bob} \\ \mathbf{0}_{2\times7} & \mathbf{J}_{\mathbf{p}1,alice} & \mathbf{J}_{\mathbf{X}1,alice} \end{bmatrix}$$

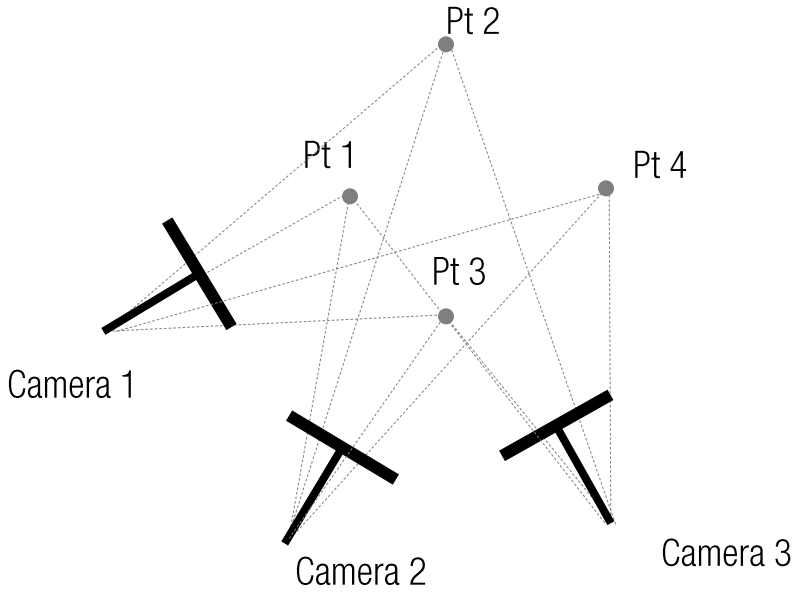
Black: given variables
Red: unknowns

Camera & Point Jacobian

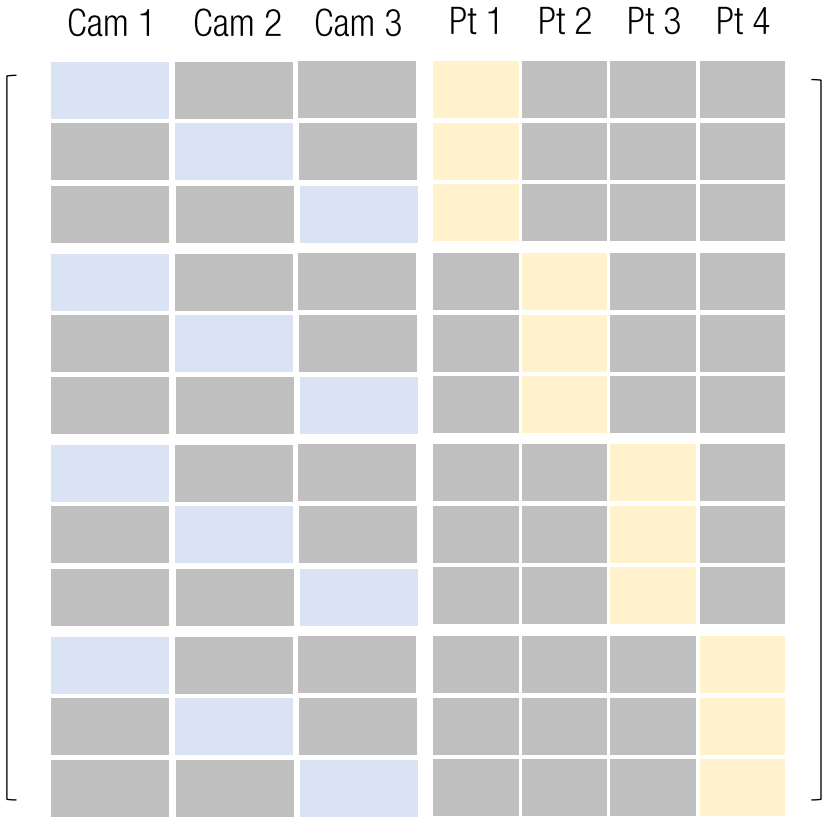


$$\mathbf{J}_{ij} = \left[\frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j} \quad \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i} \right] = \begin{bmatrix} \mathbf{J}_{\mathbf{p}ij} & \mathbf{J}_{\mathbf{X}ij} \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{\mathbf{p}1,\text{bob}} & \mathbf{0}_{2\times7} & \mathbf{J}_{\mathbf{X}1,\text{bob}} & \mathbf{0}_{2\times3} \\ \mathbf{0}_{2\times7} & \mathbf{J}_{\mathbf{p}1,\text{alice}} & \mathbf{J}_{\mathbf{X}1,\text{alice}} & \mathbf{0}_{2\times3} \\ \mathbf{J}_{\mathbf{p}2,\text{bob}} & \mathbf{0}_{2\times7} & \mathbf{0}_{2\times3} & \mathbf{J}_{\mathbf{X}2,\text{bob}} \\ \mathbf{0}_{2\times7} & \mathbf{J}_{\mathbf{p}2,\text{alice}} & \mathbf{0}_{2\times3} & \mathbf{J}_{\mathbf{X}2,\text{alice}} \end{bmatrix}$$

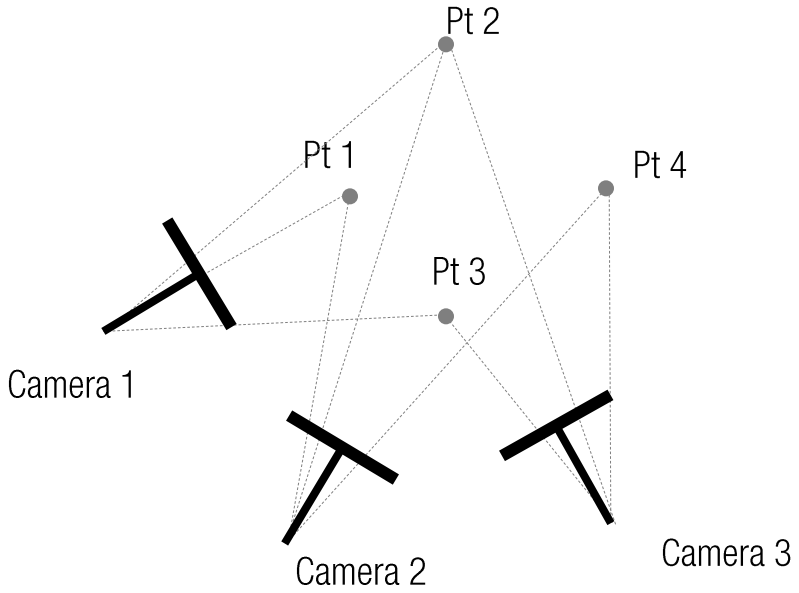


J =

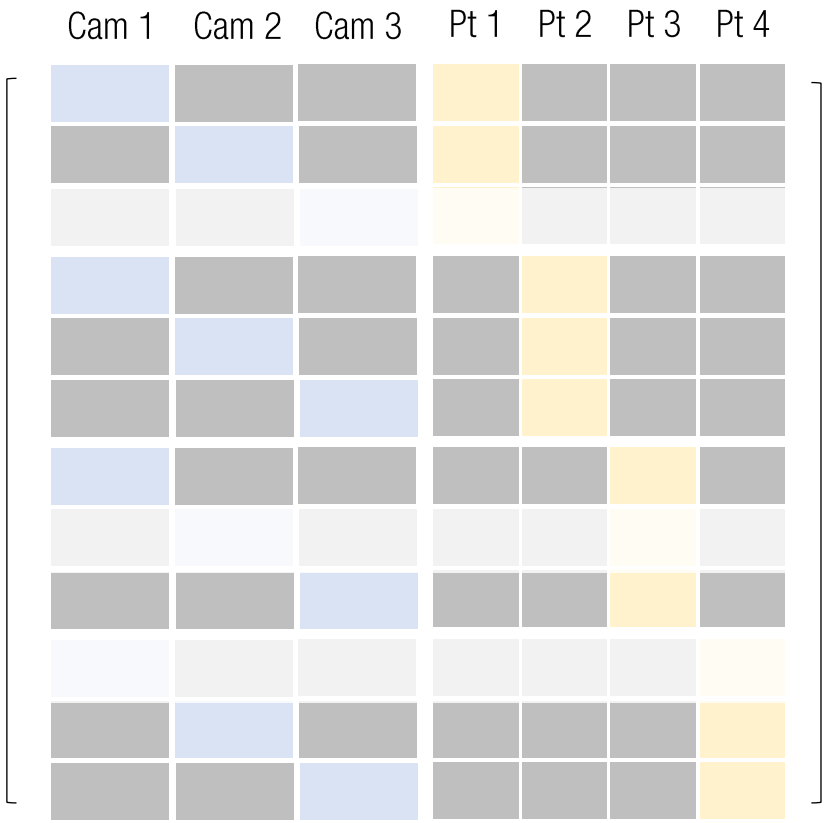


# of unknowns: 3x7+4x3

# of projections: 3x4



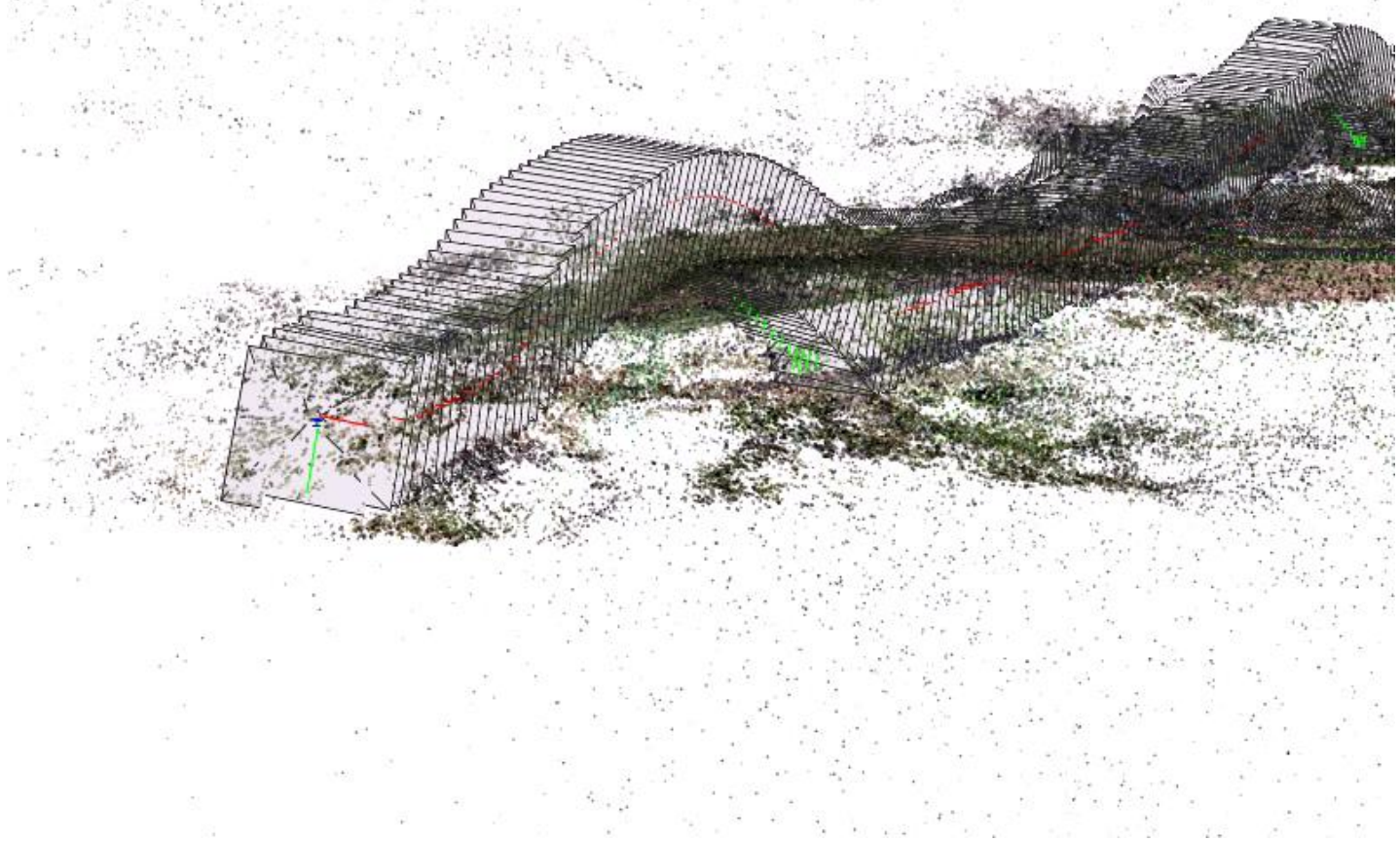
$J =$

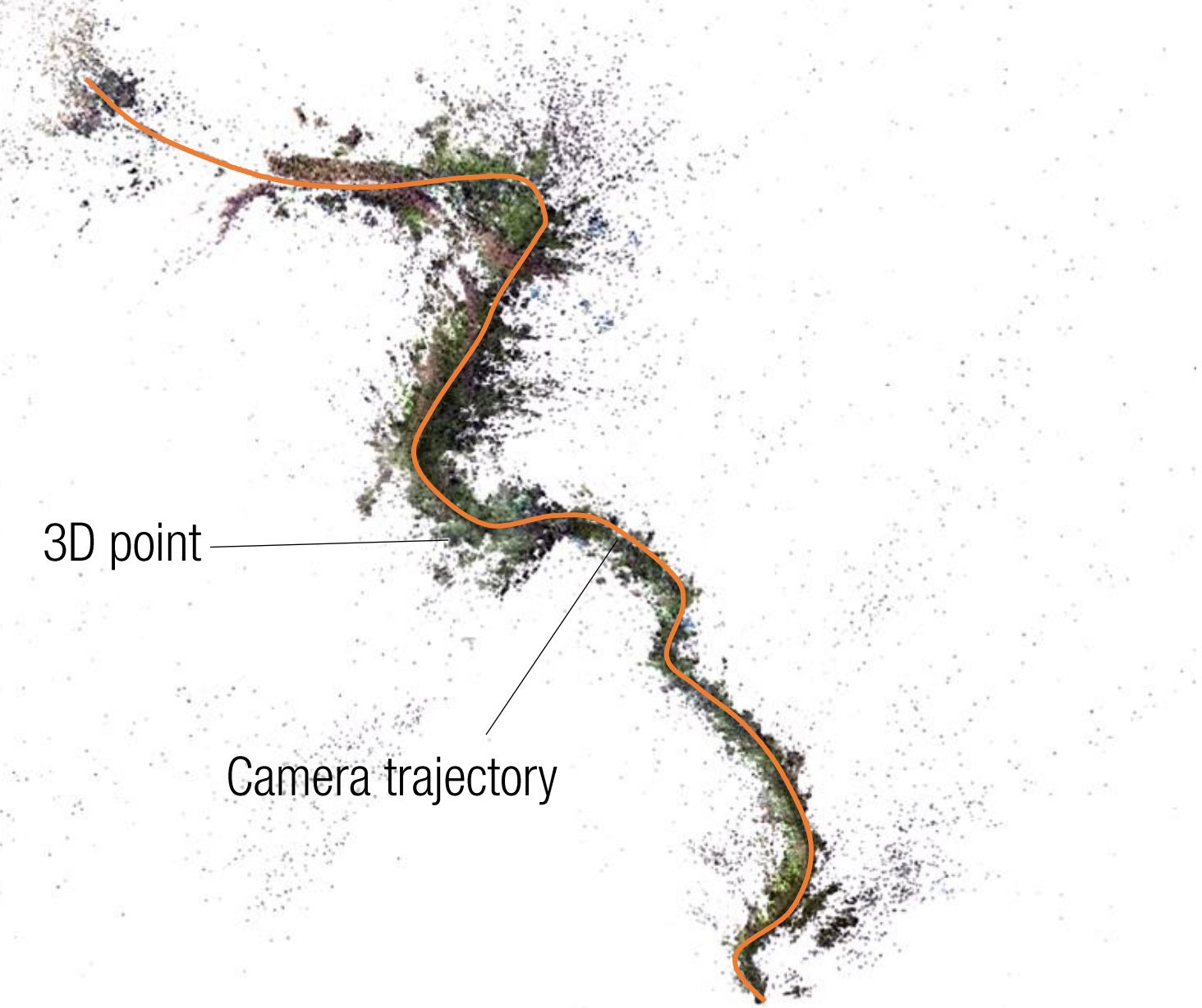


# of unknowns: 3x7+4x3

# of projections: 9 (not all points are visible from cameras)


Input: first person video
https://www.youtube.com/watch?v=bKU0Z4bj7Mw

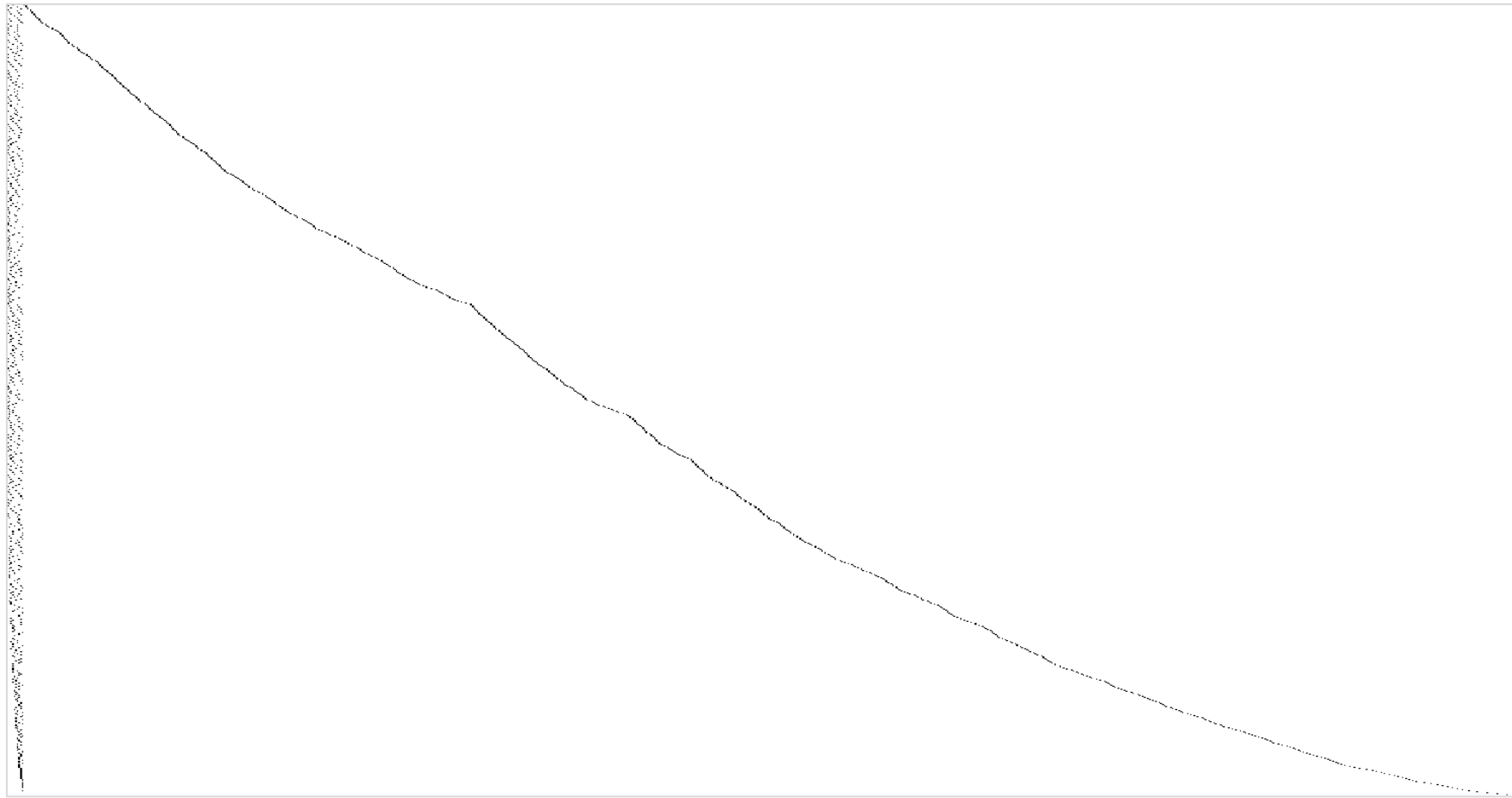




# of cameras: 3009
# of 3D points: 1,298,317

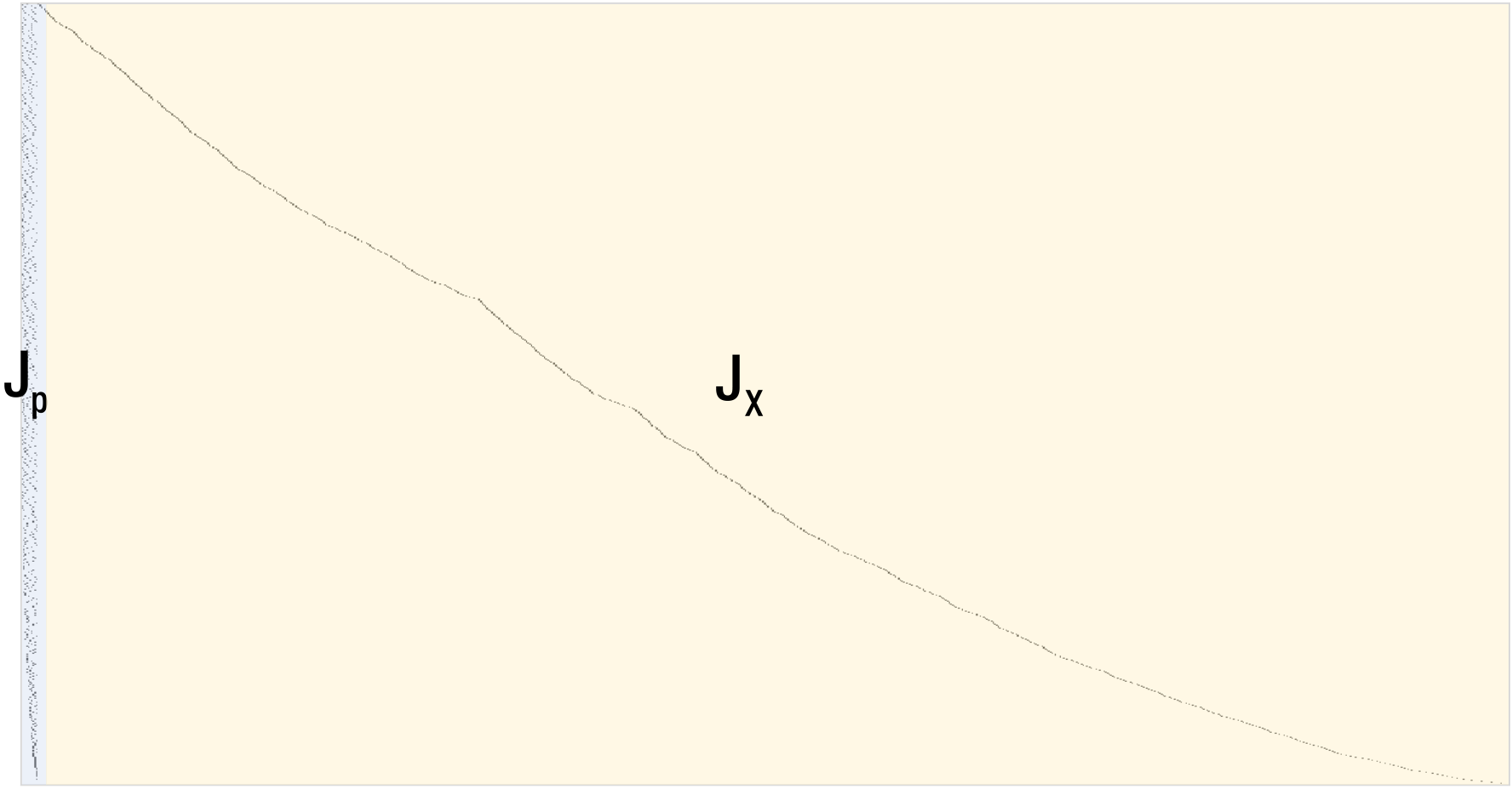
# of unknowns: 7x3009+3x1,298,317
size of $\mathbf{J}^{\top}\mathbf{J}$: 6,167,690x6,167,690

$$J =$$

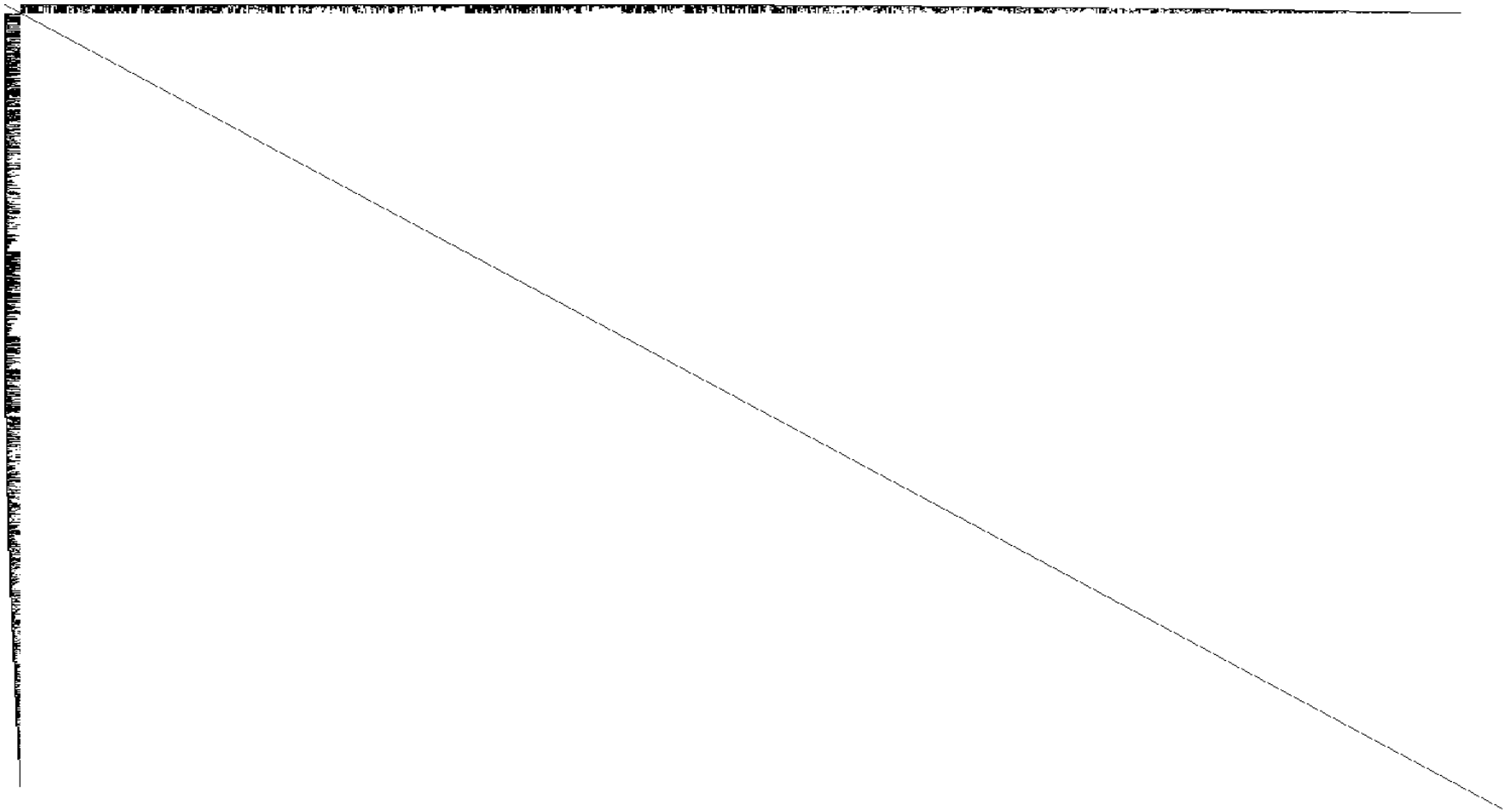


Camera 3D point

$$J = \begin{bmatrix} J_p & J_X \end{bmatrix}$$



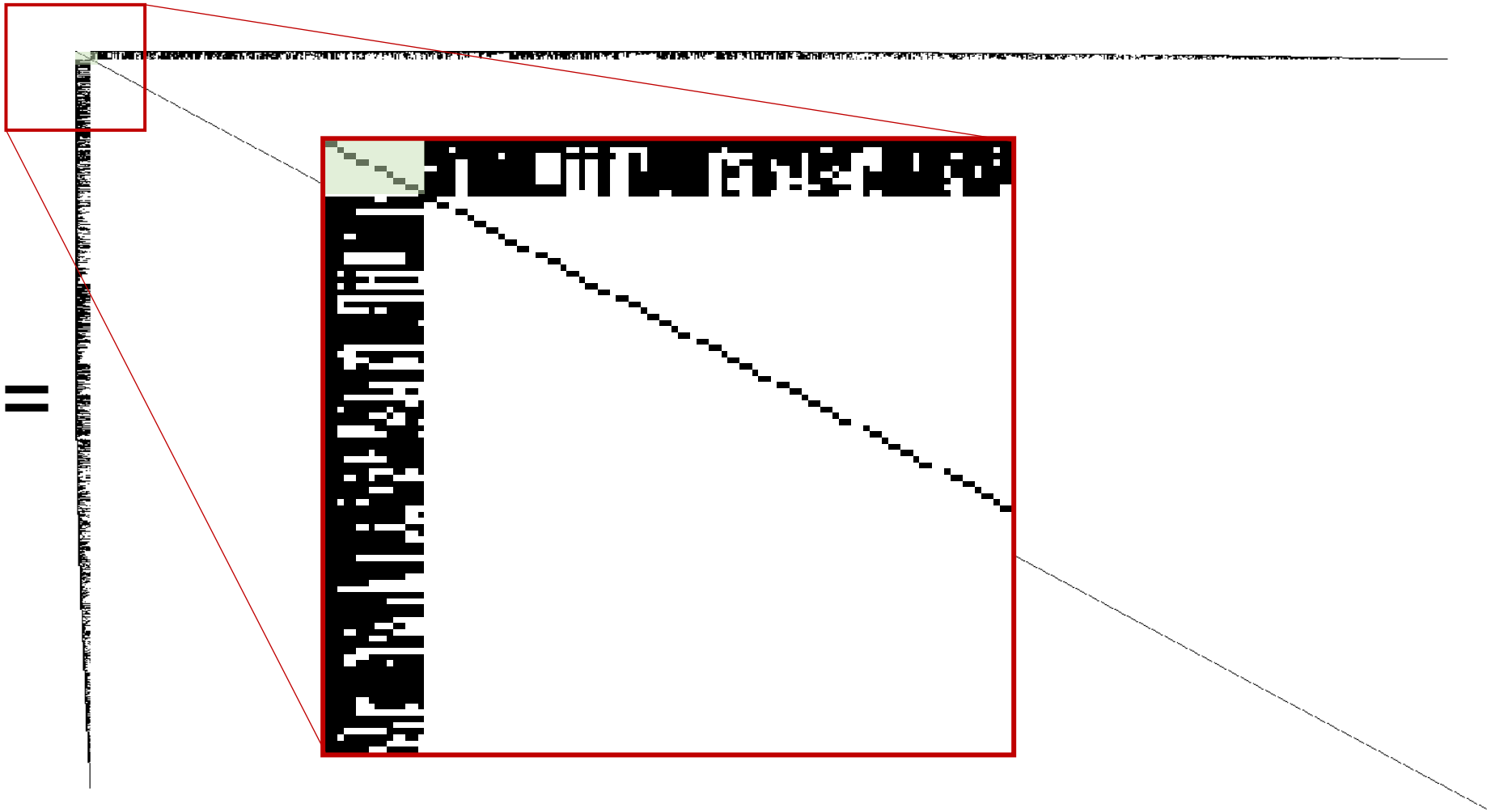
$$J^\top J = \quad$$



$$J = \begin{bmatrix} J_p & J_x \end{bmatrix}$$

$$J^\top J = \begin{bmatrix} J_p^\top J_p & J_p^\top J_x \\ J_x^\top J_p & J_x^\top J_x \end{bmatrix}$$

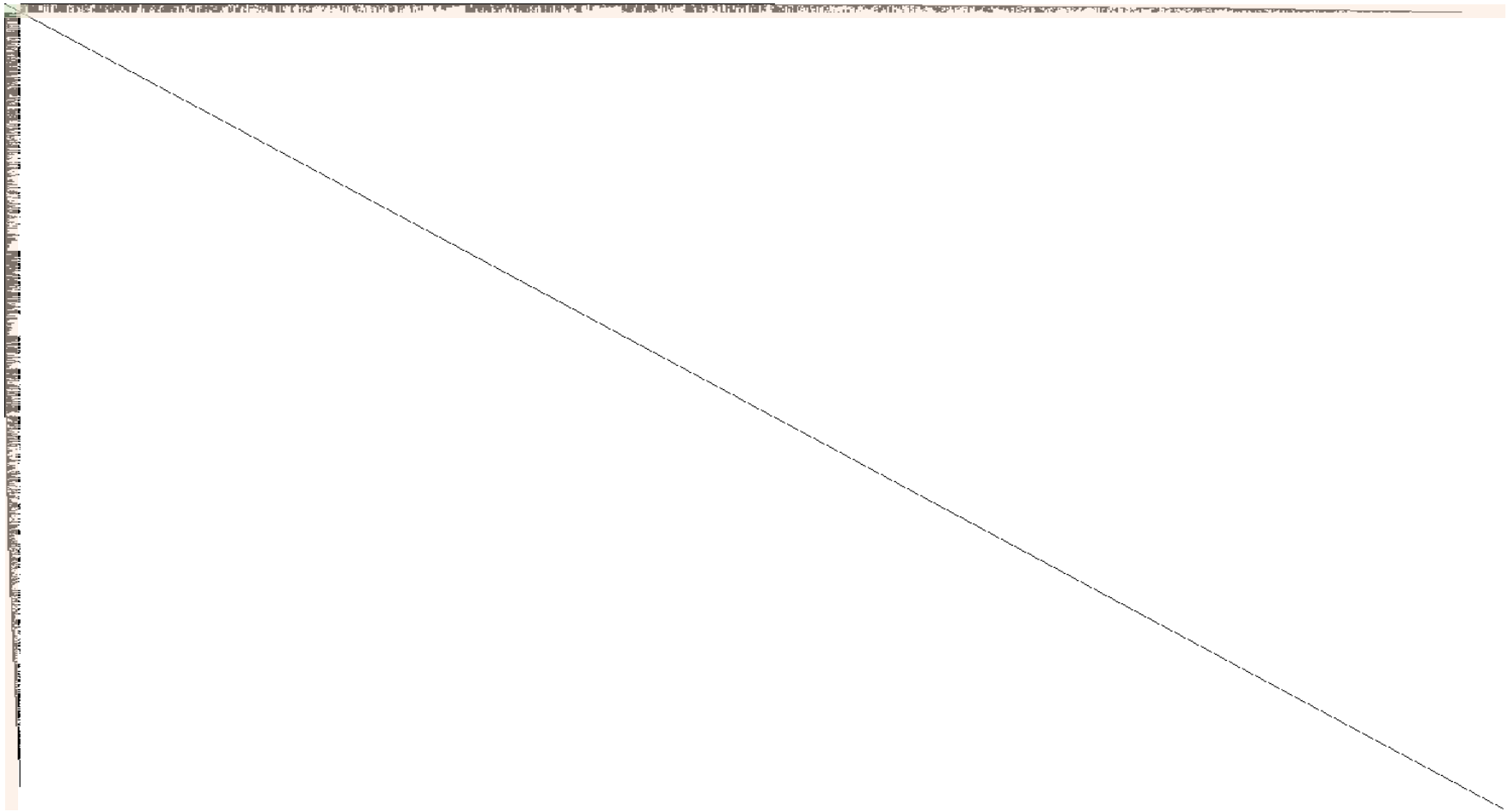
$J^\top J =$



$$J = \begin{bmatrix} J_p & J_x \end{bmatrix}$$

$$J^\top J = \begin{bmatrix} J_p^\top J_p & J_p^\top J_x \\ J_x^\top J_p & J_x^\top J_x \end{bmatrix} = \begin{bmatrix} A & B \\ B^\top & D \end{bmatrix}$$

$$J^\top J =$$

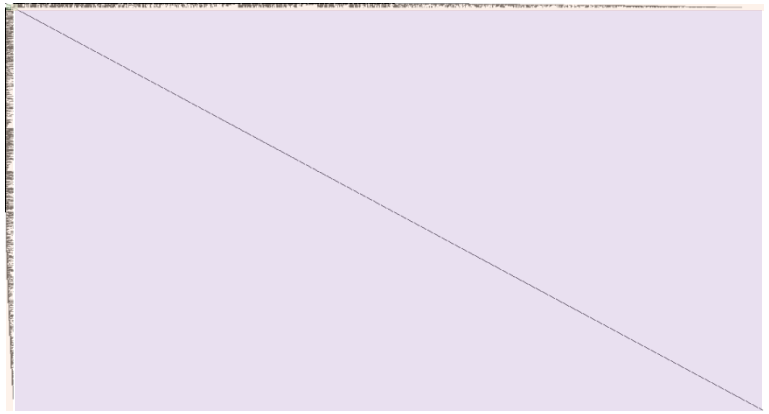


$$J = \begin{bmatrix} J_p & J_x \end{bmatrix}$$

$$J^\top J = \begin{bmatrix} J_p^\top J_p & J_p^\top J_x \\ J_x^\top J_p & J_x^\top J_x \end{bmatrix} = \begin{bmatrix} A & B \\ B^\top & D \end{bmatrix}$$

Normal equation:

$$\mathbf{J}^{\mathsf{T}}\mathbf{J}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix}=\mathbf{J}^{\mathsf{T}}\left(\mathbf{b}-f(\mathbf{X})\right)$$

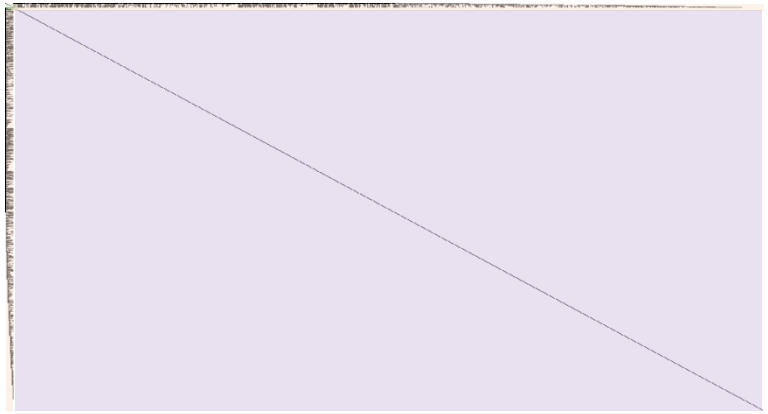
$\mathbf{J}^{\mathsf{T}}\mathbf{J} =$ 

$$\mathbf{J}^{\mathsf{T}}\mathbf{J}=\begin{bmatrix}\mathbf{J}_p^{\mathsf{T}}\mathbf{J}_p & \mathbf{J}_p^{\mathsf{T}}\mathbf{J}_X\\ \mathbf{J}_X^{\mathsf{T}}\mathbf{J}_p & \mathbf{J}_X^{\mathsf{T}}\mathbf{J}_X\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^{\mathsf{T}} & \mathbf{D}\end{bmatrix}$$

Normal equation:

$$\mathbf{J}^{\top}\mathbf{J}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix}=\mathbf{J}^{\top}\left(\mathbf{b}-f(\mathbf{X})\right)$$

$$\mathbf{J}^{\top}\mathbf{J}=$$



$$\mathbf{J}^{\top}\mathbf{J}=\begin{bmatrix}\mathbf{J}_p^{\top}\mathbf{J}_p & \mathbf{J}_p^{\top}\mathbf{J}_X\\ \mathbf{J}_X^{\top}\mathbf{J}_p & \mathbf{J}_X^{\top}\mathbf{J}_X\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^{\top} & \mathbf{D}\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}\mathbf{J}_p^{\top}\mathbf{J}_p & \mathbf{J}_p^{\top}\mathbf{J}_X\\ \mathbf{J}_X^{\top}\mathbf{J}_p & \mathbf{J}_X^{\top}\mathbf{J}_X\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix}=\begin{bmatrix}\mathbf{J}_p^{\top}\\ \mathbf{J}_X^{\top}\end{bmatrix}\left(\mathbf{b}-f(\mathbf{X})\right)=\begin{bmatrix}\mathbf{e}_p\\ \mathbf{e}_X\end{bmatrix}$$

Normal equation:

$$\mathbf{J}^{\top}\mathbf{J}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix}=\mathbf{J}^{\top}\left(\mathbf{b}-f(\mathbf{X})\right)$$

$$\mathbf{J}^{\top}\mathbf{J}=\begin{bmatrix}\mathbf{J}_p^{\top}\mathbf{J}_p & \mathbf{J}_p^{\top}\mathbf{J}_X\\ \mathbf{J}_X^{\top}\mathbf{J}_p & \mathbf{J}_X^{\top}\mathbf{J}_X\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^{\top} & \mathbf{D}\end{bmatrix}$$

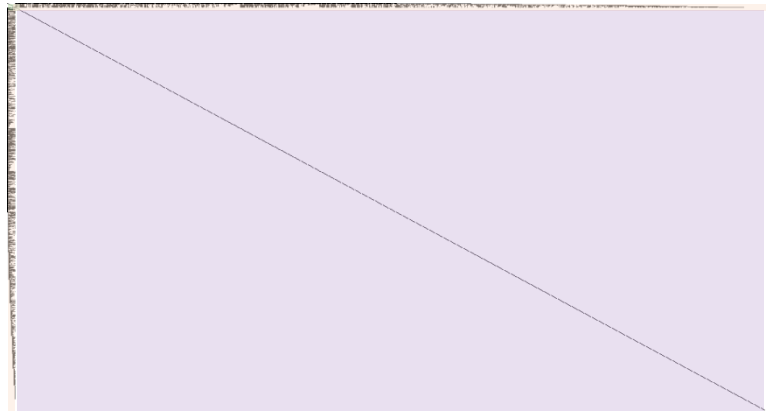
$$\longrightarrow\begin{bmatrix}\mathbf{J}_p^{\top}\mathbf{J}_p & \mathbf{J}_p^{\top}\mathbf{J}_X\\ \mathbf{J}_X^{\top}\mathbf{J}_p & \mathbf{J}_X^{\top}\mathbf{J}_X\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix}=\begin{bmatrix}\mathbf{J}_p^{\top}\\ \mathbf{J}_X^{\top}\end{bmatrix}\left(\mathbf{b}-f(\mathbf{X})\right)=\begin{bmatrix}\mathbf{e}_p\\ \mathbf{e}_X\end{bmatrix}$$

or

$$\begin{bmatrix}\mathbf{J}_p^{\top}\mathbf{J}_p+\mu\mathbf{I} & \mathbf{J}_p^{\top}\mathbf{J}_X\\ \mathbf{J}_X^{\top}\mathbf{J}_p & \mathbf{J}_X^{\top}\mathbf{J}_X+\mu\mathbf{I}\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix}=\begin{bmatrix}\mathbf{e}_p\\ \mathbf{e}_X\end{bmatrix}$$

or

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^{\top} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix}=\begin{bmatrix}\mathbf{e}_p\\ \mathbf{e}_X\end{bmatrix}$$

$\mathbf{J}^{\top}\mathbf{J}=$ 

Normal equation:

$$\mathbf{J}^T\mathbf{J}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \mathbf{J}^T\left(\mathbf{b} - f(\mathbf{X})\right)$$

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix}\mathbf{J}_p^T\mathbf{J}_p & \mathbf{J}_p^T\mathbf{J}_X\\ \mathbf{J}_X^T\mathbf{J}_p & \mathbf{J}_X^T\mathbf{J}_X\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^T & \mathbf{D}\end{bmatrix}$$

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix}\mathbf{D} & \\ & \mathbf{S}\end{bmatrix} \longrightarrow \mathbf{D} = \begin{bmatrix}\mathbf{d}_1 & & \\ & \ddots & \\ & & \mathbf{d}_M\end{bmatrix} \longrightarrow \mathbf{D}^{-1} = \begin{bmatrix}\mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1}\end{bmatrix}$$

Inversion of block diagonal matrix can be efficiently computed.

$$\longrightarrow \begin{bmatrix}\mathbf{J}_p^T\mathbf{J}_p & \mathbf{J}_p^T\mathbf{J}_X\\ \mathbf{J}_X^T\mathbf{J}_p & \mathbf{J}_X^T\mathbf{J}_X\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \begin{bmatrix}\mathbf{J}_p^T\\ \mathbf{J}_X^T\end{bmatrix}\left(\mathbf{b} - f(\mathbf{X})\right) = \begin{bmatrix}\mathbf{e}_p\\ \mathbf{e}_X\end{bmatrix}$$

or $$\begin{bmatrix}\mathbf{J}_p^T\mathbf{J}_p + \mu\mathbf{I} & \mathbf{J}_p^T\mathbf{J}_X\\ \mathbf{J}_X^T\mathbf{J}_p & \mathbf{J}_X^T\mathbf{J}_X + \mu\mathbf{I}\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \begin{bmatrix}\mathbf{e}_p\\ \mathbf{e}_X\end{bmatrix}$$

or $$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^T & \mathbf{D}\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \begin{bmatrix}\mathbf{e}_p\\ \mathbf{e}_X\end{bmatrix}$$

Normal equation:

$$\mathbf{J}^{\mathsf{T}}\mathbf{J}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \mathbf{J}^{\mathsf{T}}\left(\mathbf{b} - f(\mathbf{X})\right)$$

$$\rightarrow \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^{\mathsf{T}} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \begin{bmatrix}\mathbf{e}_{\mathbf{p}}\\ \mathbf{e}_{\mathbf{X}}\end{bmatrix}$$

$$\mathbf{J}^{\mathsf{T}}\mathbf{J} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix}\mathbf{d}_1 & & \\ & \ddots & \\ & & \mathbf{d}_M\end{bmatrix} \rightarrow \mathbf{D}^{-1} = \begin{bmatrix}\mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1}\end{bmatrix}$$

Inversion of block diagonal matrix can be efficiently computed.

$$\mathbf{J}^{\mathsf{T}}\mathbf{J} = \begin{bmatrix}\mathbf{J}_{\mathbf{p}}^{\mathsf{T}}\mathbf{J}_{\mathbf{p}} & \mathbf{J}_{\mathbf{p}}^{\mathsf{T}}\mathbf{J}_{\mathbf{X}}\\ \mathbf{J}_{\mathbf{X}}^{\mathsf{T}}\mathbf{J}_{\mathbf{p}} & \mathbf{J}_{\mathbf{X}}^{\mathsf{T}}\mathbf{J}_{\mathbf{X}}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^{\mathsf{T}} & \mathbf{D}\end{bmatrix}$$

Normal equation:

$$\mathbf{J}^T\mathbf{J}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \mathbf{J}^T\left(\mathbf{b} - f(\mathbf{X})\right)$$

$$\rightarrow \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^T & \mathbf{D}\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \begin{bmatrix}\mathbf{e}_p\\ \mathbf{e}_X\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\mathbf{I} & -\mathbf{B}\mathbf{D}^{-1}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^T & \mathbf{D}\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & -\mathbf{B}\mathbf{D}^{-1}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{e}_p\\ \mathbf{e}_X\end{bmatrix}$$

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix}\mathbf{J}_p^T\mathbf{J}_p & \mathbf{J}_p^T\mathbf{J}_X\\ \mathbf{J}_X^T\mathbf{J}_p & \mathbf{J}_X^T\mathbf{J}_X\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^T & \mathbf{D}\end{bmatrix}$$

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix}\mathbf{D} & \\ & \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix}\mathbf{d}_1 & & \\ & \ddots & \\ & & \mathbf{d}_M\end{bmatrix} \rightarrow \mathbf{D}^{-1} = \begin{bmatrix}\mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1}\end{bmatrix}$$

Inversion of block diagonal matrix can be efficiently computed.

Normal equation:

$$\mathbf{J}^T\mathbf{J}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \mathbf{J}^T\left(\mathbf{b} - f(\mathbf{X})\right)$$

$$\rightarrow \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^T & \mathbf{D}\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \begin{bmatrix}\mathbf{e}_p\\ \mathbf{e}_X\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\mathbf{I} & -\mathbf{B}\mathbf{D}^{-1}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^T & \mathbf{D}\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & -\mathbf{B}\mathbf{D}^{-1}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{e}_p\\ \mathbf{e}_X\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^T & \mathbf{0}\\ \mathbf{B}^T & \mathbf{D}\end{bmatrix}\begin{bmatrix}\Delta\mathbf{p}\\ \Delta\mathbf{X}\end{bmatrix} = \begin{bmatrix}\mathbf{e}_p - \mathbf{B}\mathbf{D}^{-1}\mathbf{e}_X\\ \mathbf{e}_X\end{bmatrix}$$

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix}\mathbf{J}_p^T\mathbf{J}_p & \mathbf{J}_p^T\mathbf{J}_X\\ \mathbf{J}_X^T\mathbf{J}_p & \mathbf{J}_X^T\mathbf{J}_X\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^T & \mathbf{D}\end{bmatrix}$$

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix}\mathbf{D} & \\ & \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix}\mathbf{d}_1 & & \\ & \ddots & \\ & & \mathbf{d}_M\end{bmatrix} \rightarrow \mathbf{D}^{-1} = \begin{bmatrix}\mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1}\end{bmatrix}$$

Inversion of block diagonal matrix can be efficiently computed.

$$J^T J = \begin{bmatrix} D & \\ & \end{bmatrix}, \quad D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_M \end{bmatrix} \longrightarrow D^{-1} = \begin{bmatrix} d_1^{-1} & & \\ & \ddots & \\ & & d_M^{-1} \end{bmatrix}$$

Inversion of block diagonal matrix can be efficiently computed.

$$J^T J = \begin{bmatrix} J_p^T J_p & J_p^T J_X \\ J_X^T J_p & J_X^T J_X \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

Normal equation:

$$J^T J \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = J^T \left(b - f(X) \right)$$

$$\longrightarrow \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} e_p \\ e_X \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} A - BD^{-1}B^T & 0 \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p - BD^{-1}e_X \\ e_X \end{bmatrix}$$

$$\longrightarrow \Delta p = \left(A - BD^{-1}B^T \right)^{-1} \left(e_p - BD^{-1}e_X \right)$$

$$\Delta X = D^{-1} \left(e_X - B^T \Delta p \right)$$

Note: $A - BD^{-1}B^T$ is Schur complement of D

$$J^T J = \begin{bmatrix} D & \\ & \end{bmatrix}, \quad D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_M \end{bmatrix} \longrightarrow D^{-1} = \begin{bmatrix} d_1^{-1} & & \\ & \ddots & \\ & & d_M^{-1} \end{bmatrix}$$

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$$\longrightarrow \begin{bmatrix} A - BD^{-1}B^T & 0 \\ B^T & D \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta X \end{bmatrix} = \begin{bmatrix} e_p - BD^{-1}e_X \\ e_X \end{bmatrix}$$

Small size matrix

$$\longrightarrow \Delta p = \left(A - BD^{-1}B^T \right)^{-1} \left(e_p - BD^{-1}e_X \right)$$

$$\Delta X = D^{-1} \left(e_X - B^T \Delta p \right)$$

Note: $A - BD^{-1}B^T$ is Schur complement of D

Algorithm 4 Bundle Adjustment

1: $\widehat{p} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} & \cdots & \mathbf{p}_I^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1^\mathsf{T} & \cdots & \mathbf{X}_M^\mathsf{T} \end{bmatrix}$
2: **for** iter $= 1 :$ nIters **do**
3: Empty $\mathbf{J_p}$, $\mathbf{J_X}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{D}_{\text{inv}}$.

$$\mathbf{J} = \begin{bmatrix} \mathbf{J_p} & \mathbf{J_X} \end{bmatrix} \qquad \left(\mathbf{b} - f(\mathbf{p})\right) \qquad \mathbf{D}^{-1} = \begin{bmatrix} \mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1} \end{bmatrix}$$

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4: **for** $i = 1 : M$ **do** ← # of points
5: $\mathbf{d} = \mathbf{0}_{3\times 3}$
6: **for** $j = 1 : I$ **do** ← # of images
7: **if** the i^{th} point is visible from the j^{th} image **then** ← if visible

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        $\mathbf{J}_1 = \mathbf{0}_{2\times 7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times 3M}$
        $\mathbf{J}_1(:, 7(j-1)+1 : 7j) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j}$ ← $\mathbf{J_p}$

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8: $\mathbf{J}_1 = \mathbf{0}_{2\times7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times3M}$
9: $\mathbf{J}_1(:, 7(j-1)+1 : 7j) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j}$ ← $\mathbf{J_p}$
10: $\mathbf{J}_2(:, 3(i-1)+1 : 3i) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$ ← $\mathbf{J_X}$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J_p} & \mathbf{J_X} \end{bmatrix} \qquad \left(\mathbf{b} - f(\mathbf{p})\right) \qquad \mathbf{D}^{-1} = \begin{bmatrix} \mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1} \end{bmatrix}$$

Algorithm 4 Bundle Adjustment

$\widehat{p} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} & \cdots & \mathbf{p}_I^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1^\mathsf{T} & \cdots & \mathbf{X}_M^\mathsf{T} \end{bmatrix}$
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  for $i = 1 : M$ **do** ← # of points
    $\mathbf{d} = \mathbf{0}_{3\times3}$
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        $\mathbf{J}_1 = \mathbf{0}_{2\times7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times3M}$
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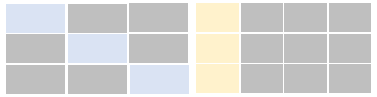
$$\mathbf{J} = \begin{bmatrix} \mathbf{J_p} & \mathbf{J_X} \end{bmatrix} \qquad \left(\mathbf{b} - f(\mathbf{p})\right) \qquad \mathbf{D}^{-1} = \begin{bmatrix} \mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1} \end{bmatrix}$$



Algorithm 4 Bundle Adjustment

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for iter $= 1 :$ nIters **do**

  Empty $\mathbf{J_p}$, $\mathbf{J_X}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{D}_{\text{inv}}$.

  for $i = 1 : M$ **do** ← # of points

    $\mathbf{d} = \mathbf{0}_{3\times 3}$

    for $j = 1 : I$ **do** ← # of images

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        $\mathbf{J}_1 = \mathbf{0}_{2\times 7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times 3M}$

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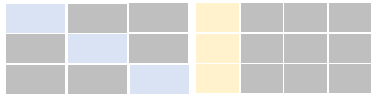
        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$ ← $\mathbf{d}$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J_p} & \mathbf{J_X} \end{bmatrix} \qquad \left(\mathbf{b} - f(\mathbf{p})\right) \qquad \mathbf{D}^{-1} = \begin{bmatrix} \mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1} \end{bmatrix}$$



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for iter $= 1 :$ nIters **do**

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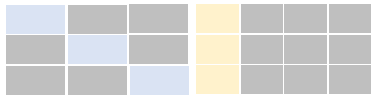
        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$ ← $\mathbf{d}$

        $\mathbf{b} = \begin{bmatrix} \mathbf{b}^\mathsf{T} & \mathbf{u}_{ij}^\mathsf{T} \end{bmatrix}$

        $\mathbf{f} = \begin{bmatrix} \mathbf{f}^\mathsf{T} & \mathbf{x}_{ij}^\mathsf{T} \end{bmatrix}$ where $\mu \begin{bmatrix} \mathbf{x}_{ij} \\ 1 \end{bmatrix} = \mathbf{R}_j \begin{bmatrix} \mathbf{I} & -\mathbf{C}_j \end{bmatrix}$ ← $\big(\mathbf{b} - f(\mathbf{p})\big)$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J_p} & \mathbf{J_X} \end{bmatrix} \qquad \big(\mathbf{b} - f(\mathbf{p})\big) \qquad \mathbf{D}^{-1} = \begin{bmatrix} \mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1} \end{bmatrix}$$



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for iter $= 1 :$ nIters **do**
  Empty $\mathbf{J_p}$, $\mathbf{J_X}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{D}_{\text{inv}}$.
  for $i = 1 : M$ **do** ← # of points
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        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$ ← $\mathbf{d}$
        $\mathbf{b} = \begin{bmatrix} \mathbf{b}^\mathsf{T} & \mathbf{u}_{ij}^\mathsf{T} \end{bmatrix}$
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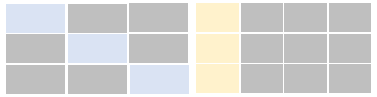
      end if
    end for
    $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$ ← $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$
    $\mathbf{D}_{\text{inv}} = \texttt{blkdiag}(\mathbf{D}_{\text{inv}}, \mathbf{d}^{-1})$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J_p} & \mathbf{J_X} \end{bmatrix} \qquad \mathbf{b} - f(\mathbf{p}) \qquad \mathbf{D}^{-1} = \begin{bmatrix} \mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1} \end{bmatrix}$$



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        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$ ← $\mathbf{d}$
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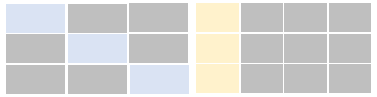
      end if
    end for
    $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$ ← $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$
    $\mathbf{D}_{\text{inv}} = \texttt{blkdiag}(\mathbf{D}_{\text{inv}}, \mathbf{d}^{-1})$
  end for
  $\mathbf{e_p} = \mathbf{J}_\mathbf{p}^\mathsf{T}(\mathbf{b} - \mathbf{f})$ ← $\begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} \\ \mathbf{J}_\mathbf{X}^\mathsf{T} \end{bmatrix} (\mathbf{b} - f(\mathbf{X})) = \begin{bmatrix} \mathbf{e}_\mathbf{p} \\ \mathbf{e}_\mathbf{X} \end{bmatrix}$
  $\mathbf{e_X} = \mathbf{J}_\mathbf{X}^\mathsf{T}(\mathbf{b} - \mathbf{f})$

$\mathbf{J} = \begin{bmatrix} \mathbf{J_p} & \mathbf{J_X} \end{bmatrix}$ $(\mathbf{b} - f(\mathbf{p}))$ $\mathbf{D}^{-1} = \begin{bmatrix} \mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1} \end{bmatrix}$



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$\widehat{p} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} & \cdots & \mathbf{p}_I^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1^\mathsf{T} & \cdots & \mathbf{X}_M^\mathsf{T} \end{bmatrix}$
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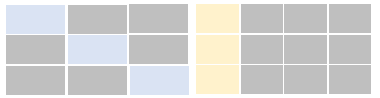
    for $j = 1 : I$ **do** ← # of images
      if the i^{th} point is visible from the j^{th} image **then** ← if visible
        $\mathbf{J}_1 = \mathbf{0}_{2\times 7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times 3M}$
        $\mathbf{J}_1(:, 7(j-1)+1 : 7j) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j}$ ← $\mathbf{J_p}$
        $\mathbf{J}_2(:, 3(i-1)+1 : 3i) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$ ← $\mathbf{J_X}$
        $\mathbf{J_p} = \begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} & \mathbf{J}_1^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\mathbf{J_X} = \begin{bmatrix} \mathbf{J}_\mathbf{X}^\mathsf{T} & \mathbf{J}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$ ←
        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$ ← $\mathbf{d}$
        $\mathbf{b} = \begin{bmatrix} \mathbf{b}^\mathsf{T} & \mathbf{u}_{ij}^\mathsf{T} \end{bmatrix}$
        $\mathbf{f} = \begin{bmatrix} \mathbf{f}^\mathsf{T} & \mathbf{x}_{ij}^\mathsf{T} \end{bmatrix}$ where $\mu \begin{bmatrix} \mathbf{x}_{ij} \\ 1 \end{bmatrix} = \mathbf{R}_j \begin{bmatrix} \mathbf{I} & -\mathbf{C}_j \end{bmatrix}$ ← $\mathbf{b} - f(\mathbf{p})$
      end if
    end for
    $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$ ← $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$
    $\mathbf{D}_{\text{inv}} = \texttt{blkdiag}(\mathbf{D}_{\text{inv}}, \mathbf{d}^{-1})$
  end for
  $\mathbf{e_p} = \mathbf{J}_\mathbf{p}^\mathsf{T}(\mathbf{b} - \mathbf{f})$ ← $\begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} \\ \mathbf{J}_\mathbf{X}^\mathsf{T} \end{bmatrix} (\mathbf{b} - f(\mathbf{X})) = \begin{bmatrix} \mathbf{e}_\mathbf{p} \\ \mathbf{e}_\mathbf{X} \end{bmatrix}$
  $\mathbf{e_X} = \mathbf{J}_\mathbf{X}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{A} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_p} + \lambda \mathbf{I}$, $\mathbf{B} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_X}$, $\mathbf{D}^{-1} = \mathbf{D}_{\text{inv}}$ ← $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\mathsf{T} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J}_\mathbf{p} + \mu \mathbf{I} & \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J}_\mathbf{X} \\ \mathbf{J}_\mathbf{X}^\mathsf{T}\mathbf{J}_\mathbf{p} & \mathbf{J}_\mathbf{X}^\mathsf{T}\mathbf{J}_\mathbf{X} + \mu \mathbf{I} \end{bmatrix}$

$\mathbf{J} = \begin{bmatrix} \mathbf{J}_\mathbf{p} & \mathbf{J}_\mathbf{X} \end{bmatrix}$ $\quad (\mathbf{b} - f(\mathbf{p}))$ $\quad \mathbf{D}^{-1} = \begin{bmatrix} \mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1} \end{bmatrix}$

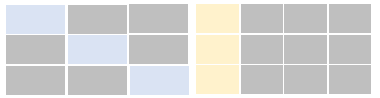


$\mathbf{D}^{-1} = \begin{bmatrix} \mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1} \end{bmatrix}$

Algorithm 4 Bundle Adjustment

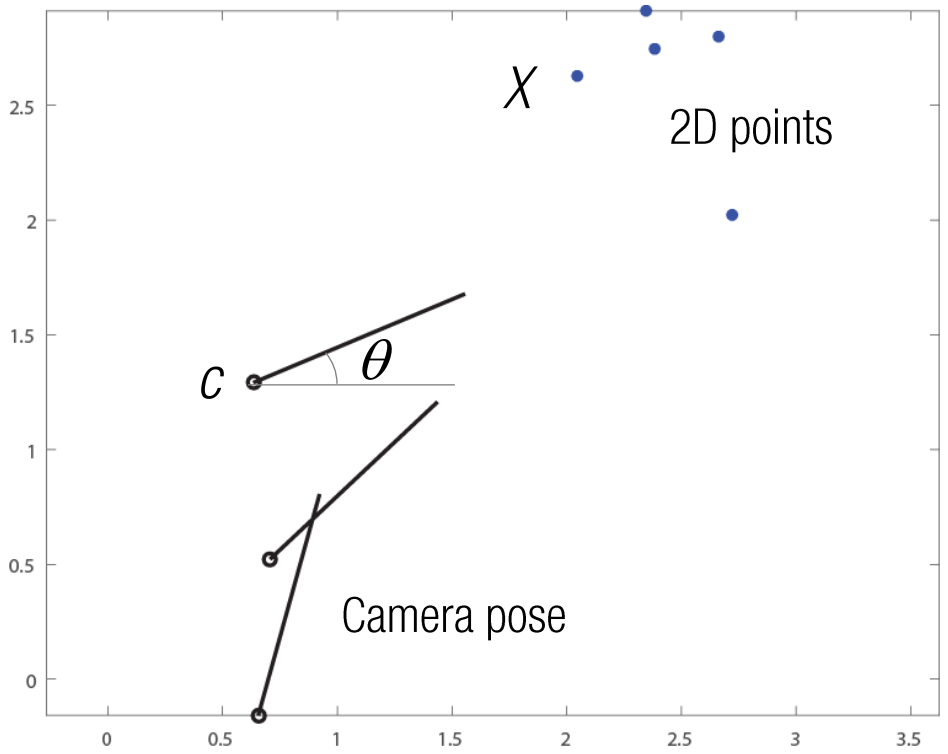
$\widehat{p} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} & \cdots & \mathbf{p}_I^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1^\mathsf{T} & \cdots & \mathbf{X}_M^\mathsf{T} \end{bmatrix}$
for iter $= 1 :$ nIters **do**
  Empty $\mathbf{J_p}$, $\mathbf{J_X}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{D}_{\text{inv}}$.
  for $i = 1 : M$ **do** ← # of points
    $\mathbf{d} = \mathbf{0}_{3\times 3}$
    for $j = 1 : I$ **do** ← # of images
      if the i^{th} point is visible from the j^{th} image **then** ← if visible
        $\mathbf{J}_1 = \mathbf{0}_{2\times 7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times 3M}$
        $\mathbf{J}_1(:, 7(j-1)+1 : 7j) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j}$ ← $\mathbf{J_p}$
        $\mathbf{J}_2(:, 3(i-1)+1 : 3i) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$ ← $\mathbf{J_X}$
        $\mathbf{J_p} = \begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} & \mathbf{J}_1^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\mathbf{J_X} = \begin{bmatrix} \mathbf{J}_\mathbf{X}^\mathsf{T} & \mathbf{J}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$ ←
        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$ ← $\mathbf{d}$
        $\mathbf{b} = \begin{bmatrix} \mathbf{b}^\mathsf{T} & \mathbf{u}_{ij}^\mathsf{T} \end{bmatrix}$
        $\mathbf{f} = \begin{bmatrix} \mathbf{f}^\mathsf{T} & \mathbf{x}_{ij}^\mathsf{T} \end{bmatrix}$ where $\mu \begin{bmatrix} \mathbf{x}_{ij} \\ 1 \end{bmatrix} = \mathbf{R}_j \begin{bmatrix} \mathbf{I} & -\mathbf{C}_j \end{bmatrix}$ ← $(\mathbf{b} - f(\mathbf{p}))$
      end if
    end for
    $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$ ← $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$
    $\mathbf{D}_{\text{inv}} = \texttt{blkdiag}(\mathbf{D}_{\text{inv}}, \mathbf{d}^{-1})$
  end for
  $\mathbf{e_p} = \mathbf{J}_\mathbf{p}^\mathsf{T}(\mathbf{b} - \mathbf{f})$ ← $\begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} \\ \mathbf{J}_\mathbf{X}^\mathsf{T} \end{bmatrix} (\mathbf{b} - f(\mathbf{X})) = \begin{bmatrix} \mathbf{e}_\mathbf{p} \\ \mathbf{e}_\mathbf{X} \end{bmatrix}$
  $\mathbf{e_X} = \mathbf{J}_\mathbf{X}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{A} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_p} + \lambda \mathbf{I}$, $\mathbf{B} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_X}$, $\mathbf{D}^{-1} = \mathbf{D}_{\text{inv}}$ ← $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\mathsf{T} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_p} + \mu \mathbf{I} & \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_X} \\ \mathbf{J}_\mathbf{X}^\mathsf{T}\mathbf{J_p} & \mathbf{J}_\mathbf{X}^\mathsf{T}\mathbf{J_X} + \mu \mathbf{I} \end{bmatrix}$
  $\Delta\widehat{\mathbf{p}} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^\mathsf{T})^{-1}(\mathbf{e_p} - \mathbf{B}\mathbf{D}^{-1}\mathbf{e_X})$ ← $\Delta\mathbf{p} = \left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^\mathsf{T}\right)^{-1}\left(\mathbf{e}_\mathbf{p} - \mathbf{B}\mathbf{D}^{-1}\mathbf{e}_\mathbf{X}\right)$
  Normalize quaternions.
  $\Delta\widehat{\mathbf{X}} = \mathbf{D}^{-1}(\mathbf{e_X} - \mathbf{B}^\mathsf{T}\Delta\widehat{\mathbf{p}})$ ← $\Delta\mathbf{X} = \mathbf{D}^{-1}\left(\mathbf{e}_\mathbf{X} - \mathbf{B}^\mathsf{T}\Delta\mathbf{p}\right)$
end for

$\mathbf{J} = \begin{bmatrix} \mathbf{J}_\mathbf{p} & \mathbf{J}_\mathbf{X} \end{bmatrix}$ $\quad (\mathbf{b} - f(\mathbf{p}))$ $\quad \mathbf{D}^{-1} = \begin{bmatrix} \mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1} \end{bmatrix}$



$\mathbf{D}^{-1} = \begin{bmatrix} \mathbf{d}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{d}_M^{-1} \end{bmatrix}$

1D Camera Bundle Adjustment

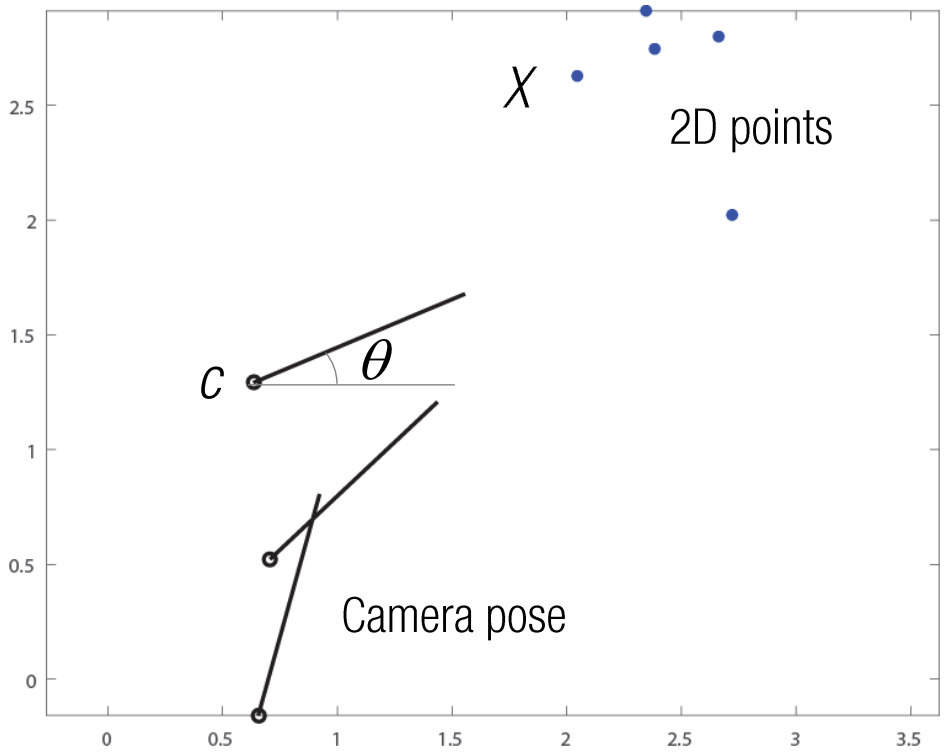


BundleAdjustment1D.m

Data generation

```
nPoints = 5;
nCameras = 3;
X = rand(nPoints, 2) + 2;
c = rand(nCameras,2);
for i = 1 : nCameras
   m = X-ones(nPoints,1)*c(i,:);
   theta = atan2(m(:,2), m(:,1));
   theta = mean(theta);

   C(i).c = c(i,:)';
   theta = theta;
   C(i).R = [sin(theta) -cos(theta); cos(theta) sin(theta)];

   u = C(i).R*[eye(2) -C(i).c] * [X'; ones(1,nPoints)];
   C(i).m = u(1,:)./u(2,:);

   C(i).c = c(i,:)'+0.3*randn(2,1);
   theta = theta + 0.3*randn(1,1);
   C(i).R = [sin(theta) -cos(theta); cos(theta) sin(theta)];
end
X = X + 0.3*randn(size(X));

[C X] = BA(C, X);
```

1D Camera Bundle Adjustment



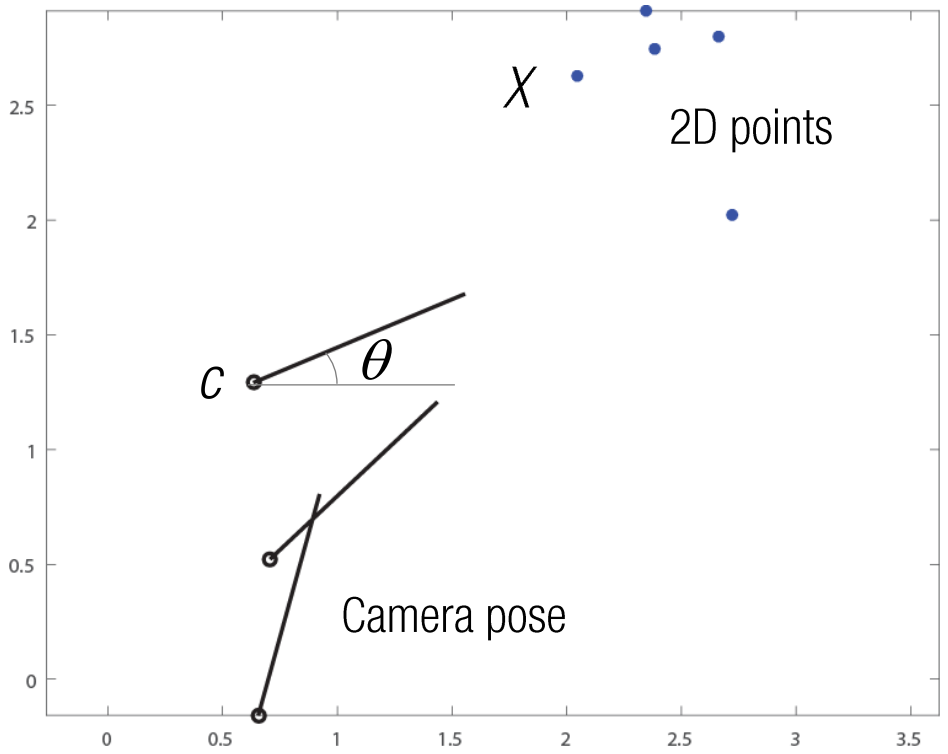
BundleAdjustment1D.m

Data generation

```
nPoints = 5;
nCameras = 3;
X = rand(nPoints, 2) + 2;
c = rand(nCameras,2);
for i = 1 : nCameras
   m = X-ones(nPoints,1)*c(i,:);
   theta = atan2(m(:,2), m(:,1));
   theta = mean(theta);

   C(i).c = c(i,:)';
   theta = theta;
   C(i).R = [sin(theta) -cos(theta); cos(theta) sin(theta)];

   u = C(i).R*[eye(2) -C(i).c] * [X'; ones(1,nPoints)];
   C(i).m = u(1,:)./u(2,:);

   C(i).c = c(i,:)'+0.3*randn(2,1);
   theta = theta + 0.3*randn(1,1);
   C(i).R = [sin(theta) -cos(theta); cos(theta) sin(theta)];
end
X = X + 0.3*randn(size(X));

[C X] = BA(C, X);
```

Ground truth projection

1D Camera Bundle Adjustment



BundleAdjustment1D.m

Data generation

```
nPoints = 5;
nCameras = 3;
X = rand(nPoints, 2) + 2;
c = rand(nCameras,2);
for i = 1 : nCameras
   m = X-ones(nPoints,1)*c(i,:);
   theta = atan2(m(:,2), m(:,1));
   theta = mean(theta);

   C(i).c = c(i,:)';
   theta = theta;
   C(i).R = [sin(theta) -cos(theta); cos(theta) sin(theta)];
```

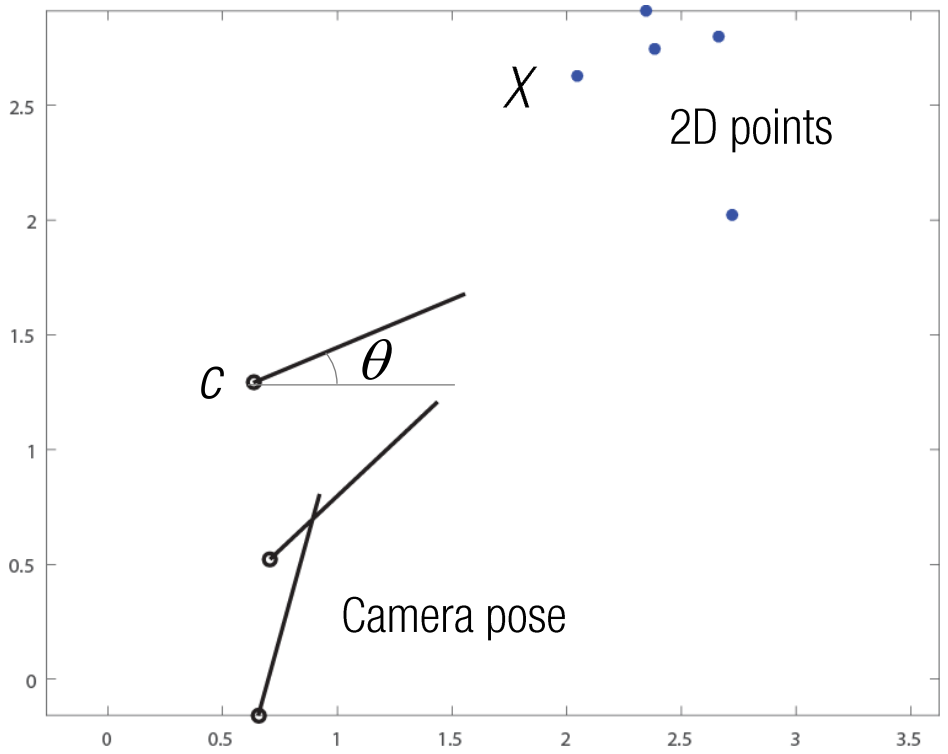
Ground truth projection

```
   u = C(i).R*[eye(2) -C(i).c] * [X'; ones(1,nPoints)];
   C(i).m = u(1,:)./u(2,:);
```

Add noise

```
   C(i).c = c(i,:)'+0.3*randn(2,1);
   theta = theta + 0.3*randn(1,1);
   C(i).R = [sin(theta) -cos(theta); cos(theta) sin(theta)];
end
X = X + 0.3*randn(size(X));

[C X] = BA(C, X);
```

1D Camera Bundle Adjustment



BundleAdjustment1D.m

```
nPoints = 5;
nCameras = 3;
X = rand(nPoints, 2) + 2;
c = rand(nCameras,2);
for i = 1 : nCameras
   m = X-ones(nPoints,1)*c(i,:);
   theta = atan2(m(:,2), m(:,1));
   theta = mean(theta);

   C(i).c = c(i,:)';
   theta = theta;
   C(i).R = [sin(theta) -cos(theta); cos(theta) sin(theta)];

   u = C(i).R*[eye(2) -C(i).c] * [X'; ones(1,nPoints)];
   C(i).m = u(1,:)./u(2,:);

   C(i).c = c(i,:)'+0.3*randn(2,1);
   theta = theta + 0.3*randn(1,1);
   C(i).R = [sin(theta) -cos(theta); cos(theta) sin(theta)];
end
X = X + 0.3*randn(size(X));

[C X] = BA(C, X);
```

Data generation

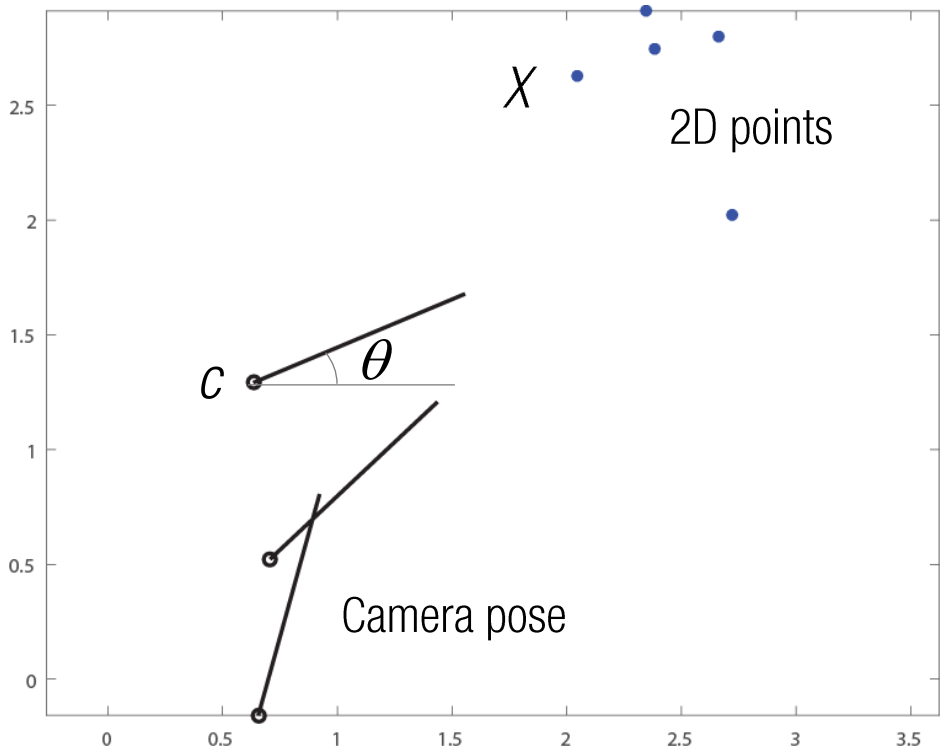
Ground truth projection

Add noise

← 1D bundle adjustment

1D Camera Bundle Adjustment

BundleAdjustment1D.m




```
function [C X] = BA(C, X)

lambda = 0.5;
nIters = 100;

xp = [];
for i = 1 : length(C)
   theta=atan2(C(i).R(2,2), C(i).R(2,1));   xp = [xp; C(i).c; theta];
end

xx = [];
for i = 1 : size(X,1)
   xx = [xx; X(i,:)'];
end
```

Algorithm 4 Bundle Adjustment

$\widehat{p} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} & \cdots & \mathbf{p}_I^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1^\mathsf{T} & \cdots & \mathbf{X}_M^\mathsf{T} \end{bmatrix}$
for iter $= 1 :$ nIters **do**
  Empty $\mathbf{J_p}$, $\mathbf{J_X}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{D}_{\text{inv}}$.
  for $i = 1 : M$ **do**
    $\mathbf{d} = \mathbf{0}_{3\times 3}$
    for $j = 1 : I$ **do**
      if the i^{th} point is visible from the j^{th} image **then**
        $\mathbf{J}_1 = \mathbf{0}_{2\times 7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times 3M}$
        $\mathbf{J}_1(:, 7(j-1)+1 : 7j) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j}$
        $\mathbf{J}_2(:, 3(i-1)+1 : 3i) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{J_p} = \begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} & \mathbf{J}_1^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\mathbf{J_X} = \begin{bmatrix} \mathbf{J}_\mathbf{X}^\mathsf{T} & \mathbf{J}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$
        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{b} = \begin{bmatrix} \mathbf{b}^\mathsf{T} & \mathbf{u}_{ij}^\mathsf{T} \end{bmatrix}$
        $\mathbf{f} = \begin{bmatrix} \mathbf{f}^\mathsf{T} & \mathbf{x}_{ij}^\mathsf{T} \end{bmatrix}$ where $\mu \begin{bmatrix} \mathbf{x}_{ij} \\ 1 \end{bmatrix} = \mathbf{R}_j \begin{bmatrix} \mathbf{I} & -\mathbf{C}_j \end{bmatrix}$
      end if
    end for
    $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$
    $\mathbf{D}_{\text{inv}} = \texttt{blkdiag}(\mathbf{D}_{\text{inv}}, \mathbf{d}^{-1})$
  end for
  $\mathbf{e_p} = \mathbf{J}_\mathbf{p}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{e_X} = \mathbf{J}_\mathbf{X}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{A} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_p} + \lambda\mathbf{I}$, $\mathbf{B} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_X}$, $\mathbf{D}^{-1} = \mathbf{D}_{\text{inv}}$
  $\Delta\widehat{\mathbf{p}} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^\mathsf{T})^{-1}(\mathbf{e_p} - \mathbf{B}\mathbf{D}^{-1}\mathbf{e_X})$
  Normalize quaternions.
  $\Delta\widehat{\mathbf{X}} = \mathbf{D}^{-1}(\mathbf{e_X} - \mathbf{B}^\mathsf{T}\Delta\widehat{\mathbf{p}})$
end for

BundleAdjustment1D.m

```
for j = 1 : nIters
   Jp = [];    Jx = [];    D_inv = [];    err = [];
   for iPoint = 1 : size(X,1)
      X1 = xx(2*(iPoint-1)+1:2*iPoint); d = zeros(2,2);
      for iC = 1 : length(C)
         c = xp(3*(iC-1)+1:3*(iC-1)+2);
         theta = xp(3*(iC-1)+3);
         R = [sin(theta) -cos(theta); cos(theta) sin(theta)];

         df_dc = JacobianC1D(R, c, X1);
         df_dR = JacobianR1D(R, c, X1)*JacobianQ1D(theta);
         df_dx = JacobianX1D(R, c, X1);

         j1 = zeros(1,3*length(C));  j1(:,3*(iC-1)+1:3*iC) = [df_dc df_dR];
         j2 = zeros(1,2*size(X,1));   j2(:,2*(iPoint-1)+1:2*iPoint) = df_dx;

         Jp = [Jp; j1];       Jx = [Jx; j2];
         d = d + df_dx'*df_dx;

         u = R * [eye(2) -c] * [X1;1];      u = u/u(2);
         u1 = C(iC).m;

         e = [u1(iPoint) - u(1)];
         err = [err; e];
      end
      d = d + lambda*eye(2);       D_inv = blkdiag(D_inv, inv(d));
   end
   ep = Jp' * err;    ex = Jx' * err;
   A = Jp'*Jp + lambda*eye(3*length(C));    B = Jp'*Jx;
   delta_p = inv(A-B*D_inv*B') * (ep-B*D_inv*ex);    delta_x = D_inv * (ex-B'*delta_p);

   xp = xp + delta_p;
   xx = xx + delta_x;
end
```

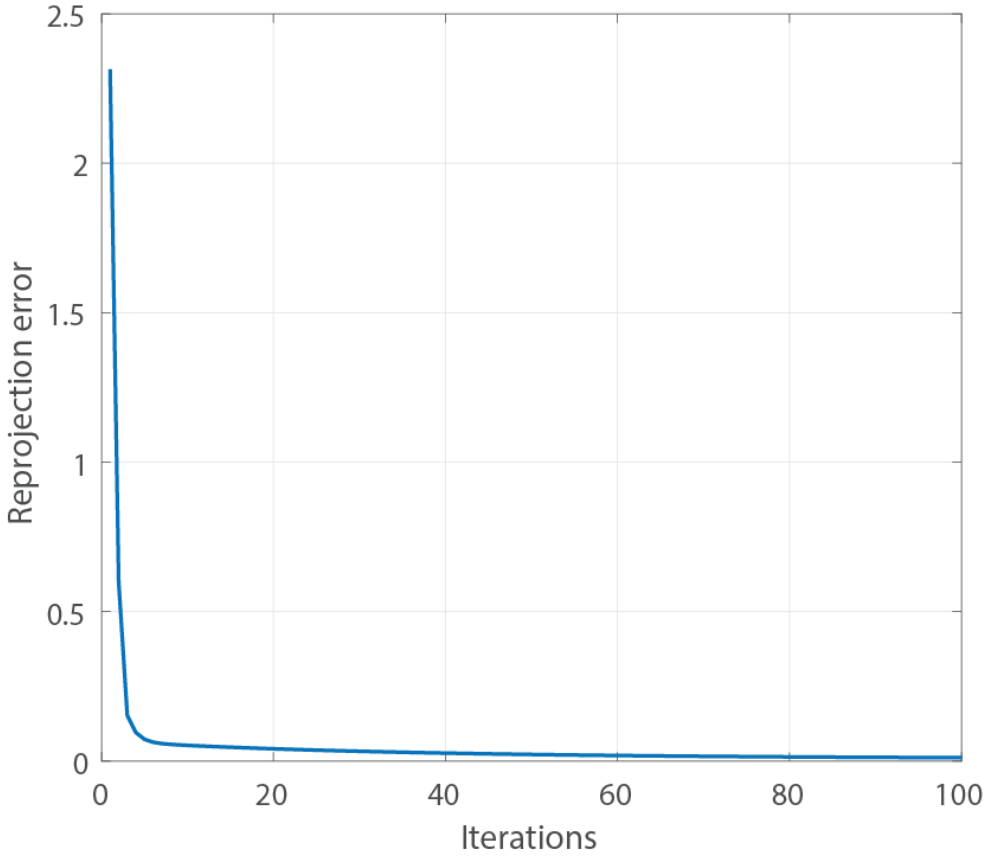
Algorithm 4 Bundle Adjustment

$\widehat{p} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} & \cdots & \mathbf{p}_I^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1^\mathsf{T} & \cdots & \mathbf{X}_M^\mathsf{T} \end{bmatrix}$
for iter $= 1 :$ nIters **do**
  Empty $\mathbf{J_p}$, $\mathbf{J_X}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{D}_{\text{inv}}$.
  for $i = 1 : M$ **do**
    $\mathbf{d} = \mathbf{0}_{3\times3}$
    for $j = 1 : I$ **do**
      if the i^{th} point is visible from the j^{th} image **then**
        $\mathbf{J}_1 = \mathbf{0}_{2\times7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times3M}$
        $\mathbf{J}_1(:, 7(j-1)+1 : 7j) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j}$
        $\mathbf{J}_2(:, 3(i-1)+1 : 3i) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{J_p} = \begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} & \mathbf{J}_1^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\mathbf{J_X} = \begin{bmatrix} \mathbf{J}_\mathbf{X}^\mathsf{T} & \mathbf{J}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$
        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{b} = \begin{bmatrix} \mathbf{b}^\mathsf{T} & \mathbf{u}_{ij}^\mathsf{T} \end{bmatrix}$
        $\mathbf{f} = \begin{bmatrix} \mathbf{f}^\mathsf{T} & \mathbf{x}_{ij}^\mathsf{T} \end{bmatrix}$ where $\mu \begin{bmatrix} \mathbf{x}_{ij} \\ 1 \end{bmatrix} = \mathbf{R}_j \begin{bmatrix} \mathbf{I} & -\mathbf{C}_j \end{bmatrix}$
      end if
    end for
    $\mathbf{d} = \mathbf{d} + \lambda\mathbf{I}$
    $\mathbf{D}_{\text{inv}} = \texttt{blkdiag}(\mathbf{D}_{\text{inv}}, \mathbf{d}^{-1})$
  end for
  $\mathbf{e_p} = \mathbf{J}_\mathbf{p}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{e_X} = \mathbf{J}_\mathbf{X}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{A} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_p} + \lambda\mathbf{I}$, $\mathbf{B} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_X}$, $\mathbf{D}^{-1} = \mathbf{D}_{\text{inv}}$
  $\Delta\widehat{\mathbf{p}} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^\mathsf{T})^{-1}(\mathbf{e_p} - \mathbf{B}\mathbf{D}^{-1}\mathbf{e_X})$
  Normalize quaternions.
  $\Delta\widehat{\mathbf{X}} = \mathbf{D}^{-1}(\mathbf{e_X} - \mathbf{B}^\mathsf{T}\Delta\widehat{\mathbf{p}})$
end for

BundleAdjustment1D.m

```
for iC = 1 : length(C)
   C(iC).c = xp(3*(iC-1)+1:3*(iC-1)+2);
   theta = xp(3*(iC-1)+3);
   C(iC).R = [sin(theta) -cos(theta); cos(theta) sin(theta)];
end

for iX = 1 : size(X,1)
   X(iX,:) = xx(2*(iX-1)+1:2*iX);
end
```

Model update



BundleAdjustment1D.m

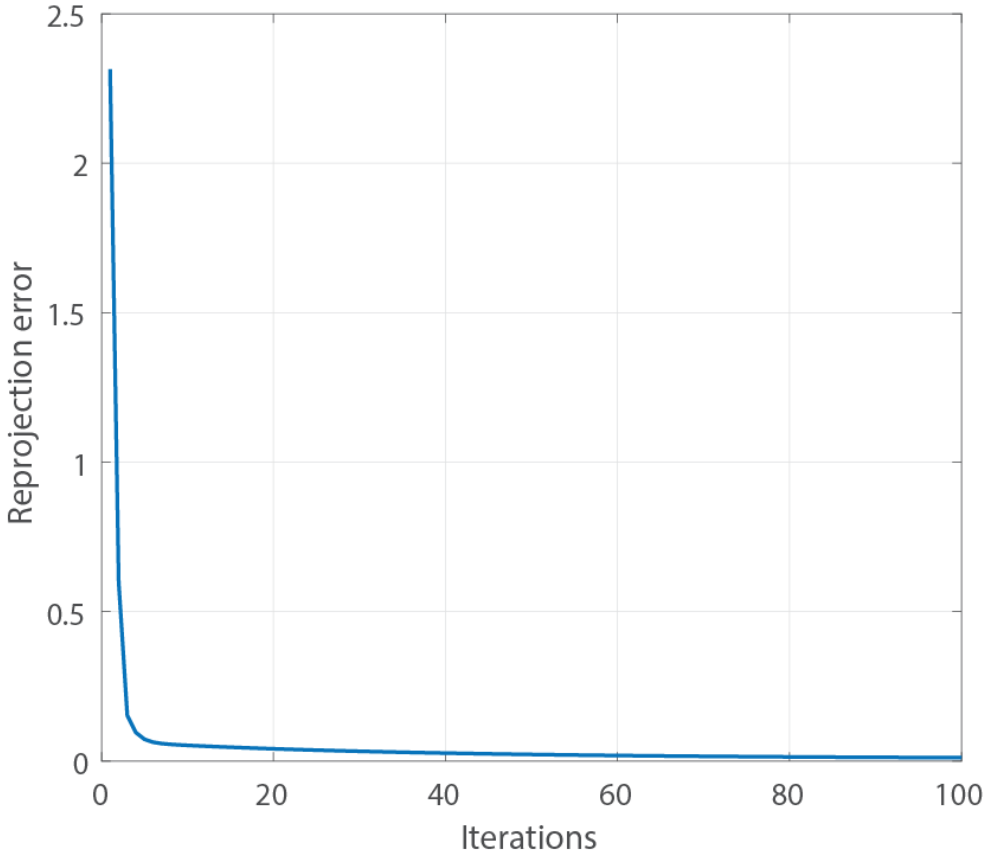
```
for iC = 1 : length(C)
   C(iC).c = xp(3*(iC-1)+1:3*(iC-1)+2);
   theta = xp(3*(iC-1)+3);
   C(iC).R = [sin(theta) -cos(theta); cos(theta) sin(theta)];
end

for iX = 1 : size(X,1)
   X(iX,:) = xx(2*(iX-1)+1:2*iX);
end
```

Model update

mean_delta_X = 0.0265

mean_delta_p = 0.2648

Why camera and point error do not converge to zero while reprojection error converges?



BundleAdjustment1D.m

```
for iC = 1 : length(C)
   C(iC).c = xp(3*(iC-1)+1:3*(iC-1)+2);
   theta = xp(3*(iC-1)+3);
   C(iC).R = [sin(theta) -cos(theta); cos(theta) sin(theta)];
end

for iX = 1 : size(X,1)
   X(iX,:) = xx(2*(iX-1)+1:2*iX);
end
```

Model update

mean_delta_X = 0.0265

mean_delta_p = 0.2648

Why camera and point error do not converge to zero while reprojection error converges?

Because the bundle adjustment is still up to scale and orietation.

SIFT detection
Reprojection

SIFT detection
Reprojection


Measurement
Linear estimate (reproj: 0.199104)
Nonlinear estimate (reproj: 0.119272)


Measurement
Linear estimate (reproj: 0.199104)
Nonlinear estimate (reproj: 0.119272)

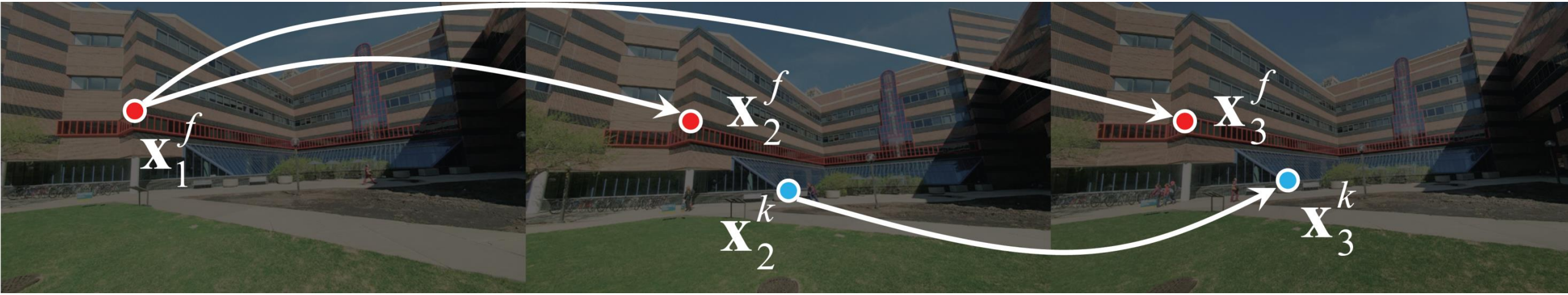
Data Capture

Algorithm 1 Structure from Motion

Build feature track ▷ `BuildFeatureTrack`
Estimate first two camera poses ▷ `EstimateCameraPose`
Initialize pose set `P`
for `i` = 2, $\cdots$, N-1 **do**
    Estimate new camera pose ▷ `PnP, PnP_nl`
    `P = P` $\cup$ `P_i`
    for `j < i` **do**
        Find new points to reconstruct ▷ `FindMissingReconstruction`
        Triangulate point ▷ `Triangulation, Triangulation_nl`
        Filter out point based on cheirality ▷ `EvaluateCheirality`
        Update 3D point
    end for
    Run bundle adjustment ▷ `RunBundleAdjustment`
end for

Algorithm 1 Structure from Motion

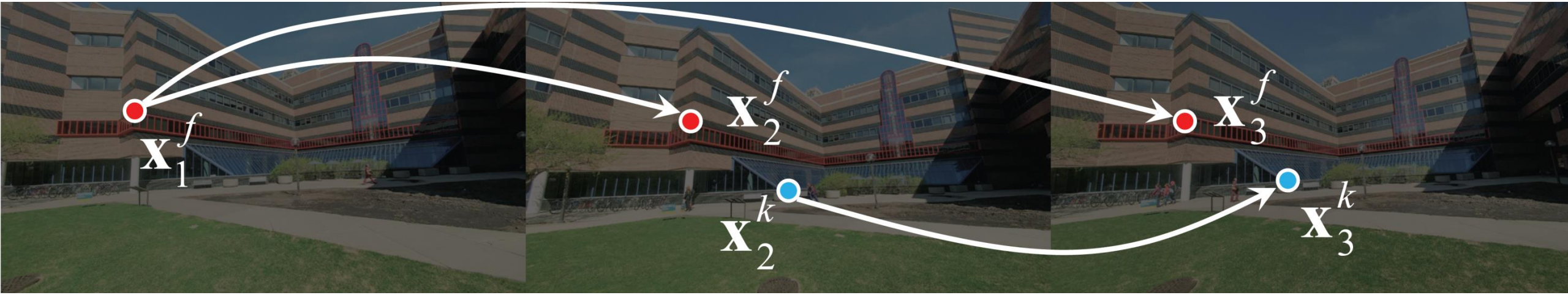
```
Build feature track                                              ▷ BuildFeatureTrack
Estimate first two camera poses                                  ▷ EstimateCameraPose
Initialize pose set P
for i = 2, ···, N-1 do
    Estimate new camera pose                                     ▷ PnP, PnP_nl
    P = P ∪ P_i
    for j < i do
        Find new points to reconstruct                           ▷ FindMissingReconstruction
        Triangulate point                                        ▷ Triangulation, Triangulation_nl
        Filter out point based on cheirality                     ▷ EvaluateCheirality
        Update 3D point
    end for
    Run bundle adjustment                                        ▷ RunBundleAdjustment
end for
```



track =

	1 pt	2 pt
1 img	$\mathbf{x}_1^1$	-1
2 img	$\mathbf{x}_1^2$	$\mathbf{x}_2^2$
3 img	$\mathbf{x}_1^3$	$\mathbf{x}_2^3$
	•	
	•	
	•	

N x *F*



track =

	1 pt	2 pt
1 img	$\mathbf{x}_1^1$	-1
2 img	$\mathbf{x}_1^2$	$\mathbf{x}_2^2$
3 img	$\mathbf{x}_1^3$	$\mathbf{x}_2^3$
	•	
	•	
	•	

N x *F* x 2

Algorithm 2 BuildFeatureTrack

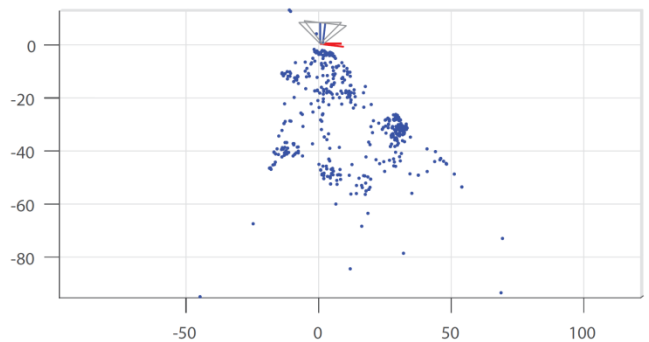
for `i` $= 0, \cdots, N-1$ **do**
    Extract SIFT descriptor of the i^{th} image, `Im[i]`
end for
for `i` $= 0, \cdots, N-1$ **do**
    Initialize `track_i` $= \text{-}1^{N \times F \times 2}$
    for `j` $= i+1, \cdots, N-1$ **do**
        Match features between the i^{th} and j^{th} images ▷ `MatchSIFT`
        Normalize coordinate by multiplying the inverse of intrinsics.
        Find inliner matches using essential matrix ▷ `EstimateE_RANSAC`
        Update `track_i` using the inlier matches.
    end for
    Remove features in `track_i` that have not been matched for $i+1, \cdots, N$.
    `track` $=$ `track` $\cup$ `track_i`
end for

Algorithm 1 Structure from Motion

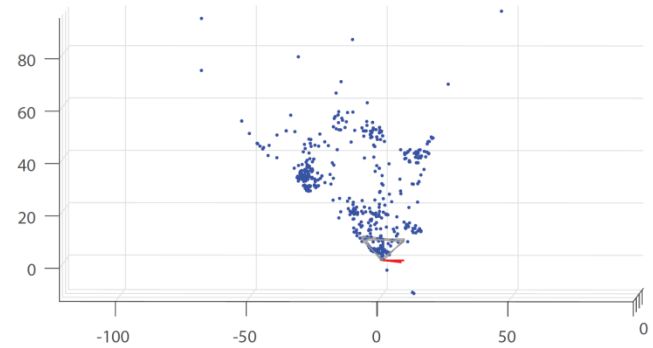
```
Build feature track                                          ▷ BuildFeatureTrack
Estimate first two camera poses                              ▷ EstimateCameraPose
Initialize pose set P
for i = 2, ···, N-1 do
    Estimate new camera pose                                 ▷ PnP, PnP_nl
    P = P ∪ P_i
    for j < i do
        Find new points to reconstruct                       ▷ FindMissingReconstruction
        Triangulate point                                    ▷ Triangulation, Triangulation_nl
        Filter out point based on cheirality                 ▷ EvaluateCheirality
        Update 3D point
    end for
    Run bundle adjustment                                    ▷ RunBundleAdjustment
end for
```

Algorithm 3 EstimateCameraPose

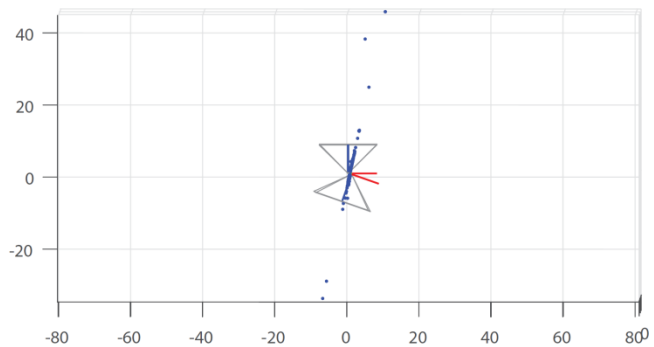
1: Compute essential matrix given tracks ▷ `EstimateE_RANSAC`
2: Estimate four configurations of poses ▷ `GetCameraPoseFromE`
3: **for** `i` $= 0, 1, 2, 3$ **do**
4: Triangulate points using for each configuration ▷ `Triangulation`
5: Evaluate cheirality for each configuration ▷ `EvaluateCheirality`
6: **end for**



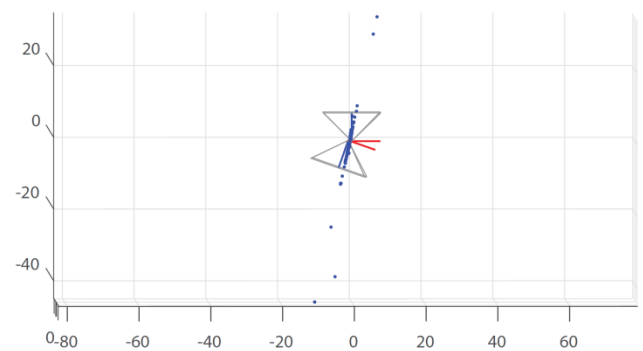
(a) nValid = 10



(b) nValid = 488



(c) nValid = 0

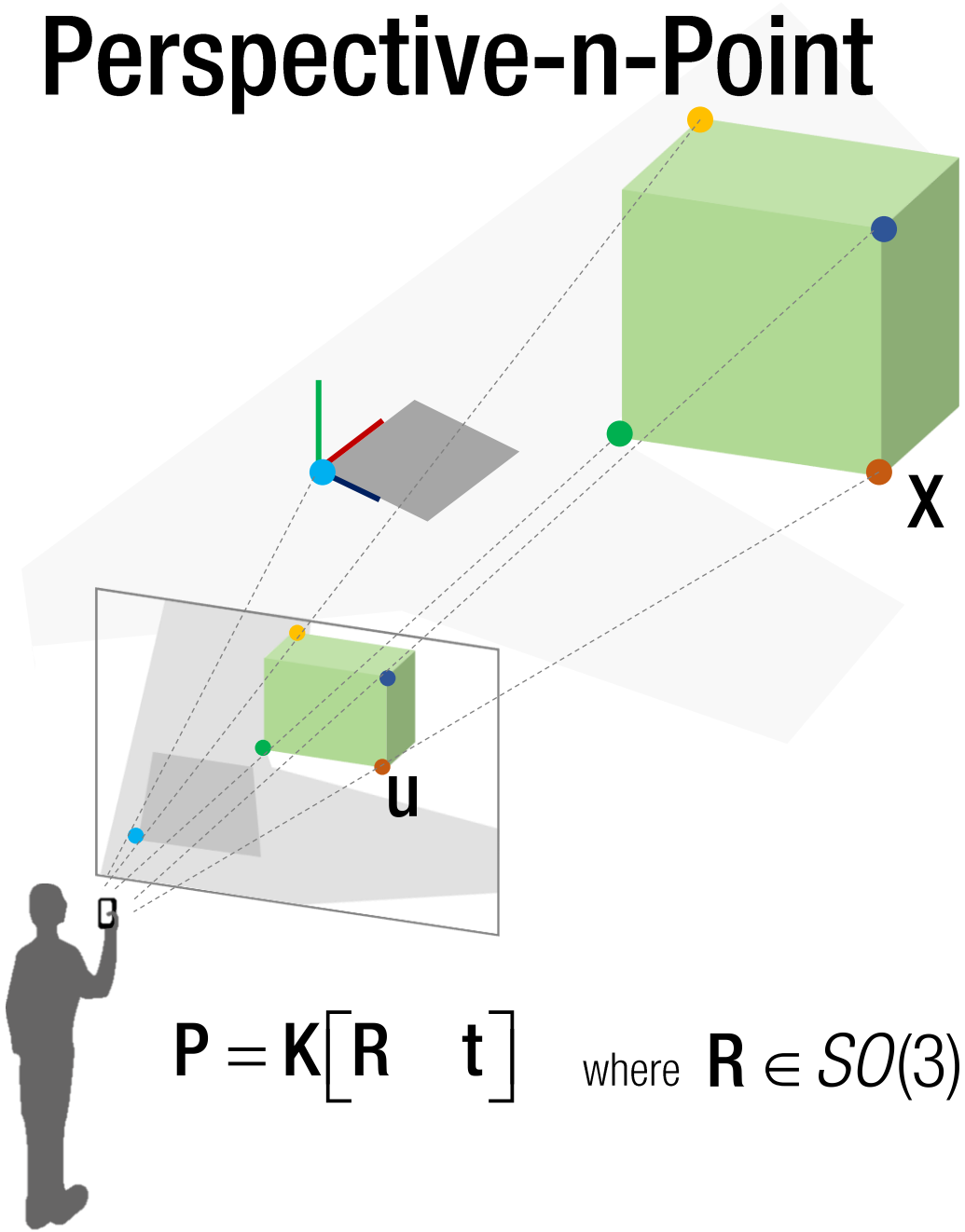


(d) nValid = 0

Algorithm 1 Structure from Motion

```
Build feature track                                              ▷ BuildFeatureTrack
Estimate first two camera poses                                  ▷ EstimateCameraPose
Initialize pose set P
for i = 2, ···, N-1 do
    Estimate new camera pose                                     ▷ PnP, PnP_nl
    P = P ∪ P_i
    for j < i do
        Find new points to reconstruct                           ▷ FindMissingReconstruction
        Triangulate point                                        ▷ Triangulation, Triangulation_nl
        Filter out point based on cheirality                     ▷ EvaluateCheirality
        Update 3D point
    end for
    Run bundle adjustment                                        ▷ RunBundleAdjustment
end for
```

Perspective-n-Point



3D-2D correspondence: $\mathbf{u} \leftrightarrow \mathbf{X}$

$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & & & & -u_1^x X & -u_1^x Y & -u_1^x Z & -u_1^x \\ & & & & X_1 & Y_1 & Z_1 & 1 & -u_1^y X & -u_1^y Y & -u_1^y Z & -u_1^y \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ X_m & Y_m & Z_m & 1 & & & & -u_m^x X & -u_m^x Y & -u_m^x Z & -u_m^x \\ & & & & X_m & Y_m & Z_m & 1 & -u_m^y X & -u_m^y Y & -u_m^y Z & -u_m^y \end{bmatrix} \mathbf{A} \quad \mathbf{x} \quad \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$2m \times 12$

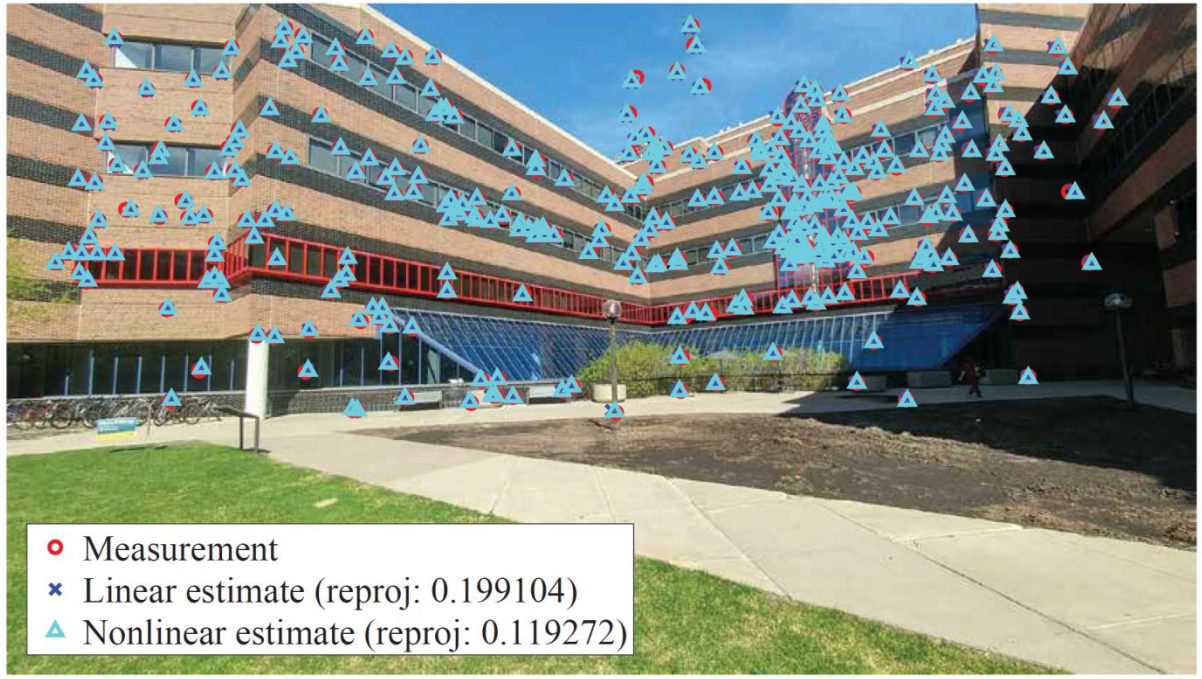
$$\mathbf{P} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \quad \text{where} \quad \mathbf{R} \in SO(3)$$

Algorithm 2 Nonlinear Camera Pose Refinement

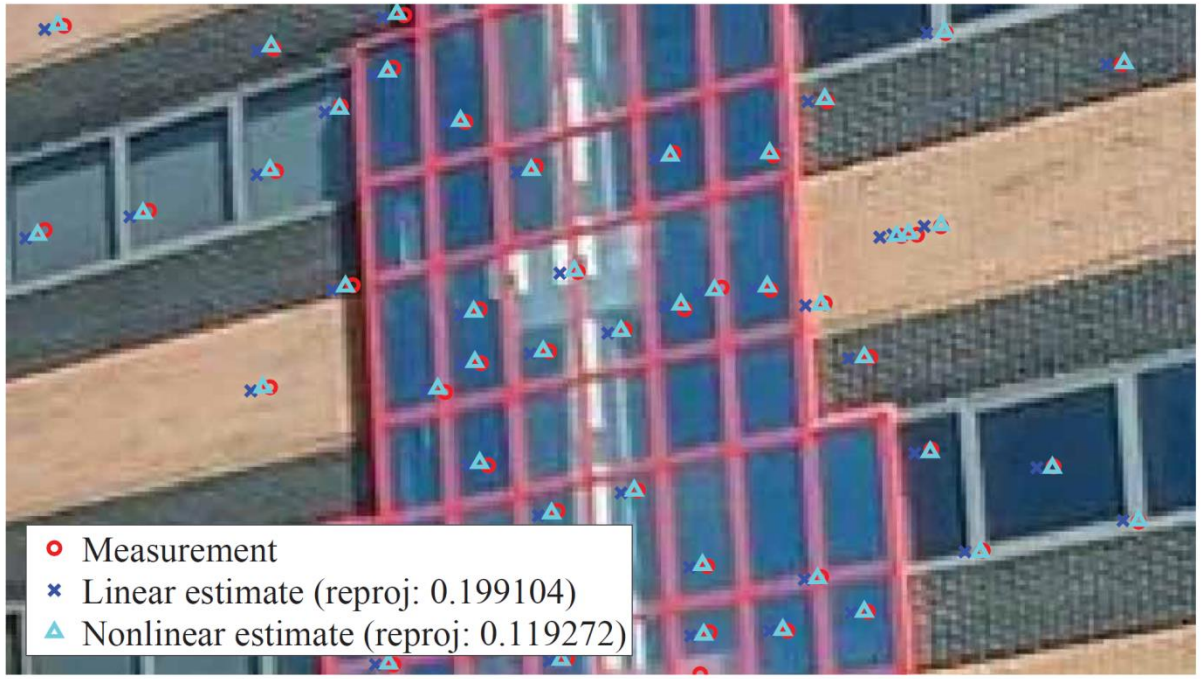
$\mathbf{p} = [\mathbf{C}^\mathsf{T}\mathbf{q}^\mathsf{T}]^\mathsf{T}$
for j = 1 : nIters **do**
    $\mathbf{C} = \mathbf{p}_{1:3}$, $\mathbf{R}$=`Quaternion2Rotation`$(\mathbf{q})$, $\mathbf{q} = \mathbf{p}_{4:7}$
    Build camera pose Jacobian for all points, $\frac{\partial f(\mathbf{p})_j}{\partial \mathbf{p}} = \left[\begin{array}{cc} \frac{\partial f(\mathbf{p})_j}{\partial \mathbf{C}} & \frac{\partial f(\mathbf{p})_j}{\partial \mathbf{q}} \end{array} \right]$.
    Compute $f(\mathbf{p})$.
    $\Delta\mathbf{p} = \left(\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}^\mathsf{T} \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}} + \lambda \mathbf{I}\right)^{-1} \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}^\mathsf{T} (\mathbf{b} - f(\mathbf{p}))$ using Equation (2).
    $\mathbf{p} = \mathbf{p} + \Delta\mathbf{p}$
    Normalize the quaternion scale, $\mathbf{p}_{4:7} = \mathbf{p}_{4:7} / \|\mathbf{p}_{4:7}\|$.
end for

$$\Delta\mathbf{p} = \left(\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}^\mathsf{T} \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}} + \lambda \mathbf{I}\right)^{-1} \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}^\mathsf{T} (\mathbf{b} - f(\mathbf{p}))$$

Camera Registration



(a) Reprojection error



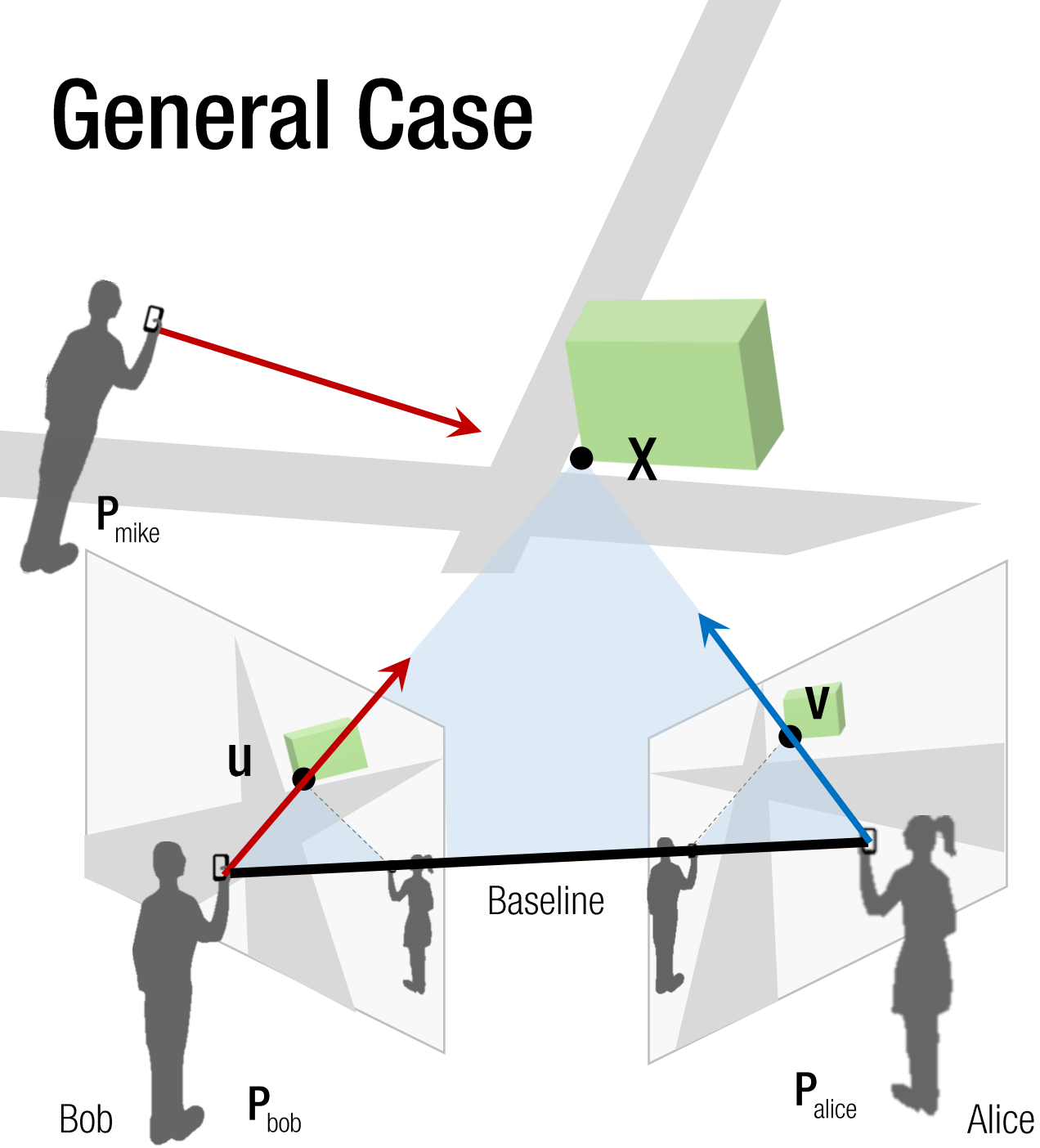
(b) Close up

Figure 4: Nonlinear refinement reduces the reprojection error (0.19→0.11).

Algorithm 1 Structure from Motion

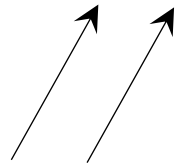
Build feature track ▷ `BuildFeatureTrack`
Estimate first two camera poses ▷ `EstimateCameraPose`
Initialize pose set `P`
for `i` = 2, $\cdots$, N-1 **do**
    Estimate new camera pose ▷ `PnP, PnP_nl`
    `P = P` $\cup$ `P_i`
    for `j < i` **do**
        Find new points to reconstruct ▷ `FindMissingReconstruction`
        Triangulate point ▷ `Triangulation, Triangulation_nl`
        Filter out point based on cheirality ▷ `EvaluateCheirality`
        Update 3D point
    end for
    Run bundle adjustment ▷ `RunBundleAdjustment`
end for

General Case



General camera pose

$$\lambda_1 \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} = \mathbf{P}_{bob} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$



Two 3D vectors are parallel.

$$\longrightarrow \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} \times \mathbf{P}_{bob} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{0}$$

$$\longrightarrow \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix}_{\times} \mathbf{P}_{bob} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}_{\times} \mathbf{P}_{alice}$$

$$\begin{bmatrix} \mathbf{w} \\ 1 \end{bmatrix}_{\times} \mathbf{P}_{mike}$$

: Knowns

: Unknowns

Algorithm 3 Nonlinear Point Refinement

1: $\mathbf{b} = \begin{bmatrix} \mathbf{u}_1^\mathsf{T} & \mathbf{u}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$
2: **for** j = 1 : nIters **do**
3: Build point Jacobian, $\frac{\partial f(\mathbf{X})_j}{\partial \mathbf{X}}$.
4: Compute $f(\mathbf{X})$.
5: $\Delta\mathbf{X} = \left(\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}}^\mathsf{T} \frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} + \lambda\mathbf{I} \right)^{-1} \frac{\partial f(\mathbf{X})}{\partial \mathbf{X}}^\mathsf{T} (\mathbf{b} - f(\mathbf{X}))$
6: $\mathbf{X} = \mathbf{X} + \Delta\mathbf{X}$
7: **end for**

$$\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}} \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix} = \begin{bmatrix} \frac{w\frac{\partial u}{\partial \mathbf{X}} - u\frac{\partial w}{\partial \mathbf{X}}}{w^2} \\ \frac{v\frac{\partial u}{\partial \mathbf{X}} - v\frac{\partial w}{\partial \mathbf{X}}}{w^2} \end{bmatrix}$$

$$\Delta\mathbf{x} = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^\mathsf{T} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^\mathsf{T} (\mathbf{b} - f(\mathbf{x}))$$

Damping factor (Levenberg-Marquardt algorithm)

$$\Delta\mathbf{x} = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^\mathsf{T} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} + \lambda\mathbf{I} \right)^{-1} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}^\mathsf{T} (\mathbf{b} - f(\mathbf{x}))$$

Algorithm 1 Structure from Motion

```
Build feature track                                              ▷ BuildFeatureTrack
Estimate first two camera poses                                  ▷ EstimateCameraPose
Initialize pose set P
for i = 2, ···, N-1 do
    Estimate new camera pose                                     ▷ PnP, PnP_nl
    P = P ∪ P_i
    for j < i do
        Find new points to reconstruct                           ▷ FindMissingReconstruction
        Triangulate point                                        ▷ Triangulation, Triangulation_nl
        Filter out point based on cheirality                     ▷ EvaluateCheirality
        Update 3D point
    end for
    Run bundle adjustment                                        ▷ RunBundleAdjustment
end for
```

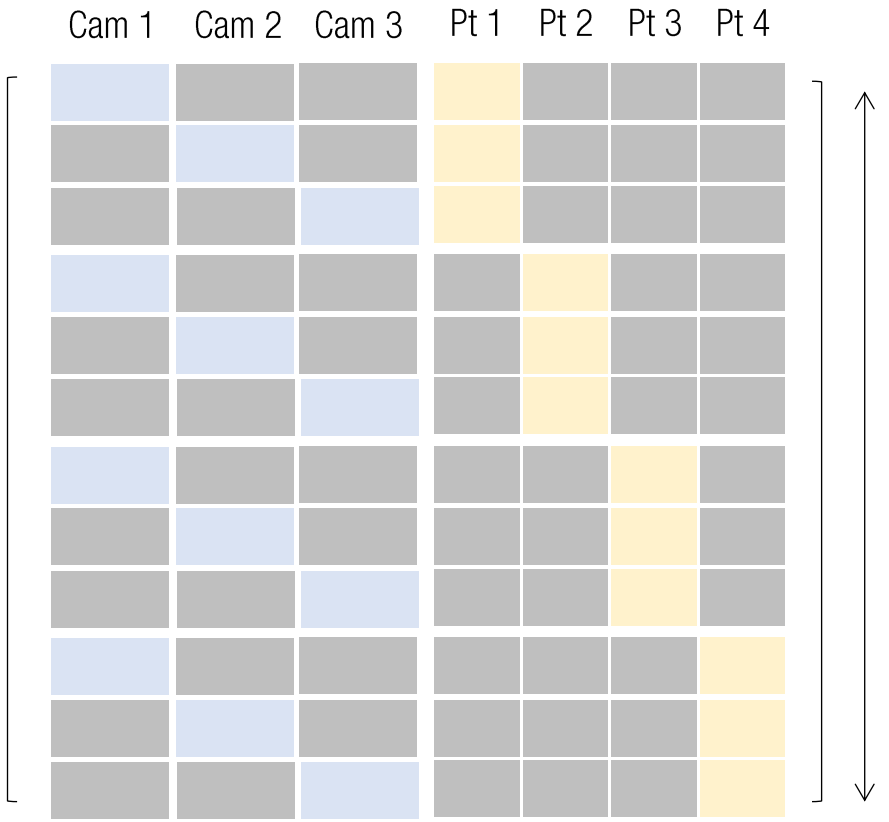
Algorithm 4 Bundle Adjustment

$\widehat{p} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} & \cdots & \mathbf{p}_I^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1^\mathsf{T} & \cdots & \mathbf{X}_M^\mathsf{T} \end{bmatrix}$
for iter $= 1 :$ nIters **do**
  Empty $\mathbf{J_p}$, $\mathbf{J_X}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{D}_{\text{inv}}$.
  for $i = 1 : M$ **do**
    $\mathbf{d} = \mathbf{0}_{3\times 3}$
    for $j = 1 : I$ **do**
      if the i^{th} point is visible from the j^{th} image **then**
        $\mathbf{J}_1 = \mathbf{0}_{2\times 7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times 3M}$
        $\mathbf{J}_1(:, 7(j-1)+1 : 7j) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j}$
        $\mathbf{J}_2(:, 3(i-1)+1 : 3i) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{J_p} = \begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} & \mathbf{J}_1^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\mathbf{J_X} = \begin{bmatrix} \mathbf{J}_\mathbf{X}^\mathsf{T} & \mathbf{J}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$
        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{b} = \begin{bmatrix} \mathbf{b}^\mathsf{T} & \mathbf{u}_{ij}^\mathsf{T} \end{bmatrix}$
        $\mathbf{f} = \begin{bmatrix} \mathbf{f}^\mathsf{T} & \mathbf{x}_{ij}^\mathsf{T} \end{bmatrix}$ where $\mu \begin{bmatrix} \mathbf{x}_{ij} \\ 1 \end{bmatrix} = \mathbf{R}_j \begin{bmatrix} \mathbf{I} & -\mathbf{C}_j \end{bmatrix}$
      end if
    end for
    $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$
    $\mathbf{D}_{\text{inv}} = \texttt{blkdiag}(\mathbf{D}_{\text{inv}}, \mathbf{d}^{-1})$
  end for
  $\mathbf{e_p} = \mathbf{J}_\mathbf{p}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{e_X} = \mathbf{J}_\mathbf{X}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{A} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_p} + \lambda \mathbf{I}$, $\mathbf{B} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_X}$, $\mathbf{D}^{-1} = \mathbf{D}_{\text{inv}}$
  $\Delta\widehat{\mathbf{p}} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^\mathsf{T})^{-1}(\mathbf{e_p} - \mathbf{B}\mathbf{D}^{-1}\mathbf{e_X})$
  Normalize quaternions.
  $\Delta\widehat{\mathbf{X}} = \mathbf{D}^{-1}(\mathbf{e_X} - \mathbf{B}^\mathsf{T}\Delta\widehat{\mathbf{p}})$
end for

$$\underset{\{\mathbf{p}_k\}\{\mathbf{X}_j\}}{\text{minimize}} \quad \sum_{k=1}^{K}\sum_{j=1}^{J} s_{k,j} \left\| f(\mathbf{p}_k, \mathbf{X}_j) - \mathbf{b}_{k,j} \right\|^2,$$

scipy.optimize.least_squares(fun, x0, jac_sparsity=S)

$\mathbf{J} =$



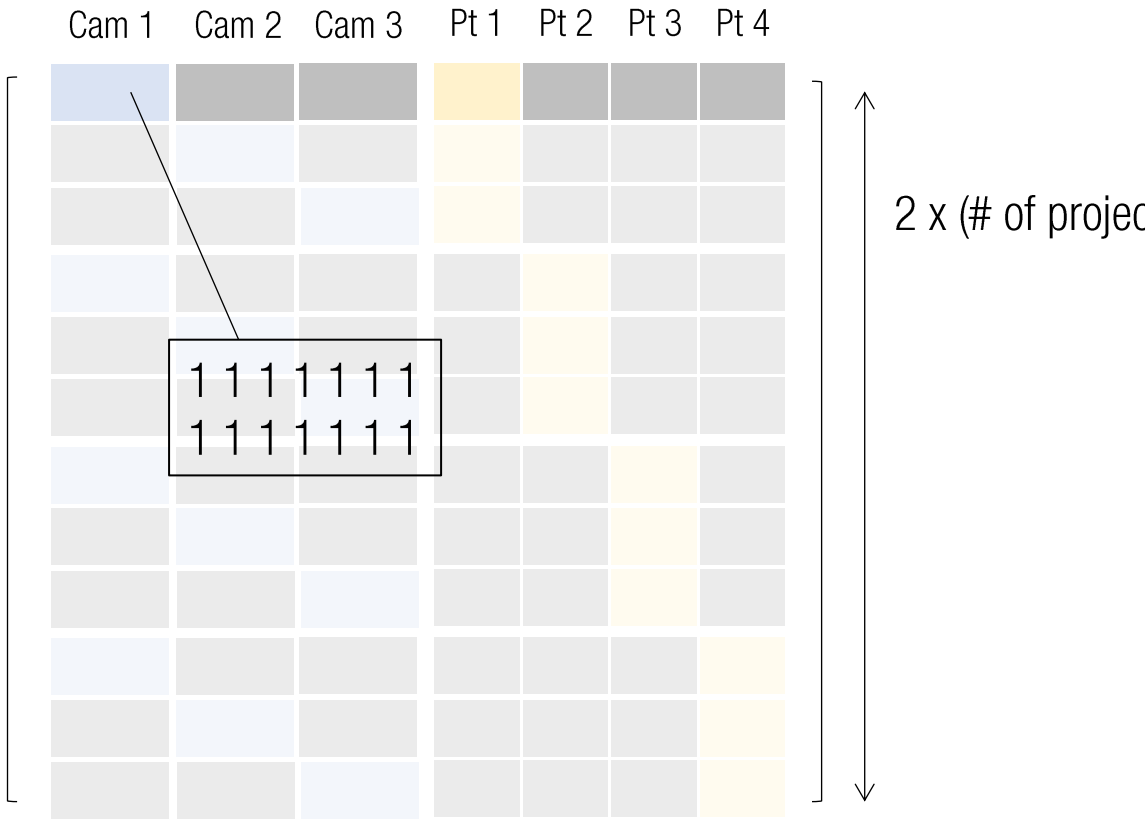
Algorithm 4 Bundle Adjustment

$\widehat{p} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} & \cdots & \mathbf{p}_I^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1^\mathsf{T} & \cdots & \mathbf{X}_M^\mathsf{T} \end{bmatrix}$
for iter $= 1 :$ nIters **do**
  Empty $\mathbf{J_p}$, $\mathbf{J_X}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{D}_{\text{inv}}$.
  for $i = 1 : M$ **do**
    $\mathbf{d} = \mathbf{0}_{3\times 3}$
    for $j = 1 : I$ **do**
      if the i^{th} point is visible from the j^{th} image **then**
        $\mathbf{J}_1 = \mathbf{0}_{2\times 7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times 3M}$
        $\mathbf{J}_1(:, 7(j-1)+1 : 7j) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j}$
        $\mathbf{J}_2(:, 3(i-1)+1 : 3i) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{J_p} = \begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} & \mathbf{J}_1^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\mathbf{J_X} = \begin{bmatrix} \mathbf{J}_\mathbf{X}^\mathsf{T} & \mathbf{J}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$
        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{b} = \begin{bmatrix} \mathbf{b}^\mathsf{T} & \mathbf{u}_{ij}^\mathsf{T} \end{bmatrix}$
        $\mathbf{f} = \begin{bmatrix} \mathbf{f}^\mathsf{T} & \mathbf{x}_{ij}^\mathsf{T} \end{bmatrix}$ where $\mu \begin{bmatrix} \mathbf{x}_{ij} \\ 1 \end{bmatrix} = \mathbf{R}_j \begin{bmatrix} \mathbf{I} & -\mathbf{C}_j \end{bmatrix}$
      end if
    end for
    $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$
    $\mathbf{D}_{\text{inv}} = \texttt{blkdiag}(\mathbf{D}_{\text{inv}}, \mathbf{d}^{-1})$
  end for
  $\mathbf{e_p} = \mathbf{J}_\mathbf{p}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{e_X} = \mathbf{J}_\mathbf{X}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{A} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_p} + \lambda\mathbf{I}$, $\mathbf{B} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_X}$, $\mathbf{D}^{-1} = \mathbf{D}_{\text{inv}}$
  $\Delta\widehat{\mathbf{p}} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^\mathsf{T})^{-1}(\mathbf{e_p} - \mathbf{B}\mathbf{D}^{-1}\mathbf{e_X})$
  Normalize quaternions.
  $\Delta\widehat{\mathbf{X}} = \mathbf{D}^{-1}(\mathbf{e_X} - \mathbf{B}^\mathsf{T}\Delta\widehat{\mathbf{p}})$
end for

$$\underset{\{\mathbf{p}_k\}\{\mathbf{X}_j\}}{\text{minimize}} \quad \sum_{k=1}^{K}\sum_{j=1}^{J} s_{k,j}\|f(\mathbf{p}_k, \mathbf{X}_j) - \mathbf{b}_{k,j}\|^2,$$

scipy.optimize.least_squares(fun, x0, jac_sparsity=S)

S =



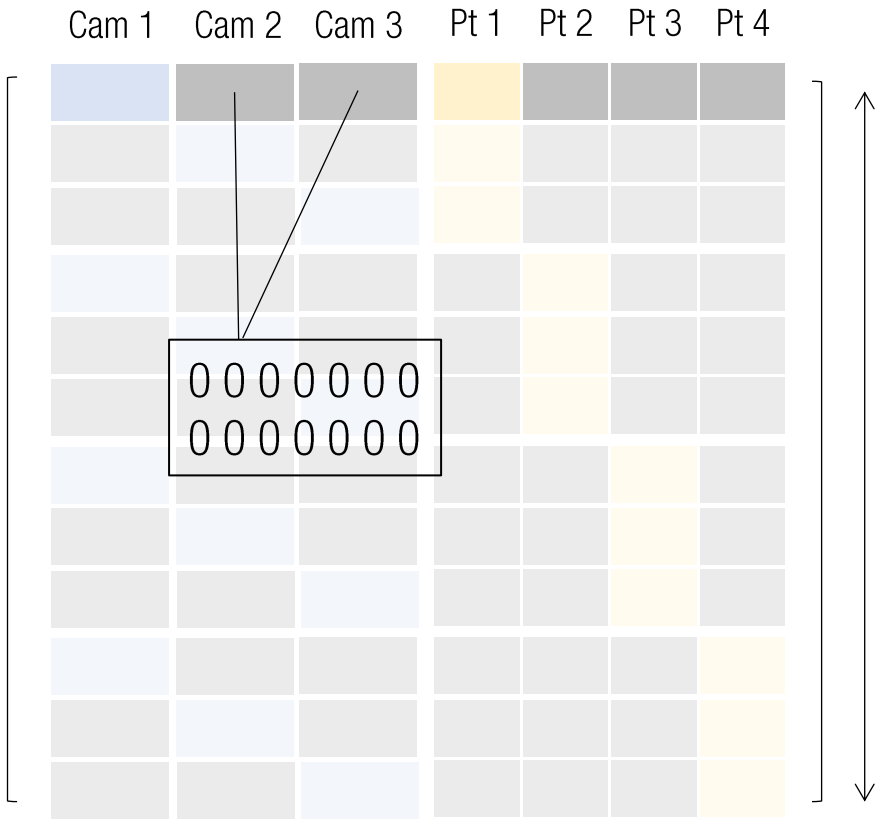
Algorithm 4 Bundle Adjustment

$\widehat{p} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} & \cdots & \mathbf{p}_I^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1^\mathsf{T} & \cdots & \mathbf{X}_M^\mathsf{T} \end{bmatrix}$
for iter $= 1 :$ nIters **do**
  Empty $\mathbf{J_p}$, $\mathbf{J_X}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{D}_{\text{inv}}$.
  for $i = 1 : M$ **do**
    $\mathbf{d} = \mathbf{0}_{3\times3}$
    for $j = 1 : I$ **do**
      if the i^{th} point is visible from the j^{th} image **then**
        $\mathbf{J}_1 = \mathbf{0}_{2\times7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times3M}$
        $\mathbf{J}_1(:, 7(j-1)+1 : 7j) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j}$
        $\mathbf{J}_2(:, 3(i-1)+1 : 3i) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{J_p} = \begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} & \mathbf{J}_1^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\mathbf{J_X} = \begin{bmatrix} \mathbf{J}_\mathbf{X}^\mathsf{T} & \mathbf{J}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$
        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{b} = \begin{bmatrix} \mathbf{b}^\mathsf{T} & \mathbf{u}_{ij}^\mathsf{T} \end{bmatrix}$
        $\mathbf{f} = \begin{bmatrix} \mathbf{f}^\mathsf{T} & \mathbf{x}_{ij}^\mathsf{T} \end{bmatrix}$ where $\mu \begin{bmatrix} \mathbf{x}_{ij} \\ 1 \end{bmatrix} = \mathbf{R}_j \begin{bmatrix} \mathbf{I} & -\mathbf{C}_j \end{bmatrix}$
      end if
    end for
    $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$
    $\mathbf{D}_{\text{inv}} = \texttt{blkdiag}(\mathbf{D}_{\text{inv}}, \mathbf{d}^{-1})$
  end for
  $\mathbf{e_p} = \mathbf{J}_\mathbf{p}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{e_X} = \mathbf{J}_\mathbf{X}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{A} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_p} + \lambda\mathbf{I}$, $\mathbf{B} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_X}$, $\mathbf{D}^{-1} = \mathbf{D}_{\text{inv}}$
  $\Delta\widehat{\mathbf{p}} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^\mathsf{T})^{-1}(\mathbf{e_p} - \mathbf{B}\mathbf{D}^{-1}\mathbf{e_X})$
  Normalize quaternions.
  $\Delta\widehat{\mathbf{X}} = \mathbf{D}^{-1}(\mathbf{e_X} - \mathbf{B}^\mathsf{T}\Delta\widehat{\mathbf{p}})$
end for

$$\underset{\{\mathbf{p}_k\}\{\mathbf{X}_j\}}{\text{minimize}} \quad \sum_{k=1}^{K}\sum_{j=1}^{J} s_{k,j} \left\| f(\mathbf{p}_k, \mathbf{X}_j) - \mathbf{b}_{k,j} \right\|^2,$$

scipy.optimize.least_squares(fun, x0, jac_sparsity=S)

S =



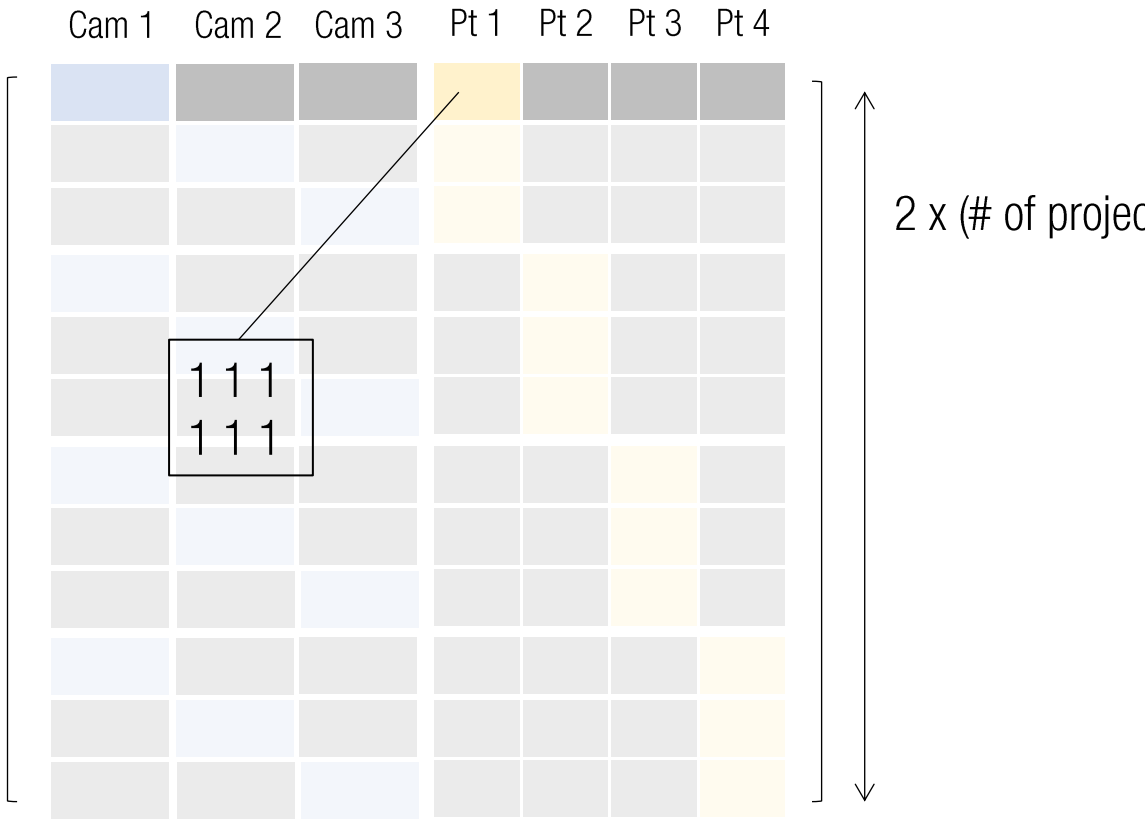
Algorithm 4 Bundle Adjustment

$\widehat{p} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} & \cdots & \mathbf{p}_I^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1^\mathsf{T} & \cdots & \mathbf{X}_M^\mathsf{T} \end{bmatrix}$
for iter $= 1 :$ nIters **do**
  Empty $\mathbf{J_p}$, $\mathbf{J_X}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{D}_{\text{inv}}$.
  for $i = 1 : M$ **do**
    $\mathbf{d} = \mathbf{0}_{3\times 3}$
    for $j = 1 : I$ **do**
      if the i^{th} point is visible from the j^{th} image **then**
        $\mathbf{J}_1 = \mathbf{0}_{2\times 7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times 3M}$
        $\mathbf{J}_1(:, 7(j-1)+1 : 7j) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j}$
        $\mathbf{J}_2(:, 3(i-1)+1 : 3i) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{J_p} = \begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} & \mathbf{J}_1^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\mathbf{J_X} = \begin{bmatrix} \mathbf{J}_\mathbf{X}^\mathsf{T} & \mathbf{J}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$
        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{b} = \begin{bmatrix} \mathbf{b}^\mathsf{T} & \mathbf{u}_{ij}^\mathsf{T} \end{bmatrix}$
        $\mathbf{f} = \begin{bmatrix} \mathbf{f}^\mathsf{T} & \mathbf{x}_{ij}^\mathsf{T} \end{bmatrix}$ where $\mu \begin{bmatrix} \mathbf{x}_{ij} \\ 1 \end{bmatrix} = \mathbf{R}_j \begin{bmatrix} \mathbf{I} & -\mathbf{C}_j \end{bmatrix}$
      end if
    end for
    $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$
    $\mathbf{D}_{\text{inv}} = \texttt{blkdiag}(\mathbf{D}_{\text{inv}}, \mathbf{d}^{-1})$
  end for
  $\mathbf{e_p} = \mathbf{J}_\mathbf{p}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{e_X} = \mathbf{J}_\mathbf{X}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{A} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_p} + \lambda\mathbf{I}$, $\mathbf{B} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_X}$, $\mathbf{D}^{-1} = \mathbf{D}_{\text{inv}}$
  $\Delta\widehat{\mathbf{p}} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^\mathsf{T})^{-1}(\mathbf{e_p} - \mathbf{B}\mathbf{D}^{-1}\mathbf{e_X})$
  Normalize quaternions.
  $\Delta\widehat{\mathbf{X}} = \mathbf{D}^{-1}(\mathbf{e_X} - \mathbf{B}^\mathsf{T}\Delta\widehat{\mathbf{p}})$
end for

$$\underset{\{\mathbf{p}_k\}\{\mathbf{X}_j\}}{\text{minimize}} \quad \sum_{k=1}^{K}\sum_{j=1}^{J} s_{k,j}\left\|f(\mathbf{p}_k, \mathbf{X}_j) - \mathbf{b}_{k,j}\right\|^2,$$

scipy.optimize.least_squares(fun, x0, jac_sparsity=S)

S =



Algorithm 4 Bundle Adjustment

$\widehat{p} = \begin{bmatrix} \mathbf{p}_1^\mathsf{T} & \cdots & \mathbf{p}_I^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\widehat{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1^\mathsf{T} & \cdots & \mathbf{X}_M^\mathsf{T} \end{bmatrix}$
for iter $= 1 :$ nIters **do**
  Empty $\mathbf{J_p}$, $\mathbf{J_X}$, $\mathbf{b}$, $\mathbf{f}$, $\mathbf{D}_{\text{inv}}$.
  for $i = 1 : M$ **do**
    $\mathbf{d} = \mathbf{0}_{3\times 3}$
    for $j = 1 : I$ **do**
      if the i^{th} point is visible from the j^{th} image **then**
        $\mathbf{J}_1 = \mathbf{0}_{2\times 7I}$ and $\mathbf{J}_2 = \mathbf{0}_{2\times 3M}$
        $\mathbf{J}_1(:, 7(j-1)+1 : 7j) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{p}_j}$
        $\mathbf{J}_2(:, 3(i-1)+1 : 3i) = \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{J_p} = \begin{bmatrix} \mathbf{J}_\mathbf{p}^\mathsf{T} & \mathbf{J}_1^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\mathbf{J_X} = \begin{bmatrix} \mathbf{J}_\mathbf{X}^\mathsf{T} & \mathbf{J}_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$
        $\mathbf{d} = \mathbf{d} + \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}^\mathsf{T} \frac{\partial f(\mathbf{p}_j, \mathbf{X}_i)}{\partial \mathbf{X}_i}$
        $\mathbf{b} = \begin{bmatrix} \mathbf{b}^\mathsf{T} & \mathbf{u}_{ij}^\mathsf{T} \end{bmatrix}$
        $\mathbf{f} = \begin{bmatrix} \mathbf{f}^\mathsf{T} & \mathbf{x}_{ij}^\mathsf{T} \end{bmatrix}$ where $\mu \begin{bmatrix} \mathbf{x}_{ij} \\ 1 \end{bmatrix} = \mathbf{R}_j \begin{bmatrix} \mathbf{I} & -\mathbf{C}_j \end{bmatrix}$
      end if
    end for
    $\mathbf{d} = \mathbf{d} + \lambda \mathbf{I}$
    $\mathbf{D}_{\text{inv}} = \texttt{blkdiag}(\mathbf{D}_{\text{inv}}, \mathbf{d}^{-1})$
  end for
  $\mathbf{e_p} = \mathbf{J}_\mathbf{p}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{e_X} = \mathbf{J}_\mathbf{X}^\mathsf{T}(\mathbf{b} - \mathbf{f})$
  $\mathbf{A} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_p} + \lambda\mathbf{I}$, $\mathbf{B} = \mathbf{J}_\mathbf{p}^\mathsf{T}\mathbf{J_X}$, $\mathbf{D}^{-1} = \mathbf{D}_{\text{inv}}$
  $\Delta\widehat{\mathbf{p}} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^\mathsf{T})^{-1}(\mathbf{e_p} - \mathbf{B}\mathbf{D}^{-1}\mathbf{e_X})$
  Normalize quaternions.
  $\Delta\widehat{\mathbf{X}} = \mathbf{D}^{-1}(\mathbf{e_X} - \mathbf{B}^\mathsf{T}\Delta\widehat{\mathbf{p}})$
end for

$$\underset{\{\mathbf{p}_k\}\{\mathbf{X}_j\}}{\text{minimize}} \quad \sum_{k=1}^{K}\sum_{j=1}^{J} s_{k,j}\left\| f(\mathbf{p}_k, \mathbf{X}_j) - \mathbf{b}_{k,j} \right\|^2,$$

scipy.optimize.least_squares(fun, x0, jac_sparsity=S)

S =