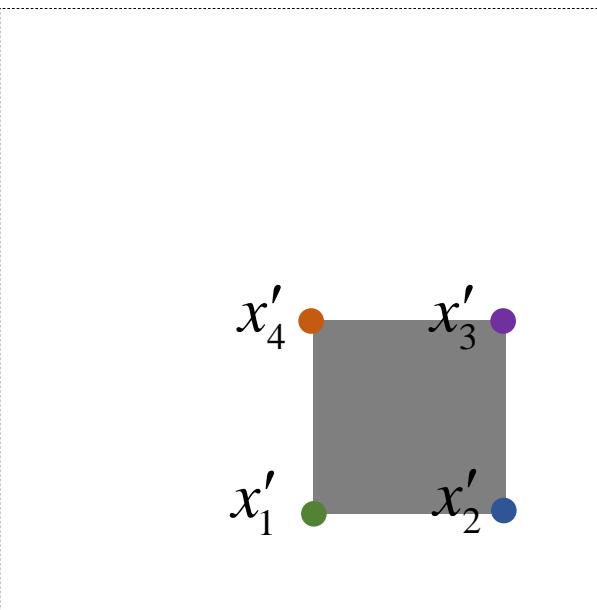


Linear Estimation

Hyun Soo Park

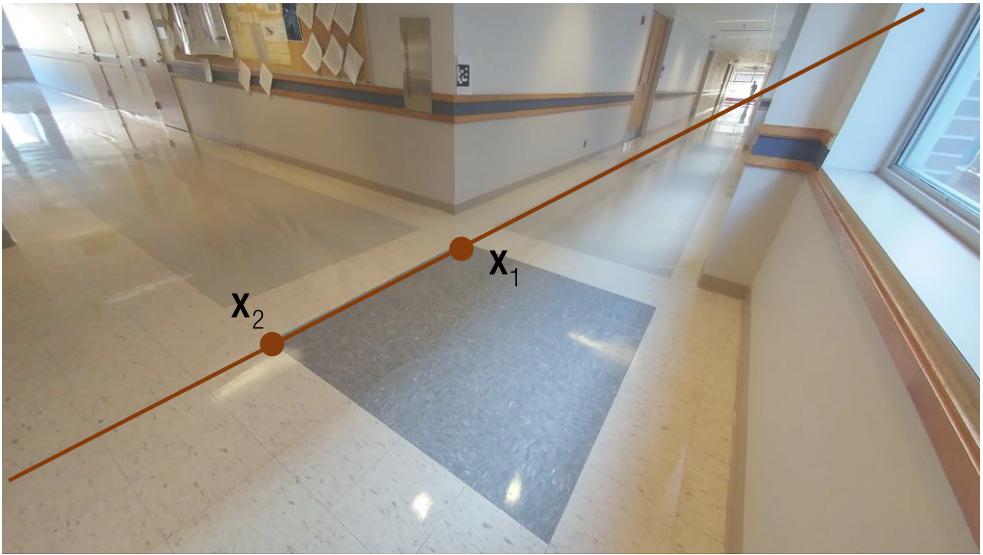
HOMOGRAPHY COMPUTATION



$$\begin{bmatrix} u_1 & v_1 & 1 & 0 & 0 & 0 & -u_1u'_1 & -v_1u'_1 \\ 0 & 0 & 0 & u_1 & v_1 & 1 & -u_1v'_1 & -v_1v'_1 \\ u_4 & v_4 & 1 & 0 & 0 & 0 & -u_4u'_4 & -v_4u'_4 \\ 0 & 0 & 0 & u_4 & v_4 & 1 & -u_4v'_4 & -v_4v'_4 \end{bmatrix} \mathbf{A} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} u'_1 \\ v'_1 \\ u'_4 \\ v'_4 \end{bmatrix} \mathbf{b}$$

$$Ax = b \quad \longrightarrow \quad x = (A^T A)^{-1} A^T b$$

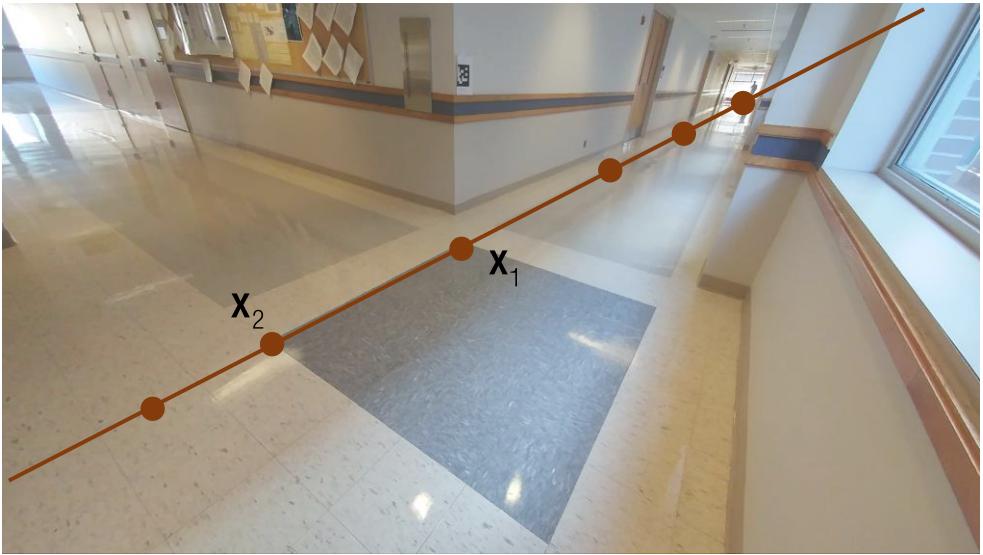
Point-Point in Image



$$\begin{bmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \end{bmatrix} \mathbf{l} = \mathbf{0}$$

$$\frac{\begin{array}{c|c} \mathbf{A} & \mathbf{l} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \hline 3 \times 2 \end{array}}{\rightarrow \mathbf{l} = \text{null}\left(\begin{array}{c|c} \mathbf{A} & \mathbf{l} \end{array}\right)} \quad \text{or} \quad \mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$

Point-Point in Image



$$\begin{array}{c|c} \text{A} & \begin{matrix} 0 \\ 0 \end{matrix} \end{array} \rightarrow \begin{matrix} \text{I} = \text{null}\left(\begin{array}{c} \text{A} \end{array}\right) \end{matrix} \quad \text{or} \quad \text{I} = \mathbf{x}_1 \times \mathbf{x}_2$$

Line Fitting

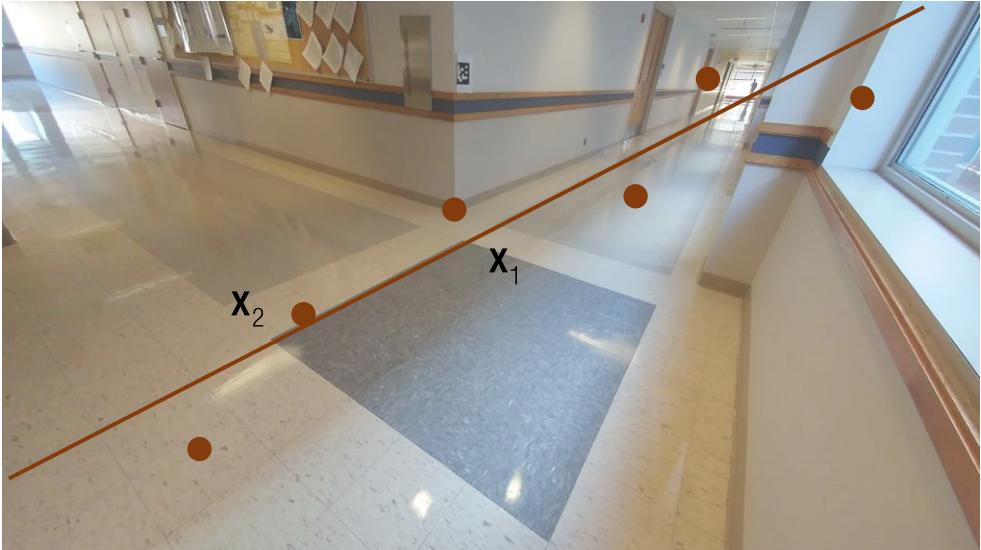


$$\begin{matrix} A \\ \hline \end{matrix} = \begin{matrix} 0 \\ 0 \end{matrix} \rightarrow ?$$

Line Fitting

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

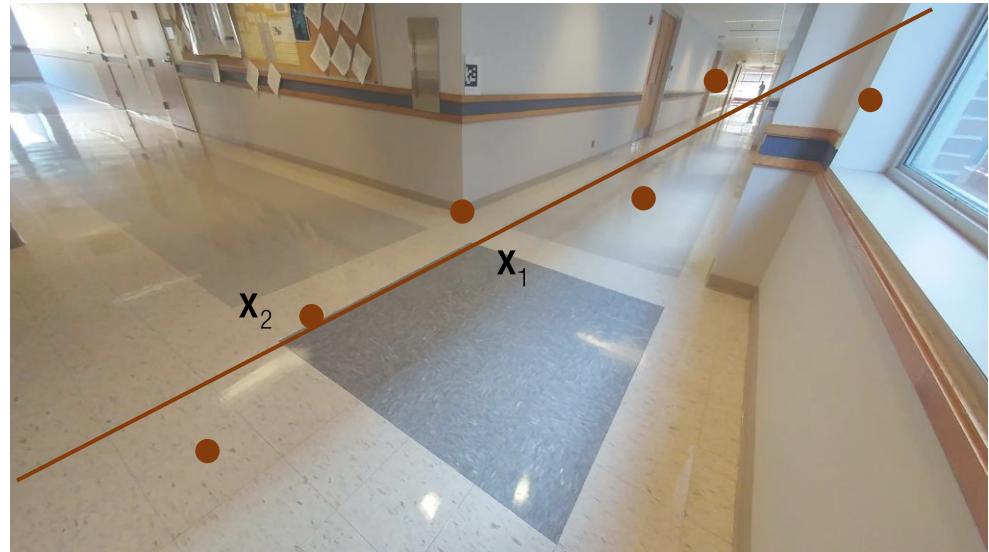
Find the best line: (a, b, c)



Line Fitting

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: (a, b, c)



$$\rightarrow au_1 + bv_1 + c \approx 0$$

$$au_2 + bv_2 + c \approx 0$$

⋮

$$au_n + bv_n + c \approx 0$$

Line Fitting

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: $(a, b, c) \longrightarrow (\underline{m}, \underline{d})$

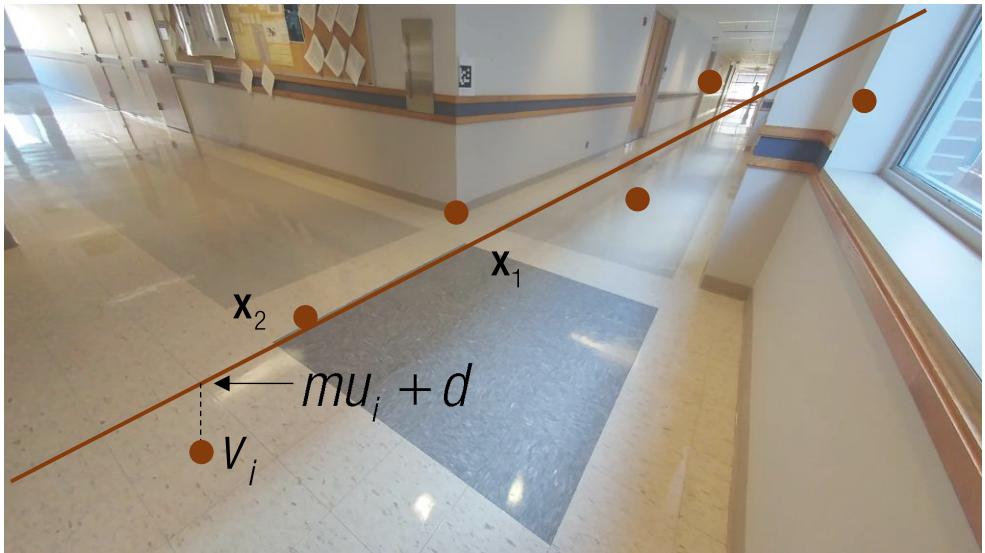
slope y-intercept

$$\longrightarrow v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

⋮

$$v_n \approx mu_n + d$$



Line Fitting

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: $(a, b, c) \longrightarrow (\underline{m}, \underline{d})$

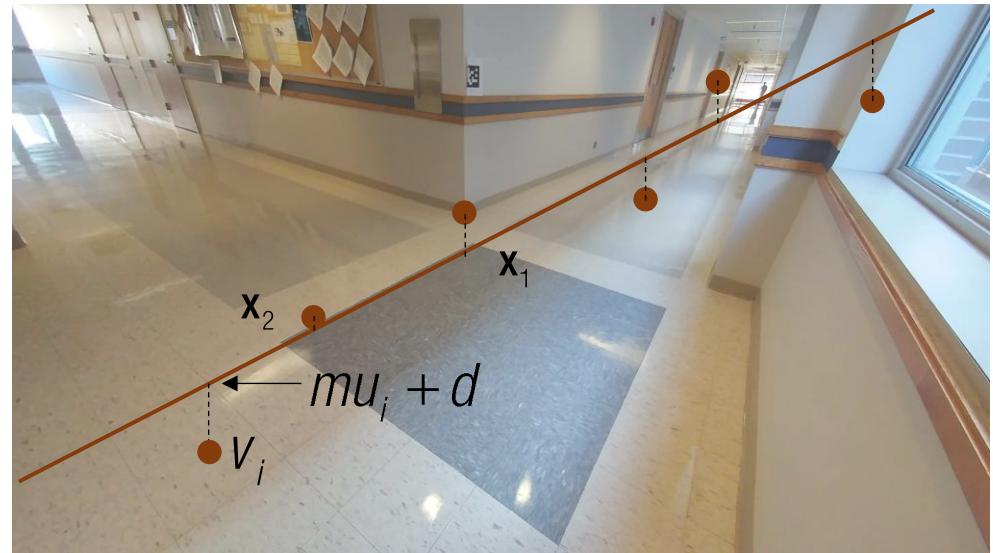
slope y-intercept

$$\longrightarrow v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

⋮

$$v_n \approx mu_n + d$$



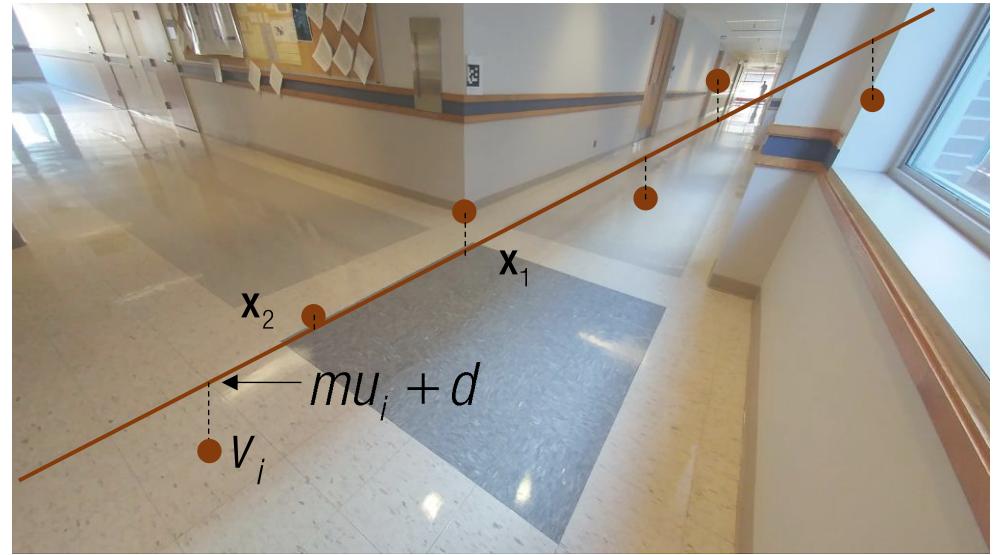
$$\text{Error: } e_i = v_i - (mu_i + d)$$

$$e_1 = v_1 - (mu_1 + d)$$

$$e_2 = v_2 - (mu_2 + d)$$

$$e_n = v_n - (mu_n + d)$$

Line Fitting



$$e_1 = v_1 - (mu_1 + d)$$

$$e_2 = v_2 - (mu_2 + d)$$

$$e_n = v_n - (mu_n + d)$$

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: $(a, b, c) \rightarrow (\underline{m}, \underline{d})$

slope y-intercept

$$\rightarrow v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

⋮

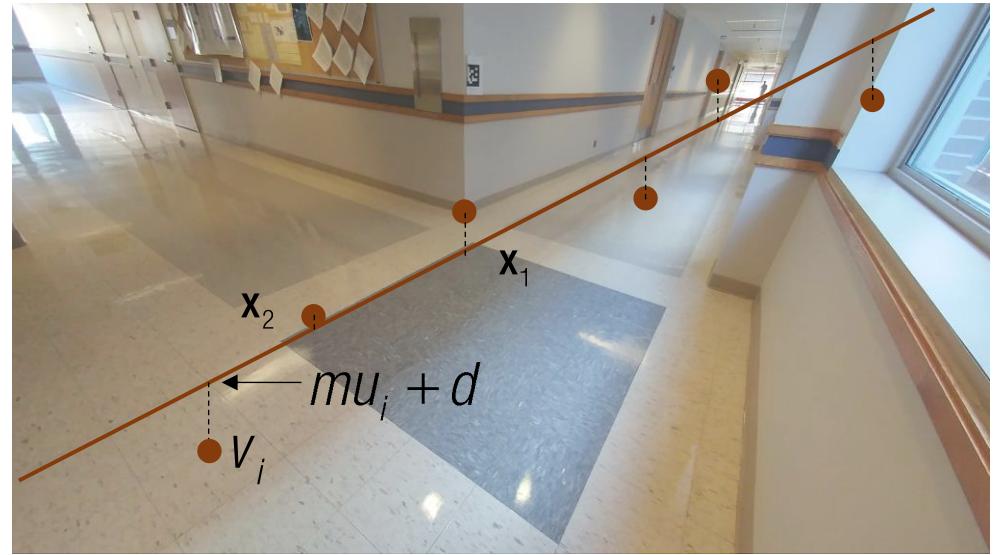
$$v_n \approx mu_n + d$$

Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

Unknowns:

Number of eq.:

Line Fitting



Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: $(a, b, c) \longrightarrow (\underline{m}, \underline{d})$

slope y-intercept

$$\longrightarrow v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

⋮

$$v_n \approx mu_n + d$$

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

$$e_1 = v_1 - (mu_1 + d)$$

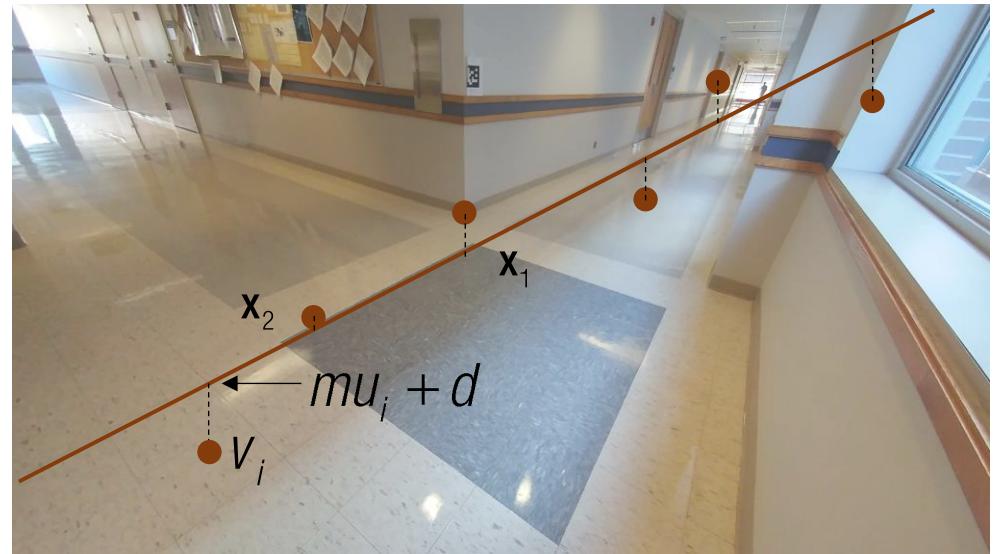
$$e_2 = v_2 - (mu_2 + d)$$

$$e_n = v_n - (mu_n + d)$$

Unknowns: m, d

Number of eq.:

Line Fitting



$$e_1 = v_1 - (mu_1 + d)$$

$$e_2 = v_2 - (mu_2 + d)$$

$$e_n = v_n - (mu_n + d)$$

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: $(a, b, c) \rightarrow (\underline{m}, \underline{d})$

slope y-intercept

$$\rightarrow v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

⋮

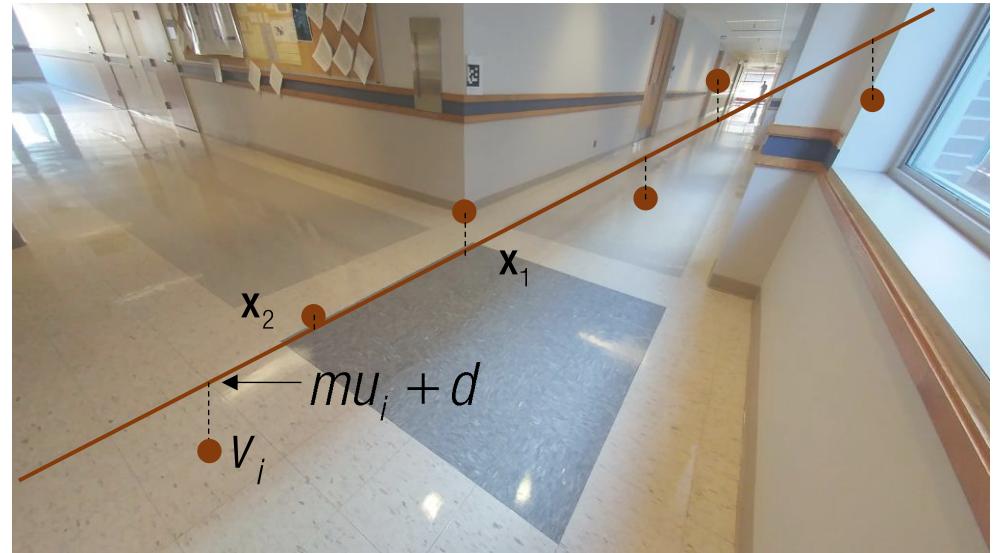
$$v_n \approx mu_n + d$$

$$\text{Total error: } E = \sum_{i=1}^n (v_i - (mu_i + d))^2$$

Unknowns: m, d

Number of eq.: n

Line Fitting



$$e_1 = v_1 - (mu_1 + d)$$

$$e_2 = v_2 - (mu_2 + d)$$

$$e_n = v_n - (mu_n + d)$$

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: $(a, b, c) \rightarrow (\underline{m}, \underline{d})$

slope y-intercept

$$\rightarrow v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

⋮

$$v_n \approx mu_n + d$$

Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

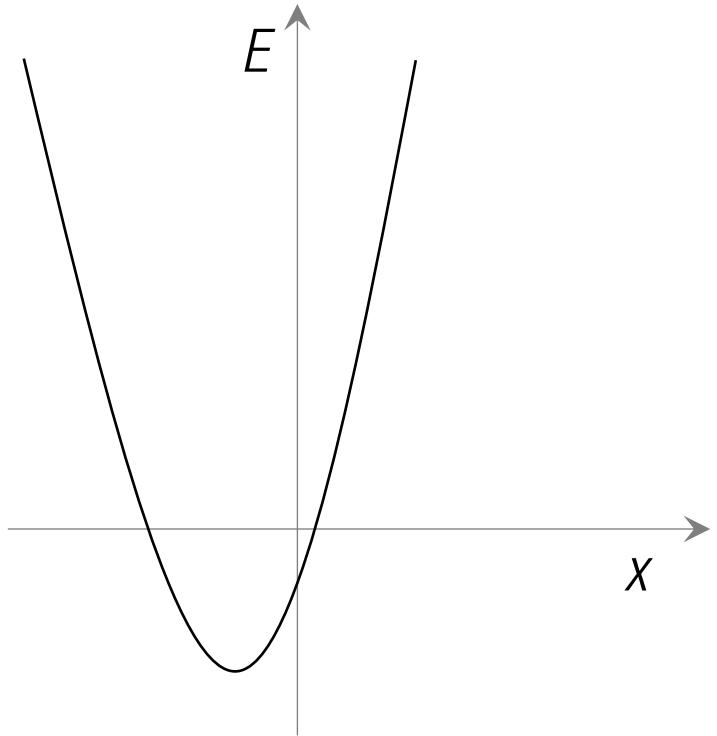
Unknowns: m, d

Number of eq.: n

$$\underset{m,d}{\text{minimize}} \sum_{i=1}^n (v_i - (mu_i + d))^2$$

How to minimize?

Line Fitting



Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: $(a, b, c) \rightarrow (\underline{m}, \underline{d})$
slope y-intercept

$$\rightarrow v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

⋮

$$v_n \approx mu_n + d$$

Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

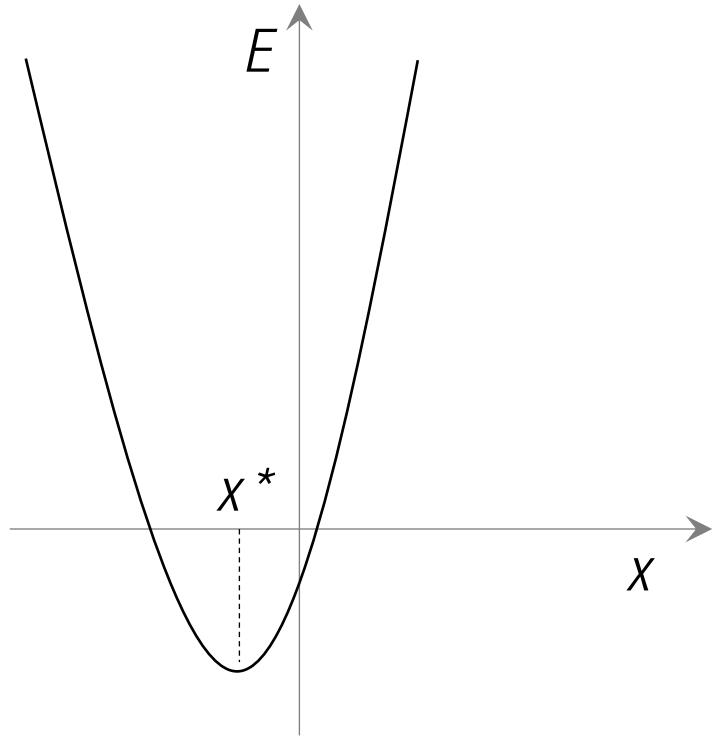
Unknowns: m, d

Number of eq.: n

$$\underset{m, d}{\text{minimize}} \sum_{i=1}^n (v_i - (mu_i + d))^2$$

How to minimize?

Line Fitting



Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: $(a, b, c) \rightarrow (\underline{m}, \underline{d})$
slope y-intercept

$$\begin{aligned} &\rightarrow v_1 \approx mu_1 + d \\ &v_2 \approx mu_2 + d \\ &\vdots \\ &v_n \approx mu_n + d \end{aligned}$$

Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

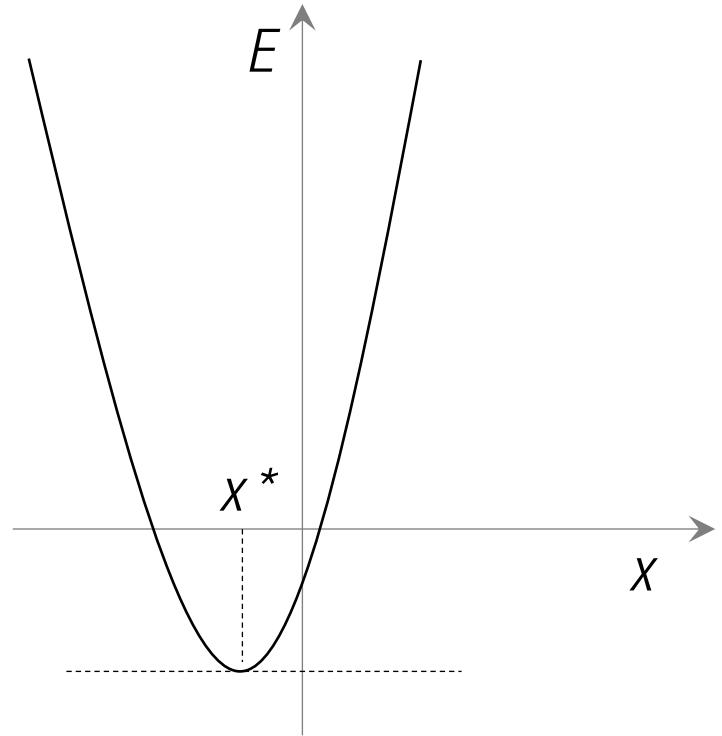
Unknowns: m, d

Number of eq.: n

$$\underset{m, d}{\text{minimize}} \sum_{i=1}^n (v_i - (mu_i + d))^2$$

How to minimize?

Line Fitting



$$\frac{dE}{dx} \Big|_{x=x^*} = 0$$

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: $(a, b, c) \rightarrow (\underline{m}, \underline{d})$

slope y-intercept

$$\rightarrow v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

⋮

$$v_n \approx mu_n + d$$

Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

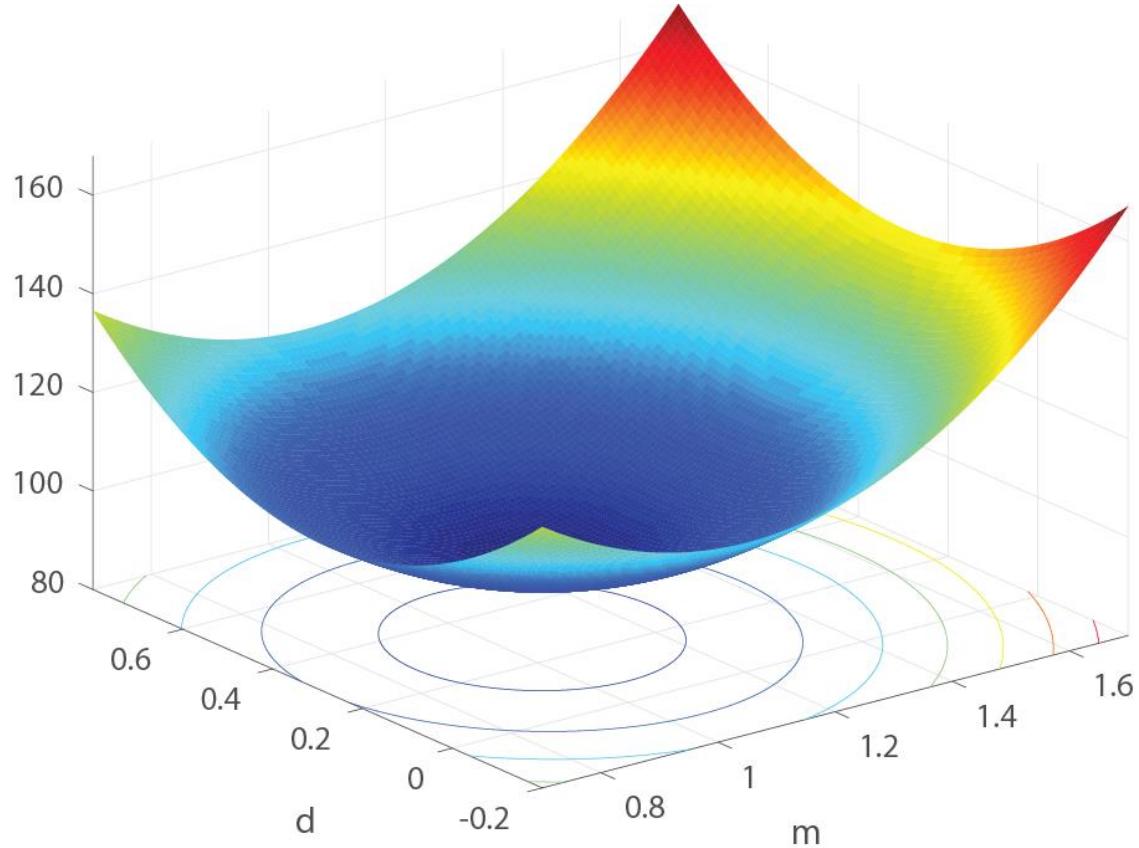
Unknowns: m, d

Number of eq.: n

$$\underset{m,d}{\text{minimize}} \sum_{i=1}^n (v_i - (mu_i + d))^2$$

How to minimize?

Line Fitting



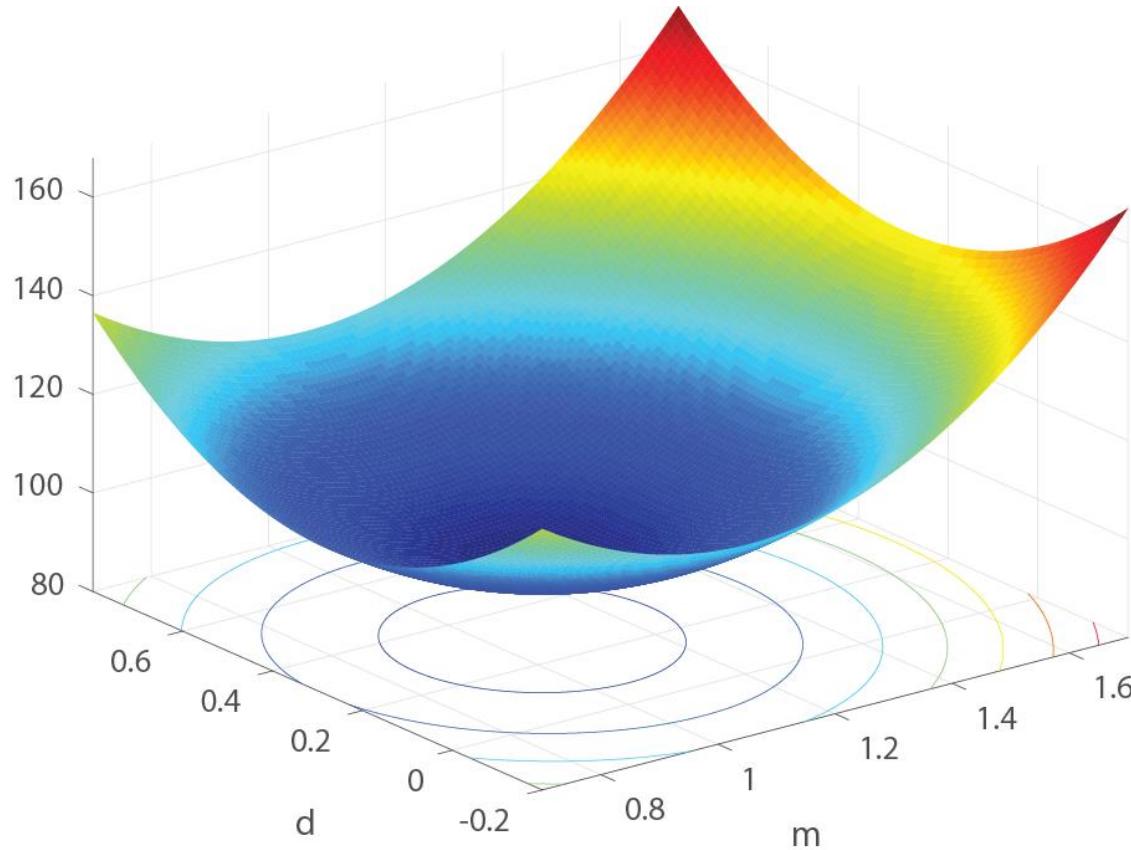
Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$\frac{\partial E}{\partial m} =$$

$$\frac{\partial E}{\partial d} =$$

Line Fitting



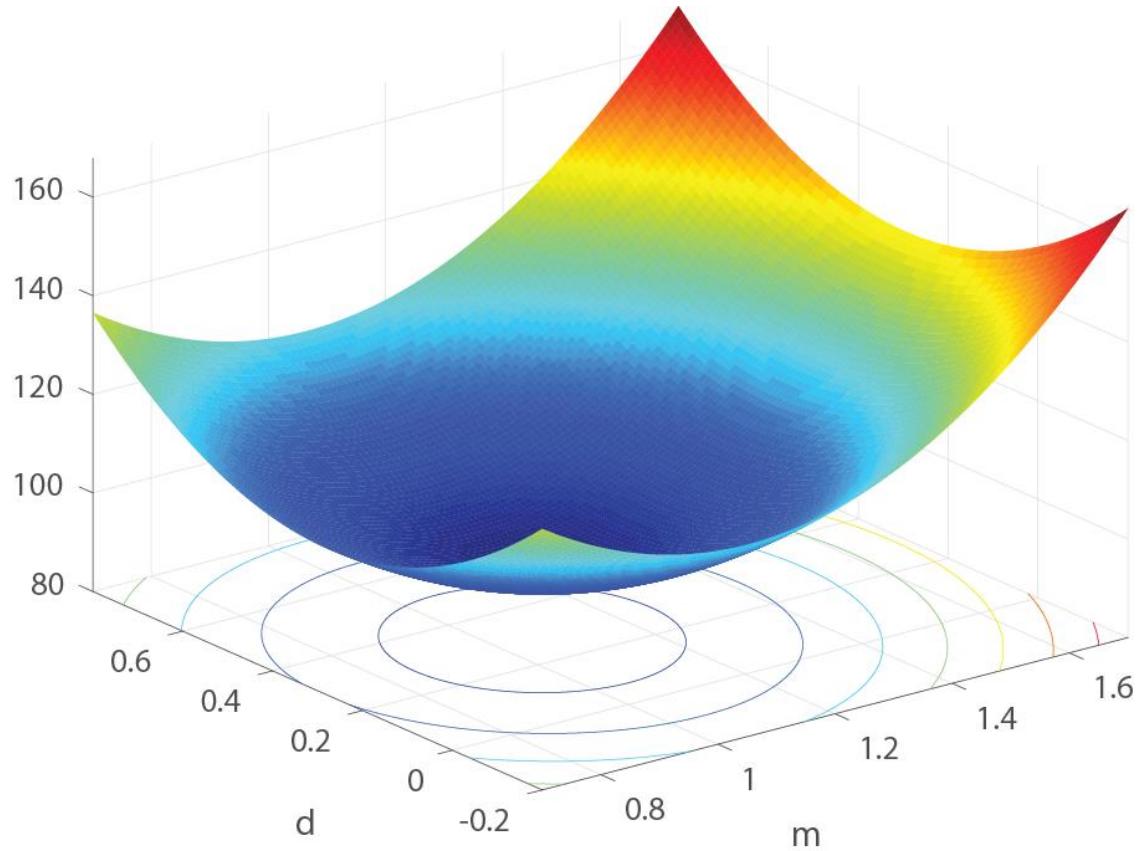
Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$\frac{\partial E}{\partial m} = -\sum_{i=1}^n 2u_i(v_i - (mu_i + d)) = 0$$

$$\frac{\partial E}{\partial d} = -\sum_{i=1}^n 2(v_i - (mu_i + d)) = 0$$

Line Fitting



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$\frac{\partial E}{\partial m} = -\sum_{i=1}^n 2u_i(v_i - (mu_i + d)) = 0$$

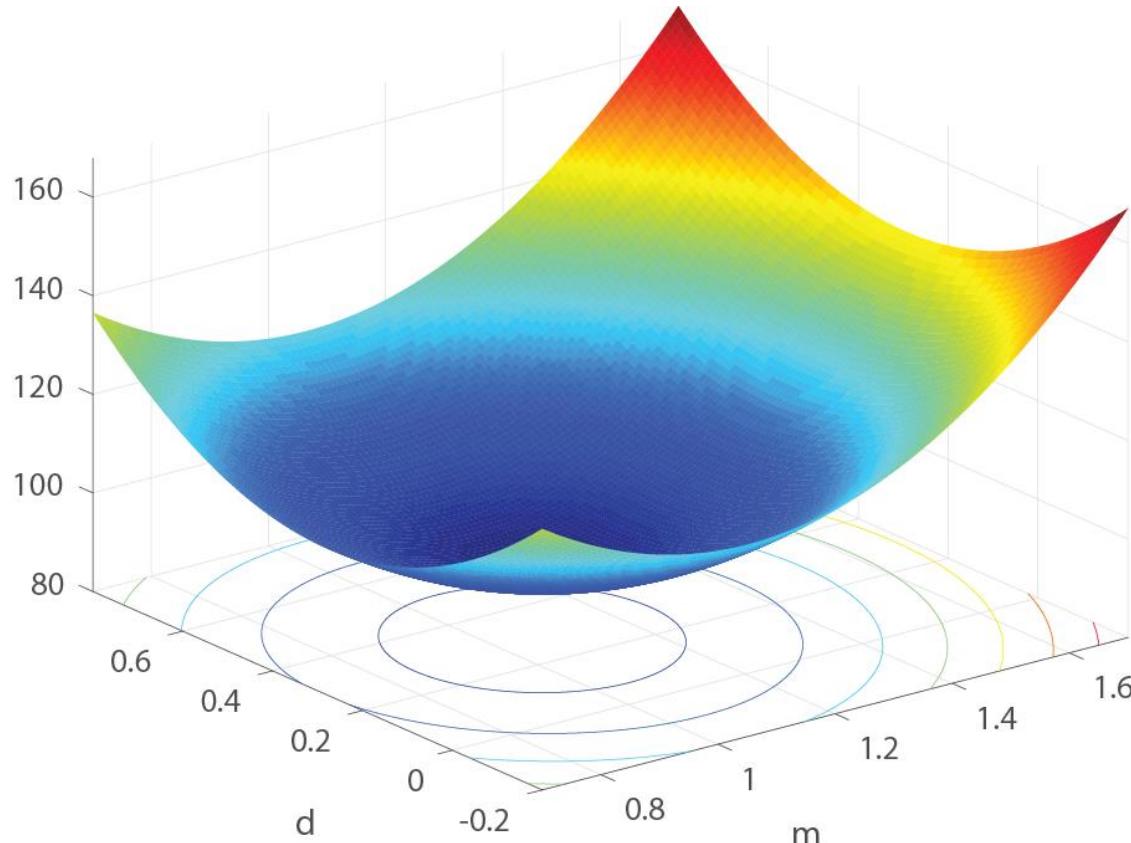
$$\frac{\partial E}{\partial d} = -\sum_{i=1}^n 2(v_i - (mu_i + d)) = 0$$



$$m \sum_{i=1}^n u_i^2 + d \sum_{i=1}^n u_i = \sum_{i=1}^n u_i v_i$$

$$m \sum_{i=1}^n u_i + n d = \sum_{i=1}^n v_i$$

Line Fitting



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$\frac{\partial E}{\partial m} = -\sum_{i=1}^n 2u_i(v_i - (mu_i + d)) = 0$$

$$\frac{\partial E}{\partial d} = -\sum_{i=1}^n 2(v_i - (mu_i + d)) = 0$$

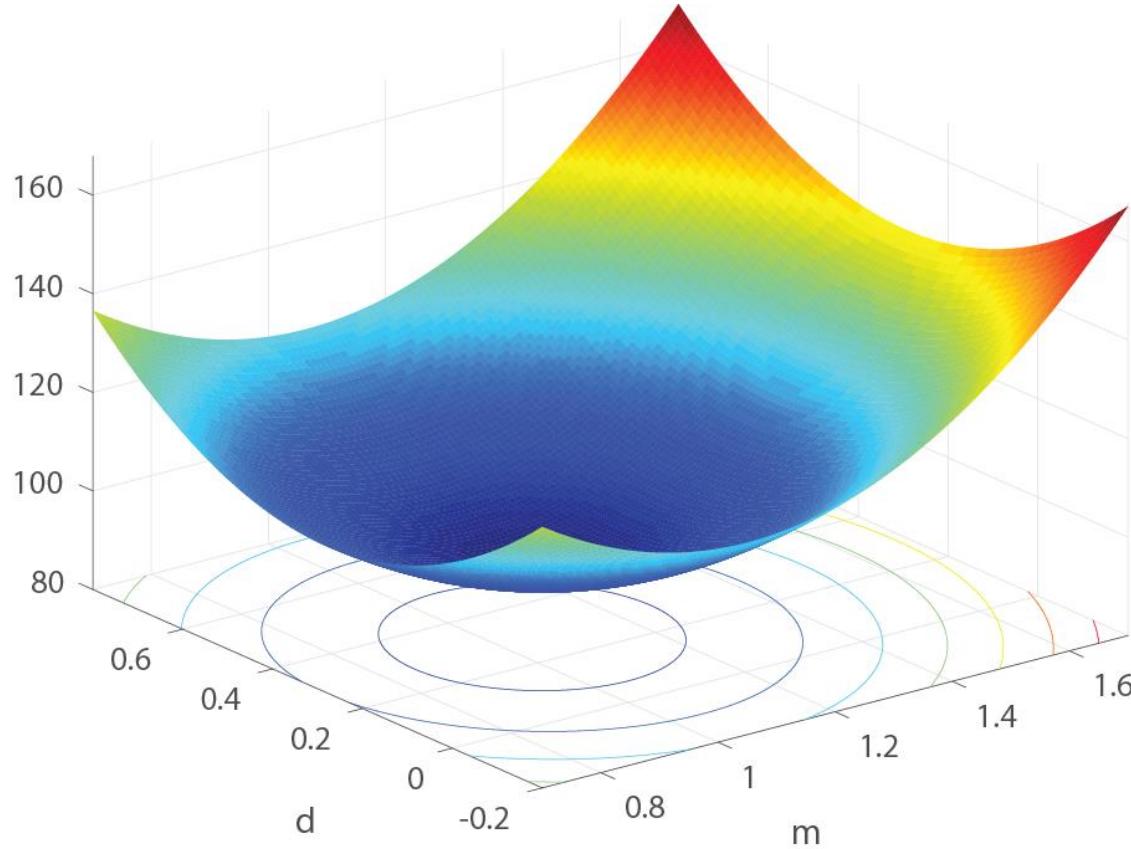
$$m \sum_{i=1}^n u_i^2 + d \sum_{i=1}^n u_i = \sum_{i=1}^n u_i v_i$$

$$m \sum_{i=1}^n u_i + n d = \sum_{i=1}^n v_i$$

$$\begin{bmatrix} \sum_{i=1}^n u_i^2 & \sum_{i=1}^n u_i \\ \sum_{i=1}^n u_i & n \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n u_i v_i \\ \sum_{i=1}^n v_i \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} m \\ d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n u_i^2 & \sum_{i=1}^n u_i \\ \sum_{i=1}^n u_i & n \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^n u_i v_i \\ \sum_{i=1}^n v_i \end{bmatrix}$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

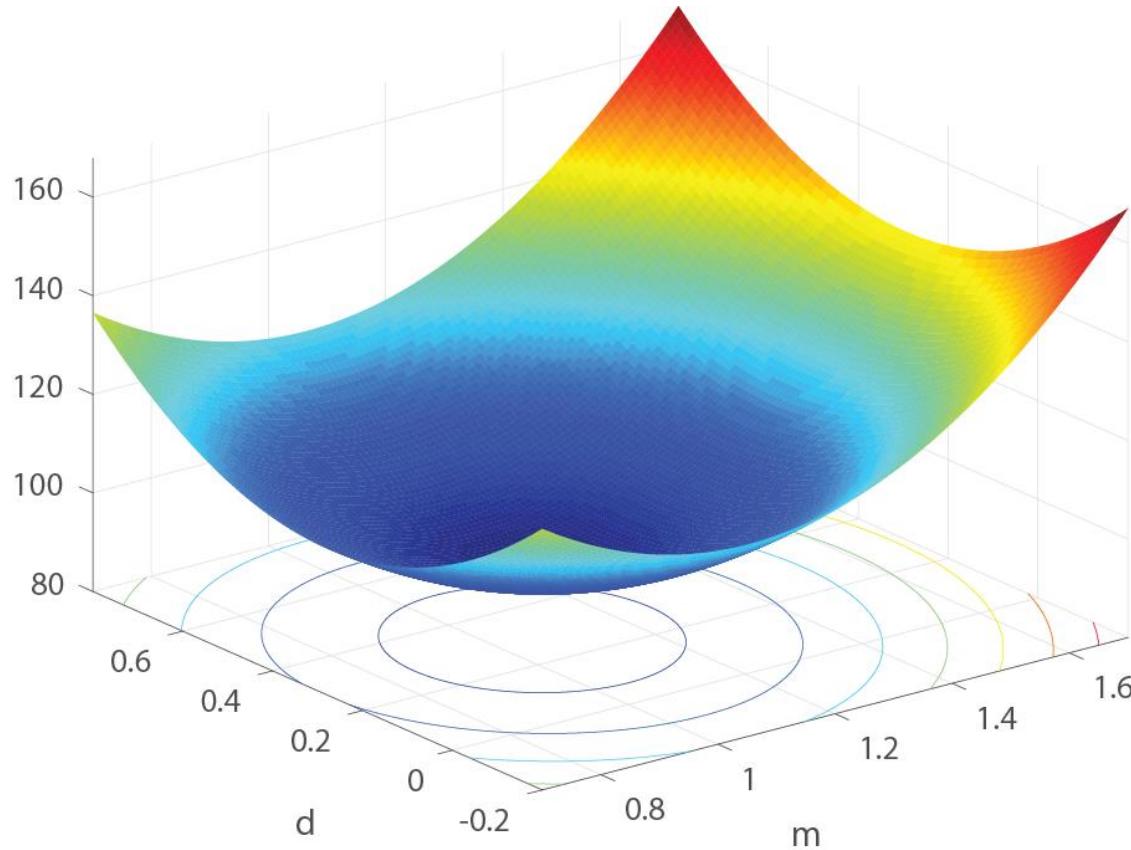
Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

$$v_n \approx mu_n + d$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

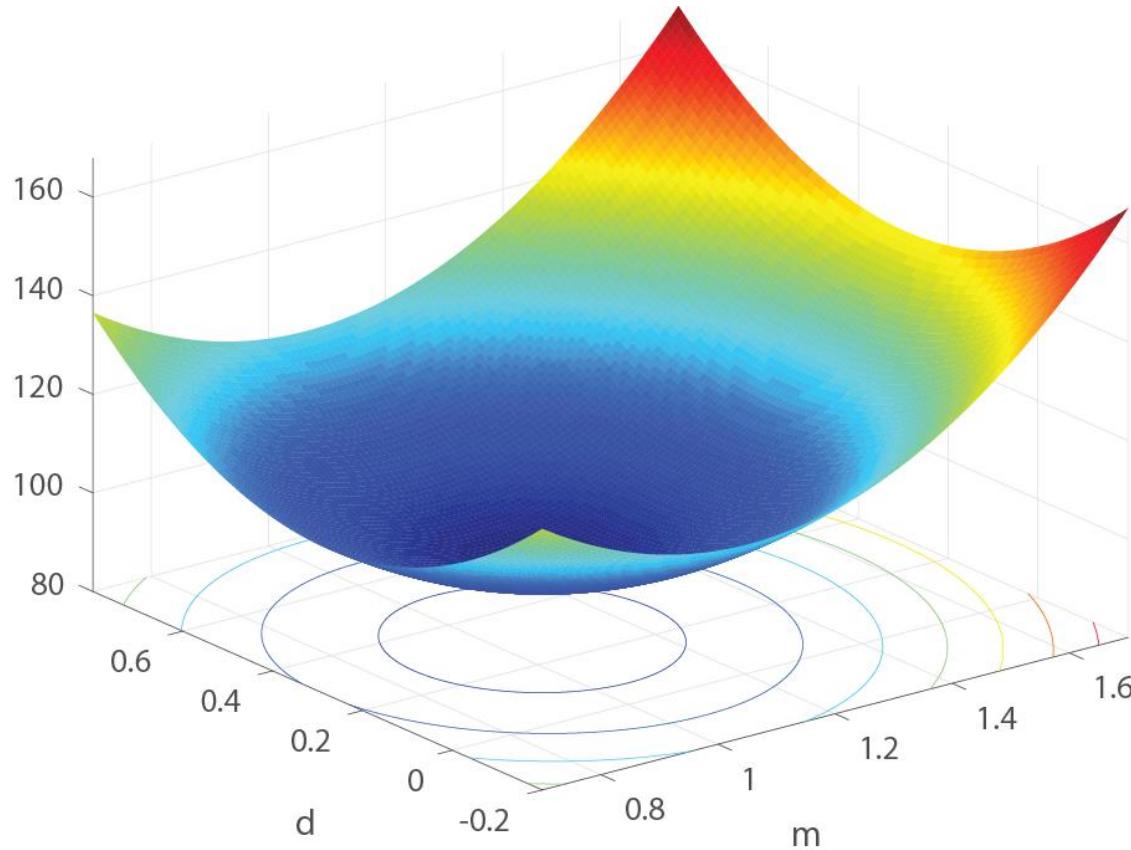
$$v_n \approx mu_n + d$$

$$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$

$$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$

$$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

$$v_n \approx mu_n + d$$

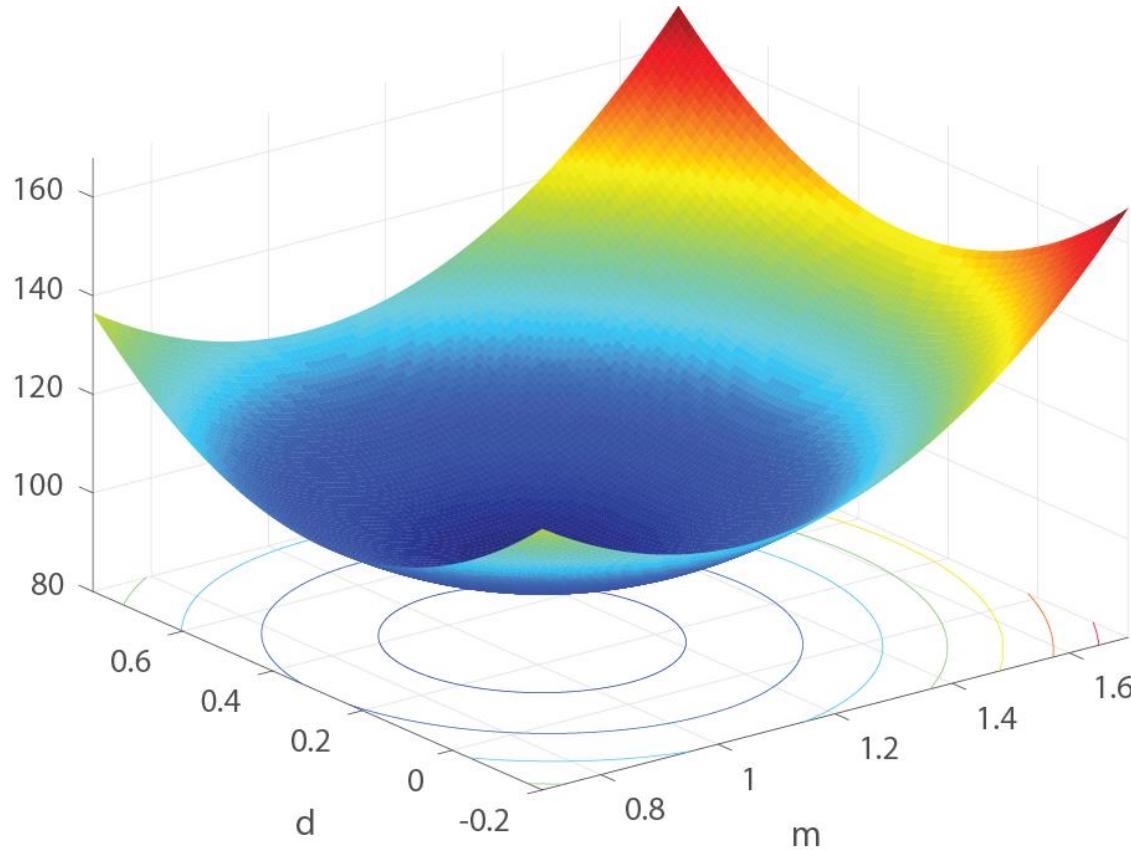
$$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$

$$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$

$$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

$$v_n \approx mu_n + d$$

$$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$

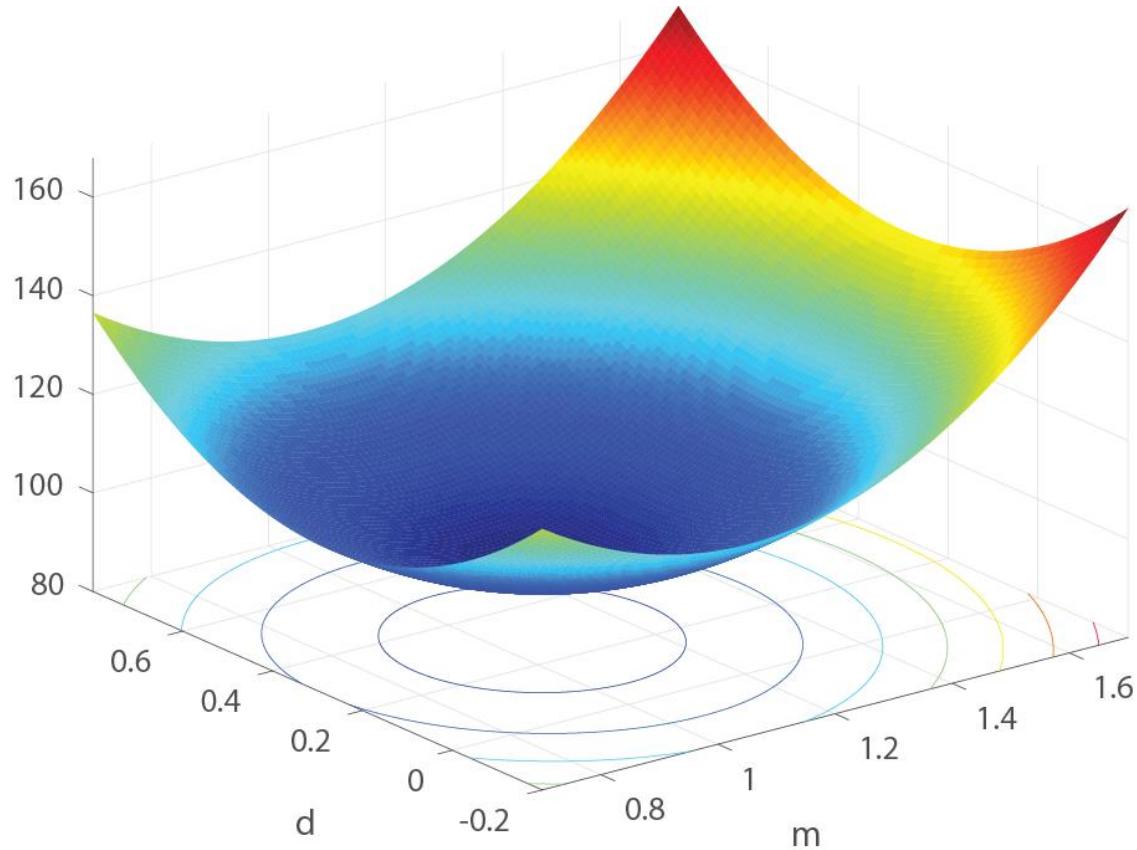
$$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$

$$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

$$\begin{bmatrix} u_1 & u_2 & \dots & u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

We can't invert **A**.

Line Fitting ($Ax=b$)

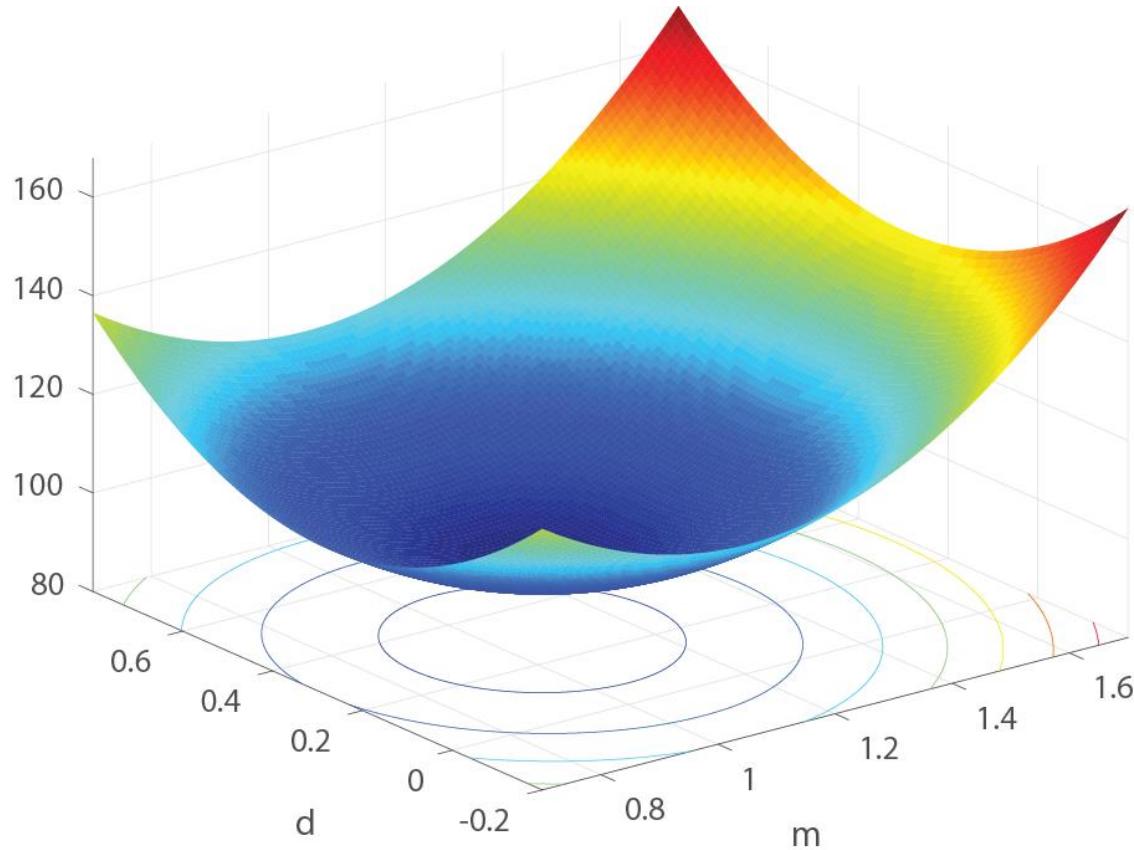


Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

$$E = (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Line Fitting ($Ax=b$)



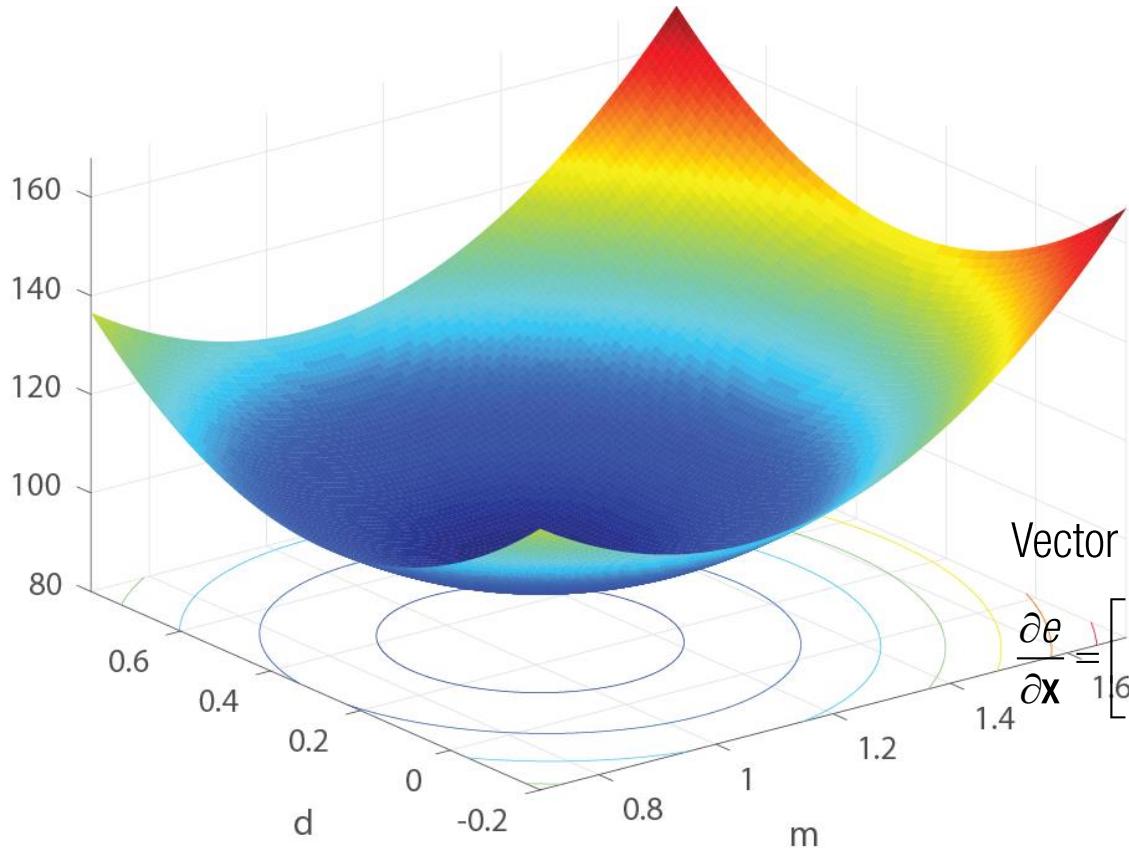
Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

$$\begin{aligned}E &= (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2 \\&= \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b}\end{aligned}$$

$$\frac{\partial E}{\partial \mathbf{x}} = ?$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \approx \begin{bmatrix} m \\ d \end{bmatrix} \mathbf{b}$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

$$\begin{aligned}E &= (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2 \\&= \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b}\end{aligned}$$

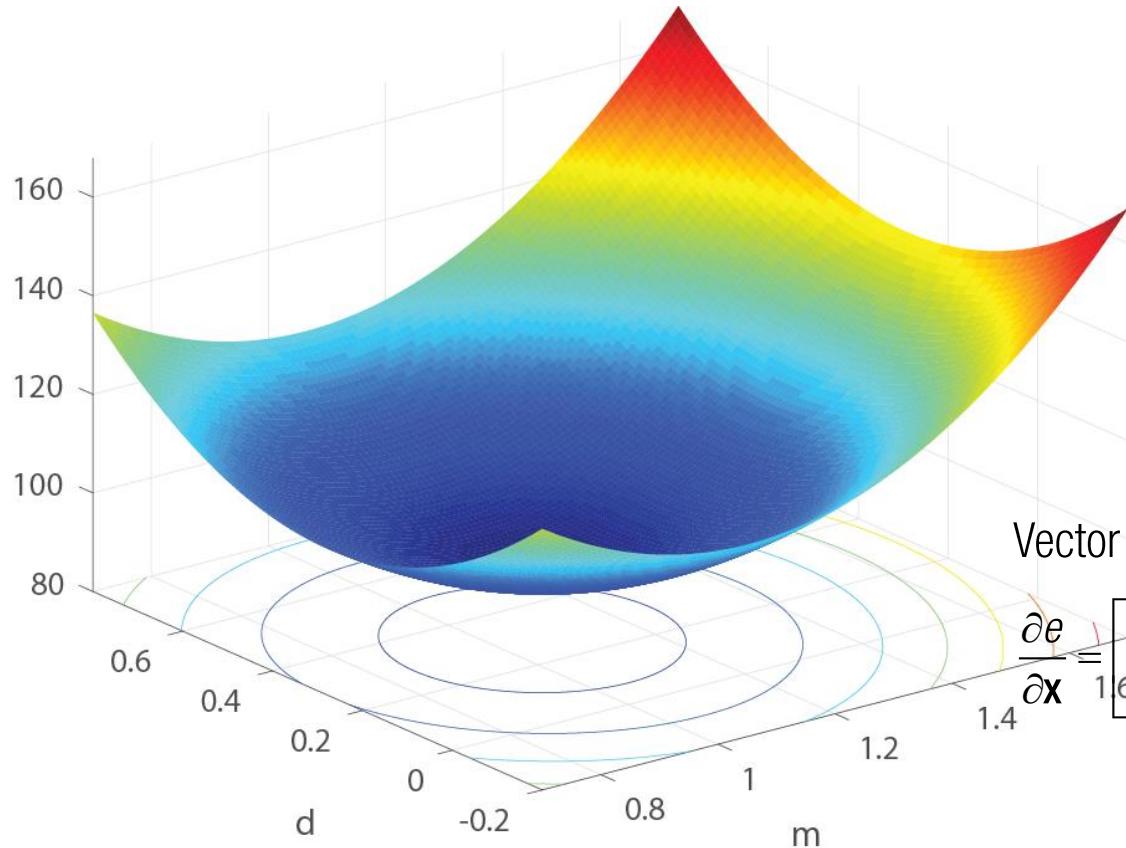
$$\frac{\partial E}{\partial \mathbf{x}} = ?$$

Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \dots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \approx \begin{bmatrix} m \\ d \end{bmatrix}$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

$$\begin{aligned} E &= (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2 \\ &= \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b} \end{aligned}$$

$$\frac{\partial E}{\partial \mathbf{x}} = ?$$

Vector derivative:

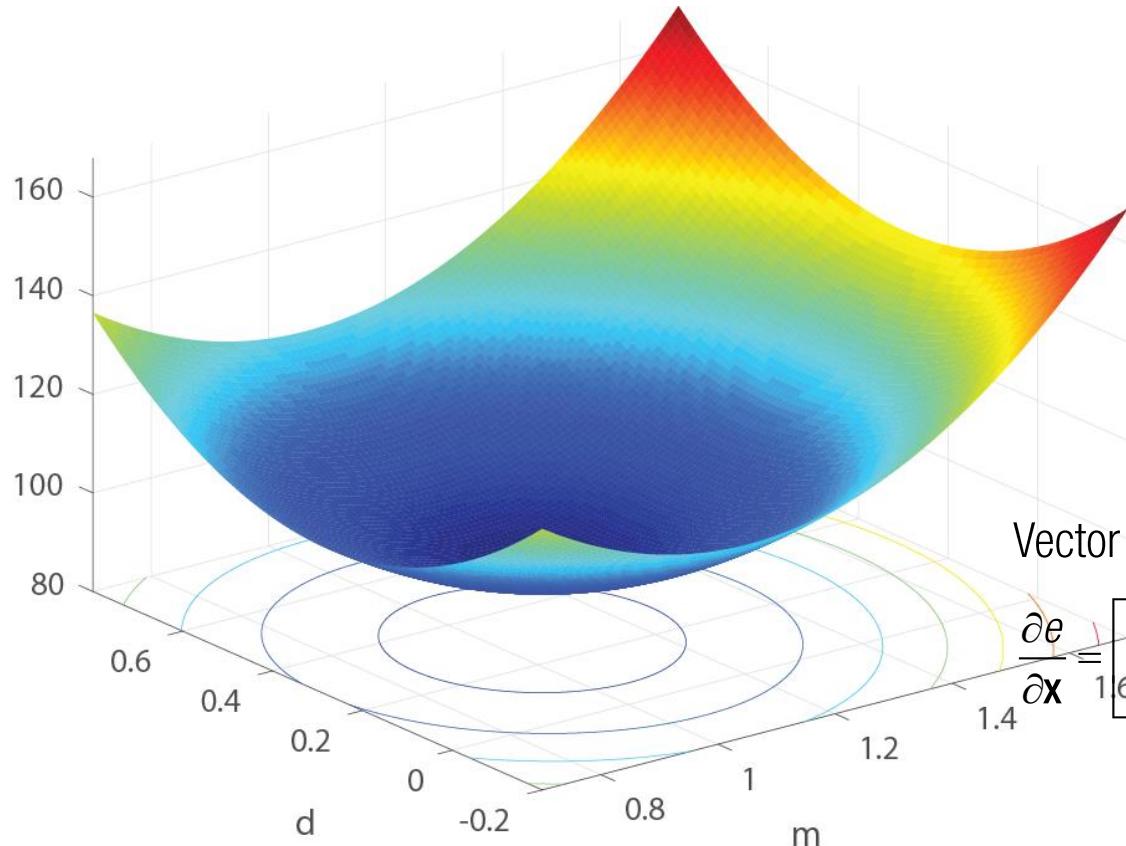
$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \dots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathbf{c}^\top \mathbf{x} = [c_1 \quad \dots \quad c_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^\top \mathbf{x}}{\partial \mathbf{x}} =$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \approx \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} \mathbf{b} \end{bmatrix}$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

$$\begin{aligned} E &= (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2 \\ &= \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b} \end{aligned}$$

$$\frac{\partial E}{\partial \mathbf{x}} = ?$$

Vector derivative:

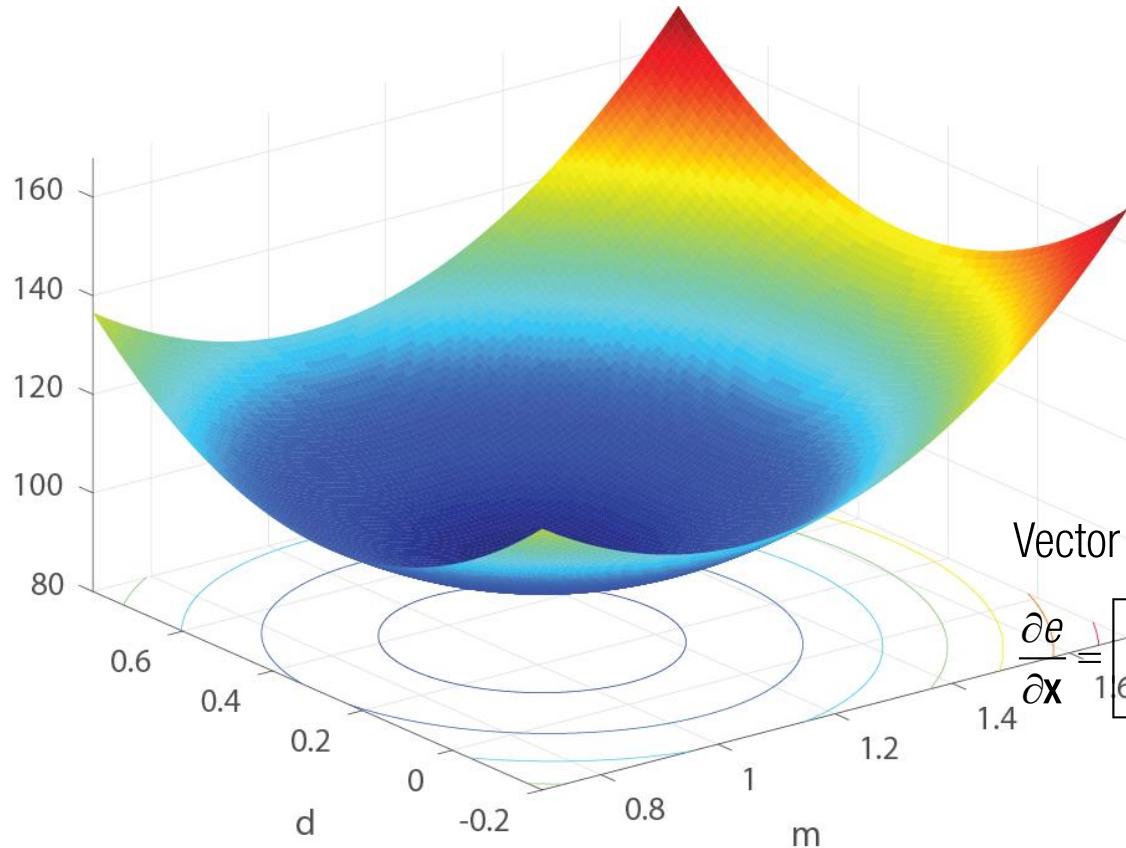
$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \dots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathbf{c}^\top \mathbf{x} = [c_1 \quad \dots \quad c_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^\top \mathbf{x}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}} (c_1 x_1 + \dots + c_n x_n)$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \approx \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

$$E = (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b}$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \approx \begin{bmatrix} m \\ d \end{bmatrix} \mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = ?$$

Vector derivative:

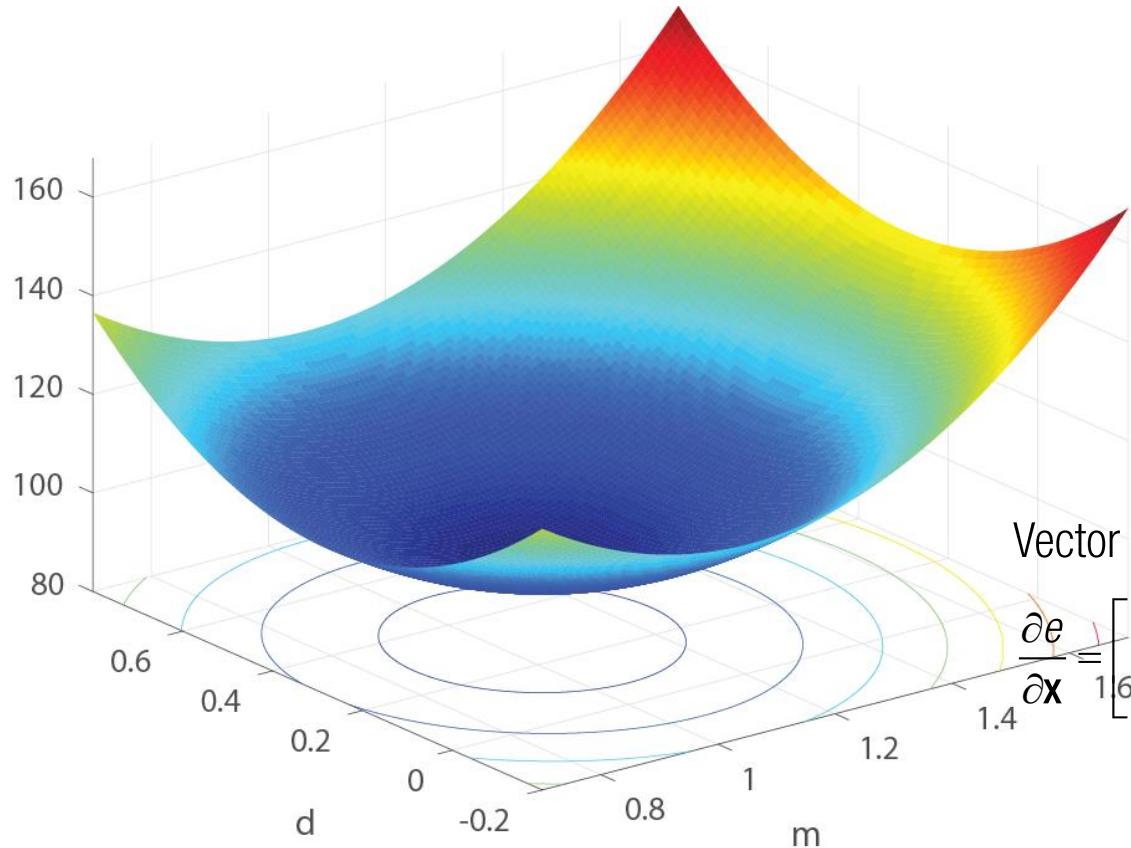
$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \dots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathbf{c}^\top \mathbf{x} = [c_1 \quad \dots \quad c_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^\top \mathbf{x}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}} (c_1 x_1 + \dots + c_n x_n)$$

$$= [c_1 \quad \dots \quad c_n]$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

$$\begin{aligned} E &= (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2 \\ &= \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b} \end{aligned}$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \approx \begin{bmatrix} m \\ d \end{bmatrix} \mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^\top \mathbf{Ax} - 2\mathbf{A}^\top \mathbf{b} = \mathbf{0}$$

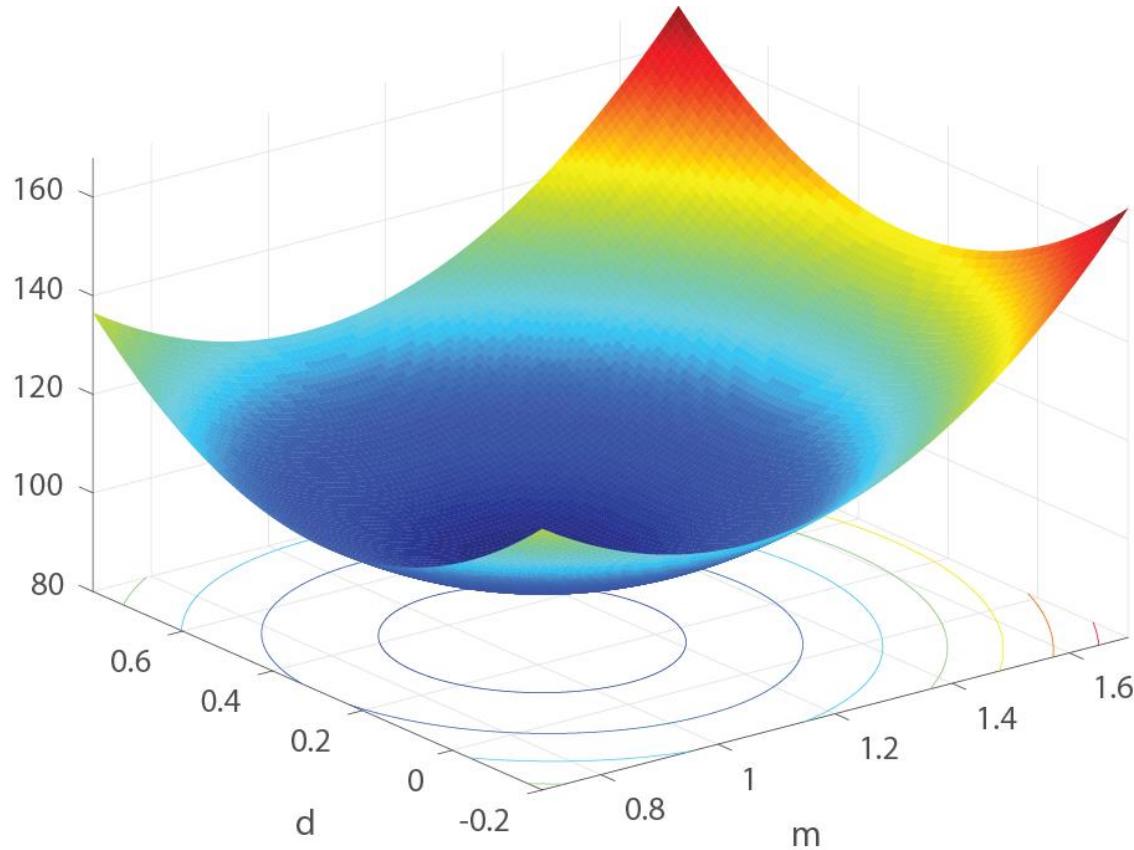
Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \dots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathbf{c}^\top \mathbf{x} = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\begin{aligned} \frac{\partial \mathbf{c}^\top \mathbf{x}}{\partial \mathbf{x}} &= \frac{\partial}{\partial \mathbf{x}} (c_1 x_1 + \dots + c_n x_n) \\ &= [c_1 \quad \dots \quad c_n] \end{aligned}$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

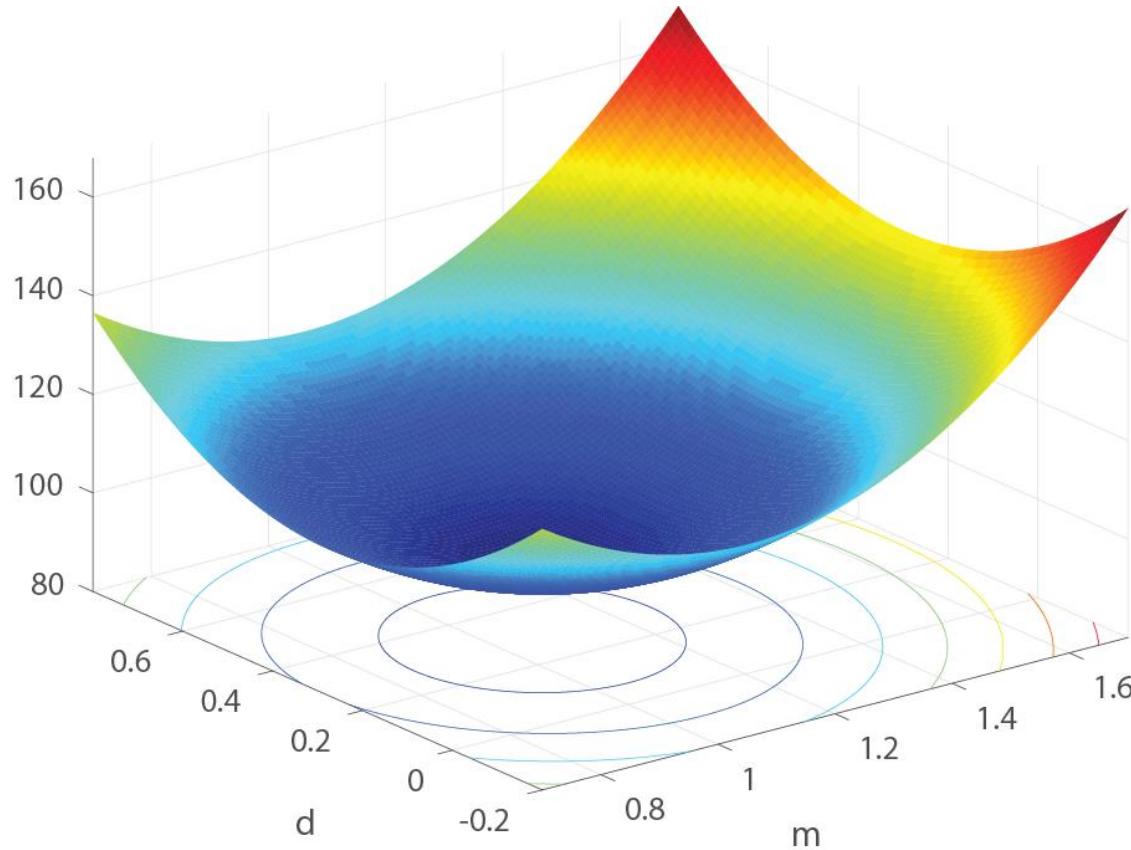
$$\begin{aligned}E &= (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2 \\&= \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b}\end{aligned}$$

$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^\top \mathbf{Ax} - 2\mathbf{A}^\top \mathbf{b} = \mathbf{0}$$

$$\rightarrow \mathbf{A}^\top \mathbf{Ax} = \mathbf{A}^\top \mathbf{b}$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A} & \\ & \mathbf{m} \\ & d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \mathbf{b}$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

$$\begin{aligned}E &= (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2 \\&= \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b}\end{aligned}$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{matrix} \mathbf{A} \\ \mathbf{x} \\ \vdots \\ \mathbf{b} \end{matrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

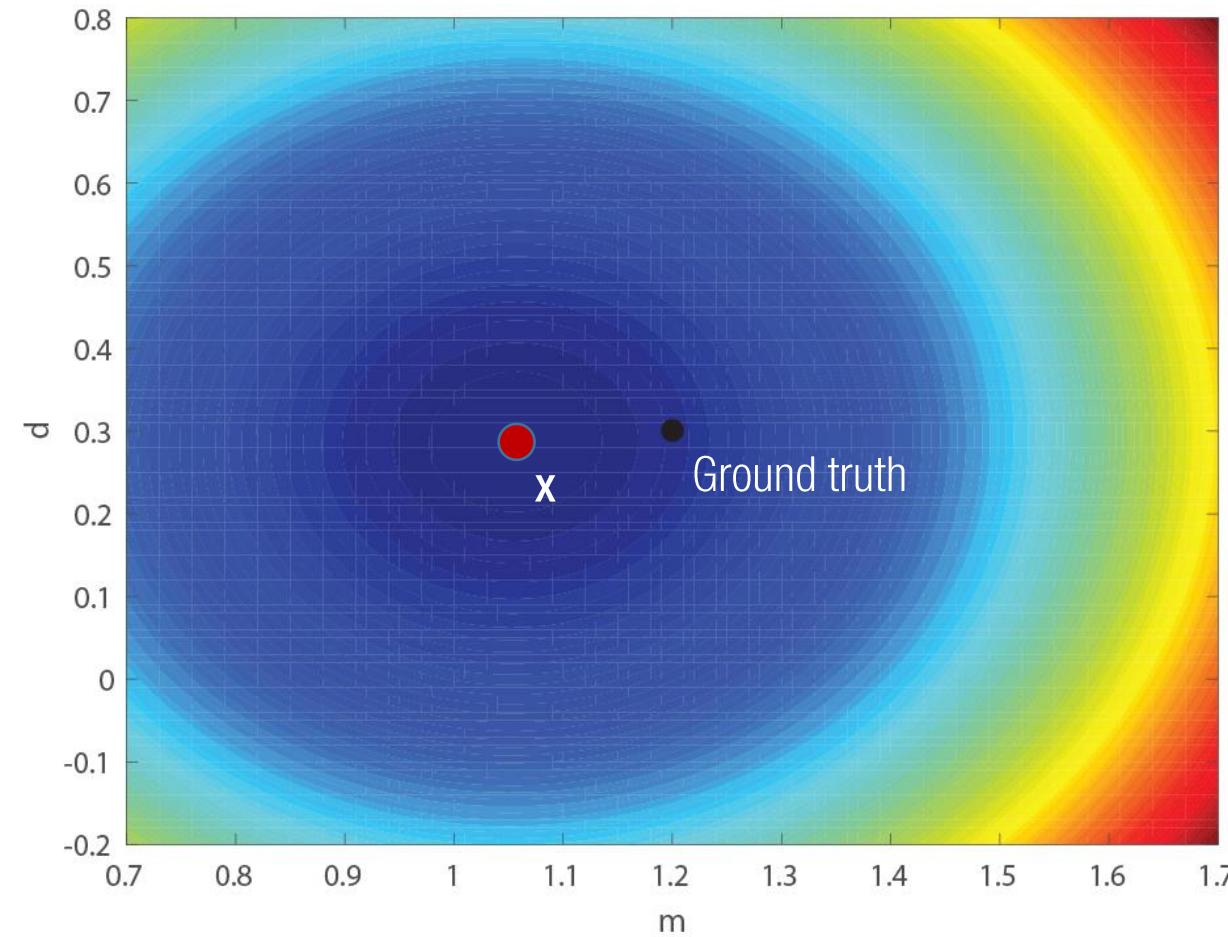
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^\top \mathbf{Ax} - 2\mathbf{A}^\top \mathbf{b} = \mathbf{0}$$

$$\rightarrow \mathbf{A}^\top \mathbf{Ax} = \mathbf{A}^\top \mathbf{b}$$

$$\begin{array}{c} \mathbf{A}^\top \\ \mathbf{A} \\ \mathbf{x} \end{array} = \begin{array}{c} \mathbf{A}^\top \\ \mathbf{A} \\ \mathbf{b} \end{array}$$

Normal equation

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

$$\begin{aligned}E &= (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2 \\&= \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b}\end{aligned}$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \approx \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} \mathbf{b} \end{bmatrix}$$

$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^\top \mathbf{Ax} - 2\mathbf{A}^\top \mathbf{b} = \mathbf{0}$$

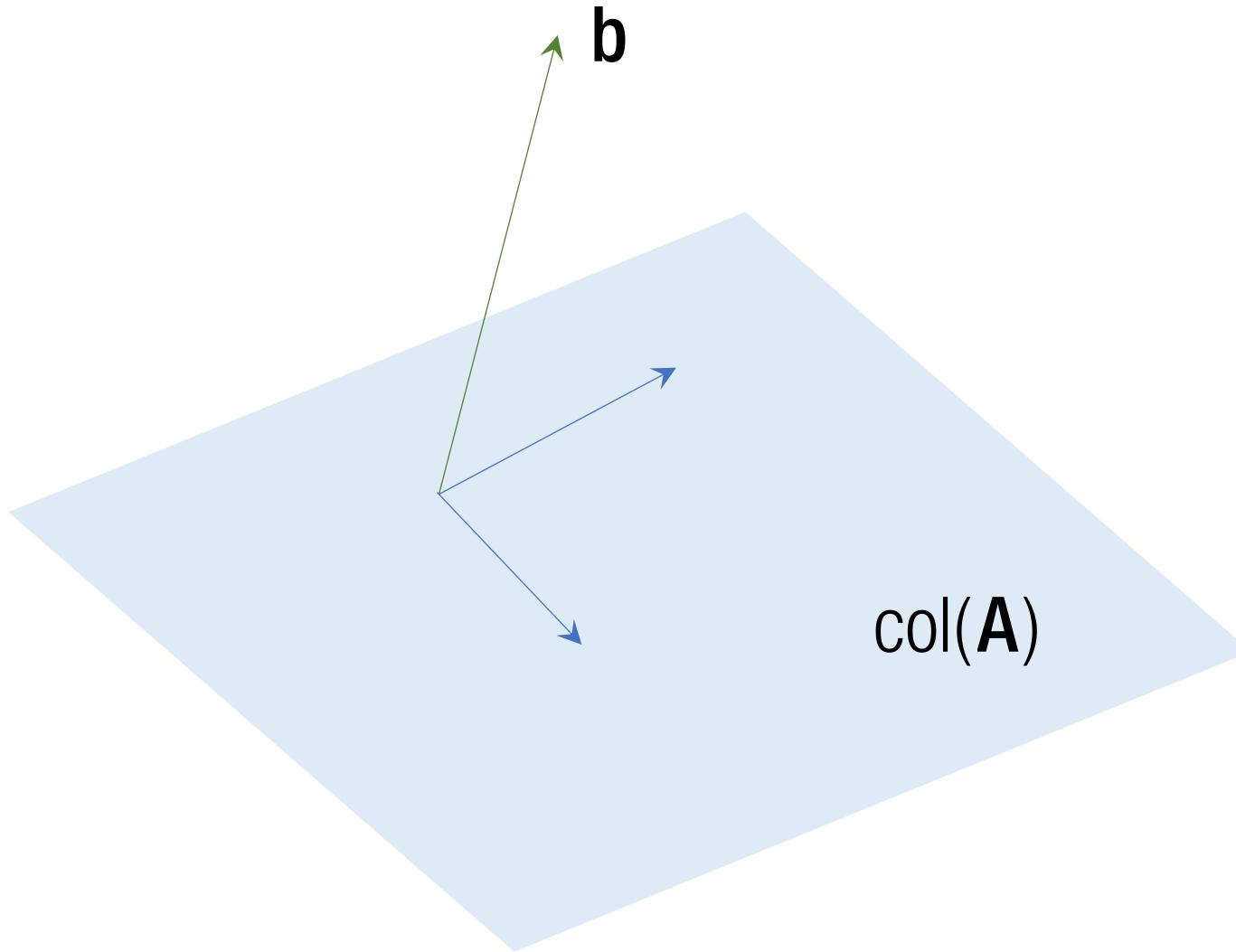
$$\rightarrow \mathbf{A}^\top \mathbf{Ax} = \mathbf{A}^\top \mathbf{b}$$

$$\begin{array}{c} \mathbf{A}^\top \\ \mathbf{A} \\ \mathbf{x} \end{array} = \begin{array}{c} \mathbf{A}^\top \\ \mathbf{b} \end{array}$$

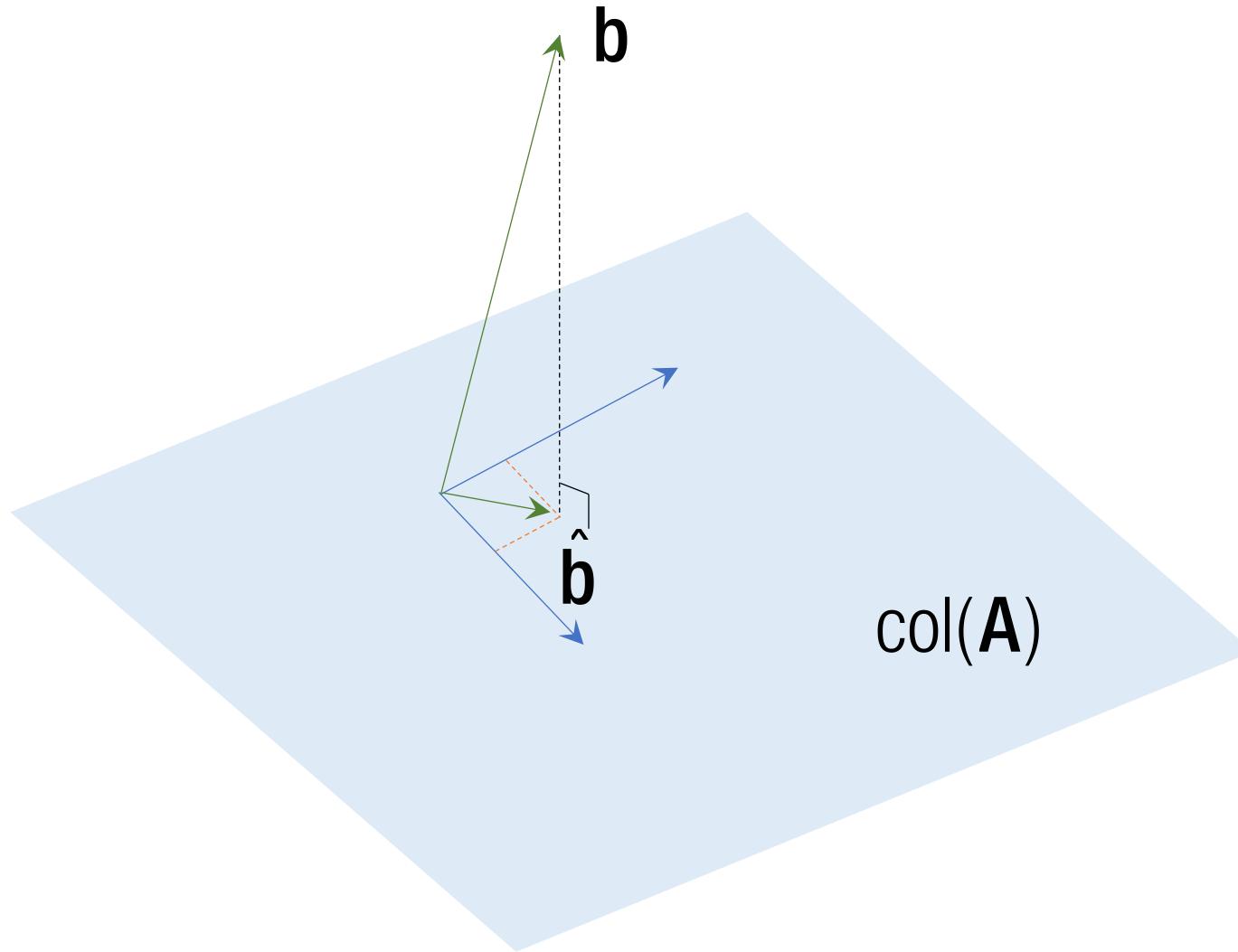
$$\mathbf{x} = \left[\begin{array}{cc} \mathbf{A}^\top & \mathbf{A} \end{array} \right]^{-1} \begin{array}{c} \mathbf{A}^\top \\ \mathbf{b} \end{array}$$

Geometric Interpretation

$$A \quad x \approx b$$



Geometric Interpretation

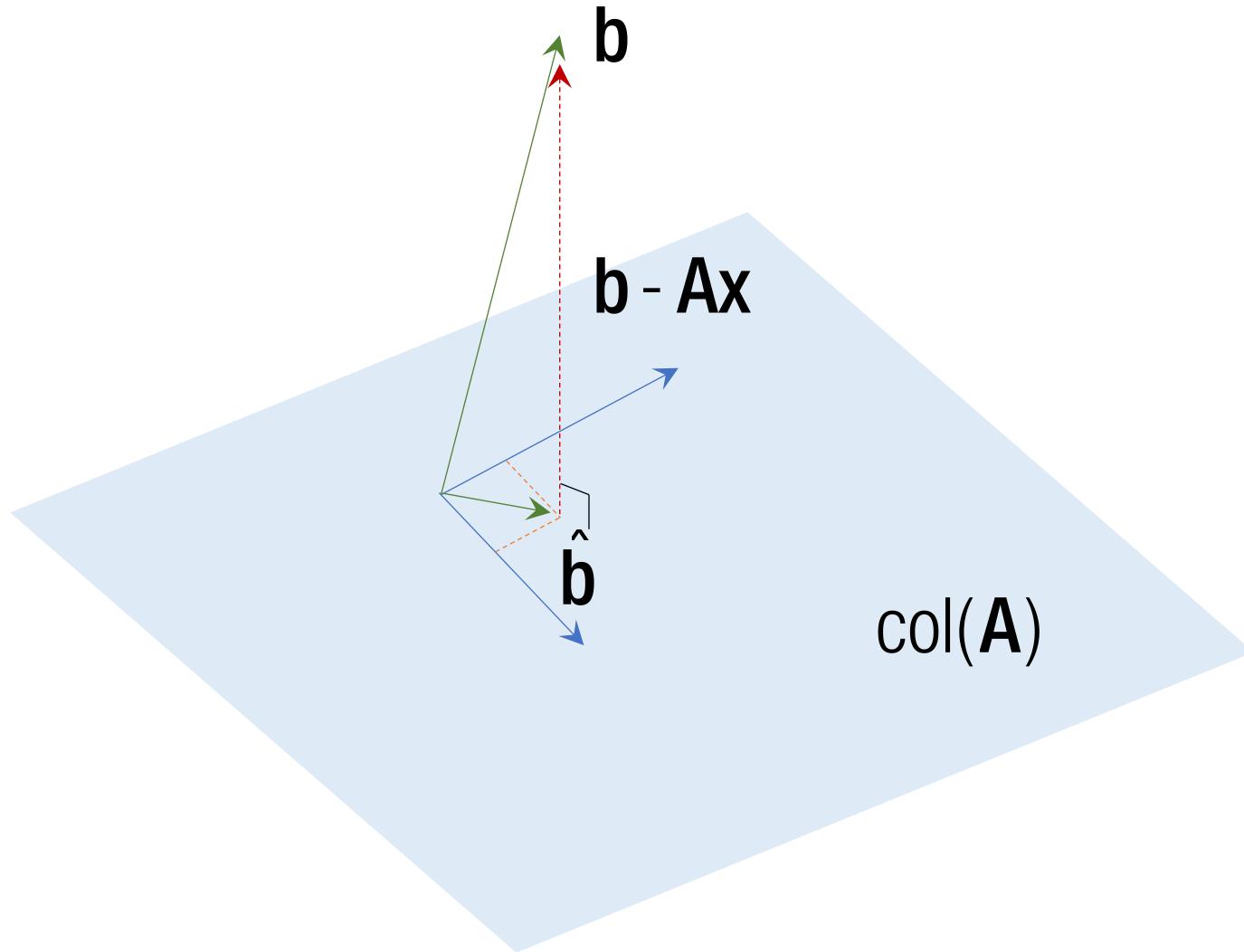


$$\mathbf{A} \quad \mathbf{x} \approx \mathbf{b}$$

$$\mathbf{A} \quad \mathbf{x} = \hat{\mathbf{b}}$$

$\hat{\mathbf{b}}$: Best approximation of \mathbf{b} spanned by $\text{col}(\mathbf{A})$

Geometric Interpretation



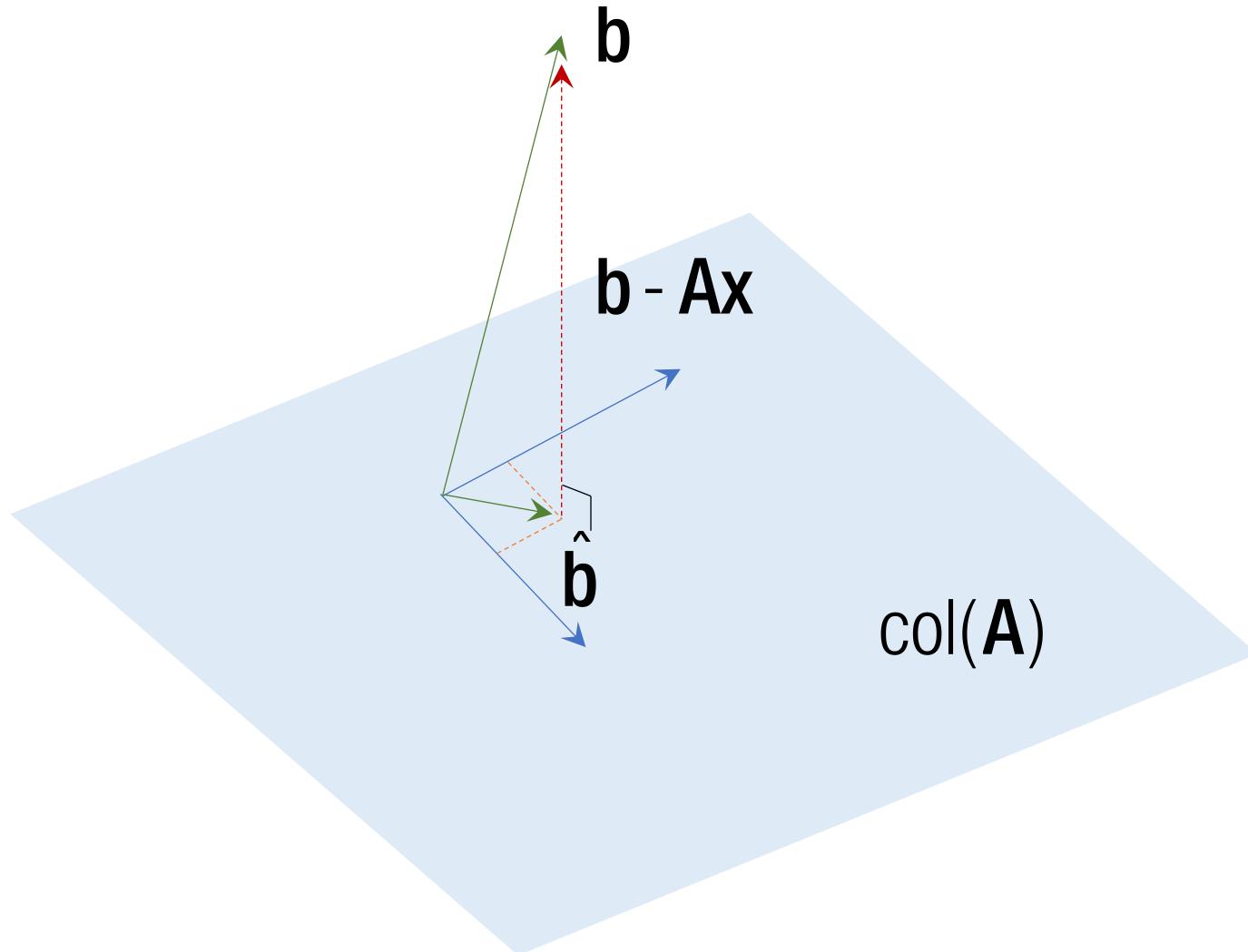
$$\mathbf{A} \quad \mathbf{x} \approx \mathbf{b}$$

$$\mathbf{A} \quad \mathbf{x} = \hat{\mathbf{b}}$$

$\hat{\mathbf{b}}$: Best approximation of \mathbf{b} spanned by $\text{col}(\mathbf{A})$

$$(\mathbf{b} - \mathbf{Ax}) \perp \text{col}(\mathbf{A})$$

Geometric Interpretation



$$\begin{array}{c|c|c} A & x & \approx b \end{array}$$

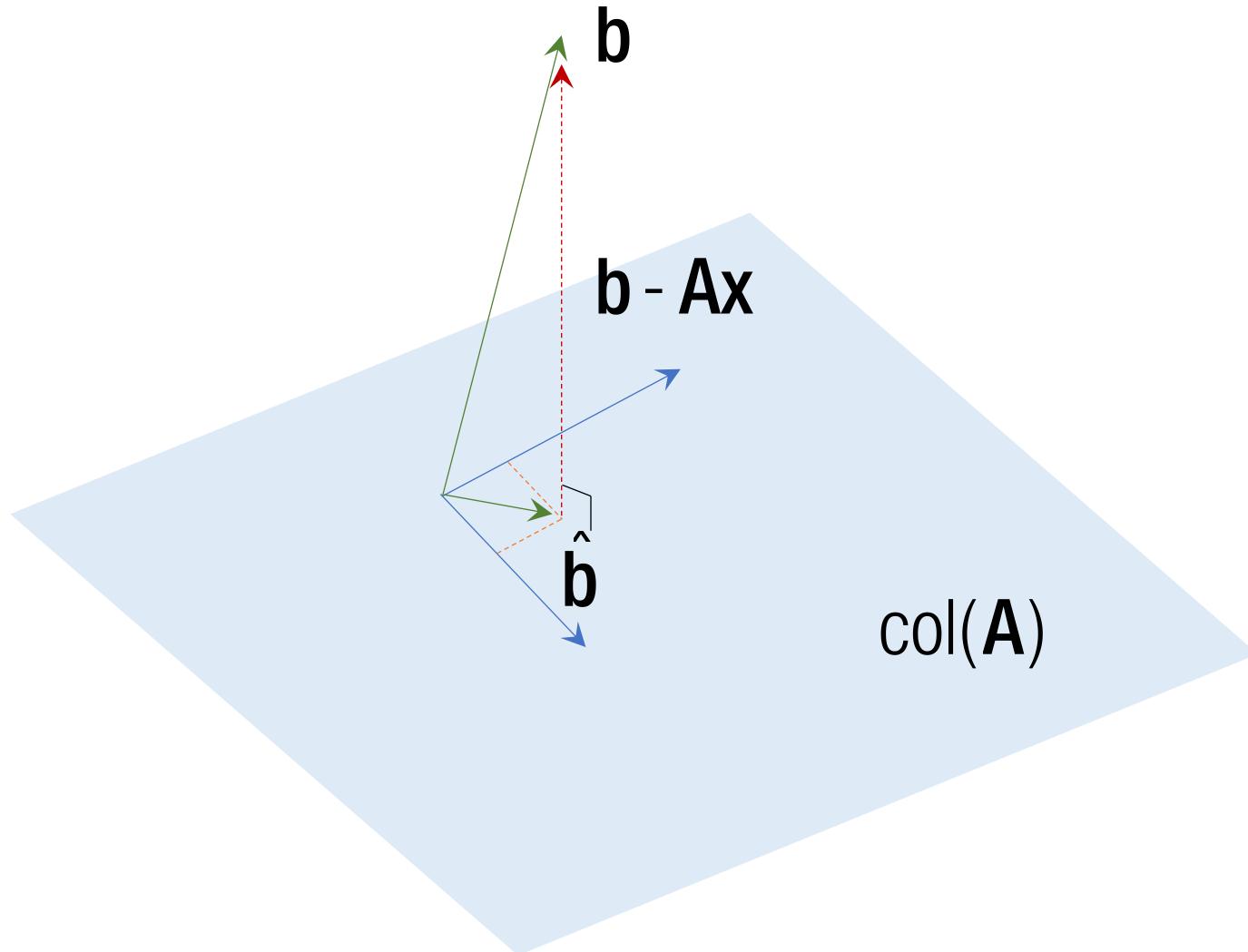
$$\begin{array}{c|c|c} A & x & = \hat{b} \end{array}$$

\hat{b} : Best approximation of b spanned by $\text{col}(A)$

$$(b - Ax) \perp \text{col}(A)$$

$$A^T(b - Ax) = 0$$

Geometric Interpretation



$$\begin{array}{c|c|c|c} A & x & \approx & b \end{array}$$

$$\begin{array}{c|c|c|c} A & x & = & \hat{b} \end{array}$$

\hat{b} : Best approximation of b spanned by $\text{col}(A)$

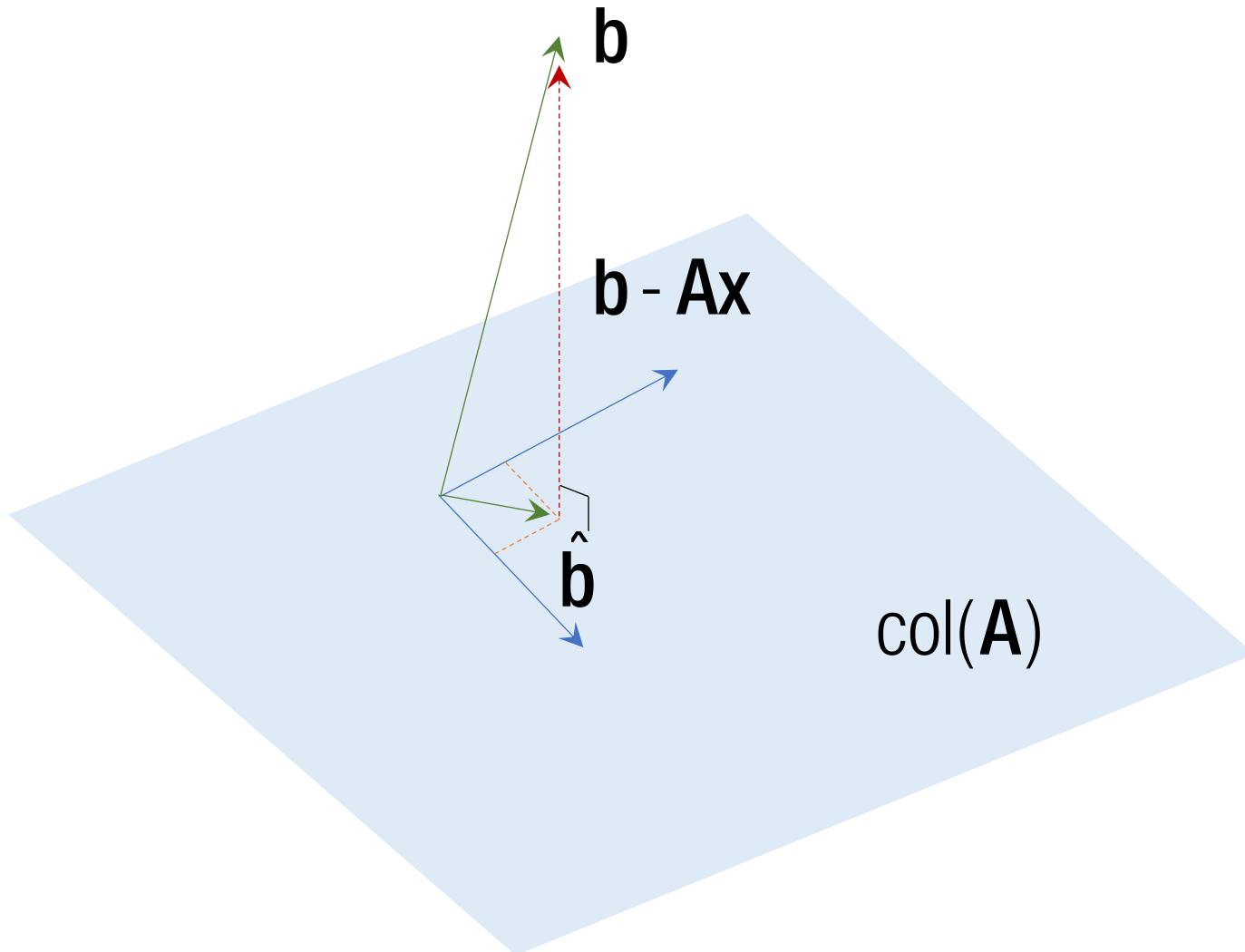
$$(b - Ax) \perp \text{col}(A)$$

$$A^T(b - Ax) = 0$$

$$\begin{array}{c|c|c|c|c} A^T & A & x & = & A^T b \end{array}$$

: Normal equation

Geometric Interpretation



$$\begin{array}{c|c|c} \text{A} & \text{x} & \approx \text{b} \end{array}$$

$$\begin{array}{c|c|c} \text{A} & \text{x} & = \hat{\text{b}} \end{array}$$

$\hat{\text{b}}$: Best approximation of b spanned by $\text{col}(\text{A})$

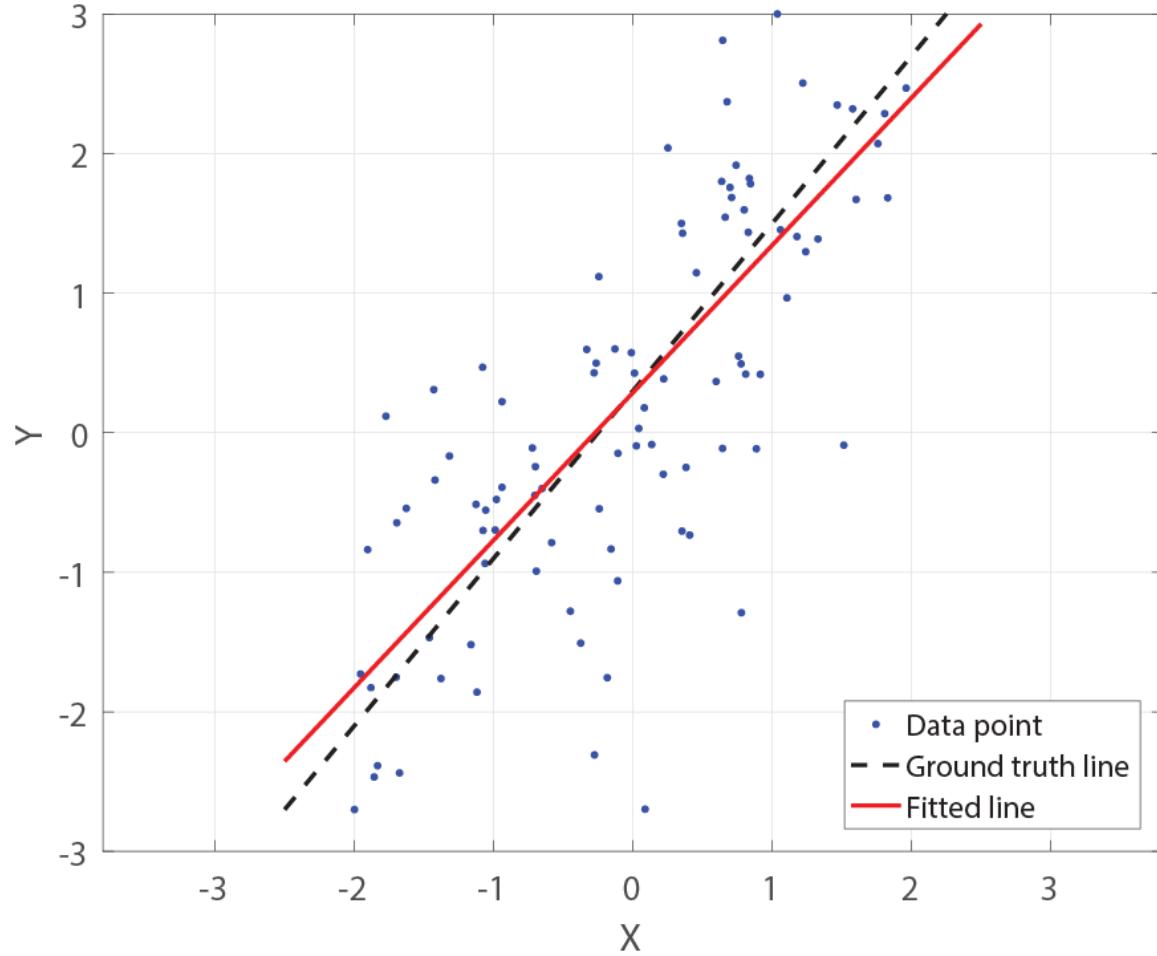
$$(\text{b} - \text{Ax}) \perp \text{col}(\text{A})$$

$$\text{A}^T(\text{b} - \text{Ax}) = 0$$

$$\begin{array}{c|c|c|c} \text{A}^T & \text{A} & \text{x} & = \text{A}^T \text{b} \end{array}$$

$$\text{x} = \left[\begin{array}{c|c} \text{A}^T & \text{A} \end{array} \right]^{-1} \text{A}^T \text{b}$$

Line Fitting ($Ax=b$)



Total error: $E = \sum_{i=1}^n (v_i - (mu_i + d))^2$

$$\begin{aligned} E &= (\mathbf{Ax} - \mathbf{b})^\top (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2 \\ &= \mathbf{x}^\top \mathbf{A}^\top \mathbf{Ax} - 2\mathbf{x}^\top \mathbf{A}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b} \end{aligned}$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \approx \begin{bmatrix} m \\ d \end{bmatrix} \mathbf{b}$$

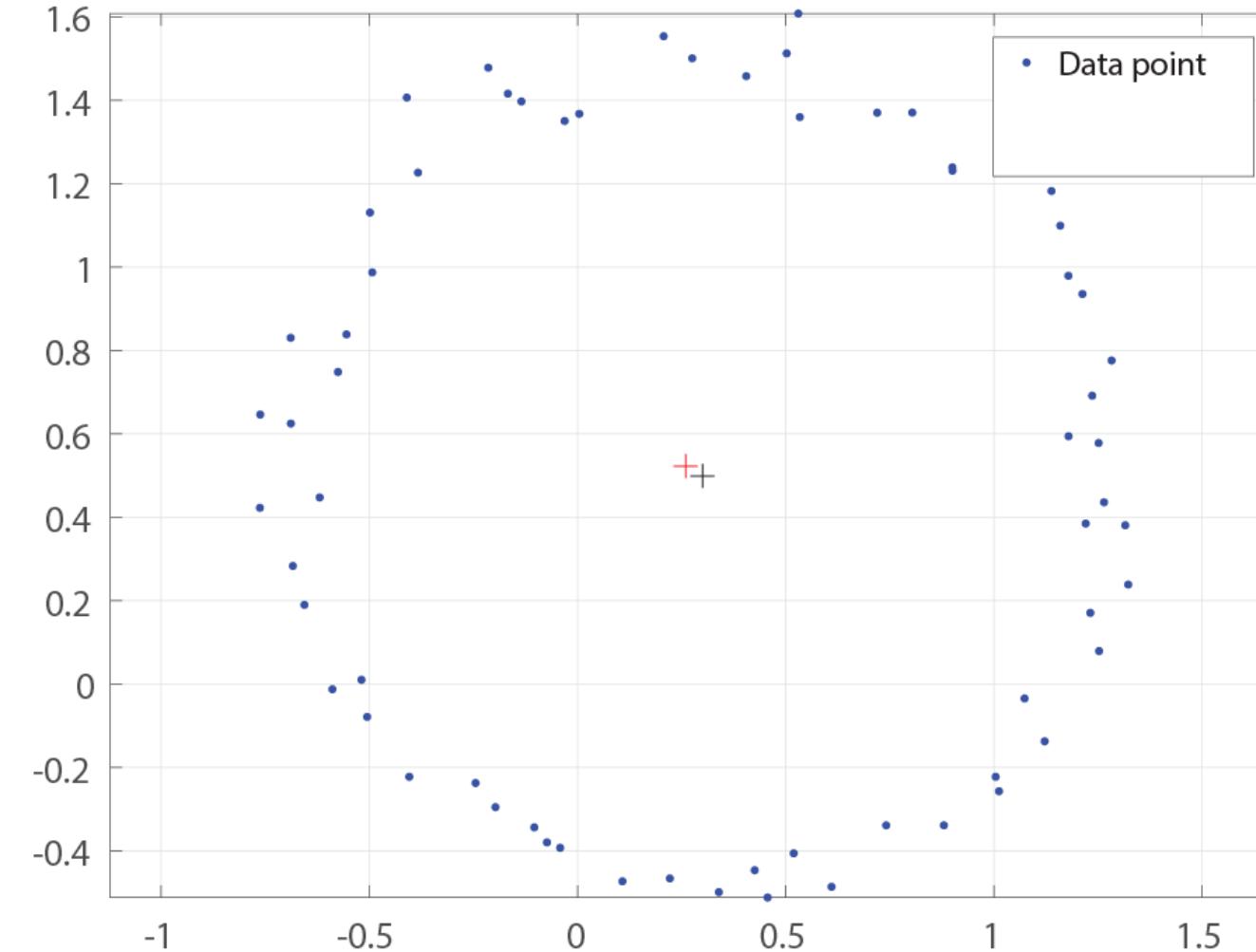
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^\top \mathbf{Ax} - 2\mathbf{A}^\top \mathbf{b} = \mathbf{0}$$

$$\rightarrow \mathbf{A}^\top \mathbf{Ax} = \mathbf{A}^\top \mathbf{b}$$

$$\mathbf{A}^\top \mathbf{A} \mathbf{x} = \mathbf{A}^\top \mathbf{b}$$

$$\mathbf{x} = [\mathbf{A}^\top \quad \mathbf{A}]^{-1} \mathbf{A}^\top \mathbf{b}$$

Circle Fitting ($Ax=b$)



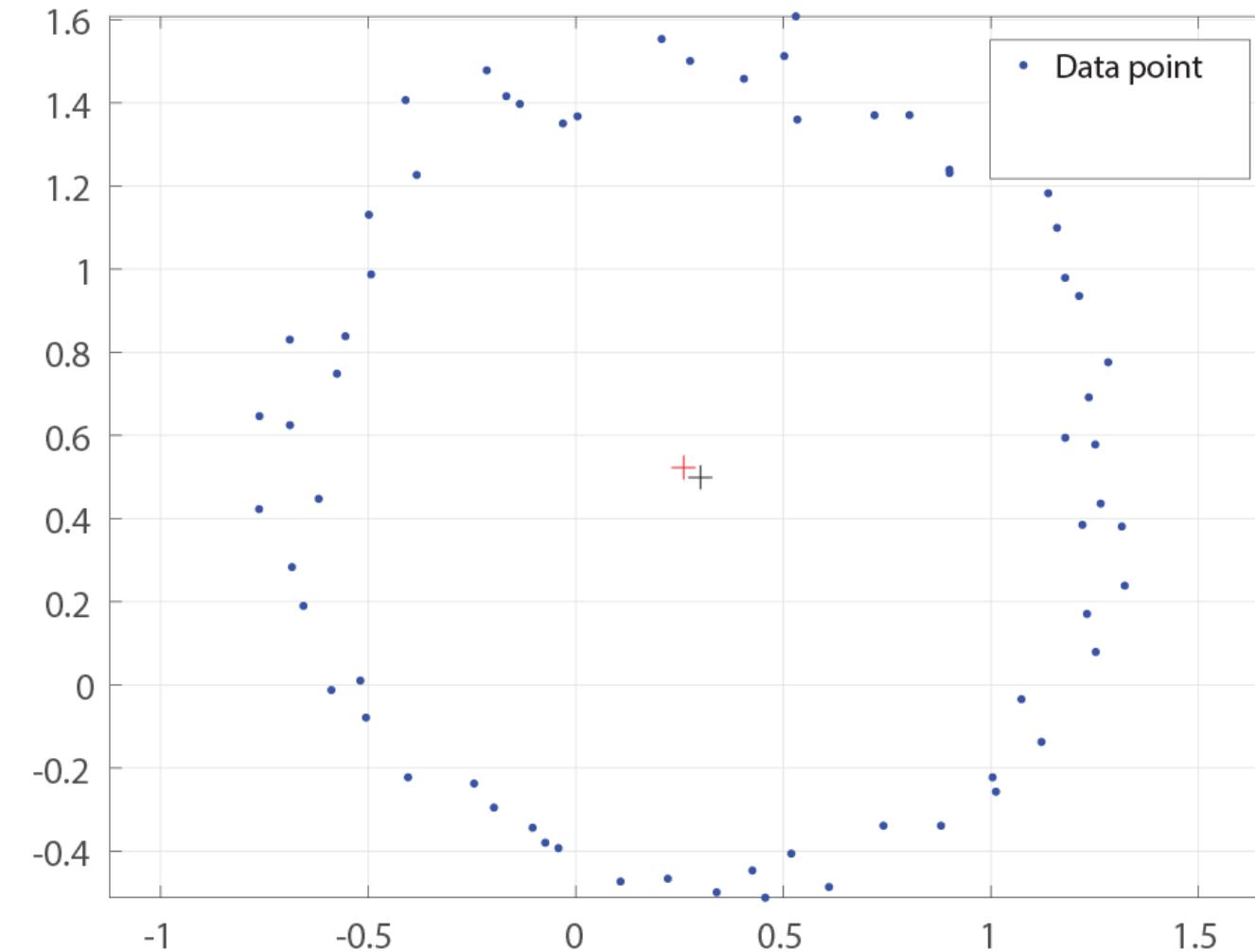
$$(x_1 - C_x)^2 + (y_1 - C_y)^2 = r^2$$

⋮

$$(x_n - C_x)^2 + (y_n - C_y)^2 = r^2$$

Unknowns: C_x, C_y, r

Circle Fitting ($Ax=b$)

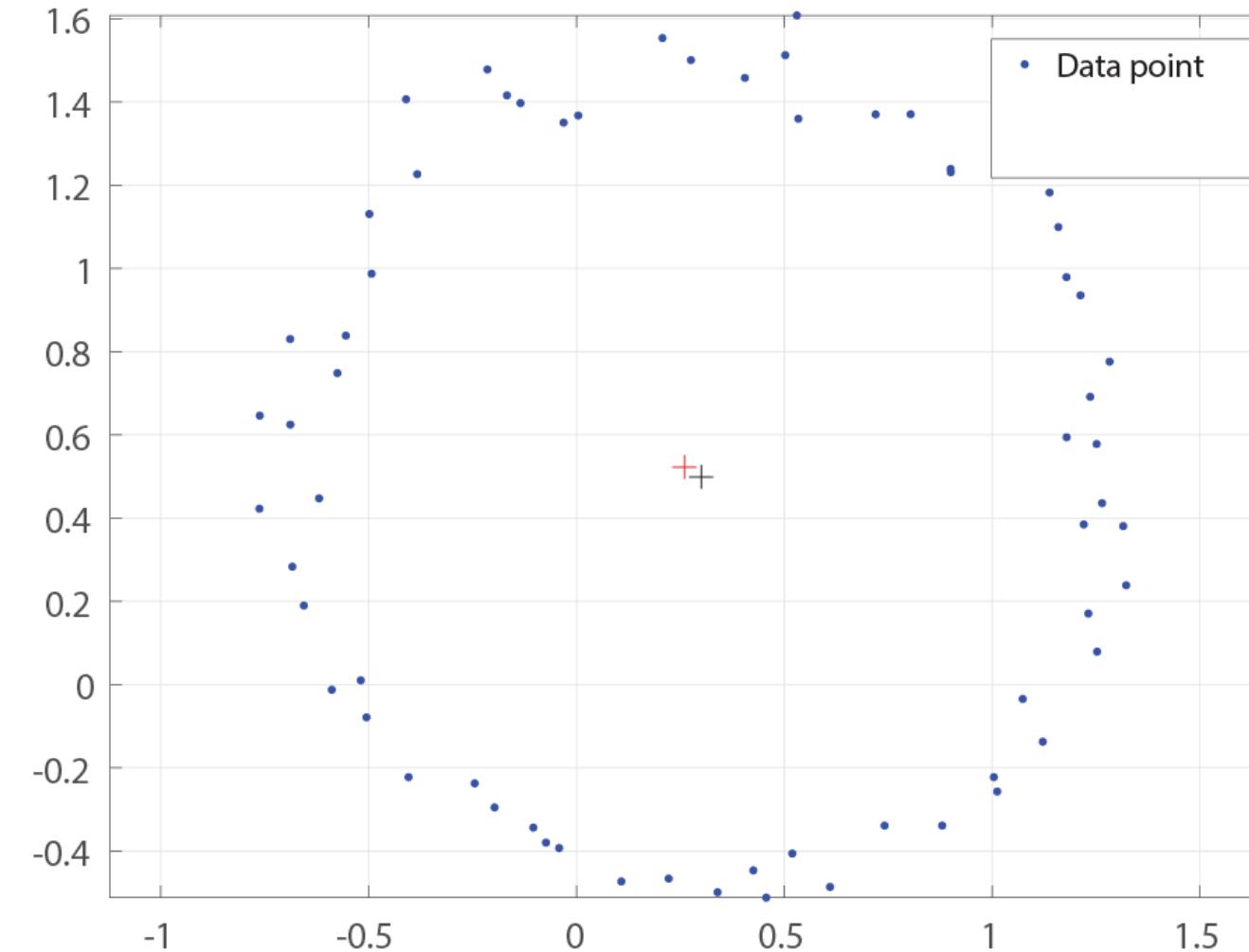


$$x_1^2 - 2x_1 C_x + C_x^2 + y_1^2 - 2y_1 C_y + C_y^2 = r^2$$

⋮

$$x_n^2 - 2x_n C_x + C_x^2 + y_n^2 - 2y_n C_y + C_y^2 = r^2$$

Circle Fitting ($Ax=b$)

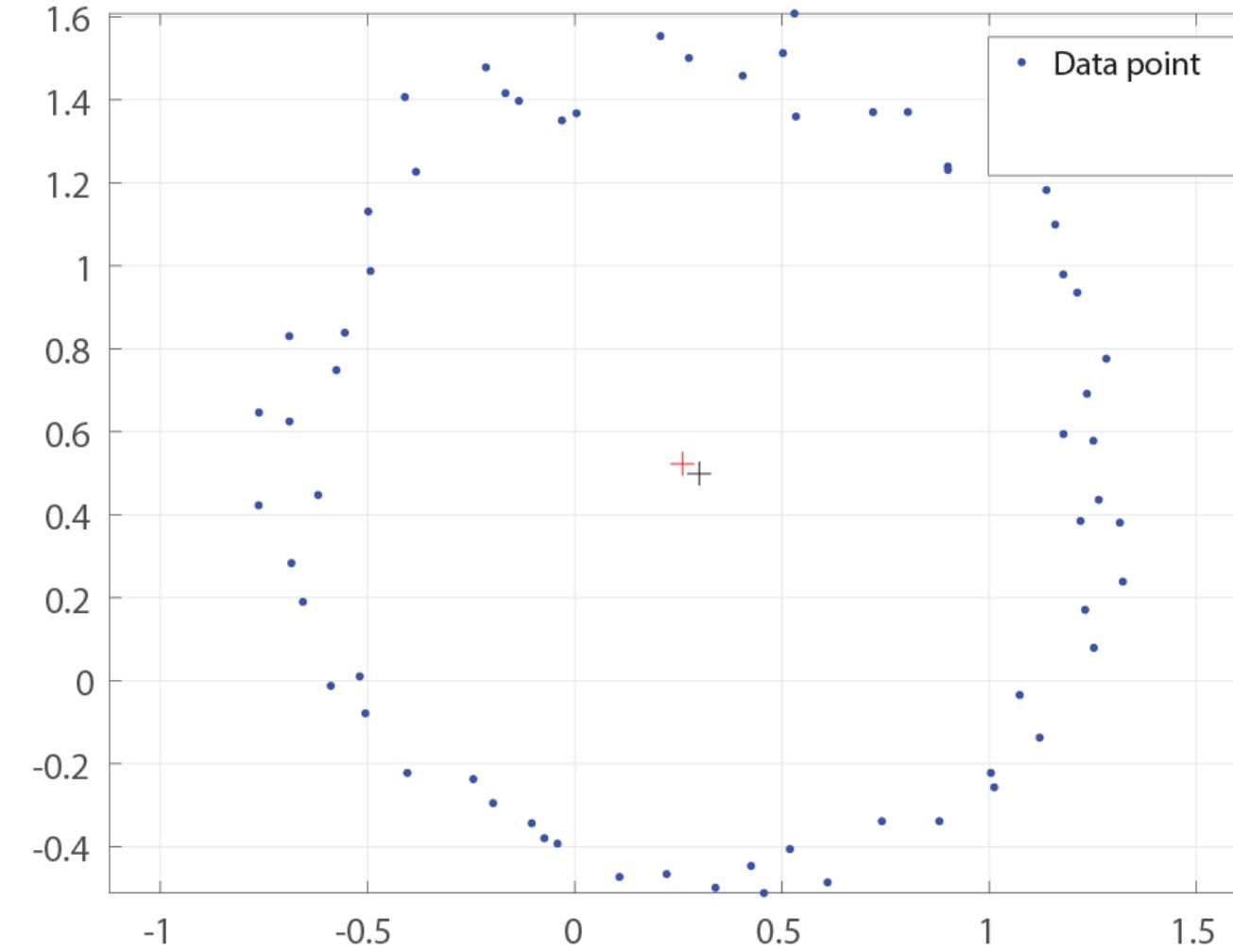


$$x_1^2 - 2x_1 C_x + C_x^2 + y_1^2 - 2y_1 C_y + C_y^2 = r^2$$

$$\vdots$$
$$x_n^2 - 2x_n C_x + C_x^2 + y_n^2 - 2y_n C_y + C_y^2 = r^2$$

$$x_i^2 - x_1^2 - 2C_x(x_i - x_1) + y_i^2 - y_1^2 - 2(y_i - y_1)C_y = 0$$

Circle Fitting ($Ax=b$)



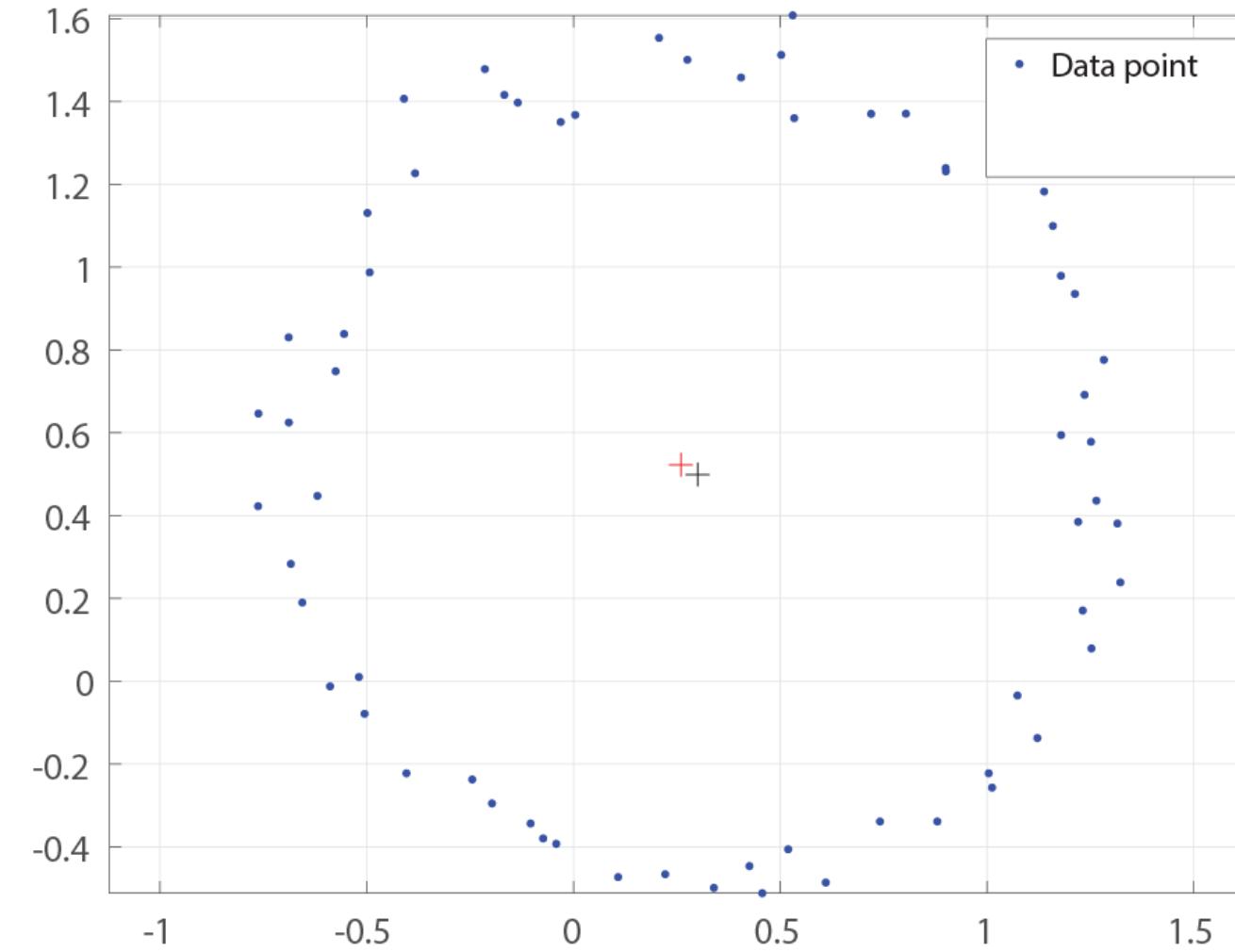
$$x_1^2 - 2x_1 C_x + C_x^2 + y_1^2 - 2y_1 C_y + C_y^2 = r^2$$

$$x_n^2 - 2x_n C_x + C_x^2 + y_n^2 - 2y_n C_y + C_y^2 = r^2$$

$$x_i^2 - x_1^2 - 2C_x(x_i - x_1) + y_i^2 - y_1^2 - 2(y_i - y_1)C_y = 0$$

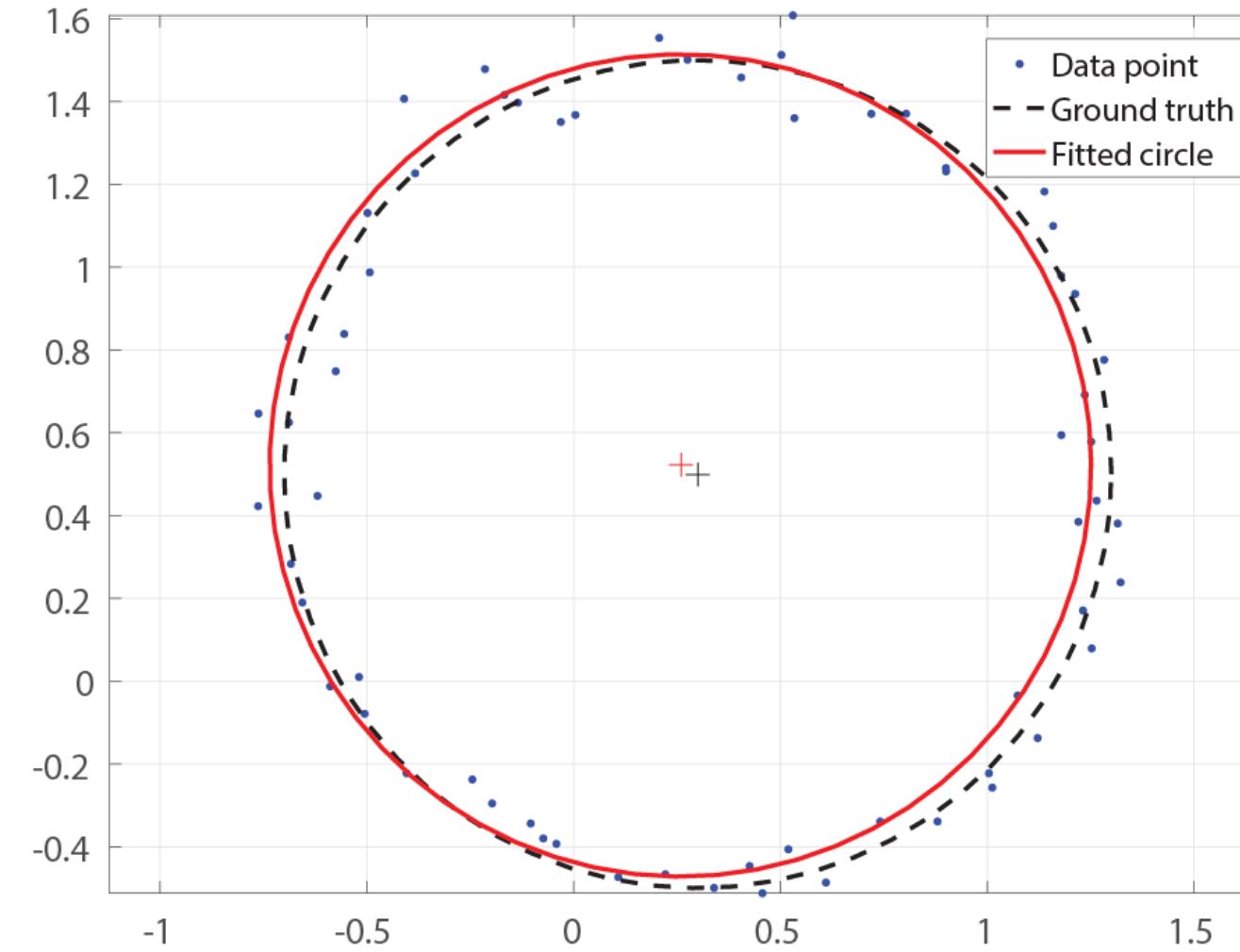
$$\begin{bmatrix} 2(x_2 - x_1) & 2(y_2 - y_1) \\ \vdots & \vdots \\ 2(x_n - x_1) & 2(y_n - y_1) \end{bmatrix} \begin{bmatrix} C_x \\ C_y \end{bmatrix} = \begin{bmatrix} x_2^2 - x_1^2 + y_2^2 - y_1^2 \\ \vdots \\ x_n^2 - x_1^2 + y_n^2 - y_1^2 \end{bmatrix}$$

Circle Fitting ($Ax=b$)



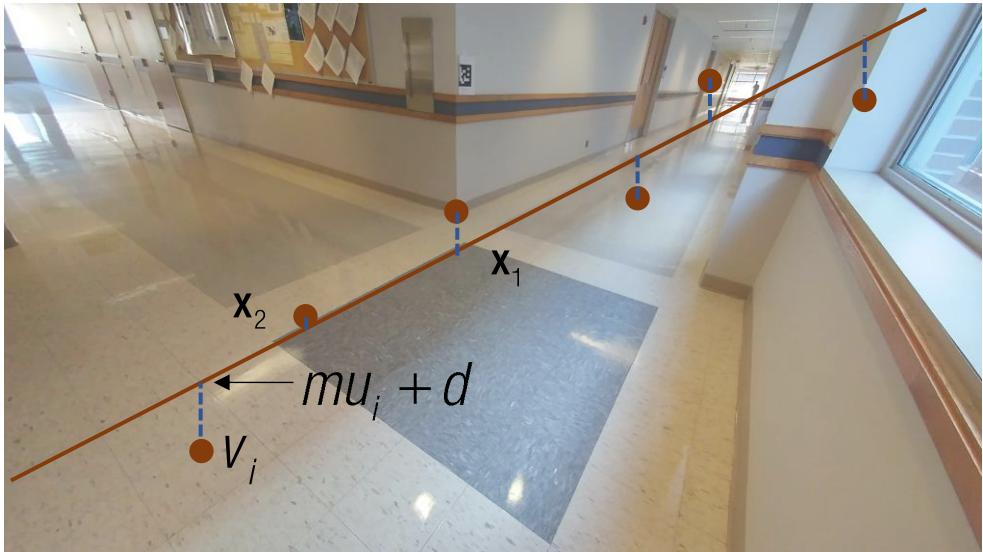
$$x_1^2 - 2x_1 C_x + C_x^2 + y_1^2 - 2y_1 C_y + C_y^2 = r^2$$
$$\vdots$$
$$x_n^2 - 2x_n C_x + C_x^2 + y_n^2 - 2y_n C_y + C_y^2 = r^2$$
$$\downarrow$$
$$x_i^2 - x_1^2 - 2C_x(x_i - x_1) + y_i^2 - y_1^2 - 2(y_i - y_1)C_y = 0$$
$$\downarrow$$
$$\begin{bmatrix} 2(x_2 - x_1) & 2(y_2 - y_1) \\ \vdots & \vdots \\ 2(x_n - x_1) & 2(y_n - y_1) \end{bmatrix} \begin{bmatrix} C_x \\ C_y \end{bmatrix} = \begin{bmatrix} x_2^2 - x_1^2 + y_2^2 - y_1^2 \\ \vdots \\ x_n^2 - x_1^2 + y_n^2 - y_1^2 \end{bmatrix}$$

Circle Fitting ($Ax=b$)



$$\begin{aligned} x_1^2 - 2x_1 C_x + C_x^2 + y_1^2 - 2y_1 C_y + C_y^2 &= r^2 \\ \vdots & \\ x_n^2 - 2x_n C_x + C_x^2 + y_n^2 - 2y_n C_y + C_y^2 &= r^2 \\ \downarrow & \\ x_i^2 - x_1^2 - 2C_x(x_i - x_1) + y_i^2 - y_1^2 - 2(y_i - y_1)C_y &= 0 \\ \downarrow & \\ \begin{bmatrix} 2(x_2 - x_1) & 2(y_2 - y_1) \\ \vdots & \vdots \\ 2(x_n - x_1) & 2(y_n - y_1) \end{bmatrix} \begin{bmatrix} C_x \\ C_y \end{bmatrix} &= \begin{bmatrix} x_2^2 - x_1^2 + y_2^2 - y_1^2 \\ \vdots \\ x_n^2 - x_1^2 + y_n^2 - y_1^2 \end{bmatrix} \end{aligned}$$

Line Fitting ($Ax=b$)



Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: $(a, b, c) \longrightarrow (\underline{m}, \underline{d})$

slope y-intercept

$$\begin{array}{l} au_1 + bv_1 + c \approx 0 \\ au_2 + bv_2 + c \approx 0 \\ \vdots \\ au_n + bv_n + c \approx 0 \end{array} \longrightarrow \begin{array}{l} v_1 \approx mu_1 + d \\ v_2 \approx mu_2 + d \\ \vdots \\ v_n \approx mu_n + d \end{array}$$

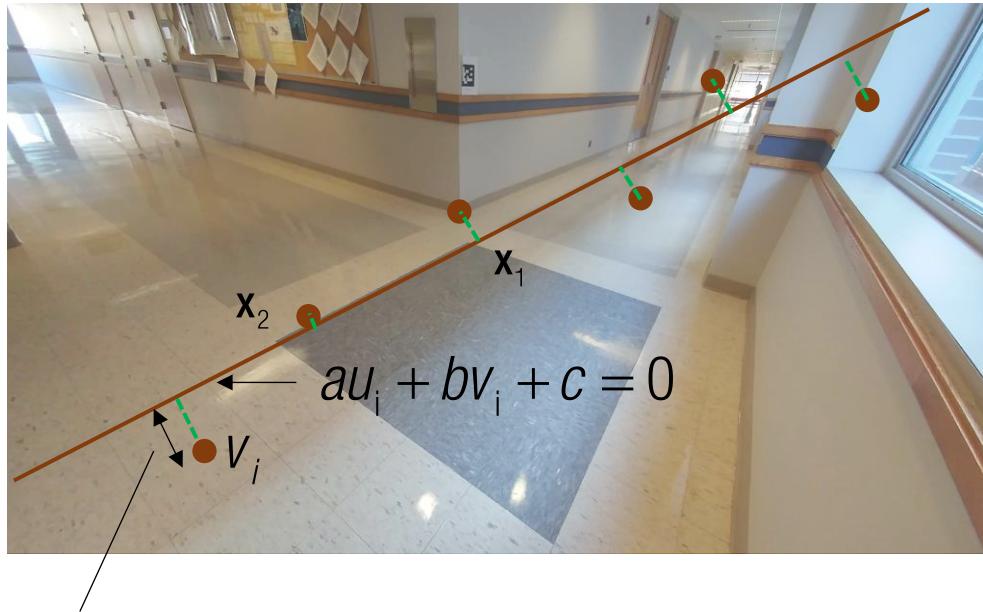
$$\mathbf{Ax = b}$$

What is different?

Line Fitting ($Ax=0$)

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: (a, b, c)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

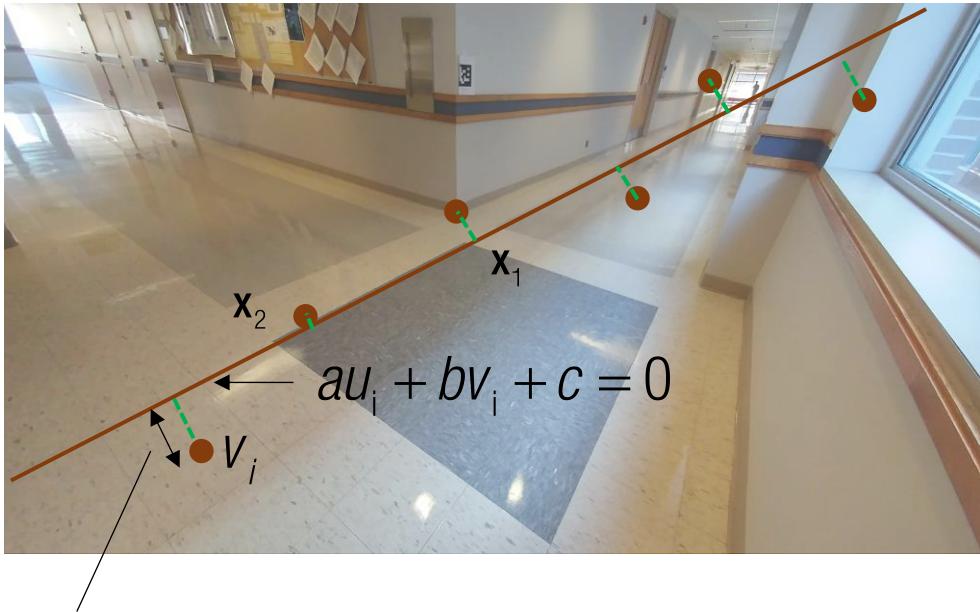
$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned}$$

$$Ax = 0$$

$$\begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$$Ax = b$$

Line Fitting ($Ax=0$)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: (a, b, c)

$$au_1 + bv_1 + c \approx 0$$

$$au_2 + bv_2 + c \approx 0$$

⋮

$$au_n + bv_n + c \approx 0$$

$$v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

⋮

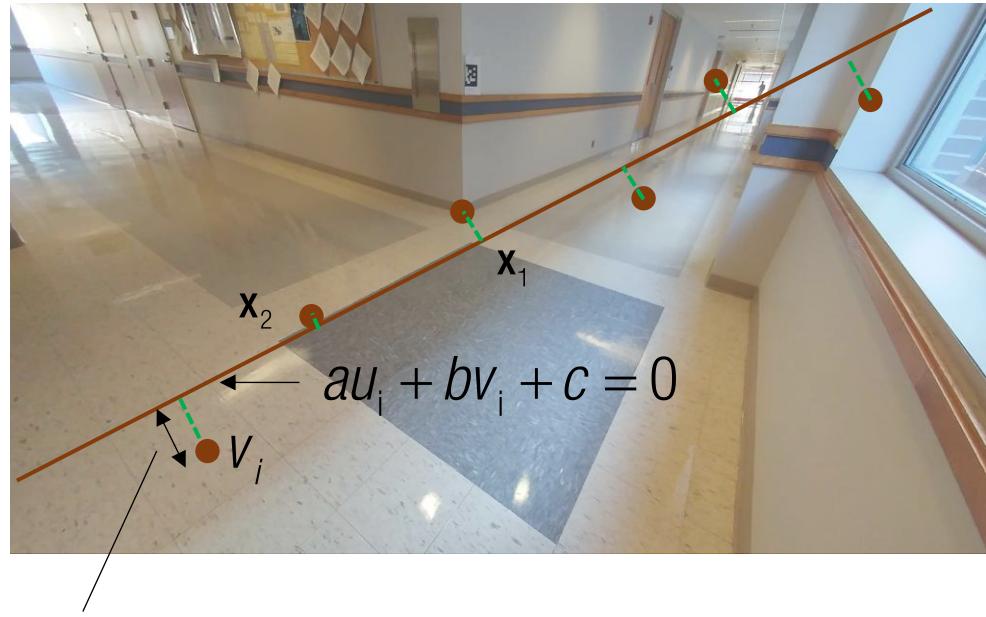
$$v_n \approx mu_n + d$$

$$\mathbf{Ax} = \mathbf{0}$$

$$\mathbf{Ax} = \mathbf{b}$$

Trivial solution: $\mathbf{x} = \mathbf{0}$

Line Fitting ($Ax=0$)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: (a, b, c)

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned}$$

$$Ax = 0$$

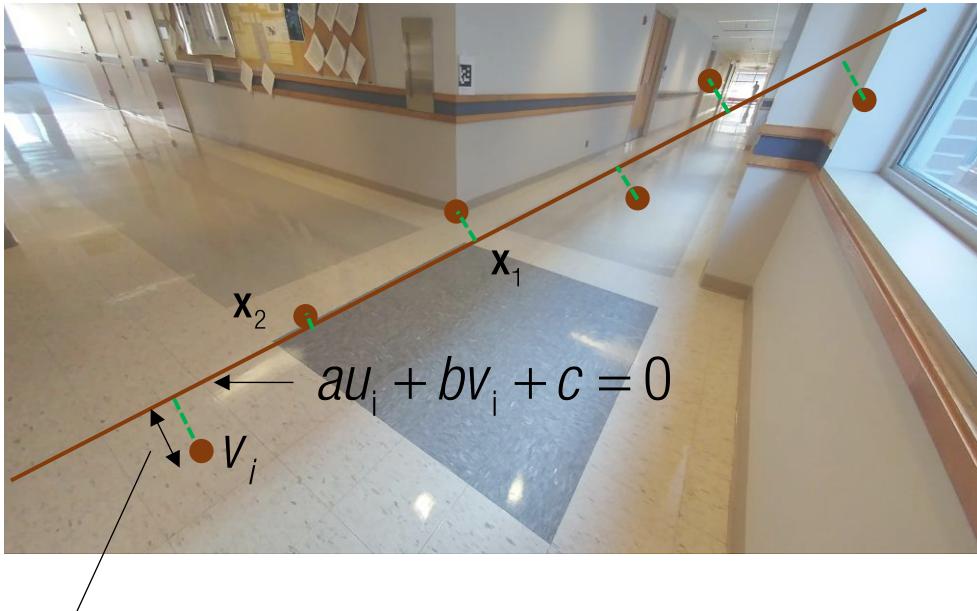
Trivial solution: $\mathbf{x} = \mathbf{0}$

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$$

$$\begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$$Ax = b$$

Line Fitting ($\mathbf{Ax} = \mathbf{0}$)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: (a, b, c)

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned}$$

$$\mathbf{Ax} = \mathbf{0}$$

Trivial solution: $\mathbf{x} = \mathbf{0}$

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$$

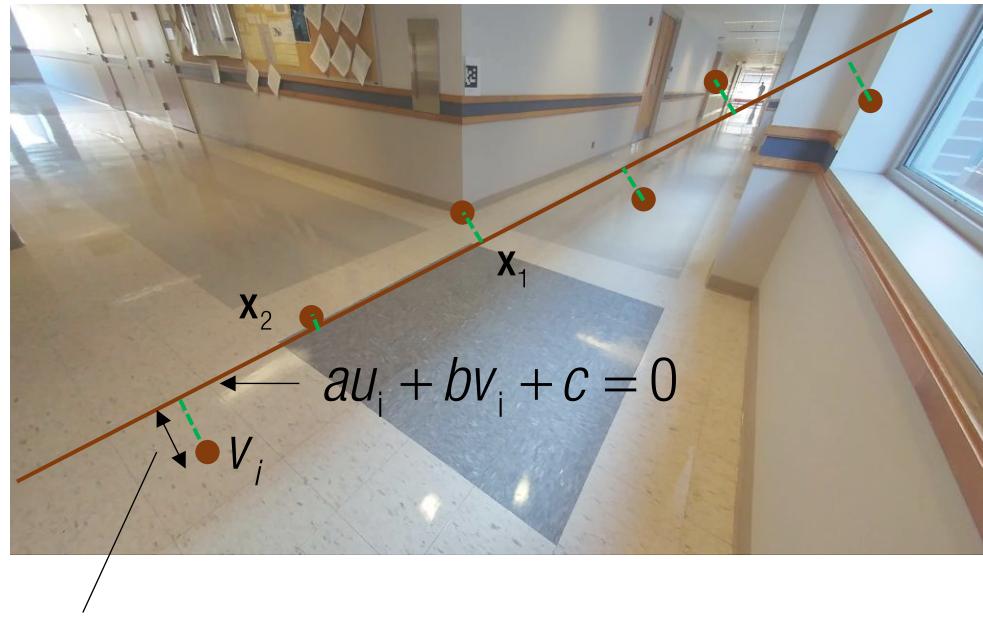
$$\text{subject to } \|\mathbf{x}\| = 1$$

Condition to avoid the trivial solution

$$\begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$$\mathbf{Ax} = \mathbf{b}$$

Line Fitting ($\mathbf{Ax} = \mathbf{0}$)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: (a, b, c)

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned}$$

$$\mathbf{Ax} = \mathbf{0}$$

$$\begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$$\mathbf{Ax} = \mathbf{b}$$

Trivial solution: $\mathbf{x} = \mathbf{0}$

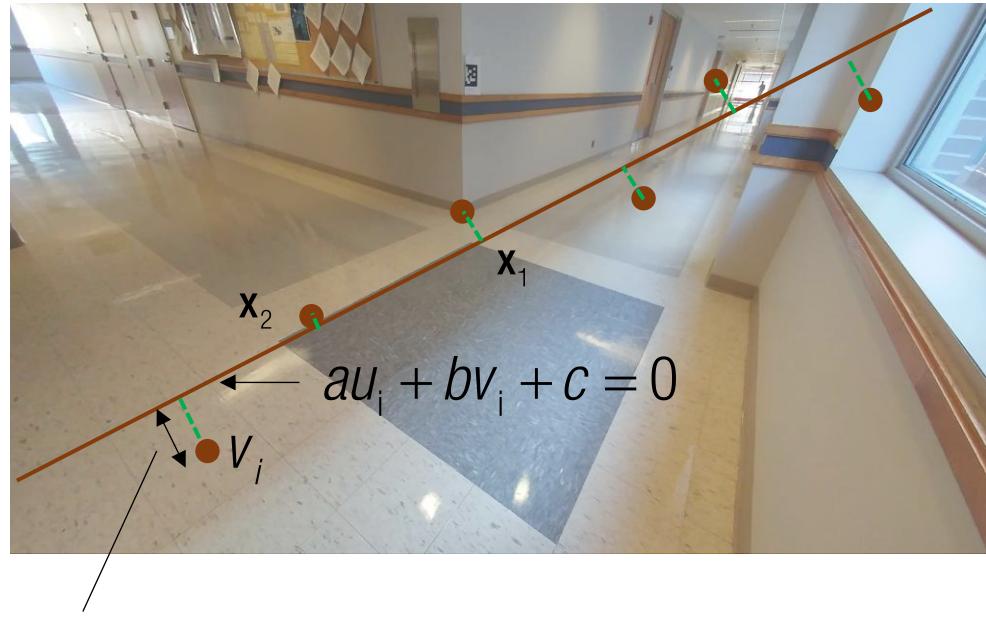
$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$$

$$\text{subject to } \|\mathbf{x}\| = 1$$

Condition to avoid the trivial solution

How to solve?

Line Fitting ($\mathbf{Ax} = 0$)



$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: (a, b, c)

$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned}$$

$$\mathbf{Ax} = \mathbf{0}$$

$$\begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$$\mathbf{Ax} = \mathbf{b}$$

Trivial solution: $\mathbf{x} = \mathbf{0}$

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \quad \text{subject to } \|\mathbf{x}\| = 1$$

Condition to avoid the trivial solution

How to solve? approximated null space $\mathbf{x} = \text{null}(\mathbf{A})$

Nullspace

eqs < # unknowns

There exist at least $n-m$ vectors that makes $\mathbf{Ax}=0$.

$$\begin{array}{c|c|c} \mathbf{A} & \mathbf{x} & = & \mathbf{0} \\ m \times n & n \times 1 & & \\ m < n & & & \end{array}$$

$$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$$

Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least $n-m$ vectors that makes $\mathbf{Ax}=0$.

$$\begin{matrix} \mathbf{A} \\ m \times n \end{matrix} \quad \begin{matrix} \mathbf{x} \\ n \times 1 \end{matrix} = \begin{matrix} \mathbf{0} \\ m \times 1 \end{matrix}$$

$$\begin{matrix} \mathbf{A} \\ m \times n \end{matrix} = \begin{matrix} \mathbf{U} \\ m \times m \end{matrix} \begin{matrix} \mathbf{D} \\ m \times n \end{matrix} \begin{matrix} \mathbf{V}^T \\ n \times n \end{matrix}$$

Orthogonal matrix Diagonal matrix Orthogonal matrix

$$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$$

Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least $n-m$ vectors that makes $\mathbf{Ax}=0$.

$$\begin{array}{c} \text{A} \\ \text{m} \times \text{n} \\ \text{m} < \text{n} \end{array} \quad \begin{array}{c} \mathbf{x} \\ \text{n} \times 1 \end{array} = \begin{array}{c} \mathbf{0} \end{array}$$
$$\begin{array}{c} \text{A} \\ \text{m} \times \text{n} \end{array} = \begin{array}{c} \text{U} \\ \text{m} \times \text{m} \end{array} \begin{array}{c} \text{D} \\ \text{m} \times \text{n} \end{array} \begin{array}{c} \text{V}^T \\ \text{n} \times \text{n} \end{array}$$

$$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$$

Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least $n-m$ vectors that makes $\mathbf{Ax}=0$.

$$\begin{matrix} \mathbf{A} \\ m \times n \end{matrix} \quad \begin{matrix} \mathbf{x} \\ n \times 1 \end{matrix} = \begin{matrix} \mathbf{0} \end{matrix}$$

$$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$$

$$\begin{matrix} \mathbf{A} \\ m \times n \end{matrix} = \begin{matrix} \mathbf{U} \\ m \times m \end{matrix} \begin{matrix} \mathbf{D} \\ m \times n \end{matrix} \begin{matrix} \mathbf{V}^T \\ n \times n \end{matrix}$$

Dimensions: $m \times m$, $m \times n$, $n \times n$.
Arrows indicate dimensions: m (vertical), $n-m$ (horizontal).

$$\begin{matrix} \mathbf{A} \\ m \times n \end{matrix} \quad \begin{matrix} \mathbf{V}_{m+1:n} \\ n \times n \end{matrix} = \begin{matrix} \mathbf{0} \end{matrix} \quad \mathbf{V}_{m+1:n} = \text{null}(\mathbf{A})$$

Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least $n-m$ vectors that makes $\mathbf{Ax}=0$.

$$\begin{array}{c|c|c} \mathbf{A} & \mathbf{x} & = & \mathbf{0} \\ m \times n & n \times 1 & & \\ \hline & & & \end{array}$$

$$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$$

$$\begin{array}{c|c|c|c} \mathbf{A} & = & \mathbf{U} & \mathbf{V}^T \\ m \times n & & m \times m & m \times n \\ \hline & & \mathbf{D} & \\ & & m & n-m \\ & & & n \times n \end{array}$$

For a unique solution, \mathbf{A} should be $m \times (m+1)$

$$\begin{array}{c|c|c} \mathbf{A} & \mathbf{V}_{:, \text{end}} & = & \mathbf{0} & \mathbf{V}_{:, \text{end}} = \text{null}(\mathbf{A}) \\ \hline & & & & \end{array}$$

Singular Value Decomposition (SVD)

eqs > # unknowns

There exist no nullspace of \mathbf{A} .

$$\begin{array}{c|c|c} \mathbf{A} & \mathbf{x} & \approx \mathbf{0} \\ m \times n & n \times 1 & \\ \hline m > n \end{array}$$

$$\begin{array}{c|c|c|c} \mathbf{A} & = & \mathbf{U} & \mathbf{V}^T \\ m \times n & & m \times n & n \times n \\ & & \mathbf{D} & \\ & & n \times n & n \times n \end{array}$$

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \text{ subject to } \|\mathbf{x}\| = 1$$

Singular Value Decomposition (SVD)

eqs > # unknowns

There exist no nullspace of \mathbf{A} .

$$\begin{array}{c|c|c} \mathbf{A} & \mathbf{x} & \approx \mathbf{0} \\ m \times n & n \times 1 & \\ \hline & m > n & \end{array}$$

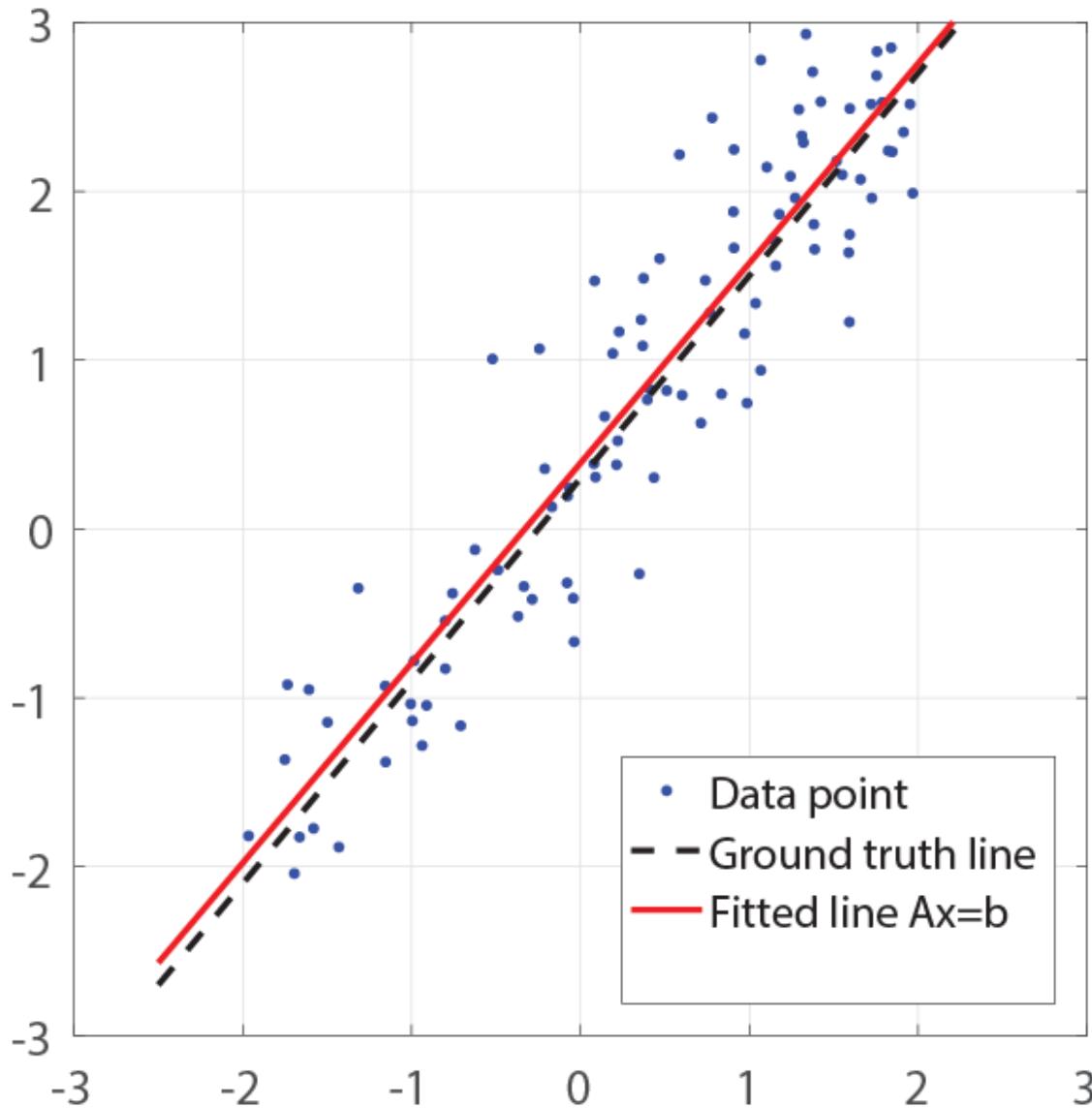
$$\begin{array}{c|c|c|c} \mathbf{A} & = & \mathbf{U} & \mathbf{V}^T \\ m \times n & & m \times n & n \times n \\ & & \mathbf{D} & \\ & & n \times n & n \times n \\ \hline & & \leftarrow \text{Last row} & \end{array}$$

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \text{ subject to } \|\mathbf{x}\| = 1$$

Approximated nullspace of \mathbf{A} :

$$\mathbf{V}_{:, \text{end}}$$

Line Fitting ($Ax=b$)

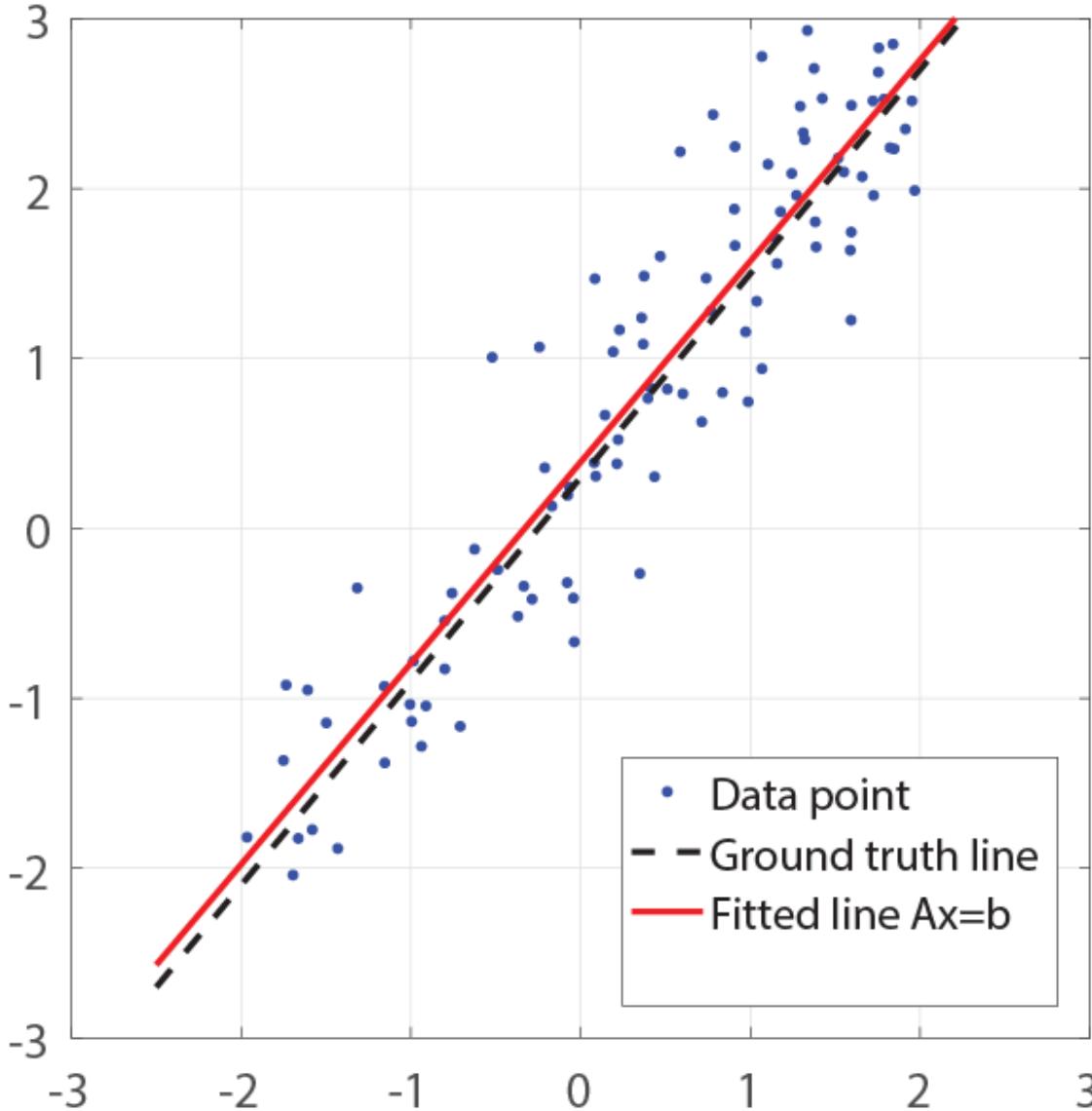


Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$
Find the best line: (m, d)

$$\begin{aligned}v_1 &\approx mu_1 + d \\v_2 &\approx mu_2 + d \\&\vdots \\v_n &\approx mu_n + d\end{aligned}$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Line Fitting ($Ax=b$)



Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: (a, b, c) (m, d)

$$au_1 + bv_1 + c \approx 0$$

$$au_2 + bv_2 + c \approx 0$$

⋮

$$au_n + bv_n + c \approx 0$$

$$v_1 \approx mu_1 + d$$

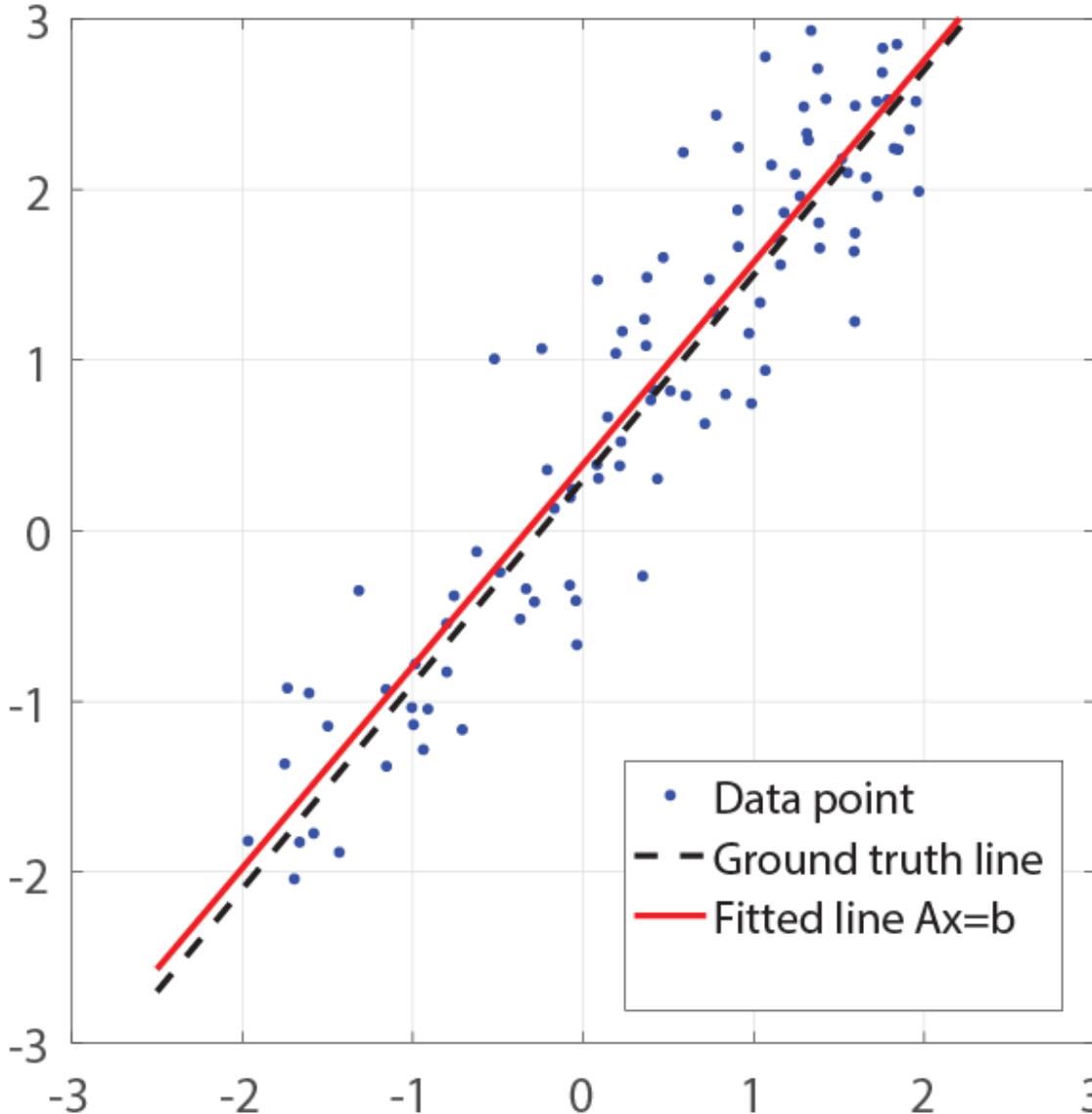
$$v_2 \approx mu_2 + d$$

⋮

$$v_n \approx mu_n + d$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Line Fitting ($Ax=b$)



Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: (a, b, c) (m, d)

$$au_1 + bv_1 + c \approx 0$$

$$au_2 + bv_2 + c \approx 0$$

⋮

$$au_n + bv_n + c \approx 0$$

$$v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

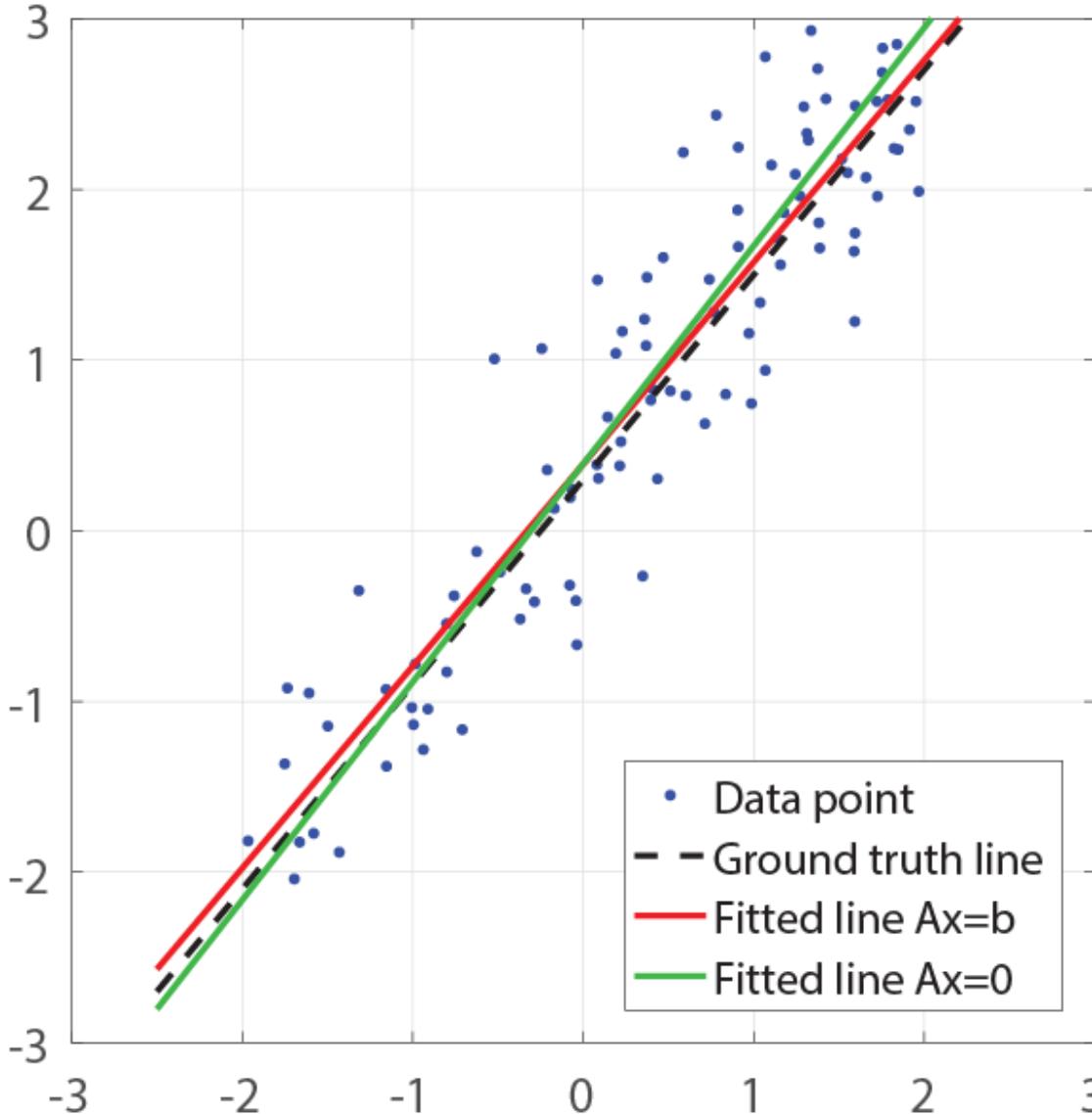
⋮

$$v_n \approx mu_n + d$$

$$\begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ \vdots & \vdots & 1 \\ u_n & v_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Line Fitting ($Ax=b$)



Given points: $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$

Find the best line: (a, b, c)

$$au_1 + bv_1 + c \approx 0$$

$$au_2 + bv_2 + c \approx 0$$

⋮

$$au_n + bv_n + c \approx 0$$

$$v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

⋮

$$v_n \approx mu_n + d$$

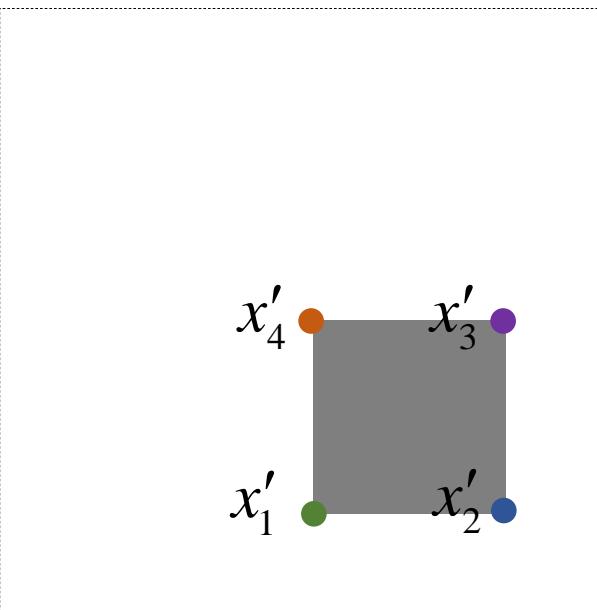
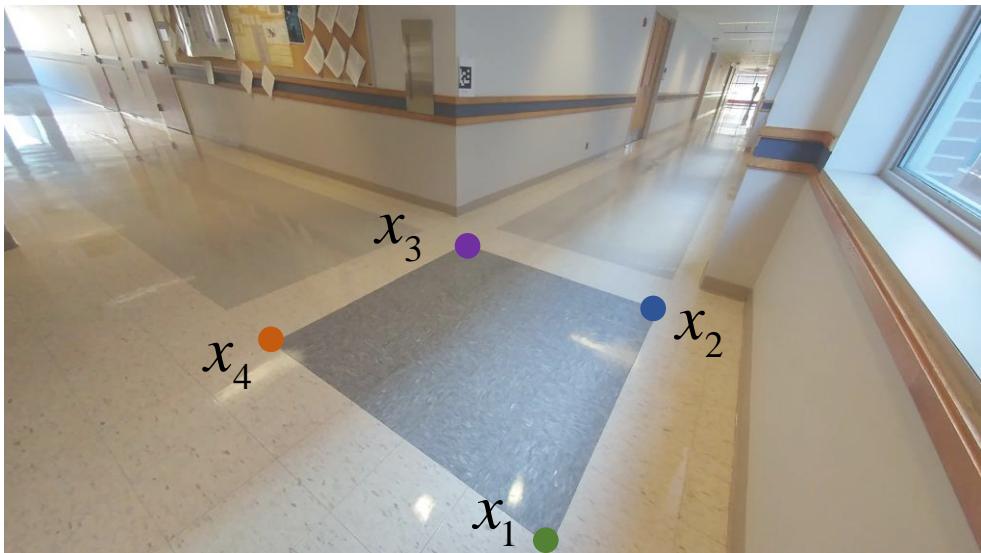
$$\begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ \vdots & \vdots & \vdots \\ u_n & v_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$



How to compute homography?

HOMOGRAPHY COMPUTATION



$$\begin{bmatrix} u_1 & v_1 & 1 & 0 & 0 & 0 & -u_1u'_1 & -v_1u'_1 \\ 0 & 0 & 0 & u_1 & v_1 & 1 & -u_1v'_1 & -v_1v'_1 \\ u_4 & v_4 & 1 & 0 & 0 & 0 & -u_4u'_4 & -v_4u'_4 \\ 0 & 0 & 0 & u_4 & v_4 & 1 & -u_4v'_4 & -v_4v'_4 \end{bmatrix} \mathbf{A} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} u'_1 \\ v'_1 \\ u'_4 \\ v'_4 \end{bmatrix} \mathbf{b}$$

$$Ax = b \quad \longrightarrow \quad x = (A^T A)^{-1} A^T b$$

Homography Computation



$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Homography Computation



$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Homography Computation



$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\begin{aligned} \rightarrow & h_{11}u_x + h_{12}u_y + h_{13} + h_{31}u_x v_x + h_{32}u_y v_x + h_{33}v_x = 0 \\ & h_{21}u_x + h_{22}u_y + h_{23} + h_{31}u_x v_y + h_{32}u_y v_y + h_{33}v_y = 0 \end{aligned}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Unknowns: h_{11}, \dots, h_{33}

Equations: 2 per correspondence

Homography Computation



$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\rightarrow h_{11}u_x + h_{12}u_y + h_{13} - h_{31}u_xv_x - h_{32}u_yv_x - h_{33}v_x = 0$$

$$h_{21}u_x + h_{22}u_y + h_{23} - h_{31}u_xv_y - h_{32}u_yv_y - h_{33}v_y = 0$$

$$\rightarrow \begin{bmatrix} u_x & u_y & 1 & -u_xv_x & -u_yv_x & -v_x \\ & & u_x & u_y & 1 & -u_xv_y & -u_yv_y & -v_y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Homography Computation



$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\rightarrow h_{11}u_x + h_{12}u_y + h_{13} - h_{31}u_xv_x - h_{32}u_yv_x - h_{33}v_x = 0$$

$$h_{21}u_x + h_{22}u_y + h_{23} - h_{31}u_xv_y - h_{32}u_yv_y - h_{33}v_y = 0$$

$$\rightarrow \begin{bmatrix} u_x & u_y & 1 & -u_xv_x & -u_yv_x & -v_x \\ & & u_x & u_y & -u_xv_y & -u_yv_y & -v_y \end{bmatrix} \mathbf{A} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2x9

Recall: Singular Value Decomposition (SVD)

eqs < # unknowns

There exist at least $n-m$ vectors that makes $\mathbf{Ax}=0$.

$$\begin{matrix} \mathbf{A} \\ m \times n \end{matrix} \quad \begin{matrix} \mathbf{x} \\ n \times 1 \end{matrix} = \begin{matrix} \mathbf{0} \end{matrix}$$

$$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$$

$$\begin{matrix} \mathbf{A} \\ m \times (m+1) \end{matrix} \quad \begin{matrix} \mathbf{v}_{:,end} \\ m+1 \times 1 \end{matrix} = \begin{matrix} \mathbf{0} \end{matrix} \quad \mathbf{v}_{:,end} = \text{null}(\mathbf{A})$$

For a unique solution, \mathbf{A} should be $m \times (m+1)$

Homography Computation

How many correspondences are needed?



$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_x & u_y & 1 \\ u_x & u_y & 1 \end{bmatrix} \begin{bmatrix} -u_x v_x & -u_y v_x & -v_x \\ -u_x v_y & -u_y v_y & -v_y \end{bmatrix} \mathbf{A} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2x9

Homography Computation

How many correspondences are needed? 4

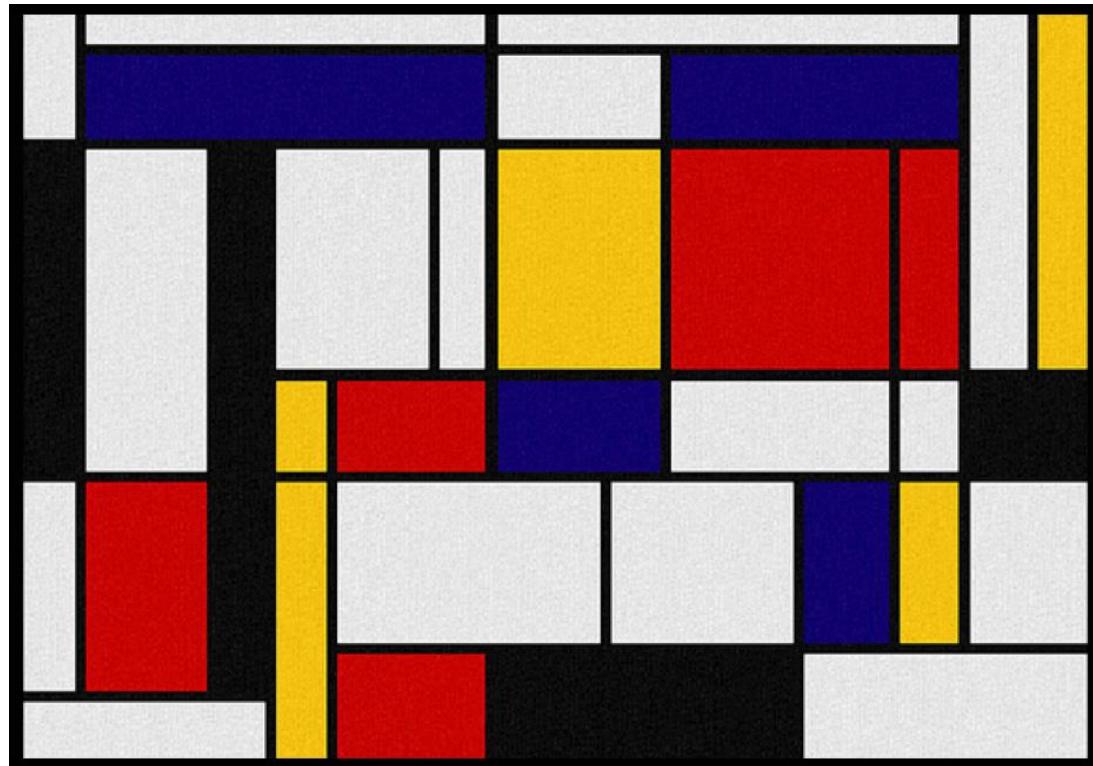


$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

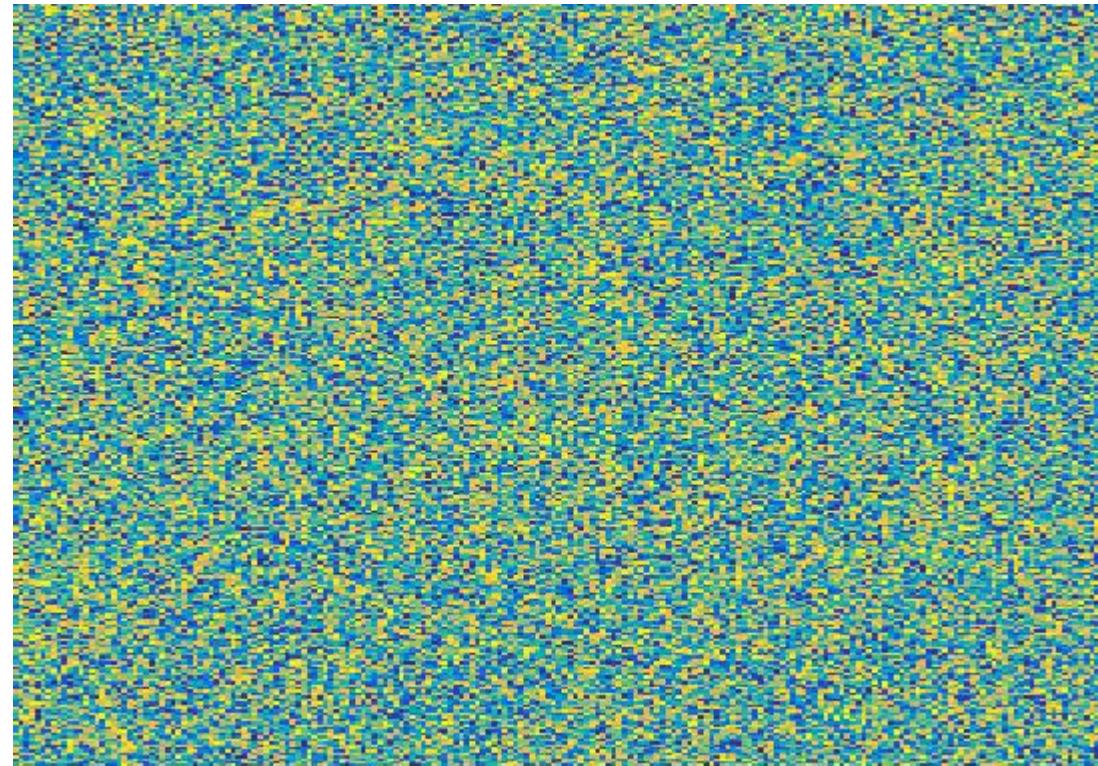
$$\begin{bmatrix} u_x & u_y & 1 \\ u_x & u_y & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} -u_x v_x & -u_y v_x & -v_x \\ -u_x v_y & -u_y v_y & -v_y \end{bmatrix} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

8x9

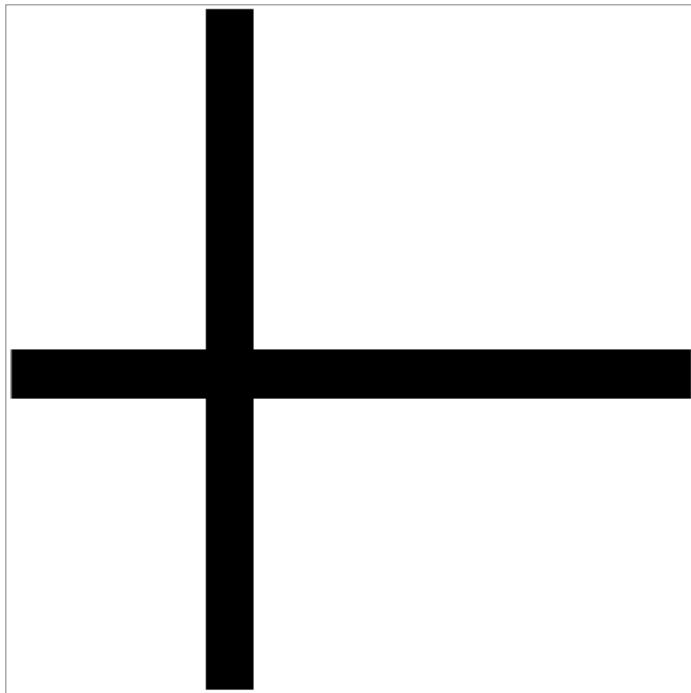
$$\mathbf{x} = \mathbf{V}_{:,end}^T = \text{null}(\mathbf{A})$$



VS

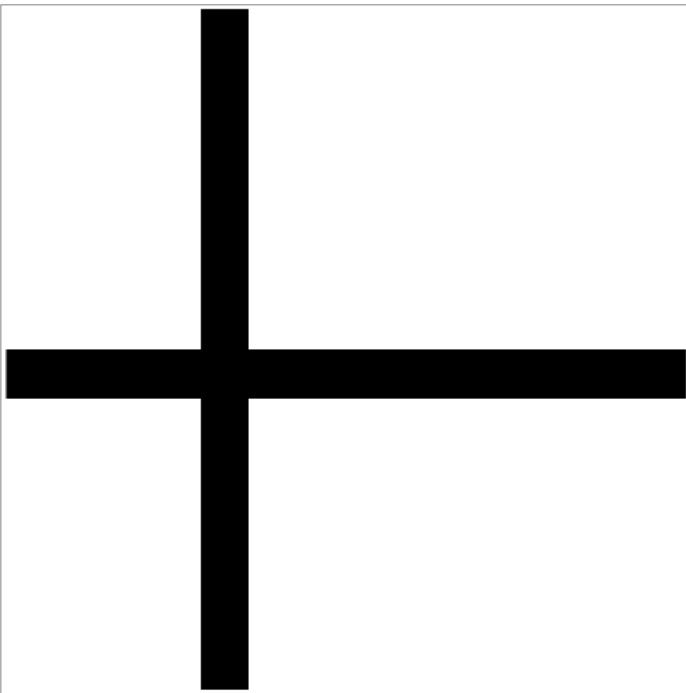


More SVD



14x14

More SVD

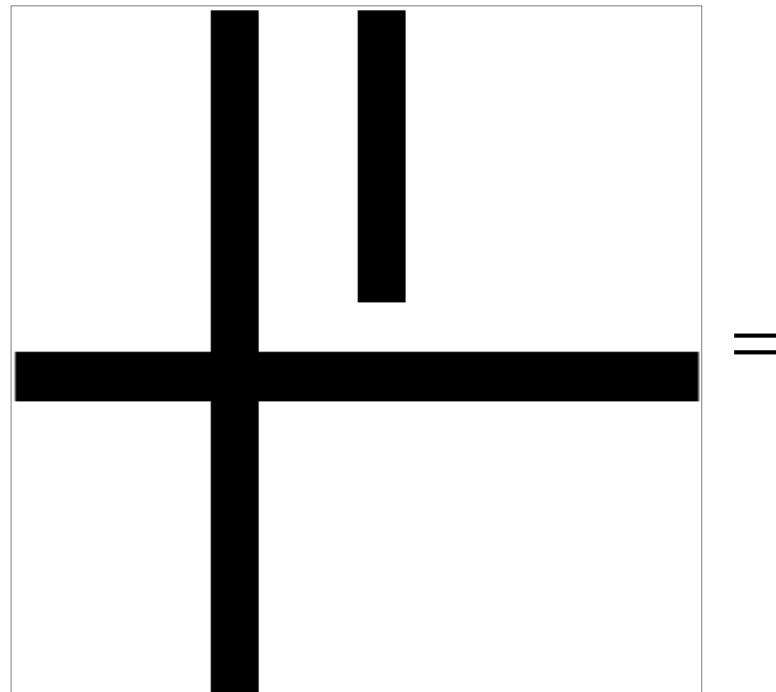


14x14

$$= \begin{matrix} & \text{red square} \\ \text{blue rectangle} & \end{matrix}$$



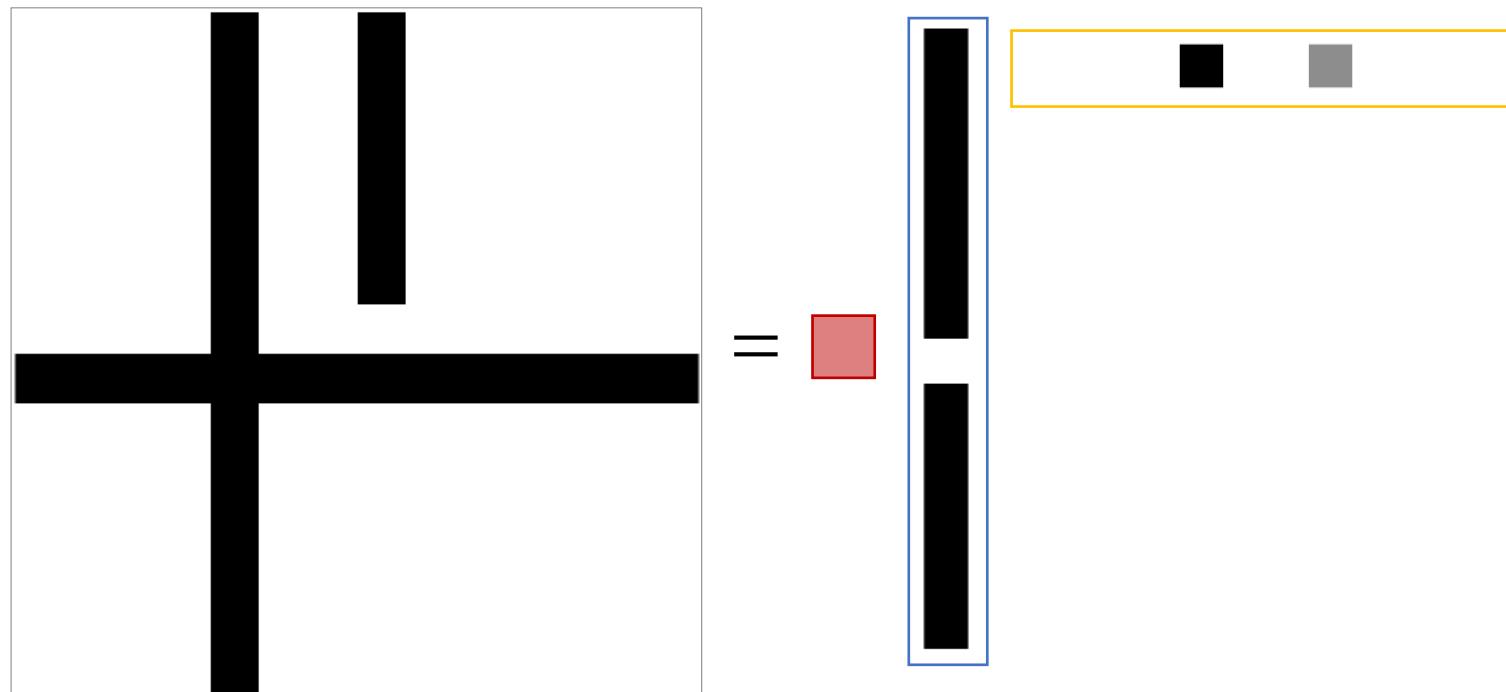
More SVD



=

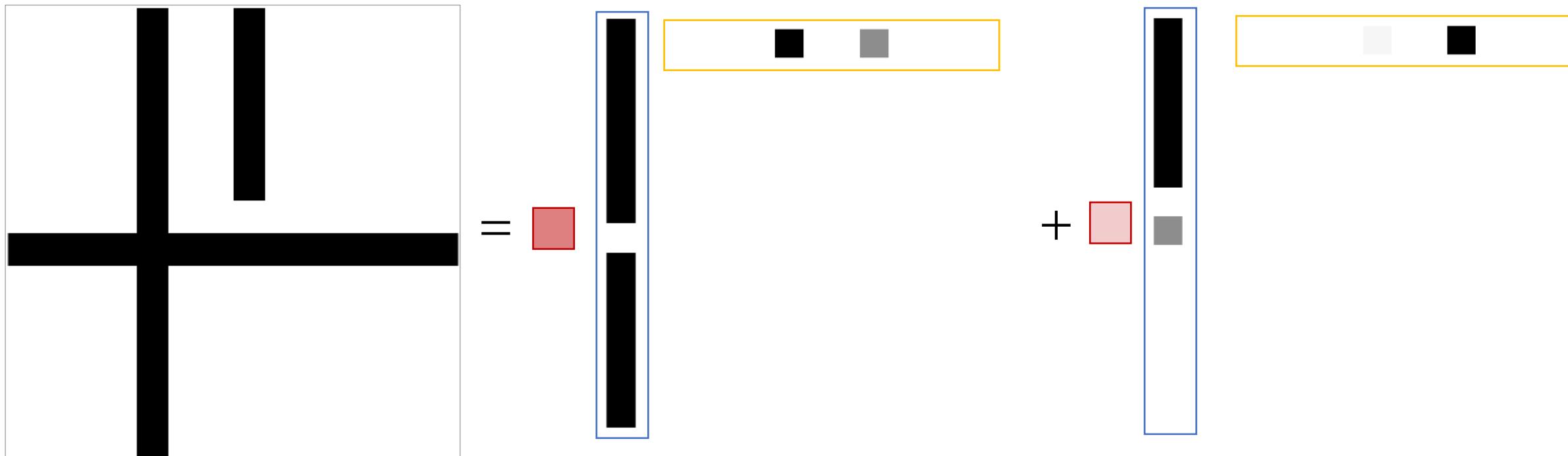
14x14

More SVD



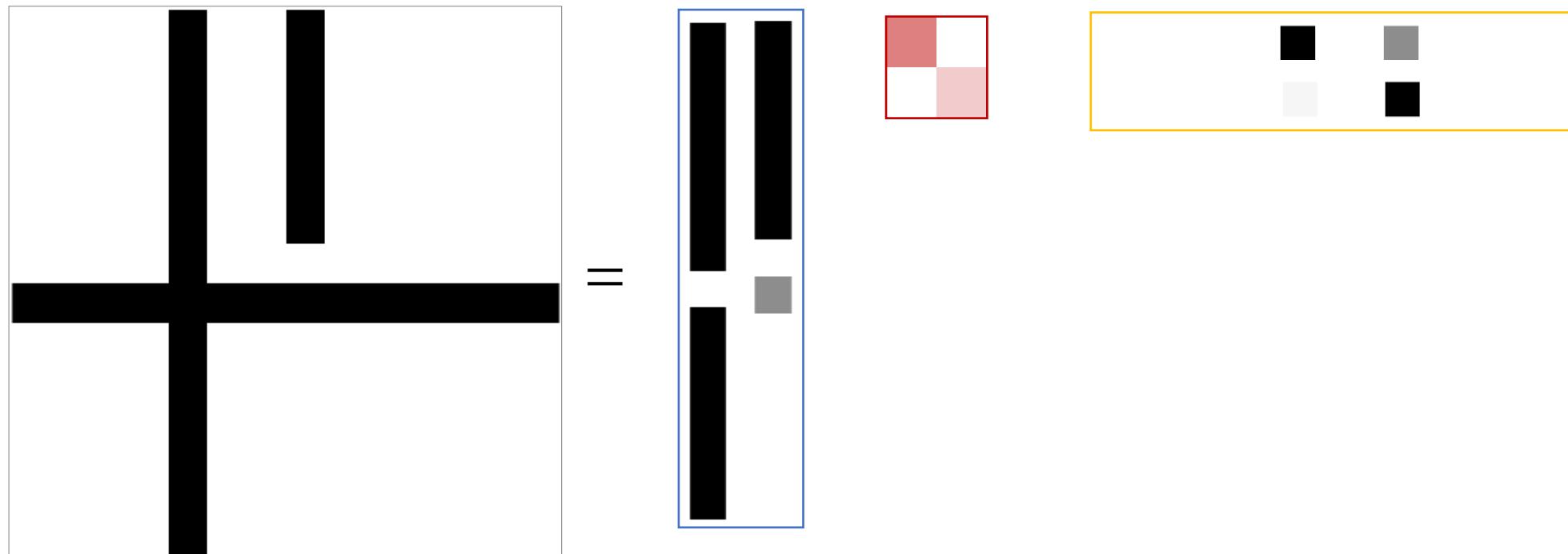
14x14

More SVD



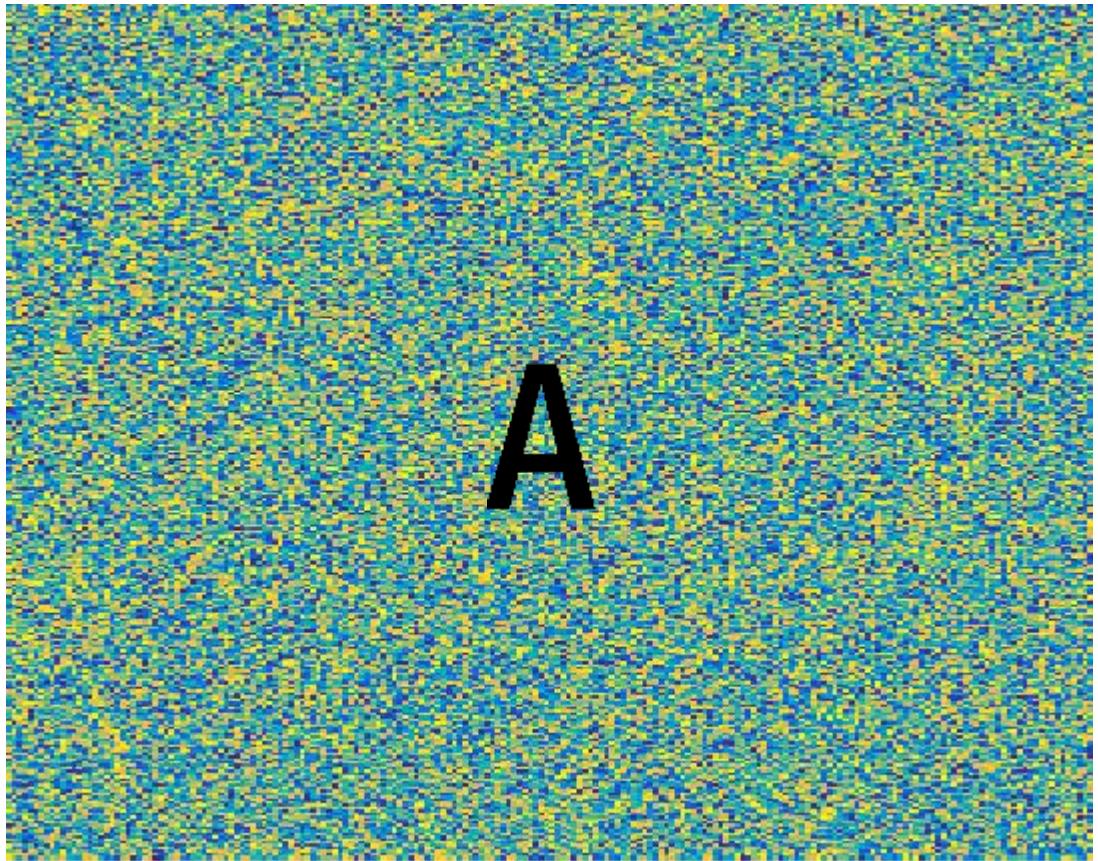
14x14

More SVD



14x14

Random Matrix SVD

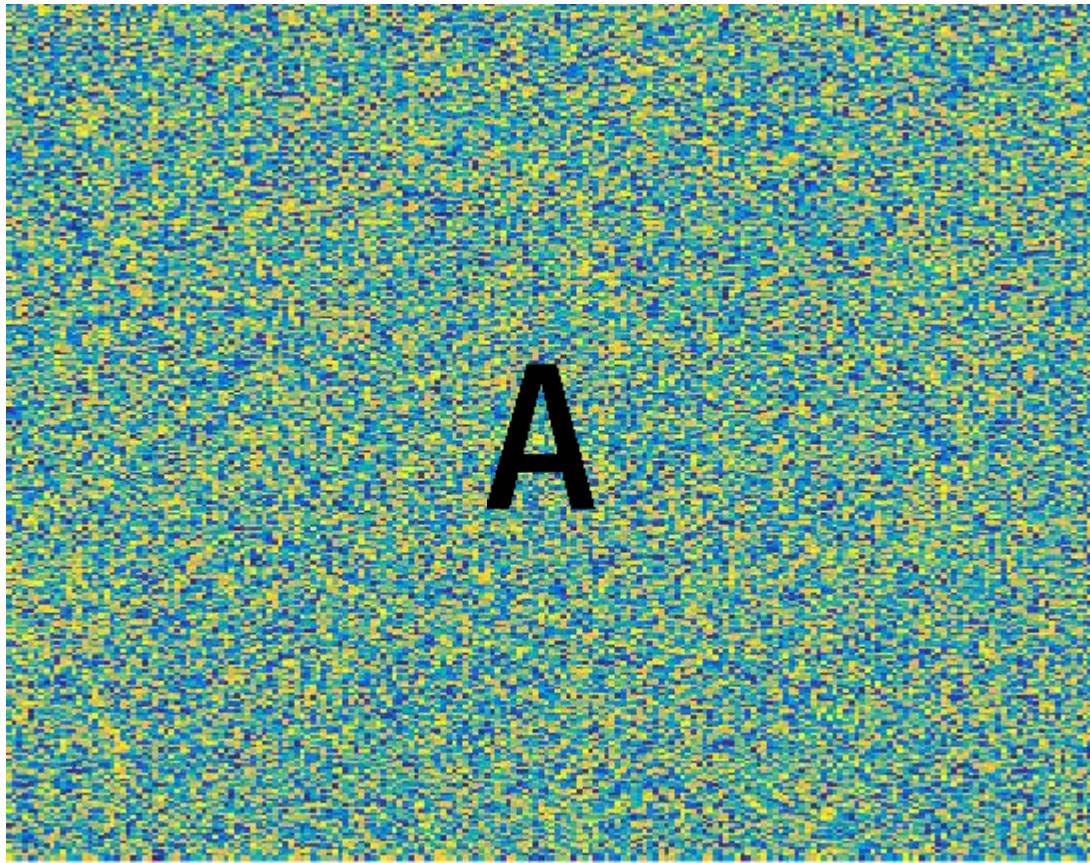


Random matrix

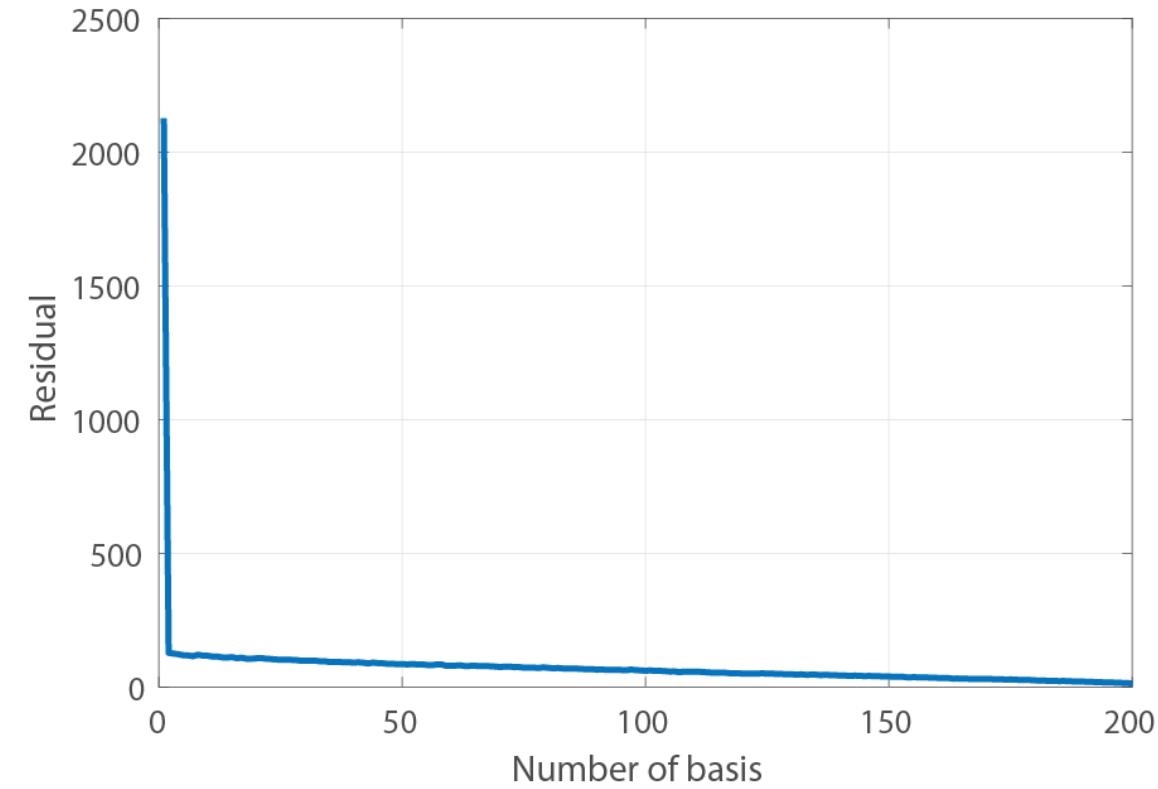
$$A = U D V^T$$

The diagram illustrates the Singular Value Decomposition (SVD) of matrix A. It shows the factorization $A = U D V^T$. The matrices U and V^T are represented by solid light blue and yellow rectangles respectively. The diagonal matrix D is shown as a white square with a red border, containing several red squares of decreasing size, with the letter 'D' in the center.

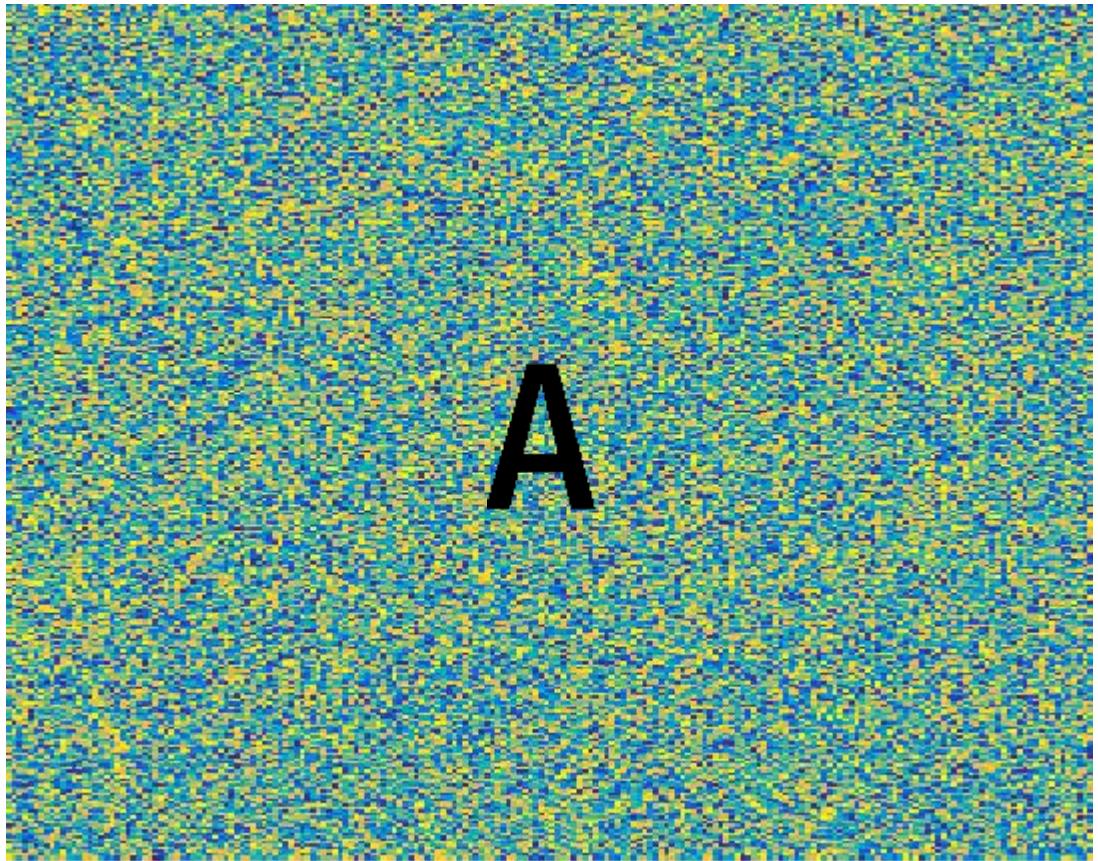
Singular Values



Random matrix



Singular Values

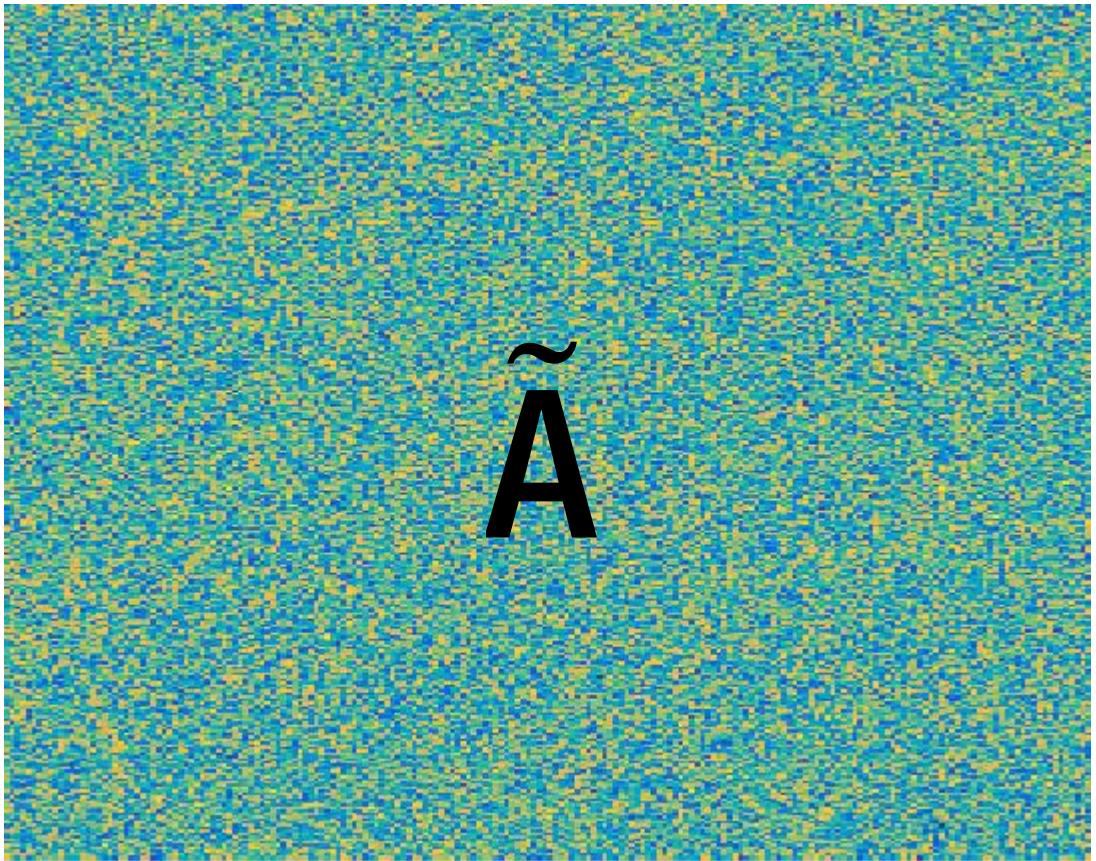


Random matrix

$$= \begin{matrix} U & D \\ V^T & \end{matrix}$$

The matrix A is decomposed into three components: U , D , and V^T . U is a unitary matrix, D is a diagonal matrix containing the singular values, and V^T is the transpose of a unitary matrix V .

SVD Matrix Approximation

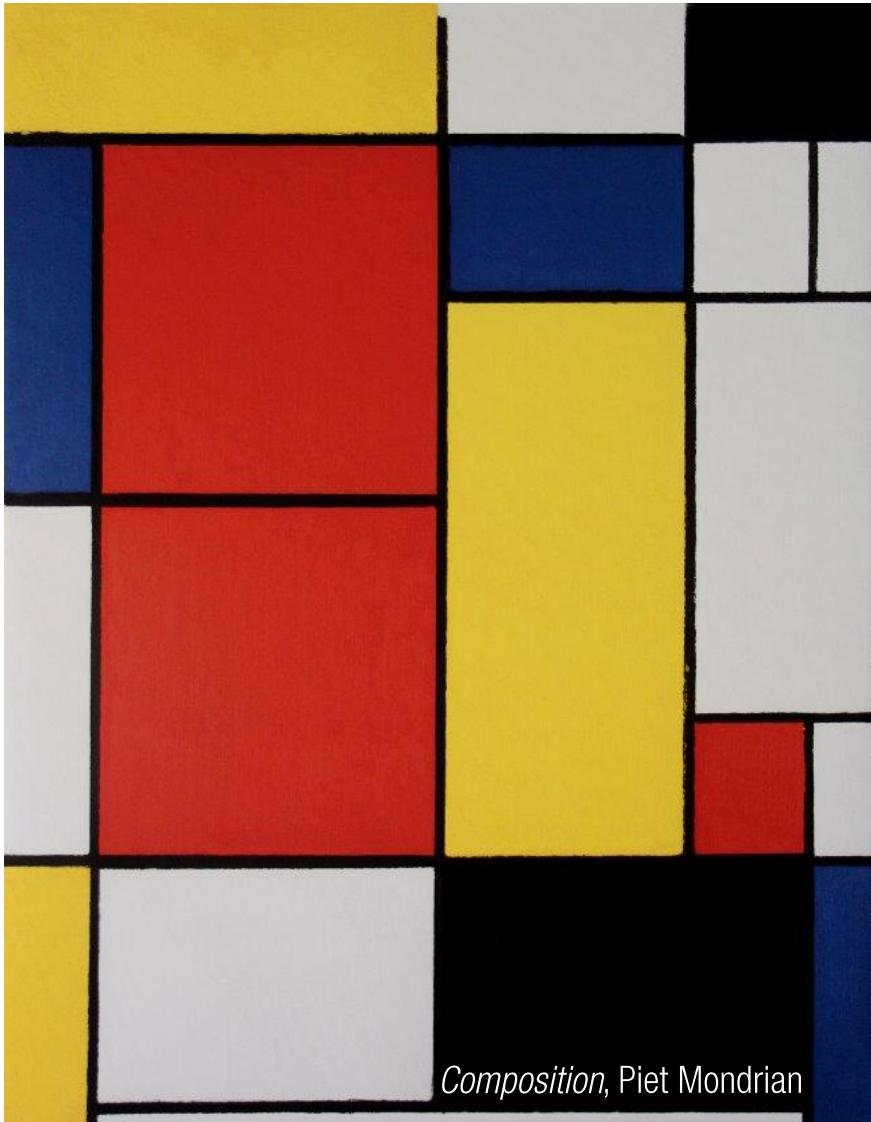


Reconstructed matrix

$$= \mathbf{U} \mathbf{D} \mathbf{V}^T$$

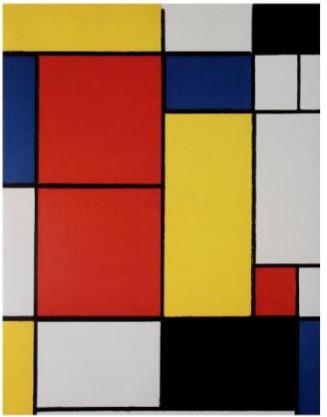
The equation shows the decomposition of the matrix $\tilde{\mathbf{A}}$ into three matrices: \mathbf{U} , \mathbf{D} , and \mathbf{V}^T .
- \mathbf{U} is a blue rectangular matrix.
- \mathbf{D} is a diagonal matrix with dark blue values along the main diagonal.
- \mathbf{V}^T is a light blue rectangular matrix.

Mondrian Painting SVD

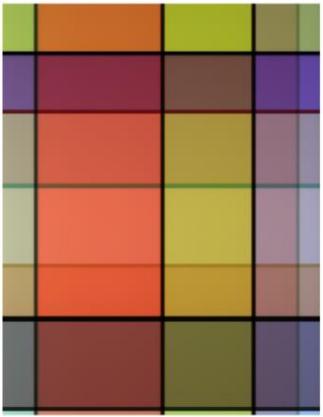


$$= \begin{matrix} U & D & V^T \\ m \times n & n \times n & n \times n \end{matrix}$$

Mondrian Painting SVD Approximation

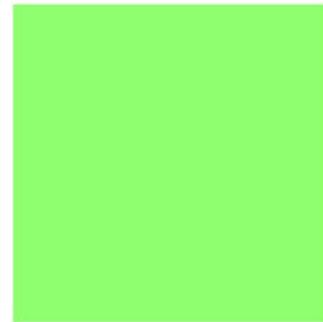


Ground truth



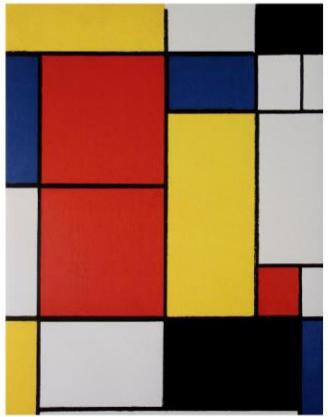
Number of basis: 1

MondrianSVD.m

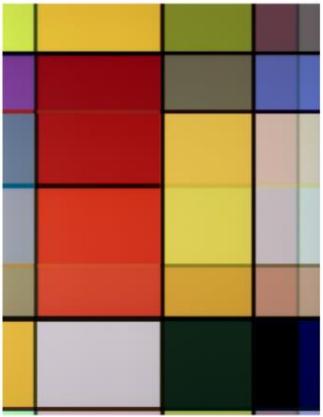


$$A = U D V^T$$

Mondrian Painting SVD Approximation

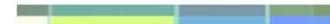
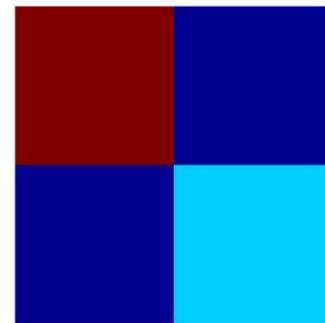
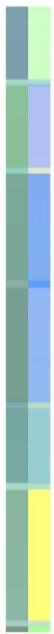


Ground truth



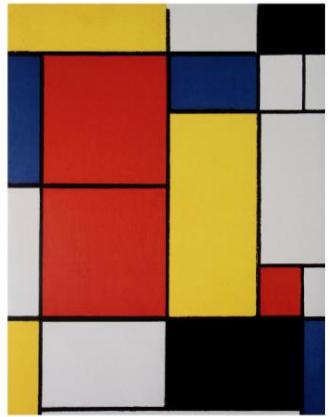
Number of basis: 2

MondrianSVD.m

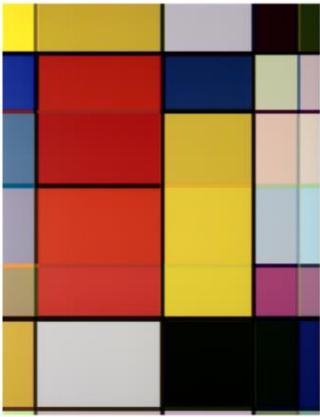


$$A = U \quad D \quad V^T$$

Mondrian Painting SVD Approximation

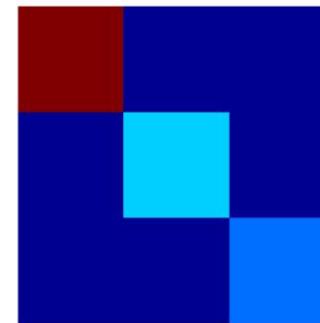
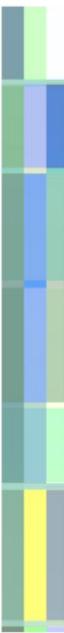


Ground truth



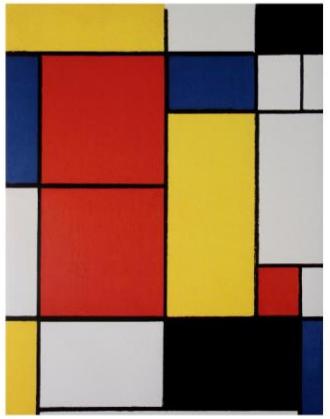
Number of basis: 3

MondrianSVD.m

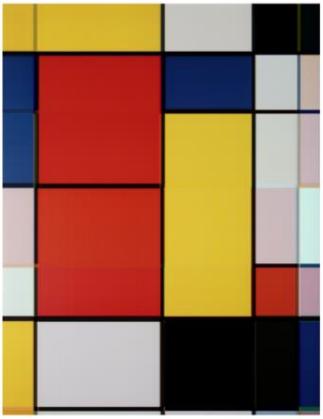


$$A = U D V^T$$

Mondrian Painting SVD Approximation

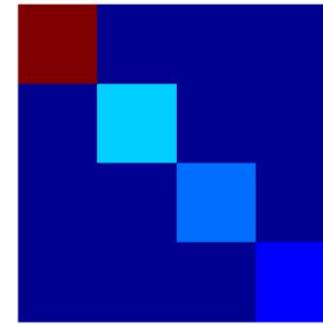


Ground truth



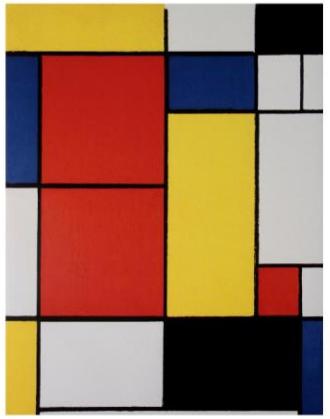
Number of basis: 4

MondrianSVD.m

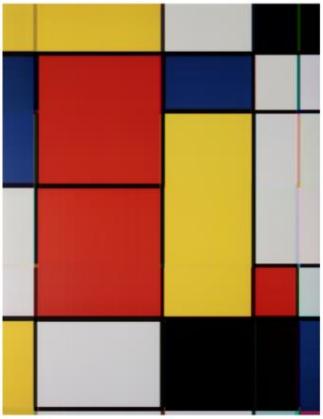


$$A = U D V^T$$

Mondrian Painting SVD Approximation

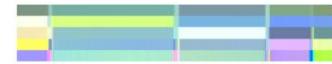
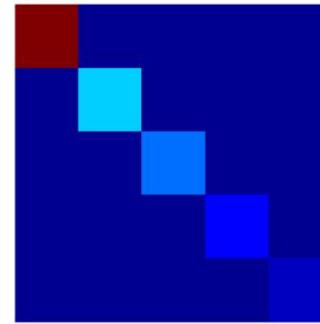
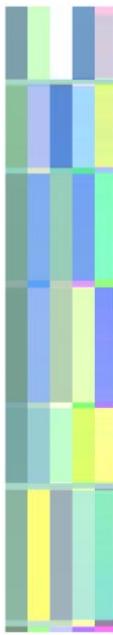


Ground truth



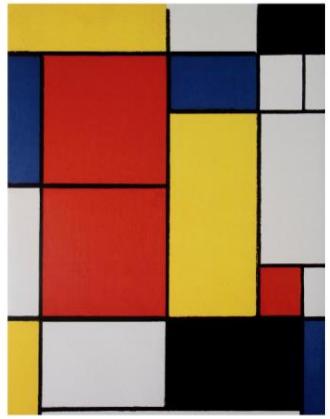
Number of basis: 5

MondrianSVD.m

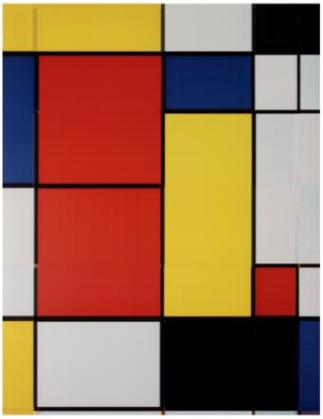


$$A = U D V^T$$

Mondrian Painting SVD Approximation

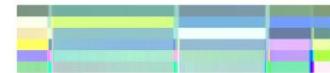
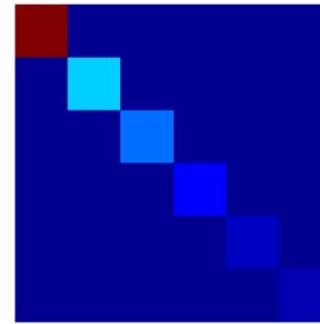
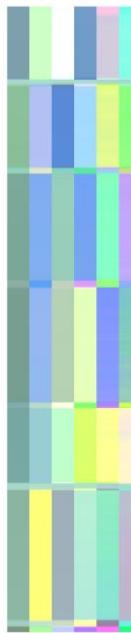


Ground truth



Number of basis: 6

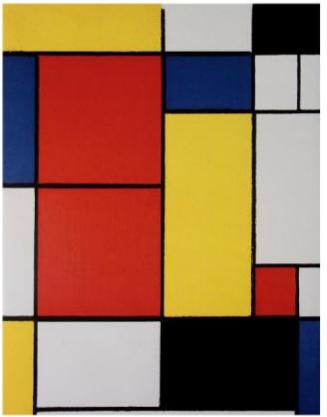
MondrianSVD.m



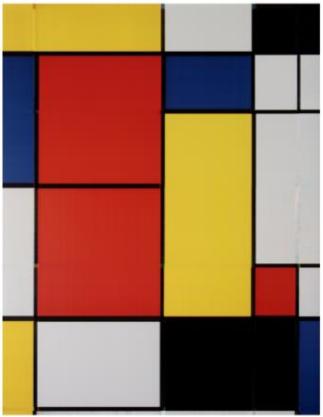
$$A = U \quad D \quad V^T$$

Mondrian Painting SVD Approximation

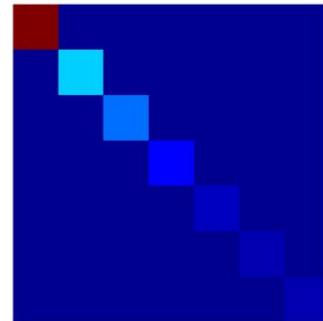
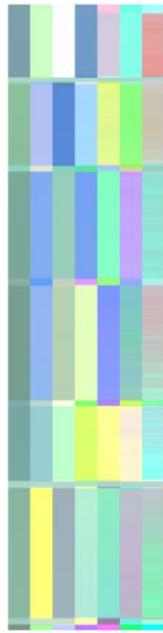
MondrianSVD.m



Ground truth

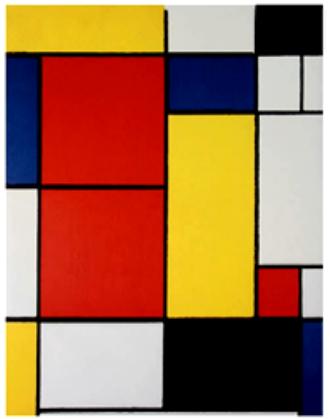


Number of basis: 7

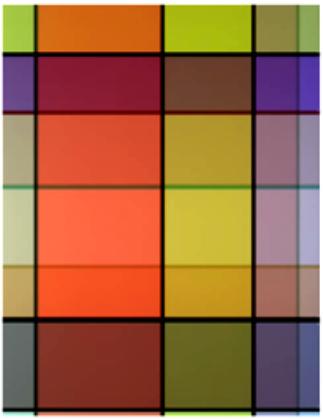


$$A = U D V^T$$

Mondrian Painting SVD Approximation

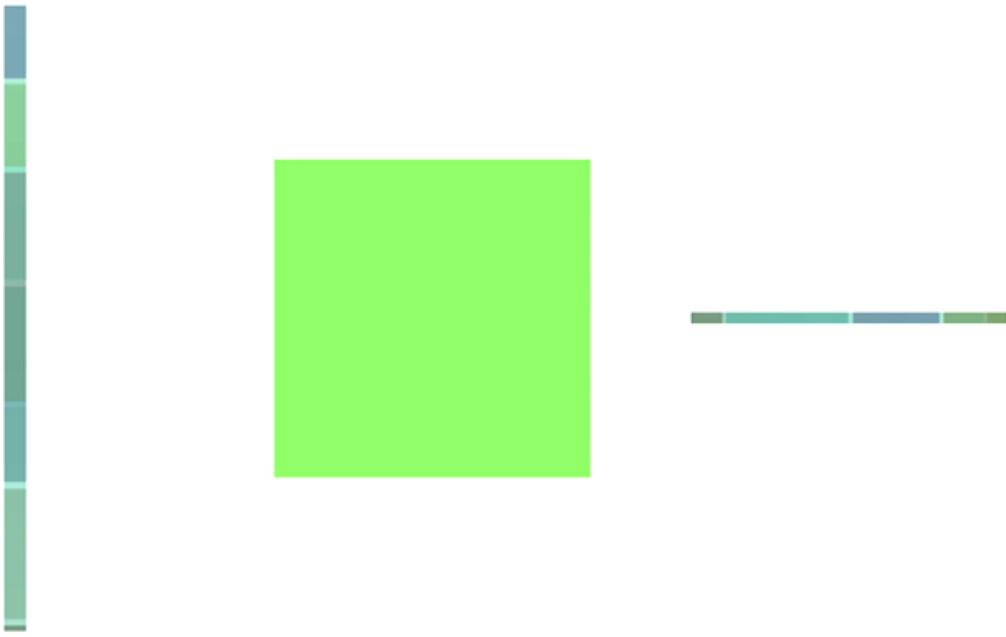


Ground truth



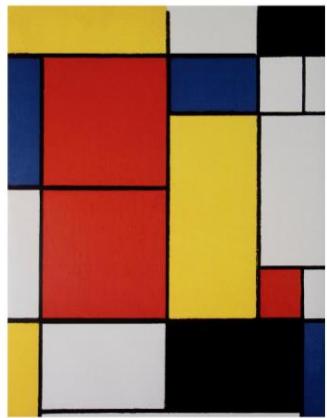
Number of basis: 1

MondrianSVD.m

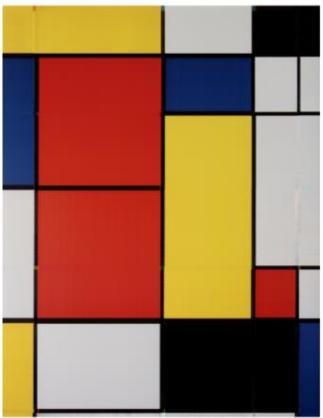


$$A = U D V^T$$

Reconstruction Error



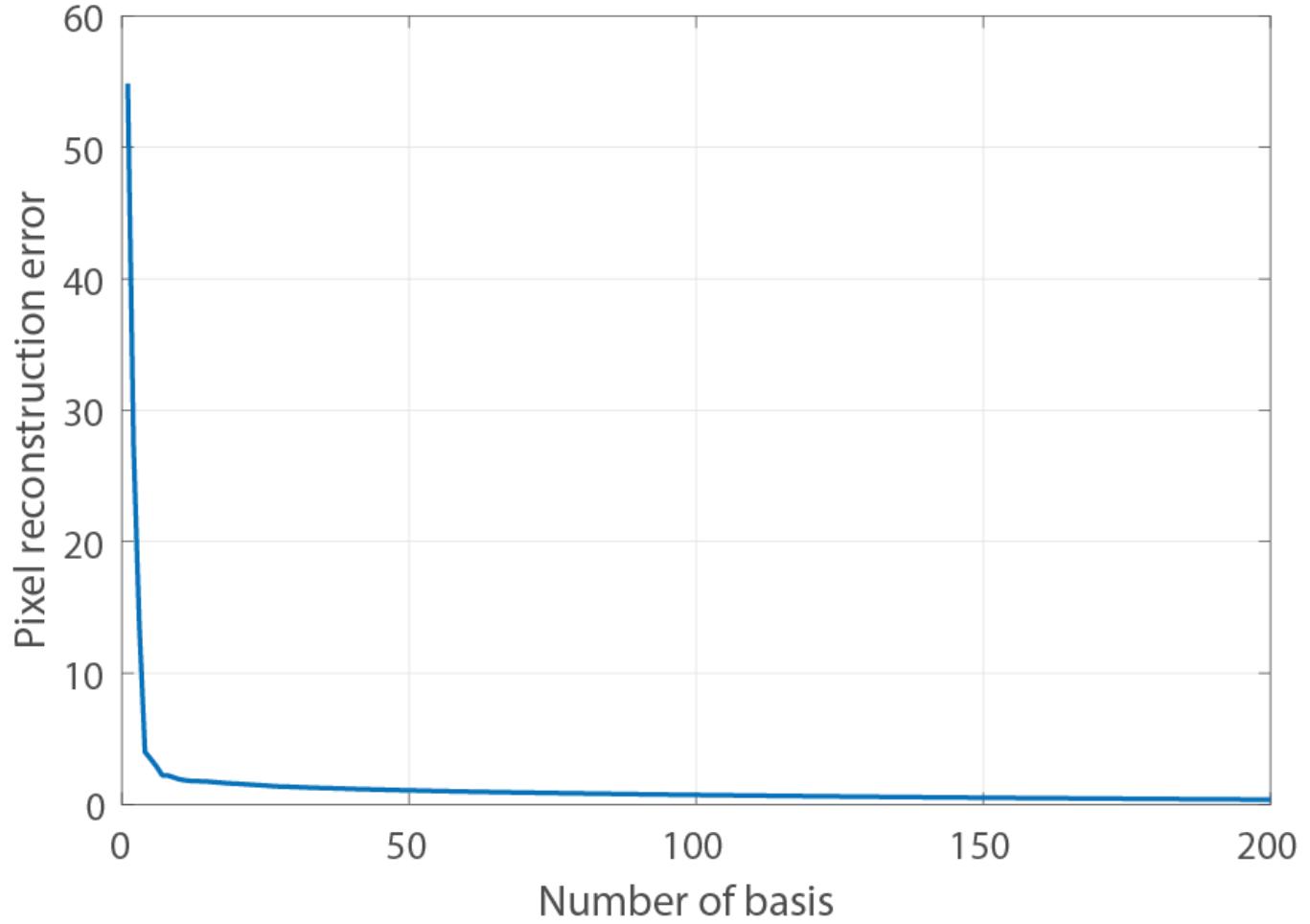
Ground truth



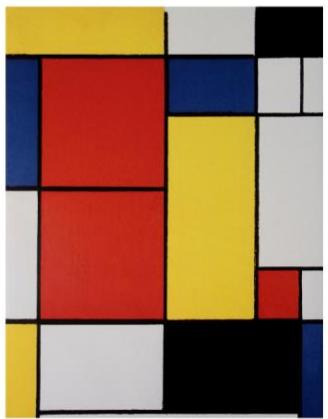
Number of basis: 7

A

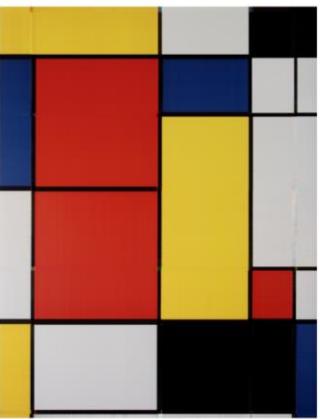
MondrianSVD.m



Rotation Matrix



Ground truth



Number of basis: 7

A

MondrianSVD.m

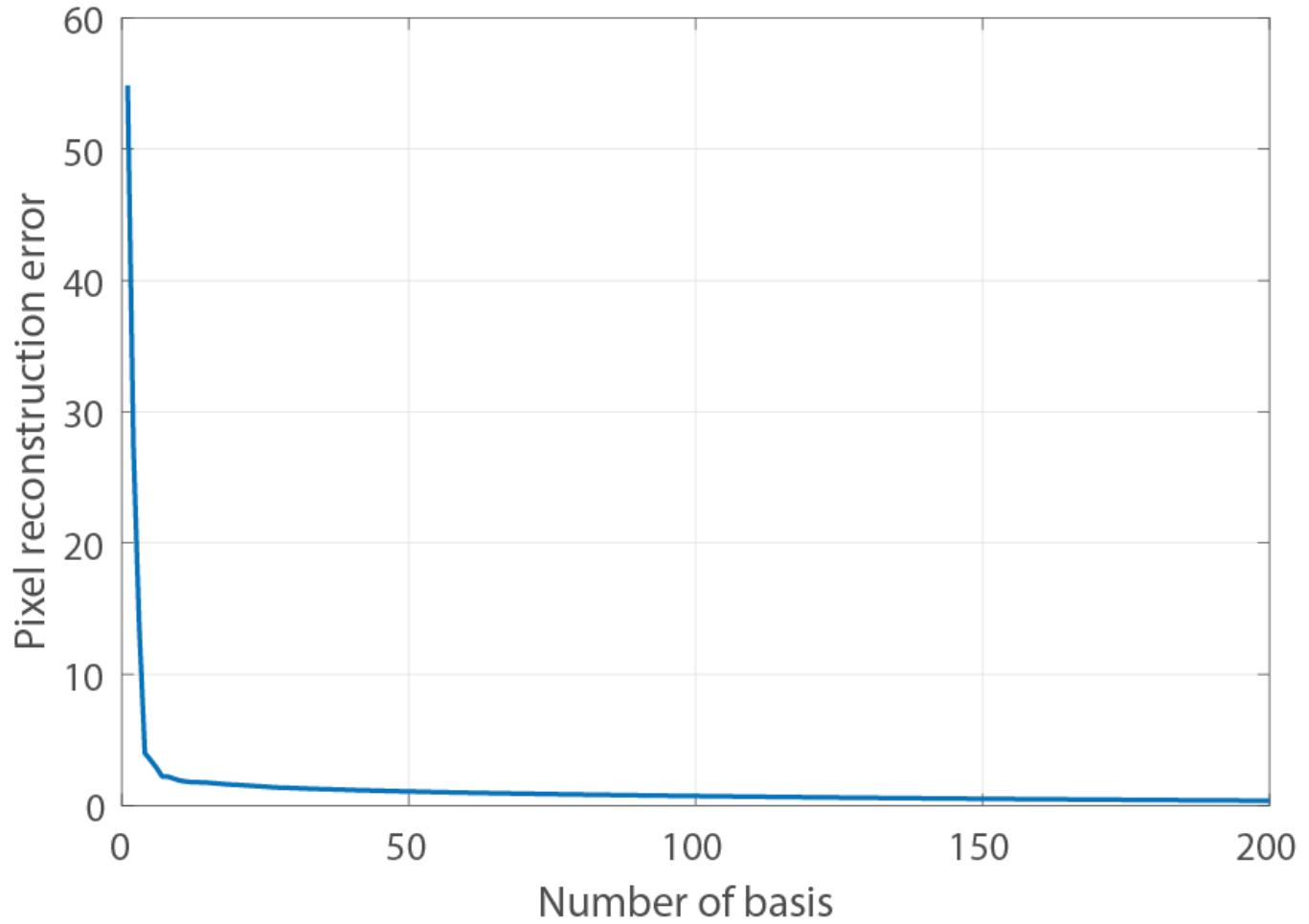
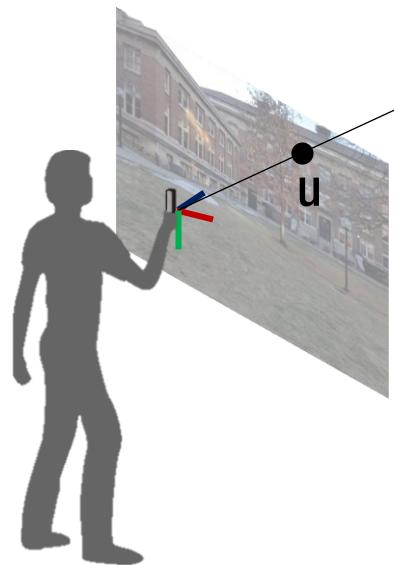
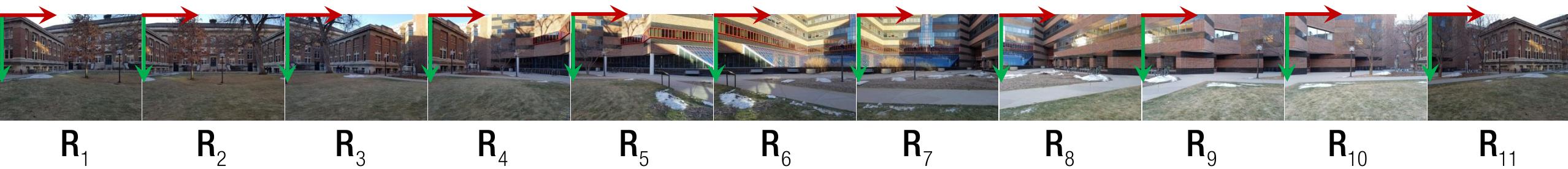


Image Panorama (Cylindrical Projection)



$$\mathbf{x} = \lambda \mathbf{K}^{-1} \mathbf{R}^T \mathbf{u}$$



$$\mathbf{R} = \mathbf{K}^{-1} \mathbf{H} \mathbf{K}$$

Rotation matrix has to be orthogonal matrix!!

$$\mathbf{R} = \mathbf{U} \mathbf{D} \mathbf{V}^T$$

$$\mathbf{R}_{orth} = \mathbf{U} \mathbf{V}^T$$