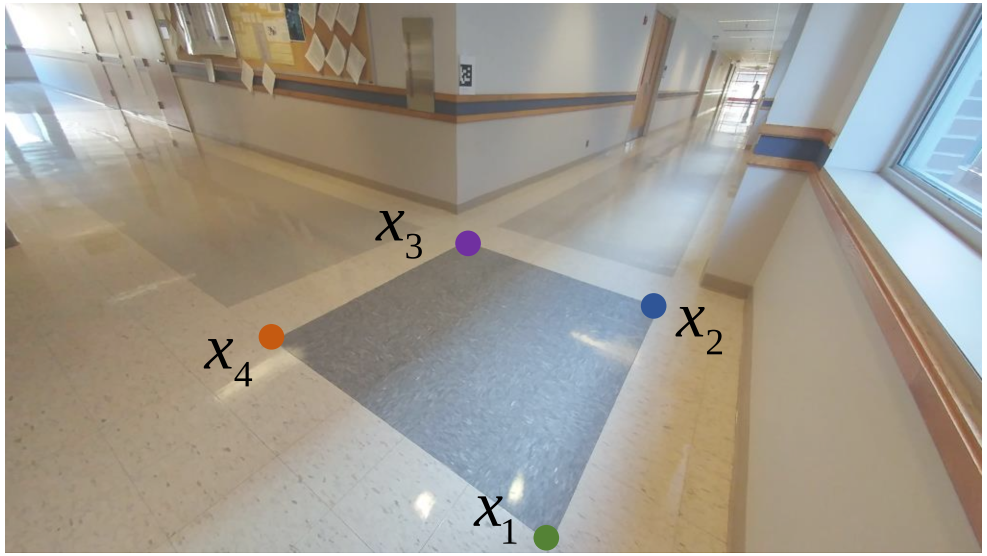
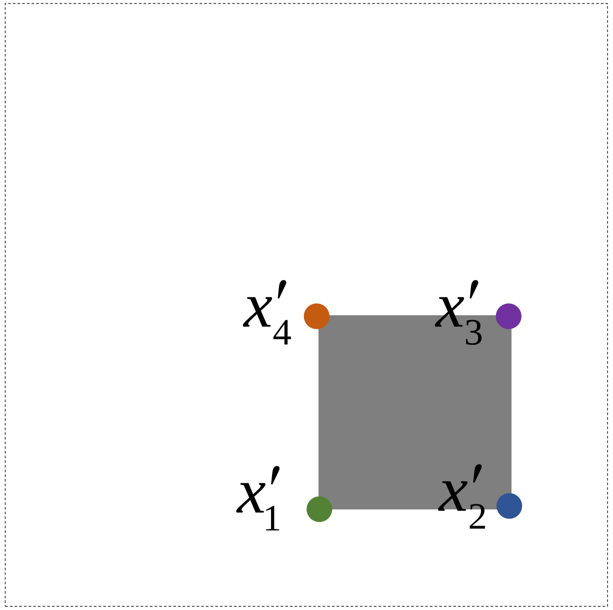
# *Linear Estimation*

Hyun Soo Park

# Homography Computation

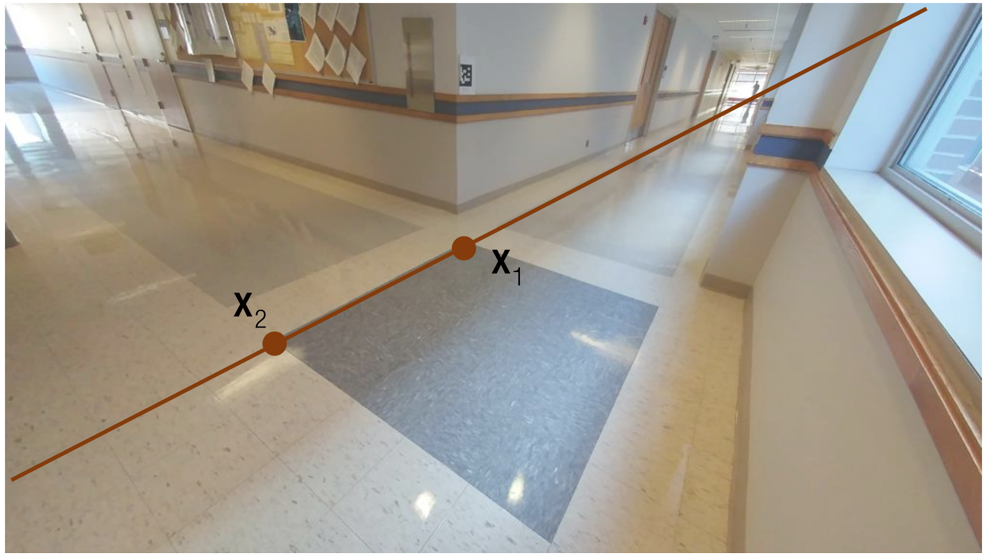



$$\underbrace{\begin{bmatrix} u_1 & v_1 & 1 & 0 & 0 & 0 & -u_1u'_1 & -v_1u'_1 \\ 0 & 0 & 0 & u_1 & v_1 & 1 & -u_1v'_1 & -v_1v'_1 \\ & & & & \vdots & & & \\ u_4 & v_4 & 1 & 0 & 0 & 0 & -u_4u'_4 & -v_4u'_4 \\ 0 & 0 & 0 & u_4 & v_4 & 1 & -u_4v'_4 & -v_4v'_4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} u'_1 \\ v'_1 \\ \vdots \\ u'_4 \\ v'_4 \end{bmatrix}}_{b}$$

$$Ax = b \longrightarrow x = \left(A^T A\right)^{-1} A^T b$$

# Point-Point in Image

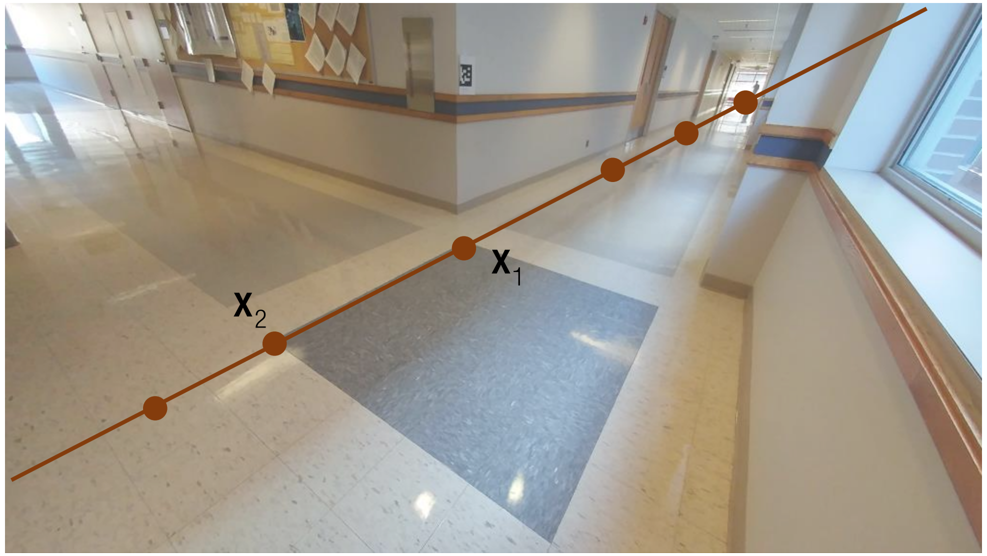


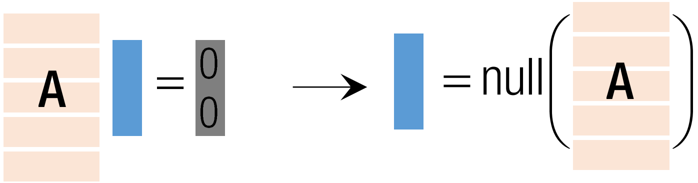

$$\begin{bmatrix}\mathbf{x}_1^{\mathsf{T}} \\ \mathbf{x}_2^{\mathsf{T}}\end{bmatrix}\mathbf{l} = \mathbf{0}$$

$$\mathbf{A}\,\blacksquare = \begin{bmatrix}0\\0\end{bmatrix} \longrightarrow \blacksquare = \text{null}\left(\mathbf{A}\right) \quad \text{or} \quad \mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$

3x2

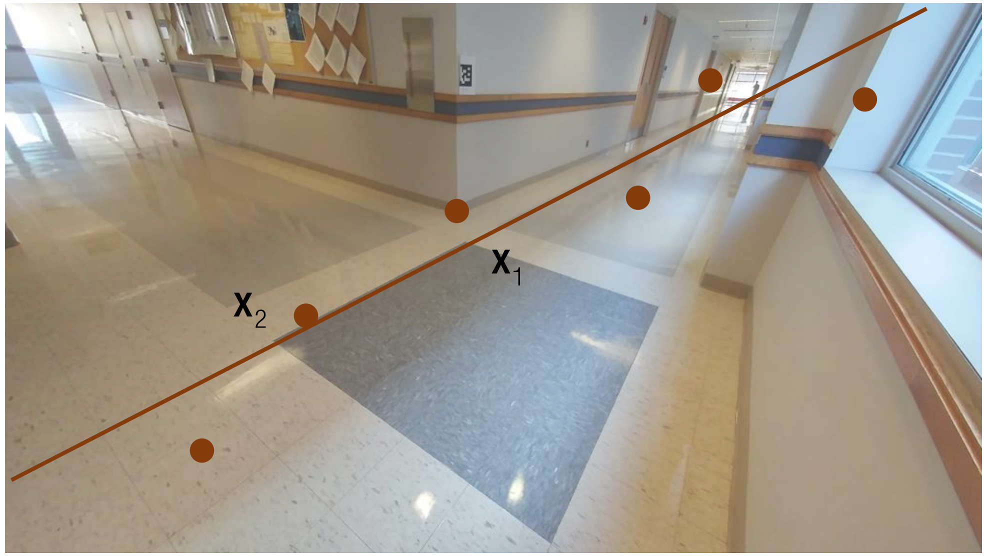
# Point-Point in Image

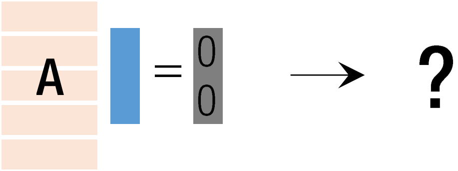



or $\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$

# Line Fitting

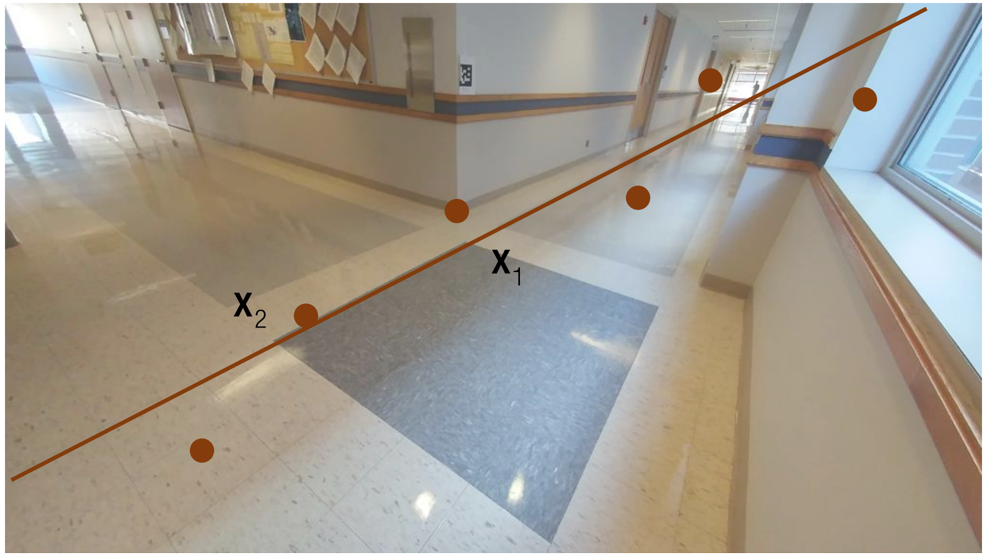


# Line Fitting

Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: $(a, b, c)$



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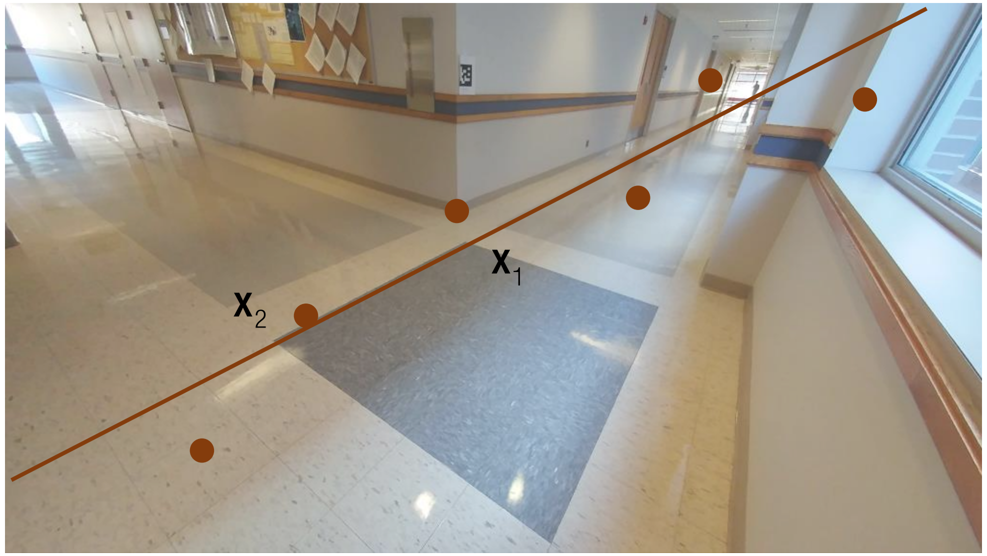


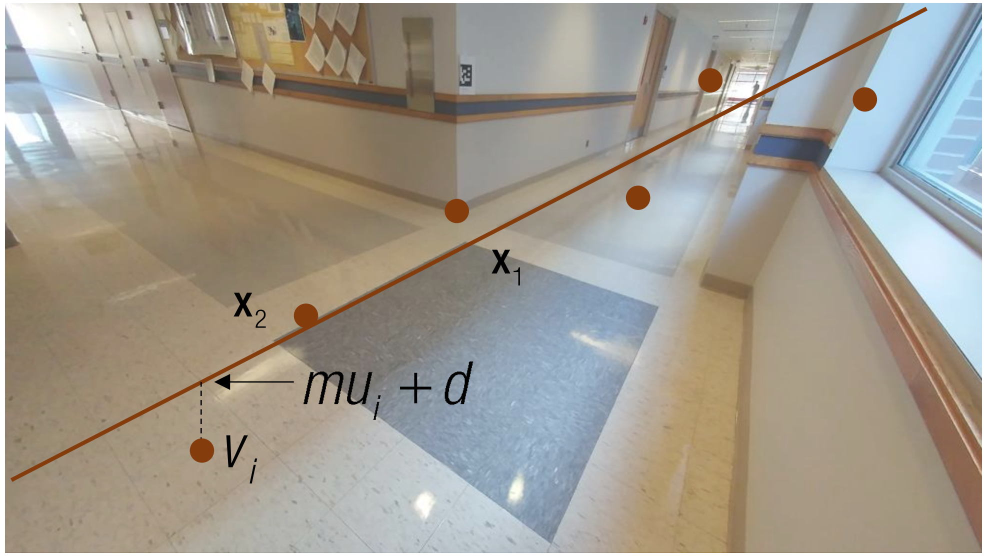

$$\longrightarrow au_1 + bv_1 + c \approx 0$$

$$au_2 + bv_2 + c \approx 0$$

$$\vdots$$

$$au_n + bv_n + c \approx 0$$

# Line Fitting




Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m},\underline{d})$

slope y-intercept

$$\longrightarrow \quad v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

$$\vdots$$

$$v_n \approx mu_n + d$$

# Line Fitting

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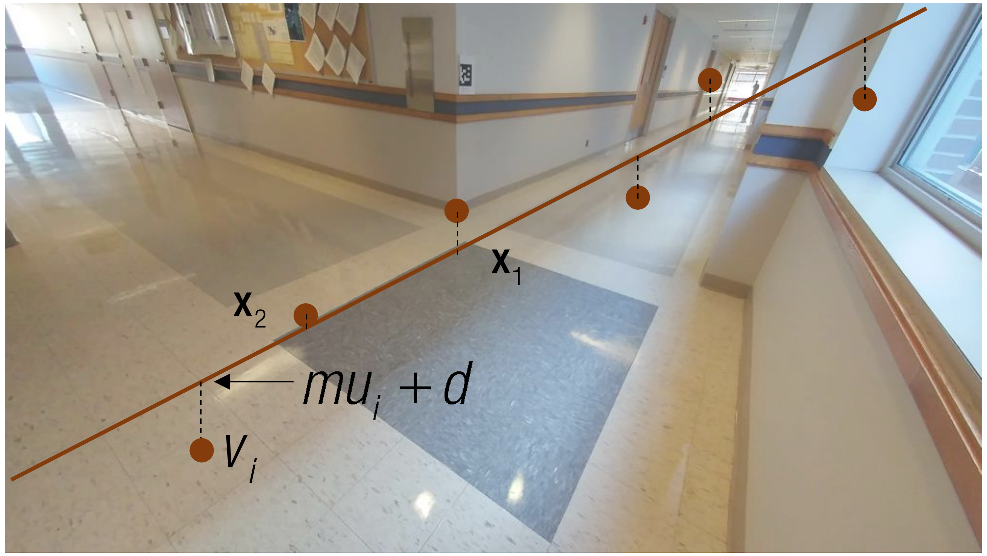


Error: $e_i = v_i - (mu_i + d)$

$$e_1 = v_1 - (mu_1 + d)$$

$$e_2 = v_2 - (mu_2 + d)$$

$$e_n = v_n - (mu_n + d)$$

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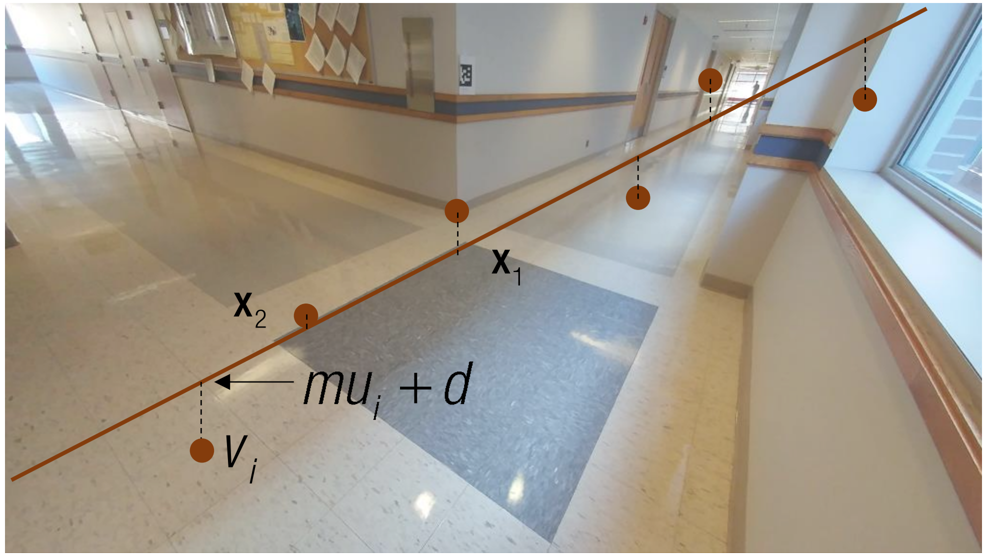


$$\longrightarrow \quad v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

$$\vdots$$

$$v_n \approx mu_n + d$$

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$e_1 = v_1 - (mu_1 + d)$

$e_2 = v_2 - (mu_2 + d)$

$e_n = v_n - (mu_n + d)$

Unknowns:

Number of eq.:

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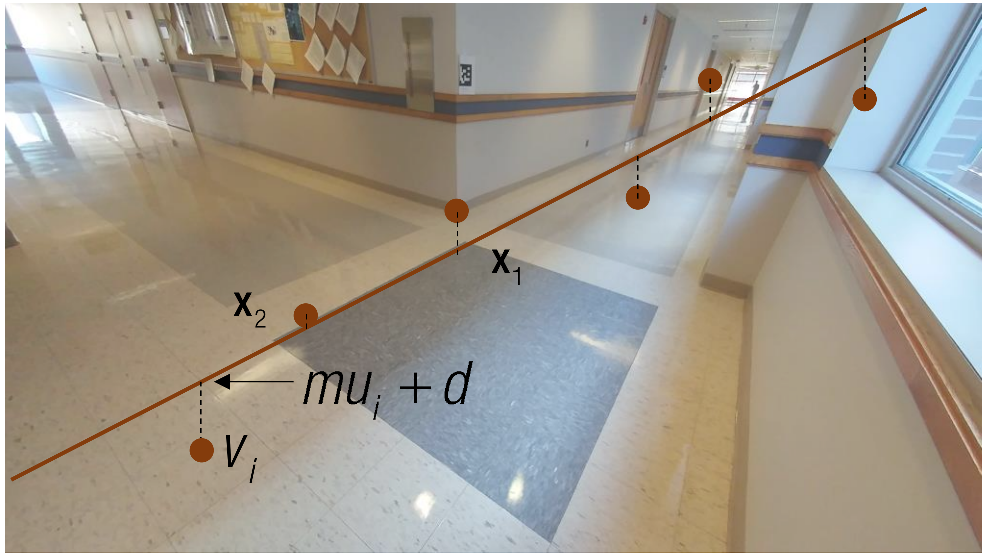


$$\longrightarrow \quad v_1 \approx mu_1 + d$$
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$$\vdots$$
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Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$e_1 = v_1 - (mu_1 + d)$

$e_2 = v_2 - (mu_2 + d)$

$e_n = v_n - (mu_n + d)$

Unknowns: $m,d$

Number of eq.:

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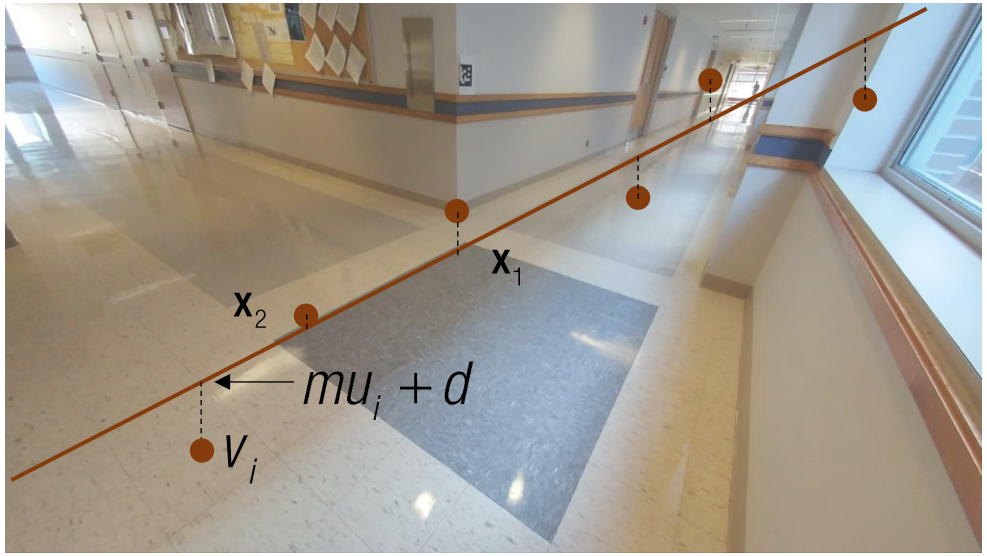


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$e_1 = v_1 - (mu_1 + d)$

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Number of eq.: $n$

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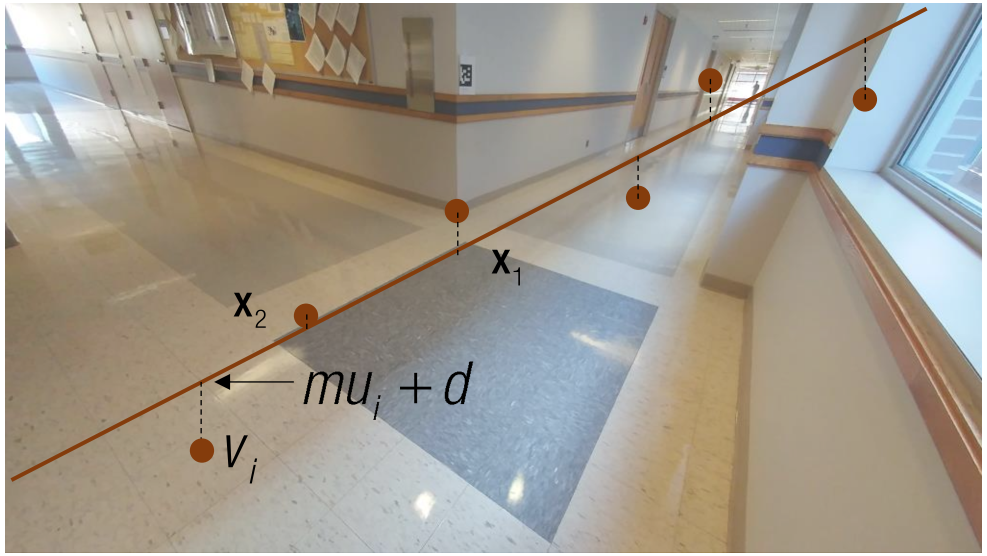



$e_1 = v_1 - (mu_1 + d)$

$e_2 = v_2 - (mu_2 + d)$

$e_n = v_n - (mu_n + d)$

$$\longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

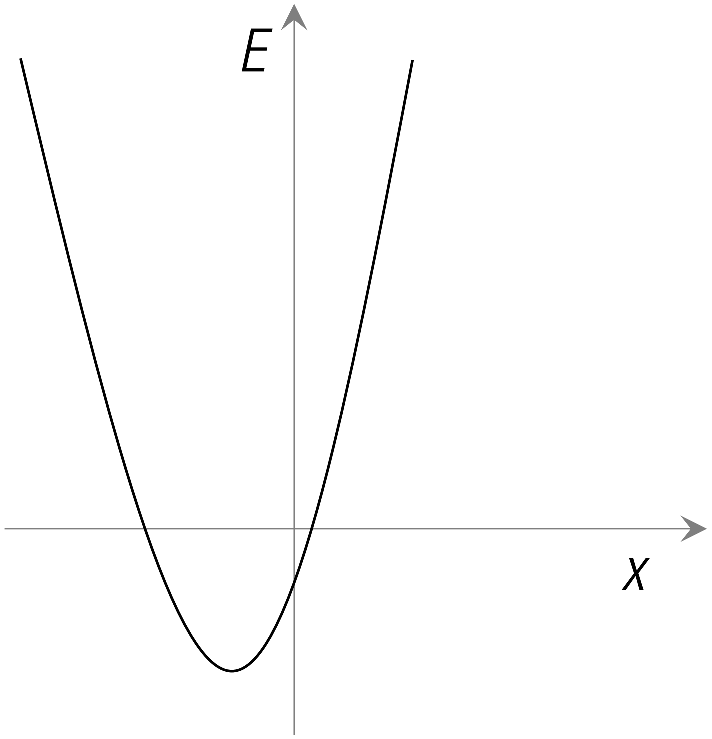
Unknowns: $m,d$

Number of eq.: $n$

$$\underset{m,d}{\text{minimize}} \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$$

How to minimize?

# Line Fitting

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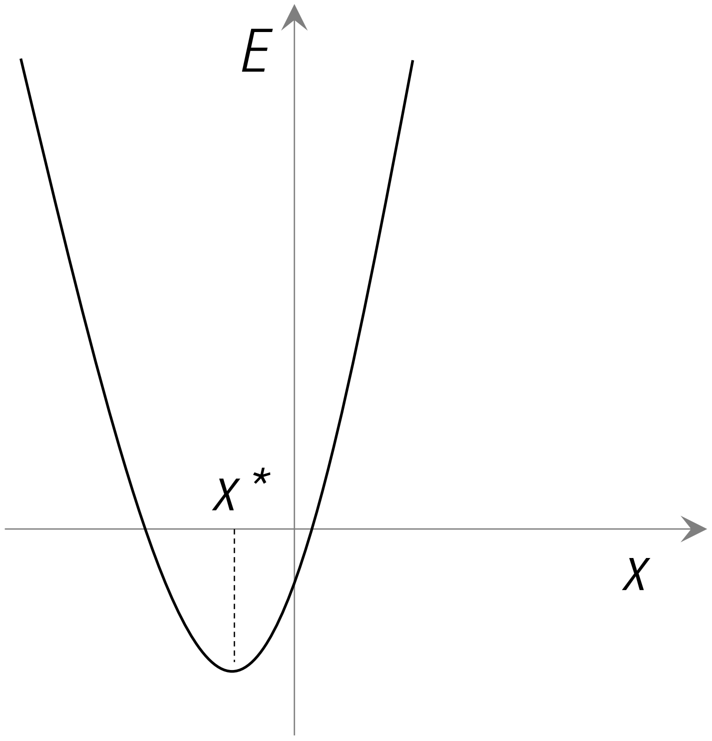


Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: $(a, b, c) \longrightarrow (\underline{m}, \underline{d})$

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$$\longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

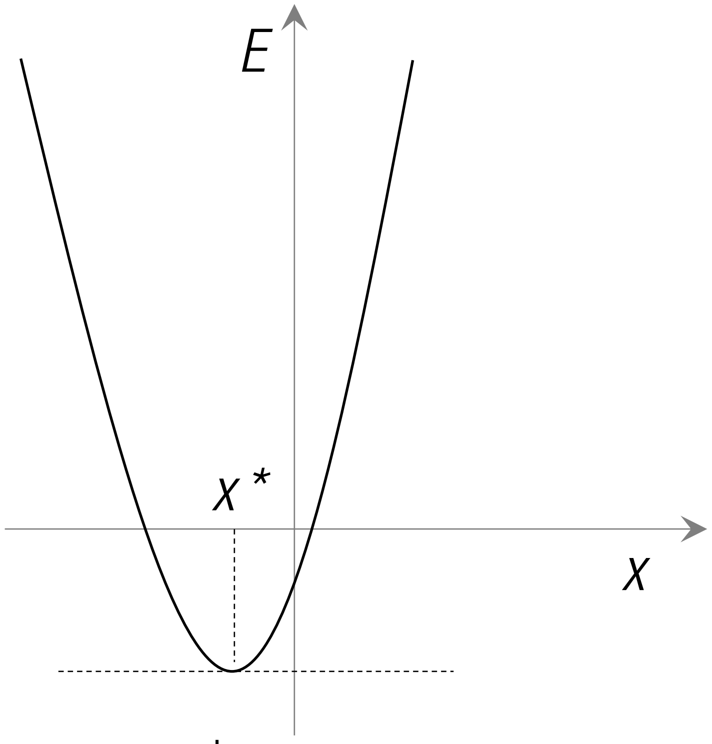
Unknowns: $m,d$

Number of eq.: $n$

$$\underset{m,d}{\text{minimize}} \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$$

How to minimize?

# Line Fitting




$$\left.\frac{dE}{dx}\right|_{x=x^*} = 0$$

Given points: $(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m}, \underline{d})$

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$$\longrightarrow \quad v_1 \approx mu_1 + d$$

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Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Unknowns: $m,d$

Number of eq.: $n$

$$\underset{m,d}{\text{minimize}} \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$$

How to minimize?

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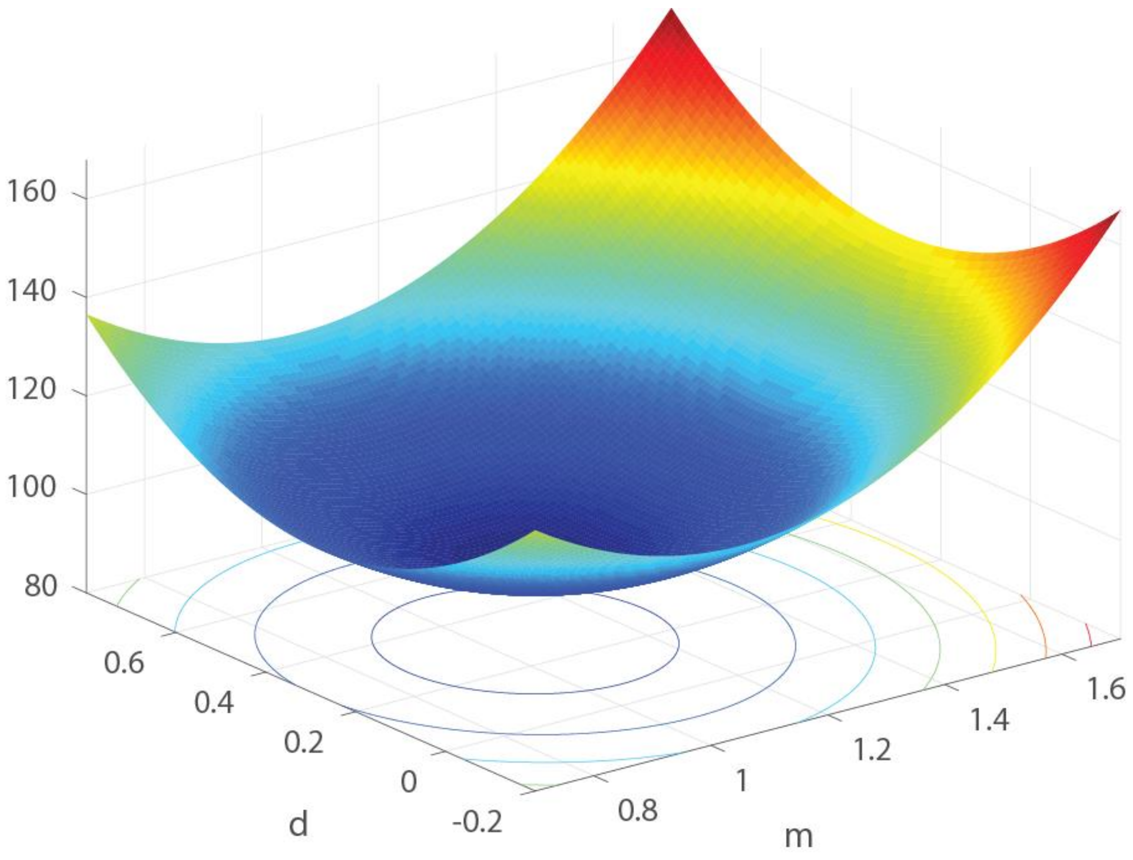


Total error: $$E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$$

Optimal point: $$\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$$

$$\frac{\partial E}{\partial m} =$$

$$\frac{\partial E}{\partial d} =$$

# Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

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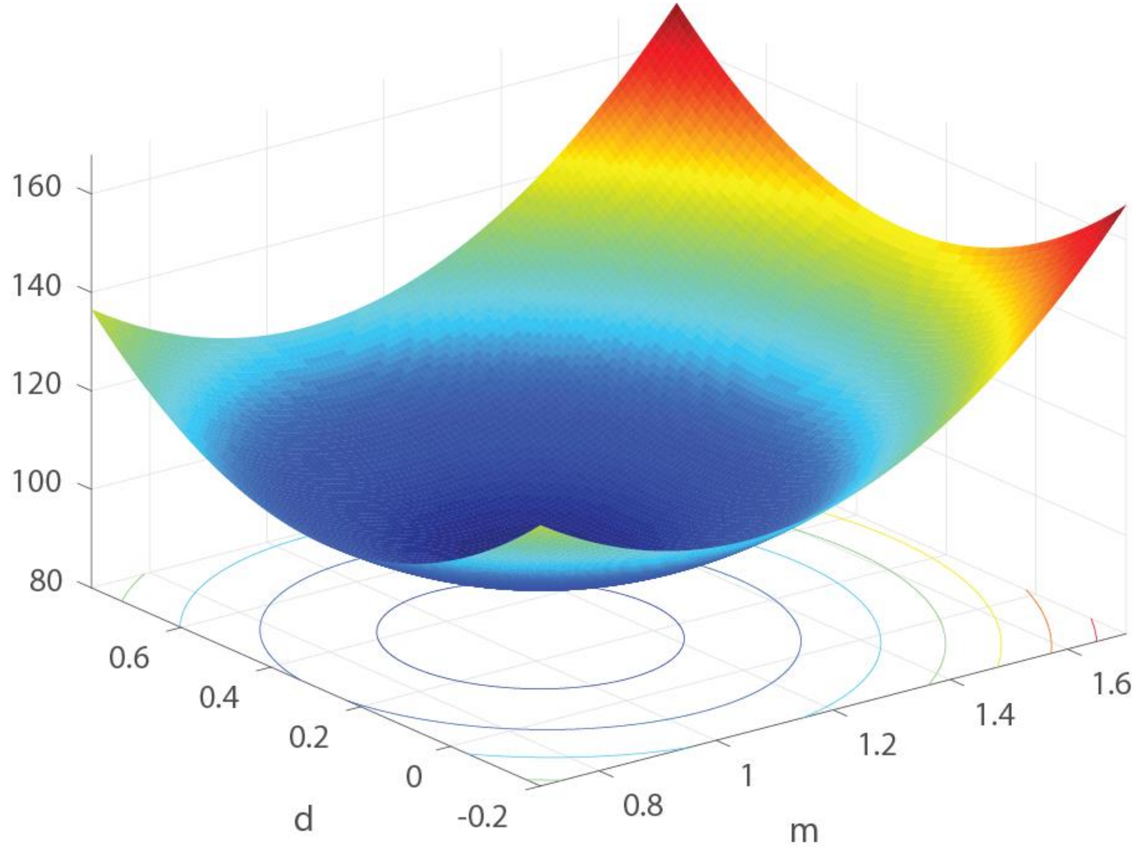


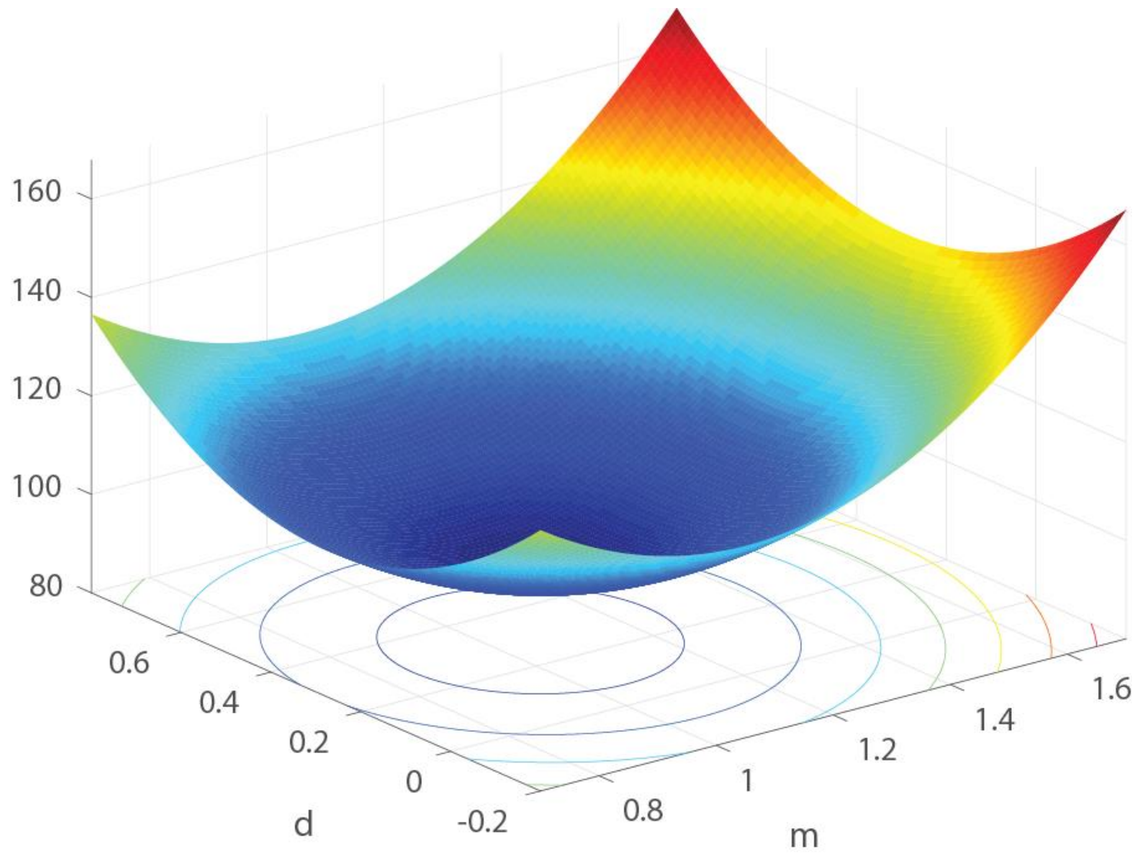

$$\frac{\partial E}{\partial m} = -\sum_{i=1}^{n} 2u_i \left(v_i - (mu_i + d)\right) = 0$$

$$\frac{\partial E}{\partial d} = -\sum_{i=1}^{n} 2\left(v_i - (mu_i + d)\right) = 0$$

# Line Fitting

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$




$$\frac{\partial E}{\partial m} = -\sum_{i=1}^{n} 2u_i \left(v_i - (mu_i + d)\right) = 0$$

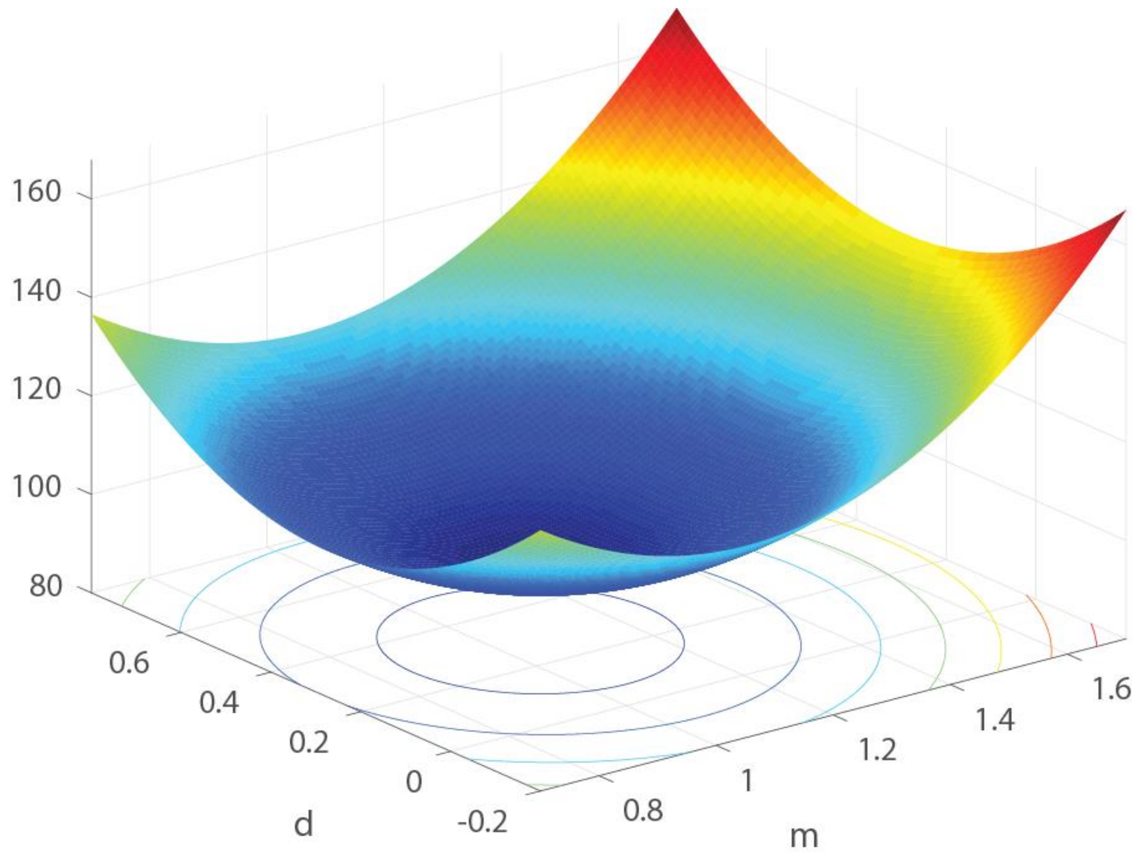
$$\frac{\partial E}{\partial d} = -\sum_{i=1}^{n} 2\left(v_i - (mu_i + d)\right) = 0$$

$\longrightarrow$

$$m\sum_{i=1}^{n} u_i^2 + d\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} u_i v_i$$

$$m\sum_{i=1}^{n} u_i + nd = \sum_{i=1}^{n} v_i$$

# Line Fitting




Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$\frac{\partial E}{\partial m} = -\sum_{i=1}^{n} 2u_i \left(v_i - (mu_i + d)\right) = 0$$

$$\frac{\partial E}{\partial d} = -\sum_{i=1}^{n} 2\left(v_i - (mu_i + d)\right) = 0$$

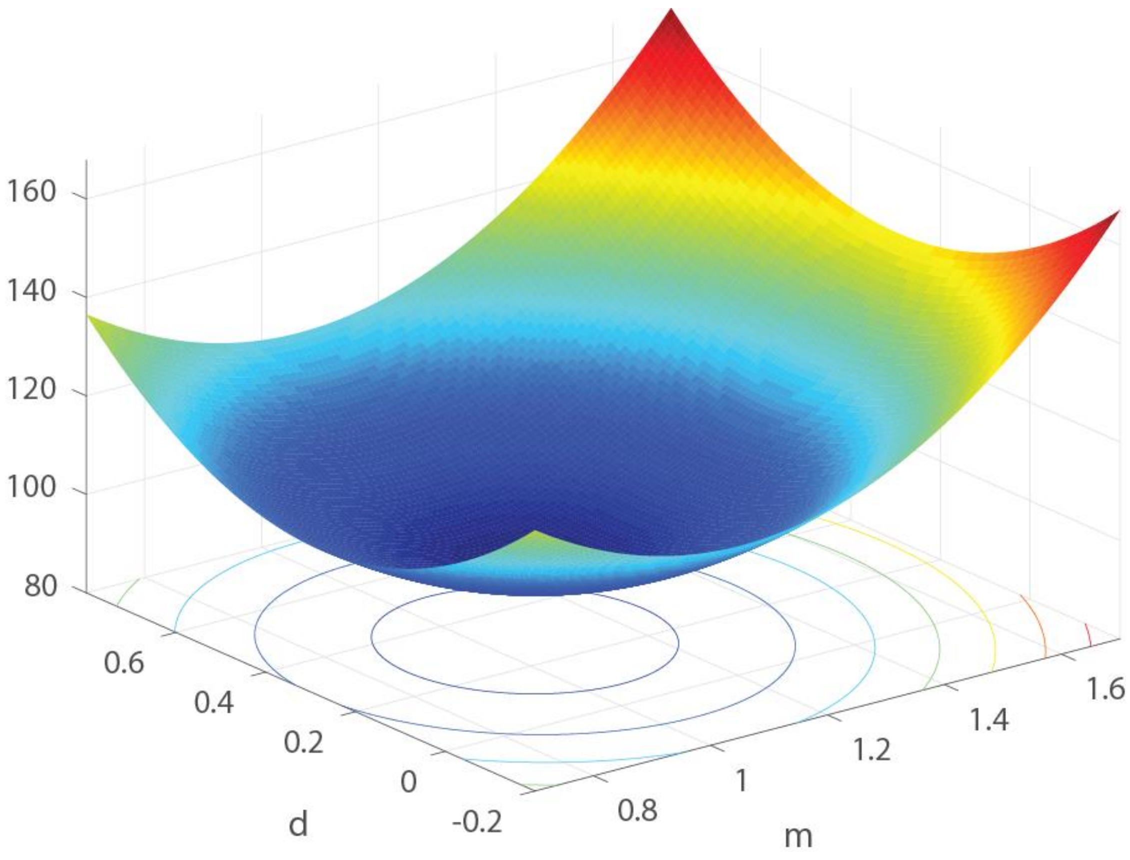
$$\longrightarrow \quad m\sum_{i=1}^{n} u_i^2 + d\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} u_i v_i$$

$$m\sum_{i=1}^{n} u_i + nd = \sum_{i=1}^{n} v_i$$

$$\begin{bmatrix} \sum_{i=1}^{n} u_i^2 & \sum_{i=1}^{n} u_i \\ \sum_{i=1}^{n} u_i & n \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} u_i v_i \\ \sum_{i=1}^{n} v_i \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} m \\ d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} u_i^2 & \sum_{i=1}^{n} u_i \\ \sum_{i=1}^{n} u_i & n \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} u_i v_i \\ \sum_{i=1}^{n} v_i \end{bmatrix}$$

# Line Fitting (Ax=b)




Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

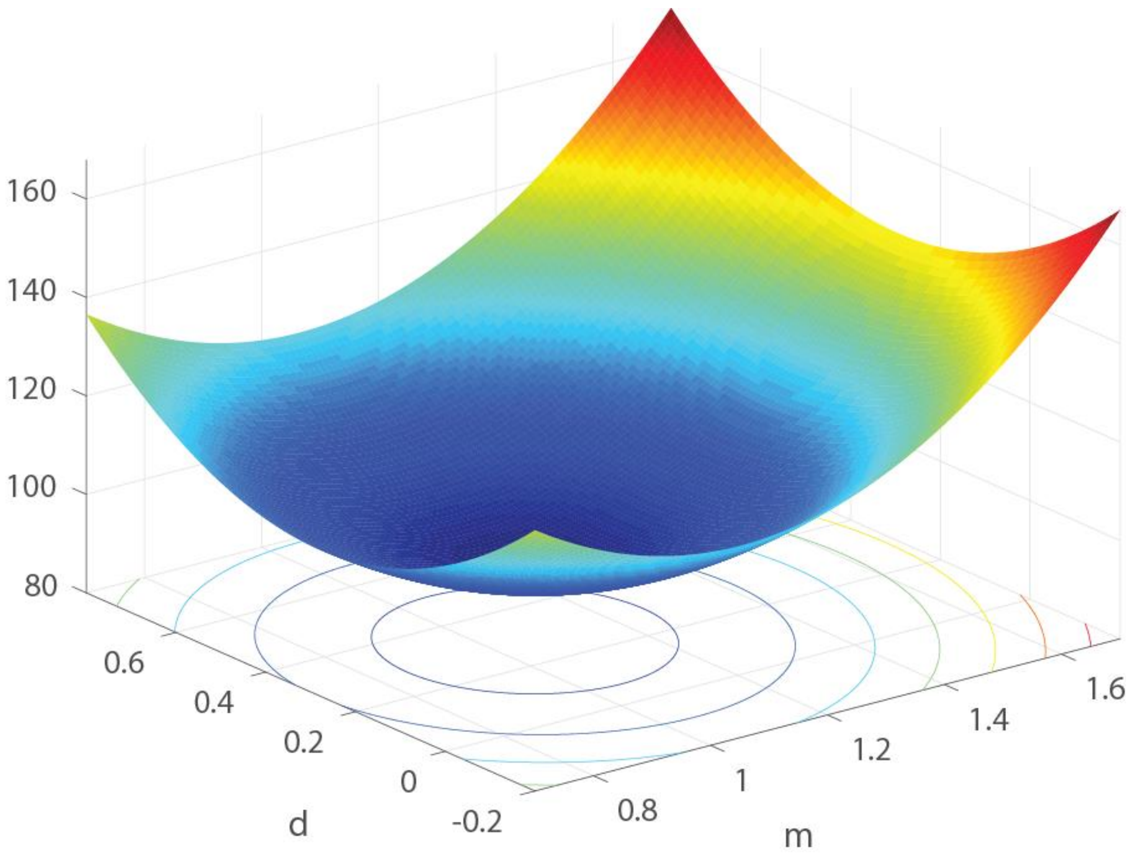
Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d$$

$$v_2 \approx mu_2 + d$$

$$v_n \approx mu_n + d$$

# Line Fitting (Ax=b)




Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$v_1 \approx mu_1 + d$ $\qquad \begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$

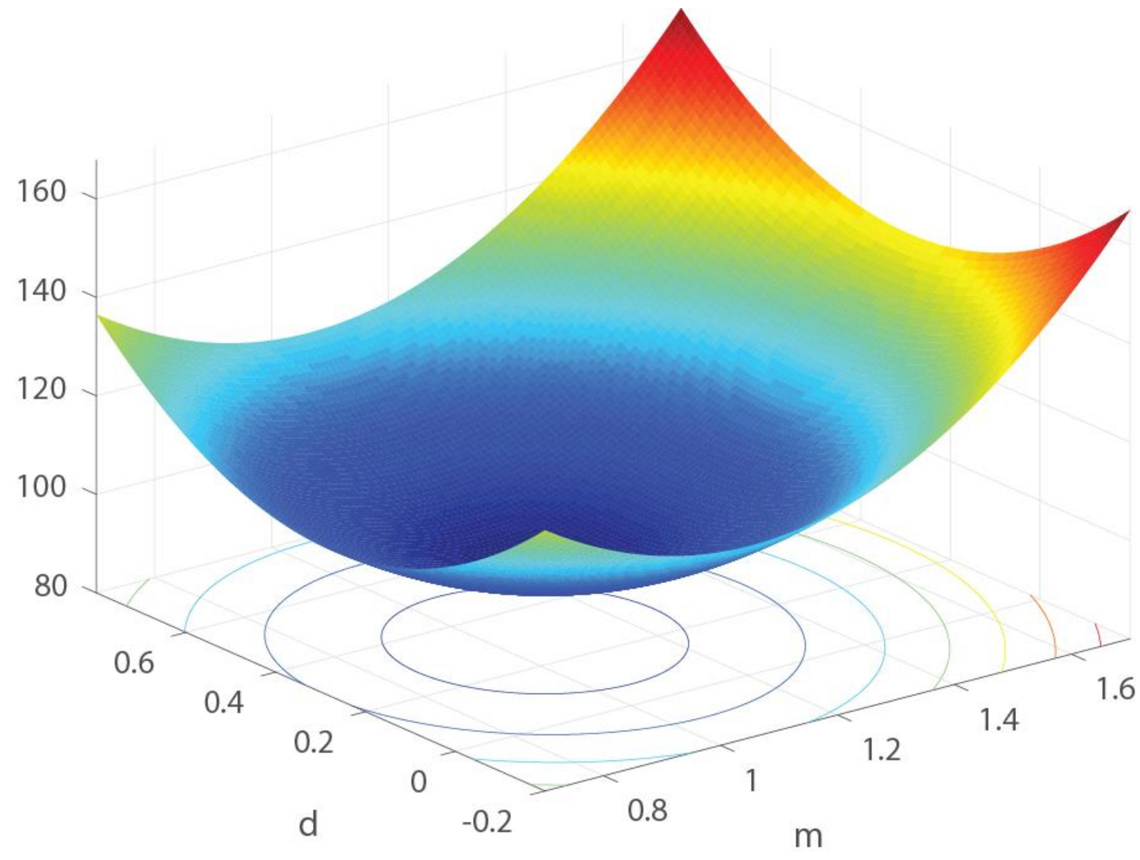
$v_2 \approx mu_2 + d$ $\quad \longrightarrow \quad \begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$

$v_n \approx mu_n + d$ $\qquad \begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$

# Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$




$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$v_n \approx mu_n + d$$

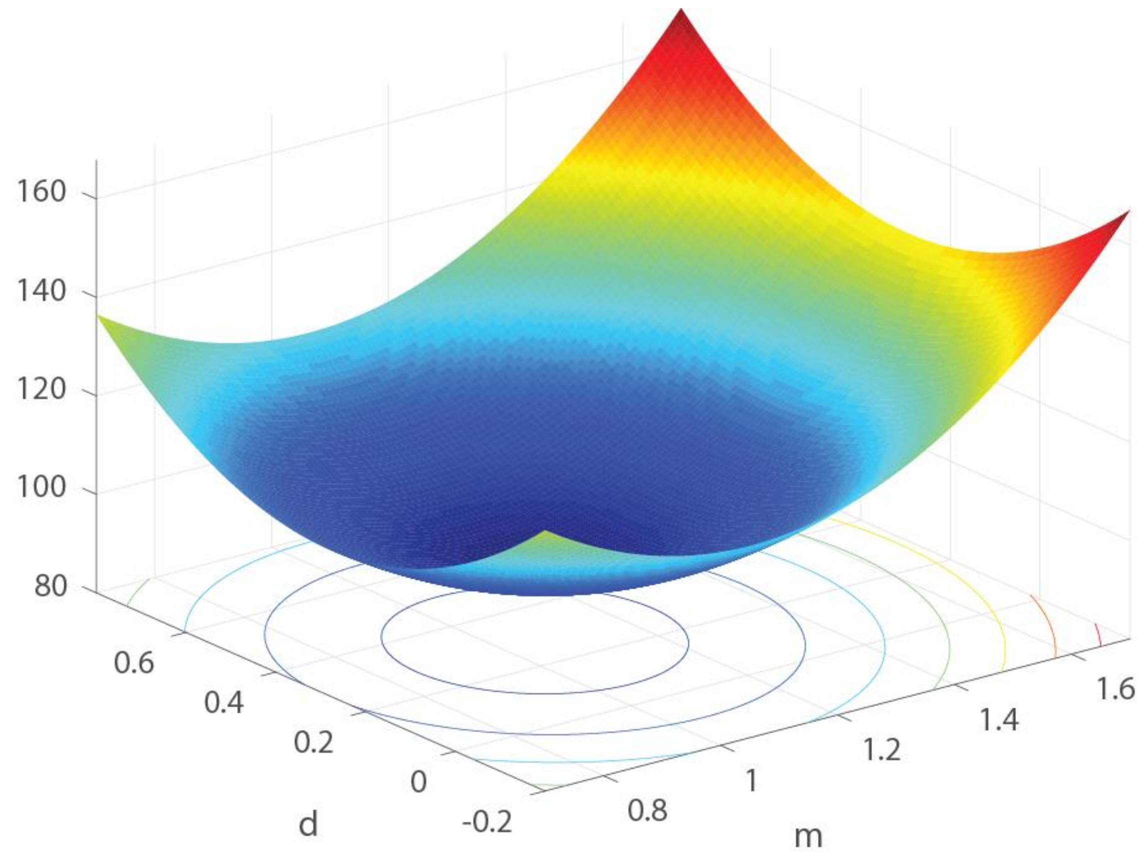
$\longrightarrow$

$$\begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$
$$\begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2$$
$$\begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

$\longrightarrow$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

# Line Fitting (Ax=b)




Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

Optimal point: $\frac{\partial E}{\partial m} = 0, \quad \frac{\partial E}{\partial d} = 0$

$$v_1 \approx mu_1 + d \qquad \begin{bmatrix} u_1 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_1$$

$$v_2 \approx mu_2 + d \quad \longrightarrow \quad \begin{bmatrix} u_2 & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_2 \quad \longrightarrow \quad \underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$v_n \approx mu_n + d \qquad \begin{bmatrix} u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx v_n$$

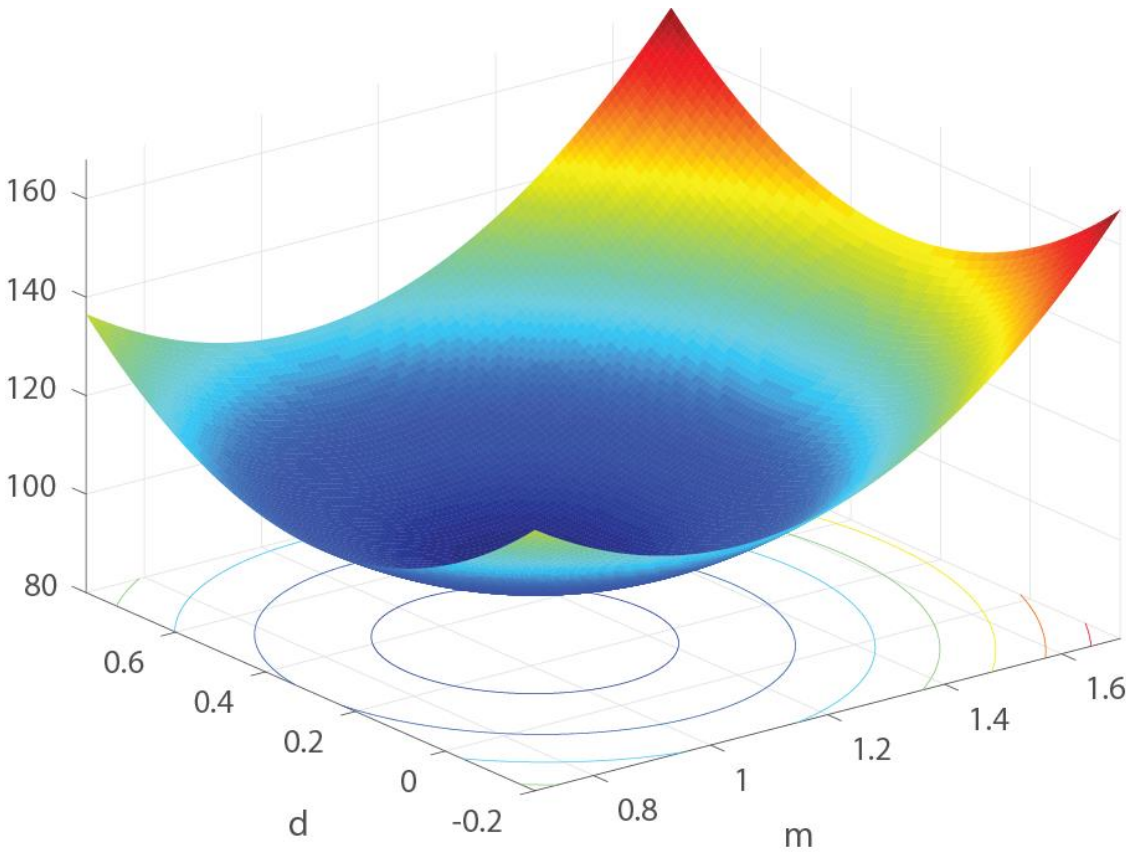
We can't invert **A**.

# Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$E = \left(\mathbf{Ax} - \mathbf{b}\right)^{\mathrm{T}} \left(\mathbf{Ax} - \mathbf{b}\right) = \left\|\mathbf{Ax} - \mathbf{b}\right\|^2$$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$



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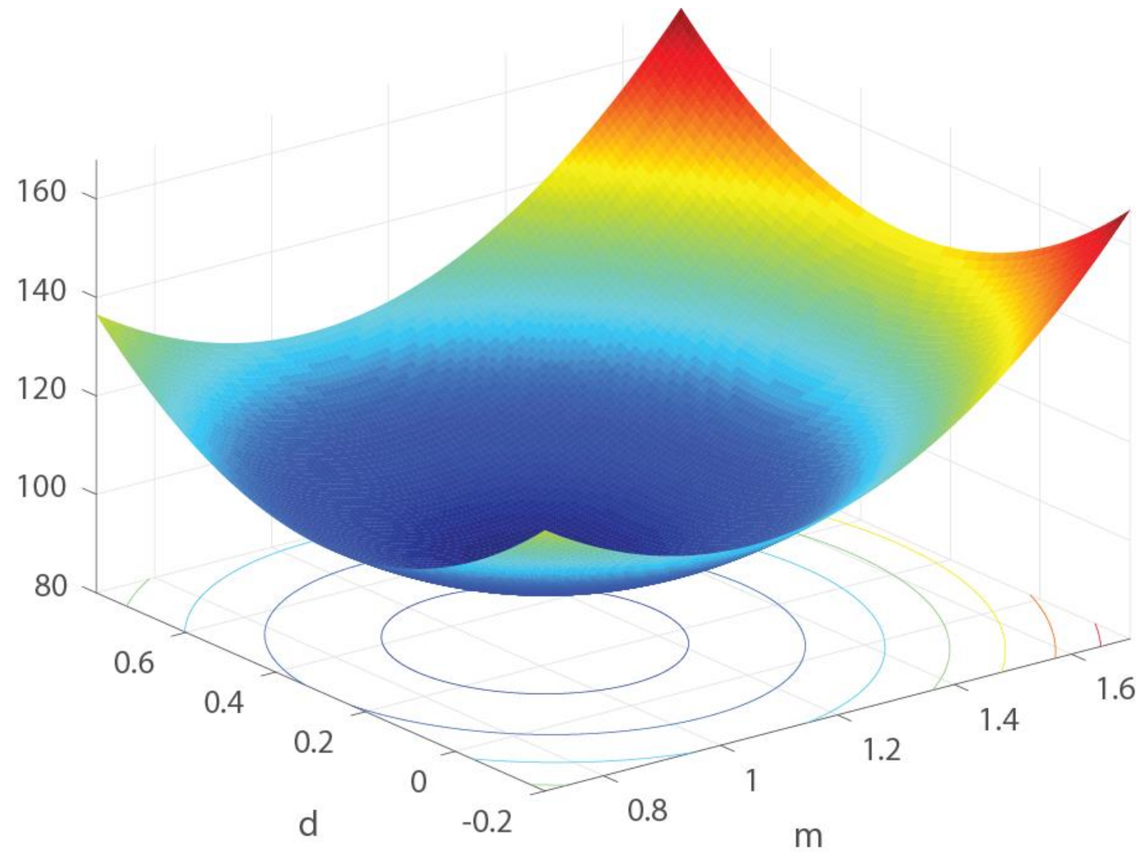
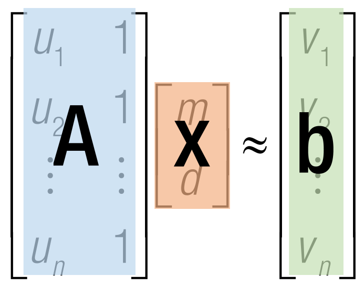



Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$E = (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} - 2\mathbf{x}^T\mathbf{A}^T\mathbf{b} + \mathbf{b}^T\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \quad ?$$

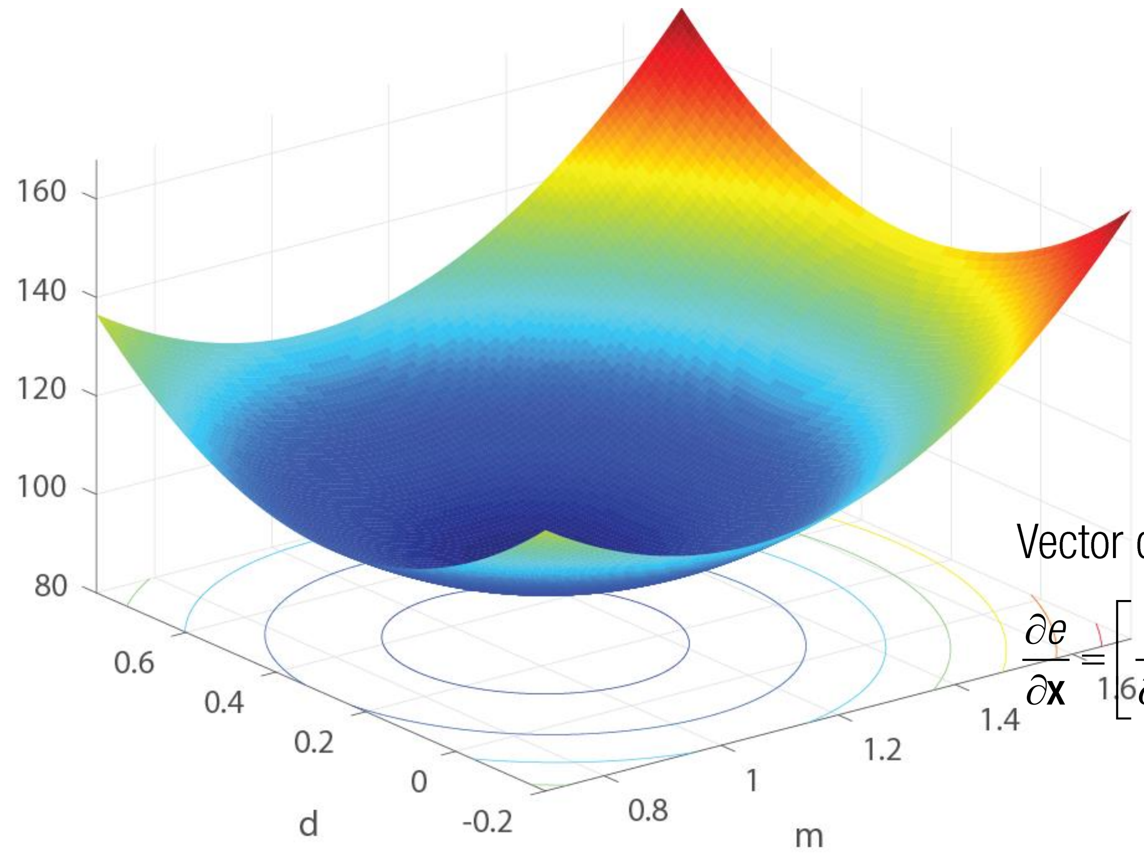


# Line Fitting (Ax=b)

Total error: $E=\sum_{i=1}^{n}\left(v_i-(mu_i+d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}\begin{bmatrix} m \\ d \end{bmatrix}\approx\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A}\mathbf{x}\approx\mathbf{b}$$




$$E=\left(\mathbf{Ax}-\mathbf{b}\right)^{\mathsf{T}}\left(\mathbf{Ax}-\mathbf{b}\right)=\left\|\mathbf{Ax}-\mathbf{b}\right\|^2$$

$$=\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax}-2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b}+\mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}}= \quad ?$$

Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}}=\begin{bmatrix}\frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n}\end{bmatrix}$$

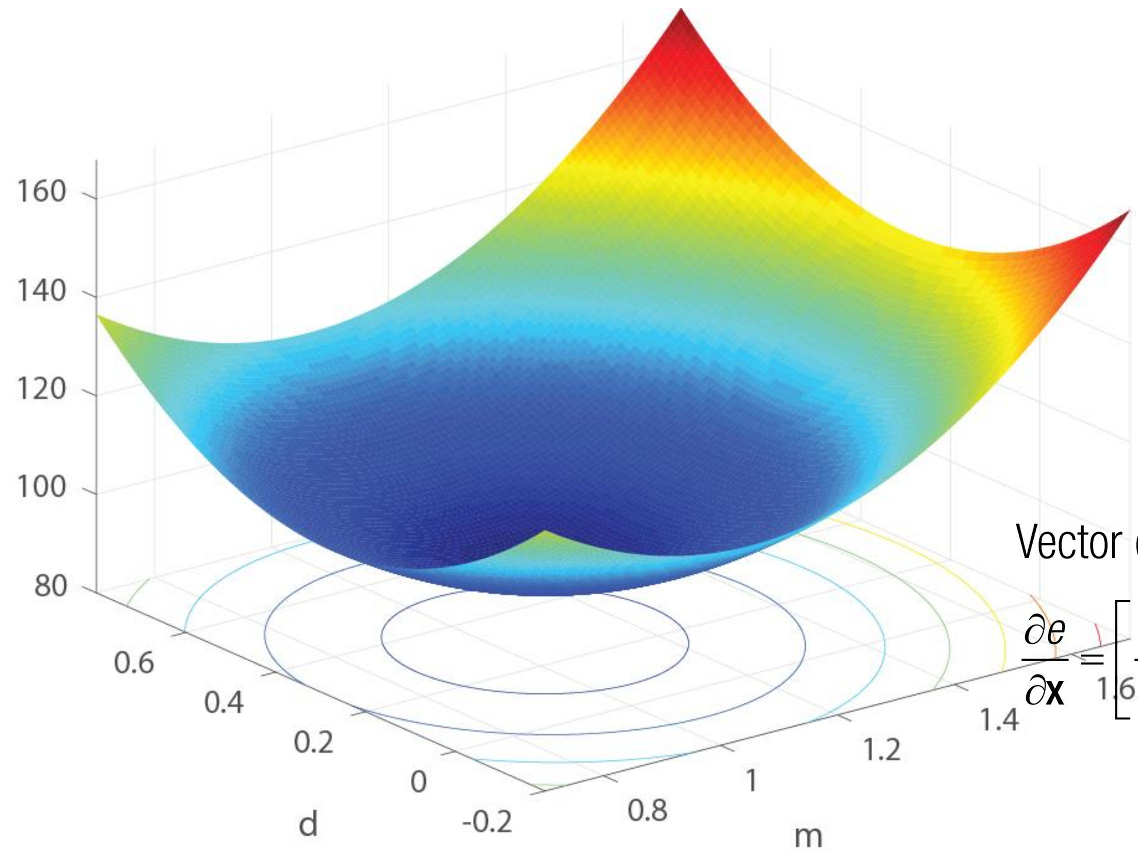
# Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n}\left(v_i - (mu_i + d)\right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$E = (\mathbf{Ax}-\mathbf{b})^{\mathrm{T}}(\mathbf{Ax}-\mathbf{b}) = \|\mathbf{Ax}-\mathbf{b}\|^2$$
$$= \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{b} + \mathbf{b}^{\mathrm{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \quad ?$$




Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathbf{c}^{\mathrm{T}}\mathbf{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^{\mathrm{T}}\mathbf{x}}{\partial \mathbf{x}} =$$

# Line Fitting (Ax=b)

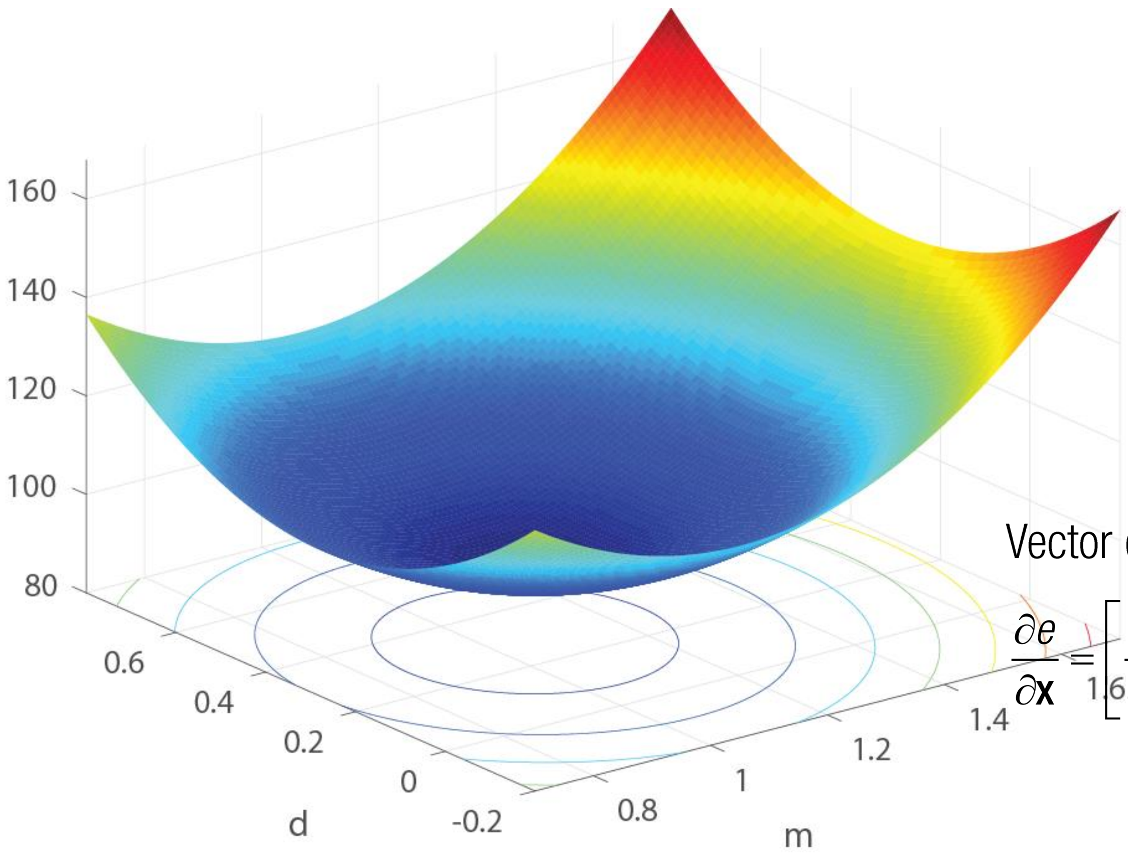
Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

$$E = (\mathbf{Ax} - \mathbf{b})^{\mathsf{T}}(\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \quad ?$$




Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathbf{c}^{\mathsf{T}}\mathbf{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^{\mathsf{T}}\mathbf{x}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\left(c_1 x_1 + \cdots + c_n x_n\right)$$

# Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

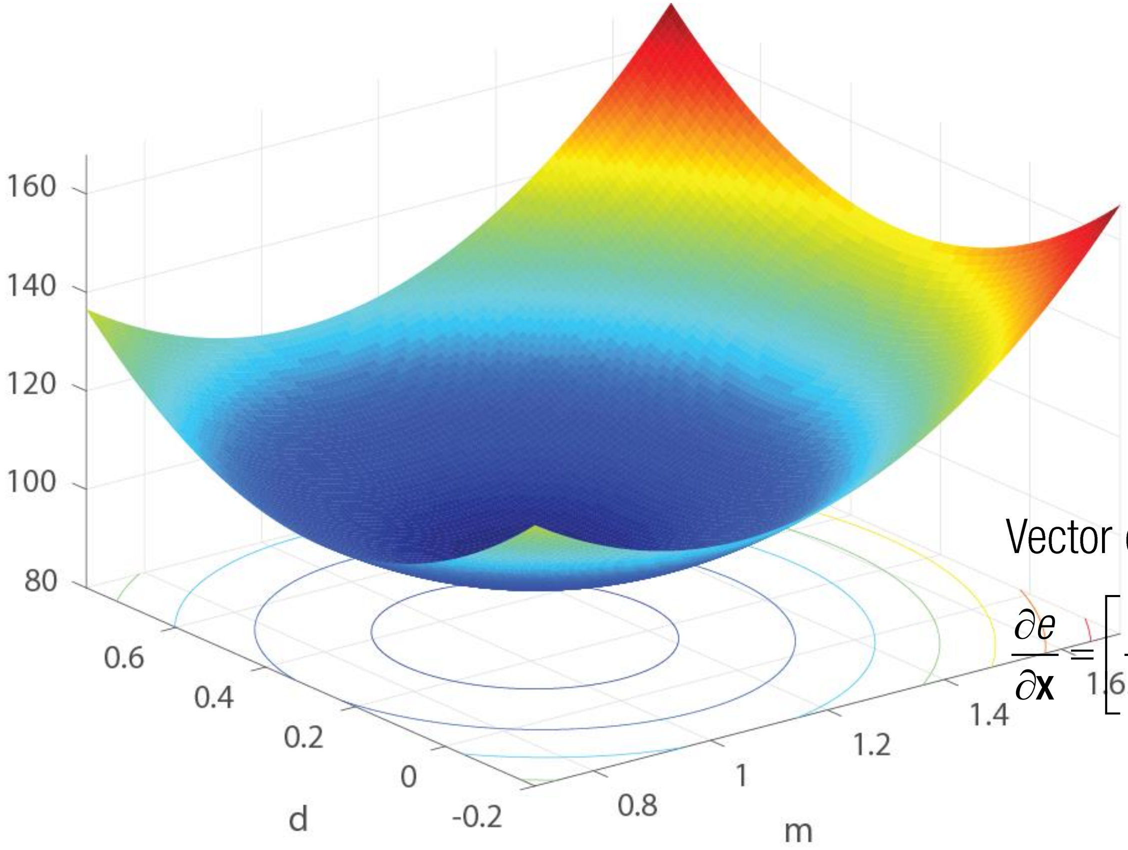
$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A} \quad \mathbf{x} \approx \mathbf{b}$$

$$E = \left(\mathbf{Ax} - \mathbf{b}\right)^{\mathsf{T}}\left(\mathbf{Ax} - \mathbf{b}\right) = \left\|\mathbf{Ax} - \mathbf{b}\right\|^2$$

$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = \qquad ?$$




Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathbf{c}^{\mathsf{T}}\mathbf{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\frac{\partial \mathbf{c}^{\mathsf{T}}\mathbf{x}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\left(c_1x_1 + \cdots + c_nx_n\right)$$

$$= \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$$

# Line Fitting (Ax=b)

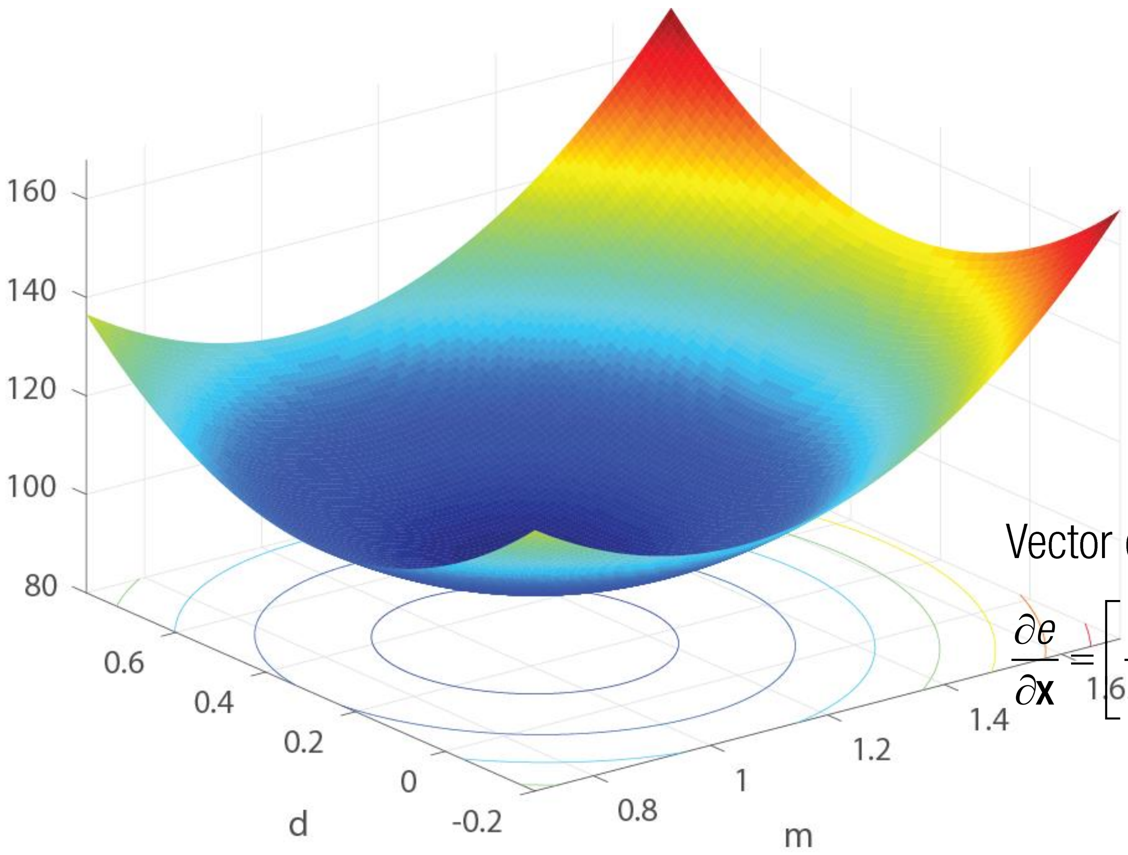
Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathbf{A}\,\mathbf{x} \approx \mathbf{b}$$

$$E = (\mathbf{Ax} - \mathbf{b})^{\mathsf{T}} (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{A}^{\mathsf{T}}\mathbf{b} = \mathbf{0}$$




Vector derivative:

$$\frac{\partial e}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial e}{\partial x_1} & \cdots & \frac{\partial e}{\partial x_n} \end{bmatrix}$$

Ex) $e = \mathbf{c}^{\mathsf{T}}\mathbf{x} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

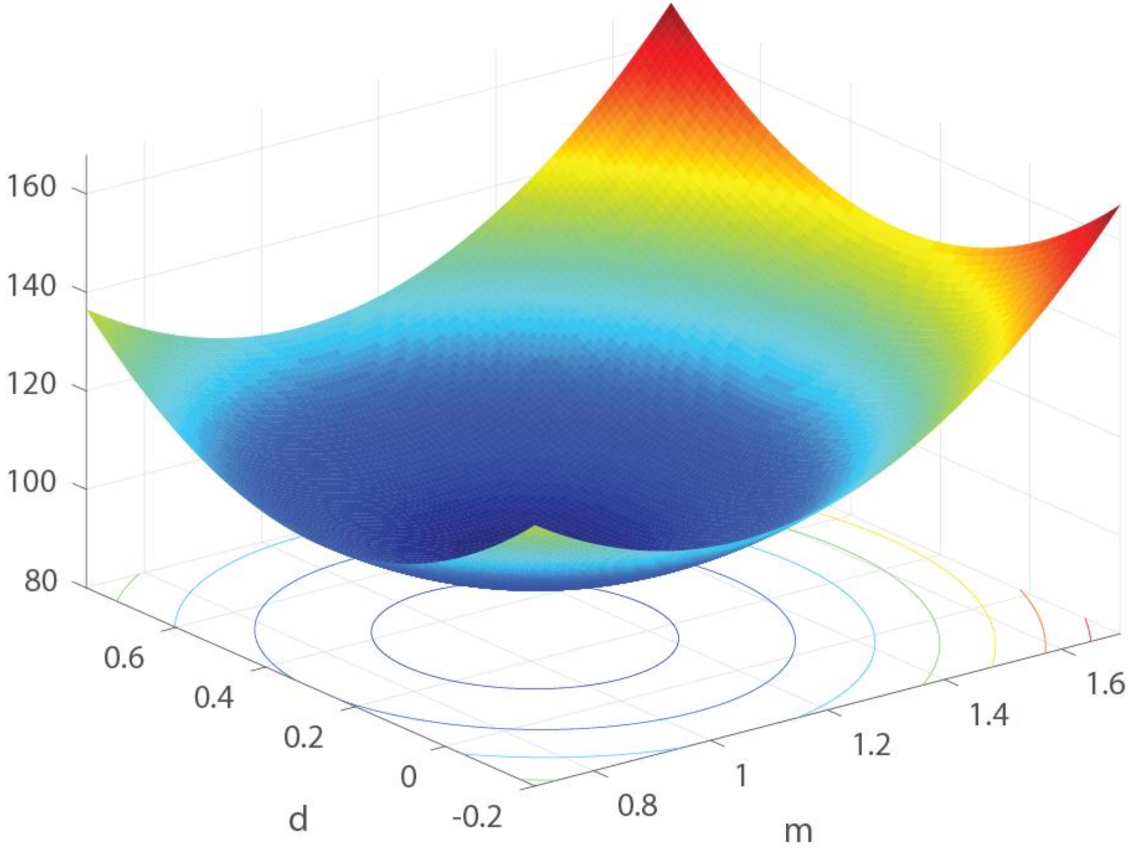
$$\frac{\partial \mathbf{c}^{\mathsf{T}}\mathbf{x}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}(c_1 x_1 + \cdots + c_n x_n)$$

$$= \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$$

# Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$




$$E = (\mathbf{Ax} - \mathbf{b})^{\mathsf{T}} (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

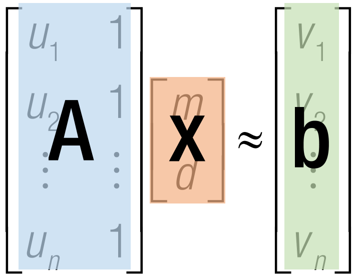
$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

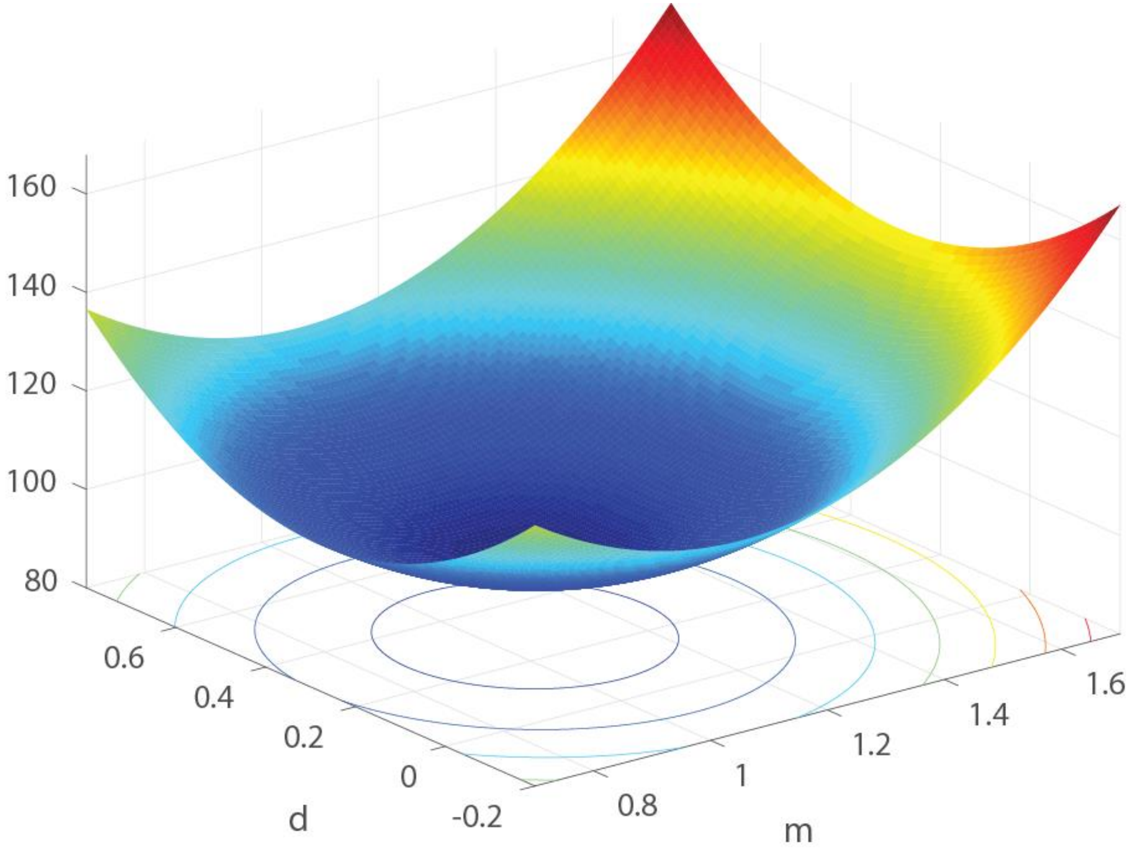
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{A}^{\mathsf{T}}\mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^{\mathsf{T}}\mathbf{Ax} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$$

# Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$



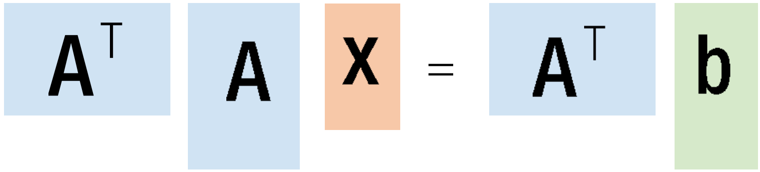




$$E = (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} - 2\mathbf{x}^T\mathbf{A}^T\mathbf{b} + \mathbf{b}^T\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^T\mathbf{A}\mathbf{x} - 2\mathbf{A}^T\mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$$



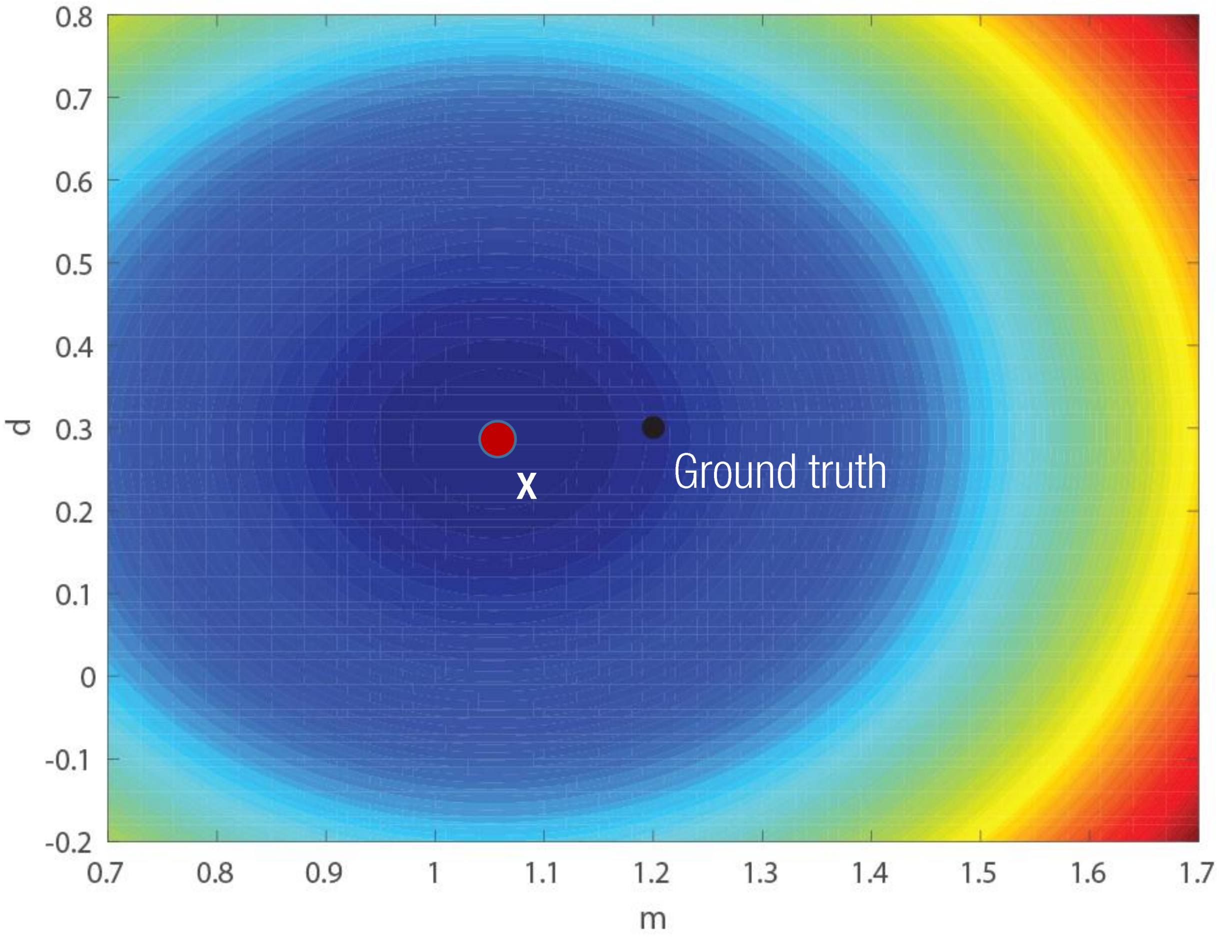

Normal equation

# Line Fitting (Ax=b)

Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A}\,\mathbf{x} \approx \mathbf{b}$$




$$E = (\mathbf{Ax} - \mathbf{b})^{\mathsf{T}}(\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{b} + \mathbf{b}^{\mathsf{T}}\mathbf{b}$$

$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - 2\mathbf{A}^{\mathsf{T}}\mathbf{b} = 0$$

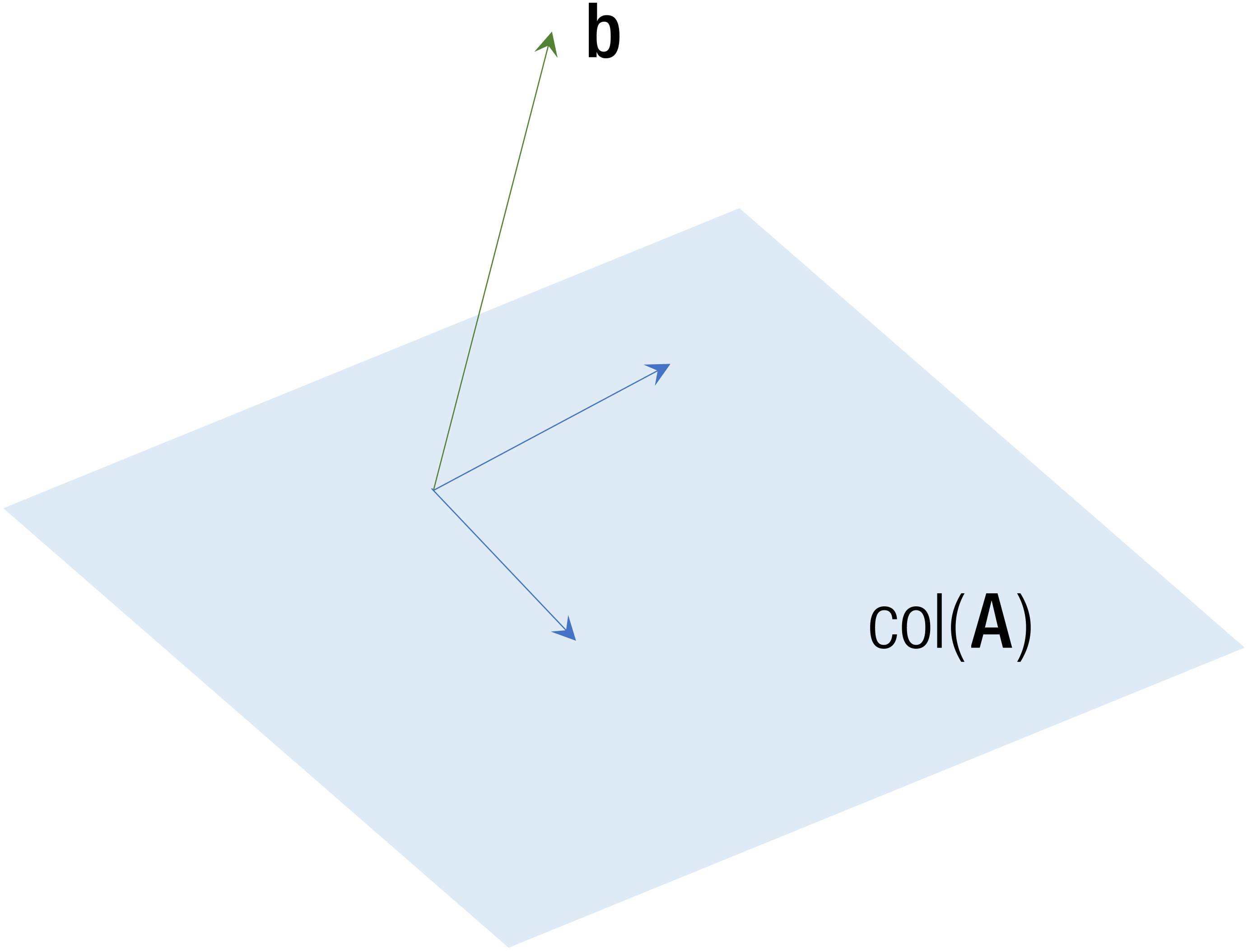
$$\longrightarrow \mathbf{A}^{\mathsf{T}}\mathbf{Ax} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$$

$$\mathbf{A}^{\mathsf{T}}\ \mathbf{A}\ \mathbf{x} = \mathbf{A}^{\mathsf{T}}\ \mathbf{b}$$

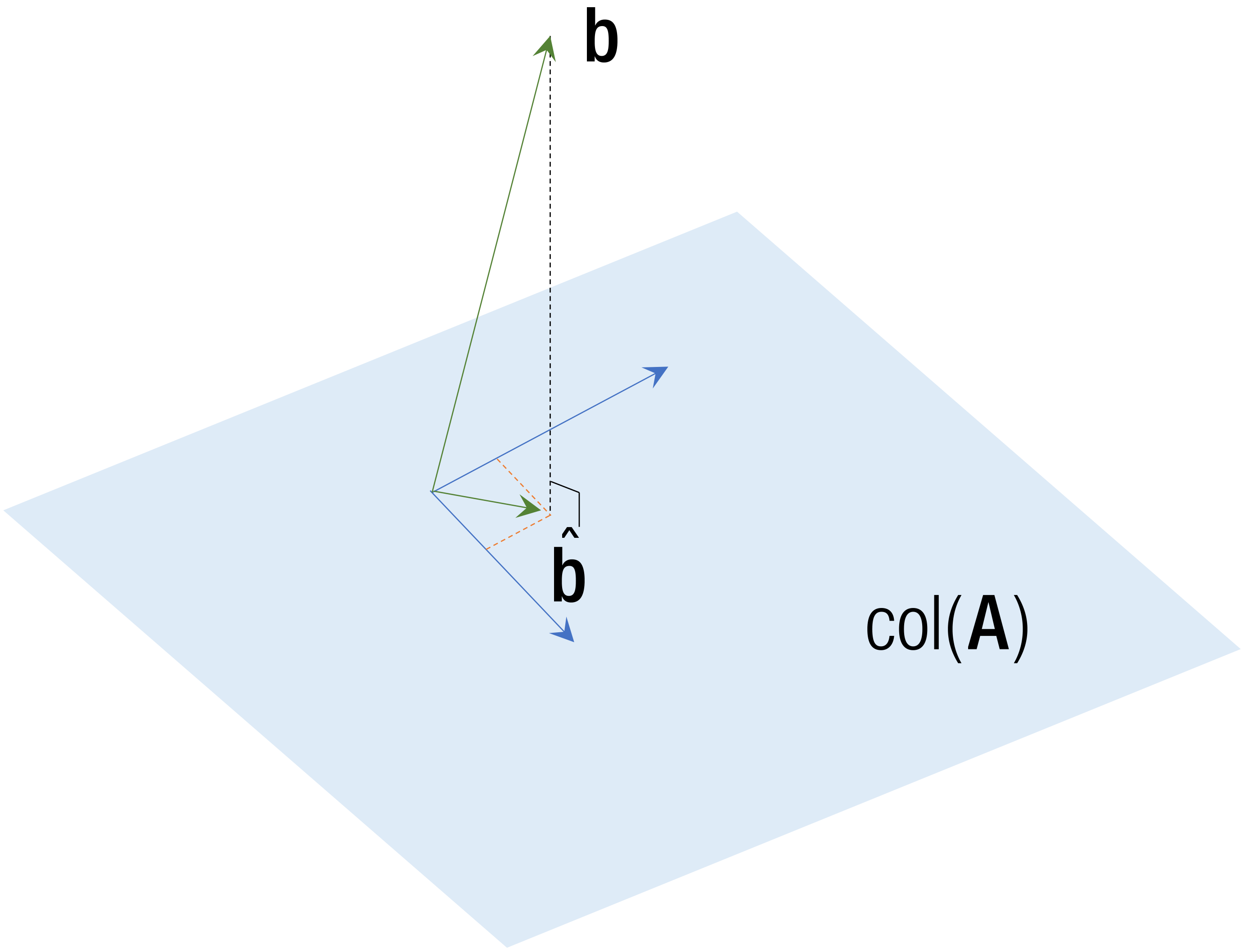
$$\mathbf{x} = \left[\mathbf{A}^{\mathsf{T}}\ \mathbf{A}\right]^{-1} \mathbf{A}^{\mathsf{T}}\ \mathbf{b}$$

# Geometric Interpretation

$\mathbf{A}\,\mathbf{x} \approx \mathbf{b}$



# Geometric Interpretation

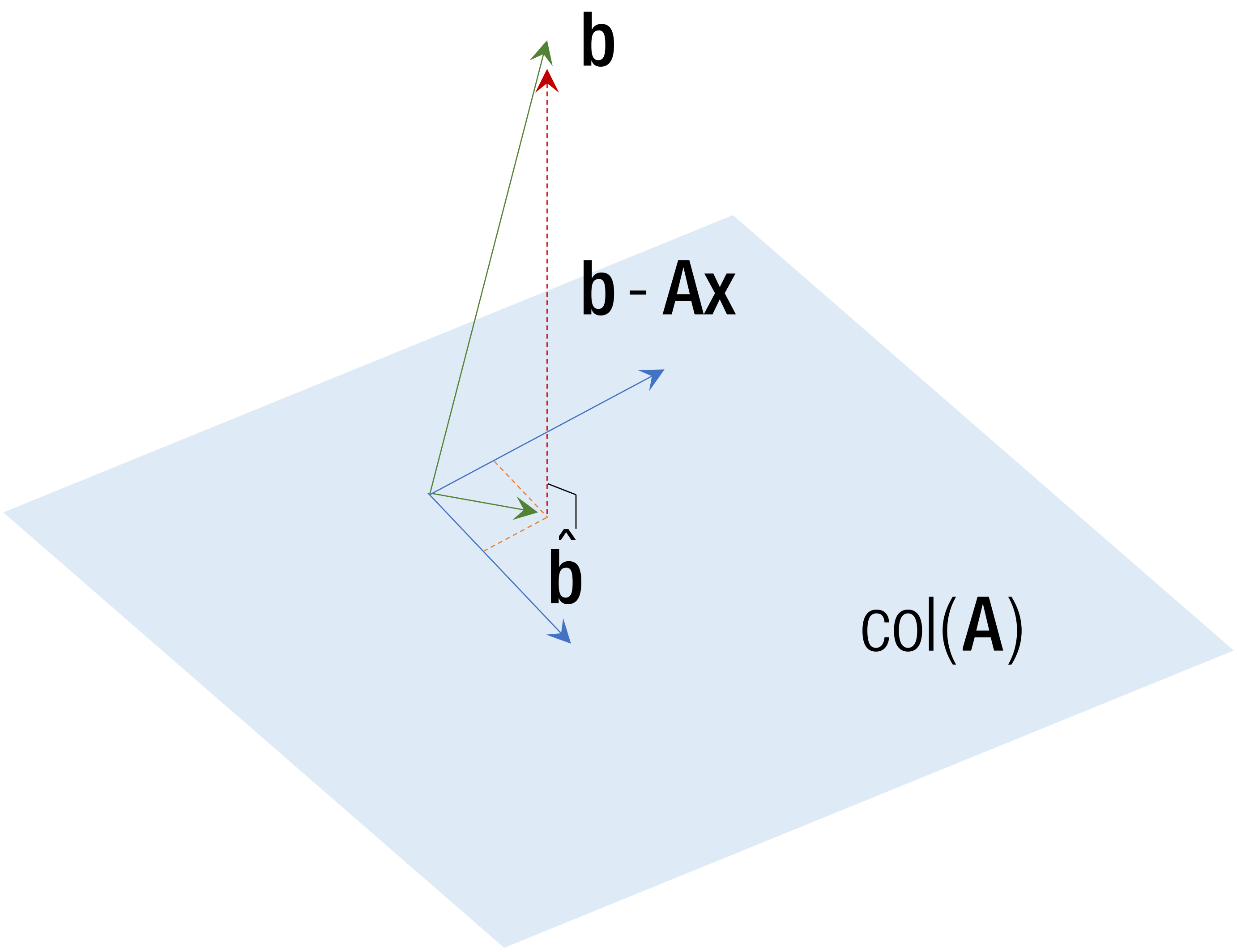



$$\mathbf{A}\,\mathbf{x} \approx \mathbf{b}$$

$$\mathbf{A}\,\mathbf{x} = \hat{\mathbf{b}}$$

$\hat{\mathbf{b}}$ : Best approximation of b spanned by col(A)

# Geometric Interpretation



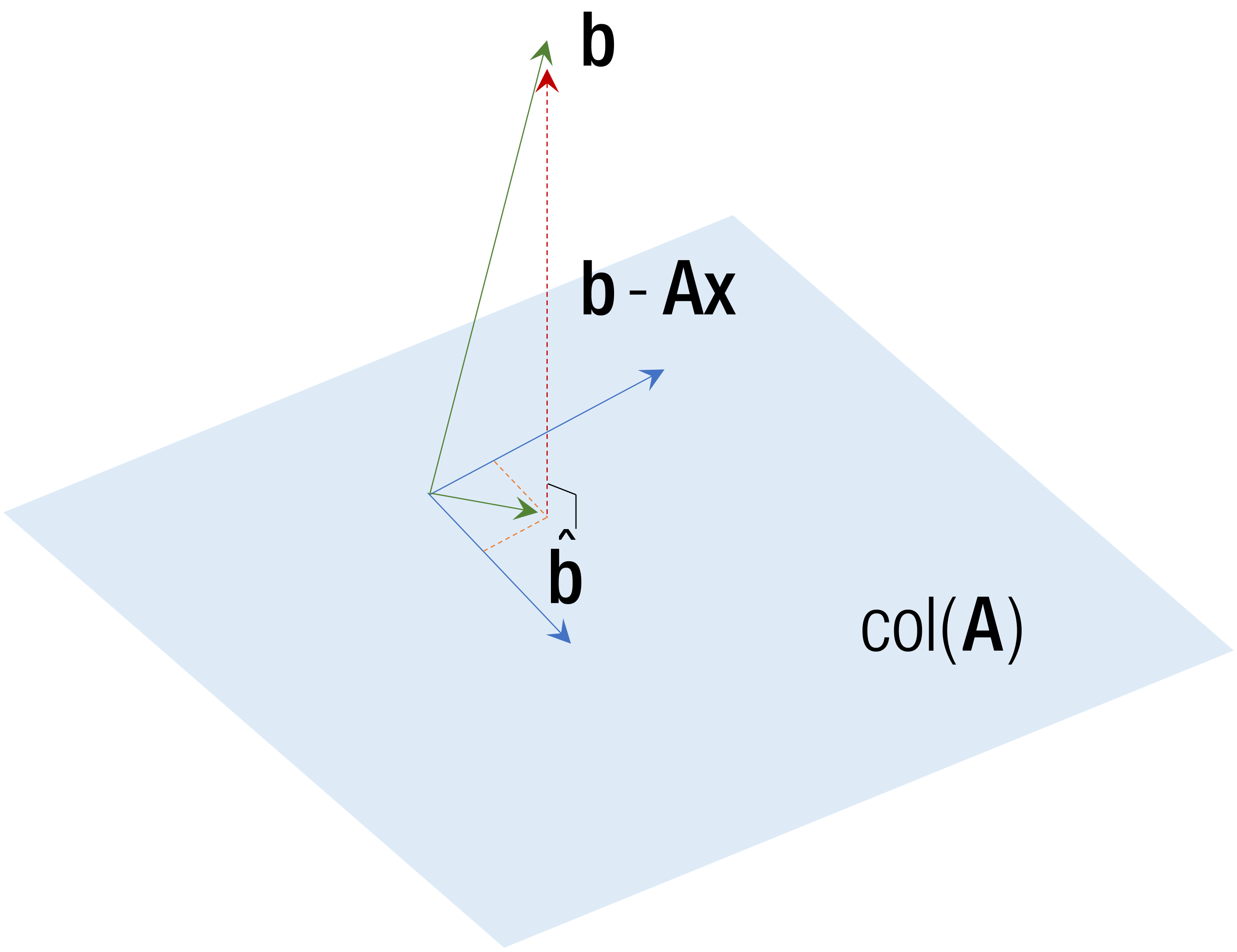

$$\mathbf{A}\,\mathbf{x} \approx \mathbf{b}$$

$$\mathbf{A}\,\mathbf{x} = \hat{\mathbf{b}}$$

$\hat{\mathbf{b}}$ : Best approximation of b spanned by col(A)

$$(\mathbf{b} - \mathbf{Ax}) \perp \text{col}(\mathbf{A})$$

# Geometric Interpretation




$$\mathbf{A}\,\mathbf{x} \approx \mathbf{b}$$

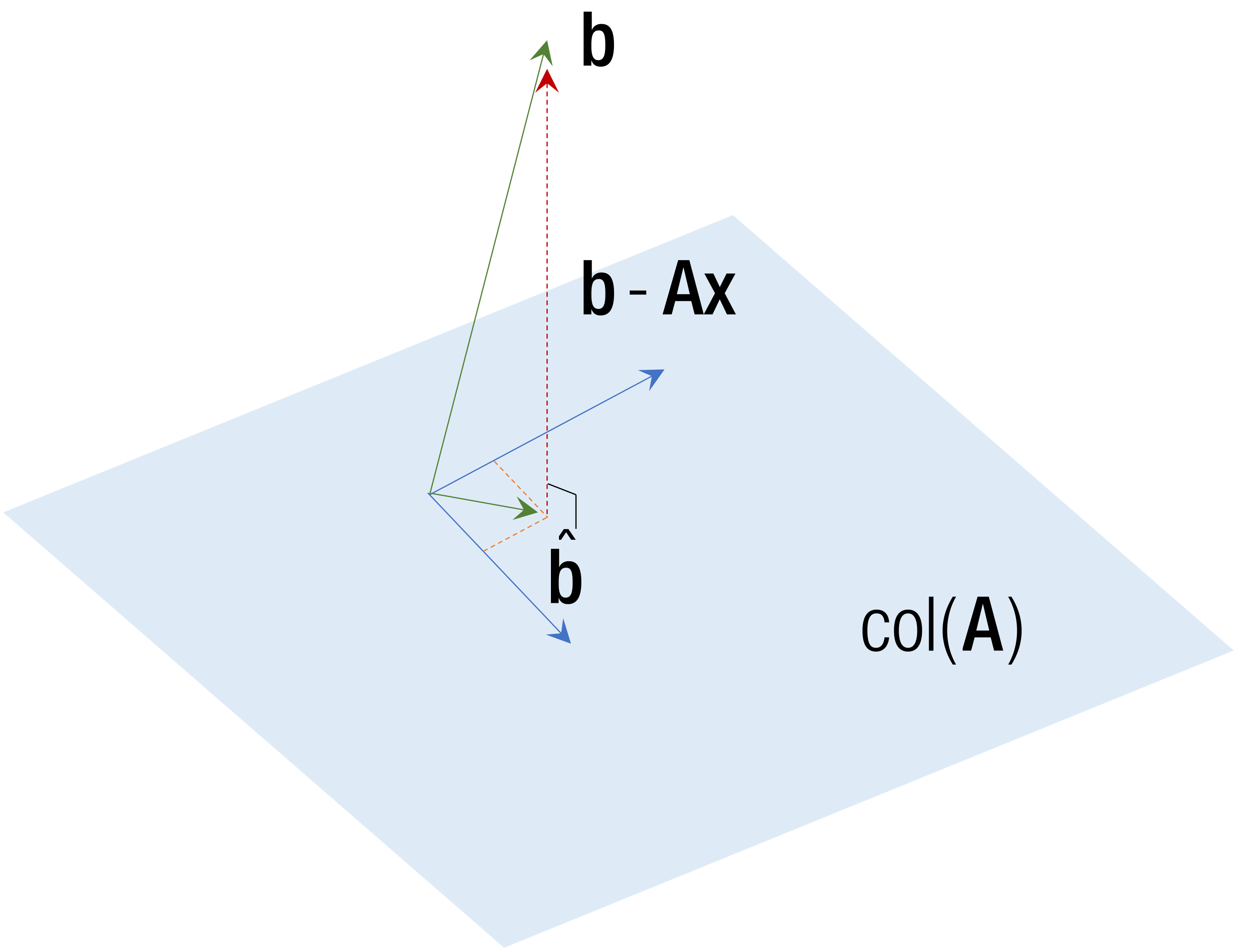
$$\mathbf{A}\,\mathbf{x} = \hat{\mathbf{b}}$$

$\hat{b}$ : Best approximation of b spanned by col(A)

$$(\mathbf{b} - \mathbf{Ax}) \perp \mathrm{col}(\mathbf{A})$$

$$\mathbf{A}^T(\mathbf{b} - \mathbf{Ax}) = \mathbf{0}$$

# Geometric Interpretation




$$\mathbf{A}\mathbf{x} \approx \mathbf{b}$$

$$\mathbf{A}\mathbf{x} = \hat{\mathbf{b}}$$

$\hat{b}$ : Best approximation of b spanned by col(A)

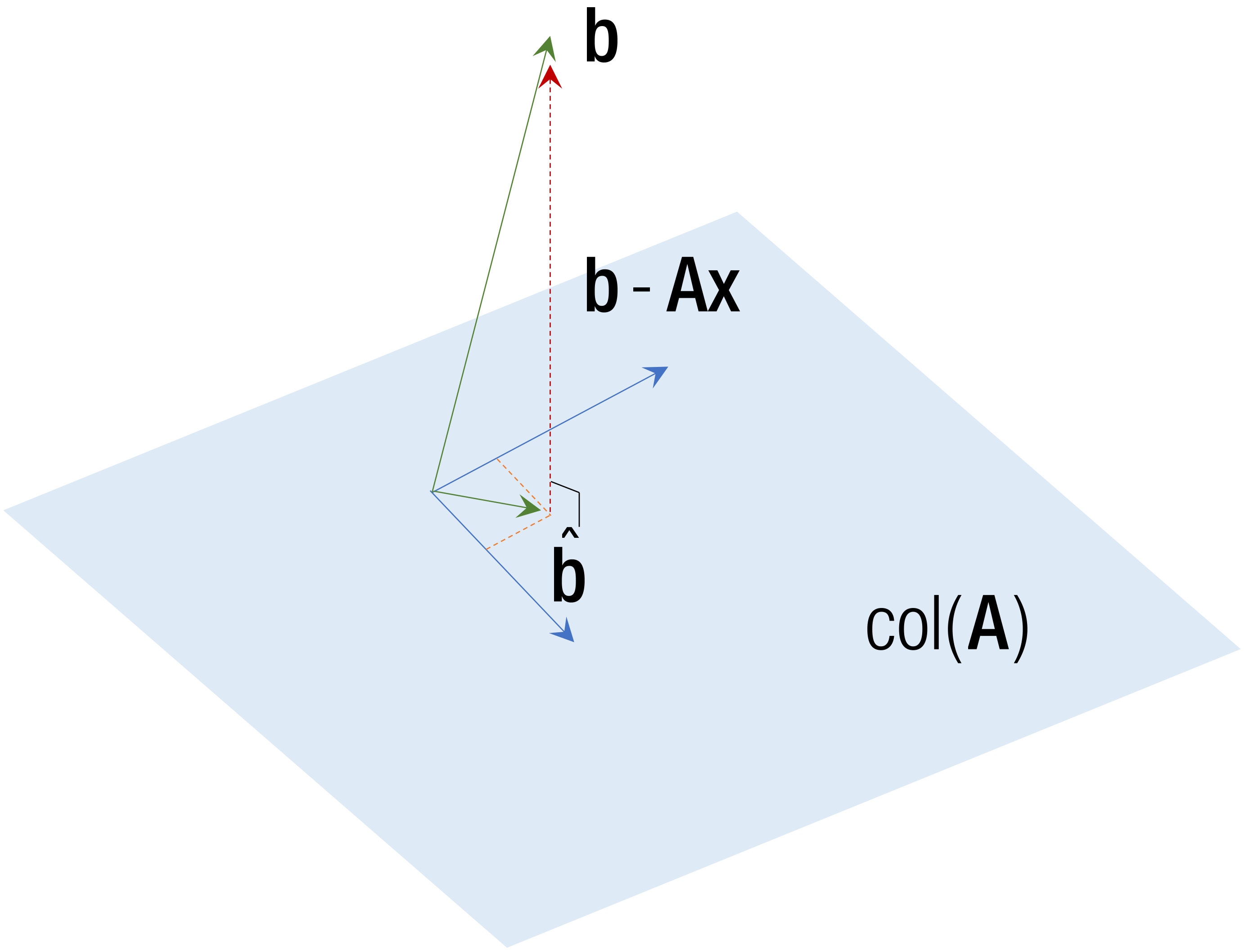
$$(\mathbf{b} - \mathbf{A}\mathbf{x}) \perp \mathrm{col}(\mathbf{A})$$

$$\mathbf{A}^T(\mathbf{b} - \mathbf{A}\mathbf{x}) = \mathbf{0}$$

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$$

: Normal equation

# Geometric Interpretation




$$A x \approx b$$

$$A x = \hat{b}$$

$\hat{b}$ : Best approximation of b spanned by col(A)

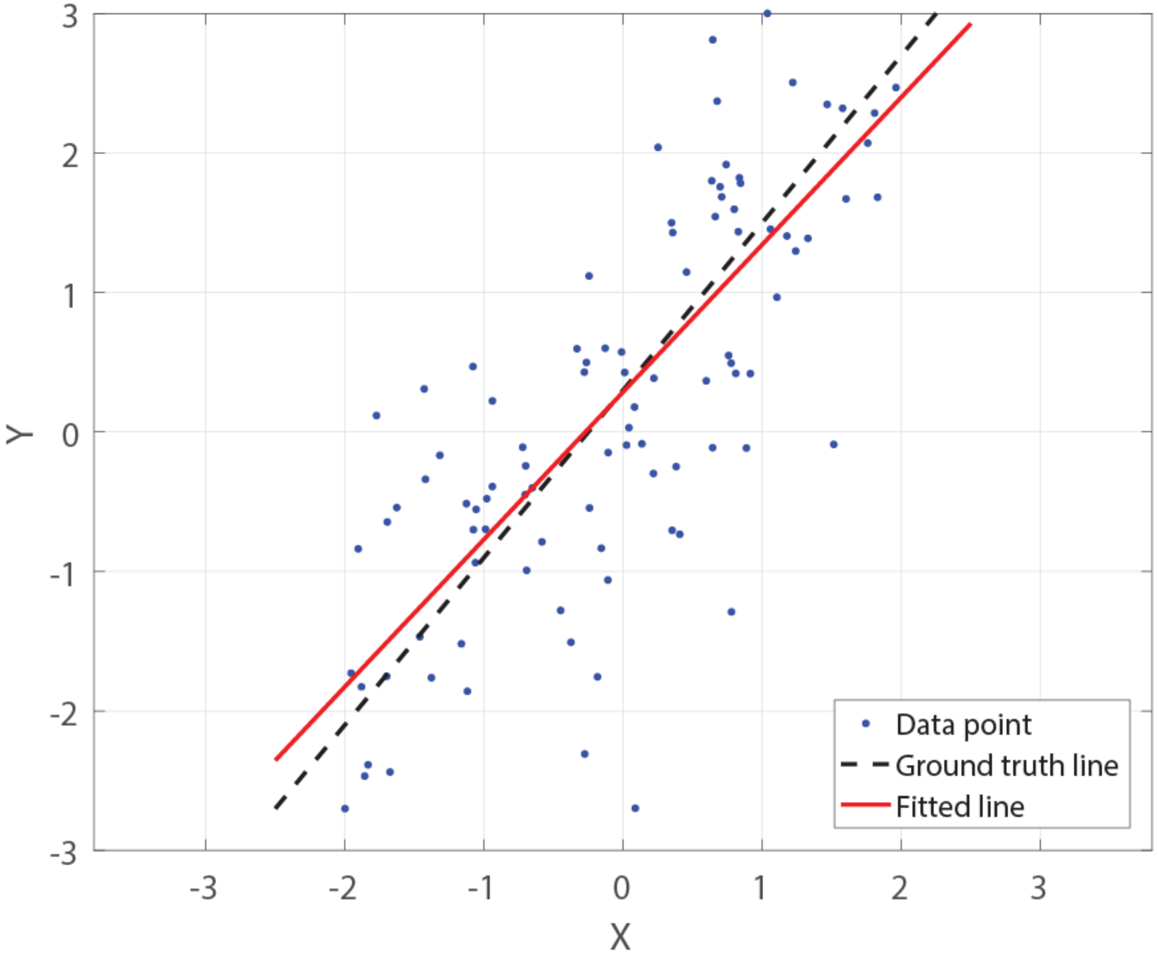
$$(b - Ax) \perp \text{col}(A)$$

$$A^T (b - Ax) = 0$$

$$A^T A x = A^T b$$

$$x = \left[ A^T A \right]^{-1} A^T b$$

# Line Fitting (Ax=b)




Total error: $E = \sum_{i=1}^{n} \left(v_i - (mu_i + d)\right)^2$

$$\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \begin{bmatrix} m \\ d \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{A} \quad \mathbf{x} \approx \mathbf{b}$$

$$E = (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$= \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} - 2\mathbf{x}^T\mathbf{A}^T\mathbf{b} + \mathbf{b}^T\mathbf{b}$$

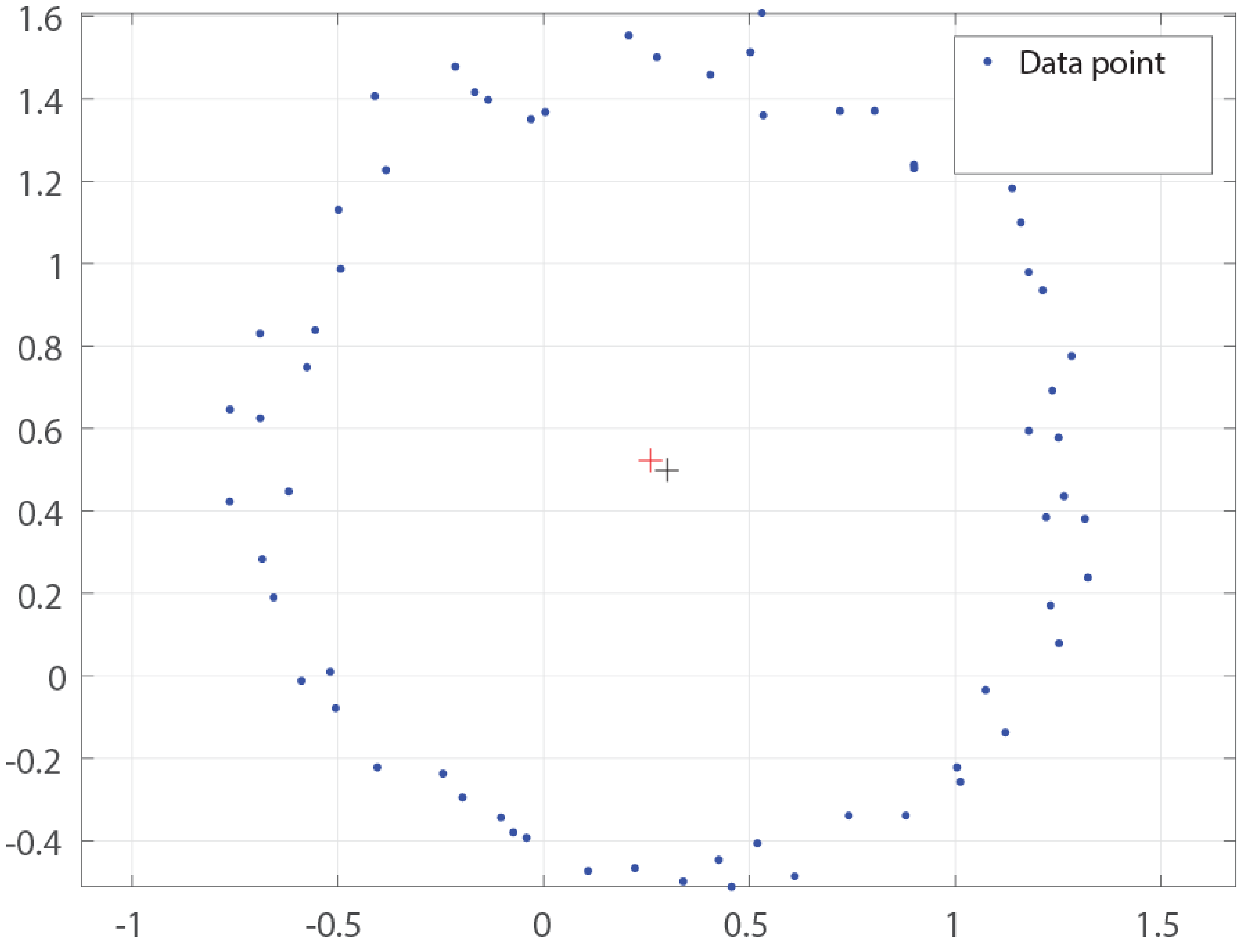
$$\frac{\partial E}{\partial \mathbf{x}} = 2\mathbf{A}^T\mathbf{Ax} - 2\mathbf{A}^T\mathbf{b} = \mathbf{0}$$

$$\longrightarrow \mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$$

$$\mathbf{A}^T \; \mathbf{A} \; \mathbf{x} = \mathbf{A}^T \; \mathbf{b}$$

$$\mathbf{x} = \left[\mathbf{A}^T \; \mathbf{A}\right]^{-1} \mathbf{A}^T \; \mathbf{b}$$

# Circle Fitting (Ax=b)



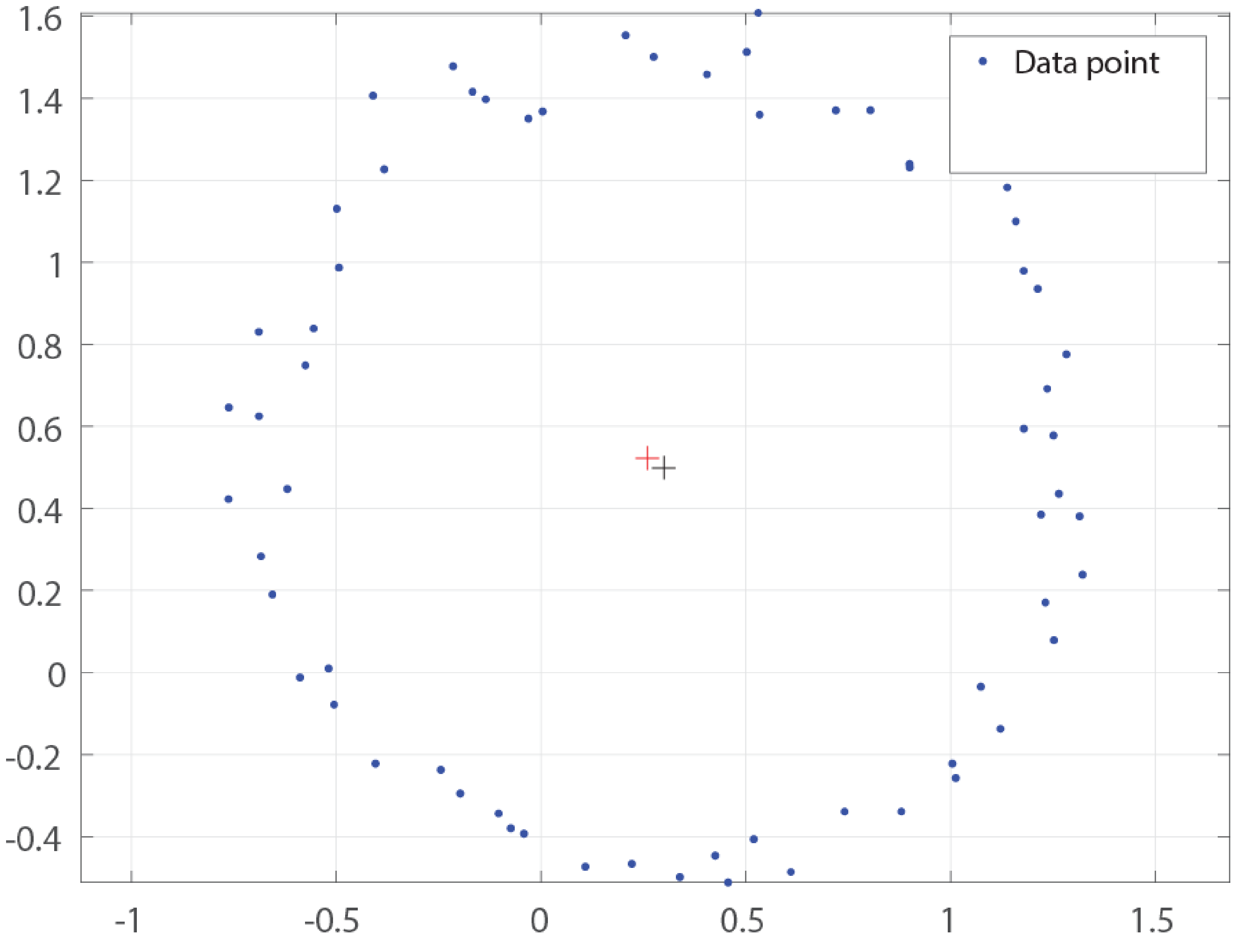

$$(x_1 - c_x)^2 + (y_1 - c_y)^2 = r^2$$

$$\vdots$$

$$(x_n - c_x)^2 + (y_n - c_y)^2 = r^2$$

Unkowns: $c_x, c_y, r$

# Circle Fitting (Ax=b)

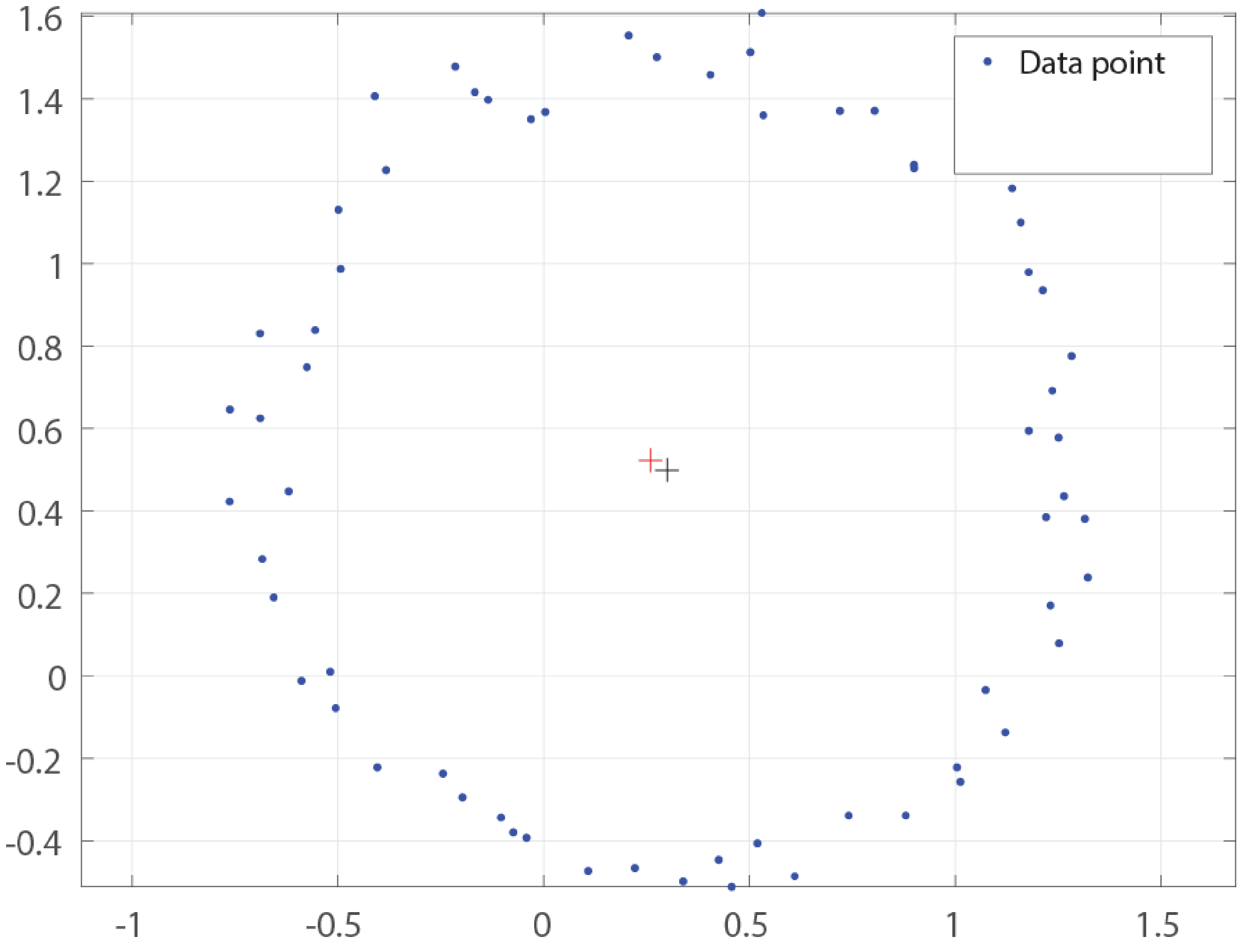



$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

# Circle Fitting (Ax=b)




$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

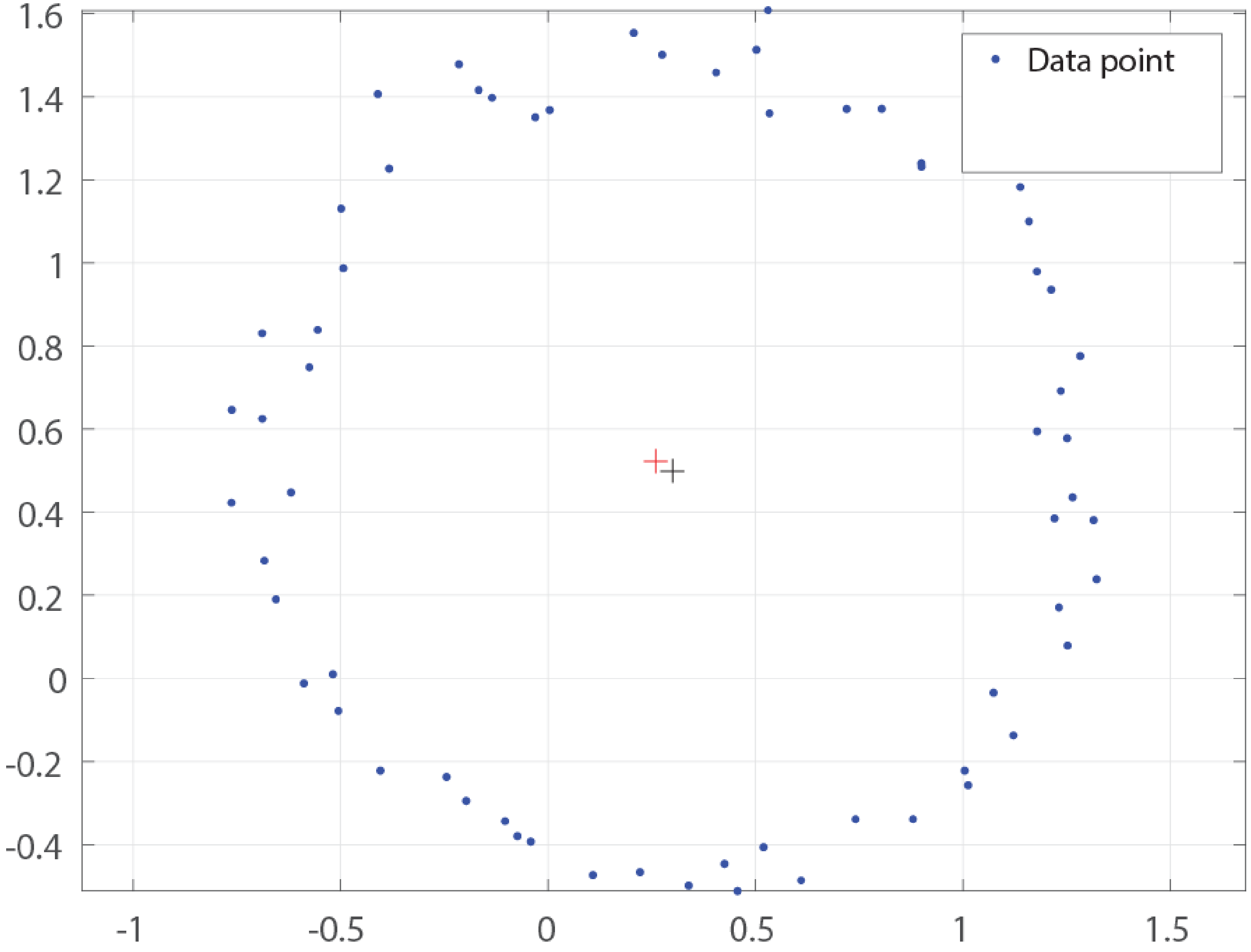
$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

$$\downarrow$$

$$x_i^2 - x_1^2 - 2c_x(x_i - x_1) + y_i^2 - y_1^2 - 2(y_i - y_1)c_y = 0$$

# Circle Fitting (Ax=b)




$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

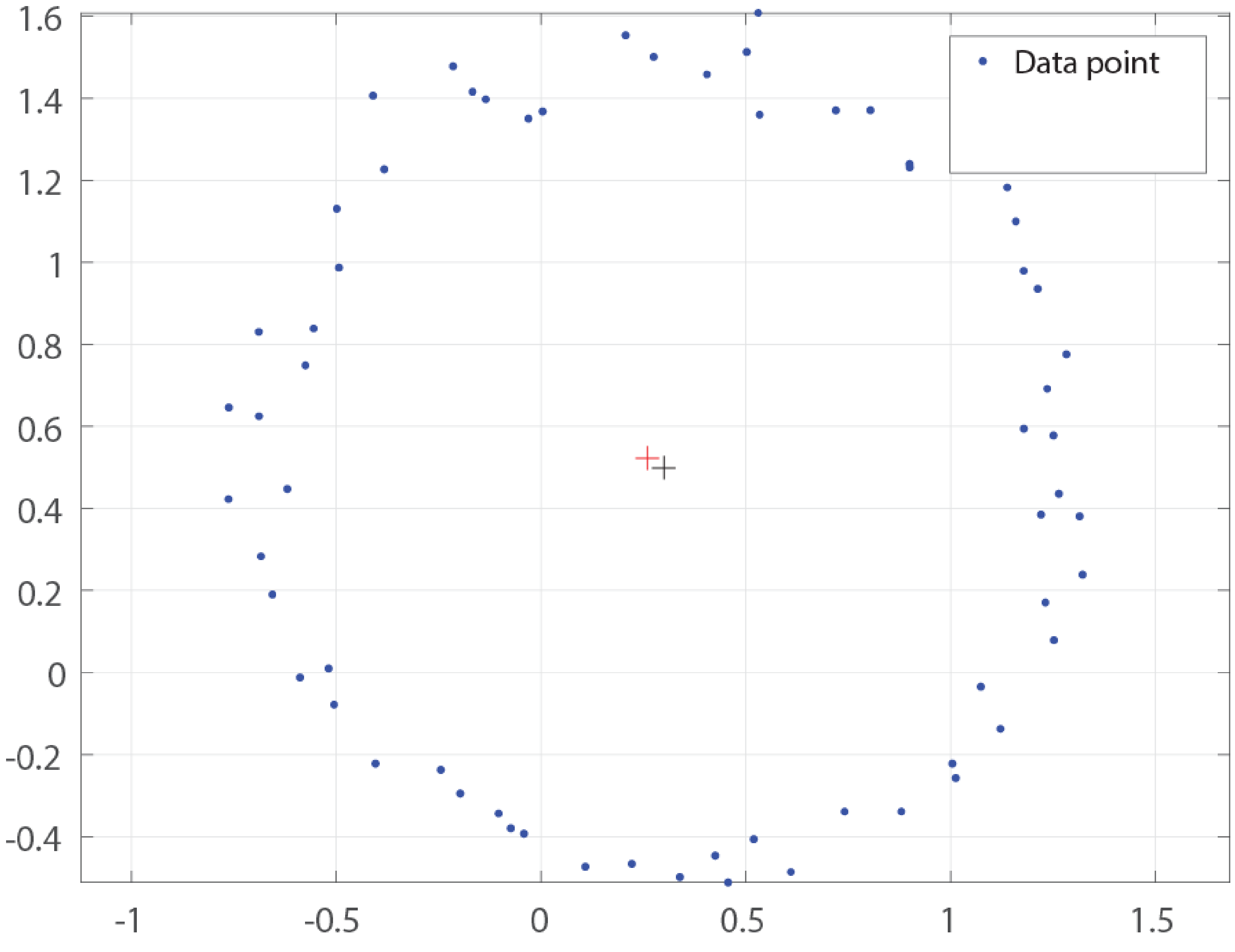
$$\downarrow$$

$$x_i^2 - x_1^2 - 2c_x\left(x_i - x_1\right) + y_i^2 - y_1^2 - 2\left(y_i - y_1\right)c_y = 0$$

$$\downarrow$$

$$\begin{bmatrix} 2\left(x_2 - x_1\right) & 2\left(y_2 - y_1\right) \\ \vdots & \vdots \\ 2\left(x_n - x_1\right) & 2\left(y_n - y_1\right) \end{bmatrix} \begin{bmatrix} c_x \\ c_y \end{bmatrix} = \begin{bmatrix} x_2^2 - x_1^2 + y_2^2 - y_1^2 \\ \\ x_n^2 - x_1^2 + y_n^2 - y_1^2 \end{bmatrix}$$

# Circle Fitting (Ax=b)




$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

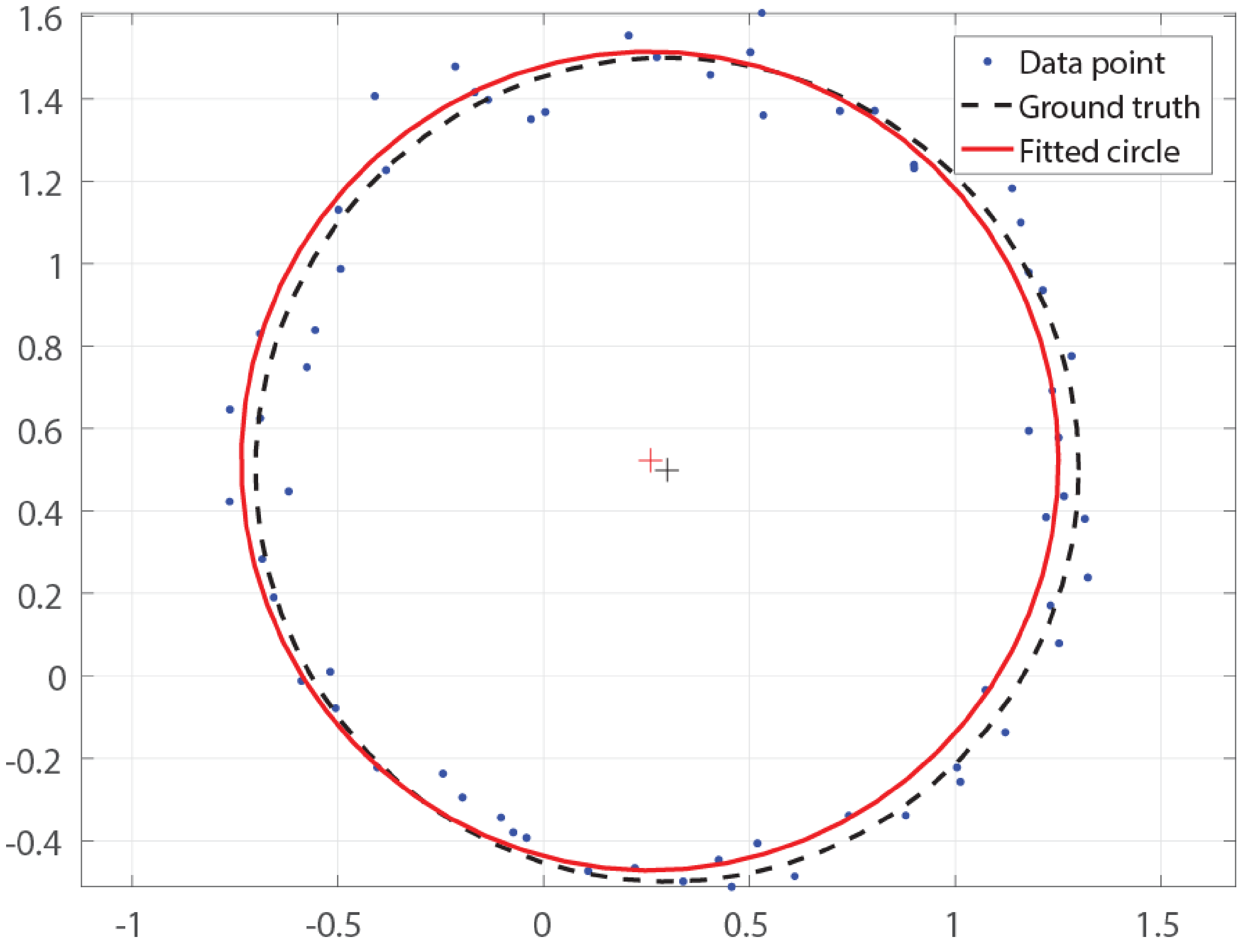
$$\downarrow$$

$$x_i^2 - x_1^2 - 2c_x\left(x_i - x_1\right) + y_i^2 - y_1^2 - 2\left(y_i - y_1\right)c_y = 0$$

$$\downarrow$$

$$\underbrace{\begin{bmatrix} 2(x_2 - x_1) & 2(y_2 - y_1) \\ \vdots & \vdots \\ 2(x_n - x_1) & 2(y_n - y_1) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} c_x \\ c_y \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} x_2^2 - x_1^2 + y_2^2 - y_1^2 \\ \vdots \\ x_n^2 - x_1^2 + y_n^2 - y_1^2 \end{bmatrix}}_{\mathbf{b}}$$

# Circle Fitting (Ax=b)




$$x_1^2 - 2x_1 c_x + c_x^2 + y_1^2 - 2y_1 c_y + c_y^2 = r^2$$

$$\vdots$$

$$x_n^2 - 2x_n c_x + c_x^2 + y_n^2 - 2y_n c_y + c_y^2 = r^2$$

$$x_i^2 - x_1^2 - 2c_x\left(x_i - x_1\right) + y_i^2 - y_1^2 - 2\left(y_i - y_1\right)c_y = 0$$

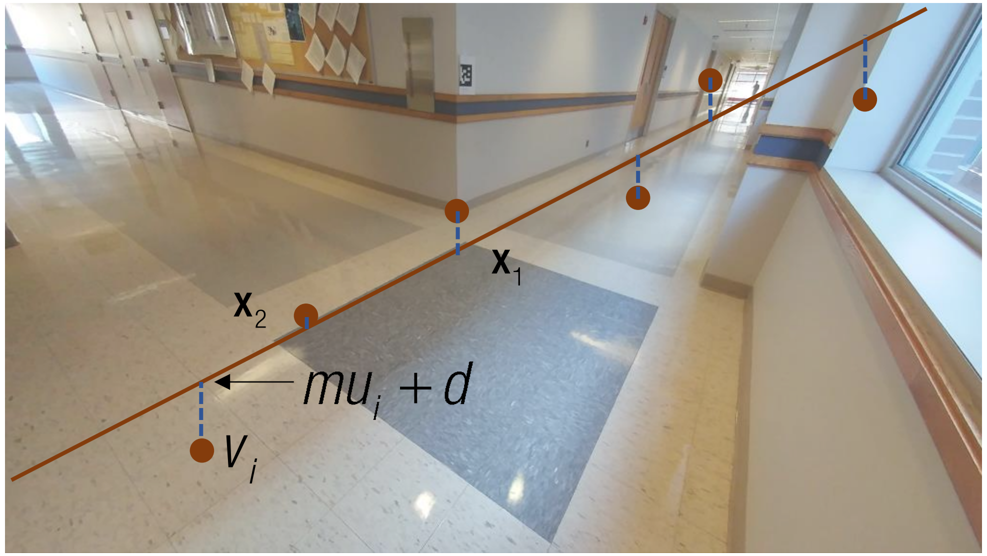
$$\underbrace{\begin{bmatrix} 2(x_2 - x_1) & 2(y_2 - y_1) \\ \vdots & \vdots \\ 2(x_n - x_1) & 2(y_n - y_1) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} c_x \\ c_y \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} x_2^2 - x_1^2 + y_2^2 - y_1^2 \\ \vdots \\ x_n^2 - x_1^2 + y_n^2 - y_1^2 \end{bmatrix}}_{\mathbf{b}}$$

# Line Fitting (Ax=b)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c) \longrightarrow (\underline{m},\underline{d})$

slope y-intercept




$$
\begin{aligned}
au_1 + bv_1 + c &\approx 0 \\
au_2 + bv_2 + c &\approx 0 \\
&\vdots \\
au_n + bv_n + c &\approx 0
\end{aligned}
\longrightarrow
\begin{aligned}
v_1 &\approx mu_1 + d \\
v_2 &\approx mu_2 + d \\
&\vdots \\
v_n &\approx mu_n + d
\end{aligned}
$$

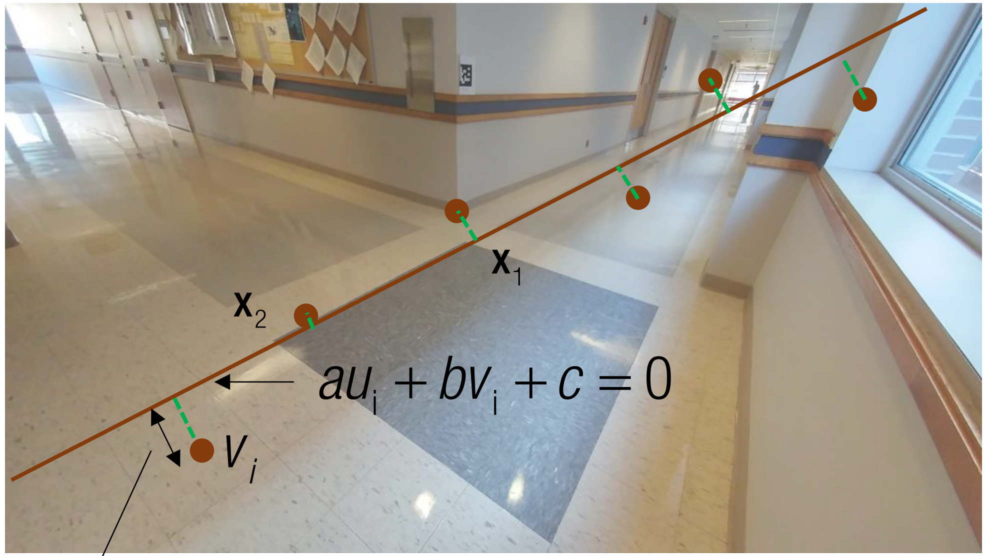
$$\mathbf{Ax} = \mathbf{b}$$

What is different?

# Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c)$




$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

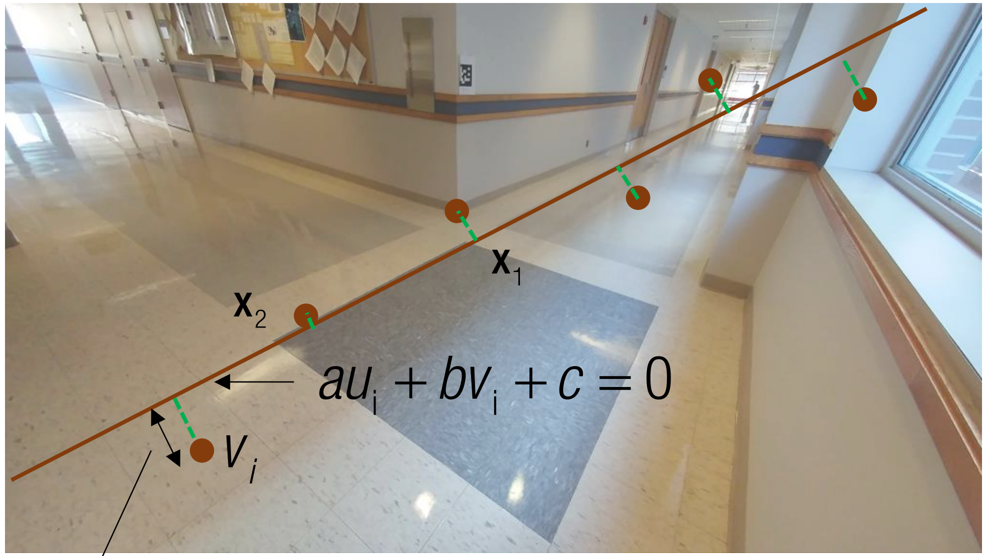
$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$\mathbf{Ax} = \mathbf{0}$ $\qquad$ $\mathbf{Ax} = \mathbf{b}$

# Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c)$

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Trivial solution: $\mathbf{x} = \mathbf{0}$

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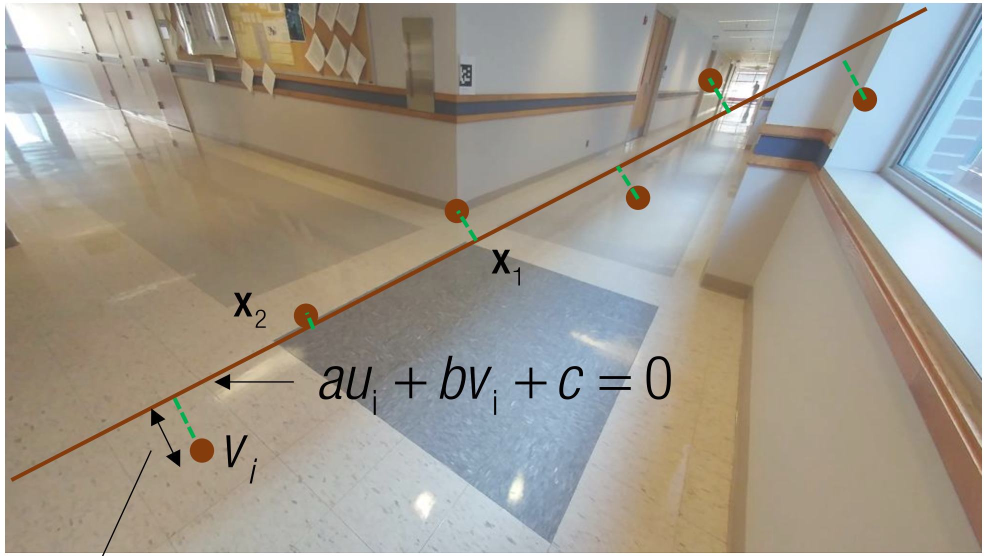


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$\mathbf{Ax} = \mathbf{0}$ $\qquad$ $\mathbf{Ax} = \mathbf{b}$

Trivial solution: $\mathbf{x} = \mathbf{0}$

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2$$

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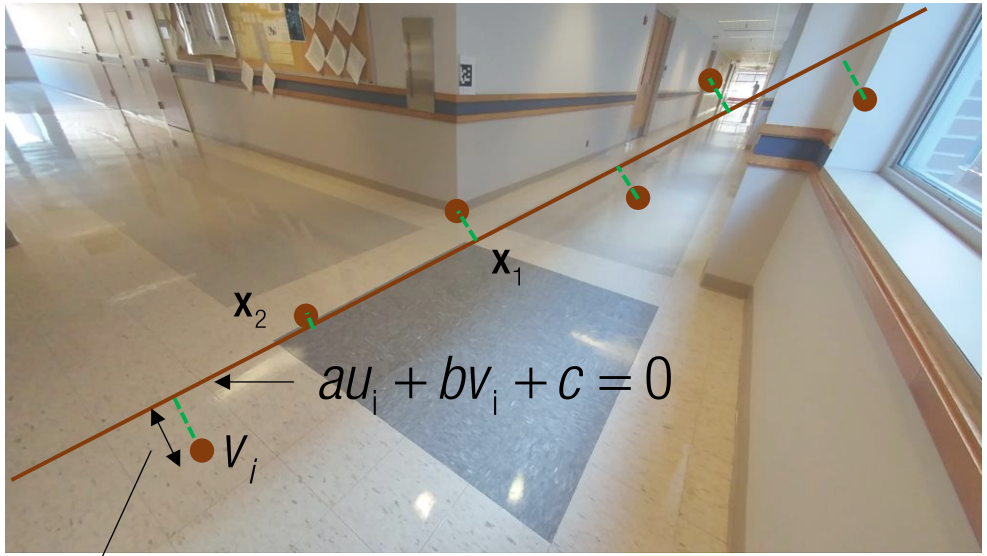
Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c)$




$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

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$\mathbf{Ax} = \mathbf{0}$ $\qquad$ $\mathbf{Ax} = \mathbf{b}$

Trivial solution: $\mathbf{x} = \mathbf{0}$

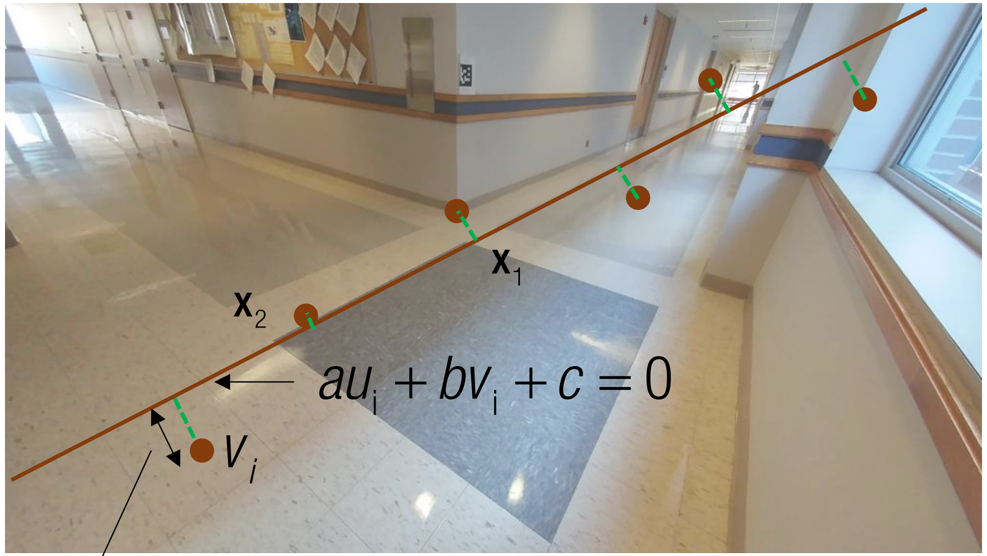
$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \qquad \text{subject to} \quad \|\mathbf{x}\| = 1$$

Condition to avoid the trivial solution

# Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c)$




$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

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$\mathbf{Ax} = \mathbf{0}$ $\qquad$ $\mathbf{Ax} = \mathbf{b}$

Trivial solution: $\mathbf{x} = \mathbf{0}$

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \quad \text{subject to} \quad \|\mathbf{x}\| = 1$$

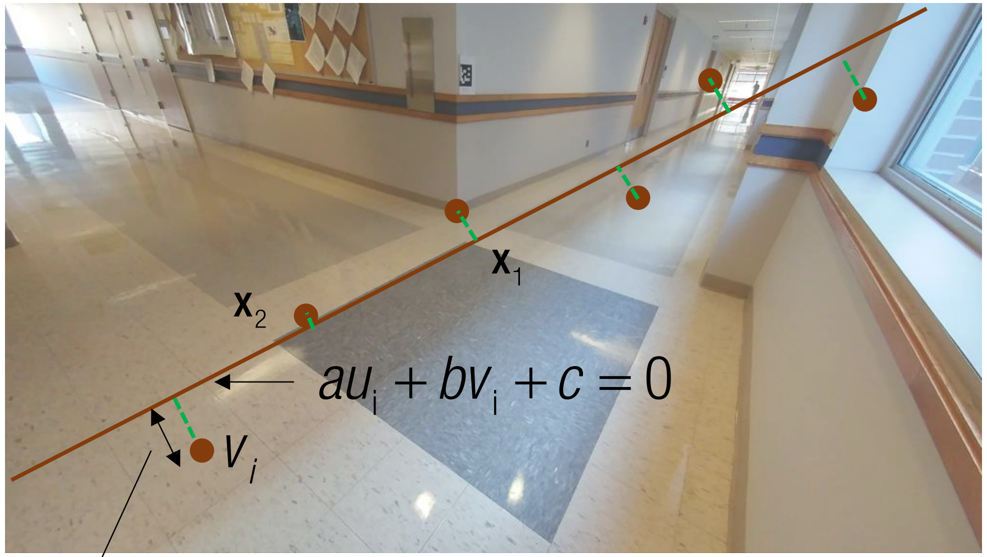
Condition to avoid the trivial solution

How to solve?

# Line Fitting (Ax=0)

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c)$

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Trivial solution: $\mathbf{x} = \mathbf{0}$

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \qquad \text{subject to} \quad \|\mathbf{x}\| = 1$$

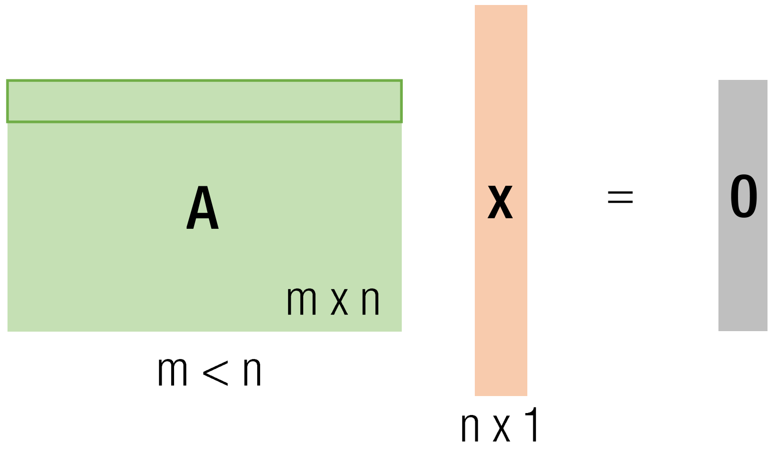
Condition to avoid the trivial solution

How to solve? approximated null space $\mathbf{x} = \text{null}(\mathbf{A})$

# Nullspace

# eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.

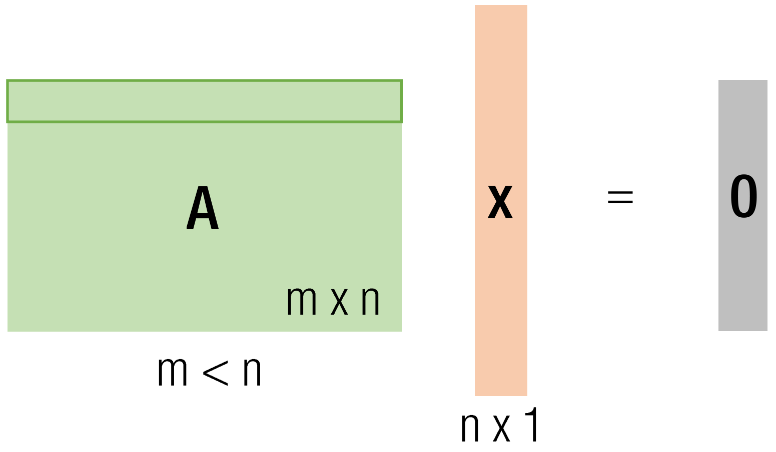


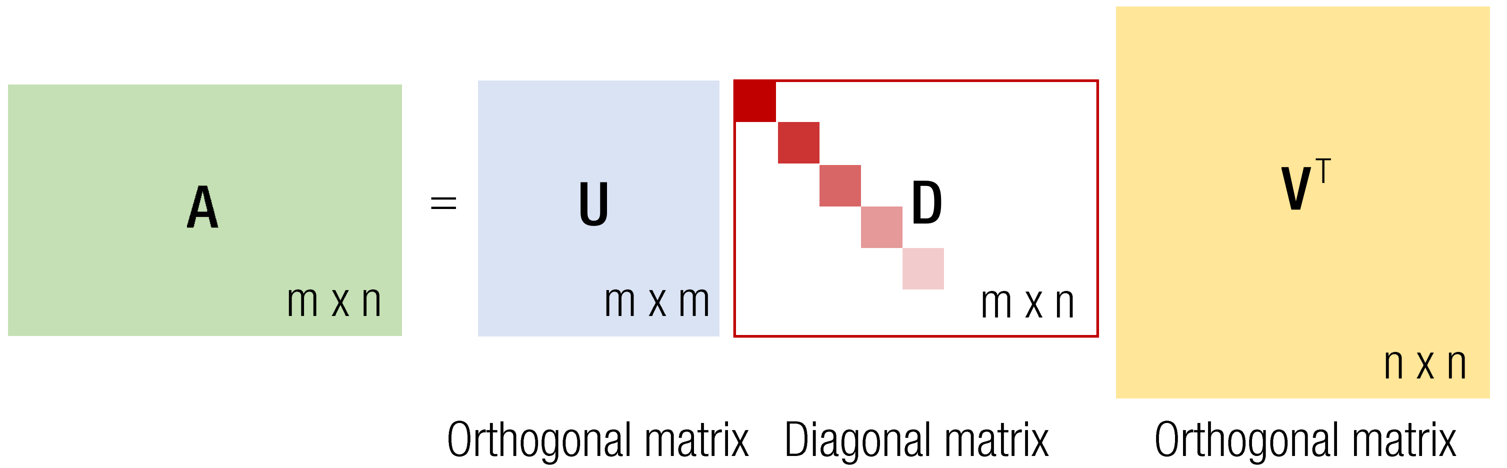

$$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$$

# Singular Value Decomposition (SVD)

# eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.



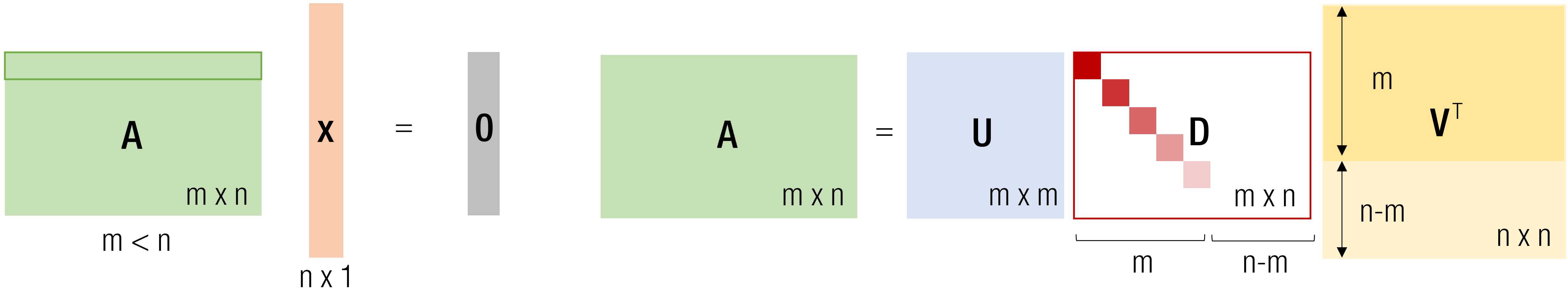




$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$

# Singular Value Decomposition (SVD)

\# eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.

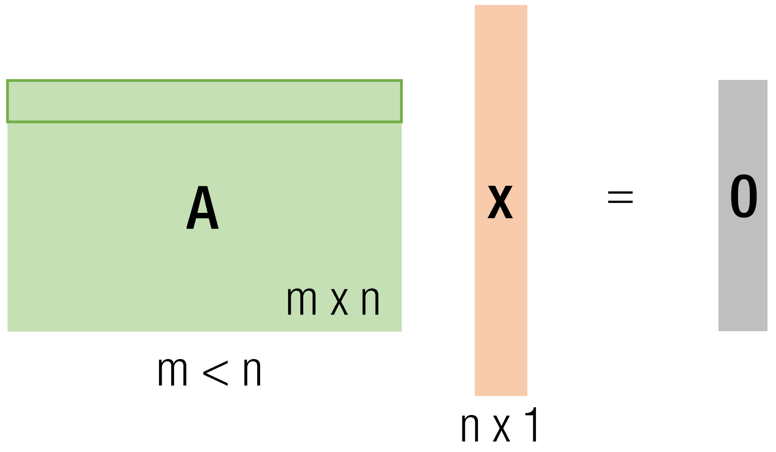


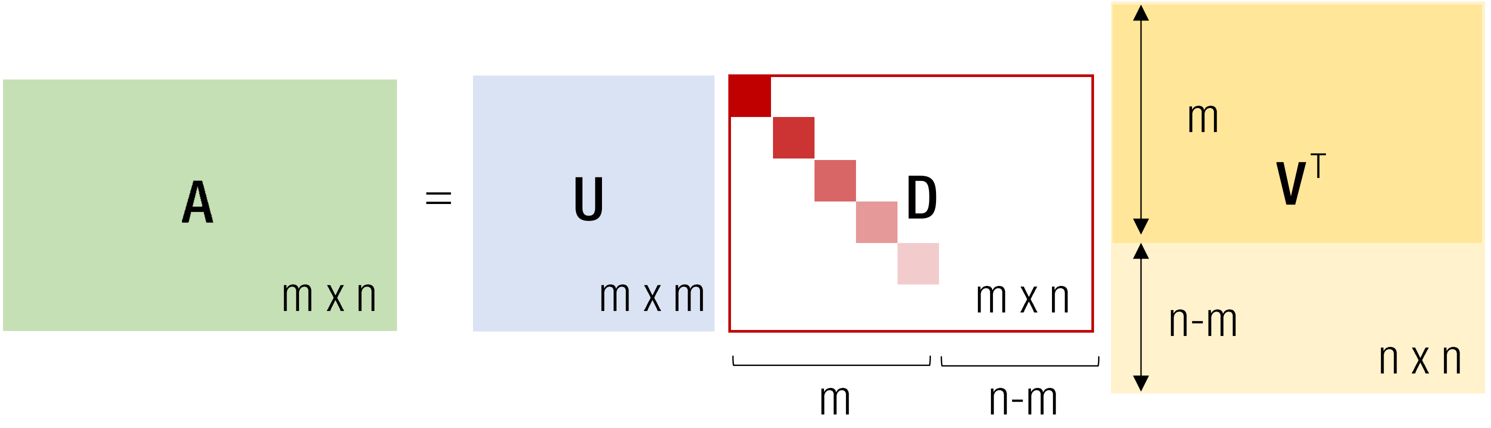

$$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$$

# Singular Value Decomposition (SVD)

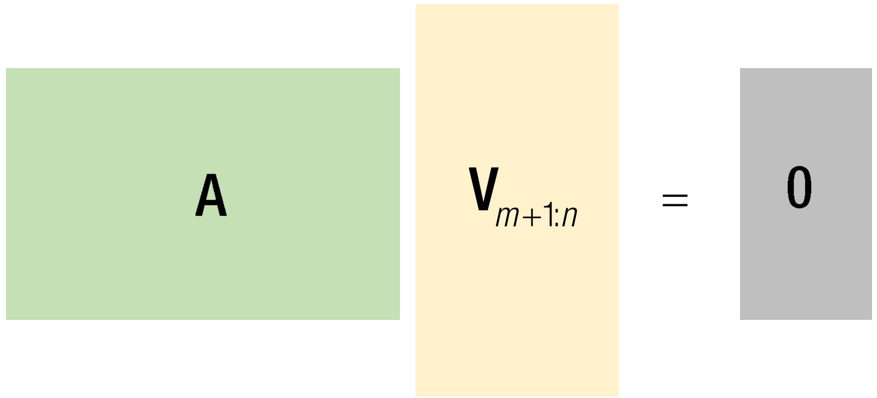
\# eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.




$$\mathbf{x} = \text{null}(\mathbf{A})$$

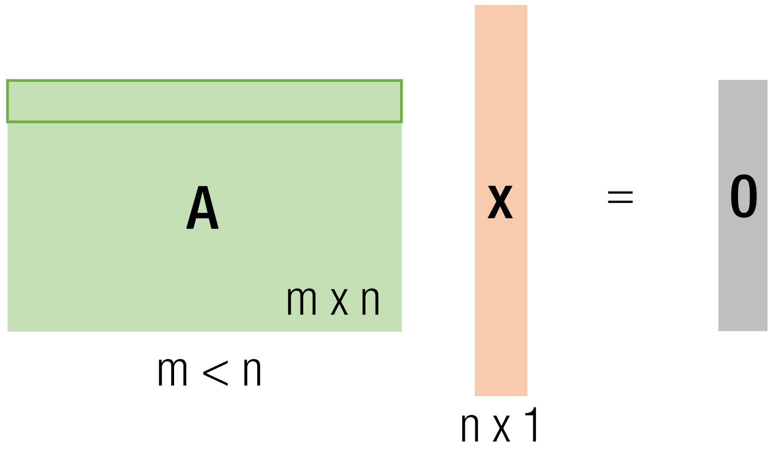


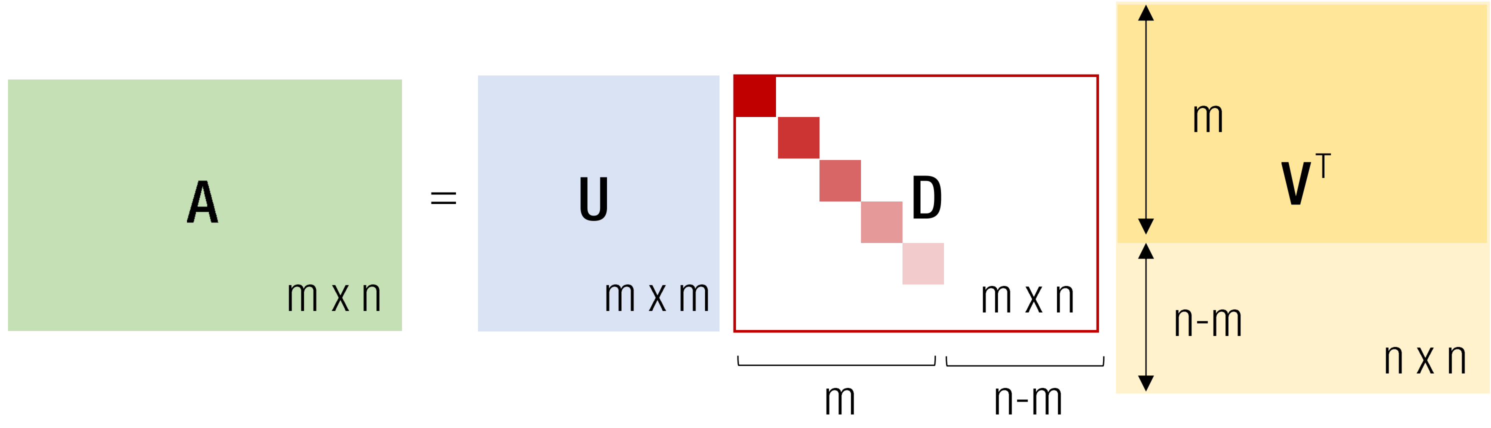

$$\mathbf{V}_{m+1:n} = \text{null}(\mathbf{A})$$

# Singular Value Decomposition (SVD)

# eqs < # unknowns

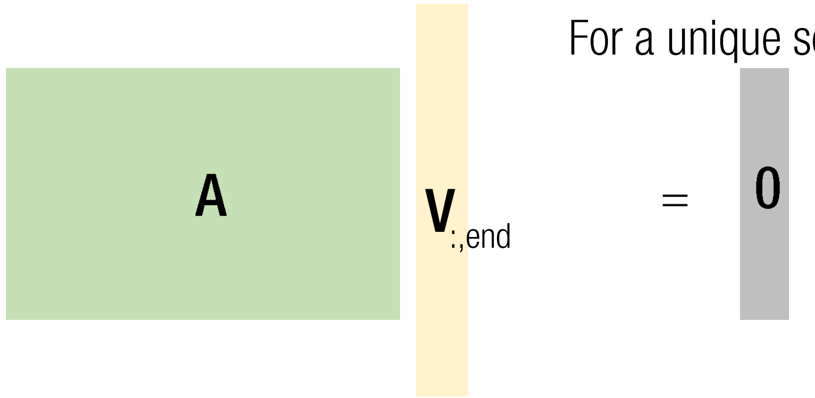
There exist at least n-m vectors that makes **Ax**=**0**.




$$\mathbf{x} = \text{null}(\mathbf{A})$$

For a unique solution, **A** should be m x (m+1)

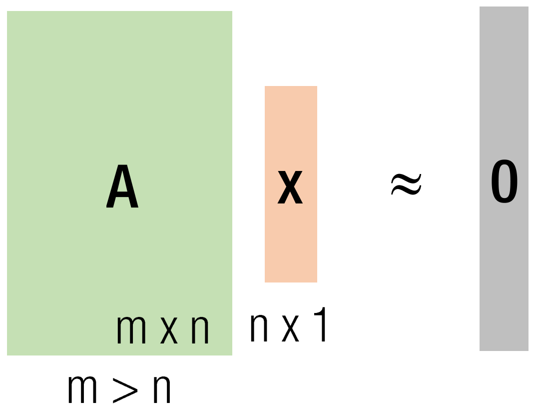


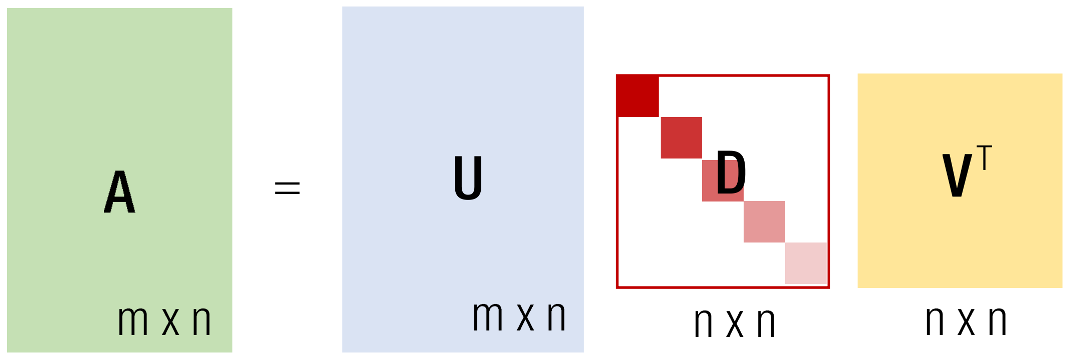

$$\mathbf{V}_{:end} = \text{null}(\mathbf{A})$$

# Singular Value Decomposition (SVD)

# eqs > # unknowns

There exist no nullspace of **A**.



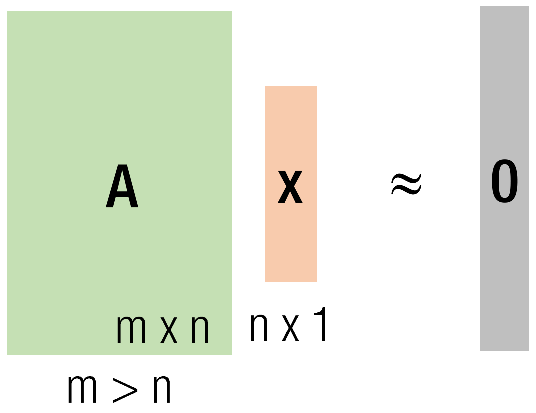



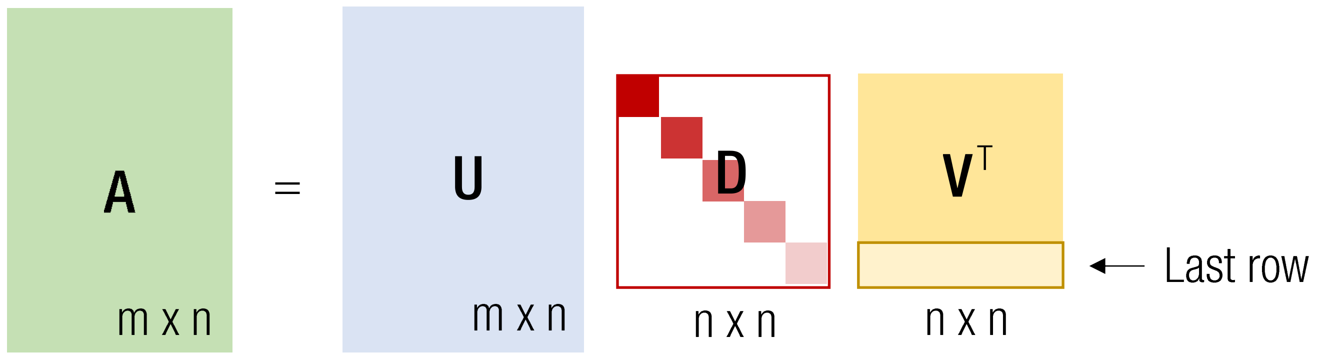

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \text{ subject to } \|\mathbf{x}\| = 1$$

# Singular Value Decomposition (SVD)

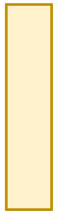
# eqs > # unknowns

There exist no nullspace of **A**.

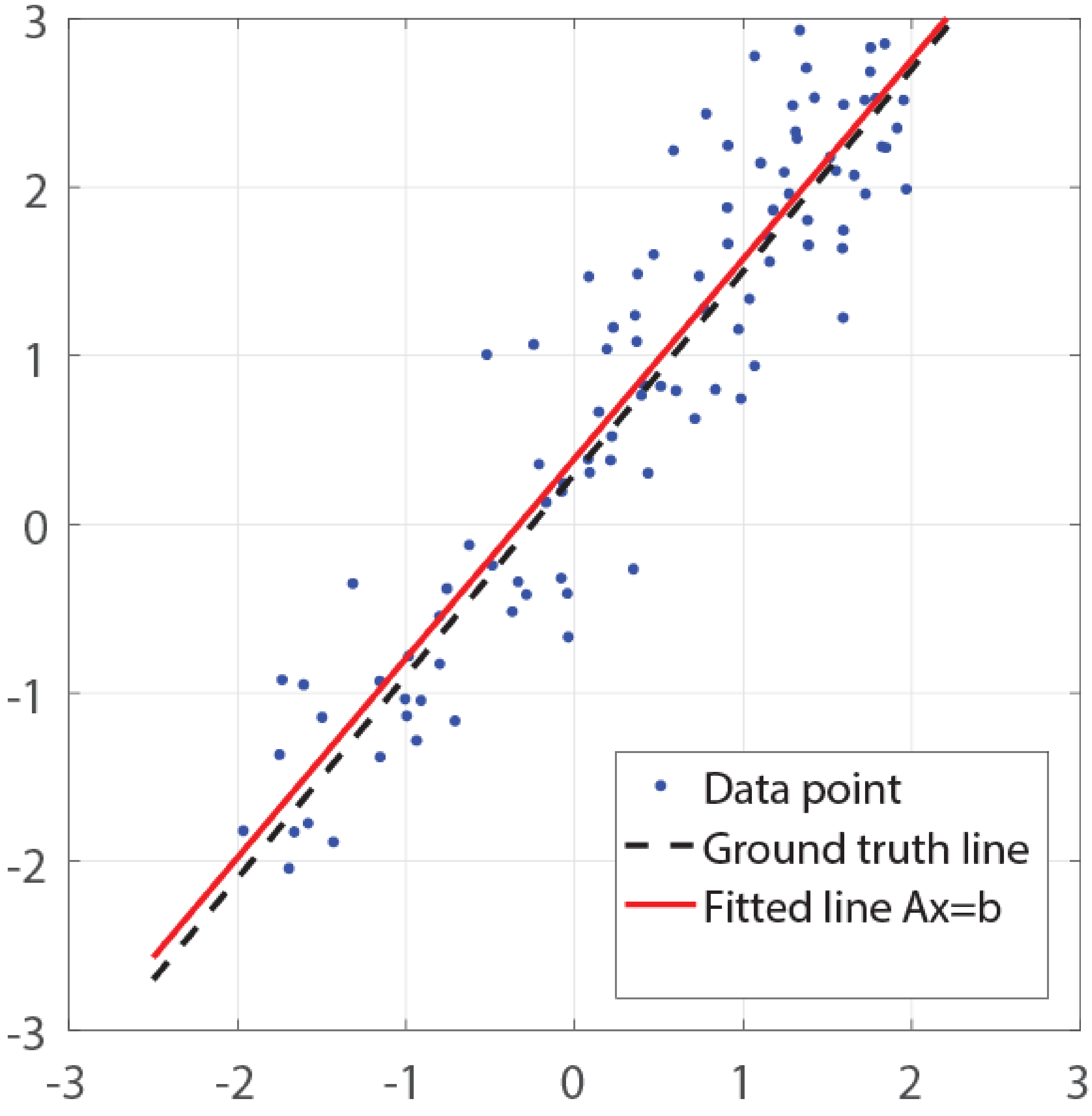



$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{Ax}\|^2 \text{ subject to } \|\mathbf{x}\| = 1$$

Approximated nullspace of **A**:  $\mathbf{V}_{:,end}$

# Line Fitting (Ax=b)



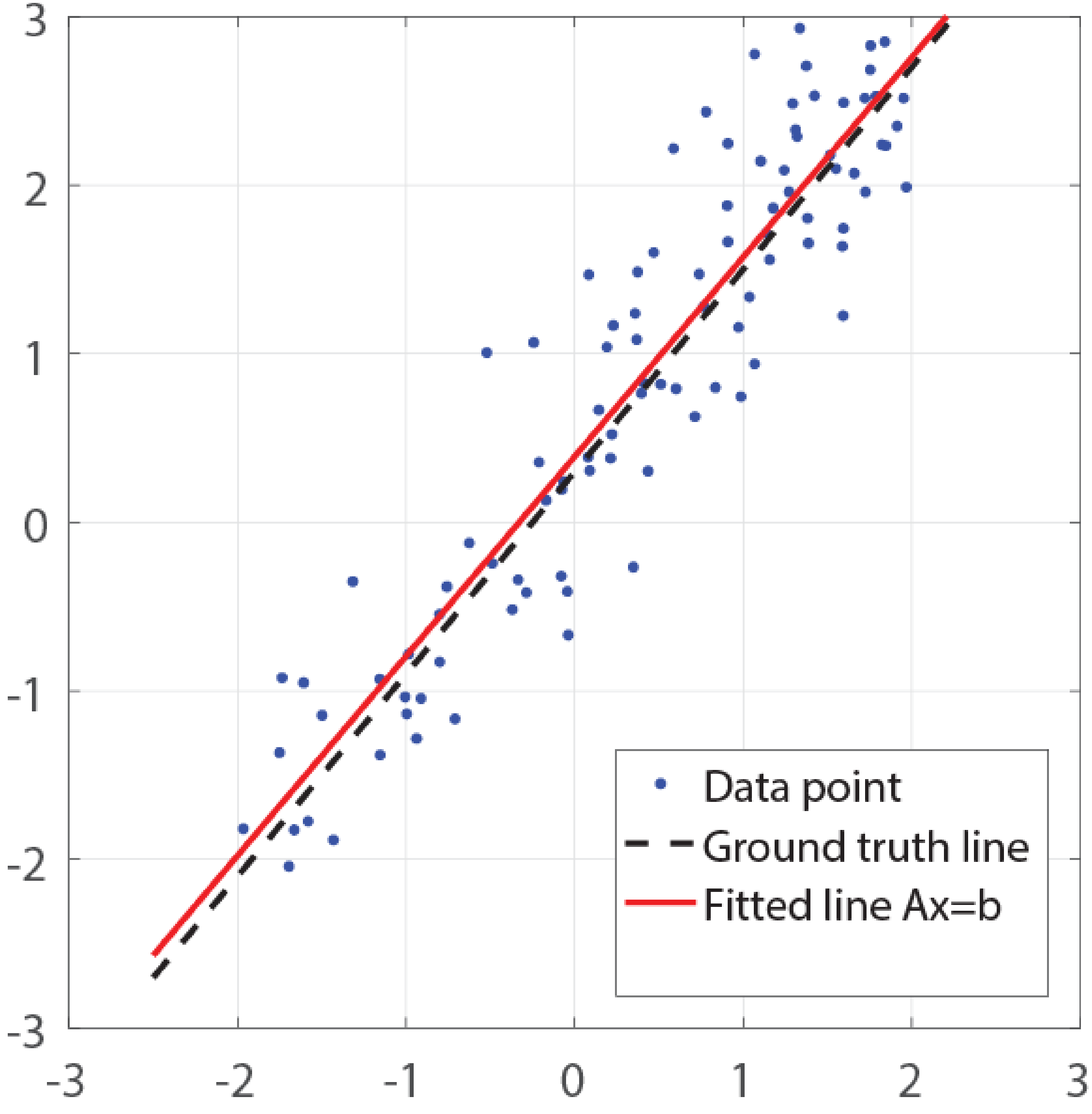

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(m,d)$

$$v_1 \approx mu_1 + d$$
$$v_2 \approx mu_2 + d$$
$$\vdots$$
$$v_n \approx mu_n + d$$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

# Line Fitting (Ax=b)



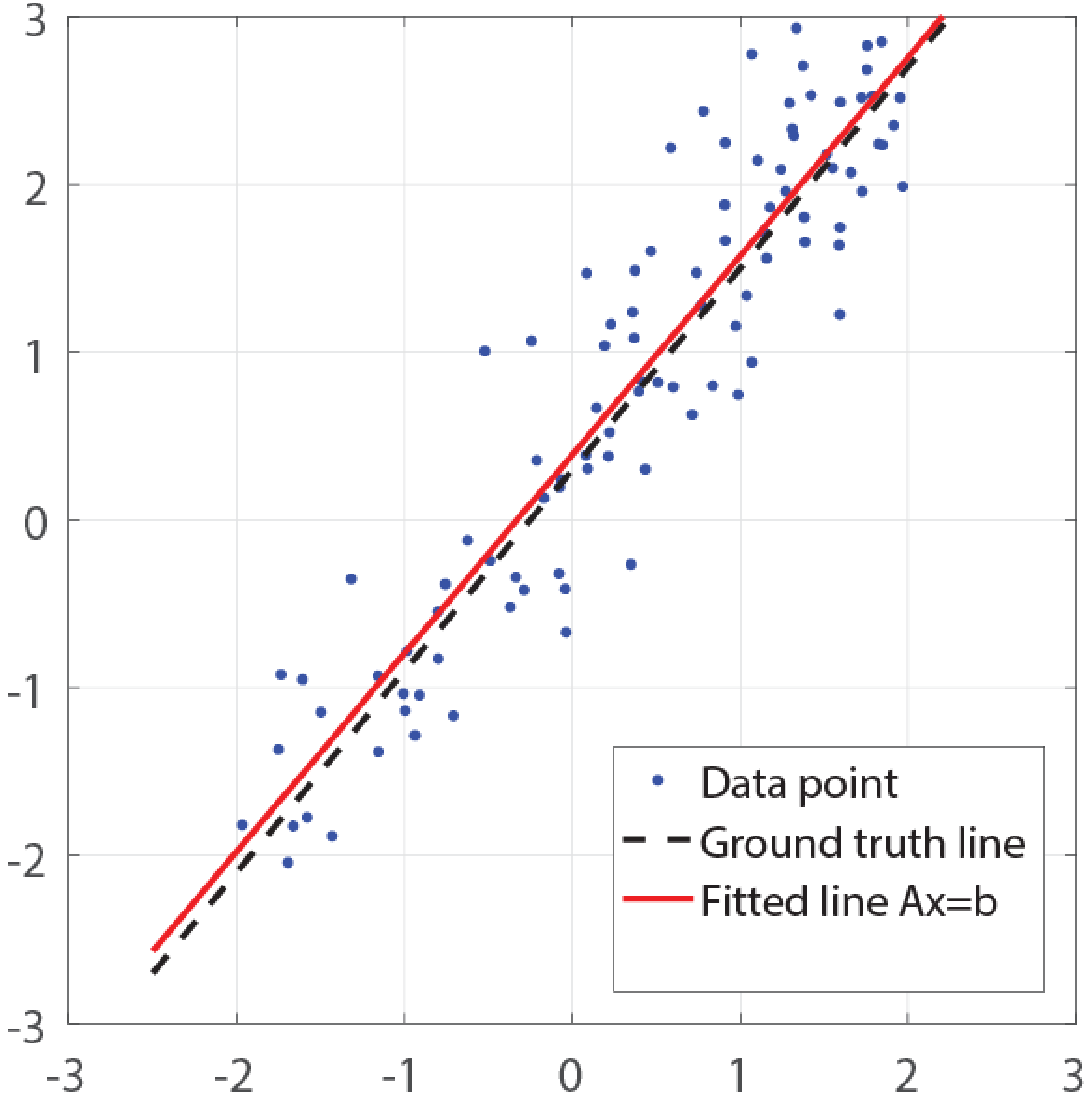

Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c)$ $(m,d)$

$$
\begin{aligned}
au_1 + bv_1 + c &\approx 0 \\
au_2 + bv_2 + c &\approx 0 \\
&\vdots \\
au_n + bv_n + c &\approx 0
\end{aligned}
\qquad
\begin{aligned}
v_1 &\approx mu_1 + d \\
v_2 &\approx mu_2 + d \\
&\vdots \\
v_n &\approx mu_n + d
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}}
\underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}}
\approx
\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}
$$

# Line Fitting (Ax=b)




Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c)$ $(m,d)$

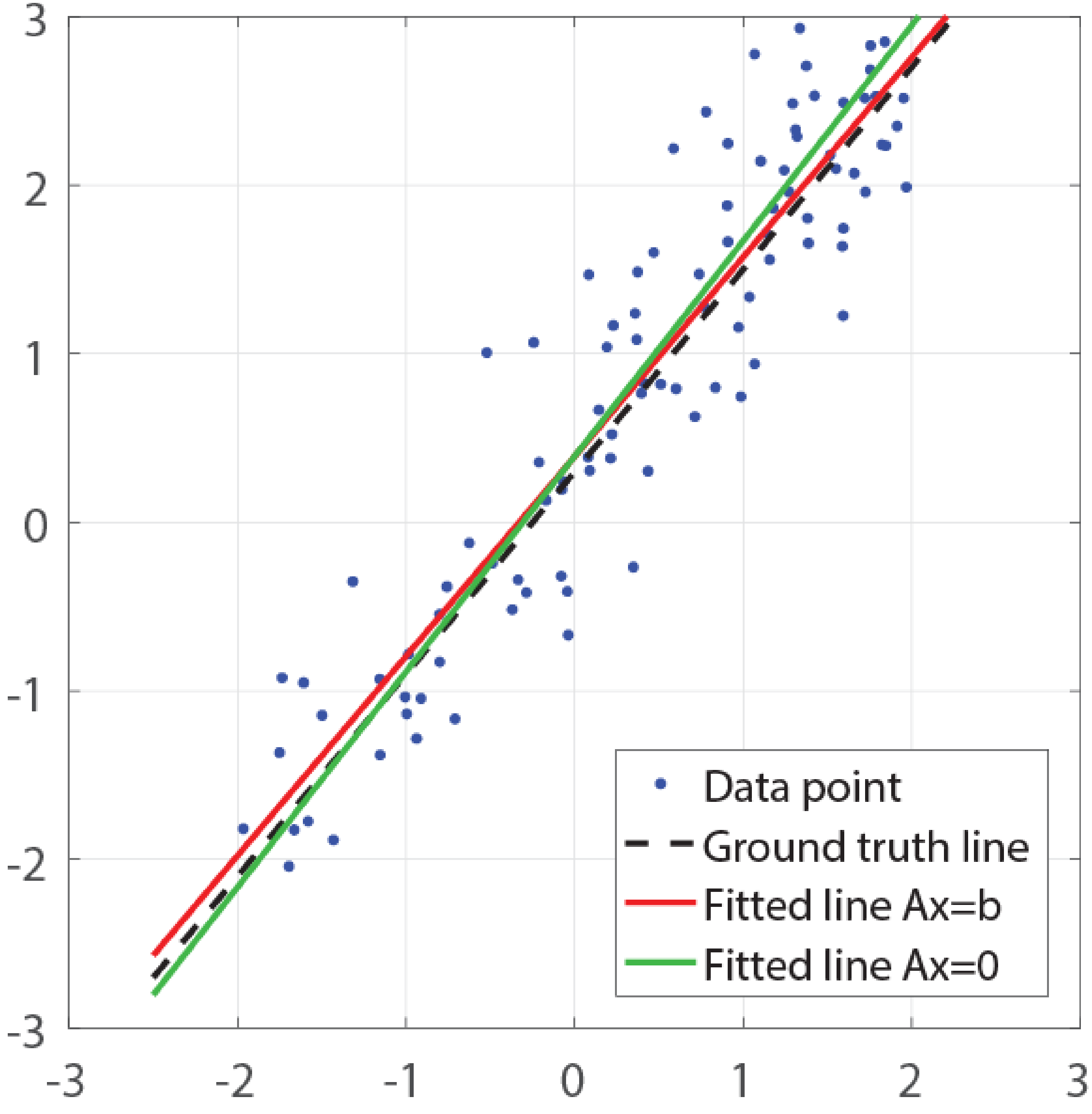
$$\begin{aligned} au_1 + bv_1 + c &\approx 0 \\ au_2 + bv_2 + c &\approx 0 \\ &\vdots \\ au_n + bv_n + c &\approx 0 \end{aligned}$$

$$\begin{aligned} v_1 &\approx mu_1 + d \\ v_2 &\approx mu_2 + d \\ &\vdots \\ v_n &\approx mu_n + d \end{aligned}$$

$$\begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ \vdots & \vdots & 1 \\ u_n & v_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

# Line Fitting (Ax=b)




Given points: $(u_1,v_1),(u_2,v_2),\cdots,(u_n,v_n)$

Find the best line: $(a,b,c)$

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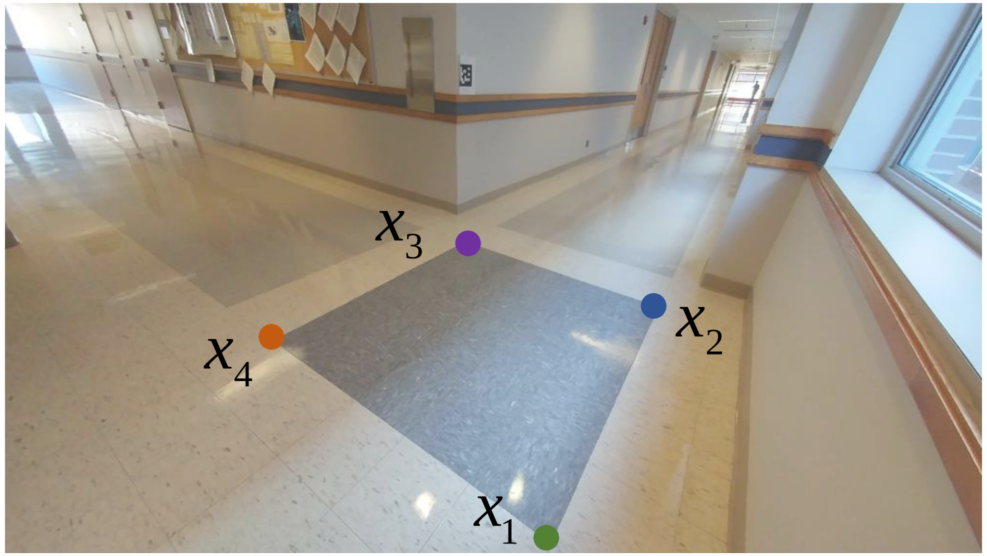
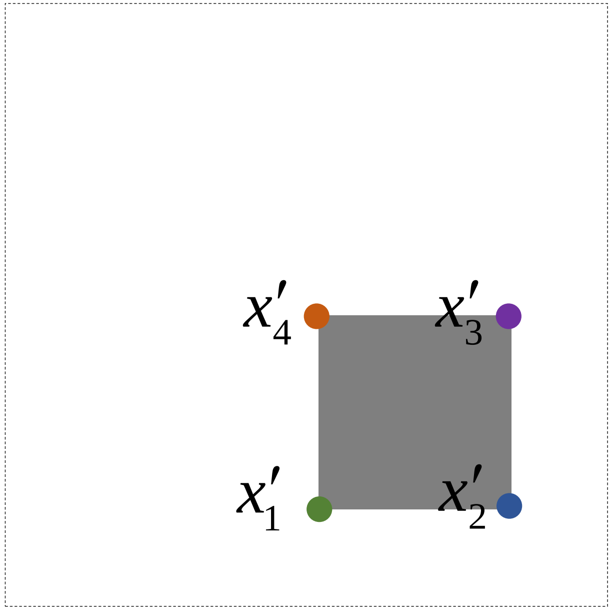
$$\underbrace{\begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ \vdots & \vdots & 1 \\ u_n & v_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{0}} \qquad \underbrace{\begin{bmatrix} u_1 & 1 \\ u_2 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} m \\ d \end{bmatrix}}_{\mathbf{x}} \approx \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{b}}$$

# How to compute homography?

# Homography Computation




$$\underbrace{\begin{bmatrix} u_1 & v_1 & 1 & 0 & 0 & 0 & -u_1u'_1 & -v_1u'_1 \\ 0 & 0 & 0 & u_1 & v_1 & 1 & -u_1v'_1 & -v_1v'_1 \\ & & & & \vdots & & & \\ u_4 & v_4 & 1 & 0 & 0 & 0 & -u_4u'_4 & -v_4u'_4 \\ 0 & 0 & 0 & u_4 & v_4 & 1 & -u_4v'_4 & -v_4v'_4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} u'_1 \\ v'_1 \\ \vdots \\ u'_4 \\ v'_4 \end{bmatrix}}_{b}$$

$$Ax = b \longrightarrow x = \left(A^T A\right)^{-1} A^T b$$

# Homography Computation



$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

# Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$



$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

# Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$



$$\longrightarrow \quad h_{11}u_x + h_{12}u_y + h_{13} + h_{31}u_x v_x + h_{32}u_y v_x + h_{33}v_x = 0$$

$$h_{21}u_x + h_{22}u_y + h_{23} + h_{31}u_x v_y + h_{32}u_y v_y + h_{33}v_y = 0$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Unknowns: $h_{11}, \cdots, h_{33}$

Equations: 2 per correspondence

# Homography Computation



$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$\longrightarrow \begin{aligned} h_{11}u_x + h_{12}u_y + h_{13} - h_{31}u_x v_x - h_{32}u_y v_x - h_{33}v_x &= 0 \\ h_{21}u_x + h_{22}u_y + h_{23} - h_{31}u_x v_y - h_{32}u_y v_y - h_{33}v_y &= 0 \end{aligned}$$

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

# Homography Computation

$$v_x = \frac{h_{11}u_x + h_{12}u_y + h_{13}}{h_{31}u_x + h_{32}u_y + h_{33}}$$

$$v_y = \frac{h_{21}u_x + h_{22}u_y + h_{23}}{h_{31}u_x + h_{32}u_y + h_{33}}$$



$$\rightarrow \quad h_{11}u_x + h_{12}u_y + h_{13} - h_{31}u_x v_x - h_{32}u_y v_x - h_{33}v_x = 0$$

$$h_{21}u_x + h_{22}u_y + h_{23} - h_{31}u_x v_y - h_{32}u_y v_y - h_{33}v_y = 0$$

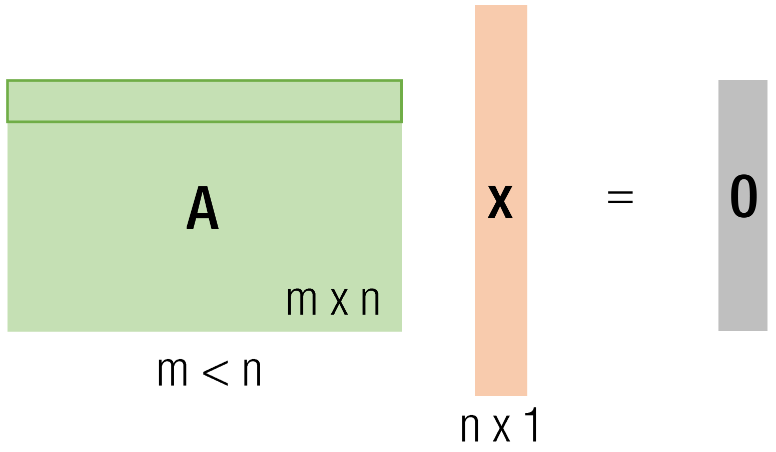
$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\rightarrow \quad \underbrace{\begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix}}_{\mathbf{A}\ (2\text{x}9)} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

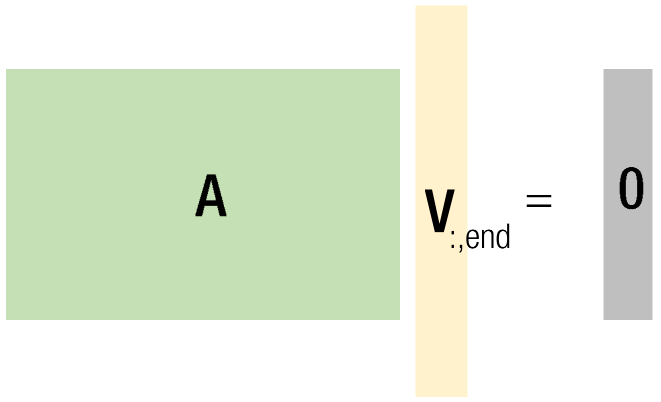
# Recall: Singular Value Decomposition (SVD)

# eqs < # unknowns

There exist at least n-m vectors that makes **Ax**=**0**.




$\longrightarrow \mathbf{x} = \text{null}(\mathbf{A})$




$\mathbf{V}_{:end} = \text{null}(\mathbf{A})$

For a unique solution, **A** should be m x (m+1)

# Homography Computation

How many correspondences are needed?

$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

2x9

# Homography Computation

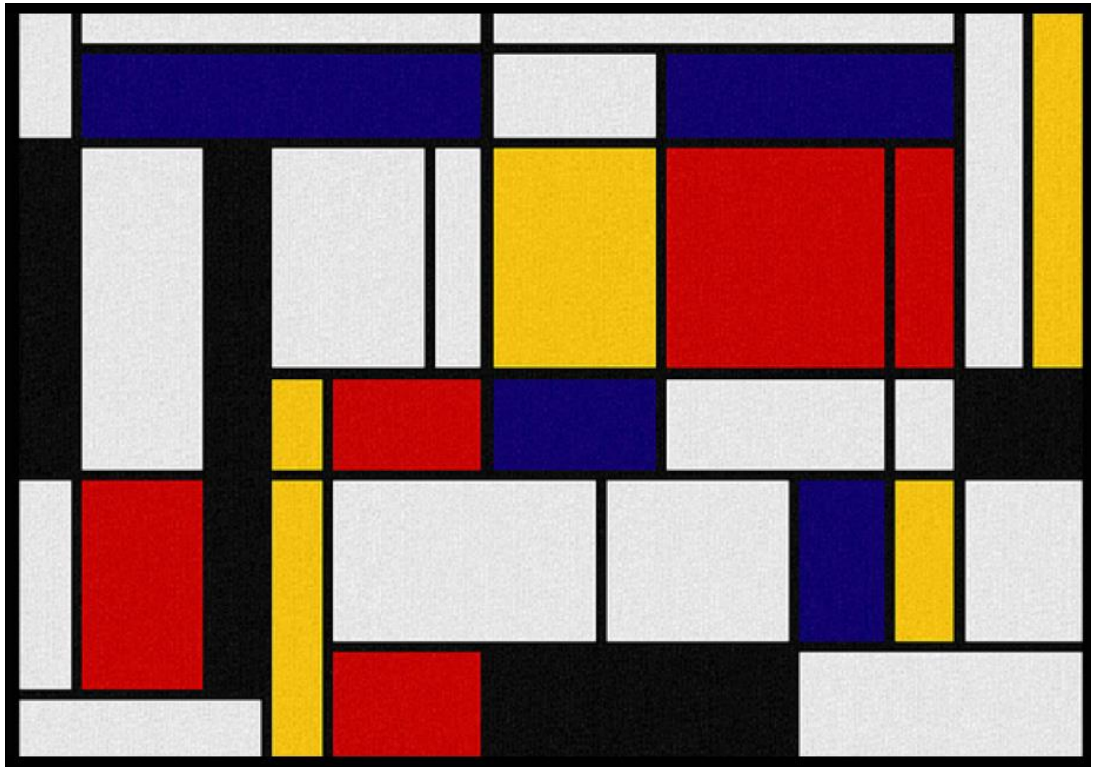
How many correspondences are needed? 4

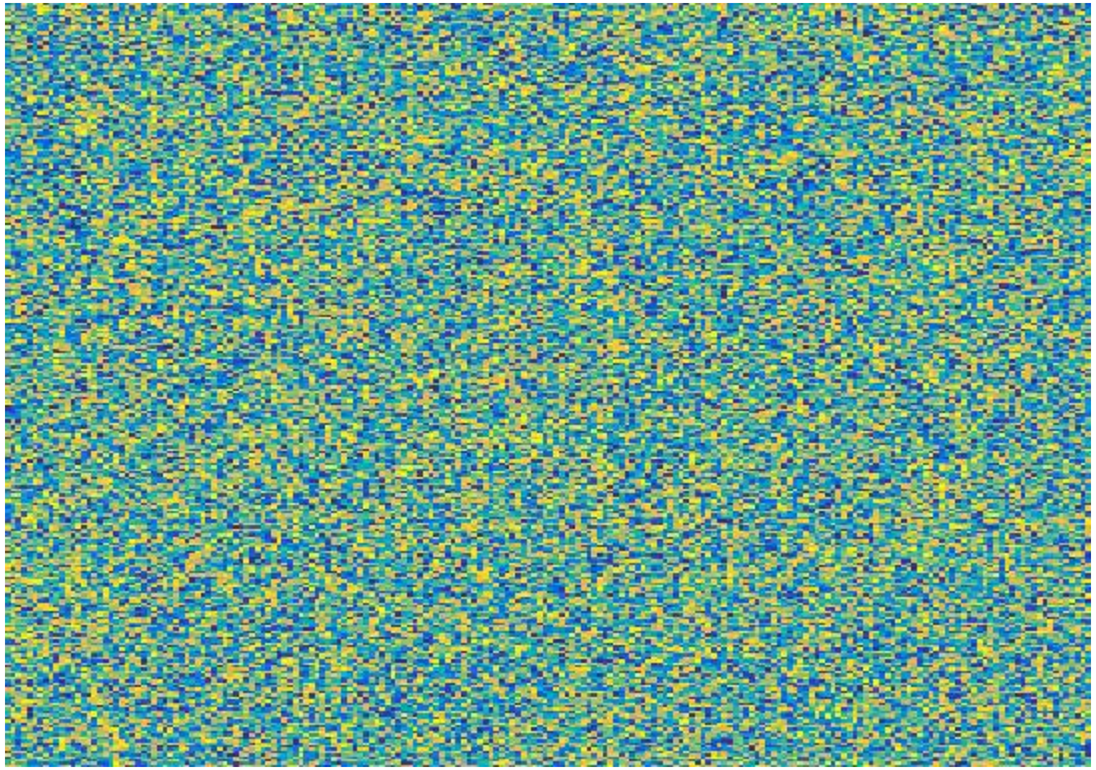
$$\lambda \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} u_x & u_y & 1 & & & & -u_x v_x & -u_y v_x & -v_x \\ & & & u_x & u_y & 1 & -u_x v_y & -u_y v_y & -v_y \end{bmatrix}}_{\mathbf{A}\ (8\text{x}9)} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

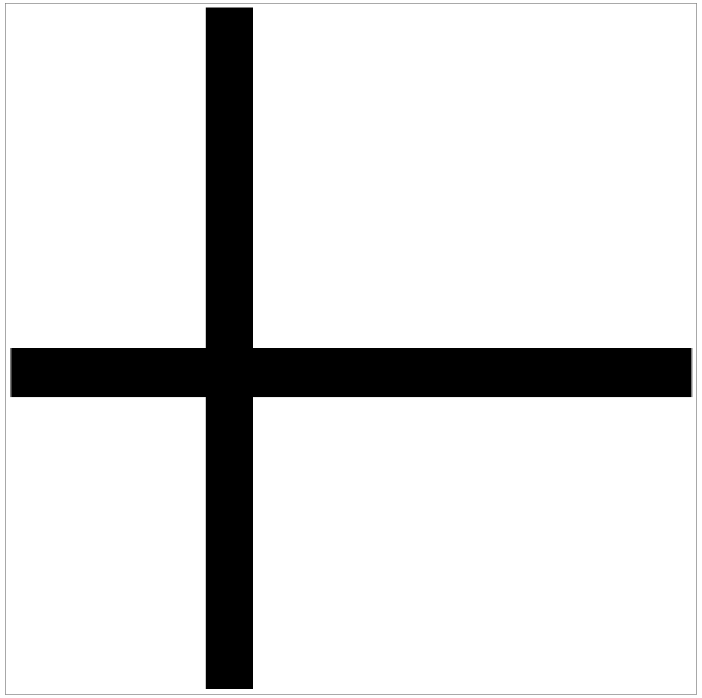
$$\mathbf{x} = \mathbf{V}_{:end} = \text{null}(\mathbf{A})$$



VS

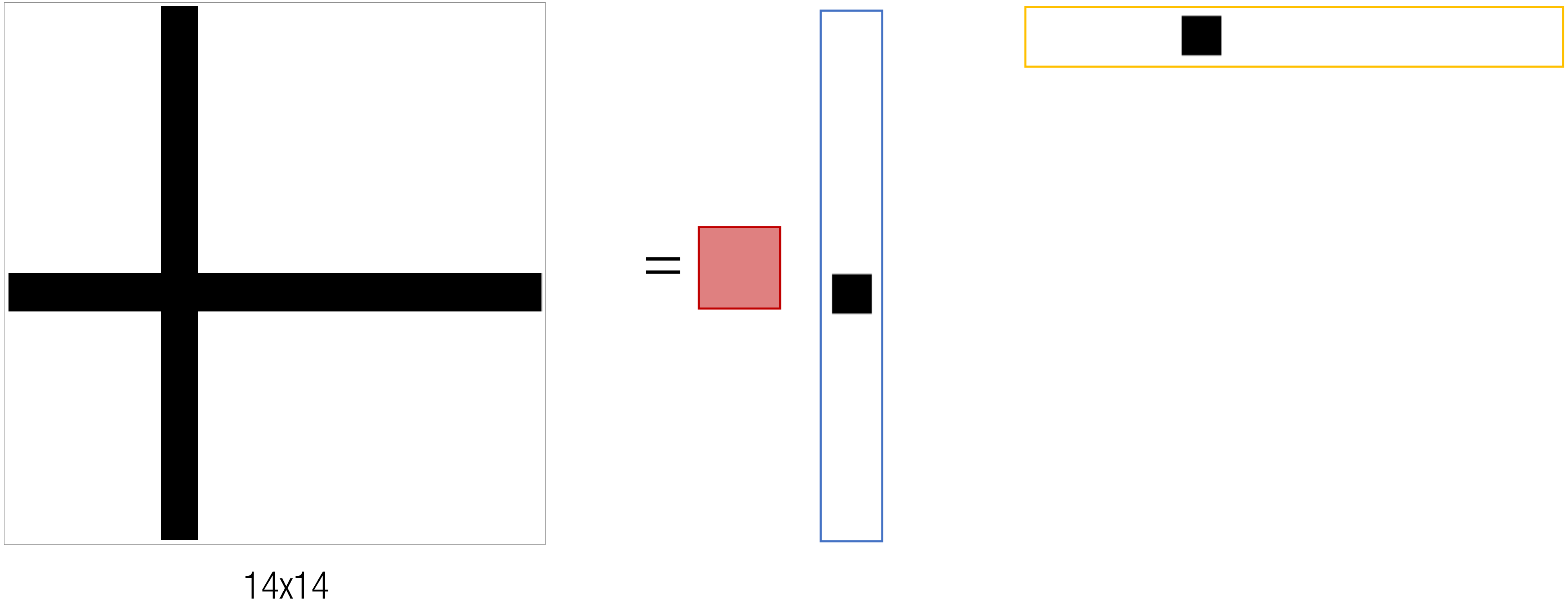


# More SVD

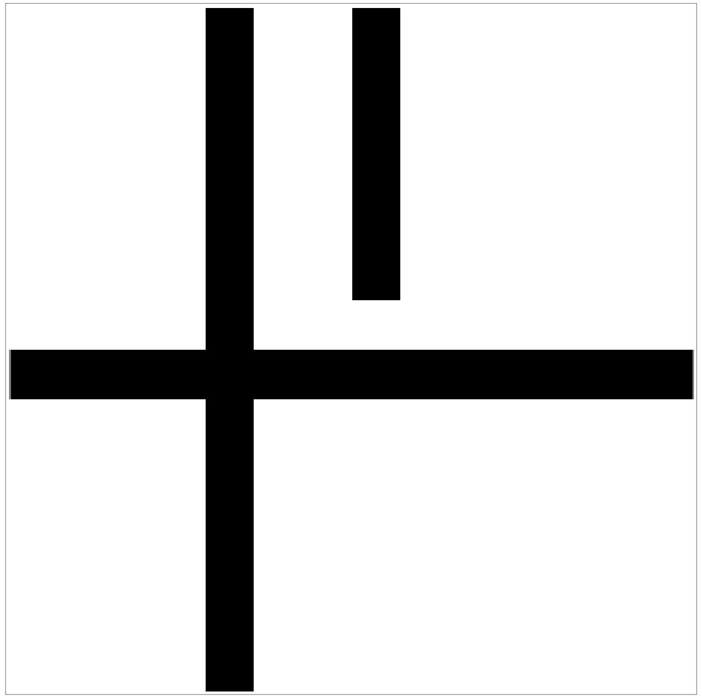


14x14

# More SVD



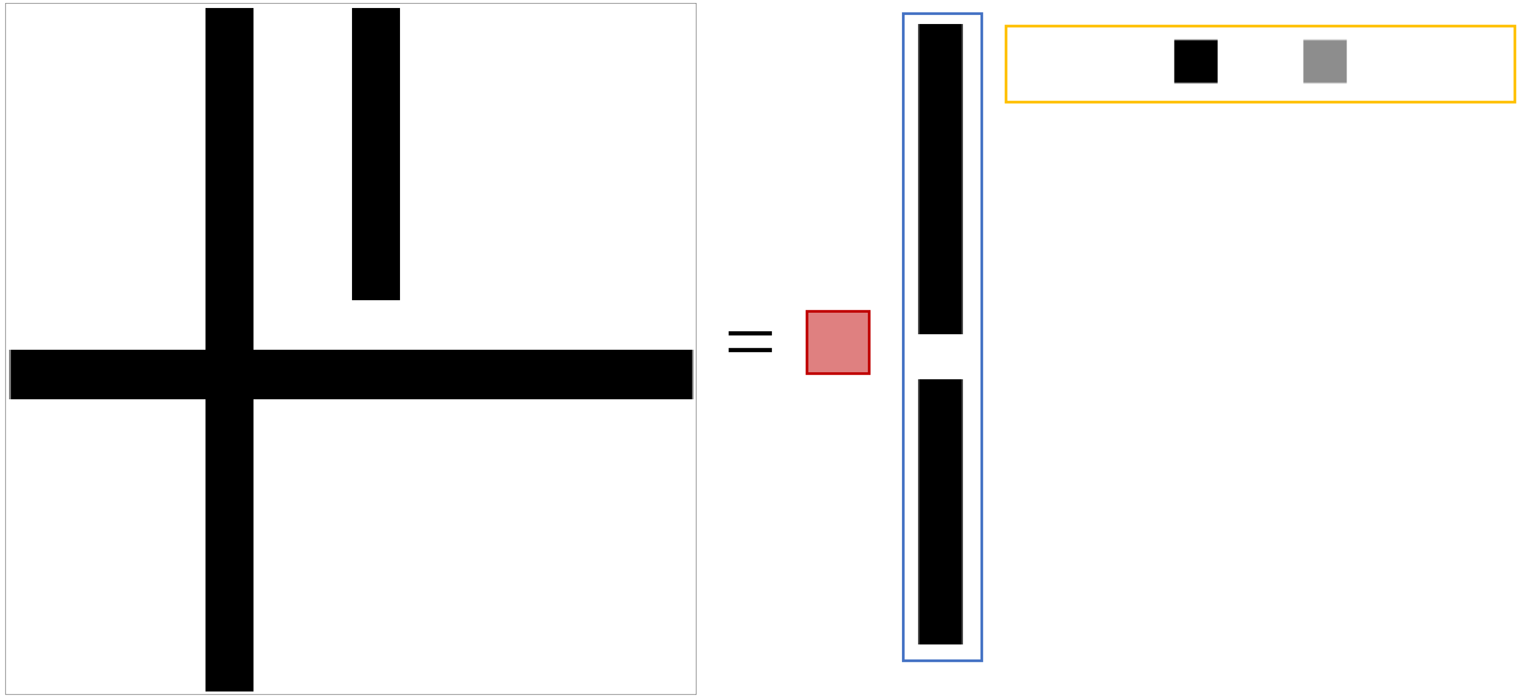
# More SVD



14x14

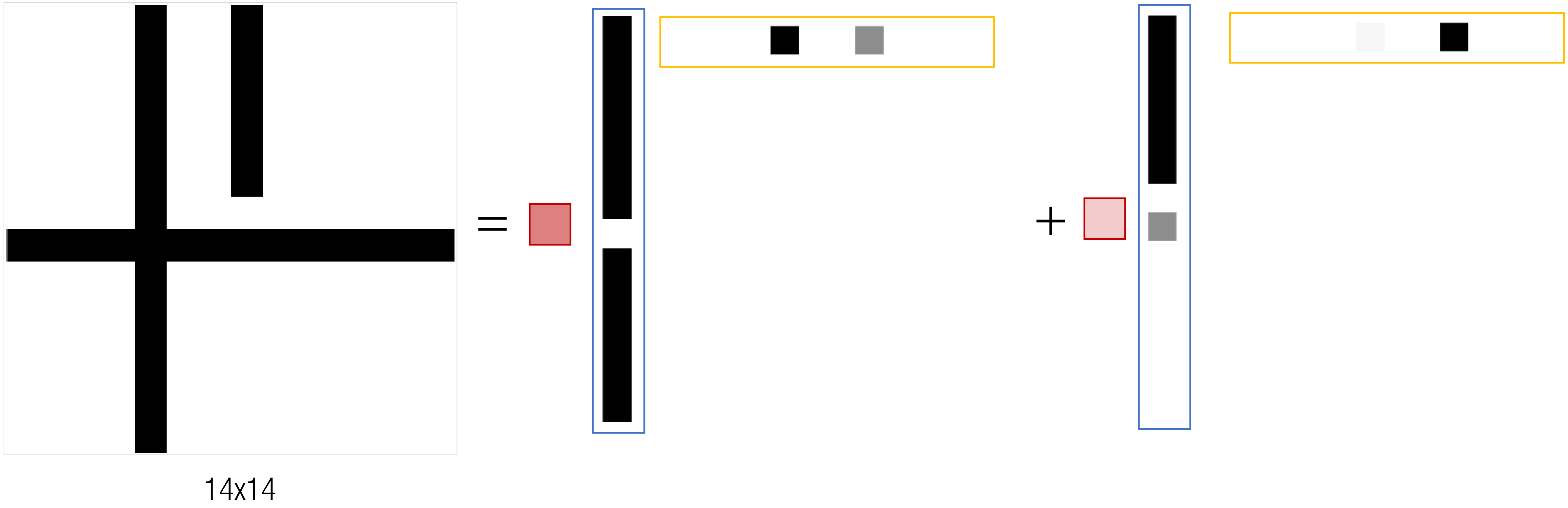
=

# More SVD

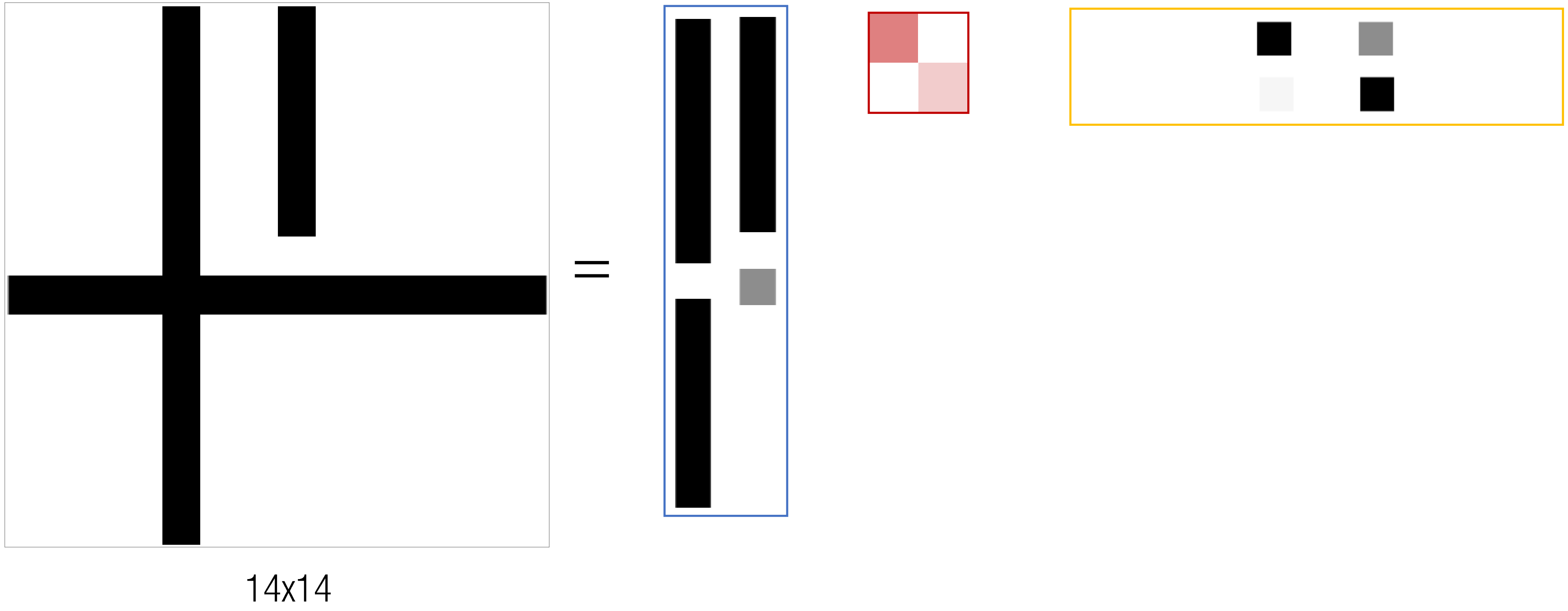


14x14

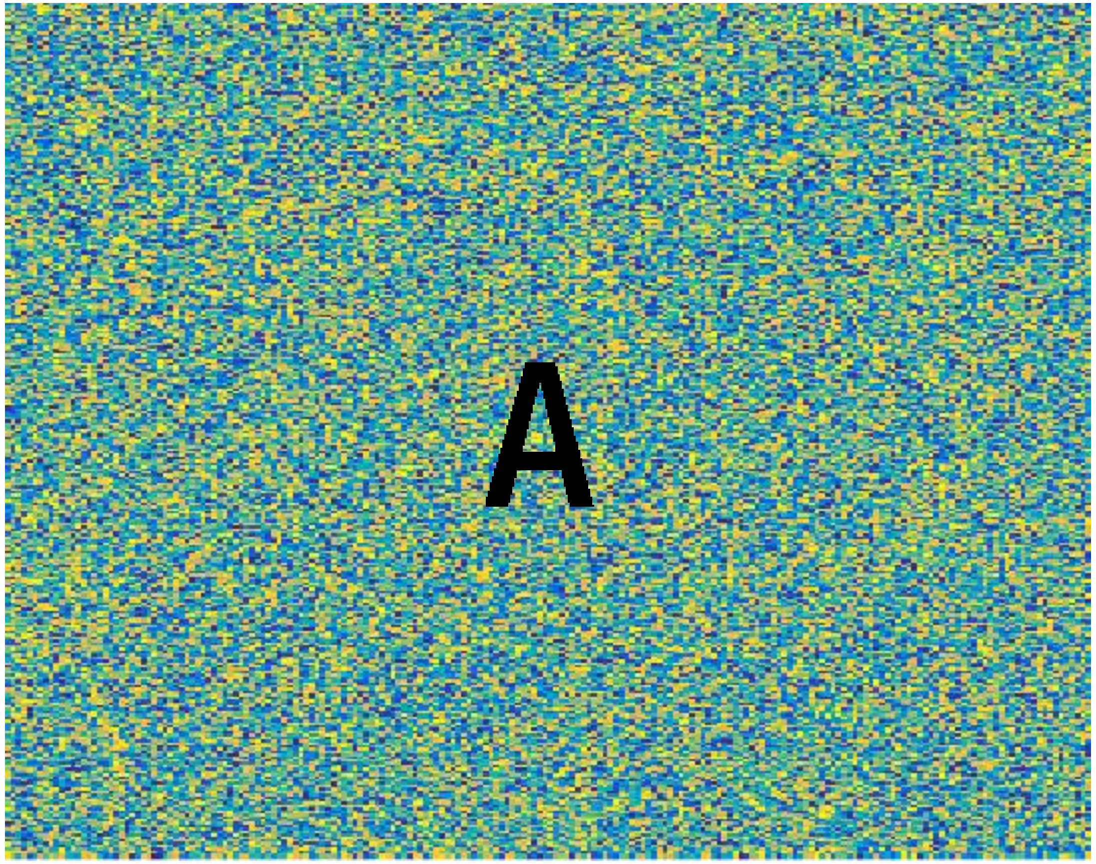
# More SVD



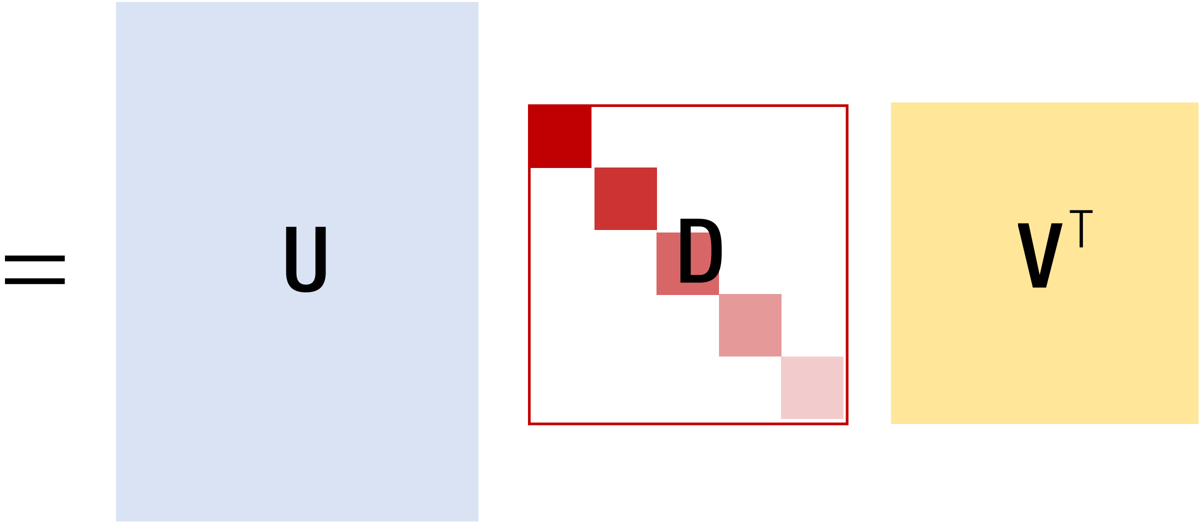
# More SVD



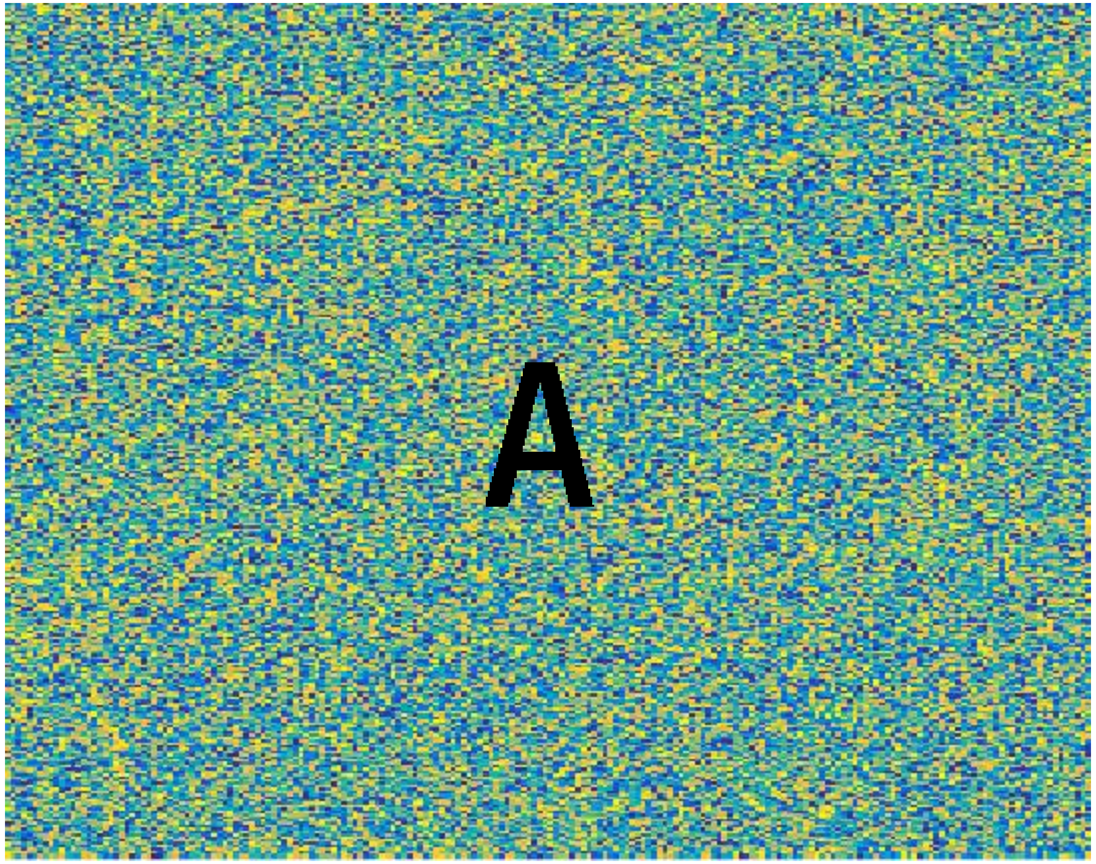
# Random Matrix SVD



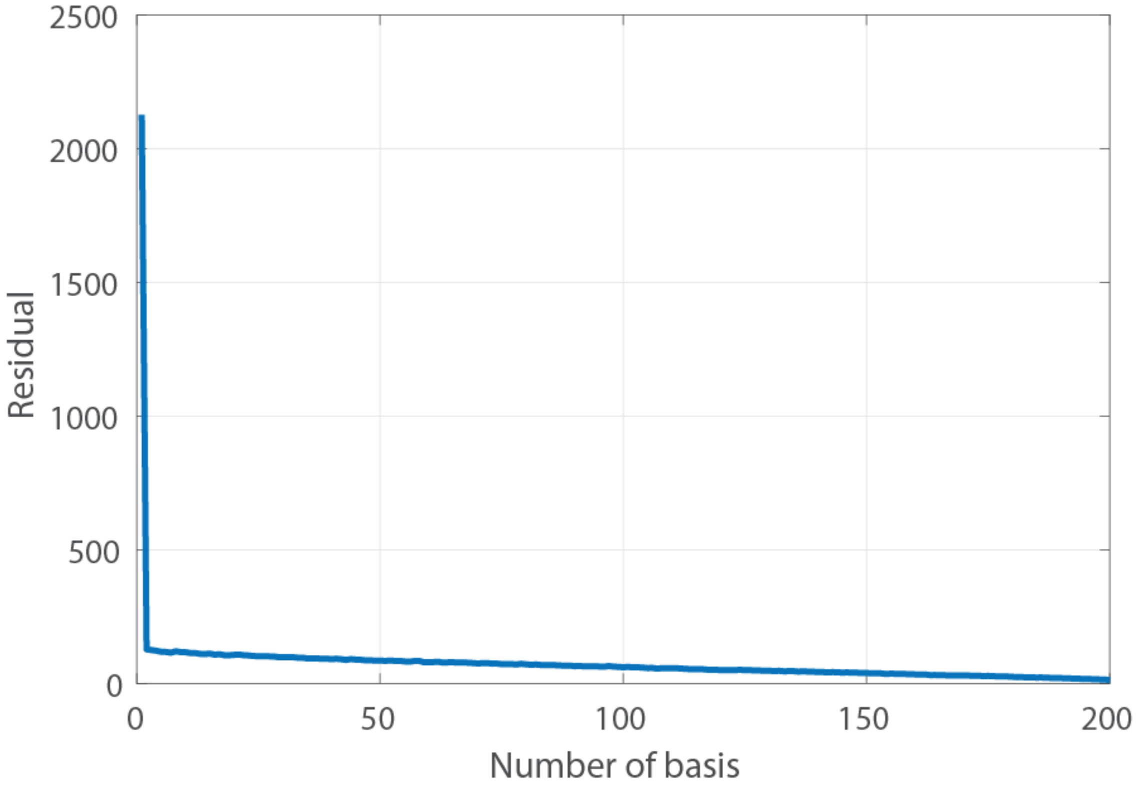

Random matrix



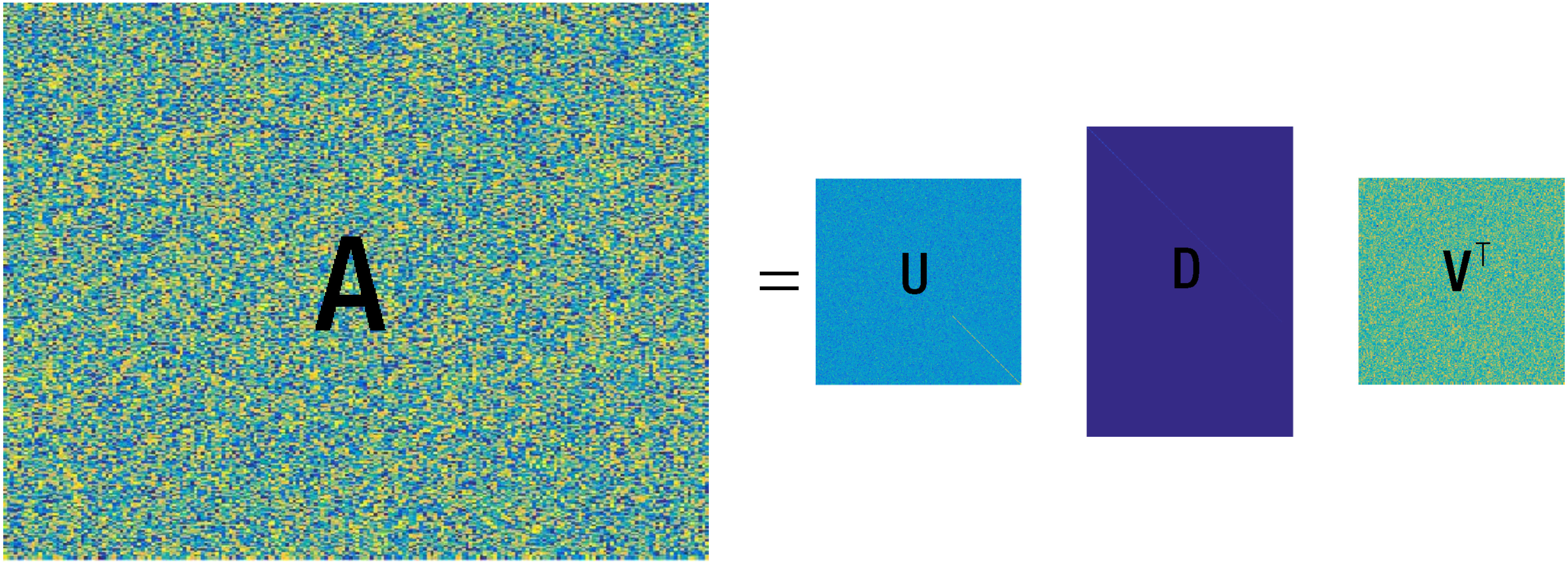
# Singular Values




Random matrix

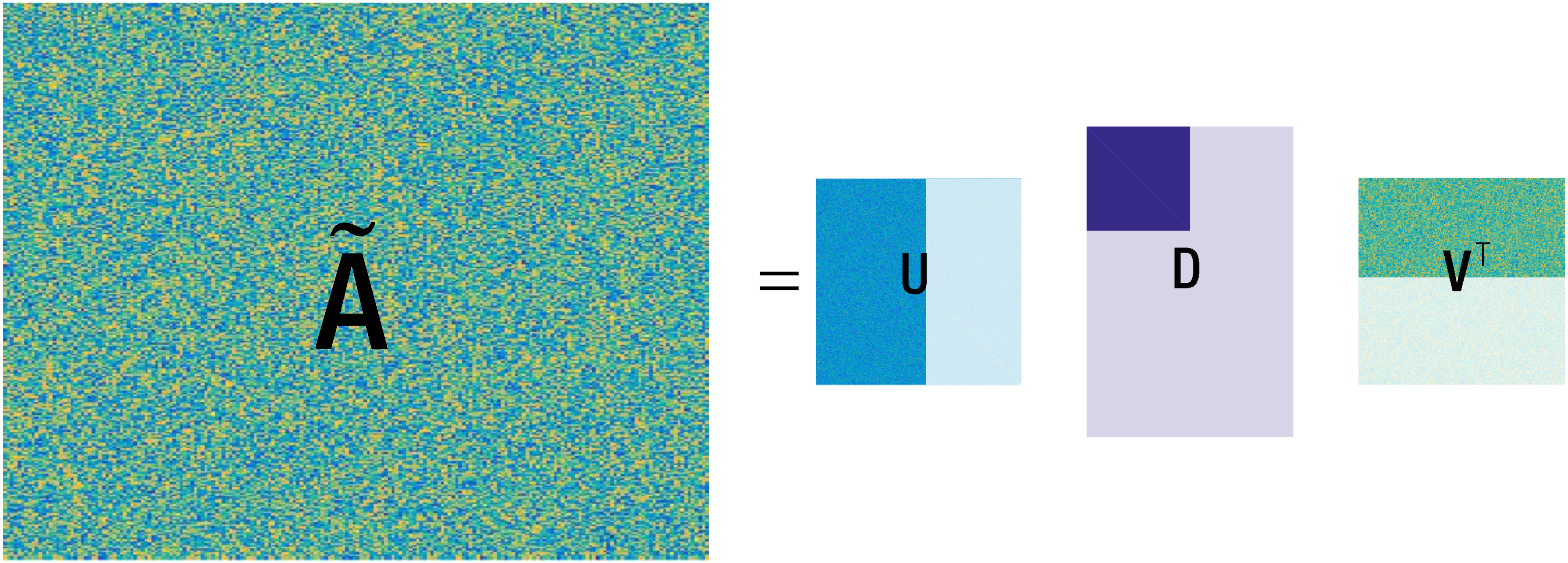


# Singular Values



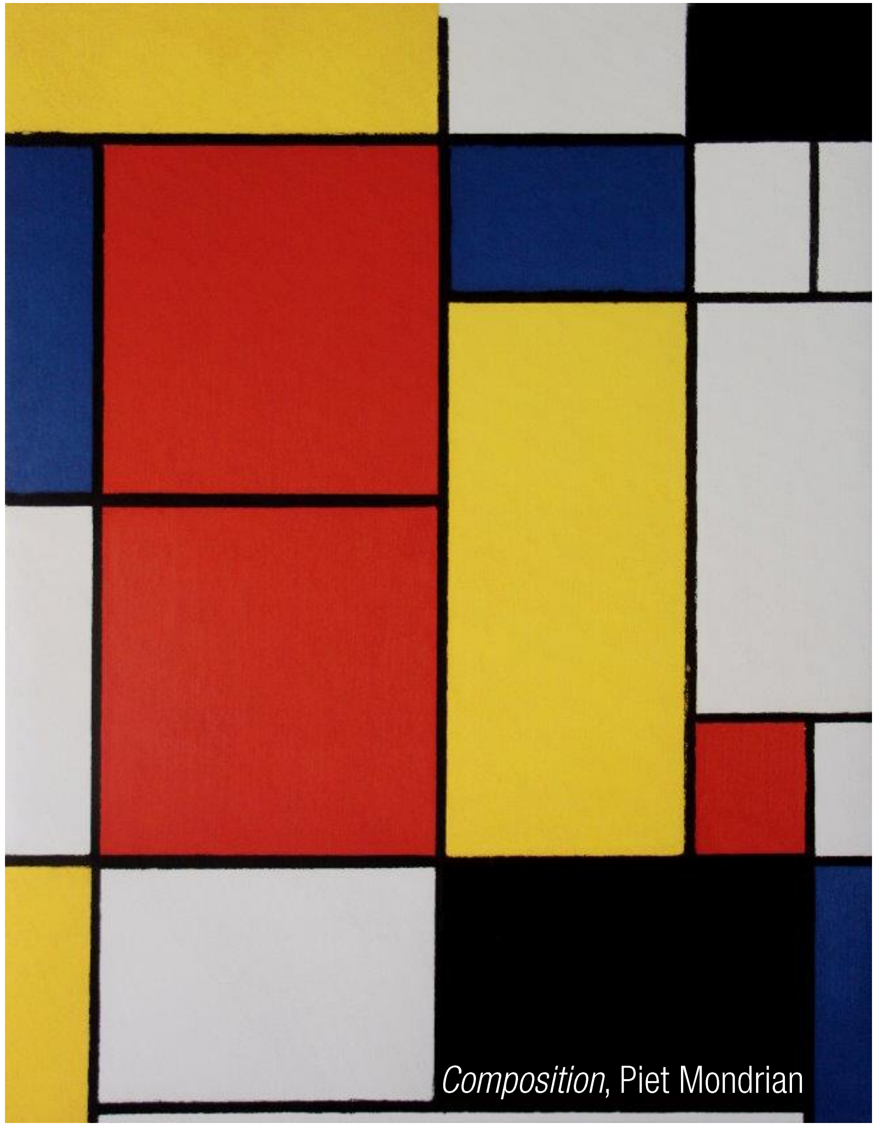

Random matrix

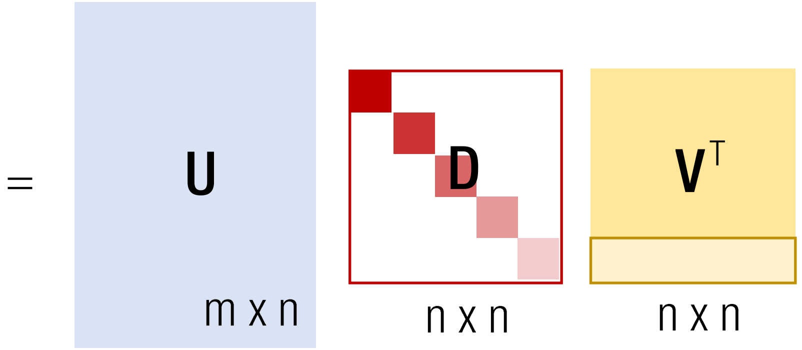
# SVD Matrix Approximation




Reconstructed matrix

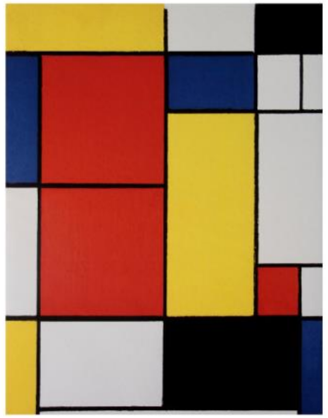
# Mondrian Painting SVD



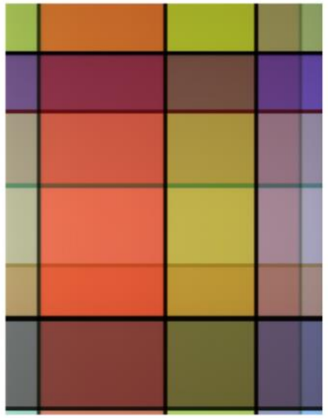



# Mondrian Painting SVD Approximation

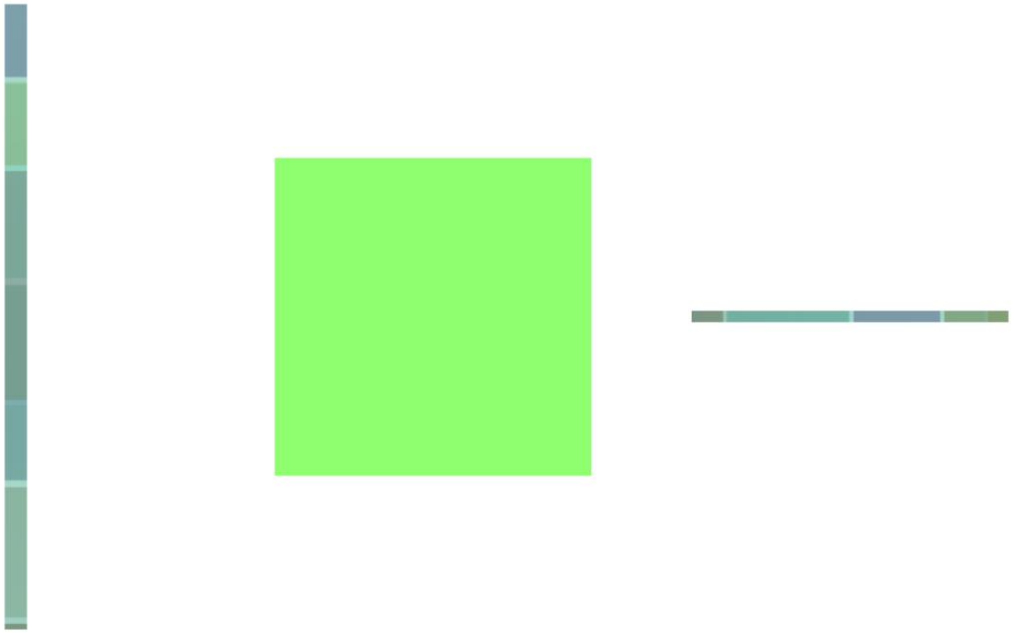
MondrianSVD.m



Ground truth



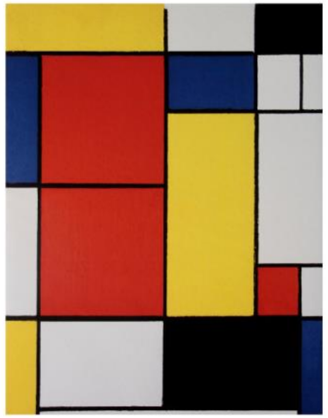
Number of basis: 1



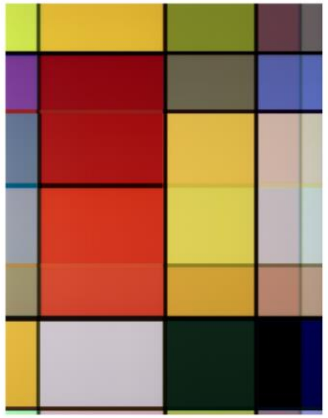
$$A = U \quad D \quad V^T$$

# Mondrian Painting SVD Approximation

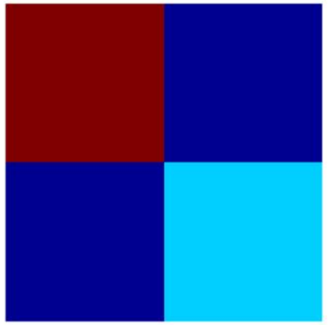
MondrianSVD.m



Ground truth



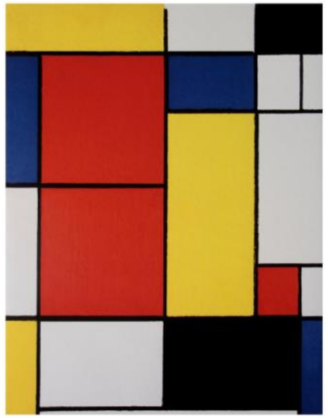
Number of basis: 2



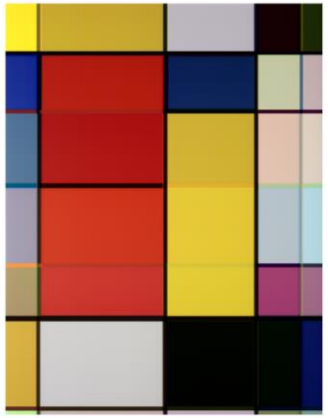
$$A = U \quad D \quad V^T$$

# Mondrian Painting SVD Approximation

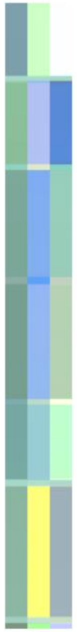
MondrianSVD.m

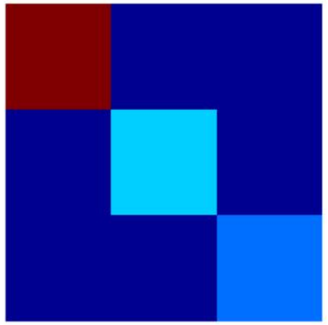


Ground truth



Number of basis: 3

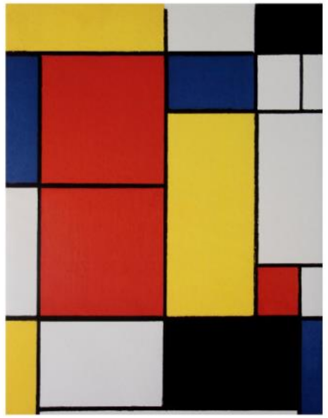




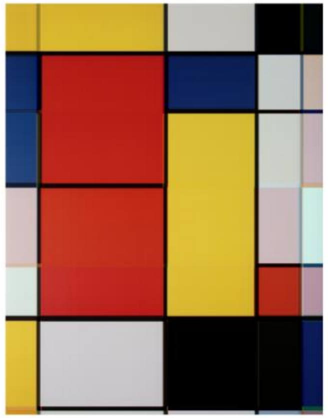
$$A = U \quad D \quad V^T$$

# Mondrian Painting SVD Approximation

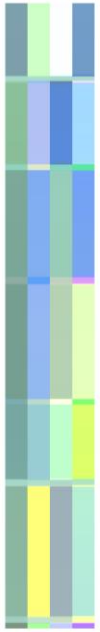
MondrianSVD.m

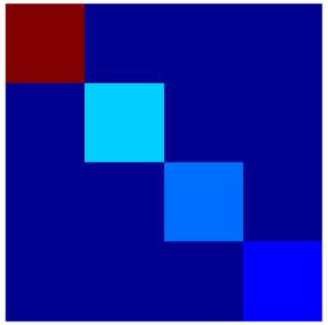


Ground truth



Number of basis: 4

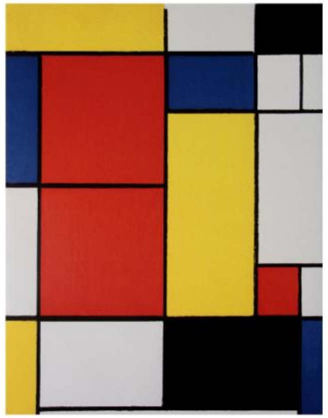




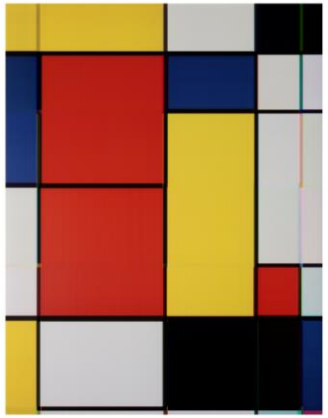
$$A = U \quad D \quad V^T$$

# Mondrian Painting SVD Approximation

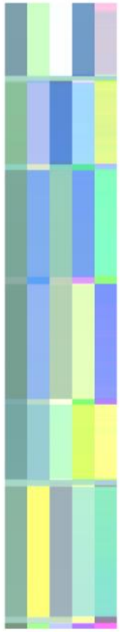
MondrianSVD.m

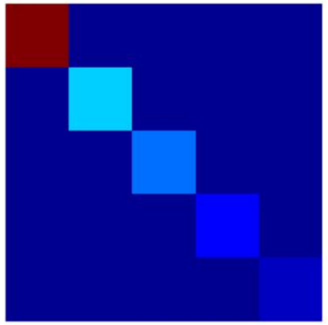


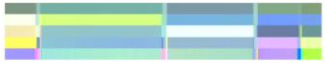
Ground truth



Number of basis: 5



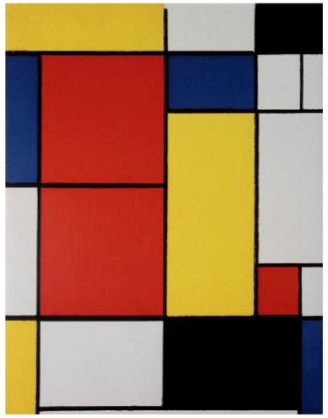




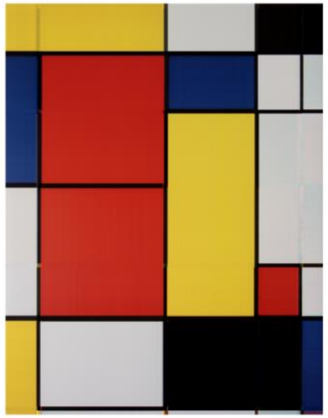
$$A = U \quad D \quad V^T$$

# Mondrian Painting SVD Approximation

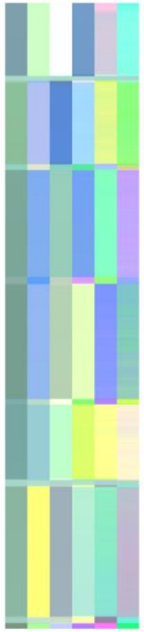
MondrianSVD.m

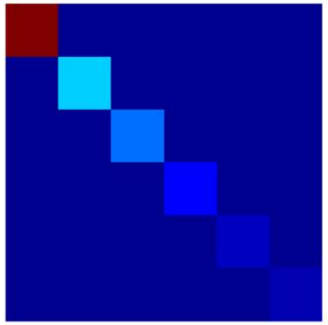


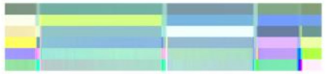
Ground truth



Number of basis: 6



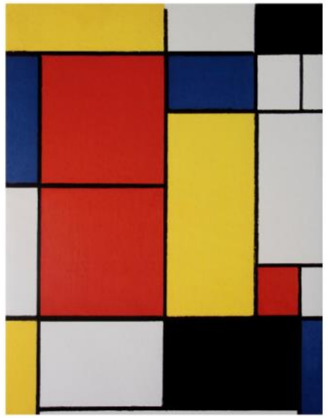




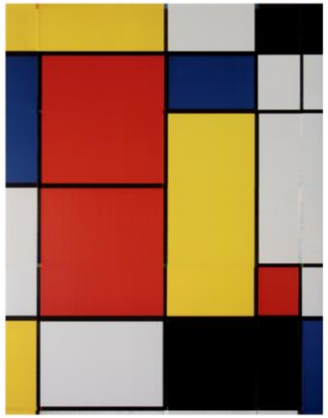
$$A = U \quad D \quad V^T$$

# Mondrian Painting SVD Approximation

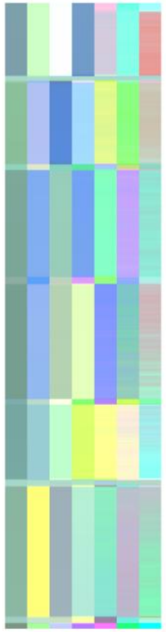
MondrianSVD.m

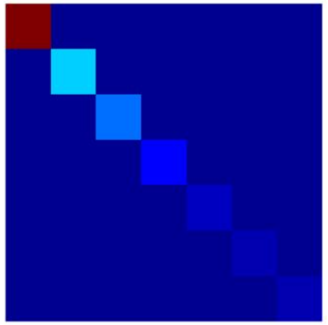


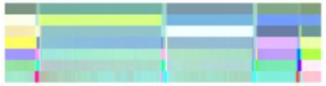
Ground truth



Number of basis: 7



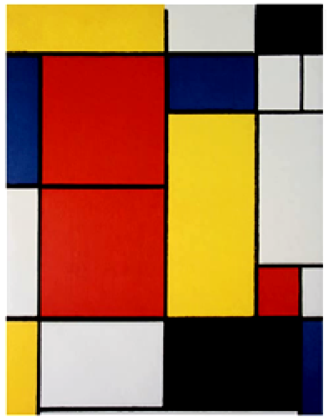




$$A = U \quad D \quad V^T$$

# Mondrian Painting SVD Approximation

MondrianSVD.m



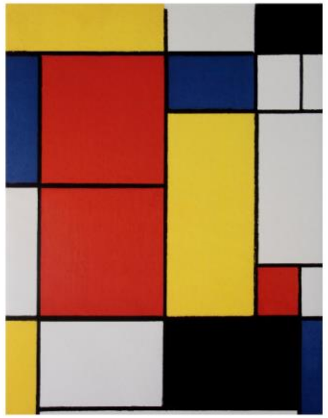
Ground truth

Number of basis: 1

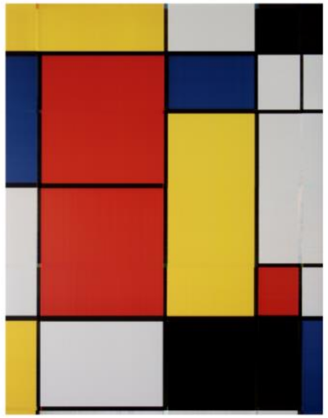


$$A = U \quad D \quad V^T$$

# Reconstruction Error



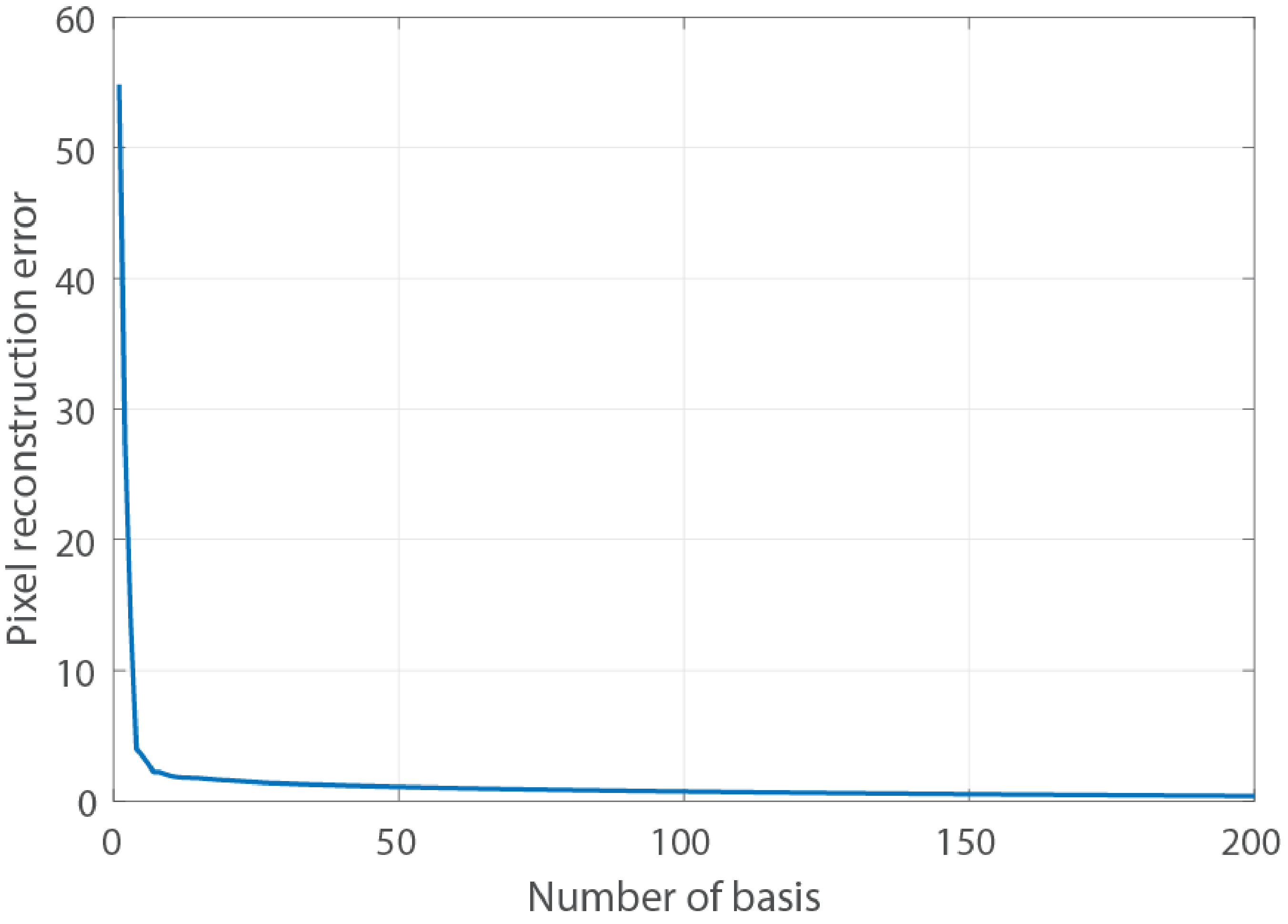
Ground truth



Number of basis: 7

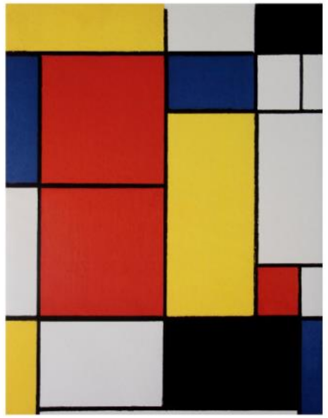
A

MondrianSVD.m

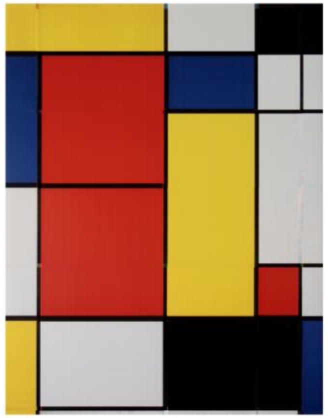


# Rotation Matrix

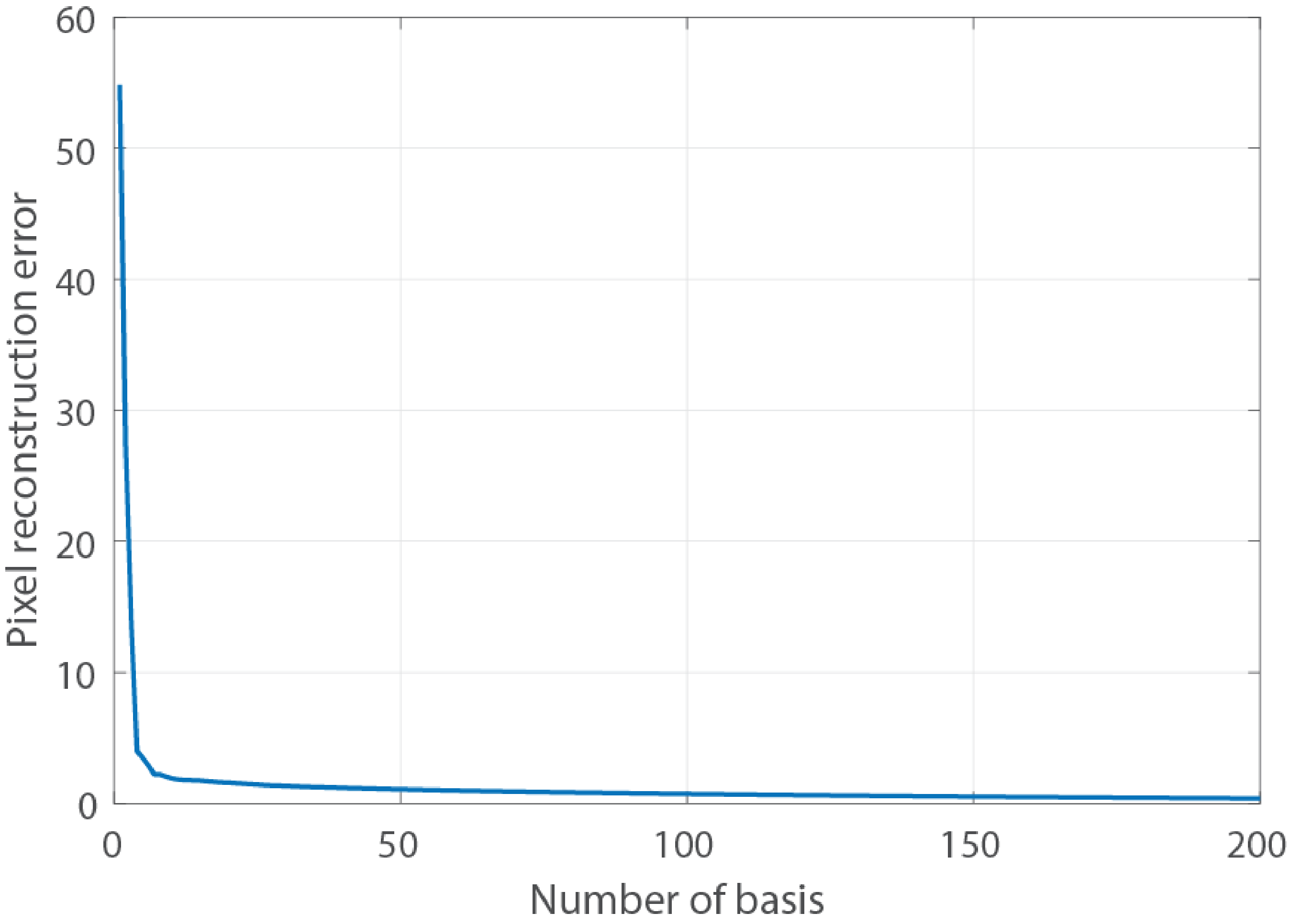
MondrianSVD.m



Ground truth



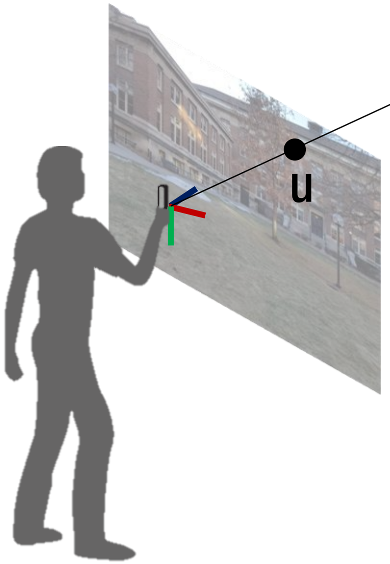
Number of basis: 7




A

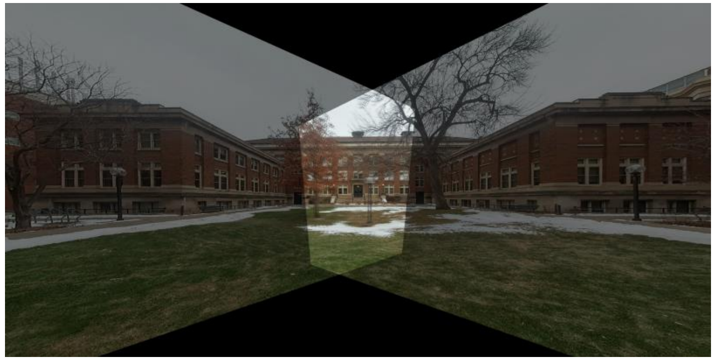
# Image Panorama (Cylindrical Projection)

$R_1$ $R_2$ $R_3$ $R_4$ $R_5$ $R_6$ $R_7$ $R_8$ $R_9$ $R_{10}$ $R_{11}$




$$X = \lambda K^{-1} R^{T} u$$

Rotation matrix has to be orthogonal matrix!!



$$R = K^{-1} H K$$

$$R = UDV^{T}$$

$$R_{orth} = UV^{T}$$