Weierstrass's non-differentiable function

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In the nineteenth century, many mathematicians held the belief that a continuous function must be differentiable at a large set of points. In 1872, Karl Weierstrass shocked the mathematical world by giving the first published example of a continuous function that is *nowhere* differentiable. His function is given by

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x).$$

In particular, Weierstrass proved the following theorem:

Theorem 1 (Weierstrass 1872)**.** Let $a \in (0,1)$, let $b > 1$ be an odd integer, and assume that

$$ab > 1 + \frac{3}{2}\pi. \tag{1}$$

Then the function W is continuous and nowhere differentiable on $\mathbb{R}$.

To see that W is continuous on $\mathbb{R}$, note that

$$|a^n \cos(b^n \pi x)| = a^n |\cos(b^n \pi x)| \leq a^n.$$

Since the geometric series $\sum a^n$ converges for $a \in (0,1)$, the Weierstrass M-test shows that the series defining W converges uniformly to W on $\mathbb{R}$. Since each function $a^n \cos(b^n \pi x)$ is continuous, each partial sum is continuous, and therefore W is continuous, being the uniform limit of a sequence of continuous functions.

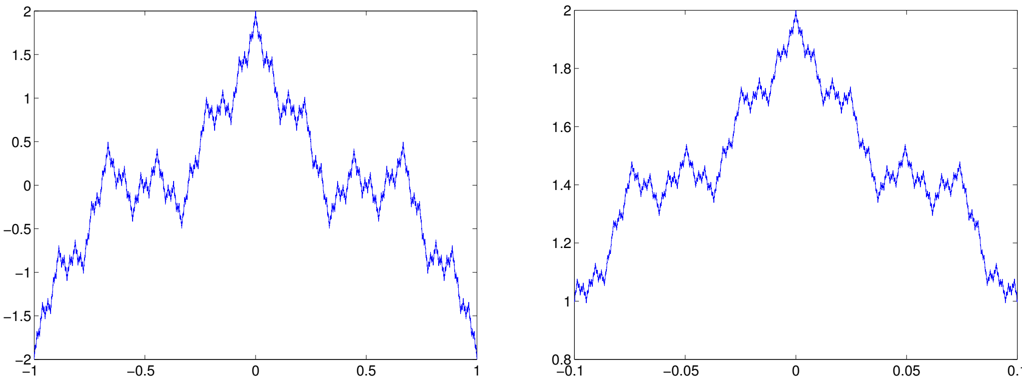
To give some motivation for the condition (1), consider the partial sums

$$W_n(x) = \sum_{k=0}^{n} a^k \cos(b^k \pi x).$$

These partial sums are differentiable functions and

$$W_n'(x) = -\sum_{k=0}^{n} \pi (ab)^k \sin(b^k \pi x).$$

If $ab < 1$, then we can again use the Weierstrass M-test to show that (W_n') converges uniformly to a continuous function on $\mathbb{R}$. In this case we can actually prove that W is differentiable and $W_n' \to W'$ uniformly. Therefore, at the very least we need $ab \geq 1$ for W to be non-differentiable. In 1916, Godfrey Hardy showed that $ab \geq 1$ is sufficient for the nowhere



(a) (b)

Figure 1: The Weierstrass function $W(x)$ for $a = 0.5$ and $b = 3$. Notice that W appears the same on the two different scales shown in (a) and (b).

differentiability of W. The more restrictive condition $ab > 1 + \frac{3}{2}\pi$ present in Weierstrass's proof is an artifact of the techniques he used. Hardy also relaxed the integral assumption on b, and allowed b to be any real number greater than 1.

Figure 1 shows a plot of the Weierstrass function for $a = 0.5$ and $b = 3$ on two different scales. Notice the similar repeating patterns on each scale. If we were to continue zooming in on W, we would continue seeing the same patterns. The Weierstrass function is an early example of a fractal, which has repeating patterns at *every scale*.

Before giving the proof, we recall a few facts that will be useful in the proof. Let $x, y \in \mathbb{R}$, and suppose $x > y$. Then by the fundamental theorem of calculus

$$\cos(x) - \cos(y) = \int_y^x -\sin(t)\,dt \leq \int_y^x 1\,dt = x - y,$$

and

$$\cos(x) - \cos(y) \geq \int_y^x -1\,dt = -(x - y).$$

Therefore

$$|\cos(x) - \cos(y)| \leq |x - y|.$$

The argument is similar when $y \geq x$, so we deduce

$$|\cos(x) - \cos(y)| \leq |x - y| \quad \text{for all } x, y \in \mathbb{R}. \tag{2}$$

Consider $\cos(n\pi + x)$ for an integer n and $x \in \mathbb{R}$. If n is even, then since cosine is 2π-periodic, $\cos(n\pi + x) = \cos(x)$. If n is odd, then $n + 1$ is even and

$$\cos(n\pi + x) = \cos((n + 1)\pi + x - \pi) = \cos(x - \pi) = -\cos(x).$$

Draw a graph of $\cos(x)$ if the last equality is unclear. Therefore we obtain

$$\cos(n\pi + x) = (-1)^n \cos(x) \quad \text{for all } x \in \mathbb{R}. \tag{3}$$

We now give the proof of Theorem 1.

Proof. Let $x_0 \in \mathbb{R}$ and let $m \in \mathbb{N}$. Let us round $b^m x_0$ to the nearest integer, and call this integer k_m. Therefore

$$b^m x_0 - \frac{1}{2} \leq k_m \leq b^m x_0 + \frac{1}{2}. \tag{4}$$

Let us also set

$$x_m = \frac{k_m + 1}{b^m}. \tag{5}$$

By (4) we see that

$$x_m \geq \frac{b^m x_0 - \frac{1}{2} + 1}{b^m} > \frac{b^m x_0}{b_m} = x_0,$$

and

$$x_m \leq \frac{b^m x_0 + \frac{1}{2} + 1}{b^m} = x_0 + \frac{3}{2b^m}.$$

Combining these equations we have

$$x_0 < x_m \leq x_0 + \frac{3}{2b^m}. \tag{6}$$

By the squeeze lemma, $\lim_{m\to\infty} x_m = x_0$.

Let us consider the difference

$$\begin{aligned} W(x_m) - W(x_0) &= \sum_{n=0}^{\infty} a^n \cos(b^n \pi x_m) - \sum_{n=0}^{\infty} a^n \cos(b^n \pi x_0) \\ &= \sum_{n=0}^{\infty} a^n \left(\cos(b^n \pi x_m) - \cos(b^n \pi x_0)\right) \\ &= A + B, \end{aligned} \tag{7}$$

where

$$A = \sum_{n=0}^{m-1} a^n \left(\cos(b^n \pi x_m) - \cos(b^n \pi x_0)\right), \tag{8}$$

and

$$B = \sum_{n=m}^{\infty} a^n \left(\cos(b^n \pi x_m) - \cos(b^n \pi x_0)\right). \tag{9}$$

The proof is now split into three steps.

1. The first step is to find an upper bound for $|A|$. Using the triangle inequality and the identity (2)

$$|A| \leq \sum_{n=0}^{m-1} a^n \left|\cos(b^n \pi x_m) - \cos(b^n \pi x_0)\right| \leq \sum_{n=0}^{m-1} a^n b^n \pi (x_m - x_0) = \pi(x_m - x_0) \sum_{n=0}^{m-1} (ab)^n.$$

Noticing the geometric series, we deduce

$$|A| \leq \pi(x_m - x_0)\frac{1-(ab)^m}{1-ab} = \pi(x_m - x_0)\frac{(ab)^m - 1}{ab-1} \leq \frac{\pi(ab)^m}{ab-1}(x_m - x_0). \tag{10}$$

In the last step we used the hypothesis that $ab > 1 + \frac{3}{2}\pi > 1$.

2. The second step is to find a lower bound for $|B|$. By the definition of x_m (5)

$$\cos(b^n \pi x_m) = \cos\left(b^n \pi \left(\frac{k_m + 1}{b^m}\right)\right) = \cos(b^{n-m}(k_m + 1)\pi).$$

For $n \geq m$, $b^{n-m}(k_m + 1)$ is an integer, and hence

$$\cos(b^n \pi x_m) = (-1)^{b^{n-m}(k_m+1)} = \left((-1)^{b^{n-m}}\right)^{k_m+1} = (-1)^{k_m+1} = -(-1)^{k_m}, \tag{11}$$

where we used the fact that b^{n-m} is *odd* so that $(-1)^{b^{n-m}} = -1$. On the other hand, we also have

$$\cos(b^n \pi x_0) = \cos\left(b^n \pi \left(\frac{k_m + b^m x_0 - k_m}{b^m}\right)\right) = \cos(b^{n-m} k_m \pi + b^{n-m} z_m \pi),$$

where $z_m = b^m x_0 - k_m$. Since $n \geq m$, $b^{n-m} k_m$ is an integer and we can use (3) to find that

$$\cos(b^n \pi x_0) = (-1)^{b^{n-m} k_m} \cos(b^{n-m} z_m \pi) = (-1)^{k_m} \cos(b^{n-m} z_m \pi), \tag{12}$$

where, as before, we used the fact that b^{n-m} is odd. We now insert (11) and (12) into (9) to obtain

$$\begin{aligned} B &= \sum_{n=m}^{\infty} a^n \left(-(-1)^{k_m} - (-1)^{k_m} \cos(b^{n-m} z_m \pi)\right) \\ &= -(-1)^{k_m} \sum_{n=m}^{\infty} a^n \left(1 + \cos(b^{n-m} z_m \pi)\right). \end{aligned}$$

Notice that $a^n > 0$ and $1 + \cos(b^{n-m} z_m \pi) \geq 0$. It follows that all the terms in the sum above are non-negative, and therefore

$$|B| = \sum_{n=m}^{\infty} a^n \left(1 + \cos(b^{n-m} z_m \pi)\right) \geq a^m \left(1 + \cos(z_m \pi)\right).$$

Recall that $z_m = b^m x_0 - k_m$. By (4), $z_m \in [-\frac{1}{2}, \frac{1}{2}]$, and therefore $\pi z_m \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. It follows that $\cos(z_m \pi) \geq 0$ and

$$|B| \geq a^m.$$

By (6), $x_m - x_0 \leq \frac{3}{2b^m}$, and thus

$$\frac{2b^m}{3}(x_m - x_0) \leq 1.$$

We can combine this with $|B| \geq a^m$ to find that

$$|B| \geq a^m \cdot 1 \geq \frac{2(ab)^m}{3}(x_m - x_0). \tag{13}$$

This is the desired lower bound on $|B|$, and completes part 2 of the proof.

3. We now combine the bounds (10) and (13) to complete the proof. Notice by (10), (13) and the reverse triangle inequality that

$$|A + B| \geq |B| - |A| \geq \frac{2(ab)^m}{3}(x_m - x_0) - \frac{\pi(ab)^m}{ab - 1}(x_m - x_0) = (ab)^m \left(\frac{2}{3} - \frac{\pi}{ab - 1}\right)(x_m - x_0).$$

By (7) we see that

$$|W(x_m) - W(x_0)| = |A + B| \geq (ab)^m \left(\frac{2}{3} - \frac{\pi}{ab - 1} \right) (x_m - x_0).$$

Since $x_m - x_0 > 0$, so that $|x_m - x_0| = x_m - x_0$, we have

$$\left| \frac{W(x_m) - W(x_0)}{x_m - x_0} \right| \geq (ab)^m \left(\frac{2}{3} - \frac{\pi}{ab - 1} \right). \tag{14}$$

We would like this difference quotient to tend to ∞ in absolute value as $m \to \infty$. For this we need $ab > 1$ and

$$\frac{2}{3} - \frac{\pi}{ab - 1} > 0.$$

Rearranging this for ab we see that we need

$$ab > \frac{3}{2}\pi + 1,$$

which is exactly the hypothesis (1). Therefore

$$\lim_{m \to \infty} \left| \frac{W(x_m) - W(x_0)}{x_m - x_0} \right| = +\infty,$$

and $x_m \to x_0$ as $m \to \infty$. This shows that W is not differentiable at x_0. □

With slight modifications to the proof, we can also show that

$$\lim_{x \to x_0} \frac{W(x) - W(x_0)}{x - x_0}$$

does not exist as a real number or $\pm\infty$. This rules out the possibility of the Weierstrass function having a vertical tangent line, or an "infinite derivative" anywhere.