Math 126 – Homework 10 (Due Monday Nov 16)

1. Show that performing Jacobi iterations is the same as solving the two-dimensional diffusion equation $v_t = v_{xx} + v_{yy}$ using centered differences for v_{xx} and v_{yy} and forward differences for v_t, with $\Delta x = \Delta y$ and $\Delta t = (\Delta x)^2/4$.

2. Find all three-dimensional plane waves; that is, all solutions of the wave equation of the form $u(\mathbf{x}, t) = f(\mathbf{k} \cdot \mathbf{x} - ct)$, where $\mathbf{k}$ is a fixed vector and f is a function of one variable.

3. Solve the wave equation in three dimensions with initial data $u(\mathbf{x}, 0) \equiv 0$ and $u_t(\mathbf{x}, 0) = y$ using Kirchhoff's formula. [Here $\mathbf{x} = (x, y, z)$.]

4. Where does a solution of the three dimensional wave equation have to vanish if its initial data $\varphi(\mathbf{x}) = u(\mathbf{x}, 0)$ and $\psi(\mathbf{x}) = u_t(\mathbf{x}, 0)$ vanish inside a sphere? What if they vanish outside of a sphere (but not inside)? [You can take the sphere to be the unit ball $B(\mathbf{0}, 1) = \{\mathbf{x} : |\mathbf{x}| \leq 1\}$ for simplicity].

5. Solve the wave equation in three dimensions with initial data $u(\mathbf{x}, 0) \equiv 0$ and $u_t(\mathbf{x}, 0) = |\mathbf{x}|^2 = x^2 + y^2 + z^2$. [Hint: It may be easier to go back and solve the Euler-Poisson-Darboux equation directly.]

6. Let u be a solution of the three dimensional wave equation $u_{tt} - c^2 \Delta u = 0$ that satisfies $u(\mathbf{x}, t) = 0$ whenever $|\mathbf{x}| \geq ct + 1$. Show that the energy

$$E(t) = \frac{1}{2} \iiint_{\mathbb{R}^3} u_t(\mathbf{x}, t)^2 + c^2 |\nabla u(\mathbf{x}, t)|^2 \, d\mathbf{x}$$

is conserved, i.e., show that $t \mapsto E(t)$ is constant. [Hint: You will need to use Green's identity at some point. To handle the boundary terms, write

$$\iiint_{\mathbb{R}^3} \cdots \, d\mathbf{x} = \iiint_{B(\mathbf{0}, ct+2)} \cdots \, d\mathbf{x},$$

which holds because $u(\mathbf{x}, t) = 0$ for $|x| \geq ct + 1$, and use Green's identity on the second term.]

7. Consider the Klein-Gordon equation

$$u_{tt} - c^2 \Delta u + m^2 u = 0.$$

Find the corresponding energy for the Klein-Gordon equation and show that it is constant. [You may make the same assumptions on u as in problem 6.]