Math 126 – Homework 11 (Due Monday Nov 23)

1. In this question, you will construct a test function that is positive on the interval $(-1, 1)$ and vanishes outside of this interval. Recall a test function must be infinitely differentiable and compactly supported.

   Define
   $$\psi(x) = \begin{cases} 0, & x \le 0 \\ e^{-\frac{1}{x^2}}, & x > 0. \end{cases}.$$

   (a) Use induction to show that for $x > 0$ *all* derivatives of ψ of all orders are of the form
   $$p\left(\tfrac{1}{x}\right) e^{-\frac{1}{x^2}},$$
   where p is a polynomial. [The polynomial p will be different for each derivative. You don't need to find p explicitly.]

   (b) For any polynomial p show that
   $$\lim_{x\to 0^+} p\left(\tfrac{1}{x}\right) e^{-\frac{1}{x^2}} = 0.$$
   [Hint: By setting $y = 1/x$, you may show instead that
   $$\lim_{y\to\infty} p(y)e^{-y^2} = 0.$$
   To prove this, first show that for any positive integer k, $\lim_{y\to\infty} y^k e^{-y^2} = 0$. There are many ways to do this. One way is to use the Taylor series for e^x to show that $e^x \ge x^k/k!$ for any $x > 0$. From this you can deduce that $e^{-y^2} \le k!/y^{2k}$.]

   (c) Conclude from (b) that ψ is infinitely differentiable.

   (d) Define
   $$\varphi(x) = \psi(x+1)\psi(1-x).$$
   Show that φ is infinitely differentiable, positive in $(-1, 1)$ and vanishes for $x \notin (-1, 1)$.

   (e) Fix $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Use part (d) to find a test function g that is positive on the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ and vanishes outside of this interval. [Hint: Scale and translate φ.]

2. Let f be any distribution. Verify that the functional f' defined by $(f', \varphi) = -(f, \varphi')$ satisfies the linearity and continuity properties and is therefore a distribution.

3. (a) Show that $f(x) = \frac{1}{2}\log(x^2)$ is locally integrable, and thus defines a distribution.

   (b) Show that the ordinary derivative $f'(x) = \frac{1}{x}$ is not locally integrable, and is therefore not a distribution.

(c) Show that the distributional derivative of f is the distribution

$$(f', \varphi) = -(f, \varphi') = \text{P.V.} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\, dx.$$

[Here, P.V. stands for the Cauchy principle value of the integral, and is defined as

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\, dx := \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx,$$

where

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx = \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}\, dx.]$$

(d) Show that the second distributional derivative of f is the distribution

$$(f'', \varphi) = (f, \varphi'') = \text{P.V.} \int_{-\infty}^{\infty} \frac{\varphi(0) - \varphi(x)}{x^2}\, dx.$$

4. Verify that $u(x,t) = H(x - ct)$ is a weak solution of the wave equation, where H is the Heaviside function.

5. Let

$$f_n(x) = \begin{cases} \frac{n}{2}, & \text{if } -\frac{1}{n} < x < \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$$

Show that $f_n \to \delta$ weakly as $n \to \infty$.

6. An infinite string, at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\delta(x)$, where V is a constant. Find the position of the string for $t > 0$.