Math 126 – Homework 2 (Due Friday Sept 11)

1. For $i = 1, 2$, let u^i be a solution of the Dirichlet problem

$$\left.\begin{aligned} u^i_t - k u^i_{xx} &= 0 && \text{for } 0 < x < l \text{ and } t > 0,\\ u^i(x,0) &= \varphi^i && \text{for } 0 \le x \le l,\\ u^i(0,t) = g^i(t) \ \text{ and } \ u^i(l,t) &= h^i(t) && \text{for } t > 0. \end{aligned}\right\} \tag{1}$$

Use the maximum principle to show that

$$\max_{\substack{0\le x\le l\\ 0\le t\le T}} |u^1 - u^2| \le \max\left\{ \max_{0\le x\le l} |\varphi^1 - \varphi^2|, \max_{0\le t\le T} |g^1 - g^1|, \max_{0\le t\le T} |h^1 - h^2| \right\}.$$

[Hint: We briefly touched on this in class; you have to fill in the details. Start by setting $w = u^1 - u^2$ and show that w satisfies a similar heat equation. Apply the maximum (and minimum) principle to w.]

2. Solve $u_{tt} = c^2 u_{xx}$ with $u(x,0) = x^2$ and $u_t(x,0) = \cos(x)$.

3. Find the general solution of $u_{xx} - 3u_{xt} - 4u_{tt} = 0$ with $u(x,0) = x^2$ and $u_t(x,0) = e^x$. [Hint: Factor the operator as we did for the wave equation.]

4. **The hammer blow:** Consider the wave equation

$$\left.\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{for } x \in \mathbb{R} \text{ and } t > 0,\\ u = 0 \ \text{ and } \ u_t &= \psi && \text{for } x \in \mathbb{R} \text{ and } t = 0, \end{aligned}\right\} \tag{2}$$

where $\psi(x) = 1$ for $|x| < 1$ and $\psi(x) = 0$ for $|x| \ge 1$. Sketch the solution at time instants $t = 1/2, 1, 3/2, 2$ and $t = 5/2$. What is the maximum displacement $\max_x u(x,t)$?

5. Solve the diffusion equation

$$\left.\begin{aligned} u_t - u_{xx} &= 0 && \text{for } x \in \mathbb{R} \text{ and } t > 0,\\ u &= e^{ax} && \text{for } x \in \mathbb{R} \text{ and } t = 0, \end{aligned}\right\} \tag{3}$$

where $a \in \mathbb{R}$.

6. 'Blow-up' for the heat equation.

   (a) Show that the function

$$u(x,t) = \frac{1}{\sqrt{1-t}} \exp\left(\frac{x^2}{4(1-t)}\right)$$

   is a solution of the heat equation

$$u_t - u_{xx} = 0 \quad \text{for } -\infty < x < \infty \text{ and } 0 < t < 1.$$

   (b) Sketch the functions $t \mapsto u(0,t)$ and $x \mapsto u(x, 1/2)$.

(c) Can you give a physical explanation for the 'blow-up' observed at $t = 1$?

7. Let u and v be functions satisfying the heat equations

$$u_t - ku_{xx} = f \quad \text{and} \quad v_t - kv_{xx} = g$$

on the rectangular strip $0 \leq x \leq l$ and $0 \leq t < \infty$ for given functions $f(x,t)$ and $g(x,t)$. Prove the following comparison principle: If $f \leq g$ on the entire rectangular strip, and $u \leq v$ on the sides $x = 0$, $x = l$ and $t = 0$, then $u \leq v$ in the entire strip $0 \leq x \leq l$ and $0 \leq t < \infty$. Can you give a physical interpretation of this comparison principle?

8. **Duhamel's principle:** Consider the heat equation with a source term $f(x,t)$

$$\left.\begin{aligned} u_t - ku_{xx} &= f \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u &= \varphi \quad \text{for } x \in \mathbb{R} \text{ and } t = 0. \end{aligned}\right\} \tag{4}$$

In this question, you will use Duhamel's principle to construct a solution u of (4).

(a) For $s \geq 0$, let $w(x,t;s)$ be the solution of the homogeneous heat equation

$$\left.\begin{aligned} w_t(x,t;s) - kw_{xx}(x,t;s) &= 0 \qquad \text{for } x \in \mathbb{R} \text{ and } t > s, \\ w(x,s;s) &= f(x,s) \quad \text{for } x \in \mathbb{R}, \end{aligned}\right\} \tag{5}$$

and define

$$w(x,t) = \int_0^t w(x,t;s)\,ds.$$

Show that w is a solution of the heat equation

$$\left.\begin{aligned} w_t - kw_{xx} &= f \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ w &= 0 \quad \text{for } x \in \mathbb{R} \text{ and } t = 0, \end{aligned}\right\} \tag{6}$$

and find a formula for w in terms of f. [Hint: You may assume that $w(x,t;s)$ is twice continuously differentiable in x and t for all $t \geq s$ and $x \in \mathbb{R}$. Also recall the identity

$$\frac{d}{dt}\int_0^t g(s,t)\,ds = g(t,t) + \int_0^t g_t(s,t)\,ds.]$$

(b) Use your answer to part (a) and linearity to find a formula for the solution u of the heat equation (4).

The technique used in part (a) is called *Duhamel's principle* and is applicable to a wide range of problems (not just the heat equation).

9. (a) Solve the diffusion equation

$$\left.\begin{aligned} u_t - u_{xx} &= 0 \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u &= x^2 \quad \text{for } x \in \mathbb{R} \text{ and } t = 0, \end{aligned}\right\} \tag{7}$$

without using the fundamental solution. [Hint: Note that u_{xxx} satisfies the same diffusion equation with zero initial condition. Therefore $u_{xxx} = 0$ and we find that $u(x,t) = A(t)x^2 + B(t)x + C(t)$. Solve for $A(t)$, $B(t)$ and $C(t)$.]

(b) Use your answer to part (a) to deduce the value of

$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp.$$

[Hint: Write an expression for $u(0,1)$ using the fundamental solution and make a substitution in the integral.]