Math 126 – Homework 7 (Due Friday Oct 23)

1. Consider the Dirichlet problem for an annulus

$$\begin{cases} \Delta u = 0 & \text{in } 0 < a^2 < x^2 + y^2 < b^2 \\ u = g(\theta) & \text{on } x^2 + y^2 = a^2 \\ u = h(\theta) & \text{on } x^2 + y^2 = b^2. \end{cases}$$

   In class we found a series representation for the solution $u(r, \theta)$. Find the coefficients of the series by substituting $r = a$ and $r = b$ into $u(r, \theta)$.

2. Solve $\Delta u = 0$ in the exterior $x^2 + y^2 > a^2$ of a disk, with boundary condition $u = 1 + 3\sin\theta$ on $x^2 + y^2 = a^2$, and the condition at infinity that u be bounded as $r \to \infty$ ($r = \sqrt{x^2 + y^2}$).

3. (a) Find the steady-state temperature distribution inside an annular plate $\{1 < r < 2\}$, whose outer edge ($r = 2$) is insulated, and on whose inner edge ($r = 1$), the temperature is maintained as $\sin^2\theta$ (find explicitly the coefficients).

   (b) Do the same with $u = 0$ on the outer edge.

4. Solve $\Delta u = 0$ in the wedge $r < a$, $0 < \theta < \beta$ with boundary conditions $u(r, \theta) = \theta$ for all points (r, θ) on the boundary of the wedge. [Hint: Look for a function that is independent of r.]

5. Suppose that $-\Delta u(x, y) \leq f(x, y)$ in the disk $x^2 + y^2 < 1$, and $u \leq 0$ on the boundary $x^2 + y^2 = 1$. Show that

$$\max_{x^2+y^2\leq 1} u \leq \frac{1}{4} \max_{x^2+y^2\leq 1} |f|.$$

   [Hint: Write $w = u + C(x^2 + y^2)$ for a constant $C > 0$. Adjust C so that $-\Delta w \leq 0$ within the disk and apply the maximum principle for subharmonic functions from the previous homework.]

6. Let $D \subseteq \mathbb{R}^3$ be open and bounded.

   (a) Prove the uniqueness up to constants of the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } D \\ \dfrac{\partial u}{\partial \mathbf{n}} = g & \text{on } \partial D, \end{cases}$$

   using energy methods.

   (b) Use energy methods to prove uniqueness for the Robin problem

$$\begin{cases} -\Delta u = f & \text{in } D \\ \dfrac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) + a(\mathbf{x})u(\mathbf{x}) = h(\mathbf{x}), & \text{for } \mathbf{x} \in \partial D, \end{cases}$$

   provided $a > 0$.

7. Consider the heat equation

$$\begin{cases} u_t - \Delta u = f & \text{in } D \times (0, T] \\ u = g & \text{on } D \times \{t = 0\} \\ u = h & \text{on } \partial D \times (0, T]. \end{cases}$$

Prove uniqueness using energy methods. [Hint: Let u and v be two solutions and set $w = u - v$. Define the energy

$$e(t) = \iiint_D w(\mathbf{x}, t)^2 \, d\mathbf{x}.$$

Show that $e'(t) \leq 0$ and $e(0) = 0$. Conclude that $e(t) = 0$ for all t.]