Math 126 Sample Final Exam

Information

- The final exam is on December 14, 7pm-10pm in 3106 Etcheverry.
- The exam is cumulative and can potentially cover any topic from class. Please see the course schedule on the Math 126 website for a list of topics covered https://math.berkeley.edu/~jcalder/126F15/schedule.html. Please also see the supplemental lecture notes available here: https://math.berkeley.edu/~jcalder/126F15/homework.html.
- The exam will have 10 questions, ranging from easy to hard. It is a good idea to look through the exam and attempt the questions you are most comfortable with first. The questions will be in no particular order.
- The exam is closed book. Books, notes, calculators, cell phones, pagers, or other similar devices are not allowed. I will provide you with scratch paper and the exam will have sufficient space to work out all the questions.

Sample problems

1. Find $u(x, y)$ satisfying $u_x + x^2 u_y = 0$ and $u(1, y) = e^y$.
2. Find $u(x, y)$ satisfying $yu_x + u_y = 0$ and $u(x, 0) = x\cos(x)$.
3. Solve the heat equation $u_t = ku_{xx}$ with initial condition $u(x, 0) = e^x$.
4. Solve the heat equation $u_t = u_{xx}$ on the half-line $x > 0$ with Neumann boundary condition $u_x(0, t) = 0$ for $t > 0$ and initial condition $u(x, 0) = x^2$ for $x > 0$. [You do not need to evaluate the integral.]
5. Solve the wave equation $u_{tt} = u_{xx}$ with initial position $u(x, 0) = \sin^2(x)$ and initial velocity $u_t(x, 0) = 2\sin(x)\cos(x)$. Simplify your solution as much as possible.
6. Solve the wave equation $u_{tt} = u_{xx}$ on the half-line $x > 0$ with Dirichlet boundary condition $u(0, t) = 1$ for $t > 0$, initial position $u(x, 0) = \cos(x)$ for $x > 0$ and initial velocity $u_t(x, 0) = e^{-x}$ for $x > 0$.
7. Solve the heat equation $u_t = ku_{xx}$ on the bounded domain $0 < x < \pi$ with mixed boundary conditions $u(0, t) = u_x(\pi, t) = 0$ and initial condition $u(x, 0) = x$ for $0 < x < \pi$.
8. Solve the wave equation $u_{tt} = c^2 u_{xx}$ on the bounded domain $0 < x < 1$ with boundary conditions $u(0, t) = 1$ and $u_x(1, t) = 0$, initial position $u(x, 0) = x$ and initial velocity $u_t(x, 0) = 1$ for $0 < x < 1$.
9. Let $f(x) = \exp(\sin(x))\sin(x)$. Explain how you can deduce, without any computations, that $A_n = 1/n^2$ and $B_n = 1/n^4$ cannot be the coefficients of the Fourier series for f

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx).$$

10. Compute the integral
$$\iint_B \log(x^2+y^2)\,dxdy,$$
where B is the ball of radius 1 centered at $(2,0)$. [Hint: Use the mean value property.]

11. Suppose that
$$\Delta u = 1$$
throughout the unit ball $B = B(0,1)$ in $\mathbb{R}^3$, and $u(x,y,z) = h(x,y,z)$ on the boundary ∂B. Show that
$$u(x,y,z) \leq \frac{1}{6}(x^2+y^2+z^2-1) + \max_{\partial B} h.$$
[Hint: Show that $v(x,y,z) = u(x,y,z) - \frac{1}{6}(x^2+y^2+z^2-1)$ is harmonic and use the maximum principle.]

12. Consider the finite difference scheme
$$\frac{u_j^{n+1}-u_j^n}{\Delta t} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x},$$
for the transport equation $u_t = u_x$, and let $s = \frac{\Delta t}{2\Delta x}$. Show that the scheme is unstable for any choice of s.

13. Use energy methods to prove uniqueness for the boundary value problem
$$\begin{cases} -\Delta u + u = f & \text{in } D \\ u = g & \text{on } \partial D, \end{cases}$$
where $D \subseteq \mathbb{R}^3$ is bounded and open. [Hint: Let u and v be two solutions of the PDE. Define $w = u - v$ and find a PDE that w satsifies. Multiply both sides of the PDE by w, integrate over D, and apply Green's identities.]

14. Use Fourier transforms to solve the ODE $-u_{xx} + a^2 u = f(x)$. [Hint: Use the convolution property.]

15. If $p(x)$ is a polynomial and $\varphi(x)$ is a test function, show that the convolution $p * \varphi$ is a polynomial. [Hint: By linearity of the convolution, it is sufficient to consider the case where $p(x) = x^k$ for $k \geq 1$. Recall the binomial theorem
$$p(x-y) = (x-y)^k = \sum_{n=0}^{k} \binom{k}{n} x^n (-y)^{k-n}.]$$

16. The convolution of a function ψ with a distribution f is defined by
$$(\psi * f, \varphi) := (f, \tilde{\psi} * \varphi),$$
where $\tilde{\psi}(x) := \psi(-x)$. Let δ be the Delta function and let ψ be any function. Show that $\delta * \psi = \psi$ in the distributional sense.

17. Find the entropy solution of Burger's equation

$$u_t + uu_x = 0, \qquad t > 0$$

satisfying $u(x,0) = a$ for $x < 0$ and $u(x,0) = b$ for $x > 0$, where $a > b$.

18. Find the entropy solution of Burger's equation

$$u_t + uu_x = 0, \qquad 0 < t < 1$$

satisfying $u(x,0) = -x$. Sketch the characteristics.