Math 1272: Calculus II
8.2 Area of surface of revolution

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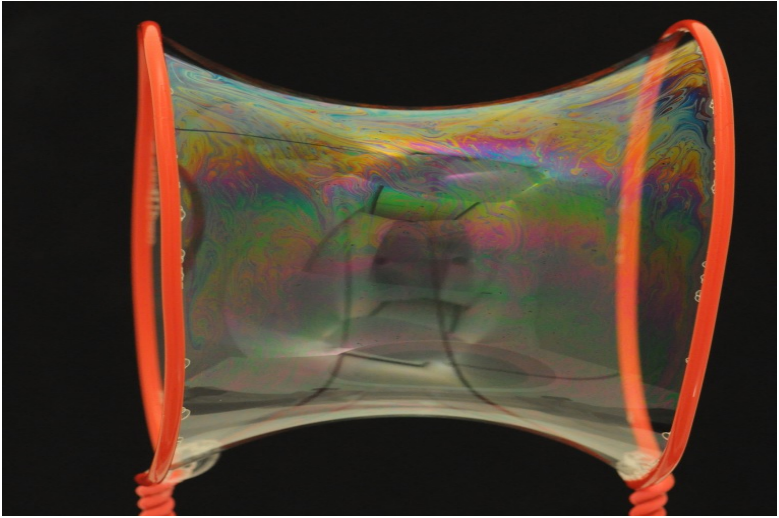
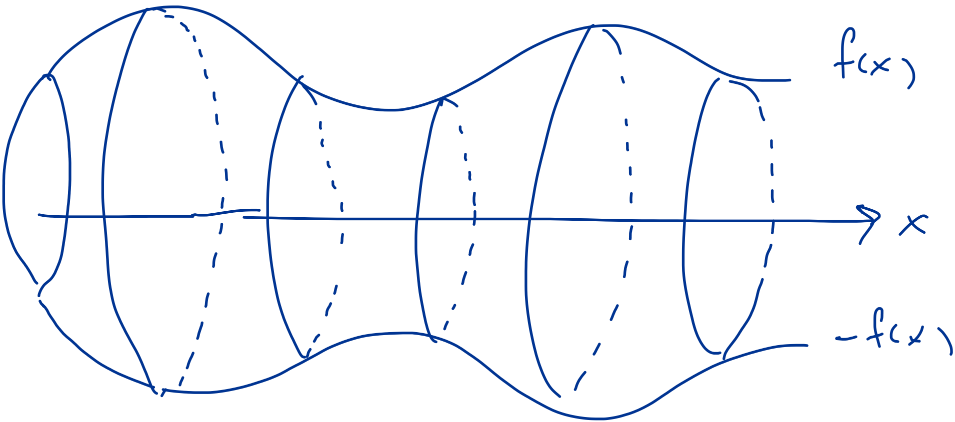
Surface of revolution



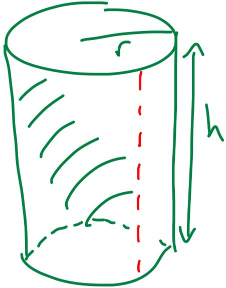
Area of surface of revolution

Consider a surface obtained by rotating the curve $y = f(x)$ about the x-axis.

What is the surface area?



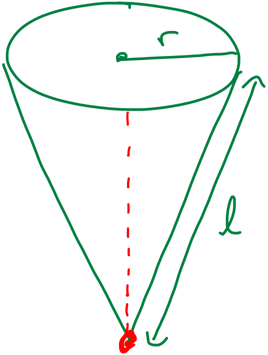
Basic shapes: Cylinders and cones

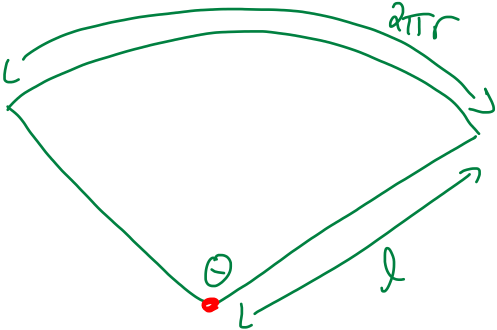


Cylinder can be unrolled into a rectangle

$$\boxed{SA = 2\pi r h}$$

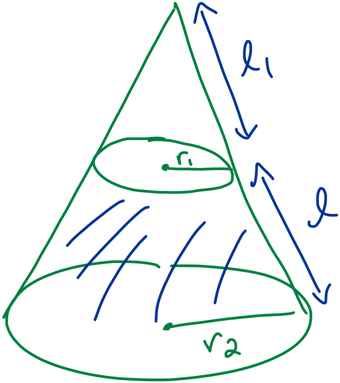
Cone can be unrolled to a section of a circle





$$\boxed{\text{Surface Area} = \left(\frac{\cancel{2\pi} r}{\cancel{2\pi}\cancel{\ell}}\right)\pi \ell^{\cancel{2}} = \pi r \ell}$$

Surface area of partial cone from r_2 to r_2.



$$SA = \underbrace{\pi r_2(\ell+\ell_1)}_{\text{Large cone}} - \underbrace{\pi r_1 \ell_1}_{\text{small cone}}$$

$$= \pi r_2 \ell + \cancel{\pi r_2 \ell_1} - \pi(\cancel{r_2 \ell_1} - r_1 \ell)$$

$$\boxed{= \pi \ell (r_1 + r_2)}$$

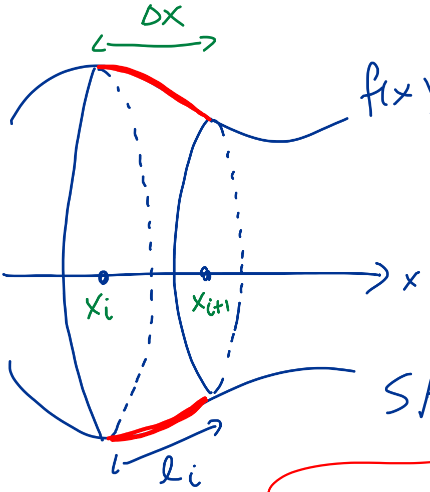
Similar triangles

$$\frac{\ell+\ell_1}{r_2} = \frac{\ell_1}{r_1} \rightsquigarrow r_1 \ell + r_1 \ell_1 = r_2 \ell_1$$

$$r_1 \ell_1 = r_2 \ell_1 - r_1 \ell$$

Area of surface of revolution

let $y = f(x)$ for $a \le x \le b$



$\Delta x = \frac{b-a}{n}$, $x_0, x_1, \ldots, x_n$

Partial cone from

x_i to x_{i+1}

$$SA_i = \pi l_i \left(f(x_i) + f(x_{i+1}) \right)$$

$$= \pi \left(f(x_i) + f(x_{i+1}) \right) \sqrt{1 + [f'(x_i)]^2} \, \Delta x$$

Sum from $i = 0, \dots, n-1$

$$SA = \sum_{i=0}^{n-1} \pi \left(f(x_i) + f(x_{i+1}) \right) \sqrt{1 + (f'(x_i))^2}\, \Delta x$$

Riemann Sum for

$$SA = \int_a^b 2\pi \underbrace{f(x)}_{\text{radius}} \underbrace{\sqrt{1 + (f'(x))^2}\, dx}_{\text{arclength}}$$

Area of surface of revolution

The surface area of a surface of rotation of the curve $y = f(x)$ about the x-axis from $x = a$ to $x = b$ is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

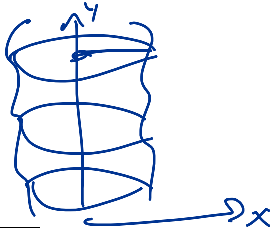
Area of surface of revolution

We can also write

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

$y = f(x)$

$$S = \int_c^d 2\pi {\color{red}x} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy,$$ (typo in book, pg 533 Eq 6)



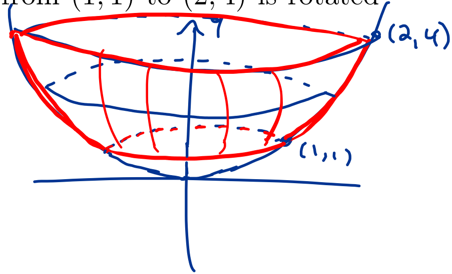
$$S = \int 2\pi y\, ds,$$

$$S = \int 2\pi x\, ds,$$

where

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Example 1. The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the surface.



$$x = \sqrt{y}$$

$$SA = 2\pi \int_1^4 x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$$

$$= 2\pi \int_1^4 \sqrt{y} \sqrt{1 + \frac{1}{4y}}\, dy$$

$$= 2\pi \int_1^4 \sqrt{y + \frac{1}{4}}\, dy$$

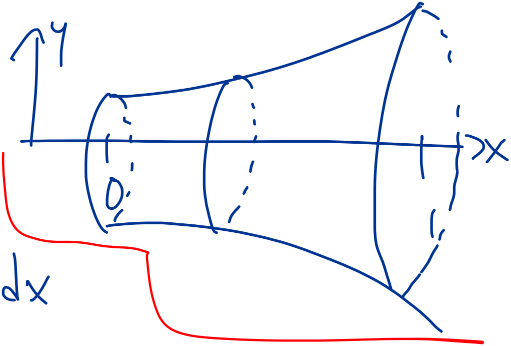
$$= 2\pi \, \frac{2}{3}\left(4+\frac{1}{4}\right)^{3/2} \Bigg]^{4}_{1}$$

$$= \frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{5}{4}\right)^{3/2}\right]$$

$$= \frac{4\pi}{3\,4^{3/2}}\left(17^{3/2} - 5^{3/2}\right)$$

$$= \frac{\pi}{3\sqrt{4}}\left(17^{3/2} - 5^{3/2}\right)$$

Example 2. Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$ about the x-axis.



$$SA = 2\pi \int_0^1 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^1 e^x \sqrt{1 + (e^x)^2}\, dx$$

$u = e^x$

$du = e^x\, dx$

$$(*) = 2\pi \int_1^e \sqrt{1+u^2}\, du$$

$u = \tan\theta$

$du = \sec^2\theta\, d\theta$

Limits

$\tan\theta = 1$, $\tan\theta = e$

$\theta = \pi/4 \qquad \theta = \tan^{-1}(e) \qquad 1 + \tan^2\theta = \sec^2\theta$

$$(\ast) = 2\pi \int_{\pi/4}^{\tan^{-1}(e)} \sqrt{1+\tan^2\theta}\, \sec^2\theta \, d\theta$$

$$= 2\pi \int_{\pi/4}^{\tan^{-1}(e)} \sec^3\theta \, d\theta$$

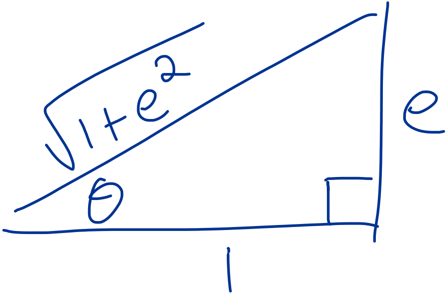
$$= 2\pi \left[\frac{1}{2} e \sec(\tan^{-1}(e)) + \frac{1}{2}\ln\left| e + \sec(\tan^{-1}(e))\right| - \frac{1}{2}\sec\left(\frac{\pi}{4}\right) - \frac{1}{2}\ln\left|1 + \sec(\pi/4)\right| \right]$$

$$\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos(\pi/4)} = \sqrt{2}$$

$\sec(\tan^{-1}(e))$

$\theta = \tan^{-1}(e)$

$e = \tan\theta$

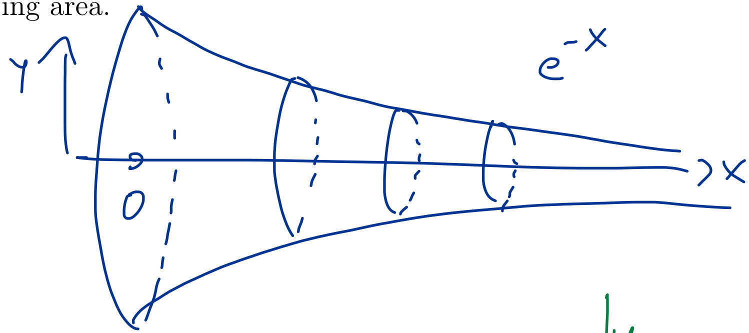


$$\sec(\tan^{-1}(e)) = \sec(\theta)$$

$$= \frac{1}{\cos(\theta)} = \sqrt{1+e^2}$$

Plus back in ....

Example 3. The infinite curve $y = e^{-x}$, $x \geq 0$ is rotated about the x-axis. Find the resulting area.



$$\frac{dy}{dx} = -e^{-x}$$

$$SA = \int_0^\infty 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^\infty 2\pi e^{-x} \sqrt{1 + (e^{-x})^2}\, dx$$

This is an improper integral. Let $t > 0$ and compute

$$\int_0^t 2\pi e^{-x} \sqrt{1 + (e^{-x})^2}\, dx$$

sub $u = e^{-x}$

$du = -e^{-x} dx$

$$= -\int_1^{e^{-t}} 2\pi \sqrt{1 + u^2}\, du$$

$$= \int_{e^{-t}}^{1} 2\pi \sqrt{1+u^2}\, du$$

sub $u = \tan\theta$

$du = \sec^2\theta \, d\theta$

$$\sqrt{1+u^2} = \sqrt{1+\tan^2\theta}$$
$$= \sqrt{\sec^2\theta}$$
$$= \sec\theta$$

$$= \int_{\tan^{-1}(e^{-t})}^{\frac{\pi}{4}} 2\pi \sec^3\theta \, d\theta$$

$$= \pi \Big[\tan\theta\sec\theta + \ln|\tan\theta + \sec\theta| \Big]_{\tan^{-1}(e^{-t})}^{\frac{\pi}{4}}$$

$$= \pi \Big[\sqrt{2} + \ln(\sqrt{2}+1) - e^{-t}\sec(\tan^{-1}(e^{-t}))$$
$$- \ln|e^{-t} + \sec(\tan^{-1}(e^{-t}))| \Big]$$

$$\lim_{t\to\infty} \sec\left(\tan^{-1}(e^{-t})\right)$$

$$= \sec\left(\lim_{t\to\infty} \tan^{-1}(e^{-t})\right) \qquad \sec x \text{ continuous}$$

$$= \sec\left(\tan^{-1}\left(\lim_{t\to\infty} e^{-t}\right)\right) \qquad \tan^{-1} \text{ continuous}$$

$$= \sec\left(\tan^{-1}(0)\right) = \sec(0) = 1$$

$$\int_0^\infty 2\pi e^{-x}\sqrt{1+(e^{-x})^2}\,dx = \lim_{t\to\infty}\int_0^t 2\pi e^{-x}\sqrt{1+(e^{-x})^2}\,dx$$

$$= \pi\left(\sqrt{2}+\ln(\sqrt{2}+1)\right)$$

= Surface area.