Math 1272: Calculus II
8.3 Applications to physics and engineering

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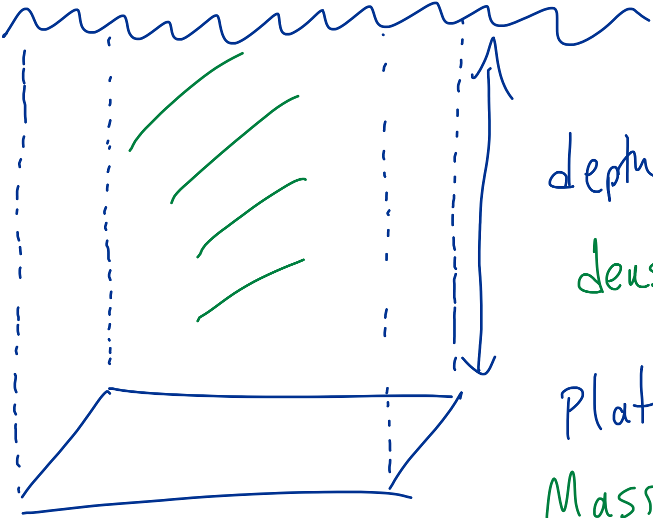
Hydrostatic pressure and force



Water

depth $= d$.

density $=$

Plate with area A

Mass $= \rho \cdot Ad$

Force $= mg = \rho A g d$

$$\boxed{\text{Pressure} = \frac{F}{A} = \rho g d}$$

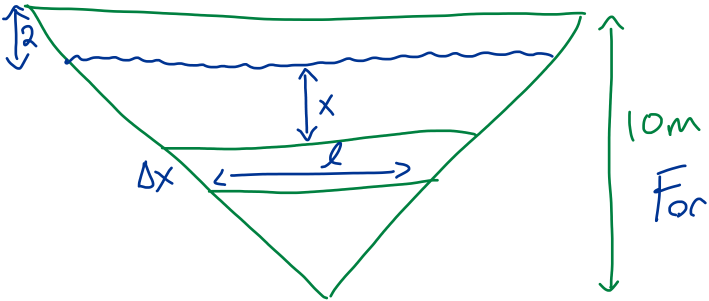
Hydrostatic pressure and force

The **pressure** at depth d in a fluid with density ρ is

$$P = \rho g d,$$

where g is acceleration due to gravity.

Example 1. A dam has the shape of an equilateral triangle with height 10m, and is filled with a fluid with density ρ up to 2m from the top. Find the hydrostatic pressure of the fluid on one side of the dam.

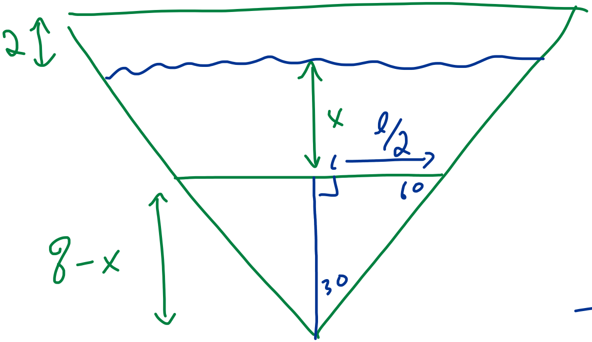


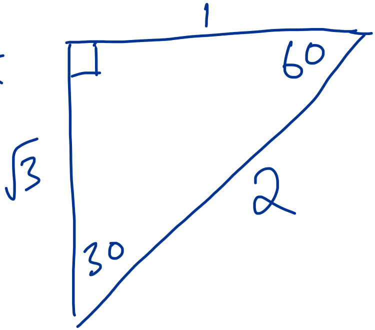
Pressure on Δx slab is $\rho g x$

Force $= P \cdot A$

$= \rho g x \, l \, \Delta x$

Express ℓ in terms of x



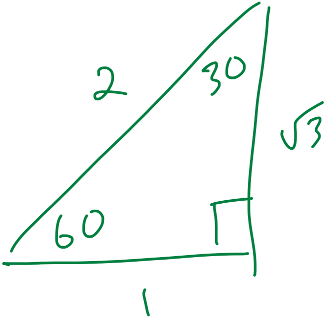


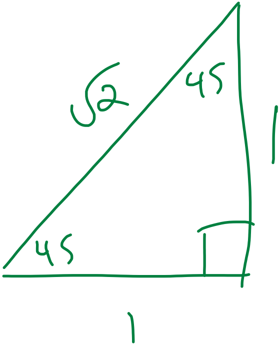
$$\frac{\ell/2}{8-x} = \tan(30)$$

$$\frac{\ell}{2} = (8-x)\tan(30)$$

$$\ell = 2(8-x)\underbrace{\tan(30)}_{=\frac{1}{\sqrt{3}}}$$

$$\boxed{\ell = \frac{2}{\sqrt{3}}(8-x)}$$





$\cos(60) = \frac{1}{2}$

$\sin(60) = \frac{\sqrt{3}}{2}$

$\tan(60) = \sqrt{3}$

$\sin(30) = \frac{1}{2}$

....

$\sin(45) = \frac{1}{\sqrt{2}}$

$\cos(45) = \frac{1}{\sqrt{2}}$

$\tan(45) = 1$

Force on Slab $= P \cdot A$

$$= \rho g x \, \ell \, \Delta x$$

$$= \rho g x \frac{2}{\sqrt{3}} (8 - x) \Delta x$$

Riemann Sum over all slabs $x_0, x_1, \ldots, x_n$

Force on dam wall $\approx \sum_{i=1}^{n} \rho g \frac{2}{\sqrt{3}} x_i (8 - x_i) \Delta x$

Send $\Delta x \longrightarrow 0$

to get

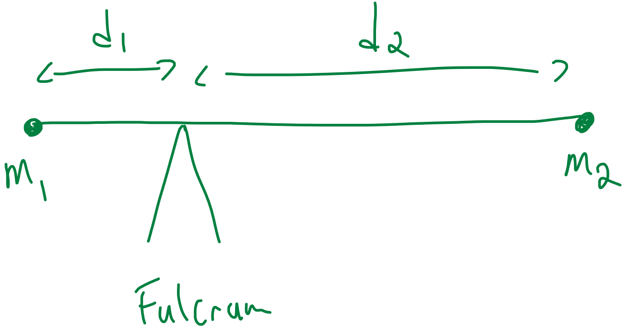
$$\text{Force} = \int_0^8 \rho g \frac{2}{\sqrt{3}} x(8-x)\,dx$$

$$= \rho g \frac{2}{\sqrt{3}} \int_0^8 8x - x^2\,dx$$

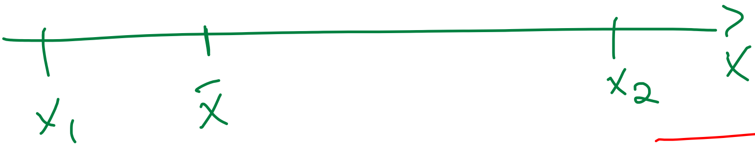
$$= \rho g \frac{2}{\sqrt{3}} \left[4x^2 - \frac{1}{3}x^3\right]_0^8$$

$$= \frac{2\rho g}{\sqrt{3}} \left[256 - \frac{512}{3}\right] \ldots$$

Moments and centers of mass



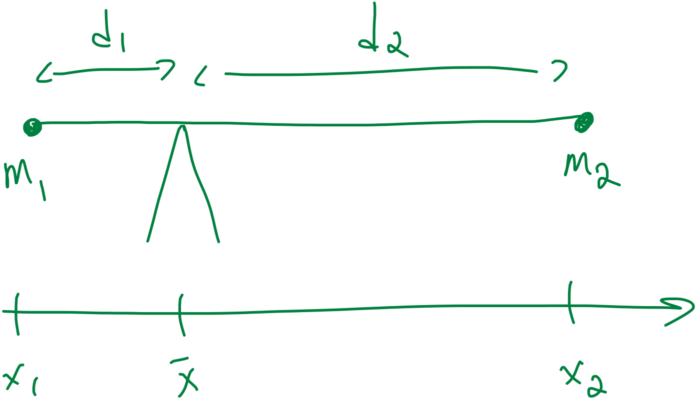
Where to put $\bar{x}$ (Fulcrum) so it balances perfectly?



$F_1 = d_1 m_1$

$F_2 = d_2 m_2$

Balance forces $\boxed{d_1 m_1 = d_2 m_2}$



$$d_2 = x_2 - \bar{x}$$

$$d_1 = \bar{x} - x_1$$

$$d_1 m_1 = d_2 m_2$$

$$(\bar{x} - x_1) m_1 = (x_2 - \bar{x}) m_2$$

$$m_1 \bar{x} - m_1 x_1 = m_2 x_2 - m_2 \bar{x}$$

$$(m_1 + m_2) \bar{x} = m_1 x_1 + m_2 x_2$$

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

Set $m = m_1 + m_2$ (Total mass)

Then

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m}}$$

$\bar{x}$ = center of mass

Suppose we have n masses $m_1, m_2, \ldots, m_n$ at positions $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{M_y}{m}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} = \frac{M_x}{m}$$

$(\bar{x}, \bar{y})$ = Centroid or center of mass.

Moments and centers of mass

Consider n masses at positions $(x_1, y_1), \ldots, (x_n, y_n)$ with masses $m_1, \ldots, m_n$. The **moment about the x-axis** is

$$M_x = \sum_{i=1}^{n} m_i y_i,$$

and the **moment about the y-axis** is

$$M_y = \sum_{i=1}^{n} m_i x_i.$$

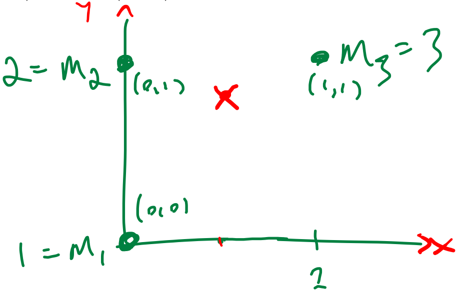
The **center of mass** $(\overline{x}, \overline{y})$ is given by

$$\boxed{\overline{x} = \frac{M_y}{m} \quad \text{and} \quad \overline{y} = \frac{M_x}{m},}$$

where $m = \sum_{i=1}^{n} m_i$ is the total mass.

Example 2. Find the moments and center of mass of the system of objects that have masses $1, 2, 3$ at positions $(0,0)$, $(0,1)$, and $(1,1)$.

$$\bar{x} = \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i}$$

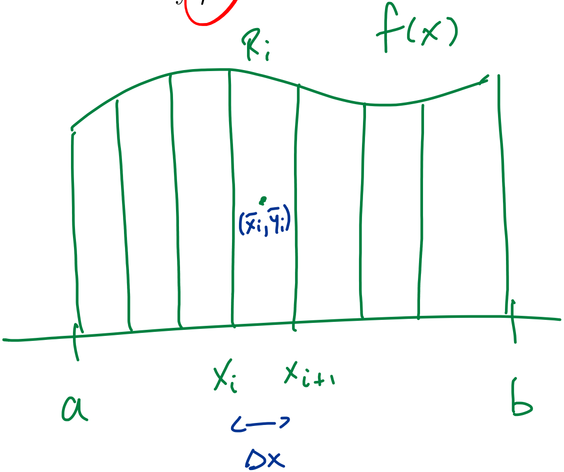


$$= \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{1\cdot 0 + 2\cdot 0 + 3\cdot 1}{1+2+3} = \frac{1}{2}$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{1\cdot 0 + 2\cdot 1 + 3\cdot 1}{1+2+3} = \frac{5}{6}$$

Moments and centers of mass of lamina

Consider the region R that lies between the lines $x = a$,$x = b$, $y = 0$ and $y = f(x)$ for a positive function f. Upon this region lies a flat plat (lamina) with density ρ.



$$\bar{x}_i = \frac{x_i + x_{i+1}}{2}$$

$$\bar{y}_i = \frac{f(\bar{x}_i)}{2}$$

$$m_i = A\rho$$

$$= f(\bar{x}_i) \cdot \Delta x \, \rho$$

$$\bar{X} \approx \frac{\sum_{i=1}^{n} m_i \bar{x}_i}{\sum_{i=1}^{n} m_i}, \quad \bar{Y} \approx \frac{\sum_{i=1}^{n} m_i \bar{Y}_i}{\sum_{i=1}^{n} m_i}$$

$$m_i = f(\bar{x}_i)\,\Delta x\,\rho, \quad \bar{x}_i = \frac{x_i + x_{i+1}}{2}, \quad \bar{Y}_i = \frac{f(\bar{x}_i)}{2}$$

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x\,\rho \approx \rho \int_a^b f(x)\,dx$$

$$\sum_{i=1}^{n} m_i \bar{x}_i = \sum_{i=1}^{n} f(\bar{x}_i)\,\bar{x}_i\,\Delta x\,\rho \approx \rho \int_a^b x f(x)\,dx$$

$$\sum_{i=1}^{n} m_i \bar{y}_i = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x\, \rho\, \frac{f(\bar{x}_i)}{2}$$

$$\approx \rho \int_a^b \frac{1}{2} f(x)^2\, dx$$

$$\bar{X} = \frac{1}{A}\int_a^b x\,f(x)\,dx$$

$$\frac{1}{2}\,\underbrace{f(x)\cdot f(x)}_{y}$$

$$\bar{Y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx$$

$$A = \int_a^b f(x)\,dx$$

$(\bar{X}, \bar{Y})$ = center of mass.

Moments and centers of mass of lamina

The moment about the y-axis is

$$M_y = \rho \int_a^b x f(x)\, dx,$$

and the moment about the x-axis is

$$M_x = \rho \int_a^b \frac{1}{2}[f(x)]^2\, dx.$$

The mass of the plate is

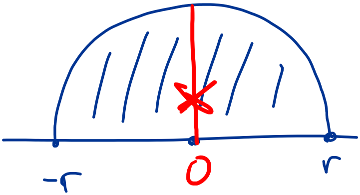
$$m = \rho \int_a^b f(x)\, dx,$$

and the center of mass is

$$\overline{x} = \frac{M_y}{m} \quad \text{and} \quad \overline{y} = \frac{M_x}{m}.$$

Example 3. Find the center of mass of a semicircular plate of radius r.

$$\boxed{A = 2\int_0^r \sqrt{r^2 - x^2}\, dx = \frac{\pi r^2}{2}}$$



Trig sub $\quad x = r\sin\theta$

$$f(x) = \sqrt{r^2 - x^2}$$

$$x^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2$$

$$y = \sqrt{r^2 - x^2}$$

$$\sqrt{r^2 - x^2} = \sqrt{r^2 - r^2\sin^2\theta}$$

$$= r\sqrt{1 - \sin^2\theta}$$

$$= r\sqrt{\cos^2\theta}$$

$$= r\cos\theta \quad , \quad dx = r\cos\theta$$

$x = r\sin\theta$, $\quad x : 0 \longrightarrow r$

$\sin\theta : 0 \longrightarrow 1$

$\theta : \sin^{-1}(0) \longrightarrow \sin^{-1}(1)$

$: 0 \longrightarrow \frac{\pi}{2}$

$$A = 2\int_0^{\pi/2} r^2 \cos^2\theta \, d\theta$$

$$= 2r^2 \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta.$$

$$= 2r^2\left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right)\Bigg]_0^{\pi/2}$$

$$= 2r^2\left(\frac{\pi}{4} + \cancel{\frac{1}{4}\sin(\pi)}^{\,0}\right) = \frac{\pi r^2}{2}$$

$$\bar{x} = \frac{1}{A}\int_{-r}^{r} x f(x)\,dx = \frac{1}{A}\int_{-r}^{r} x\sqrt{r^2-x^2}\,dx = 0$$

$$= \frac{-2(r^2-x^2)^{3/2}}{2\cdot 3}\Bigg]_{-r}^{r} = 0$$

$$\bar{Y} = \frac{1}{A}\int_{-r}^{r} \frac{1}{2}\left(f(x)\right)^2 dx$$

$$= \frac{1}{A}\int_{-r}^{r} \frac{1}{2}\left(r^2 - x^2\right) dx$$

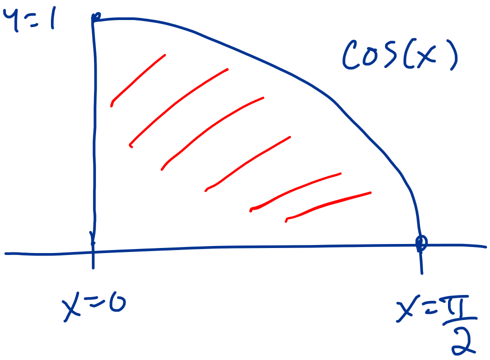
$$= \frac{1}{2A}\left(r^2 x - \frac{1}{3}x^3\right)\Bigg]_{-r}^{r}$$

$$= \frac{\cancel{2}}{\cancel{2}A}\left(r^3 - \frac{1}{3}r^3\right) = \frac{2}{3A}r^3 = \frac{r}{3\pi}$$

$$A = \frac{\pi r^2}{2}$$

$$\frac{1}{A} = \frac{2}{\pi r^2}$$

Example 4. Find the centroid of the region bounded by the curves $y = \cos x$, $y = 0$, $x = 0$, and $x = \pi/2$.



$$A = \int_0^{\pi/2} \cos x \, dx$$

$$= \sin x \Big|_0^{\pi/2} = 1$$

$$\bar{x} = \frac{1}{A} \int_0^{\pi/2} x \cos x \, dx$$

$A=1$

$u = x, \quad dv = \cos x \, dx$

$du = dx, \quad v = \sin x$

$$= x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx$$

$$= \frac{\pi}{2} + \cos x \Big|_0^{\pi/2} = \frac{\pi}{2} - 1$$

$$\bar{y} = \frac{1}{A} \int_0^{\pi/2} \frac{1}{2} (f(x))^2 \, dx$$

$$= \frac{1}{A} \int_0^{\pi/2} \frac{1}{2} \cos^2 x \, dx$$

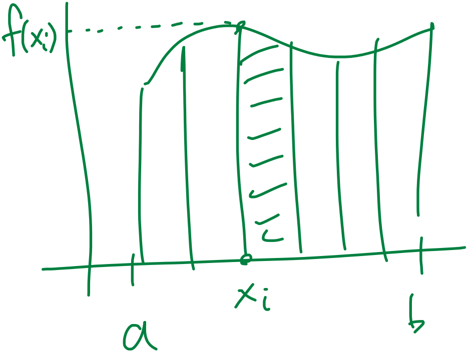
$A = 1$

$$= \frac{1}{2} \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2} \cos(2x) \right) dx$$

$$= \frac{1}{4} \left[x + \frac{1}{2} \cancel{\sin(2x)}^{\,0} \right] \Big|_0^{\pi/2} = \frac{\pi}{8}$$

Riemann Sum

$$\sum_{i=1}^{n} \underbrace{f(x_i)\,\Delta x}_{\text{area of one rectangle.}}$$



$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\,\Delta x = \text{Area under } f \text{ from } x=a \text{ to } x=b.$$

Denote Area by $\int_a^b f(x)\,dx$