Math 1272: Calculus II
9.1 Modelling with differential equations

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Basic population growth model

If $P(t)$ is the population at time t, then

$$\frac{dP}{dt} = kP,$$

for some proportionality constant $k > 0$.

A solution $P(t) = Ae^{kt}$

Check: $\frac{dP}{dt} = Ake^{kt} = kP(t)$ ✓

Solutions to differential equations involve arbitrary constants (A)

This gives a family of solutions.

$$P(t) = Ae^{kt}, \quad P(0) = A$$

Initial value problem

$$\begin{cases} \frac{dP}{dt} = kP(t) \\ P(0) = P_0 \end{cases}$$

Solution is $P(t) = P_0 e^{kt}$

If $P_0 = 100$, $P(t) = 100e^{kt}$

Improved model

The basic model grows forever: $P(t) = Ce^{kt}$.

A more realistic model:

1. $\frac{dP}{dt} \approx kP$ for small P
2. $\frac{dP}{dt} < 0$ if $P > M$.

A model that satisfies both assumptions is

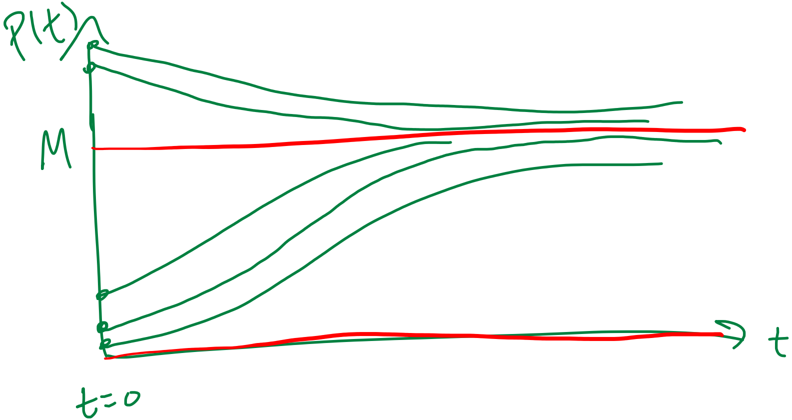
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

This is the **Logistic differential equation** proposed by Pierre-Francois Verhulst in 1840's to model population growth.

Some observations about logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

Equillibrium solutions, $P(t) = 0$, $P(t) = M$



Motion of a spring

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x.(t)$$

General differential equations

- A **differential equation** is an equation that contains an unknown function and one or more of its derivatives.

- The **order** of the differential equations is the highest derivative that occurs in the equation.

  – What order is the logistic equation?

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

First order

  – What order is the spring/mass equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

Second order.

Examples of differential equations

1. $y' = xy \quad \text{where } y = y(x).$

2. $f'(x) = xf(x)$

3. $y'(x) = x^3$

4. $y(x)^2 + y'(x)y(x) = 0.$

5. $ye^{y''}\sin(y'''') = \log(xy).$

Examples

Show that for every number c the function

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ satisfying $y(0) = 1$

Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ satisfying $y(0) = 4$.