Math 1272: Calculus II
9.3 Separable equations

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Wed office hour 12pm - 1pm

Separable equations

A **separable** differential equation has the form

$$\frac{dy}{dx} = g(x)f(y).$$

$$= \frac{g(x)}{h(y)}$$

$$h(y) = \frac{1}{f(y)}$$

$$f(y) = \frac{1}{h(y)}$$

For example

- (Separable) $y' = x \sin y$.
- (Non-separable) $y' = \sqrt{x^2 + y^2}$.

Separable equations

To solve a **separable** differential equation

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

write in **differential** form

$$h(y)dy = g(x)dx$$

and integrate both sides

$$\int h(y)\,dy = \int g(x)\,dx.$$

Solve the differential equation

$$\frac{dy}{dx} = \frac{x^3}{y^4},$$

and find the solution satisfying $y(0) = 1$.

Separate variables

$$y^4\,dy = x^3\,dx$$

$$\int y^4\,dy = \int x^3\,dx$$

$$\frac{1}{5}y^5 = \frac{1}{4}x^4 + C$$

$$y^5 = \frac{5}{4}x^4 + 5C$$

$$y(x) = \left(\frac{5}{4}x^4 + 5C\right)^{1/5}$$

Initial condition $y(0) = 1$

$$1 = y(0) = (5C)^{1/5}$$

$$1 = 5C$$

$$\boxed{y(x) = \left(\frac{5}{4}x^4 + 1\right)^{1/5}}$$

Ex: Check $\frac{dy}{dx} = \frac{x^3}{y^4}$

Solve the differential equation

$$\frac{dy}{dx} = y' = \frac{12x^3}{\cos(y) + \sec^2(y)}.$$

Separate variables

$$\int \left(\cos(y) + \sec^2(y)\right) dy = \int 12x^3 \, dx$$

$$\sin(y) + \tan(y) = 3x^4 + C$$

Implicitly defines $y(x)$.

Solve the differential equation

$$\frac{dy}{dx} = y' = xy. \longrightarrow \frac{y'}{y} = x$$

Separate variables

$$\frac{1}{y}\,dy = x\,dx$$

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

Exponentiate

$$|y(x)| = e^{\frac{1}{2}x^2 + C}$$

$$y(x) = \pm e^{\frac{1}{2}x^2 + C}$$

Solve the equation for electric current

$$\frac{dI}{dt} = 15 - 3I, \quad = f(I)g(t)$$

satisfying $I(0) = 0$.

Separate variables

$$dI = (15-3I)\,dt$$

$$\int \frac{dI}{15-3I} = \int dt = t + C$$

$$=$$

$$\int -\frac{1}{3u}\,du = -\frac{1}{3}\ln|u|$$

$u = 15 - 3I$

$du = -3\,dI$

$$-\frac{1}{3}\ln|15-3I| = t + C \qquad \boxed{= -\frac{1}{3}\ln|15-3I|}$$

$$\ln|15-3I| = -3t - 3C$$

$$|15-3I| = e^{-3t-3C} = e^{-3C}e^{-3t} = Ae^{-3t}$$

$$15-3I(t) = \pm Ae^{-3t} \qquad A = e^{-3C}$$

$$\boxed{A > 0}$$

$$15 = 3I(t) + Ae^{-3t}, \quad A \text{ any number}$$

$$5 = I(t) + \frac{A}{3}e^{-3t}$$

$$I(t) = 5 - \frac{A}{3}e^{-3t} = 5 - 5e^{-3t}$$

Initial condition $I(0) = 0$

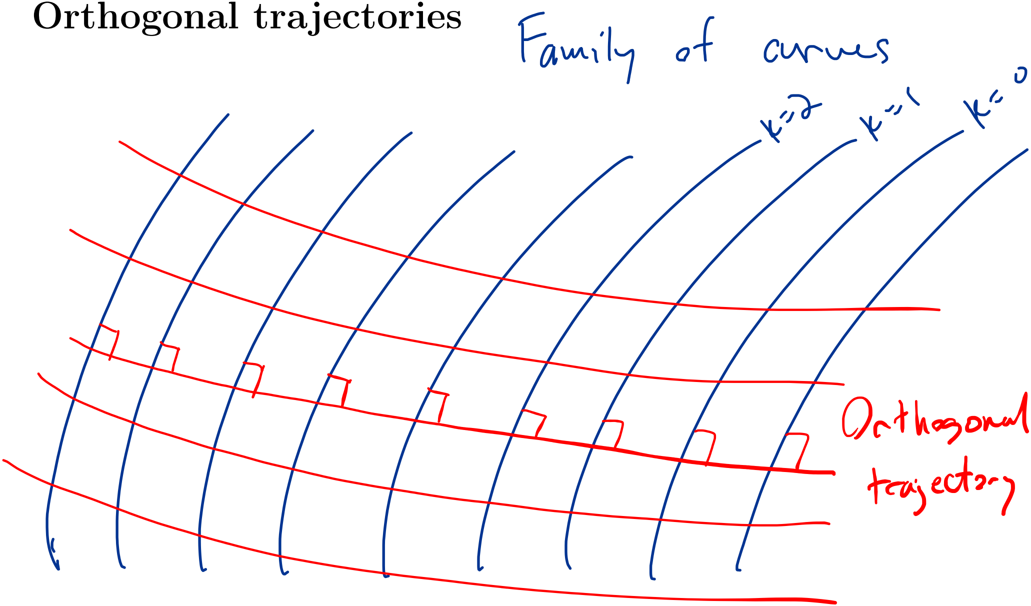
$$0 = I(0) = 5 - \frac{A}{3}e^{0} = 5 - \frac{A}{3}$$

$$\Rightarrow \frac{A}{3} = 5$$

$$\boxed{I(t) = 5(1 - e^{-3t})}$$

$$\lim_{t \to \infty} I(t) = 5$$ (Equillibrium solution)

Orthogonal trajectories



Find the orthogonal trajectories of the family of curves $x = ky^2$ where k is an arbitrary constant.

Slope $\frac{dx}{dy} = 2ky$ $\quad k = \frac{x}{y^2}$

Slopes m and $-\frac{1}{m}$ are orthogonal.

Orthogonal trajectories have slope

$$\frac{dx}{dy} = -\frac{1}{2ky}$$ (differential equation)

Eliminate k $\quad = -\frac{1}{2\left(\frac{x}{y^2}\right)y} = \frac{-1}{2\left(\frac{x}{y}\right)} = -\frac{y}{2x}$

$k = \frac{x}{y^2}$

$$\frac{dx}{dy} = \frac{-y}{2x}$$

Separate $\int 2x\,dx = \int -y\,dy$

$$x^2 = -\frac{1}{2}y^2 + C$$

$$x = \pm\sqrt{C - \frac{1}{2}y^2}$$

Family of orthogonal trajectories

Mixing problems

A tank containes 20kg of salt dissolved in 5000L of water. Brine that contains 0.03kg of salt per liter of water enters the tank at a rate of 25L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after an hour?

$S(t) = $ kg salt at time t min

$S(0) = 20$

$$\text{Flow in} = 0.03 \frac{\text{kg}}{\text{L}} \times 25 \text{ L/min}$$

$$= 0.75 \frac{\text{kg}}{\text{min}}$$

$$\text{Flow out} = \frac{S(t)\text{ kg}}{5000\text{ L}} \times 25\,\frac{\text{L}}{\text{min}}$$

$$= \frac{25}{5000} S(t) = \frac{S(t)}{200}$$

$$\frac{dS}{dt} = \text{Flow in} - \text{Flow out}$$

$$= 0.75 - \frac{S(t)}{200}$$

This is separable.

$$\int \frac{dS}{0.75 - \frac{S}{200}} = \int dt = t + C_1$$

$$-200 \ln\left|0.75 - \frac{S}{200}\right| = t + C_1$$

$$\ln\left|0.75 - \frac{S}{200}\right| = -\frac{t}{200} + C_2$$

$$0.75 - \frac{S(t)}{200} = \pm e^{-\frac{t}{200} + C_2}$$

$$= \pm A_1 e^{-\frac{t}{200}}$$

$$0.75 - \frac{S(t)}{200} = A_2 e^{-\frac{t}{200}}$$

$$\frac{S(t)}{200} = 0.75 - A_2 e^{-\frac{t}{200}}$$

$$S(t) = 150 - 200 A_2 e^{-\frac{t}{200}}$$

$$S(0) = 20 = 150 - 200 A_2$$

$$200 A_2 = 130$$

$$S(t) = 150 - 130 e^{-\frac{t}{200}}$$

After 1 hour $(t=60)$

$$S(60) = 150 - 130\,e^{-\frac{60}{200}}$$

$$= 150 - 130\,e^{-\frac{3}{10}}$$